SOME MODELS OF NONIDEAL BOSE GAS WITH BOSE-EINSTEIN CONDENSATE

Dedicated with admiration to Prof. Ludwig Streit on his 60^{th} birthday

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Abstract

Using the functional integral techniques homogeneous limits of the perturbations of thermal states (describing nonrelativistic Bose Matter at the thermal equilibrium) by bounded cocycles are constructed rigorously. Additionally some elementary properties of these limiting states are discussed and in particular the preservation of the nonpurity in the critical case is proved.

1 Introduction

1.1 The problem of Bose-Einstein condensate

One of the most spectacular achievements in the experimental low temperature physics of the past few years is the laboratory realization of the Bose-Einstein condensate (BEC) of cold atoms [1, 2, 3]. This bizarre quantum state of a Bose Matter is formed at nanokelvin temperatures and requires also high atomic densities. The BEC is formed when the quantum wave packets of atoms overlap at low temperatures and the atoms condense almost motionless, into the lowest quantum state. This means that the wave-lenght of the matter waves associated with the cold atoms [4], the de Broglie waves, become comparable in size to the mean atomic distances in a cold and dense sample.

The phenomenon of BEC was predicted by Bose [4] and Einstein [5] already in 1924-25. But even for the case of simplest ideal Bose Gas the complete mathematical proof took a long period of time ended successfully only in 70-ties by the elaborations of the Dublin group [7, 8].

Concerning the standard, gauge-invariant many body hamiltonians with realistic pair interatomic potentials the situation from the theoretical physics point of view is still very far from being clarified. It seems that the class of models which is closest to realistic systems and being tractable mathematically is that described by the so called model systems with diagonal hamiltonians [8].

In a series of papers [9, 10, 11, 12] new mathematical technologies for studying the nonrelativistic Bose Matter at thermal equilibrium and in nonzero temperature were invented. They are based on the observation that the modular structure of the free Bose Matter has the so called stochastic positivity property [11]. It is this property which enables us to use certain functional integral/random field description of these genuine noncommutative structures. Additionally those commutative analysis methods open the doors for applications of the methods of *classical* statistical mechanics for studying the perturbations of the free thermal structures by the so called thermo-field like perturbations. Such programme was initiated in [9, 10] and developed in certain directions [11, 12]. The present contribution continues the analysis of [9, 10] for the so called gentle perturbations of the free Bose Matter. A new, purely commutative strategy for studying the standard many-body hamiltonians initiated in [13] has been extended recently in [12, 16], see also the activity [17].

1.2 The thermal structure of the Ideal Bose Gas (IBG)

Let $\mathcal{W}$ be a *-algebra version of the Weyl algebra over the one-particle Hilbert space $\boldsymbol{h} = L^2(R^d, dx)$ with generic elements denoted as $\widetilde{W}_f$, $f \in \boldsymbol{h}$. The one-particle unitary dynamics U_t^0 in $\boldsymbol{h}$ is generated by $h_0^\mu = -\Delta - \mu 1$, where $-\Delta$ is the (Friedrichs) Laplacean and μ is the chemical potential. The natural lifting $\widetilde{\alpha}_t^0$ of U_t^0 is defined as $\widetilde{\alpha}_t^0 \widetilde{W}_f = \widetilde{W}_{U_t^0 f}$.

The free thermal state ω_0 on $\mathcal{W}$ is given by [18]

$$\omega_0(\widetilde{W}_f) = \exp -\frac{1}{4}\langle f| \coth \frac{\beta}{2} h_0^\mu f\rangle \tag{1}$$

and the corresponding free thermal state ω_0^{cr} for IBG containing Bose-Einstein condensate

$$\omega_0^{cr}(\widetilde{W}_f) = \exp -\frac{1}{2}c|\widehat{f}(0)|^2 \cdot \omega_0(\widetilde{W}_f)\big|_{\mu=0} \tag{2}$$

which is well defined, providing $d \geq 3$, and where $\widehat{f}$ is the Fourier transform of $f \in L^1 \cap L^2(R^d, dx)$, $c > 0$ is some constant depending on details of the thermodynamical passage and measuring to some extent the size of the condensed fraction of gas. Both states, ω_0 and ω_0^{cr} are invariant under the action of $\widetilde{\alpha}_t^0$ ($\widetilde{\alpha}_t^0 \uparrow \mu = 0$ in the case of ω_0^{cr}).

