Universität Bielefeld

Diplomarbeit

On uniqueness of ODEs perturbed by a Brownian path

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8. November 2013

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Introduction

In this thesis we study the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) \\ x(0) = x_0 \end{cases} \tag{SDE}$$

in the finite dimensional space $\mathbb{R}^d$ driven by a canonical d-dimensional Brownian motion W with a bounded Borel measurable drift f, diffusion coefficient $\sigma \in \mathbb{R} \setminus \{0\}$ and deterministic initial condition $x_0 \in \mathbb{R}^d$.

SDEs have been a very active research topic in the last decades. Several approaches were developed for example the pathwise approach where a solution x to the above equation is interpreted as a stochastic process or the mild approach. In this thesis we consider the so-called path-by-path approach where (SDE) is not considered as a stochastic differential equation. In the path-by-path picture we first plug in an $\omega \in \Omega$ into the corresponding integral equation (IE) of (SDE)

$$x(\omega, t) = x_0 + \int_0^t f(s, x(\omega, s))\,\mathrm{d}s + \sigma W(\omega, t) \tag{IE}$$

and try to find a (unique) continuous function $x(\omega, \cdot)\colon [0, T] \longrightarrow \mathbb{R}^d$ satisfying this equation, which can now be considered as an ordinary integral equation (IE), that is perturbed by a Brownian path $W(\omega)$. If such a function can be found for almost all $\omega \in \Omega$, the map $\omega \longmapsto x(\omega)$ is called a path-by-path solution to the equation (SDE). Naturally the notion of uniqueness is much stronger than in the pathwise picture. Nevertheless, we prove that the equation (SDE) even admits a path-by-path unique solution.

The main theorem of this thesis states that there exists a *unique* solution to the equation (SDE) in the path-by-path sense. With some slight variations we mainly follow A. M. Davie's proof in [Dav07]. In [Ver81] a proof for the *existence* of pathwise solutions for much more general equations than the one above has already been given.

We want to emphasize that although a pathwise solution x is not a path-by-path solution, there is always a set N of measure 0 such that x is a path-by-path solution for all $\omega \in N^c$. [Ver81] also contains a proof that the above equation has a pathwise unique solution. Pathwise uniqueness implies that for any two solutions x and y of (SDE) a zero set N can be found such that x and y coincide on $\Omega \setminus N$. In general this zero set will depend on both x and y. The notion of uniqueness in the path-by-path approach is much stronger. In this thesis we show that a zero set N can be found such that all solutions coincide for all $\omega \in \Omega \setminus N$.

Since we obtain a unique solution for almost all Brownian paths $W(\omega)$ this result can also be interpreted as a uniqueness theorem for randomly perturbed ODEs, more precisely IEs. We refer to [Fla11] for an in-depth discussion about the various notions of uniqueness for SDEs and perturbed ODEs.

Structure of this thesis

In Chapter 1 we prove an estimate which will act as a substitute for the Lipschitz continuity of f. We use the Girsanov transformation to reduce the problem to a slightly simpler one in the first section. In the second section we merely prove the estimate which will later act as a substitute for the Lipschitz condition in expectation. Based on this we obtain a version of this estimate that holds in probability in the last section of that chapter.

The claimed uniqueness of the above SDE in the path-by-path sense is proved in the second chapter. We split the proof in three parts. In the first section of that chapter we study dyadic points in the cube $[0,1]^d$ and using the results of Chapter 1 we prove an estimate for dyadic points for the substitute for the Lipschitz condition. The second section contains a technical lemma which enables us to use an approximation argument in the final proof. In the last section we finish the proof with the help of the Euler scheme and the previously established inequalities.

We discuss an application of the main results for Euler approximations in Chapter 3 and focus on the connection between the above SDE and randomly perturbed IEs.

Additionally, in Appendix A we give proofs of the basic estimates which are used in Chapter 1. For the sake of completeness we calculate the Fourier transform of the normal distribution and its second derivative in Appendix B. At last, Appendix C contains an estimate which is used in [Dav07], but not necessary for our proof in this thesis.

Outline of the proof

First, we observe that the main theorem would be trivial if f were Lipschitz continuous in the second parameter. Let x and y be two solutions of (IE). We then have

$$|x(t)-y(t)| \leq \int_0^t |f(s,x(s))-f(s,y(s))| \,\mathrm{d}s \leq \mathrm{Lip}(f) \int_0^t |x(s)-y(s)| \,\mathrm{d}s.$$

So, by Gronwall's Lemma we have $x=y$.

In the first section of the first chapter we show that by considering u defined as the difference of two solutions the main theorem can be reduced to the following problem: Let u be a continuous function which fulfills the following equation

$$u(t) = \int_0^t f(s,W(s)+u(s)) - f(s,W(s)) \,\mathrm{d}s, \qquad \forall t \in [0,T].$$

Showing that for almost all Brownian paths the only solution of the above equation is the trivial solution $u=0$ implies the proposed uniqueness.

Let us again assume that f is Lipschitz continuous. Since we have already proved the theorem in this case, it should be possible to find a simple proof for the reduced problem. Indeed, consider the interval $I = [a, b] \subseteq [0, T]$ and choose α minimal with the property

$$|u(t)| \leq \alpha, \qquad \forall t \in I.$$

We set $\beta := |u(a)|$ and using that f is assumed to be Lipschitz continuous we have

$$||u(t)| - \beta| \leq \int_a^b |f(s, W(s) + u(s)) - f(s, W(s))| \, \mathrm{d}s \leq \mathrm{Lip}(f)|I|\alpha,$$

where $|I| = b - a$ is the Lebesgue length of the interval. With the help of the reversed triangle inequality and by rearranging we conclude

$$|u(t)| \leq \beta + \mathrm{Lip}(f)|I|\alpha, \qquad \forall t \in I.$$

But, since α was chosen minimal with this property, we obtain

$$|u(t)| \leq \alpha \leq \frac{\beta}{1 - \mathrm{Lip}(f)|I|}, \qquad \forall t \in I$$

as long as $\mathrm{Lip}(f)|I| < 1$. So in particular, u vanishes on I if β is zero. This is clearly fulfilled if we choose $a = 0$. Choosing the interval I small enough such that $\mathrm{Lip}(f)|I| < 1$ holds and repeating this argument inductively, proves that u vanishes everywhere.

In this thesis we generalize the above idea. First, we need a substitute for the Lipschitz condition. In the first chapter we prove the following estimate in expectation (Theorem 1.23)

$$\mathbb{E}\left(\int_0^1 f(t, W(t) + x) - f(t, W(t)) \, \mathrm{d}t\right)^p \leq C^p (p/2)! |x|_2^p, \qquad \forall x \in \mathbb{R}^d, \ p \in 2\mathbb{N},$$

where C is independent of f which is due to the fact that f is assumed to be bounded by 1. This estimate is obtained by a careful analysis of the Gaussian kernel and its first and second derivatives. Using this result, we prove versions of this estimate which hold in probability (Corollary 1.25, 1.26 and 1.28):

$$\mathbb{P}[|\sigma_{a,b}(x)| > \lambda\sqrt{b-a}|x|_2 | \mathcal{F}_s] \leq 2e^{-\lambda^2/(2C^2)}, \qquad \forall x \in \mathbb{R}^d, \ \lambda > 0,$$

and

$$\mathbb{E}[|\sigma_{a,b}(x)|^p | \mathcal{F}_s] \leq C^p |b-a|^{p/2} (p/2)! |x|_2^p, \qquad \forall x \in \mathbb{R}^d, \ \lambda > 0, \ p \in 2\mathbb{N},$$

where

$$\sigma_{a,b}(x) := \int_a^b f(t, W(t) + x) - f(t, W(t)) \, \mathrm{d}t, \qquad \forall a, b \in \mathbb{R}$$

and $0 \leq s \leq a < b \leq 1$. These two estimates follow from the above "in expectation" estimate by applying the Chebychev inequality. The fact that we can take the conditional expectation (conditional probability respectively) w.r.t. $\mathcal{F}_s$, as long as $s \leq a$, is due to the Markov property of W.

In the second chapter we improve on this and show an “almost sure version” namely the bound

$$\left| \int_{k2^{-n}}^{(k+1)2^{-n}} f(t, W(t)+x) - f(t, W(t)) \, \mathrm{d}t \right| \leq C\sqrt{n} 2^{-n/2} \max\left(|x|_\infty, 2^{-2^n}\right)$$

if $x \in \mathbb{R}^d$ is a dyadic point, $n \in \mathbb{N}$ and $k \in \{0, ..., 2^n - 1\}$ (Lemma 2.5). This estimate will become the most important ingredient for the rest of the proof.

Next, we approximate u by step functions u_ℓ and write

$$\int_I f(t, W(t)+u(t)) - f(t, W(t)) \, \mathrm{d}t = \lim_{\ell\to\infty} \int_I f(t, W(t)+u_\ell(t)) - f(t, W(t)) \, \mathrm{d}t$$

$$= \int_I f(t, W(t)+u_m(t)) - f(t, W(t)) \, \mathrm{d}t + \sum_{\ell=m}^{\infty} \int_I f(t, W(t)+u_{\ell+1}(t)) - f(t, W(t)+u_\ell(t)) \, \mathrm{d}t.$$

These step functions are constant on the interval I and hence we are able to use the above bound (Lemma 2.5). Moreover, we prove a technical lemma (Lemma 2.9) to be able to approximate u by step functions (Lemma 2.10) in order to make the above argument rigorous. Furthermore, since we want to estimate terms of the form

$$\sum_{\ell=m}^{\infty} \int_I f(t, W(t)+u_{\ell+1}(t)) - f(t, W(t)+u_\ell(t)) \, \mathrm{d}t$$

we need better bounds than the ones described above. In Lemma 2.14 we use the Euler approximation scheme to obtain a bound for the term

$$\sum_{q=1}^{r} \int_I f(t, W(t)+x_{q-1}) - f(t, W(t)+x_q) \, \mathrm{d}t$$

where $x_{q+1} = x_q + \sigma_{q2^{-n},(q+1)2^{-n}}(x_q)$ is the Euler approximation for the point x_{q+1} given the previous one x_q. Comparing arbitrary points $y_0, ..., y_r$ with the Euler approximation (Lemma 2.15) yields an estimate for

$$\sum_{q=1}^{r} \int_I f(t, W(t)+y_{q-1}) - f(t, W(t)+y_q) \, \mathrm{d}t.$$

With this, the analog of the above proof is carried out in Lemma 2.16. Instead of $|u(t)| \leq \alpha$ we have the more complicated condition (2.16.2). Since our estimates are weaker than in the Lipschitz case we will not be able to immediately conclude that u vanishes, but if $|u(j2^{-m})| \leq \beta$ is “small” we have

$$|u((j+1)2^{-m})| \leq \beta\left(1 + K2^{-m}\log_2(1/\beta)\right)$$

with some constant K. By letting m go to infinity, this is enough to conclude the main result as shown in Theorem 2.17.

Differences between Davie's proof and this thesis

In this thesis we give an extended and more detailed version of Davie's proof in [Dav07]. Nevertheless, for the sake of simplicity and clarity we made some slight changes.

Proposition 1.9, Theorem 1.23 and the subsequent corollaries are extended to the case $p = 1$ (cf. [Dav07] Proposition 2.1 and 2.2). In addition, we give a simple generalization of Davie's Corollary 2.6 in our Corollary 1.28.

In the second chapter we skip Davie's Lemma 3.1. For a detailed proof of this estimate we refer to Appendix C (Lemma C.2). As a substitute we extend Lemma 2.5 (cf. [Dav07, Lemma 3.2]) in Corollary 2.6. This replacement has some marginal influence on the proof of Lemma 2.9. Furthermore, we were not able to obtain the factor $2^{-n/4}$ in front of the error term in Lemma 2.15, as Davie did in Lemma 3.6. This is partially because of our replacement of Davie's Lemma 3.1. Nonetheless, we still obtain the factor $2^{-n/8}$ to control the error term. It turns out that this is sufficient to complete the proof of the main theorem.

1 Preliminaries

In this chapter we reduce the main theorem (Theorem 1.5) to a slightly simpler problem using the Girsanov transformation (Lemma 1.8). In the second section we obtain an estimate, which will later act as a substitute for the Lipschitz condition (Theorem 1.23). In the last section we prove different versions of this estimate which hold in probability. These estimates play a crucial role in the proof of the main theorem.

1.1 Reduction via Girsanov transformation

Let d be a positive integer, $T > 0$ and $\Omega := \mathcal{C}([0,T],\mathbb{R}^d)$. Let $\mathbb{P}$ be the classical Wiener measure on Ω. Note that $\pi(\omega,t) = \omega(t)$ is a d dimensional Brownian motion with $\pi(0) = 0$ $\mathbb{P}$-a.s.. Define $\mathbb{P}_x[A] := \mathbb{P}[A - x]$ then $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ are measures such that $((\pi)_{t\in[0,T]}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ is a universal Markov process (cf. [Bau96] Definition 42.15). Let $\mathcal{F}_t$ be the natural filtration of $(\pi_t)_{t\in[0,T]}$ i.e. $\mathcal{F}_t$ is $\mathbb{P}$-complete (in the sense that every zero set of $\mathcal{F}_T$ is in $\mathcal{F}_0$) and right-continuous. In this thesis we consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}x(t) = f(t,x(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\pi(t) \\ \;x(0) = x_0 \end{cases} \tag{1.0}$$

where $f\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a bounded Borel function, $x_0 \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}\setminus\{0\}$.

Definition 1.1 (path-by-path solution)

A map $x\colon \Omega \longrightarrow \Omega$ is called a path-by-path solution to equation (1.0) if there exists $N_x \subseteq \Omega$ with $\mathbb{P}[N_x] = 0$ such that x fulfills the corresponding integral equation

$$x(\omega,t) = x_0 + \int_0^t f(s,x(\omega,s))\,\mathrm{d}s + \sigma\omega(t) \tag{1.1}$$

for every $t \in [0,T]$ and $\omega \in \Omega\setminus N_x$. Notation: $x \in \mathcal{S}(\mathcal{C}([0,T],\mathbb{R}^d), f, \sigma, x_0)$, where $\mathcal{S}(\mathcal{C}([0,T],\mathbb{R}^d), f, \sigma, x_0)$ denotes the set of path-by-path solutions to equation (1.0).

Proposition 1.2 (Existence of path-by-path solutions)

Let $f\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel function, $\sigma \in \mathbb{R}\setminus\{0\}$, $x_0 \in \mathbb{R}^d$ and π a canonical Brownian motion. Then there exists a map $x\colon \Omega \longrightarrow \Omega$ and a set $N_x \subseteq \Omega$ with $\mathbb{P}[N_x] = 0$ such that x is a path-by-path solution of (1.0) in the sense of Definition 1.1 i.e. $x \in \mathcal{S}(\Omega, f, \sigma, x_0)$.

Proof

Define $W\colon \Omega \longrightarrow \Omega$ by

$$W(\omega,t) := \pi(\omega,t) - \sigma^{-1}\int_0^t f(s, x_0 + \sigma\pi(\omega,s))\ \mathrm{d}s, \qquad \forall \omega \in \Omega,\ t \in [0,T].$$

Then for all $\omega \in \Omega$ and $t \in [0,T]$ set

$$\begin{aligned} x(W(\omega),t) := x_0 + \sigma\pi(\omega,t) &= x_0 + \int_0^t f(s, x_0 + \sigma\pi(\omega,s))\ \mathrm{d}s + \sigma W(\omega,t) \\ &= x_0 + \int_0^t f(s, x(W(\omega,s)))\ \mathrm{d}s + \sigma W(\omega,t). \end{aligned}$$

Now by Girsanov's Theorem (cf. [Shr04], Theorem 5.2.3) for

$$\phi_T(\omega) := \exp\left[\int_0^T \left\langle \sigma^{-1} f(s, x_0 + \sigma\pi(\omega,s)),\ \mathrm{d}\pi(s)\right\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^T \sigma^{-2} f(s, x_0 + \sigma\pi(\omega,s))^2\ \mathrm{d}s\right]$$

we have $(\phi_T \cdot \mathbb{P}) \circ W^{-1} = \mathbb{P}$ and $\phi_T \cdot \mathbb{P} \approx \mathbb{P}$. Defining $N_x := \Omega \setminus W(\Omega)$, we have

$$\begin{aligned} \mathbb{P}[N_x] &= \mathbb{P}[\Omega] - \mathbb{P}[W(\Omega)] \\ &= \mathbb{P}[\Omega] - (\phi_T \cdot \mathbb{P})[\Omega] = 0 \end{aligned}$$

and for all $\omega \in \Omega \setminus N_x$ and $t \in [0,T]$

$$x(\omega,t) = x_0 + \int_0^t f(s, x(\omega,s))\ \mathrm{d}s + \sigma\omega(t).$$

Hence x (with N_x as above) is a path-by-path solution.

