UNBOUNDED PERTURBATIONS OF THE BOSE-EINSTEIN CONDENSATE

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Abstract. New, commutative probability tools for studying the problem of the Bose-Einstein condensate preservation under thermofield and standard gauge-invariant perturbations are presented. In particular, a new result on the stability of a Bose-Einstein condensate under the thermofield perturbations of a polynomial type is presented.

I. Introduction: The problem of the Bose-Einstein condensate.

The problem of the Bose-Einstein condensate preservation that do arises in the free Bose Matter under switching on any realistic two-body interactions between particles has still remained open despite of its long history and efforts of many theoretists. Even in the case of noninteracting Bose particles, the process of providing a complete and rigorous (from the mathematical physics point of view) proof of Einstein's heuristic indications [1] took a very long time period during which Mark Kac's fundamental contribution [2] played a crucial role. It is due to the long-time activity of Dublin's group that Mark Kac's ideas have been cleaned up and, at present, a very clear, general understanding of the condensation phenomenon in noninteracting Bose systems is available [3].

Let (Λ_n) be a monotonous family of bounded regions in the Euclidean space $\mathbb{R}^d$, $d \geq 3$, such that $\bigcup_n \Lambda_n = \mathbb{R}^d$ and let (h_n) be a family of one particle kinetic energy operators that possess purely discrete spectra

$$\sigma(h_n) = \{\lambda_n(k)\}_k$$

with the lowest egenvalues $\lambda_n(1) \geq 0$ for all n. We define also

$$\sigma_n(k) = \lambda_n(k) - \lambda_n(1), \quad \text{for} \quad k = 2, 3, \dots$$

Then we will say that the family (Λ_n, h_n) is admissible iff:

$$\text{(ad)}: \qquad \begin{cases} \text{for all } \beta > 0, \ \lim\limits_n \dfrac{1}{|\Lambda_n|} \displaystyle\sum_{k=1}^{\infty} \exp -\beta\sigma_n(k) \equiv \phi^\infty(\beta) \\ \text{exists and for some } \ \beta \in (0,\infty), \ \phi^\infty(\beta) \neq 0 \end{cases} \tag{1}$$

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The following result (formulated informally) seems to be the most general one for noninteracting systems [3]. Let us define the partition function

$$Z_n(\beta,\mu) = \mathrm{Tr}\,_{\Gamma(L^2(\Lambda_n,dx))} \exp -\beta(d\Gamma(h_n+\mu 1) \tag{2}$$

and the finite volume free energy density

$$p_n(\beta,\mu) = \frac{1}{|\Lambda_n|}\ln Z_n(\beta,\mu) \tag{3}$$

where $|\Lambda|$ stands for the volume of Λ and $\Gamma(L^2(\Lambda,dx))$ is the bosonic Fock space on $L^2(\Lambda,dx)$.

Theorem ([3]). *Let (Λ_n,h_n) be an admissible family. Then*

(1) *the unique limit*

$$p_\infty(\beta,\mu) = \lim_n p_n(\beta,\mu)$$

exists

(2) *the equation*

$$\frac{d}{d\mu}p_\infty(\beta,\mu) = \rho$$

has the unique solution for any $\rho \in (o,\infty)$ and such that for any $\rho \geq \rho_{cr}(\beta)$, where

$$\rho_{cr}(\beta) = \int\limits_{(0,\infty)} (e^{\beta\lambda}-1)^{-1}dF(\lambda) \tag{4}$$

the corresponding $\mu = 0$.

The density of states dF in (4) is uniquelly determined by

$$\phi^\infty(\beta) = \int\limits_{(0,\infty)} e^{-\beta\lambda}dF(\lambda) \tag{5}$$

Now, let Λ_n be a tori and let Δ_n^p be the corresponding periodic b.c. Laplace operator. The standard, gauge-invariant, many body interactions with an interparticle two-body potential V [3, 4] can be decomposed as

$$H_n^{int} = H_n^{diag} + \delta H_n$$

where, the so called diagonal part H_n^{diag} of H_n^{int} is given explicitely as

$$H_n^{diag} = \frac{\widehat{V}(0)}{2|\Lambda_n|}(N^2-N) + \frac{1}{2|\Lambda_n|}\sum_k\sum_{k'\neq k}\widehat{V}(k-k')n_k n_{k'} \tag{6}$$

where N, n_k are the corresponding number operators and $\widehat{V}$ stands for the Fourier transform of V.