Applications of the GNS construction lead to the well known Araki-Wood thermal modules $(\mathcal{H}_0^{(cr)}, \Omega_0^{(cr)}, \pi^{0,(cr)}, U_t^{0,(cr)})$. We define the corresponding von Neumann algebras $\mathcal{M}_0$ (resp. $\mathcal{M}_0^{cr}$) as weak closures of $\pi^0(\mathcal{W})$ (resp. of $\pi^{0,(cr)}(\mathcal{W})$). Then the systems

$$(\mathcal{M}_0, \alpha_t^0, \omega_0) \equiv \mathcal{A}_0, \quad (\mathcal{M}_0^{cr}, \alpha_t^{0,cr}, \omega_0^{cr}) \equiv \mathcal{A}_0^{cr}$$

form W*-KMS systems (and where $\alpha_t^{0,(cr)}$ are the corresponding extensions of $U_t^{0,(cr)}$). The presence of the Bose-Einstein condensate in $\mathcal{A}_0^{cr}$ is manifested throughout the nonpurity of $\mathcal{A}_0^{cr}$. The central decomposition of ω_0^{cr} is well known since Araki-Wood work [19].

In the present contribution we shall study the (homogeneous limits of) perturbations of $\mathcal{A}_0^{(cr)}$ throughout the unitary cocycles perturbations [18] and in particular the question on the preservation of the nonpurity of the limiting thermal states (in the critical regime) will be settled up positively. The techniques employed are based on the functional integral approach invoked in [9, 10, 11].

2 The main result

Let $(\chi_\epsilon)_{\epsilon>0}$ be a net from $C_0(R^d)$ (the space of continuous functions with compact support), such that $\chi_\epsilon \geq 0$ pointwise and

$$\lim_{\epsilon\downarrow 0} \chi_\epsilon = \delta$$

in the weak sense. For $x \in R^d, \epsilon > 0$ and $\alpha \in R$ we define

$$W_\epsilon(\alpha, x) = W_{\alpha\cdot\chi_\epsilon(\cdot - x)}$$

where W_f is now the representative of $\widetilde{W}_f$ in the corresponding free thermal module. Let $d\rho$ be a (complex in general) Borel measure on R, with compact support and such that $d\rho(-\alpha) = \overline{d\rho(\alpha)}$. For $\Lambda \subset R^d$ being bounded region we define

$$H_\Lambda^L = \lambda \int d\rho(\alpha) \int_\Lambda dx\, W_\epsilon(\alpha, x) \tag{3}$$

where $\lambda \in R$ is called coupling constant, and

$$H_\Lambda^{nL} = \lambda \int d\rho(\alpha) \int d\rho(\beta) \int_\Lambda dx \int_\Lambda dy\, W_\epsilon(\alpha, x)\Gamma(x-y)W_\epsilon(\beta, y) \tag{4}$$

where $\Gamma \in L^1(R^d)$ and integrals are defined in the σ-weak topology of the corresponding W*-algebras. From the simple estimates

$$\begin{aligned} ||H_\Lambda^L|| &\leq |\lambda| Var(\rho)|\Lambda| \\ ||H_\Lambda^{nL}|| &\leq |\lambda| \cdot |Var(\rho)|^2 ||\Gamma||_1 \cdot |\Lambda| \end{aligned}$$

where $Var(\rho)$ is the variation of ρ and $|\Lambda|$ means the volume of Λ it follows that the unitary cocycles perturbations theory [18] for studying the perturbed (by $H_\Lambda^{\#}$) free dynamics can be applied. For this goal we define:

• the unitary cocycle $\Gamma_t^{\#,\Lambda}$:

$$\Gamma_t^{\#,\Lambda} \equiv 1 + \sum_{n\geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \alpha_{t_n}^{0,(cr)}(H_\Lambda^\#) \cdots \alpha_{t_1}^{0,(cr)}(H_\Lambda^\#) \tag{5}$$

where $\#$ stands for L or nL and the free dynamics α_t^0 acts as

$$\alpha_t^0(H_\Lambda^L) = \lambda \int d\rho(\alpha) \int_\Lambda dx W_{\alpha U_t^0 \chi_\epsilon(\cdot - x)}$$

and similarly for H_Λ^{nL},

• the perturbed dynamics $\alpha_t^{\#,\Lambda}$:

$$\begin{aligned} \alpha_t^{\#,\Lambda}(A) &= \Gamma_t^{\#,\Lambda}(A)\Gamma_t^{\#,\Lambda\,*} \\ &= \alpha_t^{0,(cr)}(A) + \sum_{n\geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ &\quad [\alpha_{t_n}^{0,(cr)}(H_\Lambda^\#), [\cdots [\alpha_{t_1}^{0,(cr)}(H_\Lambda^\#), A] \cdots]\,] \end{aligned} \tag{6}$$

for $A \in \mathcal{A}_0^{(cr)}$,

• the perturbed thermal vacuum $\Omega_\Lambda^\#$:

$$\begin{aligned} \Omega_\Lambda^{\#,(cr)} &= \Omega_0^{(cr)} + \int_{0\leq s_1 \leq \cdots \leq s_n \leq \frac{\beta}{2}} ds_1 \cdots ds_n \\ &\quad e^{-s_n H_0} H_\Lambda^\# e^{-(s_n - s_{n-1})H_0} \cdots H_\Lambda^\# \Omega_0^{(cr)} \end{aligned} \tag{7}$$

where $H_0^{(cr)}$ is the generator of $\alpha_t^{0,(cr)}$ in $\mathcal{H}_0^{(cr)}$ and again the integrals and series are weakly convergent in $\mathcal{H}_0^{(cr)}$.

Remark 1 From the Araki theorem [18, 20] it follows that there is a holomorphic map:

$$R^n + iT_n^{\beta/2} \ni (z_1, \ldots, z_n) \to e^{iz_n H_0} H_\Lambda^\# e^{i(z_n - z_{n-1})H_0} \cdots H_\Lambda^\# \Omega_0^{(cr)} \in \mathcal{H}_0^{(cr)}$$

where

$$T_n^{\beta/2} = \{(s_1, \ldots, s_n) \,|\, 0 < s_1 < \cdots < s_n < \beta/2\},$$

continuous on $R^n + i\overline{T}_n^{\beta/2}$ and additionally obeying the estimate (uniformly on $R^n + i\overline{T}_n^{\beta/2}$)

$$\begin{aligned} ||e^{iz_n H_0} H_\Lambda^\# & e^{i(z_n - z_{n-1})H_0} \cdots H_\Lambda^\# \Omega_0^{(cr)}|| \\ &\leq |\lambda|^n (Var(\rho))^n \cdot |\Lambda|^n \quad \text{for} \quad \# = L \\ &\leq |\lambda|^n (Var(\rho))^{2n} \cdot ||\Gamma||_1^n \cdot ||\Lambda||^n \quad \text{for} \quad \# = nL \end{aligned}$$

From the general theory [18] it follows that the systems

$$\mathcal{A}_\Lambda^{\#,(cr)} \equiv \{\mathcal{H}_0^{(cr)}, ||\Omega_\Lambda^{\#}||^{-1} \cdot \Omega_\Lambda^{\#}, \pi_0, \alpha_t^{\#,\Lambda}, \mathcal{A}_0^{(cr)}\}$$

form a β-KMS structures and such that the modular dynamics corresponding to the vector states $\omega_\Lambda^{\#}$ given by $\Omega_\Lambda^{\#}$ are exactly equal to $\alpha_t^{\#,\Lambda}$.

In the following we shall sometimes drop out the superscript # in the notation and the following abbrevations will be used:

$$\int_\Lambda d(\tau,\alpha,x)_1^n = \int_\Lambda dx_1 \cdots \int_\Lambda dx_n \int d\tau_1 \cdots \int d\tau_n \int d\rho(\alpha_1) \cdots \int d\rho(\alpha_n), \quad \text{etc.}$$

Now we are ready to formulate main results of this contribution.

Theorem 1 (The noncritical case) *1. There exists small* λ_0 *(depending on* $\beta, \#, \Gamma, \rho$*) such that for all* $|\lambda| < \lambda_0$ *the unique thermodynamic limit*

$$\lim_\Lambda \omega_\Lambda^{\#}(\lambda) \equiv \omega^{\#}(\lambda)$$

exists in the sense of weak convergence. The limiting state $\omega^{\#}(\lambda)$ *is faithfull on* $\mathcal{A}_0$ *and is ergodic with respect to the (natural) action of the Euclidean motions on* $\mathcal{A}_0$*. The limiting state is analytic in* λ *and entire analytic on* $\mathcal{A}_0$*.*
2. If additionally $d\rho(\alpha) = d\rho(-\alpha), \quad \lambda \geq 0$ *and* $\Gamma \geq 0$ *(pointwise) in the case* $\# = nL$*, then the unique thermodynamic limit*

$$\omega^{\#}(\lambda) = \lim_\Lambda \omega_\Lambda^{\#}(\lambda)$$

exists for all $\lambda \geq 0$ *and is faithfull and Euclidean invariant.*

Here is the main result:

Theorem 2 (The critical case) *There exists* λ_0 *(depending on* $\rho, \#, \Gamma, \ldots$*) such that the unique thermodynamic limit*

$$\lim_\Lambda \omega^{cr}(\lambda) \equiv \omega^{cr}(\lambda)$$

exists for all $|\lambda| < \lambda_0$*. The limiting state* $\omega^{cr}(\lambda)$ *is faithfull, Euclidean invariant and satisfies: there exists a Borel measure* $d\lambda_{ren}(\alpha,\theta)$ *on* $[0,\infty) \times [0,2\pi)$ *and a family of faithfull and ergodic (with respect to the translations) states* $\omega_{r,\theta}(\lambda)$*, indexed by* $[0,\infty) \times [0,2\pi)$ *such that*

$$\omega^{(cr)}(\lambda) = \int_0^\infty \int_0^{2\pi} d\lambda_{ren}(r,\theta)\omega_{r,\theta}(\lambda)$$

Moreover the measure $d\lambda_{ren}(r,\theta)$ *is not concentrated on a single point.*

3 Proofs: the main ideas

For $f, g \in \boldsymbol{h}$ such that $f = \overline{f}$, $g = \overline{g}$ and for any $\tau \in [0, \beta/2)$ it follows by the straightforward computations that [9, 10]:

$$\langle \Omega_0^{(cr)} | W_f e^{-\tau H_0^{(cr)}} W_g \Omega_0^{(cr)} \rangle = \exp -\frac{1}{2} S_0^{\beta,(cr)}(\tau, f \otimes g) \tag{8}$$

where

$$S_0^{\beta,(cr)}(\tau, f \otimes g) = S_0^{\beta}(\tau, f \otimes g) + (c \int f(x)dx \int g(x)dx) \tag{9}$$

and

$$S_0^{\beta}(\tau, f \otimes g) = \langle f | \frac{e^{-\tau(-\Delta-\mu)} + e^{-(\beta-\tau)(-\Delta-\mu)}}{1 - e^{-\beta(-\Delta-\mu)}} g \rangle_{\boldsymbol{h}} \tag{10}$$

Extending $S^{\beta,(cr)}(\tau, \cdot, \cdot)$ to R^1 by reflection invariance and periodicity it follows that $S^{\beta,(cr)}(\tau, \cdot, \cdot)$ is stochastically positive and reflection positive on $K_\beta = (-\beta/2, \beta/2)$ continuous kernel. Therefore, there exists a Gaussian process $\xi^{0,(cr)}$ with values in $\mathcal{S}'(R^d)$, which is faithfull, stochastically continuous, periodic and reflection positive, and satisfies

$$\mathbb{E}(\xi_\tau^{0,(cr)}, \cdot)(\xi_0^{0,(cr)}, \cdot) = S_0^{\beta,(cr)}(\tau, \cdot, \cdot) \tag{11}$$

The law of this process can be identified with the Gaussian random field $\mu_0^{\beta,(cr)}$ on the space $\mathcal{S}'(K_\beta \times R^d)$ such that

$$\mathbb{E}_{\mu_0^{\beta,(cr)}} \varphi(\tau, x)\varphi(0, y) = S_0^{\beta,(cr)}(\tau, x - y) \tag{12}$$