□

Proposition 1.3 (Scaling invariance)

Let $f\colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel function with $|f|_\infty \neq 0$, $\sigma \in \mathbb{R} \setminus \{0\}$, $x_0 \in \mathbb{R}^d$. Define

$$\tilde{f}\colon [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad (t,u) \longmapsto |f|_\infty^{-1} f(|f|_\infty^{-2}\sigma^2 t, x_0 + |f|_\infty^{-1}\sigma^2 u).$$

Let $\tilde{x}$ (with $N_{\tilde{x}}$) be a path-by-path solution to equation (1.1) with f, σ and x_0 replaced by $\tilde{f}$, 1 and 0, i.e. $\tilde{x} \in \mathcal{S}(\mathcal{C}([0,T],\mathbb{R}^d), \tilde{f}, 1, 0)$. Define

$$\begin{aligned} y\colon \quad & \Omega = \mathcal{C}([0,T],\mathbb{R}^d) \longrightarrow \mathcal{C}([0,|f|_\infty^{-2}\sigma^2 T],\mathbb{R}^d) =: \Omega_{|f|_\infty^{-2}\sigma^2 T} \\ & y(\omega,t) := x_0 + |f|_\infty^{-1}\sigma^2 \tilde{x}(\omega, |f|_\infty^2 \sigma^{-2} t), \qquad \forall \omega \in \Omega,\ t \in [0,|f|_\infty^{-2}\sigma^2 T] \end{aligned}$$

and

$$\begin{aligned}\varphi_{|f|_\infty,\sigma}\colon \Omega = \mathcal{C}([0,T],\mathbb{R}^d) &\longrightarrow \mathcal{C}([0,|f|_\infty^{-2}\sigma^2 T],\mathbb{R}^d) = \Omega_{|f|_\infty^{-2}\sigma^2 T}\\ (t\longmapsto \omega(t)) &\longmapsto \big(t\longmapsto |f|_\infty^{-1}\sigma\omega(|f|_\infty^2\sigma^{-2}t)\big)\end{aligned}$$

which is a homomorphism from Ω to $\Omega_{|f|_\infty^{-2}\sigma^2 T}$. Then $x\colon \Omega_{|f|_\infty^{-2}\sigma^2 T}\longrightarrow \Omega_{|f|_\infty^{-2}\sigma^2 T}$ defined by

$$x(\omega,t) := y(\varphi_{|f|_\infty,\sigma}^{-1}(\omega),t), \qquad \forall \omega\in\Omega,\ t\in[0,|f|_\infty^{-2}\sigma^2 T]$$

is a path-by-path solution of equation (1.1) with $N_x := \varphi_{|f|_\infty,\sigma}(N_{\tilde{x}})$. This means we have a one to one correspondence between the sets

$$\mathcal{S}(\mathcal{C}([0,T],\mathbb{R}^d),\tilde{f},1,0)\longleftrightarrow \mathcal{S}(\mathcal{C}([0,|f|_\infty^{-2}\sigma^2T],\mathbb{R}^d),f,\sigma,x_0)$$

Proof

Observe that

$$\begin{aligned} y(\omega,t) &= x_0 + |f|_\infty^{-1}\sigma^2\int_0^{|f|_\infty^2\sigma^{-2}t}\tilde{f}(s,\tilde{x}(\omega,s))\,\mathrm{d}s + |f|_\infty^{-1}\sigma^2\omega(|f|_\infty^2\sigma^{-2}t)\\ &= x_0 + |f|_\infty^{-1}\sigma^2\int_0^{|f|_\infty^2\sigma^{-2}t}|f|_\infty^{-1}f(|f|_\infty^{-2}\sigma^2 s, x_0+|f|_\infty^{-1}\sigma^2\tilde{x}(\omega,s))\,\mathrm{d}s + |f|_\infty^{-1}\sigma^2\omega(|f|_\infty^2\sigma^{-2}t)\\ &= x_0 + \int_0^t f(s,x_0+|f|_\infty^{-1}\sigma^2\tilde{x}(\omega,|f|_\infty^2\sigma^{-2}s))\,\mathrm{d}s + |f|_\infty^{-1}\sigma^2\omega(|f|_\infty^2\sigma^{-2}t)\\ &= x_0 + \int_0^t f(s,y(\omega,s))\,\mathrm{d}s + \sigma\varphi_{|f|_\infty,\sigma}(\omega)(t). \end{aligned}$$

Hence for all $\omega\in\varphi_{|f|_\infty,\sigma}(\Omega\setminus N_{\tilde{x}})$ and all $t\in[0,|f|_\infty^{-2}\sigma^2T]$

$$x(\omega,t) = x_0 + \int_0^t f(s,x(\omega,s))\,\mathrm{d}s + \sigma\omega(t).$$

But $\mathbb{P}\circ\varphi_{|f|_\infty,\sigma}^{-1} = \mathbb{P}$, hence setting $N_x := \varphi_{|f|_\infty,\sigma}(N_{\tilde{x}})$ we have $\mathbb{P}[N_x]=0$. Hence x with N_x is a path-by-path solution of equation (1.1).

□

Remark 1.4

Since $T>0$ was arbitrary, as a result of this scaling property its sufficient to consider the SDE (1.0) in the case where $x_0=0$, $\sigma=1$ and f is bounded by 1 or 1/2.

Theorem 1.5 (Main result)

There exists $N \subseteq \Omega$ with $\mathbb{P}[N] = 0$ such that all path-by-path solutions of (1.0) in the sense of Definition 1.1 coincide for every $\omega \in \Omega \setminus N$. In particular if x and y are two path-by-path solutions then $x(\omega) = y(\omega)$ in the sense of continuous functions for every $\omega \in \Omega \setminus (N_x \cup N_y \cup N)$.

Corollary 1.6

There exists a path-by-path solution x with $N_x \subseteq \Omega$ such that every other path-by-path solution y with $N_y \subseteq \Omega$ coincides with x on $\Omega \setminus (N_x \cup N_y)$.

Proof

By Proposition 1.2 there exists a path-by-path solution x (with $N_x \subseteq \Omega$) to equation (1.0). Invoking Theorem 1.5 and replacing N_x with the set $N_x \cup N$ implies that x with $N_x \cup N$ is the unique (in the sense of Theorem 1.5) path-by-path solution of equation (1.0).

□

Lemma 1.7

Fix an $\omega \in \Omega$. Let $x(\omega)$ be a function that fulfills equation (1.1) with $x_0 = 0$, $\sigma = 1$ and $|f|_\infty \leq 1$. If the only solution of

$$u(t) = \int_0^t f(s, x(\omega, s) + u(s)) - f(s, x(\omega, s)) \, \mathrm{d}s, \qquad \forall t \in [0, T]$$

is $u = 0$, then $x(\omega)$ is the only solution to (1.1).

Proof

Let $\omega \in \Omega$. Let $x'(\omega)$ be another solution of (1.1) for the path ω. Subtracting $x(\omega)$ from $x'(\omega)$ results in

$$x'(\omega, t) - x(\omega, t) = \int_0^t f(s, x'(\omega, s)) - f(s, x(\omega, s)) \, \mathrm{d}s.$$

We set $u(t) := x'(\omega, t) - x(\omega, t)$ and obtain

$$u(t) = \int_0^t f(s, x(\omega, s) + u(s)) - f(s, x(\omega, s)) \, \mathrm{d}s.$$

By assumption we deduce $u = 0$ and therefore $x(\omega) = x'(\omega)$.

□

Lemma 1.8

Suppose there exists a set $\tilde{N} \subseteq \Omega$ with $\mathbb{P}[\tilde{N}] = 0$ such that for every $\omega \in \Omega \setminus \tilde{N}$ and every function $u\colon [0,T] \longrightarrow \mathbb{R}^d$ satisfying

$$u(t) = \int_0^t f(s, \omega(s) + u(s)) - f(s, \omega(s)) \,\mathrm{d}s, \qquad \forall t \in [0,T] \tag{1.8}$$

we have $u = 0$.
Then the conclusion of Theorem 1.5 holds.

Proof

We define

$$\phi_T := \exp\left[\int_0^T \langle f(s,\pi(s)),\ \mathrm{d}\pi(s)\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^T f(s,\pi(s))^2 \,\mathrm{d}s\right]$$

and set $\mu := \phi_T \cdot \mathbb{P}$. Note that $\mu \approx \mathbb{P}$. By Girsanov's Theorem (cf. [Shr04], Theorem 5.2.3) for

$$W(\omega,t) := \pi(\omega,t) - \int_0^t f(s,\pi(\omega,s)) \,\mathrm{d}s, \qquad \forall \omega \in \Omega,\ \forall t \in [0,T]$$

we have $\mu \circ W^{-1} = \mathbb{P}$. Furthermore

$$x(W(\omega,\,\cdot\,),t) := \pi(\omega,t) = \int_0^t f(s,\pi(\omega,s)) \,\mathrm{d}s + W(\omega,t), \qquad \forall \omega \in \Omega,\ \forall t \in [0,T].$$

Let $\omega \in \Omega \setminus \tilde{N}$. Then by assumption and the above the condition in Lemma 1.8 is fulfilled for the "input" path $W(\omega,\,\cdot\,)$ and the corresponding solution $\pi(\omega)$ ("output path"). Hence for the set

$$W(\Omega \setminus \tilde{N}) \subseteq \mathcal{C}([0,T],\mathbb{R}^d)$$

we have for all $z \in W(\Omega \setminus \tilde{N})$ that the corresponding solution $\pi(\omega) = x(z)$ is unique. To see that then Theorem 1.5 holds for $N := \Omega \setminus W(\Omega \setminus \tilde{N})$ it remains to be shown that

$$\mathbb{P}[W(\Omega \setminus \tilde{N})] = 1.$$

But

$$1 \geq \mathbb{P}[W(\Omega \setminus \tilde{N})] = (\mu \circ W^{-1})[W(\Omega \setminus \tilde{N})] \geq \mu[\Omega \setminus \tilde{N}] = 1.$$

□

1.2 Substitute for Lipschitz condition

The aim of this section is to prove Theorem 1.23 below, which is the essential ingredient of the main proof. The main objective of this section is Proposition 1.9. The proof of Theorem 1.23 will be an application of Proposition 1.9. For the rest of this thesis let W be a Brownian motion on $\mathbb{P}$ and $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ measures such that the universal Markov property (cf. [Bau96] equation 42.18) holds.

Proposition 1.9

There exists $C \in \mathbb{R}$ such that for all compactly supported, smooth, real-valued functions g on $[0,1]\times\mathbb{R}$ with $|g| \le 1$ everywhere and g' is bounded, where g' denotes the derivative w.r.t. the second variable, and every $n \in \mathbb{N}$, we have

$$\mathbb{E}\left[\left(\int_0^1 g'(t,W(t))\ \mathrm{d}t\right)^n\right] \le C^n \Gamma\left(\frac{n}{2}+1\right).$$

Where W is a one-dimensional continuous Brownian motion with $W(0) = 0$ $\mathbb{P}$-a.s. and Γ the gamma function.

Proof

Expanding the integral of the left-hand side of the inequality leads to

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 g'(t,W(t))\ \mathrm{d}t\right)^n\right] &= \mathbb{E}\prod_{j=1}^n \int_0^1 g'(t_j,W(t_j))\ \mathrm{d}t_j \\ &= \mathbb{E}\int_0^1 \dots \int_0^1 \prod_{j=1}^n g'(t_j,W(t_j))\ \mathrm{d}t_1 \dots \mathrm{d}t_n.\end{aligned}$$

Changing the set of integration from $[0,1]^n$ to $\{0 < t_{\sigma(1)} < \dots < t_{\sigma(n)} < 1\}$ for every $\sigma \in \mathrm{Per}(n)$, where $\mathrm{Per}(n)$ denotes the set of all permutations of $\{0,\dots,n\}$, yields

$$= \mathbb{E}\sum_{\sigma\in\mathrm{Per}(n)}\ \int_{0<t_{\sigma(1)}<\dots<t_{\sigma(n)}<1}\ \prod_{j=1}^n g'(t_{\sigma(j)},W(t_{\sigma(j)}))\ \mathrm{d}t_{\sigma(1)}\dots\mathrm{d}t_{\sigma(n)}.$$

Since $\prod_{j=1}^n g'(t_j,W(t_j))$ is a symmetric function, we have

$$= n!\ \mathbb{E}\int_{0<t_1<\dots<t_n<1}\ \prod_{j=1}^n g'(t_j,W(t_j))\ \mathrm{d}t_1\dots\mathrm{d}t_n.$$

Using Fubini's Theorem and the joint distribution of $W(t_1), ..., W(t_n)$ we rewrite the integral as

$$= n! \int\limits_{0<t_1<...<t_n<1} \int\limits_{\mathbb{R}^n} \prod_{j=1}^{n} g'(t_j, z_j)\, \mathrm{E}(t_j - t_{j-1}, z_j - z_{j-1})\ \mathrm{d}z_1 ... \mathrm{d}z_n\, \mathrm{d}t_1 ... \mathrm{d}t_n$$

where $t_0 := 0,\ z_0 := 0$ and

$$\mathrm{E}(t, z) := (2\pi t)^{-1/2} e^{-z^2/2t}, \qquad \forall t > 0.$$

For $1 \leq k \leq n$ we define

$$J_n^{(k)}(t_{k-1}, z_{k-1}) := \int\limits_{t_{k-1}<t_k<...<t_n<1} \int\limits_{\mathbb{R}^{n-k+1}} \prod_{j=k}^{n} g'(t_j, z_j)\, \mathrm{E}(t_j - t_{j-1}, z_j - z_{j-1})\ \mathrm{d}z_k ... \mathrm{d}z_n\, \mathrm{d}t_k ... \mathrm{d}t_n.$$

Observe that the left-hand side of the proposed inequality is therefore $n! J_n^{(1)}(0, 0)$. We stop the proof at this point in order to introduce some useful definitions and notations. We continue this proof on page 17.

Definition 1.10

Let $\mathrm{E}(t, z) := (2\pi t)^{-1/2} e^{-z^2/2t}$ for all $t > 0$. We set B and D as the first and second derivative of E w.r.t. the second variable.

$$\begin{aligned} \mathrm{B}(t, z) &:= \partial_z \mathrm{E}(t, z) = -(2\pi t)^{-1/2} \frac{z}{t} e^{-z^2/2t}, \\ \mathrm{D}(t, z) &:= \partial_z^2 \mathrm{E}(t, z) = (2\pi t)^{-1/2} \frac{z^2 - t}{t^2} e^{-z^2/2t}. \end{aligned}$$

Definition 1.11

We set

$$\mathcal{S}_k := \{\mathrm{E}, \mathrm{B}, \mathrm{D}\}^k, \qquad \forall k \in \mathbb{N}.$$

We call S a string if there exists a (unique) $k \in \mathbb{N}$ such that $\mathrm{S} \in \mathcal{S}_k$. We say S has length k. Notation: $\#\mathrm{S} := k$. Observe that $\emptyset$ is also a string (of length 0).

Definition 1.12

Let k and ℓ be positive integers. We define the composition of two strings via concatenation.

$$\circ\colon \mathcal{S}_k \times \mathcal{S}_\ell \longrightarrow \mathcal{S}_{k+\ell}, \qquad (\mathrm{S}_0, ..., \mathrm{S}_{k-1}) \circ (\mathrm{S}'_0, ..., \mathrm{S}'_{\ell-1}) := (\mathrm{S}_0, ..., \mathrm{S}_{k-1}, \mathrm{S}'_0, ..., \mathrm{S}'_{\ell-1})$$

and set $\mathrm{S} \circ \emptyset := \mathrm{S}$, $\emptyset \circ \mathrm{S} := \mathrm{S}$ for every string S. We often write $\mathrm{S}_1\mathrm{S}_2$ instead of $\mathrm{S}_1 \circ \mathrm{S}_2$. In the same way i.e. by repeated concatenation we define the expressions S^k where we again set $\mathrm{S}^0 := \emptyset$. We additionally define $\mathrm{S} \cdot 1 := \mathrm{S}$ and $\mathrm{S} \cdot 0 = \emptyset$ with $0, 1 \in \mathbb{N}$.

Definition 1.13

We define the reduction map $\mathcal{R}$ as

$$\mathcal{R}\colon \bigcup_{k=0}^{\infty} \mathcal{S}_k \longrightarrow \bigcup_{k=0}^{\infty} \mathcal{S}_k, \qquad (\mathrm{S}_0, ..., \mathrm{S}_{k-1}) \longmapsto \prod_{i=0}^{k-1} \mathrm{S}_i \, \mathbb{1}_{\{\mathrm{E},\mathrm{D}\}}(\mathrm{S}_i).$$

$\mathcal{R}$ is the map that removes all Bs from a string.

Definition 1.14

Let S be a string. We call S valid if there exists $r \in \mathbb{N}$ such that

$$\mathcal{R}(\mathrm{S}) = (\mathrm{ED})^r.$$

Again 0 is a valid choice for r.