It is worthwhile to mention here that a variety of models, such as Huang-Yang-Luttinger model [6], mean field like models [8] etc. that were analysed intensively in the past correspond to certain approximations of H_n^{diag}. It seems that the result of [4] on the stability of the Bose-Einstein condensation under perturbation of the free Bose gas hamiltonian by the diagonal part of H_n^{int} is one of main achievements of the (mentioned above) Dublin group activity and in a sense it is a result which is closest to realistic interactions. Passing throughout the proofs in [4,7] it becomes clear that the opportunity to express the diagonal part of H_n^{int} as a bilinear form of the commuting random variables n_k is the basic feature which made the corresponding analysis so succesfull.

It is the main objective of the present contribution to present a new commutative tools for analysis of the perturbations of the free Bose matter by thermofield like interactions (next paragraph) and also by the standard gauge invariant many body interactions.

II. The Gaussian analysis: Thermofield perturbations.

Let $\mathfrak{W}$ be the Weyl algebra on the Hilbert space $L^2(\mathbb{R}^d, dx)$, the generic element of which we shall denote as $W_f,\ f \in L^2(\mathbb{R}^d, dx)$. The following faithfull states:

$$\omega_0^\beta(W_f) = \exp -\frac{1}{4}\langle f|\coth\frac{\beta}{2}h_\mu f\rangle_2, \quad h_\mu = -\Delta + \mu \tag{7}$$

and (for $d \geq 3$)

$$\omega_{0,cr}^\beta(W_f) = \exp -c|\widehat{f}(0)|^2 \exp -\frac{1}{4}\langle f|\coth\frac{\beta}{2}h_0 f\rangle_2 \tag{8}$$

$c > 0$, and $\widehat{f}$ is the Fourier transforms of f, are invariant under the following evolution

$$\alpha_t f = e^{-ith_\mu} f$$

and give rise by the GNS construction to Araki-Wood thermal modules [4]

$$\mathfrak{N}_{(cr)} \equiv (\mathfrak{H}^\omega_{(cr)}, \Omega^\omega_{(cr)}, \mathfrak{W}^\omega_{(cr)}, \alpha_t^\omega, \beta)$$

defined as free, noncritical (resp. critical) modular structures of noninteracting Bose matter. It was shown in [8, 9] that both $\mathfrak{N}$ and $\mathfrak{N}_{cr}$ are stochastically positive and stochastically determined i.e. there exists processes ξ_τ (resp. ξ_τ^{cr}) with values in the space $\mathcal{S}'(\mathbb{R}^d)$ ($\equiv$ the space of tempered distributions) and such that for $f_1, \dots, f_n$ real, $0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_n \leq \beta$

$$\mathbb{E} e^{i(\xi_{\tau_1}, f_1)} \dots e^{i(\xi_{t_n}, f_n)} = \omega^\beta_{0,(cr)}(\alpha_{i\tau_1} W_{f_1} \cdots \alpha_{i\tau_n} W_{f_n}) \tag{9}$$

and where on the right hand side there stand the corresponding Euclidean time Green function of the thermal structure $\mathfrak{N}_{(cr)}$. The process(es) $\xi_\tau^{(cr)}$ is (are) periodic, reflection positive on the circle K_β of radius β, shift invariant and Markovian on K_β. The corresponding law(s) can be realized as a Gaussian measures $d\mu^\beta_{0,(cr)}$ on the space of continuous loops

$$\mathcal{L}_c^\beta(\mathcal{S}'(\mathbb{R}^d)) = \{\varphi : K_\beta \to \mathcal{S}'(\mathbb{R}^d) \,|\, \varphi \quad \text{continuous}\}$$

and with the covariance(s) given explicitely as:

$$\begin{aligned}&\int\limits_{\mathcal{L}_c^\beta(\mathcal{S}'(\mathbb{R}^d))} d\mu_0^\beta(\varphi)\varphi(f\otimes\delta_0)\varphi(g\otimes\delta_\tau)\\&=\int dx\int dy\, f(x)\,\frac{e^{-\tau h_\mu}+e^{-(\beta-\tau)h_\mu}}{1-e^{-\beta h_\mu}}\,g(y)\end{aligned}\tag{10}$$

and, respectively in the critical case:

$$\begin{aligned}&\int\limits_{\mathcal{L}_c^\beta(\mathcal{S}'(\mathbb{R}^d))} d\mu_{0,cr}^\beta(\varphi)\varphi(f\otimes\delta_0)\varphi(g\otimes\delta_\tau)\\&=c'_\delta\widehat{f}(0)\widehat{g}(0)+\langle f|\frac{e^{-\tau h_0}+e^{-(\beta-\tau)h_0}}{1-e^{-\beta h_0}}\,g\rangle_2\end{aligned}\tag{11}$$

It is important to point out here that the canonical procedure of reconstructing the whole noncommutative modular structure of the free Bose matter from the process(es) ξ_τ (ξ_τ^{cr}) is well known [8, 10, 11]. It is the non ergodicity of the process ξ_τ^{cr} that reflects the presence of Bose-Einstein condensate, see i.e. [3, 5] in the $\mathfrak{N}_{cr}$ (the spontaneous breaking of $U(1)$-gauge invariance). In the papers [8,9,12] gentle perturbations of the free thermal structure(s) $\mathfrak{N}_{(cr)}$ have been studied using the idea that the central, affilated with $\mathfrak{W}^{\omega\,\prime\prime}$ operators acts as multiplication in the Gaussian space $L^2(\mathcal{L}_c^\beta(\mathcal{S}'(\mathbb{R}^d)),d\mu_0^\beta)$. In particular the preservation of the nonergodicity (in the thermodynamic limit) of the perturbed critical modular structure (thus confirming the stability of Bose-Einstein condensate under such perturbations) was demonstrated there. Here we shall report on the recent stability result for perturbations of polynomial types.

Let a_x (resp. a_x^+) denotes the corresponding annihilation (resp. creation) operators (in the Araki-Wood module(s) $\mathfrak{N}_{(cr)}$). The Local Polynomial interaction term is defined as

$$(\mathrm{LP})_\Lambda:\quad\begin{cases}H_\Lambda^{int}=-\lambda\displaystyle\int\limits_\Lambda dx\mathcal{P}(a_x^\epsilon+a_x^{+\,\epsilon})\\ \lambda\geq 0,\ \Lambda\subset\mathbb{R}^d\quad\text{bounded, } \mathcal{P}(x)\quad\text{polynomial bounded}\\ \text{from below, } a_x^\epsilon,\ a_x^{+\epsilon}\quad\text{are regularized properly bosonic}\\ \text{annihilation and creation operators}\end{cases}$$

The nonLocal Polynomial interaction term is defined as:

$$(\mathrm{nLP})_\Lambda:\quad\begin{cases}H_\Lambda^{int}=-\lambda\displaystyle\int\limits_\Lambda dx\,dy\,\mathcal{P}(a_x^\epsilon+a_x^{+\,\epsilon})F(x-y)\mathcal{P}(a_y^\epsilon+a_y^{+\,\epsilon})\\ \text{where } \lambda\geq 0, \text{ the kernel } F\in L^1(\mathbb{R}^d)\\ \text{and is positive definite}\end{cases}$$

The perturbed by local or nonlocal polynomial interactions as in $(\mathrm{LP})_\Lambda$ or $(\mathrm{nLP})_\Lambda$ thermal measure(s) μ^β_Λ is given by:

$$\mu^\beta_{\Lambda,(cr)}(d\varphi) = (Z_{\Lambda,(cr)}(\beta))^{-1} \exp \int_0^\beta d\tau \int_\Lambda dx\, H^{int}_\Lambda(\varphi)(\tau,x) \cdot d\mu^\beta_{0,(cr)}(\varphi) \tag{12}$$

where

$$Z_{\Lambda,(cr)}(\beta) = \int_{\mathcal{S}'(K_\beta\times\mathbb{R}^d)} d\mu^\beta_{0,(cr)}(\varphi) \exp \int_0^\beta d\tau \int_\Lambda dx\, H^{int}_\Lambda(\varphi)(\tau,x) \tag{13}$$

for the case of $(\mathrm{LP})_\Lambda$ and similarly in the case of $(\mathrm{nLP})_\Lambda$. As the good controll on the limits $\lim_\Lambda \mu^\beta_\Lambda$ may be obtained by applying the high temperature cluster expansion [12] the problem of constructing $\lim_\Lambda \mu^\beta_{\Lambda,(cr)}$ is much more complicated due to the long-range nature of the corresponding potential [13].