We define the following functionals of $\mu_0^{\beta,(cr)}$:

$$U_\Lambda^L(\varphi) = \lambda \int_0^\beta d\tau \int d\rho(\alpha) \int dx : e^{i\alpha\varphi_\epsilon(\tau,x)} : \tag{13}$$

and

$$U_\Lambda^{nL} = \lambda \int_0^\beta d\tau \int d\rho(\alpha) \int d\rho(\beta) \int_\Lambda dx \int_\Lambda dy e^{i\alpha\varphi_\epsilon(\tau,x)} \Gamma(x-y) e^{i\beta\varphi_\epsilon(\tau,y)} \tag{14}$$

where $\varphi_\epsilon(\tau, x) = (\varphi * \chi_\epsilon)(\tau, x)$

The finite volume Gibbsian perturbed measures $d\mu_\Lambda^{\beta,(cr)}$ are defined as

$$d\mu_\Lambda^{\beta,(cr)}(\varphi) = (Z_\Lambda^{\beta,(cr)})^{-1} \exp U_\Lambda^{\#}(\varphi) \, d\mu_0^{\beta,(cr)}(\varphi)$$

and their corresponding canonical correlation functions $\rho_\Lambda^{(cr)}(\tau_1, \alpha_1, x_1, \dots, \tau_n, \alpha_n, x_n)$ as

$$\rho_\Lambda^{(cr)}(\tau_1, \alpha_1, x_1, \dots, \tau_n, \alpha_n, x_n) = \lambda^n \mathbb{E}_{\mu_\Lambda} e^{i\alpha_1 \varphi_\epsilon(\tau_1, x_1)} \dots e^{i\alpha_n \varphi_\epsilon(\tau_n, x_n)}$$

Recall, that their thermodynamic limits in various ranges of couplings were controlled rigorously in previous publications [9, 10].

Using definition of ω_Λ it follows by straitforward computations that

$$Z_\Lambda^{\beta,(cr)} = \langle \Omega_\Lambda^{(cr)} | \Omega_\Lambda^{(cr)} \rangle_{\mathcal{H}_0^{(cr)}} \tag{15}$$

and for $f \in \boldsymbol{h}$

$$\begin{aligned} \omega_\Lambda^{(cr)}(W_f) &= \langle \Omega_\Lambda^{(cr)} | W_f \Omega_\Lambda^{(cr)} \rangle = \exp -\frac{1}{2} S_0^{\beta,(cr)}(0, f \otimes f) \\ &\cdot \sum_{n \geq 0} [\frac{1}{n!} \int_{\Lambda,\beta} d(\tau, \alpha, x)_1^n \prod_{j=1}^n e^{iIm\langle f | \alpha_j \chi_{\epsilon, i\tau_j} \rangle (x_j)} \\ &\prod_{j=1}^n (e^{-\alpha_j (S_{0,\epsilon}^{\beta,(cr)} * f)(\tau_j, x_j)} - 1) \rho_\Lambda^{(cr)}(\tau, \alpha, x)_1^n] \end{aligned} \tag{16}$$

(where $S_{0,\epsilon}^{\beta,(cr)} = (\chi_\epsilon \otimes \chi_\epsilon) * S_0^{\beta,(cr)}$) for U_Λ^L case, and similarly (albeit more complicated) in the case of U_Λ^{nL}:

$$\begin{aligned} &\omega_\Lambda^{(cr)}(W_f) = \exp -\frac{1}{2} S_0^{\beta,(cr)}(0, f \otimes f) \\ &\cdot \sum_{N,M,L \geq 0} \frac{1}{N!M!L!} \Big[\int_{\Lambda,\beta} d(\tau^1, \alpha^1, x^1)_1^N \, d(\sigma^1, \beta^1, y^1)_1^N \\ &\int_{\Lambda,\beta} d(\tau^2, \alpha^2, x^2)_1^M \, d(\sigma^2, \beta^2, y^2)_1^M \int_{\Lambda,\beta} d(\tau^3, \alpha^3, x^3)_1^L \, d(\sigma^3, \beta^3, y^3)_1^L \\ &\cdot \prod_{j=1}^N \Big\{ \exp(iIm\langle f | \alpha_j^1 \chi_{\epsilon, i\tau_j^1}(\cdot - x_j^1) \rangle) \exp(iIm\langle f | \beta_j^1 \chi_{\epsilon, i\sigma_j^1}(\cdot - y_j^1) \rangle \Gamma(x_j^1 - y_j^1) \\ &\big[\exp(-\alpha_j^1 S_{0,\epsilon}^{\beta,(cr)} * f(x_j^1)) - 1 \big] \big[\exp(-\beta_j^1 S_{0,\epsilon}^{\beta,(cr)} * f(y_j^1)) - 1 \big] \Big\} \\ &\cdot \prod_{j=1}^M \Big\{ \exp(iIm\langle f | \alpha_j^2 \chi_{\epsilon, i\tau_j^2}(\cdot - x_j^2) \rangle) \exp(iIm\langle f | \beta_j^2 \chi_{\epsilon, i\sigma_j^2}(\cdot - y_j^2) \rangle \Gamma(x_j^2 - y_j^2) \\ &\big[\exp(-\alpha_j^2 S_{0,\epsilon}^{\beta,(cr)} * f(x_j^2)) - 1 \big] \Big\} \\ &\cdot \prod_{j=1}^L \Big\{ \exp(iIm\langle f | \alpha_j^3 \chi_{\epsilon, i\tau_j^3}(\cdot - x_j^3) \rangle) \exp(iIm\langle f | \beta_j^3 \chi_{\epsilon, i\sigma_j^3}(\cdot - y_j^3) \rangle \Gamma(x_j^3 - y_j^3) \end{aligned}$$