Example 1.15

The valid strings of length three are

$$\mathrm{BBB},\ \mathrm{EDB},\ \mathrm{EBD},\ \mathrm{BED}\,.$$

Also note that there are exactly 2^{n-1} valid strings of length n, but this will not be of any importance for the proof.

Proof (continued)

Let $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ and $\mathrm{S} = (\mathrm{S}_0, ..., \mathrm{S}_{\#\mathrm{S}-1})$ a string of length at least $n-k+1$. We define

$$I_{\mathrm{S}}^{(k,n)}(t_{k-1}, z_{k-1}) := \int\limits_{t_{k-1}<t_k<...<t_n<1} \int\limits_{\mathbb{R}^{n-k+1}} \prod_{j=k}^{n} g(t_j, z_j)\, \mathrm{S}_{j-k}(t_j - t_{j-1}, z_j - z_{j-1}) \; \mathrm{d}z_k ... \mathrm{d}z_n \, \mathrm{d}t_k ... \mathrm{d}t_n$$

and

$$I_{\mathrm{S}}(t_0, z_0) := I_{\mathrm{S}}^{(1,\#\mathrm{S})}(t_0, z_0).$$

Claim:

$$J_n^{(k)}(t_{k-1}, z_{k-1}) = \sum_{\ell=1}^{2^{n-k}} \pm I_{\mathrm{S}^{(\ell)}}^{(k,n)}(t_{k-1}, z_{k-1}), \qquad \forall 1 \leq k \leq n,\ t_{k-1} \in [0,1],\ z_{k-1} \in \mathbb{R}^d \tag{1.9.1}$$

where $\mathrm{S}^{(\ell)}$ is for every ℓ a valid string of length $n-k+1$. Fix some $n \in \mathbb{N}$. We prove (1.9.1) by induction on k. Let $k = n$ we then have

$$J_n^{(n)}(t_{n-1}, z_{n-1}) = \int_{t_{n-1}}^{1} \mathrm{d}t_n \int_{\mathbb{R}} g'(t_n, z_n)\, \mathrm{E}(t_n - t_{n-1}, z_n - z_{n-1})\ \mathrm{d}z_n.$$

By integration by parts and since g is compactly supported, we obtain

$$= - \int_{t_{n-1}}^{1} \mathrm{d}t_n \int_{\mathbb{R}} g(t_n, z_n)\, \mathrm{B}(t_n - t_{n-1}, z_n - z_{n-1})\ \mathrm{d}z_n.$$

Observe that B is a valid string, so (1.9.1) is true for $k = n$. Now, assuming (1.9.1) for some $k > 1$, we have

$$J_n^{(k-1)}(t_{k-2}, z_{k-2}) = \int_{t_{k-2}}^{1} \mathrm{d}t_{k-1} \int_{\mathbb{R}} g'(t_{k-1}, z_{k-1})\, \mathrm{E}(t_{k-1} - t_{k-2}, z_{k-1} - z_{k-2}) J_n^{(k)}(t_{k-1}, z_{k-1})\ \mathrm{d}z_{k-1}.$$

Again, integration by parts yields

$$\begin{aligned} = &- \int_{t_{k-2}}^{1} \mathrm{d}t_{k-1} \int_{\mathbb{R}} g(t_{k-1}, z_{k-1})\, \mathrm{B}(t_{k-1} - t_{k-2}, z_{k-1} - z_{k-2}) J_n^{(k)}(t_{k-1}, z_{k-1})\ \mathrm{d}z_{k-1} \\ &- \int_{t_{k-2}}^{1} \mathrm{d}t_{k-1} \int_{\mathbb{R}} g(t_{k-1}, z_{k-1})\, \mathrm{E}(t_{k-1} - t_{k-2}, z_{k-1} - z_{k-2}) \partial_{z_{k-1}} J_n^{(k)}(t_{k-1}, z_{k-1})\ \mathrm{d}z_{k-1}. \end{aligned}$$

Where the last partial derivative can be easily calculated using the induction hypothesis

$$\partial_{z_{k-1}} J_n^{(k)}(t_{k-1}, z_{k-1}) \overset{(1.9.1)}{=} \sum_{\ell=1}^{2^{n-k}} \pm \partial_{z_{k-1}} I_{\mathrm{S}^{(\ell)}}^{(k,n)}(t_{k-1}, z_{k-1}).$$

Since g, E, B and D are all smooth functions we can differentiate inside the integral.

$$\begin{aligned} &= \sum_{\ell=1}^{2^{n-k}} \pm \int_{t_{k-1}<t_k<\ldots<t_n<1} \int_{\mathbb{R}^{n-k+1}} \prod_{j=k}^{n} g(t_j, z_j) \partial_{z_{k-1}} \mathrm{S}_{j-k}^{(\ell)}(t_j - t_{j-1}, z_j - z_{j-1})\ \mathrm{d}z_k \ldots \mathrm{d}z_n\, \mathrm{d}t_k \ldots \mathrm{d}t_n \\ &= \sum_{\ell=1}^{2^{n-k}} \mp \int_{t_{k-1}<t_k<\ldots<t_n<1} \int_{\mathbb{R}^{n-k+1}} \prod_{j=k}^{n} g(t_j, z_j)\, \tilde{\mathrm{S}}_{j-k}^{(\ell)}(t_j - t_{j-1}, z_j - z_{j-1})\ \mathrm{d}z_k \ldots \mathrm{d}z_n\, \mathrm{d}t_k \ldots \mathrm{d}t_n \\ &= \sum_{\ell=1}^{2^{n-k}} \mp I_{\tilde{\mathrm{S}}^{(\ell)}}^{(k,n)}(t_{k-1}, z_{k-1}), \end{aligned}$$

where $\tilde{\mathrm{S}}$ is defined as

$$\tilde{\mathrm{S}} = \begin{cases} \mathrm{BS}^*, & \text{if } \mathrm{S} = \mathrm{ES}^* \\ \mathrm{DS}^*, & \text{if } \mathrm{S} = \mathrm{BS}^* \end{cases}$$

Because S is a valid string $\tilde{\mathrm{S}}$ is well-defined. Also observe that $\tilde{\mathrm{S}}$ is no longer a valid string, but $\mathrm{E}\tilde{\mathrm{S}}$ is again valid. Applying this to the above equation results in

$$J_n^{(k-1)}(t_{k-2}, z_{k-2}) = \sum_{\ell=1}^{2^{n-k}} \mp I_{\mathrm{BS}^{(\ell)}}^{(k-1,n)}(t_{k-2}, z_{k-2}) + \sum_{\ell=1}^{2^{n-k}} \pm I_{\mathrm{E}\tilde{\mathrm{S}}^{(\ell)}}^{(k-1,n)}(t_{k-2}, z_{k-2}).$$

This proves claim (1.9.1). We still need to prove that $n! J_n^{(1)}(0,0) \leq C^n \Gamma\left({}^n\!/_2 + 1\right)$. To this end we use (1.9.1) and estimate terms of the form $I_{\mathrm{S}}^{(k,n)}$ where S is a valid string. Our strategy will be the following: First we proof a rather general estimate in Proposition 1.16 using a discretization argument and Fourier transformation. We apply this inequality to obtain estimates for the strings ED and BD (Corollary 1.17). These estimates will be improved in Proposition 1.19 to enable us to use an induction argument over the length of the string. The induction is carried out in Proposition 1.22, which yields estimate (1.22.1) thus enabling us to finish the proof of this proposition on page 37.

Proposition 1.16

There exists a constant $C \in \mathbb{R}$ such that for all real-valued Borel functions ϕ and h on $[0,1] \times \mathbb{R}$ with $|\phi(s,z)| \leq e^{-z^2/3s}$ for all $(s,z) \in [{}^1\!/_4, 1] \times \mathbb{R}$ and $|h(t,y)| \leq 1$ everywhere the estimate

$$\int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(s,z) h(t,y)\, \mathrm{D}(t-s, y-z)\ \mathrm{d}y\, \mathrm{d}z \leq C$$

holds.

Proof

Let I denote the above integral. For ℓ, $m \in \mathbb{Z}$ define

$$\begin{aligned} \phi_\ell(s,z) &:= \mathbb{1}_{[\ell,\ell+1[}(z)\phi(s,z), \\ h_m(t,y) &:= \mathbb{1}_{[m,m+1[}(y)h(t,y). \end{aligned}$$

With $I_{\ell,m}$ we denote the integral I where ϕ, h is replaced by ϕ_ℓ, h_m, respectively. We then have $I = \sum_{\ell,m\in\mathbb{Z}} I_{\ell,m}$ as long as the sum converges absolutely. We show the convergence of the sum in two steps.

Case 1: $|\ell - m| =: k \geq 2$

If $|\ell - m| =: k \geq 2$ we have

$$1 \leq k-1 \leq |y-z| \leq k+1, \qquad \forall z \in [\ell, \ell+1[, y \in [m, m+1[. \tag{1.16.1}$$

With this we obtain the following bound on D

$$|\,\mathrm{D}(t-s, y-z)| \overset{1.10}{=} \left| \frac{1}{\sqrt{2\pi}} \frac{(y-z)^2 - (t-s)}{(t-s)^{5/2}} e^{-(y-z)^2/2(t-s)} \right|.$$

For $t \in [1/2, 1]$, $s \in [t/2, t[$ we have $t - s \in]0, 1/2]$. Since $(y-z)^2 \geq 1$ by (1.16.1) we have

$$= \frac{1}{\sqrt{2\pi}} \frac{(y-z)^2 - (t-s)}{(t-s)^{5/2}} e^{-(y-z)^2/2(t-s)}.$$

And because of the fact that $k - 1 \leq |y - z| \leq k + 1$ we get

$$\leq \frac{1}{\sqrt{2\pi}} \frac{(k+1)^2 - (t-s)}{(t-s)^{5/2}} e^{-(k-1)^2/2(t-s)}.$$

Invoking Proposition A.2 with $x := t - s \in]0, 1/2]$ leads us to

$$\overset{A.2}{\leq} C_1 e^{(k-3/2)/(t-s)} e^{-(k-1)^2/2(t-s)} = C_1 e^{-(k-2)^2/2(t-s)} \leq C_1 e^{-(k-2)^2}$$

with $C_1 = C(2\pi)^{-1/2}$ where C is the constant from Proposition A.2. Using this estimate we deduce the following bound on $I_{\ell,m}$

$$\begin{aligned} I_{\ell,m} &\leq \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \int_{\mathbb{R}} |\phi_\ell(s,z) h_m(t,y)\, \mathrm{D}(t-s, y-z)| \;\mathrm{d}y\, \mathrm{d}z \\ &\leq C_1 \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\ell}^{\ell+1} \mathrm{d}z \int_{m}^{m+1} \mathrm{d}y \; e^{-z^2/3s} e^{-(k-2)^2} \leq C_1 \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \; e^{-\ell^2/3s} e^{-(k-2)^2} \\ &\leq C_1 \int_{1/2}^{1} \mathrm{d}t \; \frac{t}{2} e^{-\ell^2/3t} e^{-(k-2)^2} \leq \frac{C_1}{4} e^{-\ell^2/3} e^{-(k-2)^2}. \end{aligned}$$

And hence

$$\begin{aligned} \sum_{|\ell-m|\geq 2} I_{\ell,m} &= \sum_{k=2}^{\infty} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \{\ell-k, k-\ell\}} I_{\ell,m} \quad \leq \frac{C_1}{2} \sum_{k=2}^{\infty} \sum_{\ell \in \mathbb{Z}} e^{-\ell^2/3} e^{-(k-2)^2} \\ &= \frac{C_1}{2} \sum_{k=2}^{\infty} e^{-(k-2)^2} \sum_{\ell \in \mathbb{Z}} e^{-\ell^2/3} \leq C_2. \end{aligned}$$

For some $C_2 \in \mathbb{R}$ since both sums converge.

Case 2: $|\ell - m| \leq 1$

Let $\mathcal{F}$ denote the Fourier transformation. We define

$$\begin{aligned}
\hat{\phi}_\ell(s,\xi) &:= \mathcal{F}[\phi_\ell(s,\,\cdot\,)](\xi),\\
\hat{h}_m(t,\xi) &:= \mathcal{F}[h_m(t,\,\cdot\,)](\xi),\\
\hat{\mathrm{D}}(t,\xi) &:= \mathcal{F}[\mathrm{D}(t,\,\cdot\,)](\xi).
\end{aligned}$$

Since $\phi_\ell(s,\,\cdot\,)$, $\hat{\phi}_\ell(s,\,\cdot\,)$, $h_m(t,\,\cdot\,)$, $\hat{h}_m(t,\,\cdot\,) \in L^2(\mathbb{R})$ w.r.t. the second variable for $s \in [1/4, 1]$ and $t \in [0,1]$ we can use the Plancherel Theorem to obtain

$$\int_{\mathbb{R}} \hat{\phi}_\ell(s,\xi)^2 \,\mathrm{d}\xi = \int_{\mathbb{R}} \phi_\ell(s,z)^2 \,\mathrm{d}z \leq \int_{\ell}^{\ell+1} e^{-2z^2/3s} \,\mathrm{d}z \leq e^{-2\ell^2/3}, \qquad \forall s \in [1/4, 1] \tag{1.16.2}$$

$$\int_{\mathbb{R}} \hat{h}_m(t,-\xi)^2 \,\mathrm{d}\xi = \int_{\mathbb{R}} h_m(t,-z)^2 \,\mathrm{d}z \leq 1, \qquad \forall t \in [0,1]. \tag{1.16.3}$$

Using these estimates we will now prove the boundedness of $I_{\ell,m}$. Observe that the innermost integration can be written as the convolution of h_m with D.

$$\begin{aligned}
I_{\ell,m} &= \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\ell(s,z) h_m(t,y)\, \mathrm{D}(t-s, y-z) \,\mathrm{d}y\,\mathrm{d}z\\
&= \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \phi_\ell(s,z) \int_{\mathbb{R}} h_m(t,y)\, \mathrm{D}(t-s, z-y) \,\mathrm{d}y\,\mathrm{d}z\\
&= \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \phi_\ell(s,z) (h_m(t,\,\cdot\,) * \mathrm{D}(t-s,\,\cdot\,))(z) \,\mathrm{d}z.
\end{aligned}$$

Since $\mathrm{D}, \hat{\mathrm{D}} \in L^2(\mathbb{R})$ using Parseval's Theorem together with the convolution theorem yields

$$= \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \hat{\phi}_\ell(s,\xi) \overline{\hat{h}_m(t,\xi) \hat{\mathrm{D}}(t-s,\xi)} \,\mathrm{d}\xi.$$

Calculating the Fourier transformation of D (see Proposition B.2 for details) yields

$$= \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \hat{\phi}_\ell(s,\xi) \hat{h}_m(t,-\xi) (-4\pi^2\xi^2) e^{-2\pi^2(t-s)\xi^2} \,\mathrm{d}\xi.$$

Using the Young inequality $ab \le \frac{1}{2}(a^2c + b^2c^{-1})$ with $a = |\hat{\phi}_l(s,\xi)|$, $b = |\hat{h}_m(t,-\xi)|$ and $c = e^{\ell^2/3}$ we get

$$I_{\ell,m} \le \underbrace{2\pi^2 \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \hat{\phi}_\ell(s,\xi)^2 e^{\ell^2/3} \xi^2 e^{-2\pi^2(t-s)\xi^2}\, \mathrm{d}\xi}_{=:\mathrm{A}_1}$$
$$+ \underbrace{2\pi^2 \int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}} \hat{h}_m(t,-\xi)^2 e^{-\ell^2/3} \xi^2 e^{-2\pi^2(t-s)\xi^2}\, \mathrm{d}\xi}_{=:\mathrm{A}_2}.$$

We estimate A_1 and A_2 separately. Let us first consider A_1. Using Fubini's Theorem we can switch the t with the s integration.

$$\mathrm{A}_1 = 2\pi^2 \int_{1/4}^{1} \mathrm{d}s \int_{1/2 \vee s}^{2s \wedge 1} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{\phi}_\ell(s,\xi)^2 e^{\ell^2/3} \xi^2 e^{-2\pi^2(t-s)\xi^2}$$

To estimate this integral we first integrate w.r.t. t.

$$= 2\pi^2 \int_{1/4}^{1} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{\phi}_\ell(s,\xi)^2 e^{\ell^2/3} \xi^2 e^{2\pi^2 s\xi^2} \int_{1/2 \vee s}^{2s \wedge 1} \mathrm{d}t\ e^{-2\pi^2 t\xi^2}$$
$$= \int_{1/4}^{1} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{\phi}_\ell(s,\xi)^2 e^{\ell^2/3} e^{2\pi^2 s\xi^2} \Big[e^{-2\pi^2(1/2 \vee s)\xi^2} \underbrace{- e^{-2\pi^2(2s\wedge 1)\xi^2}}_{\le 0} \Big]$$
$$\le \int_{1/4}^{1} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{\phi}_\ell(s,\xi)^2 e^{\ell^2/3} e^{2\pi^2(s - 1/2 \vee s)\xi^2}$$

Since $s - (1/2 \vee s) \le 0$ for $s \in [1/4, 1]$ the last factor can be estimated by 1. Using the above estimate (1.16.2) results in

$$\le e^{\ell^2/3} \int_{1/4}^{1} \mathrm{d}s \int_{\mathbb{R}} \hat{\phi}_\ell(s,\xi)^2\, \mathrm{d}\xi \le e^{\ell^2/3} e^{-2\ell^2/3} = e^{-\ell^2/3}.$$

Let us now consider the second summand A_2. Integrating first w.r.t. s yields

$$\begin{aligned}\mathrm{A}_2 &= 2\pi^2 \int_{1/2}^{1} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{h}_m(t,-\xi)^2 e^{-\ell^2/3} \xi^2 e^{-2\pi^2 t\xi^2} \int_{t/2}^{t} \mathrm{d}s\ e^{2\pi^2 s\xi^2} \\ &= \int_{1/2}^{1} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{h}_m(t,-\xi)^2 e^{-\ell^2/3} e^{-2\pi^2 t\xi^2} \Big[e^{2\pi^2 t\xi^2} \underbrace{-e^{\pi^2 t\xi^2}}_{\leq 0} \Big] \\ &\leq e^{-\ell^2/3} \int_{1/2}^{1} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\xi\ \hat{h}_m(t,-\xi)^2.\end{aligned}$$

By applying the estimate (1.16.3) we deduce

$$\mathrm{A}_2 \leq e^{-\ell^2/3}.$$

We therefore have

$$I_{\ell,m} \leq \mathrm{A}_1 + \mathrm{A}_2 \leq 2e^{-\ell^2/3}.$$

This implies

$$\sum_{|\ell-m|\leq 1} I_{\ell,m} = \sum_{\ell\in\mathbb{Z}} \sum_{m\in\{\ell,\ell\pm 1\}} I_{\ell,m} \leq 6 \sum_{\ell\in\mathbb{Z}} e^{-\ell^2/3} \leq C_3$$

for some $C_3 \in \mathbb{R}$ which concludes the proof.