Let $\omega_{\Lambda,(cr)}$ stands for the state on $\mathfrak{W}''_{cr}$ obtained as the unitary-like cocycle perturbation of $\omega_{0,cr}$ where the underlying cocycle $\Gamma_t(\Lambda)$ is given (informally) by

$$\Gamma_t(\Lambda) = \exp\, it(H^0 + H^{int}_\Lambda) \exp -itH^0 \tag{14}$$

where H^0 is the free Bose gas hamiltonian. The main result of [12] is the following:

Theorem 2.1. *If*

$$\sup_\Lambda \int d\mu^\beta_{\Lambda,cr}(\varphi) e^{\varphi(f)} < \infty$$

then for sufficintly small λ (depending on $\beta, F, \dots$) there exists an unique limit $\lim_\Lambda \omega_{\Lambda,cr} \equiv \omega^\lambda_{cr}$ (as a weak limit) and the limiting state is faithfull and is not pure state on $\mathfrak{W}''_{cr}$.

The main ideas of the proof.

Step 1. From the Araki-Wood paper [14] (see also [5]) we know that

$$\omega^{cr}_0 = \int d\lambda_0(r,\theta)\omega^{cr}_{r,\theta} \tag{15}$$

where $d\lambda_o(r,\theta)$ is the spectral measure on $[0,\infty)\times K_{2\pi}$ and $\omega^{cr}_{r,\theta}$ are pure quasi-free β–KMS states on $\mathfrak{N}^\omega_{cr}$ given by:

$$\omega^{cr}_{r,\theta}(W_f) = e^{ic^{1/2}\cos\theta\cdot\widehat{f}(0)} e^{-\frac{1}{2}S^\beta_{\mu=0}(f\otimes f,0)} \tag{16}$$

where $S^\beta_{\mu=0}$ is given by (10) with $\mu=0$. Restricting $\omega^{cr}_{r,\theta}$ to the abelian sector of $\mathfrak{M}^\omega_{cr}$ it follows that for $f=\overline{f}$ there exist a Gaussian probability measures $d\mu^\beta_{r,\theta}$ on $\mathcal{S}'(K_\beta\times\mathbb{R}^d)$ such that

$$\omega^{cr}_{r,\theta}(W_f) = \int_{\mathcal{S}'(K_\beta\times\mathbb{R}^d)} d\mu^\beta_{r,\theta}(\varphi)e^{i(\varphi,f)} \tag{17}$$

and then

$$\omega_0^{cr}(W_f) = \int d\lambda_0(r,\theta) \int_{\mathcal{S}'(K_\beta \times \mathbb{R}^d)} d\mu^\beta_{r,\theta}(\varphi) e^{i(\varphi,f)} \tag{18}$$

From (15) it follows easily that

$$\omega_{\Lambda,cr}(W_f) = \int d\lambda_0(r,\theta) \int d\mu^{(r,\theta)}_{\Lambda,cr}(\varphi) e^{i(\varphi,f)} \cdot \frac{Z^{(r,\theta)}}{Z_{\Lambda,cr}} \tag{19}$$

where

$$d\mu_\Lambda^{(r,\theta)}(\varphi) = \frac{1}{Z_\Lambda^{r,\theta}} e^{H_\Lambda^{int}(\varphi)} d\mu^\beta_{r,\theta}(\varphi) \tag{20}$$

The Gaussian measure(s) $d\mu^\beta_{r,\theta}(\varphi)$ have a fast decay of correlations, in contrast to $d\mu^\beta_{0,cr}$ and this enables us to prove the convergence of the corresponding expansions.