$$\left[\exp(-\beta^3_j S^{\beta,(cr)}_{0,\epsilon} * f(y^3_j)) - 1\right]\Big\}$$

$$\cdot\rho^{(cr)}_{\Lambda}((\tau^1, \alpha^1, x^1)^N_1, \ldots, (\sigma^3, \beta^3, y^3)^L_1)\Big] \tag{17}$$

where

$$\chi_{\epsilon,i\tau}(\cdot - x) = U^0_{i\tau}\chi_\epsilon(\cdot - x) \tag{18}$$

and

$$\rho^{(cr)}_{\Lambda}((\tau^1, \alpha^1, x^1)^N_1, \ldots, (\sigma^3, \beta^3, y^3)^L_1) =$$

$$\lambda^{N+M+L} \int\limits_{\mathcal{S}'(R^d \times K_\beta)} d\mu^{(cr)}_{\Lambda}(\varphi)\Big\{\prod_{j=1}^{N}\left(e^{i\alpha^1_j\varphi_\epsilon(\tau^1_j,x^1_j)}e^{i\beta^1_j\varphi_\epsilon(\tau^1_j,y^1_j)}\right)$$

$$\prod_{j=1}^{M}\left(e^{i\alpha^2_j\varphi_\epsilon(\tau^2_j,x^2_j)}e^{i\beta^2_j\varphi_\epsilon(\tau^2_j,y^2_j)}\right)\prod_{j=1}^{L}\left(e^{i\alpha^3_j\varphi_\epsilon(\tau^3_j,x^3_j)}e^{i\beta^3_j\varphi_\epsilon(\tau^3_j,y^3_j)}\right)\Big\} \tag{19}$$

Formulae (16) and (17) provide a link between the analysis on the abelian sectors described in the earlier papers [9, 10] and the corresponding states $\omega^{(cr)}_{\Lambda}$ on the whole algebra(s) of observables $\mathcal{M}^{(cr)}_0$. This is why we propose to call them the reduction formula(e) for the state(s).

The essential volume dependence of $\omega^{(cr)}_{\Lambda}$ is that of the canonical correlation functions entering in formulas (16) and (17). To control the thermodynamic limits $\lim_{\Lambda}\omega^{(cr)}_{\Lambda}$ we have to divide the analysis into two parts, the first (easier) devoted to the noncritical case and the second dealing with the critical one.

3.1 The case of noncritical IBG

The following results have been proved in [9]: There exists a small λ_0 (depending on $z, \beta, \ldots$) such that the unique thermodynamic limits

$$\lim_{\Lambda}\rho^{(nL)}_{\Lambda}(\tau, \alpha, x)^N_1 = \rho^{(nL)}_{\lambda}(\tau, \alpha, x)^N_1$$

exists for all N and $|\lambda| < \lambda_0$. The limiting correlation functions $\rho^{(nL)}_{\lambda}(\tau, \alpha, x)^N_1$ are Euclidean invariant, possess the cluster decomposition property and are analytic in the disc $|\lambda| < \lambda_0$. Moreover $\rho^{(nL)}_{\Lambda} \to \rho^{(nL)}_{\lambda}$ locally uniformly. See Prop. 3.2 and Prop. 2.4 in [9].

Applying the formulated results, the proof of the first part of Theorem 1 follows whence using the reduction formulae (16) and (17). If additionaly we assume that the measure $d\rho$ is real and even, $\lambda \geq 0$ and $\Gamma \geq 0$ pointwise (in the case of nL), then we can use Theorem 3.9 of [9] to controll the limit $\lim_{\Lambda}\rho^{(nL)}_{\Lambda} = \rho^{(nL)}_{\lambda}$. Modulo the cluster decomposition property, the limiting canonical correlation functions possess most of the properties as in the local perturbations case and this enables us to prove the second part of Theorem 1 similarly as above.