□

Below we apply this proposition to obtain estimates for the term I_{S} where S is either ED or BD. Note that DD can never be part of any valid string.

Corollary 1.17

Let g and h be real-valued Borel functions on $[0,1]\times\mathbb{R}$ bounded by 1. Then

(i) $$\int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{E}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)\ \mathrm{d}y\,\mathrm{d}z\ \leq C$$

(ii) $$\int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)\ \mathrm{d}y\,\mathrm{d}z\ \leq C$$

holds, where C is 8 times the constant from Proposition 1.16.

Proof

(i):

Since

$$|g(s,z)\,\mathrm{E}(s,z)| \overset{1.10}{\leq} (2\pi s)^{-1/2} e^{-z^2/2s} \leq (\pi/2)^{-1/2} e^{-z^2/2s} \leq e^{-z^2/2s} \leq e^{-z^2/3s}, \qquad \forall s \in [1/4, 1]$$

the assertion follows from Proposition 1.16 with $\phi(s,z) := g(s,z)\,\mathrm{E}(s,z)$.

(ii):

With a similar calculation as in (i) we obtain

$$\begin{aligned} |g(s,z)\,\mathrm{B}(s,z)| &\overset{1.10}{\leq} (2\pi s)^{-1/2}(|z|/s)e^{-z^2/2s} \leq (\pi/2)^{-1/2} 4|z|e^{-z^2/2s}, && \forall s \in [1/4, 1] \\ &\leq 4|z|e^{-z^2/2s} \leq 8e^{z^2/6}e^{-z^2/2s} \leq 8e^{z^2/6s}e^{-z^2/2s} = 8e^{-z^2/3s}, && \forall s \in [1/4, 1] \end{aligned}$$

Again, the assertion follows from Proposition 1.16 with $\phi(s,z) := \frac{1}{8} g(s,z)\,\mathrm{B}(s,z)$.

□

To be able to prove Proposition 1.9 via induction we need a much stronger estimate than the one in Corollary 1.17. In the induction we will get terms of the form $g\,\mathrm{B}\,h\,\mathrm{D}(1-t)^r$ and we need to control the dependence on r as precisely as possible. Also, since we are integrating over the set $\{t_0 < t_1 < ... < t_n < 1\}$ we need an estimate that reflects the dependence on t_0. We will obtain such an estimate in Proposition 1.19. The following lemma is needed to prove this improved estimate.

Lemma 1.18

We have the following bounds.

(i) $$\int_{\mathbb{R}} |\,\mathrm{B}(s,z)|\,\mathrm{d}z \leq \sqrt{2/\pi}\, s^{-1/2}$$

(ii) $$\int_{\mathbb{R}} |\,\mathrm{D}(t,z)|\,\mathrm{d}z \leq 2t^{-1}$$

Proof

(i):

Using the symmetry of the integrand we can easily calculate

$$\int_{\mathbb{R}} |\,\mathrm{B}(s,z)|\,\mathrm{d}z \overset{1.10}{=} (2\pi s)^{-1/2} s^{-1} \int_{\mathbb{R}} |z| e^{-z^2/2s}\,\mathrm{d}z = (2\pi s)^{-1/2} 2 s^{-1} \underbrace{\int_0^\infty z e^{-z^2/2s}\,\mathrm{d}z}_{=s} = \sqrt{2/\pi}\, s^{-1/2}.$$

(ii):

Triangle inequality and integration by parts yield

$$\begin{aligned}\int_{\mathbb{R}} |\,\mathrm{D}(t,z)|\;\mathrm{d}z &\overset{1.10}{=} (2\pi t)^{-1/2}t^{-2}\int_{\mathbb{R}} |z^2-t|e^{-z^2/2t}\;\mathrm{d}z\\ &\leq (2\pi t)^{-1/2}t^{-2}\int_{\mathbb{R}} z^2e^{-z^2/2t}\;\mathrm{d}z + (2\pi t)^{-1/2}t^{-1}\int_{\mathbb{R}} e^{-z^2/2t}\;\mathrm{d}z\\ &= (2\pi t)^{-1/2}t^{-1}\int_{\mathbb{R}} e^{-z^2/2t}\;\mathrm{d}z + t^{-1} = 2t^{-1}.\end{aligned}$$

□

We are now ready to prove the crucial estimate which is necessary to complete the proof of Proposition 1.9.

Proposition 1.19

There exists $C \in \mathbb{R}$ such that for all real-valued Borel functions g, h on $[0,1]\times\mathbb{R}$ bounded by 1 everywhere, $t_0 \in [0,1]$ and for all $r \geq 0$

$$\text{(i)}\quad \int_{t_0}^{1}\mathrm{d}t\int_{t_0}^{t}\mathrm{d}s\int_{\mathbb{R}^2} g(s,z)\,\mathrm{E}(s-t_0,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z \;\leq C\frac{(1-t_0)^{r+1}}{1+r}$$

$$\text{(ii)}\quad \int_{t_0}^{1}\mathrm{d}t\int_{t_0}^{t}\mathrm{d}s\int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s-t_0,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z \;\leq C\frac{(1-t_0)^{r+1/2}}{(1+r)^{1/2}}$$

holds.

Proof

For the proof we use the following strategy: First we note that by a simple transformation we only need to prove this estimate in the case of $t_0 = 0$ (step 3). We split the integral over s in two parts.

For the case $t/2 \leq s \leq t$ (step 1) we split the set of the t integration $[0,1]$ into the sets $[2^{-k-1}, 2^{-k}]$ for $k \in \mathbb{N}$. This enables us to use Corollary 1.17. Instead of a constant we will get a quite complicated sum on the right-hand side of the inequality. A careful analysis of this sum is carried out in Appendix A. In part (i) of the proof this is mainly done by estimating the sum by its integral (Proposition A.3). In part (ii) the sum is more complicated. By estimating the sum by its integral we will arrive at the beta function which we estimate using Stirling's formula (Lemma A.5, Proposition A.4).

For the case $0 \leq s \leq t/2$ the previous Lemma 1.18 will be applied to obtain the required bound. Again, part (ii) will be more complicated since the beta function will turn up again.

We now turn to the details of the proof.

(i):

Step 1: Estimate for $t/2 \leq s \leq t$

We use Corollary 1.17.(i) to get

$$\int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(2^{-k}s, 2^{-k/2}z)\, \mathrm{E}(s,z) h(2^{-k}t, 2^{-k/2}y)\, \mathrm{D}(t-s, y-z)\ \mathrm{d}y\, \mathrm{d}z \leq C_1$$

for every $k \in \mathbb{N}$ and some $C_1 \in \mathbb{R}$. Consider the following transformation

$$t' = 2^{-k}t, \qquad s' = 2^{-k}s, \qquad y' = 2^{-k/2}y, \qquad z' = 2^{-k/2}z.$$

An easy calculation shows

$$\mathrm{E}(s,z) = 2^{-k/2}\, \mathrm{E}(s',z'),$$
$$\mathrm{D}(t-s, y-z) = 2^{-3k/2}\, \mathrm{D}(t'-s', y'-z').$$

Applying the transformation results in

$$\Rightarrow \int_{2^{-k-1}}^{2^{-k}} \mathrm{d}t' \int_{t'/2}^{t'} \mathrm{d}s' \int_{\mathbb{R}^2} 2^k g(s',z')\, \mathrm{E}(s',z') h(t',y')\, \mathrm{D}(t'-s', y'-z')\ \mathrm{d}y' \mathrm{d}z' \leq C_1.$$

Multiplying with 2^{-k} and putting $(1-t)^r$ inside the integral yields

$$\Rightarrow \int_{2^{-k-1}}^{2^{-k}} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\, \mathrm{E}(s,z) h(t,y)\, \mathrm{D}(t-s, y-z)(1-t)^r\ \mathrm{d}y\, \mathrm{d}z$$
$$\leq C_1 2^{-k} \sup_{t \in [2^{-k-1}, 2^{-k}]} (1-t)^r = C_1 2^{-k}(1-2^{-k-1})^r.$$

We sum over $k \in \mathbb{N}$ to obtain

$$\int_{0}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\, \mathrm{E}(s,z) h(t,y)\, \mathrm{D}(t-s, y-z)(1-t)^r\ \mathrm{d}y\, \mathrm{d}z \leq C_1 \sum_{k=0}^{\infty} 2^{-k}(1-2^{-k-1})^r.$$

Using the estimate of Proposition A.3 we get

$$\int_{0}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\, \mathrm{E}(s,z) h(t,y)\, \mathrm{D}(t-s, y-z)(1-t)^r\ \mathrm{d}y\, \mathrm{d}z \leq C_2(1+r)^{-1}$$

where $C_2 = C_1 C$ with C being the constant in Proposition A.3.

Step 2: Estimate for $0 \leq s \leq t/2$

Let us now turn to the case of $0 \leq s \leq t/2$. We have

$$\int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{E}(s,z) h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r \,\mathrm{d}y\,\mathrm{d}z$$

$$\leq \int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\ (1-t)^r \int_{\mathbb{R}} \mathrm{E}(s,z) \int_{\mathbb{R}} |\,\mathrm{D}(t-s,y-z)|\,\mathrm{d}y\,\mathrm{d}z.$$

Applying Lemma 1.18.(ii) gives us

$$\leq 2 \int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\ (1-t)^r \int_{\mathbb{R}} \mathrm{E}(s,z)(t-s)^{-1}\,\mathrm{d}z.$$

Using that E is a probability density results in

$$= 2 \int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\ (1-t)^r \underbrace{(t-s)^{-1}}_{\leq (t/2)^{-1}} \leq 2 \int_0^1 (1-t)^r \,\mathrm{d}t = 2(1+r)^{-1}.$$

Combining the estimates of step 1 and 2 yields the required bound proving the assertion in the case $t_0 = 0$.

Step 3: Reduction to $t_0 = 0$

In the case of $t_0 > 0$ consider the following transformation

$$t' = \frac{t-t_0}{1-t_0}, \qquad s' = \frac{s-t_0}{1-t_0}, \qquad y' = y(1-t_0)^{-1/2}, \qquad z' = z(1-t_0)^{-1/2}.$$

The same calculations as in step 1 yield

$$(1-t)^r = (1-t')^r(1-t_0)^r,$$
$$\mathrm{E}(s-t_0,z) = (1-t_0)^{-1/2}\,\mathrm{E}(s',z'),$$
$$\mathrm{D}(t-s,y-z) = (1-t_0)^{-3/2}\,\mathrm{D}(t'-s',y'-z').$$

We set

$$\tilde{g}(s',z') := g(s'(1-t_0)+t_0, z'(1-t_0)^{1/2}),$$
$$\tilde{h}(t',y') := h(t'(1-t_0)+t_0, y'(1-t_0)^{1/2}).$$

Observe that $\tilde{g}$ and $\tilde{h}$ are still bounded by 1 everywhere. Using that, we can rewrite the following integral to

$$\int_{t_0}^{1} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{E}(s-t_0,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r(1-t_0)^{-r-1}\;\mathrm{d}y\,\mathrm{d}z$$
$$= \int_{0}^{1} \mathrm{d}t' \int_{0}^{t'} \mathrm{d}s' \int_{\mathbb{R}^2} \tilde{g}(s',z')\,\mathrm{E}(s',z')\tilde{h}(t',y')\,\mathrm{D}(t'-s',y'-z')(1-t')^r\;\mathrm{d}y'\mathrm{d}z'.$$

Therefore, it is sufficient to show the assertion for $t_0 = 0$. This completes the first part of the proof.

(ii):

Step 1: Estimate for $t/2 \le s \le t$

We use Corollary 1.17.(ii) to get

$$\int_{1/2}^{1} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(2^{-k}s, 2^{-k/2}z)\,\mathrm{E}(s,z)h(2^{-k}t, 2^{-k/2}y)\,\mathrm{D}(t-s,y-z)\;\mathrm{d}y\,\mathrm{d}z \le C_1$$

for any $k \in \mathbb{N}$ and some $C_1 \in \mathbb{R}$. Consider the following transformation

$$t' = 2^{-k}t, \qquad s' = 2^{-k}s, \qquad y' = 2^{-k/2}y, \qquad z' = 2^{-k/2}z.$$

A similar calculation as in part (i) yields

$$\mathrm{B}(s,z) = 2^{-k}\,\mathrm{B}(s',z'),$$
$$\mathrm{D}(t-s,y-z) = 2^{-3k/2}\,\mathrm{D}(t'-s',y'-z').$$

Applying the transformation results in

$$\Rightarrow \int_{2^{-k-1}}^{2^{-k}} \mathrm{d}t' \int_{t'/2}^{t'} \mathrm{d}s' \int_{\mathbb{R}^2} 2^{k/2} g(s',z')\,\mathrm{B}(s',z')h(t',y')\,\mathrm{D}(t'-s',y'-z')\;\mathrm{d}y'\,\mathrm{d}z' \le C_1.$$

Multiplying with $2^{-k/2}$ and putting $(1-t)^r$ inside the integral yields

$$\Rightarrow \int_{2^{-k-1}}^{2^{-k}} \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z$$
$$\le C_1 2^{-k/2} \sup_{t\in[2^{-k-1},2^{-k}]} (1-t)^r = C_1 2^{-k/2}(1-2^{-k-1})^r.$$

We sum over $k \in \mathbb{N}$ to obtain

$$\int_0^1 \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z \le C_1 \sum_{k=0}^{\infty} 2^{-k/2}(1-2^{-k-1})^r.$$

Using the estimate shown in Proposition A.5 we get

$$\int_0^1 \mathrm{d}t \int_{t/2}^{t} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z \le C_2(1+r)^{-1/2}$$

where $C_2 = C_1 C$ with C being the constant from Proposition A.5.