Proposition 2.2. *There exists a value of* λ_0 *(depending on* $\beta, \dots$*) such that for all* $\lambda \in (0, \lambda_0)$ *the corresponding high temperature cluster expansions for* $\mu_\Lambda^{(r,\theta)}$ *are convergent uniformly in the parameters* (r,θ)*. The limiting measures* $\mu_{cr}^{r,\theta}(\lambda)$ *are ergodic with respect to the translations.*

Step 2. From the hypothesis $\sup_\Lambda \omega_{\Lambda,cr}(W_f) < \infty$ and the uniform convergence $\mu_\Lambda^{(r,\theta)} \to \mu_{cr}^{(r,\theta)}(\lambda)$ it follows that there exists a limit (in the sense of measures)

$$\lim_\Lambda d\lambda_0(r,\theta) \frac{Z^{(r,\theta)}_{\Lambda,cr}}{Z_{\Lambda,cr}} \equiv d\lambda_{ren}(r,\theta)$$

Moreover the limiting measure $\lim_\Lambda d\mu_{\Lambda,cr} = d\mu^\lambda_{cr}$ is given by

$$\mu^\lambda_{cr} = \int d\lambda_{ren}(r,\theta)\, \mu^{(r,\theta)}_{cr}(\lambda) \tag{21}$$

That the measure $d\lambda_{ren}(r,\theta)$ is not concentrated at one point follows by the integration by parts formula as in the case of gentle perturbations studied in [8,9].
Step 3. Using certain reduction formula derived in [13] (in the spirit similar to those presented in [12]) it follows from Step 2 that for any $f \in L^1 \cap L^2(\mathbb{R}^d),\ d \geq 3$ there exists an unique thermodynamic limit

$$\lim_\Lambda \omega_\Lambda^{cr}(W_f) \equiv \omega_\lambda^{cr}(W_f)$$

providing λ is sufficiently small, and moreover the limiting state ω^{cr}_λ can be decomposed into the pure sates as follows

$$\omega^{cr}_\lambda = \int d\lambda_{ren}(r,\theta) \omega_\lambda^{(r,\theta)} \tag{22}$$

where $\omega_\lambda^{(r,\theta)}$ are the corresponding extensions of the measures $\mu_\lambda^{(r,\theta)}$ to the whole Weyl algebra.

III. The Poissonian analysis: perturbations by standard gauge invariant hamiltonians.

In this section we will present a construction of the diagonalizing space for the standard many body hamiltonians

$$H_\Lambda^{int} = \int_\Lambda a^+(x)a^+(y)V(x-y)a(x)a(y)\,dx\,dy \tag{23}$$

where the two-particle potential V obeys standard requirements (see bolow). Although the underlying construction may be given in the pure Poissonian analysis language [18, 19] we display it in terms of the corresponding Generalized Random Fields (GRF). For this goal let us denote by $\mathcal{O}'_r(p_\beta;\Lambda)$ the (closed) subspace of $\mathcal{D}'(\mathbb{R}_+ \times \Lambda)$ ($\equiv$ the space of real Schwartz distributions with support in $\mathbb{R}_+ \times \Lambda$). The subspace $\mathcal{O}'_r(p_\beta;\Lambda)$ is defined as these $\varphi \in \mathcal{D}'(\mathbb{R}_+ \times \Lambda)$ such that

(i) the map $\mathbb{R}_+ \ni\to \varphi(t,\cdot) \in \mathcal{D}'(\Lambda)$ is continuous, and
(ii) there exists $j \in \mathbb{N}$ such that $\mathbb{R}_+ \ni t \to \varphi(t,\cdots) \in \mathcal{D}'(\Lambda)$ is periodic with period $j\beta$.