3.2 The case of critical IBG

The main difficulty here is that $S_0^{\beta,cr}$ has no long range decay and this is why the high temperature/low density methods (Kirkwood-Salsburg identities, cluster expansions) do not apply straightforwardly to study the limit(s) $\lim_{\Lambda} \rho_\Lambda^{cr}$. However, such an analysis is possible in the pure phases. For this goal, let $K_{2\pi}$ be the circle of radius 2π and let $d\lambda_0$ be the spectral measure of the state ω_0^{cr} on $K_{2\pi} \times (0, \infty)$ i.e.

$$\omega_0^{cr} = \int d\lambda_0(r, \theta)\omega_{r,\theta}^{cr} \tag{20}$$

where $\omega_{r,\theta}^{cr}$ are pure states on $\mathcal{M}_0^{cr}$ characterised by

$$\omega_{r,\theta}^{cr}(W_f) = e^{ic^{1/2}r^{1/2}\cos\theta \cdot \widehat{f}(0)} e^{-\frac{1}{2}S_0^{\beta}(0, f\otimes f)} \tag{21}$$

The explicite form of $d\lambda_0$ is well known [18, 19]. It was observed in [10] that the states $\omega_{r,\theta}^{cr}$ are stochastically positive and the underlying periodic processes $\xi_\tau^{(r,\theta)}$ with values in $\mathcal{S}'(R^d)$ are Gaussian processes with covariances given by (informally):

$$\mathbb{E}\xi_\tau^{(r,\theta)}(x)\xi_0^{(r,\theta)}(y) = \frac{e^{-|\tau|(-\Delta)} + e^{-(\beta-|\tau|)(-\Delta)}}{1 - e^{-\beta(-\Delta)}} \tag{22}$$

and means

$$\mathbb{E}\xi_\tau^{(r,\theta)}(x) = c^{1/2}r^{1/2}\cos\theta \tag{23}$$

The corresponding random fields $\mu_{r,\theta}^{\beta,cr}$ on the space $\mathcal{S}'(K_\beta \times R^d)$ are Gaussian and with second moments given by (informally):

$$\int\limits_{\mathcal{S}'(K_\beta \times R^d)} \varphi(0, x)\varphi(\tau, y)d\mu_{r,\theta}^{\beta,cr}(\varphi) = S_0^{\beta}(\tau, x - y) \tag{24}$$

and means:

$$\int\limits_{\mathcal{S}'(K_\beta \times R^d)} \varphi(o, x)d\mu_{r,\theta}^{\beta,cr}(\varphi) = c^{1/2}r^{1/2}\cos\theta \tag{25}$$

The perturbed, finite-volume Gibbs measures $d\mu_\Lambda^{(r,\theta)}$ are defined as:

$$d\mu_\Lambda^{(r,\theta)}(\varphi) = (Z_\Lambda^{(r,\theta)})^{-1} \exp U_\Lambda^{\#}(\varphi) d\mu_{r,\theta}^{\beta,cr}(\varphi) \tag{26}$$

where

$$Z_\Lambda^{(r,\theta)} = \int\limits_{\mathcal{S}'(K_\beta \times R^d)} \exp U_\Lambda^{\#}(\varphi) d\mu_\Lambda^{(r,\theta)}(\varphi) \tag{27}$$

The thermodynamic limits

$$\lim_\Lambda \mu_\Lambda^{(r,\theta)} = \mu_\lambda^{(r,\theta)}$$

were controlled rigorously in [10] and in particular the existence of the thermodynamic limits of the correlation functions $\rho_\Lambda^{(r,\theta)}((\tau,\alpha,x)_1^n)$ defined as

$$\rho_\Lambda^{(r,\theta)}((\tau,\alpha,x)_1^n) = \lambda^n \int\limits_{\mathcal{S}'(K_\beta \times R^d)} \prod_{j=1}^n e^{i\alpha_j \varphi(\tau_j, x_j)} d\mu_\Lambda^{(r,\theta)} \tag{28}$$

and for small values of couplig constant λ were proved in [10].