Step 2: Estimate for $0 \le s \le t/2$

Let us now turn to the case of $0 \le s \le t/2$. We have

$$\begin{aligned}
&\int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s \int_{\mathbb{R}^2} g(s,z)\,\mathrm{B}(s,z)h(t,y)\,\mathrm{D}(t-s,y-z)(1-t)^r\;\mathrm{d}y\,\mathrm{d}z \\
&\quad\le \int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\;(1-t)^r \int_{\mathbb{R}} |\,\mathrm{B}(s,z)| \int_{\mathbb{R}} |\,\mathrm{D}(t-s,y-z)|\;\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Using Lemma 1.18.(ii) yields

$$\le 2\int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\;(1-t)^r \int_{\mathbb{R}} |\,\mathrm{B}(s,z)|(t-s)^{-1}\;\mathrm{d}z.$$

We apply Lemma 1.18.(i) in order to obtain

$$\begin{aligned}
&\le 2\sqrt{2/\pi}\int_0^1 \mathrm{d}t \int_0^{t/2} \mathrm{d}s\;(1-t)^r \underbrace{(t-s)^{-1}}_{\le 2/t} s^{-1/2} \le 2\sqrt{2/\pi}\int_0^1 \mathrm{d}t\;(1-t)^r \frac{2}{t} \underbrace{\int_0^{t/2} s^{-1/2}\;\mathrm{d}s}_{=\sqrt{2t}} \\
&= 8\pi^{-1/2}\int_0^1 (1-t)^r t^{-1/2}\;\mathrm{d}t = 8\pi^{-1/2}\beta\left(\frac{1}{2}, r+1\right) = 8\pi^{-1/2}\frac{\Gamma(1/2)\Gamma(r+1)}{\Gamma(r+3/2)} \\
&= \frac{8\Gamma(r+1)}{\Gamma(r+3/2)}.
\end{aligned}$$

Where β is the beta function. We use Lemma A.4 with $\alpha = \frac{1}{2}$ to estimate the gamma function, leading us to

$$\le 8e^{7/12}(r+1)^{-1/2}.$$

Combining the estimates of step 1 and 2 yield the required bound proving the assertion in the case $t_0 = 0$.

Step 3: Reduction to $t_0 = 0$

In the case of $t_0 > 0$ consider the following transformation

$$t' = \frac{t - t_0}{1 - t_0}, \qquad s' = \frac{s - t_0}{1 - t_0}, \qquad y' = y(1 - t_0)^{-1/2}, \qquad z' = z(1 - t_0)^{-1/2}.$$

The same calculations as in step 1 yield

$$\begin{aligned}
&(1 - t)^r = (1 - t')^r (1 - t_0)^r, \\
&\mathrm{B}(s - t_0, z) = (1 - t_0)^{-1} \,\mathrm{B}(s', z'), \\
&\mathrm{D}(t - s, y - z) = (1 - t_0)^{-3/2} \,\mathrm{D}(t' - s', y' - z').
\end{aligned}$$

We set

$$\begin{aligned}
&\tilde{g}(s', z') := g(s'(1 - t_0) + t_0, z'(1 - t_0)^{1/2}), \\
&\tilde{h}(t', y') := h(t'(1 - t_0) + t_0, y'(1 - t_0)^{1/2}).
\end{aligned}$$

Observe that $\tilde{g}$ and $\tilde{h}$ are still bounded by 1 everywhere. Using that, we can rewrite the following integral to

$$\begin{aligned}
&\int\limits_{t_0}^{1} \mathrm{d}t \int\limits_{t_0}^{t} \mathrm{d}s \int\limits_{\mathbb{R}^2} g(s, z) \,\mathrm{B}(s - t_0, z) h(t, y) \,\mathrm{D}(t - s, y - z)(1 - t)^r (1 - t_0)^{-r-1/2} \,\mathrm{d}y \,\mathrm{d}z \\
&\qquad = \int\limits_{0}^{1} \mathrm{d}t' \int\limits_{0}^{t'} \mathrm{d}s' \int\limits_{\mathbb{R}^2} \tilde{g}(s', z') \,\mathrm{B}(s', z') \tilde{h}(t', y') \,\mathrm{D}(t' - s', y' - z')(1 - t')^r \,\mathrm{d}y' \mathrm{d}z'.
\end{aligned}$$

Therefore it is sufficient to show the assertion for $t_0 = 0$. This completes the last part of the proof.

□

As a corollary we will trivially generalize this to the case where h is only bounded by some constant.

Corollary 1.20

There exists $C \in \mathbb{R}$ such that for all real-valued bounded Borel functions g, h on $[0,1] \times \mathbb{R}$ with $|g(s,z)| \leq 1$ everywhere, $t_0 \in [0,1]$ and for all $r \geq 0$

$$\text{(i)} \quad \int_{t_0}^{1} \mathrm{d}t_1 \int_{t_1}^{1} \mathrm{d}t_2 \int_{\mathbb{R}^2} g(t_1,z_1)\,\mathrm{E}(t_1-t_0,z_1)h(t_2,z_2)\,\mathrm{D}(t_2-t_1,z_2-z_1)(1-t_2)^r \; \mathrm{d}z_1\,\mathrm{d}z_2 \leq C\|h\|_\infty \frac{(1-t_0)^{r+1}}{1+r}$$

$$\text{(ii)} \quad \int_{t_0}^{1} \mathrm{d}t_1 \int_{t_1}^{1} \mathrm{d}t_2 \int_{\mathbb{R}^2} g(t_1,z_1)\,\mathrm{B}(t_1-t_0,z_1)h(t_2,z_2)\,\mathrm{D}(t_2-t_1,z_2-z_1)(1-t_2)^r \; \mathrm{d}z_1\,\mathrm{d}z_2 \leq C\|h\|_\infty \frac{(1-t_0)^{r+1/2}}{(1+r)^{1/2}}$$

holds.

Proof

The assertion is trivial for $\|h\|_\infty = 0$. Assume $\|h\|_\infty \neq 0$. We set

$$\tilde{h}(t,z) := \|h\|_\infty^{-1} h(t,z).$$

Using the fact that

$$\{(t,s) \in \mathbb{R}^2 | t \in [t_0,1],\; s \in [t_0,t]\} = \{(t_2,t_1) \in \mathbb{R}^2 | t_1 \in [t_0,1],\; t_2 \in [t_1,1]\}$$

and invoking Proposition 1.19 with $\tilde{h}$ instead of h concludes the proof.

□

We are now able to obtain the bounds on I_{S} which are required to complete the proof of Proposition 1.9. Before turning to the proof we first prove the following lemma which is necessary to simplify one term in the proof of Proposition 1.22.

Lemma 1.21

Let $m \in \mathbb{N}$ with $m \geq 1$, $n \in \mathbb{N}$ such that $n \geq m$ and $t_0 \in [0,1]$. Then the following identity holds.

$$\int_{t_0<\ldots<t_m<1} (1-t_m)^{(n-m-1)/2} \prod_{i=2}^{m} (t_i - t_{i-1})^{-1/2} \; \mathrm{d}t_1 \ldots \mathrm{d}t_m = \frac{\pi^{(m-1)/2}\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}$$

Proof

We proof the assertion via induction over m.

Base case: $m = 1$

Since $\frac{-2}{n}(1-t_1)^{n/2}$ is an anti-derivative of the integrand, a simple calculation shows

$$\int_{t_0}^{1} (1-t_1)^{(n-2)/2}\,\mathrm{d}t_1 = \frac{2}{n}(1-t_0)^{n/2} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\frac{n}{2}}(1-t_0)^{n/2} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.$$

Inductive step: $m \longrightarrow m+1$

Assume the assertion holds for some $m \in \mathbb{N}$. Rewriting the left-hand side of the assertion yields

$$\begin{aligned}
&\int\limits_{t_0<\ldots<t_{m+1}<1} (1-t_{m+1})^{(n-m-2)/2} \prod_{i=2}^{m+1} (t_i - t_{i-1})^{-1/2}\,\mathrm{d}t_1 \ldots \mathrm{d}t_{m+1} \\
&= \int\limits_{t_0<\ldots<t_m<1} \prod_{i=2}^{m} (t_i - t_{i-1})^{-1/2} \int_{t_m}^{1} (1-t_{m+1})^{(n-m-2)/2} (t_{m+1}-t_m)^{-1/2}\,\mathrm{d}t_{m+1}\,\mathrm{d}t_1 \ldots \mathrm{d}t_m.
\end{aligned}$$

We use the transformation

$$t'_{m+1} = \frac{t_{m+1}-t_m}{1-t_m}.$$

An easy calculation shows

$$(1-t'_{m+1})(1-t_m) = 1-t_{m+1}, \qquad \partial_{t_{m+1}} t'_{m+1} = (1-t_m)^{-1}.$$

By applying the transformation we obtain

$$= \int\limits_{t_0<\ldots<t_m<1} (1-t_m)^{(n-m-1)/2} \prod_{i=2}^{m} (t_i - t_{i-1})^{-1/2} \int_{0}^{1} (1-t'_{m+1})^{(n-m-2)/2}\, t'^{-1/2}_{m+1}\,\mathrm{d}t'_{m+1}\,\mathrm{d}t_1 \ldots \mathrm{d}t_m.$$

Using the induction hypothesis and rewriting the last integral with the help of the beta function results in

$$\begin{aligned}
&= \frac{\pi^{(m-1)/2}\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \beta\left(\frac{1}{2}, \frac{n-m}{2}\right)(1-t_0)^{n/2} = \frac{\pi^{(m-1)/2}\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\sqrt{\pi}\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}(1-t_0)^{n/2} \\
&= \frac{\pi^{m/2}\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.
\end{aligned}$$

□

We will estimate terms of the form I_{S} and finish the proof of Proposition 1.9.

Proposition 1.22

There exists $M \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ and for every valid string S of length n

$$|I_{\mathrm{S}}(t_0, z_0)| \le \frac{M^n}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}, \qquad \forall t_0 \in [0,1],\ z_0 \in \mathbb{R} \tag{1.22.1}$$

holds.

Proof

We prove the assertion by induction on n. The case $n = 1$ is clear by Lemma 1.18.(i) with $M = \sqrt{2}$, so let $n > 1$ and assume (1.22.1) holds for every valid string S of length less than n. We split the proof in the following three cases:

$$\begin{aligned} &\text{Case 1:}\ \mathrm{S} = \mathrm{BS}', && && \#\,\mathrm{S}' = n-1 \\ &\text{Case 2:}\ \mathrm{S} = \mathrm{EDS}', && && \#\,\mathrm{S}' = n-2 \\ &\text{Case 3:}\ \mathrm{S} = \mathrm{EB}^m\,\mathrm{DS}', && m \ge 1, && \#\,\mathrm{S}' = n-m-2 \end{aligned}$$

Observe that S' is in every case a valid string.

Case 1: $\mathrm{S} = \mathrm{BS}'$

We have

$$|I_{\mathrm{S}}(t_0, z_0)| = \left| \int_{t_0}^{1} \mathrm{d}t_1 \int_{\mathbb{R}} g(t_1, z_1)\, \mathrm{B}(t_1 - t_0, z_1 - z_0) I_{\mathrm{S}'}(t_1, z_1)\ \mathrm{d}z_1 \right|.$$

Using the inductive hypothesis and $|g| \le 1$ results in

$$\le \frac{M^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{t_0}^{1} \mathrm{d}t_1\ (1-t_1)^{(n-1)/2} \int_{\mathbb{R}} |\,\mathrm{B}(t_1 - t_0, z_1 - z_0)|\ \mathrm{d}z_1.$$

And with the help of Lemma 1.18.(i) we obtain

$$\le \frac{M^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)} \sqrt{2/\pi} \int_{t_0}^{1} (1-t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2}\ \mathrm{d}t_1.$$

We use the transformation

$$t_1' = \frac{t_1 - t_0}{1 - t_0}$$

to transform the integral into

$$= \frac{\sqrt{2}M^{n-1}}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}(1-t_0)^{n/2}\int_0^1 t_1'^{-1/2}(1-t_1')^{(n-1)/2}\,\mathrm{d}t_1'$$

$$= \frac{\sqrt{2}M^{n-1}}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}(1-t_0)^{n/2}\beta\left(\frac{1}{2},\frac{n+1}{2}\right) = \frac{\sqrt{2}M^{n-1}}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}(1-t_0)^{n/2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}.$$

If M is sufficiently large we obtain

$$= \frac{\sqrt{2}M^{n-1}}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2} \leq \frac{M^n}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.$$

Case 2: $\mathrm{S} = \mathrm{EDS}'$

We have

$$|I_{\mathrm{S}}(t_0,z_0)| = \Bigg|\int_{t_0}^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \int_{\mathbb{R}^2} g(t_1,z_1)\,\mathrm{E}(t_1-t_0,z_1-z_0) \cdot g(t_2,z_2)\,\mathrm{D}(t_2-t_1,z_2-z_1)I_{\mathrm{S}'}(t_2,z_2)\;\mathrm{d}z_1\,\mathrm{d}z_2\Bigg|.$$

Define

$$h(t,z) := g(t,z)I_{\mathrm{S}'}(t,z)(1-t)^{-(n-2)/2}.$$

So that by the inductive hypothesis and $|g| \leq 1$ we have the following bound on h

$$|h(t,z)| \leq \left|\frac{M^{n-2}}{\Gamma\left(\frac{n}{2}\right)}(1-t)^{(n-2)/2}(1-t)^{-(n-2)/2}\right| = \frac{M^{n-2}}{\Gamma\left(\frac{n}{2}\right)}.$$

By definition of h we establish that

$$|I_{\mathrm{S}}(t_0,z_0)| = \Bigg|\int_{t_0}^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \int_{\mathbb{R}^2} g(t_1,z_1)\,\mathrm{E}(t_1-t_0,z_1-z_0) \cdot h(t_2,z_2)\,\mathrm{D}(t_2-t_1,z_2-z_1)(1-t_2)^{(n-2)/2}\;\mathrm{d}z_1\,\mathrm{d}z_2\Bigg|.$$

Using Corollary 1.20.(i) we deduce

$$\leq C\frac{M^{n-2}}{\Gamma\left(\frac{n}{2}\right)}\frac{(1-t_0)^{n/2}}{\frac{n}{2}} = C\frac{M^{n-2}}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}$$

for some $C \in \mathbb{R}$. If M is sufficiently large we obtain

$$\leq \frac{M^{n}}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.$$

Case 3: $\mathrm{S} = \mathrm{E}\mathrm{B}^m\,\mathrm{D}\mathrm{S}'$

We have

$$\begin{aligned}|I_{\mathrm{S}}(t_0,z_0)| = \Bigg| \int\limits_{t_0<\ldots<t_{m+2}<1} \mathrm{d}t_1 \ldots \mathrm{d}t_{m+2} \int\limits_{\mathbb{R}^{m+2}} & g(t_1,z_1)\,\mathrm{E}(t_1-t_0, z_1-z_0) \\ & \cdot \prod_{i=2}^{m+1} g(t_i,z_i)\,\mathrm{B}(t_i-t_{i-1}, z_i-z_{i-1}) \\ & \cdot g(t_{m+2},z_{m+2})\,\mathrm{D}(t_{m+2}-t_{m+1}, z_{m+2}-z_{m+1}) \\ & \cdot I_{\mathrm{S}'}(t_{m+2},z_{m+2})\ \mathrm{d}z_1 \ldots \mathrm{d}z_{m+2}\Bigg|.\end{aligned}$$

Define

$$h(t,z) := g(t,z) I_{\mathrm{S}'}(t,z)(1-t)^{-(n-m-2)/2}.$$

So that by the inductive hypothesis and $|g| \leq 1$ we have the following bound on h

$$|h(t,z)| \leq \left|\frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)}(1-t)^{(n-m-2)/2}(1-t)^{-(n-m-2)/2}\right| = \frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)}.$$

By setting

$$\begin{aligned}\Omega(t,z) := \int_t^1 \mathrm{d}t_{m+1} \int_{t_{m+1}}^1 \mathrm{d}t_{m+2} \int\limits_{\mathbb{R}^2} & g(t_{m+1},z_{m+1})\,\mathrm{B}(t_{m+1}-t, z_{m+1}-z) \\ & \cdot h(t_{m+2},z_{m+2})(1-t_{m+2})^{(n-m-2)/2} \\ & \cdot \mathrm{D}(t_{m+2}-t_{m+1}, z_{m+2}-z_{m+1})\ \mathrm{d}z_{m+1}\,\mathrm{d}z_{m+2}\end{aligned}$$

we can use Corollary 1.20.(ii) to deduce

$$|\Omega(t,z)| \leq C\frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)}\frac{(1-t)^{(n-m-1)/2}}{\left(\frac{n-m}{2}\right)^{1/2}} = C\sqrt{\frac{2}{n-m}}\frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)}(1-t)^{(n-m-1)/2}. \qquad (1.22.2)$$

We can now rewrite $I_{\mathrm{S}}(t_0, z_0)$ as

$$|I_{\mathrm{S}}(t_0,z_0)| = \Bigg| \int\limits_{t_0<\ldots<t_m<1} \mathrm{d}t_1 \ldots \mathrm{d}t_m \int\limits_{\mathbb{R}^m} g(t_1,z_1)\, \mathrm{E}(t_1-t_0, z_1-z_0) \cdot \prod_{i=2}^{m} g(t_i,z_i)\, \mathrm{B}(t_i-t_{i-1}, z_i-z_{i-1}) \Omega(t_m,z_m)\ \mathrm{d}z_1 \ldots \mathrm{d}z_m \Bigg|.$$