On the space $\mathcal{D}(\mathbb{R}_+ \times \Lambda)$ we define the following characteristic functional Γ_Λ:

$$\Gamma_\Lambda(f) \equiv \exp \sum_{j=1}^{\infty} \frac{z^j}{j} \int_\Lambda dx \int d_\Lambda W_{x|x}^{j\beta,\sigma}(\omega^j)(\exp i \int_0^{j\beta} f(\tau,\omega^j(\tau))\,d\tau - 1) \tag{24}$$

where $dW_{x|x}^{j\beta,\sigma}$ stands for the σ-conditioned Brownian bridge measure of lenght $j\beta$ and the conditioning is given by the classical b.c. one i.e. $\sigma \in C(\partial\Lambda)$ and $\sigma \geq 0$ pointwise on $\partial\Lambda$ (see e.g. [5] for this). By Minlos theorem there exists a measuredP_Λ on the space $\mathcal{D}'(\mathbb{R}_+ \times \Lambda)$ (called free functional Poisson measure) such that

$$\int_{\mathcal{D}'(\mathbb{R}_+\times\Lambda)} dP_\Lambda^\sigma(\varphi) e^{i(\varphi,f)} = \Gamma_\Lambda(f)$$

Actually it can be shown that the set $\mathcal{O}'_r(p_\beta;\Lambda)$ is the carrier set for the measure dP_Λ. Other elementary properties of dP_Λ can be found in [G1].

The two-particle potential V in (15) is assumed to obey the following (standard [20]) assumptions:

c_0) V is central and $V \in C(\mathbb{R}^d \setminus \{0\})$,
c_1) V is stable,
c_3) there exists $r_0 \geq 0$ such that $\int_{r_0}^{\infty} V(r)\,dr < \infty$.

Proposition 3.1. *Let V fulfills c_0 – c_3. Then the following equality is valid:*

$$\begin{aligned} Z_\Lambda(z,\beta) &= Tr_{\,\Gamma(L^2(\Lambda,dx))} \exp -\beta[d\Gamma(-\Delta_\Lambda^\sigma + \mu 1) + H_\Lambda^{int}] \\ &= \int_{\mathcal{O}'_r(p_\beta;\Lambda)} dP_\Lambda^\sigma(\varphi) \exp -\frac{\beta}{2} \iint_\Lambda dx\,dy :\varphi(x)V(x-y)\varphi(y): \end{aligned} \tag{25}$$

where (informally)

$$:\varphi(x)V(x-y)\varphi(y): = \varphi(x)V(x-y)\varphi(y) - V(0)\#\{\varphi\}$$

and $\#\{\varphi\}$ *is the "loop number" functional defined (for* dP_Λ^σ *a.e.* φ*) in* [15,16].

Proof. See [17,18]

Thus in a certain sense the space $L^2(dP_\Lambda^\sigma)$ plays the role of diagonalizing space for H_Λ^{int}. A Gibbsian type perturbations dG_Λ^σ of dP_Λ^σ is defined by:

$$dG_\Lambda^\sigma(\varphi) = Z_\Lambda(z,\beta)^{-1} \exp -\frac{\beta}{2} \iint\limits_\Lambda dx\, dy\ :\varphi(x)V(x-y)\varphi(y): dP_\Lambda^\sigma \tag{26}$$

Theorem 3.2. *Let* V *obeys* c_0–c_2. *Then, for sufficiently small* z *(depending on* $\beta, V, \dots$*), the unique thermodynamic limits*

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \int dG_\Lambda^\sigma(\varphi) \prod_{i=1}^n (\varphi, f_i) \equiv \int dG(\varphi) \prod_{i=1}^n (\varphi, f_i) \tag{27}$$

exists for any $n \in \mathbb{N}$, $f_i \in \mathcal{D}(\mathbb{R}_+ \times \Lambda)$ *and the limits on l.h.s of (27) determine uniquelly a measure* $dG(\varphi)$ *entering on r.h.s. of (27). Moreover the limiting measure* dG *does not depends on the particular choise of* σ.

The main ideas of the proof.
Step 1. The following formula for integration by parts is valid:

$$\int dP_\Lambda^\sigma(\varphi) :<\varphi, f> F(\varphi) :_P G(\varphi) = \sum_{j=1}^\infty \frac{z^j}{j} \int\limits_\Lambda dx \int d_\Lambda W_{x|x}^{j\beta,\sigma}(\omega^j) \int\limits_0^{j\beta} d\tau f(\tau, \omega^j(\tau))$$
$$\cdot \int dP_\Lambda^\sigma(\varphi) : F(\varphi) :_P G(\varphi + \delta(\cdot - \omega^j(\tau)) \tag{28}$$