Similarly as in the noncritical case the following reduction formula (writttten only for the local perturbations only) holds:

$$\begin{aligned} \omega_\Lambda^{(r,\theta)}(W_f) \quad = \quad & \text{similar as in (16) but with } \rho_\Lambda^{(r,\theta)} \text{ instead of } \rho_\Lambda \\ & \text{and with } S^{(r,\theta)} \text{ instead of } S_0^{\beta,(cr)} \end{aligned} \tag{29}$$

and where $\omega_\Lambda^{(r,\theta)}$ is the state obtained from $\omega_{r,\theta}^{(cr)}$ by the unitary cocycle(s) perturbations as in the noncritical case. Using the reduction formula (29) and results of [10] the existence of the unique limits

$$\lim_\Lambda \omega_\Lambda^{(r,\theta)} = \omega_\lambda^{(r,\theta)}$$

as a weak limits on $\mathcal{M}_0^{cr}$ and for all $(r,\theta) \in (0,\infty) \times K_\beta$ follows. In particular the limiting states $\omega_\lambda^{(r,\theta)}$ are Euclidean invariant, pure and faithfull. Using the central decomposition (20) the following formula can be derived easily:

$$\omega_\Lambda^{cr}(W_f) = \int d\lambda_0(r,\theta) \omega_{r,\theta}^{cr}(W_f) \frac{||\Omega_\Lambda^{(r,\theta)}||^2}{||\Omega_\Lambda^{cr}||^2} \tag{30}$$

where $\Omega_\Lambda^{(r,\theta)}$, Ω_Λ^{cr} are the corresponding vacuums given by the formula (7). From the reduction formulae (29) and (30) it follows easily that

$$\sup_\Lambda \omega_\Lambda^{cr}(W_f) < \infty$$

for any $f \in \mathcal{S}(R^d)$. Using this observation it follows by a similar arguments to that presented in [10] that there exists

$$\lim_\Lambda d\lambda_0(r,\theta) \frac{||\Omega_\Lambda^{(r,\theta)}||^2}{||\Omega_\Lambda^{cr}||^2} = d\lambda_{ren}(r,\theta) \tag{31}$$

in the sense of measures. That the measure $d\lambda_{ren}(r,\theta)$ is not concentrated on a single point was proved in .

Summarizing this discussion we have obtained the following result:

$$\omega_\lambda^{cr}(W_f) = \int\limits_{K_\beta\times(0,\infty)} d\lambda_{ren}(r,\theta)\omega_\lambda^{(r,\theta)}((W_f) \tag{32}$$

for sufficiently small λ and the measure $d\lambda_{ren}(r,\theta)$ is not concentrated on a single point which means that the limiting state ω_λ^{cr} is not the pure state. Moreover the states $\omega_\lambda^{(r,\theta)}$ appearing in (32) are pure states on $\mathcal{M}_0^{cr}$.

4 Concluding remarks

From the Tomita-Takesaki theory it follows that there exists a canonical (modular) dynamics α_t^{TT} on $\mathcal{M}_0^{(cr)}$ such that the constructed states $\omega_\lambda^{(cr)}$ are KMS states with respect to α_t^{TT}. On the other hand from the corresponding abelian euclidean-time Green functions some (a priori) another W*-KMS structure(s) can be constructed [9, 10]. This yields the difficult (and therefore very interesting) question whether these two a priori different W*-KMS structures do coincide. In the finite volume perturbation case it has been proved in [15] that these W*-KMS structures coincide. Concerning the infinite volume situation the following result has been proved in [15]:
Theorem *Let ξ_t^λ be the corresponding to the infinite volume limits of the perturbations considered in this contribution thermal processes. Then, these processes are Markovian diffusions on the circle K_β providing λ is sufficiently small.*
From the above theorem it follows that the corresponding thermal vacuum Ω_λ^A constructed from the abelian euclidean-time Green functions (see [9, 10] for details) has the following cyclicity property:
the linear hull of the vectors

$$W_f^A e^{-\tau H_\lambda^A} W_g^A \Omega_\lambda^A, \quad f=\overline{f}, g=\overline{g}, \tau\in[0,\beta/2)$$

in the corresponding Hilbert space $\mathcal{H}^A$ is dense.
However, the still missing point of the identification of the canonical Tomita-Takesaki W*-KMS structure with those obtained in [9, 10] is the question whether the vacuum Ω_λ^A is cyclic under the action of the time - 0 Weyl algebra $\mathcal{W}(\boldsymbol{h})$ in $\mathcal{H}^A$.

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