Using (1.22.2) and the fact that $|g| \leq 1$ we get

$$\leq C\sqrt{\frac{2}{n-m}} \frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)} \int\limits_{t_0<\ldots<t_m<1} \mathrm{d}t_1 \ldots \mathrm{d}t_m \int\limits_{\mathbb{R}^m} \mathrm{E}(t_1-t_0, z_1-z_0) \prod_{i=2}^{m} |\, \mathrm{B}(t_i-t_{i-1}, z_i-z_{i-1})| \cdot (1-t_m)^{(n-m-1)/2}\ \mathrm{d}z_1 \ldots \mathrm{d}z_m.$$

With the help of Lemma 1.18.(i) and using the fact that E is a probability density we obtain

$$\leq C\sqrt{\frac{2}{n-m}} \frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)} (2/\pi)^{(m-1)/2} \int\limits_{t_0<\ldots<t_m<1} (1-t_m)^{(n-m-1)/2} \prod_{i=2}^{m} (t_i-t_{i-1})^{-1/2}\ \mathrm{d}t_1 \ldots \mathrm{d}t_m.$$

Invoking Lemma 1.21 yields

$$= C\sqrt{\frac{2}{n-m}} \frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)} (2/\pi)^{(m-1)/2} \frac{\pi^{(m-1)/2}\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} (1-t_0)^{n/2}$$

$$= C\sqrt{\frac{2^m}{n-m}} \frac{M^{n-m-2}}{\Gamma\left(\frac{n-m}{2}\right)} \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} (1-t_0)^{n/2}.$$

Setting $x := (n-m)/2$ and using Stirling's formula as in the proof of Lemma A.4 we obtain

$$\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} \leq \frac{(x+1/2)^x e^{-x-1/2} e^{\frac{1}{12(x+1/2)}}}{x^{x-1/2}e^{-x}} \overset{x\geq 0}{\leq} e^{-1/2}e^{1/6}\sqrt{x}\left(\frac{x+1/2}{x}\right)^x$$

$$= e^{-1/2}e^{1/6}\sqrt{x}e^{x\ln\left(\frac{x+1/2}{x}\right)}.$$

Applying the basic estimate $\ln x \leq x-1$ yields

$$\leq e^{1/6}\sqrt{x}.$$

With the help of this estimate we finally get

$$|I_{\mathrm{S}}(t_0,z_0)| \leq Ce^{1/6}\sqrt{\frac{2^m}{n-m}}\sqrt{\frac{n-m}{2}} \frac{M^{n-m-2}}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2} = Ce^{1/6}2^{(m-1)/2}\frac{M^{n-m-2}}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.$$

If M is sufficiently large we obtain

$$\leq \frac{M^n}{\Gamma\left(\frac{n}{2}+1\right)}(1-t_0)^{n/2}.$$

Looking back at these three cases, if we set M as

$$M := (1+C)e^{1/6}\sqrt{2}$$

where C is the constant from Corollary 1.20, M is "sufficiently large".

□

Finally, we are able to complete the proof of Proposition 1.9.

Proof (continued)

Recall that

$$J_n^{(k)}(t_0, z_0) = \sum_{\ell=1}^{2^{n-k}} \pm I_{S^{(\ell)}}^{(k,n)}(t_0, z_0), \qquad \forall k \leq n. \tag{1.9.1}$$

Setting $t_0 = z_0 = 0$, $k = 1$ and using Proposition 1.22 results in

$$|J_n^{(1)}(0,0)| \leq \sum_{\ell=1}^{2^{n-1}} |I_{S^{(\ell)}}(0,0)| \leq \frac{2^{n-1}M^n}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Since for $n \in \mathbb{N}$ we have $n! \leq 2^n\Gamma\left(\frac{n}{2}+1\right)^2$ we obtain

$$n!|J_n^{(1)}(0,0)| \leq \frac{2^{2n-1}M^n\Gamma\left(\frac{n}{2}+1\right)^2}{\Gamma\left(\frac{n}{2}+1\right)} \leq C^n\Gamma\left(\frac{n}{2}+1\right)$$

with $C := 4M$, completing the proof of Proposition 1.9.

□

With this bound we can now obtain our estimate which will act as a substitute for the Lipschitz condition. We first prove this substitute in expectation. This is clearly not enough to prove the main result. In the next section we will therefore use the Chebychev inequality to get an almost sure version of this Lipschitz condition, in order to tackle the main theorem in the next chapter.

Theorem 1.23

There exists $C \in \mathbb{R}$ such that for every real-valued Borel function g on $[0,1] \times \mathbb{R}^d$ bounded by 1 everywhere, $x \in \mathbb{R}^d$, every even integer $p \in \mathbb{N}$ and $p = 1$, we have

$$\mathbb{E}\left(\int_0^1 g(t, W(t)+x) - g(t, W(t))\ \mathrm{d}t\right)^p \leq C^p \Gamma\left(\frac{p}{2}+1\right) |x|_2^p,$$

where $|\cdot|_2$ denotes the Euclidean norm of $\mathbb{R}^d$ and W standard d-dimensional Brownian motion with $W(0) = 0$ $\mathbb{P}$-a.s..

Proof

Step 1: $d = 1$

Let g be a real-valued Borel function with $|g| \leq 1$ everywhere. There exists a sequence $g_n\colon [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ of smooth, compactly supported functions with $|g_n| \leq 1$ everywhere such that g_n converges to g almost everywhere, i.e.

$$g_n(t,x) \overset{n\to\infty}{\longrightarrow} g(t,x), \qquad \forall t \in [0,1] \setminus M,\ \forall x \in \mathbb{R} \setminus \tilde{N}_t$$

where M and N_t are Lebesgue zero sets for all $t \in [0,1]$, respectively. And since the distribution of $W(t)$ is absolutely continuous w.r.t. Lebesgue measure we have

$$g_n(t, W(t)) \overset{n\to\infty}{\longrightarrow} g(t, W(t)) \quad \text{and} \quad g_n(t, W(t)+x) \overset{n\to\infty}{\longrightarrow} g(t, W(t)+x)$$

for all $t \in [0,1] \setminus M$ and $\omega \in \Omega \setminus N_t$, where $N_t := W(t)^{-1}(\tilde{N}_t) \cup (W(t)+x)^{-1}(\tilde{N}_t)$ and x is a point in $\mathbb{R}$. We use the fundamental theorem of calculus and the transformation rule to obtain

$$\mathbb{E}\left(\int_0^1 g_n(t, W(t)+x) - g_n(t, W(t))\ \mathrm{d}t\right)^p = \mathbb{E}\left(\int_0^1\int_0^1 x g_n'(t, W(t)+ux)\ \mathrm{d}u\,\mathrm{d}t\right)^p.$$

Here $'$ again denotes the derivative w.r.t. the second variable. We swap the order of integration by Fubini's Theorem and since p is even or 1 we can apply Jensen's inequality

$$= x^p\,\mathbb{E}\left(\int_0^1\int_0^1 g_n'(t, W(t)+ux)\ \mathrm{d}t\,\mathrm{d}u\right)^p \leq x^p\,\mathbb{E}\int_0^1\left(\int_0^1 g_n'(t, W(t)+ux)\ \mathrm{d}t\right)^p \mathrm{d}u.$$

We define $h_n(t,s) := g_n(t, s+ux)$ for every $u \in [0,1]$. Observe that h_n satisfies all conditions of Proposition 1.9, so we deduce that

$$= x^p\,\mathbb{E}\int_0^1\left(\int_0^1 h_n'(t, W(t))\ \mathrm{d}t\right)^p \mathrm{d}u \leq |x|^p \int_0^1 C^p\Gamma\left(\frac{p}{2}+1\right)\ \mathrm{d}u = C^p\Gamma\left(\frac{p}{2}+1\right)|x|^p.$$

We now let $n \to \infty$. Using the boundedness of g_n the result follows by Fatou's inequality

$$\mathbb{E}\left(\int_0^1 g(t, W(t)+x) - g(t, W(t))\ \mathrm{d}t\right)^p$$

$$= \int_0^1 \dots \int_0^1 \mathbb{E} \prod_{j=1}^p g(t_j, W(t_j) + x) - g(t_j, W(t_j)) \, \mathrm{d}t_1 \dots \mathrm{d}t_p$$

$$= \int_0^1 \dots \int_0^1 \mathbb{E} \prod_{j=1}^p \mathbb{1}_{[0,1]\setminus M}(t_j) \mathbb{1}_{\Omega\setminus N_{t_j}} \left[g(t_j, W(t_j) + x) - g(t_j, W(t_j)) \right] \, \mathrm{d}t_1 \dots \mathrm{d}t_p$$

$$\leq \liminf_{n\to\infty} \int_0^1 \dots \int_0^1 \mathbb{E} \prod_{j=1}^p \mathbb{1}_{[0,1]\setminus M}(t_j) \mathbb{1}_{\Omega\setminus N_{t_j}} \left[g_n(t_j, W(t_j) + x) - g_n(t_j, W(t_j)) \right] \, \mathrm{d}t_1 \dots \mathrm{d}t_p$$

$$= \liminf_{n\to\infty} \mathbb{E} \left(\int_0^1 g_n(t, W(t) + x) - g_n(t, W(t)) \, \mathrm{d}t \right)^p \leq C^p \Gamma\left(\frac{p}{2} + 1\right) |x|^p$$

as in the proof of Proposition 1.9.

Step 2: Reduction to $d = 1$

We will prove the assertion by reducing it to the case of $d = 1$. Let x and g be as specified in the assertion. Let Φ be a rotation on $\mathbb{R}^d$ satisfying $\Phi(\alpha, 0, ..., 0)^\top = x$ with $\alpha := |x|_2$. Define $\tilde{\Phi} := \mathrm{id}_{[0,1]} \times \Phi$. We have

$$\mathbb{E} \left(\int_0^1 g(t, W(t) + x) - g(t, W(t)) \, \mathrm{d}t \right)^p .$$

Setting $\tilde{W}(t) = \Phi^{-1} W(t)$ results in

$$= \mathbb{E} \left(\int_0^1 g(\tilde{\Phi}(t, \tilde{W}(t) + (\alpha, 0, .., 0)^\top)) - g(\tilde{\Phi}(t, \tilde{W}(t))) \, \mathrm{d}t \right)^p .$$

Using Lebesgue's transformation rule and $\det \mathrm{D}\tilde{\Phi} = 1$ yields

$$= \mathbb{E} \left(\int_0^1 g(t, \tilde{W}(t) + (\alpha, 0, .., 0)^\top) - g(t, \tilde{W}(t)) \, \mathrm{d}t \right)^p .$$

And since Brownian motion is invariant under rotations, $\tilde{W}$ is still a Brownian motion. Applying the transformation formula results in

$$= \int_{\mathcal{C}([0,1],\mathbb{R}^d)} \left(\int_0^1 g(t, f(t) + (\alpha, 0, .., 0)^\top) - g(t, f(t)) \, \mathrm{d}t \right)^p \mathrm{d}\mathcal{W}(f)$$

where $\mathcal{W}$ denotes the Wiener measure on $\mathcal{C}([0,1], \mathbb{R}^d)$. Since the components of W are independent $\mathcal{W}$ is of product type and we have

$$= \int_{\mathcal{C}([0,1],\mathbb{R})} \dots \int_{\mathcal{C}([0,1],\mathbb{R})} \left(\int_0^1 g(t, f(t) + (\alpha, 0, .., 0)^\top) - g(t, f(t)) \, \mathrm{d}t \right)^p \mathrm{d}\mathcal{W}_1(f_1) \dots \mathrm{d}\mathcal{W}_d(f_d)$$

where $f = (f_1, ..., f_d)^\top$. For fixed paths $f_2, ..., f_d$ we define

$$h(t, u) := g(t, (u, f_2(t), ..., f_d(t))^\top), \qquad \forall t \in [0,1],\ u \in \mathbb{R}.$$

And step 1 yields

$$\int\limits_{\mathcal{C}([0,1],\mathbb{R})} \left(\int\limits_0^1 g(t, f(t) + (\alpha, 0, .., 0)^\top) - g(t, f(t)) \,\mathrm{d}t \right)^p \mathrm{d}\mathcal{W}_1(f_1)$$
$$= \int\limits_{\mathcal{C}([0,1],\mathbb{R})} \left(\int\limits_0^1 h(t, f_1(t) + \alpha) - h(t, f_1(t)) \,\mathrm{d}t \right)^p \mathrm{d}\mathcal{W}_1(f_1) \le C^p \Gamma\left(\frac{p}{2} + 1\right) |\alpha|^p$$

for fixed paths $f_2, ..., f_d$. Averaging over $f_2, ..., f_d$ and using that $|x|_2 = |\alpha|$ holds, completes the proof.

□

Using Theorem 1.23 we will obtain a different version of this estimate, which holds in probability instead of in expectation in the next section. This improved estimate will eventually lead to Lemma 2.5, which acts as our substitute for the Lipschitz condition.

1.3 Lipschitz condition in probability

The aim of this section is to improve the previous estimate (Theorem 1.23) in the sense that we get an estimate which holds in probability instead of in expectation. The main ingredient for archiving this is the Chebychev inequality. The version we obtain here will also be local in character i.e. we no longer restrict ourselves to integrals over the interval $[0, 1]$. Integrating only over some interval $[a, b]$ also enables us to prove the estimates in conditional expectation w.r.t. $\mathcal{F}_s$ as long as $s \le a$. To this end recall that we have measures $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$ such that the universal Markov property (cf. [Bau96] equation 42.18) holds.

Definition 1.24

Let g be a bounded, real-valued Borel function on $[0, 1] \times \mathbb{R}^d$, $0 \le a < b \le 1$ and $x \in \mathbb{R}^d$. We define

$$\sigma_{a,b}(x; g, W) := \int\limits_a^b g(t, W(t) + x) - g(t, W(t)) \,\mathrm{d}t.$$

If it is clear from context we drop the g and W in the notation and write $\sigma_{a,b}(x)$ instead of $\sigma_{a,b}(x; g, W)$.

Corollary 1.25

Let g be a real-valued Borel function on $[0, 1] \times \mathbb{R}^d$ bounded by 1 everywhere. Let $0 \le s \le a < b \le 1$, $\ell := b - a$. We then have

$$\mathbb{P}[|\sigma_{a,b}(x)| > \lambda\sqrt{\ell}|x|_2 | \mathcal{F}_s] \le 2e^{-\lambda^2/(2C^2)}, \qquad \mathbb{P}\text{-a.s.},\ \forall x \in \mathbb{R}^d,\ \forall \lambda > 0,$$

where C is the constant from Theorem 1.23.

Proof

The assertion is trivial in the case $x = 0$, so we assume $x \neq 0$.