for any cylindrical and $L^1(dP_\Lambda)$ functionals F and G and where $:\ :_P$ means the Poissonian normal ordering (see e.g. [17, 18]).
Step 2. Applying Step 1 it follows that:

$$\int dG_\Lambda^\sigma(\varphi) : \prod_{i=1}^n (\varphi, f_i) :_P = \sum_{j_1,\dots,j_n \geq 1} \frac{z^{j_1+\dots+j_n}}{j_1\cdots j_n} \prod_{\alpha=1}^n \int\limits_\Lambda dx_\alpha \int d_\Lambda W_{x_\alpha|x_\alpha}^{j_\alpha\beta,\sigma}(\omega^{j_\alpha})$$
$$\cdot \int\limits_0^{j_\alpha\beta} f(\tau, \omega^{j_\alpha}(\tau))\, d\tau\ \cdot \sigma_\Lambda^\bullet(\omega^{j_1}, \dots, \omega^{j_n}) \tag{29}$$

where

$$\sigma_\Lambda^\bullet(\omega^{j_1}, \dots, o^{j_n}) = e^{-\mathcal{E}_V^\beta(\omega^{j_1},\dots,\omega^{j_n})} \int dP_\Lambda^\sigma(\varphi) \exp -\mathcal{E}_V^\beta(\omega^{j_1}, \dots, \omega^{j_n})|\varphi) \tag{30}$$

$$\mathcal{E}_V^\beta(\omega^{j_1},\dots,\omega^{j_n}) = \sum_{1\le k<l\le n} \int_0^\beta d\tau \sum_{\alpha_k=0}^{j_k-1} \sum_{\alpha_l=0}^{j_l-1} V(\omega^{j_k}(\tau+\alpha_k\beta);\omega^{j_l}(\tau+\alpha_l\beta)) \tag{31}$$

$$\mathcal{E}_V^\beta(\omega^{j_1},\dots,\omega^{j_n}|\varphi) = \sum_{k=1}^n \mathcal{E}_V^\beta(\omega^{j_k};\varphi) \tag{32}$$

and $\mathcal{E}_V^\beta(\omega^{j_k};\varphi)$ is defined (dP_Λ^σ a.e.) as in [15,16]. Now, let $\mathcal{P}_\Lambda^\beta(j_1,\dots,j_n)$ be the space consisting of n-tuples of paths $(\eta^{j_1},\dots,\eta^{j_n})$ where

$$\eta^{j_k} \in C([0,j_k\beta] \to \overline{\Lambda})$$

The extensions of the functionals $\mathcal{E}_V^\beta(\cdot|\varphi)$ to the spaces $\mathcal{P}_\Lambda^\beta(j_1,\dots,j_n)$ and the corresponding extensions of the functionals $\sigma_\Lambda^\bullet$ given by (30) will be denoted by the same symbols.

Step 3. Let $\sigma_\Lambda^\bullet$ be the extensions of the functionals (22). Then the standard, σ-conditioned (see e.g. [5]) reduced density matrices (the kernels of) $\rho_\Lambda^\bullet$ are given by:

$$\begin{aligned}\rho_\Lambda^\bullet(x_1,\dots,x_n|y_1,\dots,y_n) = \\ z^n \sum_{\pi\in S_n}\sum_{j_1=1}^\infty \cdots \sum_{j_n=1}^\infty \int d_\Lambda W_{x_1|\pi(y_1)}^{j_1\beta,\sigma}(\omega^{j_1})\cdots \int d_\Lambda W_{x_n|\pi(y_n)}^{j_n\beta,\sigma}(\omega^{j_n}) \\ \cdot \frac{z^{j_1+\dots+j_n}}{j_1\cdots j_n}\sigma_\Lambda^\bullet(\omega^{j_1},\dots,\omega^{j_n})\end{aligned}$$

Therefore the (extended to deal with the case of arbitrary classical b.c.) classical analysis of Ginibre [21, 22] can be applied to controll the limits $\lim_\Lambda \sigma_\Lambda^\bullet(\cdot)$ rigorously.

Step 4. Comparison of the corresponding Kirkwood-Salsburg resolvent expansions leads to the statement on the σ-independence of the limiting Gibbs measures dG.

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