Step 1: $s = a = 0$, $b = 1$

Define $\alpha := (2C^2|x|_2^2)^{-1}$ with C being the constant from Theorem 1.23. We then have

$$\mathbb{E}[e^{\alpha|\sigma_{0,1}(x)|^2}] = \mathbb{E}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}|\sigma_{0,1}(x)|^{2k} = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\mathbb{E}\left[\sigma_{0,1}(x)^{2k}\right].$$

Using Theorem 1.23 with $p = 2k$ we deduce

$$\overset{1.23}{\leq}\sum_{k=0}^{\infty}\alpha^k C^{2k}|x|_2^{2k} = \sum_{k=0}^{\infty}2^{-k} = 2.$$

We can conclude the proof with the help of the Chebychev inequality.

$$\mathbb{P}[|\sigma_{0,1}(x)| > \lambda|x|_2] = \mathbb{P}[e^{\alpha|\sigma_{0,1}(x)|^2} > e^{\alpha\lambda^2|x|_2^2}] \leq e^{-\alpha\lambda^2|x|_2^2}\mathbb{E}[e^{\alpha|\sigma_{0,1}(x)|^2}] \leq 2e^{-\alpha\lambda^2|x|_2^2} = 2e^{-\lambda^2/(2C^2)}.$$

Step 2: General case

Fix a version of the conditional expectation. With the help of the universal Markov property (cf. [Bau96] equation (42.18)) we obtain

$$\mathbb{E}\left[\int_a^b g(t, W(t)+x) - g(t, W(t))\,\mathrm{d}t\,\middle|\,\mathcal{F}_a\right] = \mathbb{E}_{W(a)}\left[\int_a^b g(t, W(t-a)+x) - g(t, W(t-a))\,\mathrm{d}t\right]$$

where $\mathbb{E}_x$ denotes the expectation w.r.t. the measure $\mathbb{P}_x$. Consider the following transformation

$$t' := \frac{t-a}{\ell}, \qquad \partial_t t' = \ell^{-1}.$$

This leads us to

$$= \mathbb{E}_{W(a)}\left[\ell\int_0^1 g(\ell t' + a, W(\ell t') + x) - g(\ell t' + a, W(\ell t'))\,\mathrm{d}t'\right].$$

Define

$$\tilde{W}(t) := \ell^{-1/2}W(\ell t)$$

and observe that $\tilde{W}$ is again a Brownian motion under $\mathbb{P}_x$ starting in x for every $x \in \mathbb{R}^d$. We now have

$$= \mathbb{E}_{W(a)}\left[\ell\int_0^1 g(\ell t' + a, \sqrt{\ell}\tilde{W}(t') + x) - g(\ell t' + a, \sqrt{\ell}\tilde{W}(t'))\,\mathrm{d}t'\right].$$

Fix $\omega_0 \in \Omega$. Set $Q := \mathbb{P}_{W(\omega_0,a)}$ and note that $\tilde{W} - W(\omega_0, a)$ is a Brownian motion under Q starting in 0 Q-a.s.. We define

$$h(s,u) := g\left(\ell s + a, \sqrt{\ell}\,(u + W(\omega_0, a))\right), \qquad \forall s \in [0,1],\ u \in \mathbb{R}^d.$$

So we finally have

$$\begin{aligned}
&\mathbb{E}\left[|\sigma_{a,b}(x; g, W)||\,\mathcal{F}_a\right](\omega_0)\\
&= \mathbb{E}_Q\left[\ell\left|\int_0^1 h\left(t', \tilde{W}(t') - W(\omega_0,a) + \frac{x}{\sqrt{\ell}}\right) - h\left(t', \tilde{W}(t') - W(\omega_0,a)\right)\,\mathrm{d}t'\right|\right]\\
&= \mathbb{E}_Q\left[\ell\left|\sigma_{0,1}\left(\frac{x}{\sqrt{\ell}}; h, \tilde{W} - W(\omega_0,a)\right)\right|\right],
\end{aligned}$$

where $\mathbb{E}_Q$ denotes the expectation w.r.t. the measure Q. With this calculation we express the conditional probability as

$$\mathbb{P}[|\sigma_{a,b}(x; g, W)| > \lambda\sqrt{\ell}|x|_2|\mathcal{F}_a](\omega_0) = Q\left[\left|\sigma_{0,1}\left(\frac{x}{\sqrt{\ell}}; h, \tilde{W} - W(\omega_0,a)\right)\right| > \lambda\frac{|x|_2}{\sqrt{\ell}}\right].$$

In consequence of the fact that $\tilde{W} - W(\omega_0, a)$ is a Brownian motion starting in 0 w.r.t. Q we are able to apply the conclusion of step 1 and deduce

$$\mathbb{P}[|\sigma_{a,b}(x; g, W)| > \lambda\sqrt{\ell}|x|_2|\mathcal{F}_a](\omega_0) \le 2e^{-\lambda^2/(2C^2)}, \qquad \forall \omega_0 \in \Omega.$$

Since ω_0 was arbitrary this inequality holds for all $\omega_0 \in \Omega$. Because of the relation $\mathcal{F}_s \subseteq \mathcal{F}_a$ taking the conditional expectation w.r.t. $\mathcal{F}_s$ results in

$$\mathbb{P}[|\sigma_{a,b}(x; g, W)| > \lambda\sqrt{\ell}|x|_2|\mathcal{F}_s] \le 2e^{-\lambda^2/(2C^2)}.$$

which concludes the proof of the assertion.

□

Using the technique of the last proof we also obtain the following bound for the conditional expectation of $\sigma^p_{a,b}$.

Corollary 1.26

Let g be a real-valued Borel function on $[0,1] \times \mathbb{R}^d$ bounded by 1 everywhere. Let $0 \le s \le a < b \le 1$, $\ell := b - a$ and p an even integer or $p = 1$. We then have

$$\mathbb{E}[|\sigma_{a,b}(x)|^p|\mathcal{F}_s] \le C^p \ell^{p/2}\Gamma\left(\frac{p}{2} + 1\right)|x|_2^p, \qquad \mathbb{P}\text{-a.s. } \forall x \in \mathbb{R}^d.$$

where C is the constant from Theorem 1.23.

Proof

Fix a version of the conditional expectation. With the same calculation as in the last proof we have

$$\mathbb{E}[|\sigma_{a,b}(x;g,W)|^p|\mathcal{F}_a](\omega_0) = \mathbb{E}_{W(\omega_0,a)}\left[\ell^p \left|\sigma_{0,1}\left(\frac{x}{\sqrt{\ell}};h,\tilde{W}-W(\omega_0,a)\right)\right|^p\right]$$

where h and $\tilde{W}$ are defined as in the proof of Corollary 1.25. Since $\mathcal{F}_s \subseteq \mathcal{F}_a$ taking the conditional expectation w.r.t. $\mathcal{F}_s$ and applying Theorem 1.23 results in

$$\mathbb{E}[|\sigma_{a,b}(x;g,W)|^p|\mathcal{F}_s] \overset{1.23}{\leq} \ell^p C^p \Gamma\left(\frac{p}{2}+1\right)\left|\frac{|x|_2}{\sqrt{\ell}}\right|^p = C^p \ell^{p/2}\Gamma\left(\frac{p}{2}+1\right)|x|_2^p.$$

□

Definition 1.27

Let g be a bounded, real-valued Borel function on $[0,1]\times\mathbb{R}^d$, $0\leq a<b\leq 1$ and x, $y\in\mathbb{R}^d$. We define

$$\rho_{a,b}(x,y;g,W) := \sigma_{a,b}(x;g,W)-\sigma_{a,b}(y;g,W) = \int_a^b g(t,W(t)+x)-g(t,W(t)+y)\,\mathrm{d}t.$$

If it is clear from context we drop the g and W in the notation and write $\rho_{a,b}(x,y)$ instead of $\rho_{a,b}(x,y;g,W)$.

Note that $\rho_{a,b}(x,0)=\sigma_{a,b}(x)$. Hence, it is natural to ask whether the previous estimates for $\sigma_{a,b}$ can be translated to estimates for $\rho_{a,b}$. The next corollary gives an affirmative answer to this question.

Corollary 1.28

Let g be a real-valued Borel function on $[0,1]\times\mathbb{R}^d$ bounded by 1 everywhere. Let $0\leq s\leq a<b\leq 1$, $\ell := b-a$ and p an even integer or $p=1$. We then have

(i) $\mathbb{P}[|\rho_{a,b}(x,y)|>\lambda\sqrt{\ell}|x-y|_2|\mathcal{F}_s]\leq 2e^{-\lambda^2/(2C^2)}$, $\quad\mathbb{P}$-a.s., $\forall x,y\in\mathbb{R}^d$, $\forall\lambda>0$

(ii) $\mathbb{E}[|\rho_{a,b}(x,y)|^p|\mathcal{F}_s]\leq C^p\ell^{p/2}\Gamma\left(\frac{p}{2}+1\right)|x-y|_2^p$, $\quad\mathbb{P}$-a.s., $\forall x,y\in\mathbb{R}^d$

where C is the constant from Theorem 1.23.

Proof

We set

$$h(t,u) := g(t,u+y), \qquad \forall t\in[0,1],\ u\in\mathbb{R}^d.$$

and immediately obtain

$$\rho_{a,b}(x,y;g,W) = \sigma_{a,b}(x-y;h,W).$$

Applying Corollary 1.25 leads us to

$$\mathbb{P}[|\rho_{a,b}(x,y;g,W)| > \lambda\sqrt{\ell}|x-y|_2|\mathcal{F}_s] = \mathbb{P}[|\sigma_{a,b}(x-y;h,W)| > \lambda\sqrt{\ell}|x-y|_2|\mathcal{F}_s] \overset{1.25}{\leq} 2e^{-\lambda^2/(2C^2)}$$

proving claim (i). With the help of Corollary 1.26 we obtain in the same way

$$\mathbb{E}[|\rho_{a,b}(x,y;g,W)|^p|\mathcal{F}_s] = \mathbb{E}[|\sigma_{a,b}(x-y;h,W)|^p|\mathcal{F}_s] \overset{1.26}{\leq} C^p\ell^{p/2}\Gamma\left(\frac{p}{2}+1\right)|x-y|_2^p.$$

which completes the proof.

□

Lemma 1.29

Let $p > 1 + \frac{d}{2}$ and $g \in L^p([0,1]\times\mathbb{R}^d)$ then

$$\mathbb{E}\int_0^1 g(t,W(t))\,\mathrm{d}t \leq C(p,d)\|g\|_{L^p([0,1]\times\mathbb{R}^d)}$$

where W denotes standard d-dimensional Brownian motion with $W(0) = 0$ $\mathbb{P}$-a.s., $q = \frac{p}{p-1}$ and

$$C(p,d) := \left(q^{-d/2}\cdot\frac{(2\pi)^{(1-q)d/2+1}}{(1-q)\frac{d}{2}+1}\right)^{1/q}.$$

Proof

We set $q := \frac{p}{p-1}$. Let E_d be the Lebesgue density of the d-dimensional normal distribution

$$\mathrm{E}_d(t,z) := (2\pi t)^{-d/2}e^{-|z|_2^2/(2t)}, \qquad \forall t>0,\ z\in\mathbb{R}^d.$$

We then have

$$\int_0^1\int_{\mathbb{R}^d}\mathrm{E}_d(t,z)^q\,\mathrm{d}z\,\mathrm{d}t = \int_0^1(2\pi t)^{-qd/2}\int_{\mathbb{R}^d}e^{-|z|_2^2/(2t/q)}\,\mathrm{d}z\,\mathrm{d}t$$
$$= \int_0^1(2\pi t)^{-qd/2}(2\pi t/q)^{d/2}\,\mathrm{d}t = q^{-d/2}\int_0^1(2\pi t)^{(1-q)d/2}\,\mathrm{d}t.$$

Since $p > 1 + \frac{d}{2}$ we have $(1-q)d/2 > -1$. The integrand is therefore integrable and we obtain

$$= q^{-d/2} \frac{(2\pi)^{(1-q)d/2+1}}{(1-q)\frac{d}{2}+1} = C(p,d)^q.$$

So by Fubini's Theorem and Hölder's inequality we deduce

$$\mathbb{E} \int_0^1 g(t, W(t))\ \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^d} g(t,z)\, \mathrm{E}_d(t,z)\ \mathrm{d}z\, \mathrm{d}t \leq \|g\|_{L^p([0,1]\times\mathbb{R}^d)} \cdot \|\, \mathrm{E}_d \,\|_{L^q([0,1]\times\mathbb{R}^d)}$$

$$= C(p,d)\|g\|_{L^p([0,1]\times\mathbb{R}^d)}.$$

□

2 Proof of the main theorem

In this chapter we will provide a proof for the main Theorem 1.5. In the first section we discuss dyadic points and obtain an "almost sure version" of Theorem 1.23 (Lemma 2.5). For the main proof it is vital to do an approximation. The question of convergence of this particular approximation is answered in the second section. Using this, the last section contains the proof of the main result.

2.1 Dyadic points

In this section we introduce the space of dyadic points and dyadic neighbors. Using a quite general approximation technique (Lemma 2.4) we deduce an "almost sure version" of Theorem 1.23. At the end of this section we generalize this estimate in Corollary 2.6.

Definition 2.1

Let $n \in \mathbb{N}$. For any $k \in \{0, ..., 2^n - 1\}$ we set

$$I_{n,k} := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[.$$

Definition 2.2

Let $1 \leq d < \infty$ and $|x|_\infty := \max_{1\leq i\leq d} |x_i|$. We call

$$Q := \{x \in \mathbb{R}^d \colon |x|_\infty \leq 1,\ x = (x_1, ..., x_d)^\top,\ \forall i \colon \exists k \colon 2^k x_i \in \mathbb{Z}\}$$

the space of dyadic points. We call $x, y \in Q$ dyadic neighbors of order $m \in \mathbb{N}$ if

$$|x - y|_\infty = 2^{-m} \qquad \text{and} \qquad 2^m x,\ 2^m y \in \mathbb{Z}^d.$$

Notation: $x \sim_m y$. Additionally, we call $x, y \in Q$ dyadic neighbors (Notation: $x \sim y$) if there exists some $m \in \mathbb{N}$ such that $x \sim_m y$.

Definition 2.3

Let g be a bounded, real-valued or $\mathbb{R}^d$-valued Borel function on $[0,1] \times \mathbb{R}^d$ and $x,\ y \in \mathbb{R}^d$. Analog to Definition 1.24 and 1.27 we define

$$\sigma_{n,k}(x; g, W) := \int_{I_{n,k}} g(t, W(t) + x) - g(t, W(t))\ \mathrm{d}t,$$

$$\rho_{n,k}(x, y; g, W) := \sigma_{n,k}(x; g, W) - \sigma_{n,k}(y; g, W) = \int_{I_{n,k}} g(t, W(t) + x) - g(t, W(t) + y)\ \mathrm{d}t.$$

If it is clear from the context we drop g and W in the notation and write $\sigma_{n,k}(x)$ instead of $\sigma_{n,k}(x; g, W)$ and $\rho_{n,k}(x, y)$ instead of $\rho_{n,k}(x, y; g, W)$.

Lemma 2.4

Let x, $y \in Q$ with $0 < |x-y|_\infty < 1$. For every integer $r \geq 0$ choose $x_r \in 2^{-r}\mathbb{Z}^d$ such that

$$|x - x_r|_\infty = \min_{z \in 2^{-r}\mathbb{Z}^d \cap Q} |x - z|_\infty.$$

We define y_r in the same way. Let $m \geq 0$ be the largest integer such that $|x-y|_\infty < 2^{-m}$ holds. Then the following statements hold

(i)	$x_m \sim_m y_m$	or	$x_m = y_m$
(ii)	$x_{r+1} \sim_{r+1} x_r$	or	$x_{r+1} = x_r$
(iii)	$\exists N_0 \in \mathbb{N}\colon \forall n \geq N_0$:		$x_{n+1} = x_n$

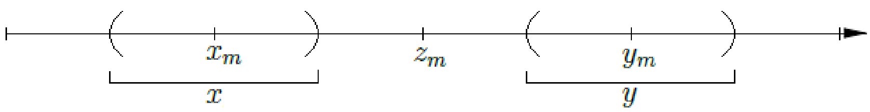
Proof

(i)

Assume $x_m \not\sim_m y_m$ and $x_m \neq y_m$. Let us first consider the case $d = 1$. W.l.o.g. we have $x_m < y_m$. We therefore have $y_m - x_m > 2^{-m}$. The distance between x and x_m (respectively y and y_m) is at most 2^{-m-1}. This implies that $x \leq y$. Since x_m and y_m are not dyadic neighbors of order m there exists z_m with $2^m z_m \in \mathbb{Z}$ such that $x_m < z_m < y_m$. Because x_m and y_m are closer to x and y than z_m, respectively, we have

$$x \leq z_m \leq y.$$

Since x is closer to x_m than to z_m we have $z_m - x \geq 2^{-m-1}$. The same holds for y, so we have $y - z_m \geq 2^{-m-1}$. See the following picture for an overview of the situation.



Hence

$$y - x = y - z_m + z_m - x \geq 2^{-m}.$$

Contradicting $|x-y|_\infty < 2^{-m}$. Now, consider the case $d > 1$. Let π_i be the projection to the i-th coordinate. There exists $i \in \{1, ..., d\}$ such that

$$|\pi_i(x_m) - \pi_i(y_m)| = |x_m - y_m|_\infty \notin \{0, 2^{-m}\}.$$

So we have $\pi_i(x_m) \not\sim_m \pi_i(y_m)$ and $\pi_i(x_m) \neq \pi_i(y_m)$. By the $d = 1$ case we get $|\pi_i(y) - \pi_i(x)| \geq 2^{-m}$. This implies $|x-y|_\infty \geq 2^{-m}$ concluding the first part of the proof.

(ii)

Assume $x_{r+1} \not\sim_{r+1} x_r$ and $x_{r+1} \neq x_r$. Let us again first consider the case $d = 1$. Since $x_r \in 2^{-r}\mathbb{Z}$ we have $x_r = p2^{-r}$ with $p \in \{-2^r, ..., 2^r\}$. W.l.o.g. $x_r < x_{r+1}$. Set $z = (2p \pm 1)2^{-r-1}$ such that $z \in Q$. W.l.o.g. we only consider the case $x_r < z$ i.e. $z = (2p+1)2^{-r-1}$. Since

$$z - x_r = \frac{2p+1}{2^{r+1}} - \frac{p}{2^r} = 2^{-r-1}$$

holds, z and x_r are dyadic neighbors of order $r+1$. This means that $x_r < z < x_{r+1}$. But by the definition of x_r the distance between x and x_r is at most 2^{-r-1}. We therefore have $x \leq z < x_{r+1}$ which is a contradiction to the definition of x_{r+1}. Now, consider the case $d > 1$. Let π_i be the projection to the i-th coordinate. There exists $i \in \{1, ..., d\}$ such that

$$|\pi_i(x_{r+1}) - \pi_i(x_r)| = |x_{r+1} - x_r|_\infty \notin \{0, 2^{-r-1}\}.$$

This implies $\pi_i(x_{r+1}) \not\sim_{r+1} \pi_i(x_r)$ and $\pi_i(x_{r+1}) \neq \pi_i(x_r)$. By the $d = 1$ case we conclude the second part of the proof.

(iii)

Since $x \in Q$ there exists $N_0 \in \mathbb{N}$ such that $x = 2^{-N_0}(k_1, ..., k_d)^\top$ with $k_i \in \{-2^{N_0}, ..., 2^{N_0}\}$. Let $n \geq N_0$ then $x_n = x$ since

$$x = 2^{-N_0}(k_1, ..., k_d)^\top = 2^{-n}(k_1 2^{n-N_0}, ..., k_d 2^{n-N_0})^\top \in 2^{-n}\mathbb{Z}^d.$$

□

Lemma 2.5

For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for every real-valued Borel function g on $[0,1] \times \mathbb{R}$ bounded by 1 everywhere, we have

$$|\sigma_{n,k}(x)| \leq C(\varepsilon)\sqrt{n}2^{-n/2} \max\left(|x|_\infty, 2^{-2^n}\right)$$

for all dyadic points $x \in Q$, $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$ and $\omega \in A_\varepsilon^c$.

Proof

Step 1:

For $r \in \mathbb{N}$ we define $Q_r := \{x \in Q : |x|_\infty \leq 2^{-r}\}$. Let m be an integer with $m \geq r$ and $x, y \in Q_r$ be dyadic neighbors of order m. Applying Corollary 1.28.(i) with $\lambda = \lambda'(\sqrt{n} + \sqrt{m-r})$ for some $\lambda' > 0$, $s = 0$ and using that $\sqrt{d}|x-y|_\infty \geq |x-y|_2$ yields

$$\begin{aligned}
&\mathbb{P}[|\rho_{n,k}(x,y)| > \lambda'\sqrt{d}(\sqrt{n} + \sqrt{m-r})2^{-m-n/2}] \\
&\leq \mathbb{P}[|\rho_{n,k}(x,y)| > \lambda'(\sqrt{n} + \sqrt{m-r})|x-y|_2 2^{-n/2}] \\
&\leq 2e^{-\lambda'^2(n+m-r)/(2C^2)} = 2e^{-\lambda'^2 n/(2C^2)} e^{-\lambda'^2(m-r)/(2C^2)}.
\end{aligned}$$

Choose λ' large enough such that $\lambda'^2 \geq 4dC^2 + \frac{\lambda'^2}{2n}$ holds, and henceforth we get

$$\leq 2e^{-n\left(4C^2+\frac{\lambda'^2}{2n}\right)/(2C^2)}e^{-2dC^2(m-r)/(2C^2)} = 2e^{-2n}e^{-\lambda'^2/(4C^2)}e^{-d(m-r)}.$$

Using the above inequality we obtain

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{n=1}^{\infty}\bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty}\bigcup_{\substack{x,y\in Q_r\\ x\sim_m y}}\bigcup_{k=0}^{2^n-1}|\rho_{n,k}(x,y)| > \lambda'\sqrt{d}(\sqrt{n}+\sqrt{m-r})2^{-m-n/2}\right]\\
&\leq \sum_{n=1}^{\infty}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\sum_{\substack{x,y\in Q_r\\ x\sim_m y}}\sum_{k=0}^{2^n-1} 2e^{-2n}e^{-d(m-r)}e^{-\lambda'^2/(4C^2)}\\
&= \sum_{n=1}^{\infty}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}\underbrace{\#\{(x,y)\in Q_r^2 | x\sim_m y\}}_{\leq (2\cdot 2^{m-r}+1)^d 3^d} 2\cdot 2^n e^{-2n}e^{-d(m-r)}e^{-\lambda'^2/(4C^2)}\\
&\leq 2\sum_{n=1}^{\infty}\sum_{r=0}^{2^n}\sum_{m=r}^{\infty}(3\cdot 2^{m-r}3)^d\left(\frac{2}{e^2}\right)^n e^{-d(m-r)}e^{-\lambda'^2/(4C^2)}\\
&= 2\cdot 3^{2d}\sum_{n=1}^{\infty}\sum_{r=0}^{2^n}\sum_{m=0}^{\infty}2^{dm}\left(\frac{2}{e^2}\right)^n e^{-dm}e^{-\lambda'^2/(4C^2)}\\
&\leq 3^{2d+2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{4}{e^2}\right)^n\left(\frac{2}{e}\right)^{dm} e^{-\lambda'^2/(4C^2)}\\
&\leq 3^{2d+2}\underbrace{\sum_{n=1}^{\infty}\left(\frac{4}{e^2}\right)^n}_{\leq 2}\underbrace{\sum_{m=0}^{\infty}\left(\frac{2}{e}\right)^m}_{\leq 4} e^{-\lambda'^2/(4C^2)}\\
&\leq 3^{2d+4}e^{-\lambda'^2/(4C^2)} \overset{\lambda'\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Since the left-hand side converges to zero as λ' approaches infinity, we have proved that

$$\lim_{\lambda'\to\infty}\mathbb{P}\left[\bigcup_{n=1}^{\infty}\bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty}\bigcup_{\substack{x,y\in Q_r\\ x\sim_m y}}\bigcup_{k=0}^{2^n-1}|\rho_{n,k}(x,y)| > \lambda'\sqrt{d}(\sqrt{n}+\sqrt{m-r})2^{-m-n/2}\right] = 0.$$

Let $\varepsilon > 0$. Then there exists $C(\varepsilon) > 0$ such that

$$\mathbb{P}\underbrace{\left[\bigcup_{n=1}^{\infty}\bigcup_{r=0}^{2^n}\bigcup_{m=r}^{\infty}\bigcup_{\substack{x,y\in Q_r\\ x\sim_m y}}\bigcup_{k=0}^{2^n-1}|\rho_{n,k}(x,y)| > C(\varepsilon)(\sqrt{n}+\sqrt{m-r})2^{-m-n/2}\right]}_{=:A_\varepsilon} \leq \varepsilon.$$

Note that the probability of the event A_ε^c is at least $1-\varepsilon$.

So, for $r \in \{0, ..., 2^n\}$ we have

$$\begin{aligned} &|\rho_{n,k}(x,y)| \le C(\varepsilon)(\sqrt{n} + \sqrt{m-r})2^{-m-n/2}, \\ &\forall x, y \in Q_r,\ x \sim_m y,\ m \ge r,\ n \ge 1,\ k \in \{0, ..., 2^n - 1\}. \end{aligned} \tag{2.5.1}$$

as long as $\omega \in A_\varepsilon^c$.

With this inequality we will now prove the asserted inequality using an approximation argument based on Lemma 2.4.

Let $x \in Q_r$ with r as above. For every integer $i \ge r$ choose $x_i \in Q_r$ as in Lemma 2.4 i.e. $x_i \in Q_r \cap 2^{-i}\mathbb{Z}^d$ such that x_i minimizes the distance to x. By the triangle inequality we immediately get

$$|\sigma_{n,k}(x)| \le |\rho_{n,k}(x_r, 0)| + \sum_{i=r}^{\infty} |\rho_{n,k}(x_{i+1}, x_i)|.$$

Observe that $x_r \sim_r 0$ or $x_r = 0$ and $x_{i+1} \sim_{i+1} x_i$ or $x_{i+1} = x_i$ by Lemma 2.4.(ii). The sum converges trivially by Lemma 2.4.(iii). This enables us to use the above estimate.

$$\begin{aligned} &\overset{(2.5.1)}{\le} C(\varepsilon)\sqrt{n}2^{-r-n/2} + \sum_{i=r}^{\infty} C(\varepsilon)(\sqrt{n} + \sqrt{i+1-r})2^{-i-1-n/2} \\ &= C(\varepsilon)2^{-n/2} \sum_{i=r}^{\infty} (\sqrt{n} + \sqrt{i-r})2^{-i} \\ &= C(\varepsilon)2^{-n/2} \left[2^{1-r}\sqrt{n} + 2^{-r} \sum_{i=1}^{\infty} \sqrt{i}2^{-i} \right] \\ &\le C(\varepsilon)2^{-n/2}2^{-r} \left[2\sqrt{n} + \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i \right] \\ &= C(\varepsilon)2^{-n/2}2^{-r}[2\sqrt{n} + 2] \overset{n\ge 1}{\le} 4C(\varepsilon)2^{-n/2}2^{-r}\sqrt{n}. \end{aligned}$$

Step 2:

For a fixed $n \in \mathbb{N}$ let $x \in Q$ such that $|x|_\infty > 2^{-2^n}$. We set

$$r := \left\lfloor \log_2 |x|_\infty^{-1} \right\rfloor \le \left\lfloor \log_2 2^{2^n} \right\rfloor \le 2^n.$$

And hence we have

$$2^{-r-1} = 2^{-\left\lfloor \log_2 |x|_\infty^{-1} \right\rfloor - 1} \le 2^{-\log_2 |x|_\infty^{-1}} = |x|_\infty. \tag{2.5.2}$$

Additionally, we have $x \in Q_r$, because of the fact that

$$|x|_\infty = 2^{-\log_2 |x|_\infty^{-1}} \le 2^{-r}.$$

So we can apply step 1 to get

$$|\sigma_{n,k}(x)| \le 4C(\varepsilon)2^{-n/2}2^{-r}\sqrt{n} \overset{(2.5.2)}{\le} 8C(\varepsilon)\sqrt{n}2^{-n/2}|x|_\infty.$$

Step 3:

Again for a fixed $n \in \mathbb{N}$ let $x \in Q$ such that $|x|_\infty \leq 2^{-2^n}$. Then $x \in Q_r$ with $r = 2^n$, so we have

$$|\sigma_{n,k}(x)| \leq 4C(\varepsilon)2^{-n/2}2^{-r}\sqrt{n} = 4C(\varepsilon)\sqrt{n}2^{-n/2}2^{-2^n}.$$

This concludes the proof.

□

Using the relation between $\sigma_{n,k}$ and $\rho_{n,k}$ we generalize the last lemma in a similar way as Corollary 1.28 is proved.

Corollary 2.6

For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for every real-valued Borel function g on $[0,1] \times \mathbb{R}$ bounded by 1

$$|\rho_{n,k}(x,y)| \leq C(\varepsilon)\sqrt{n}2^{-n/2} \max\left(|x-y|_\infty, 2^{-2^n}\right)$$

holds for all dyadic points $x, y \in Q$, $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$ and $\omega \in A_\varepsilon^c$.

Proof

Let x, $y \in Q$. We set

$$h(t,u) := g\left(t, u+y\right), \qquad r(t,u) := h\left(t, u + \frac{x-y}{2}\right), \qquad \forall t \in [0,1],\ u \in \mathbb{R}^d$$

and immediately obtain

$$\begin{aligned}\rho_{n,k}(x,y;g,W) &= \rho_{n,k}\left(x-y,0;h,W\right) = \rho_{n,k}\left(x-y, \frac{x-y}{2}; h, W\right) + \rho_{n,k}\left(\frac{x-y}{2}, 0; h, W\right) \\ &= \sigma_{n,k}\left(\frac{x-y}{2}; r, W\right) + \sigma_{n,k}\left(\frac{x-y}{2}; h, W\right).\end{aligned}$$

Using the fact that $(x-y)/2 \in Q$ and invoking Lemma 2.5 readily results in

$$\begin{aligned}|\rho_{n,k}(x,y;g,W)| &\leq \left|\sigma_{n,k}\left(\frac{x-y}{2}; r, W\right)\right| + \left|\sigma_{n,k}\left(\frac{x-y}{2}; h, W\right)\right| \\ &\leq C(\varepsilon)\sqrt{n}2^{-n/2}\max\left(|x-y|_\infty, 2^{-2^n}\right), \qquad \forall \omega \in A_\varepsilon^c.\end{aligned}$$

□

2.2 Approximation via step functions

In the last section we obtained the estimate

$$|\sigma_{n,k}(x)| \leq C(\varepsilon)\sqrt{n}2^{-n/2}\max\left(|x|_\infty, 2^{-2^n}\right)$$

for dyadic points $x \in Q$. We would like to replace x by a Lipschitz continuous function $t \longmapsto u(t)$. To this end we approximate u by Q-valued step functions u_n which are constant on the interval $I_{n,k}$. In this section we show that the approximants converge in the right sense (Lemma 2.10) and use this result to generalize the above estimate (Corollary 2.12).

Definition 2.7

Define

$$\begin{aligned}
\Phi &:= \{u\colon [0,1] \longrightarrow [-1,1]^d\colon |u(s)-u(t)|_\infty \leq |s-t|,\ \forall s,t \in [0,1]\},\\
\Phi_n &:= \left\{u\colon [0,1] \longrightarrow [-1,1]^d \,\middle|\, \begin{array}{c} \forall 0 \leq k < 2^n\colon \forall s,t \in I_{n,k}\colon u(s) = u(t), \\ \forall m,\ell \in \mathbb{Z}\cap[0,2^n]\colon |u(m2^{-n}) - u(\ell 2^{-n})|_\infty \leq |m-\ell|2^{-n} \end{array}\right\},\\
\Phi^* &:= \bigcup_{n=1}^{\infty}\Phi_n \cup \Phi.
\end{aligned}$$

Note that elements in Φ are continuous, since functions in Φ are Lipschitz continuous (with Lipschitz constant at most 1). Φ_n will be used to approximate elements in Φ. Also note that Φ and Φ_n are separable w.r.t. the maximum norm and hence Φ^* is separable.

Lemma 2.8

Let $u \in \Phi^*$ and $n \in \mathbb{N}$. We then have

$$\sum_{k=0}^{2^n-1}\left|u(k2^{-n}) - u((k+1)2^{-n})\right|_\infty \leq 1.$$

Proof

Let $u \in \Phi^*$ and $n \in \mathbb{N}$ be as in the assertion. If $u \in \Phi$ the inequality follows immediately from the Lipschitz continuity of u. Let $u \in \Phi_m$ for some $m \in \mathbb{N}$.

Case 1: $m \geq n$

$$\begin{aligned}
\sum_{k=0}^{2^n-1}\left|u(k2^{-n}) - u((k+1)2^{-n})\right|_\infty &= \sum_{k=0}^{2^n-1}\left|u(k2^{m-n}2^{-m}) - u((k+1)2^{m-n}2^{-m})\right|_\infty\\
&\overset{u\in\Phi_m}{\leq} \sum_{k=0}^{2^n-1} 2^{m-n}2^{-m} = 1
\end{aligned}$$

Case 2: $m < n$

$$\sum_{k=0}^{2^n-1} \left|u(k2^{-n}) - u((k+1)2^{-n})\right|_\infty$$

Since u is constant on $I_{m,k}$ this sum simplifies to

$$= \sum_{k=0}^{2^m-1} \left|u(k2^{-m}) - u((k+1)2^{-m})\right|_\infty$$

and because of the fact that $u \in \Phi_m$ we have

$$\leq \sum_{k=0}^{2^m-1} 2^{-m} = 1.$$

□

The following technical lemma is needed to prove the main lemma (Lemma 2.10) of this section.

Lemma 2.9

For every $\varepsilon > 0$ there exist $\delta > 0$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that if $U \subseteq [0,1] \times \mathbb{R}^d$ is open with $|U| \leq \delta$, then

$$\int_0^1 \mathbb{1}_U(t, W(t) + u(t))\ \mathrm{d}t \leq \varepsilon, \qquad \forall u \in \Phi^*,\ \omega \in A_\varepsilon^c$$

holds, where $|U|$ is the mass of U w.r.t. Lebesgue measure.

Proof

Let $\varepsilon > 0$. By Corollary 2.6 there exists $C(\varepsilon) \in \mathbb{R}$ such that

$$\left| \int_{I_{n,k}} \phi(t, W(t) + x) - \phi(t, W(t) + y)\ \mathrm{d}t \right| \leq C(\varepsilon)\sqrt{n}2^{-n/2}\left(|x-y|_\infty + 2^{-2^n}\right) \tag{2.9.1}$$
$$\forall x, y \in Q,\ n \geq 1,\ k \in \{0, .., 2^n - 1\}.$$

holds for every real-valued Borel function ϕ on $[0,1] \times \mathbb{R}^d$ satisfying $|\phi| \leq 1$ everywhere with probability at least $1 - \varepsilon/2$. Choose $m \in \mathbb{N}$ such that

$$5C(\varepsilon)\sum_{n=m}^{\infty} \sqrt{n+1}2^{-n/2} \leq \frac{\varepsilon}{2}. \tag{2.9.2}$$

Define the finite set

$$\Lambda := \{x \in Q | x \in 2^{-m}\mathbb{Z}^d\}.$$

Set $p := 1 + d$ and $\eta := \frac{\varepsilon^2}{4 \cdot 2^{2m} \#(\Lambda) \cdot C(p,d)}$ where $C(p,d)$ is the constant from Lemma 1.29. Then by the Chebychev inequality for every bounded, real-valued Borel function ϕ with $\|\phi\|_{L^p([0,1]\times\mathbb{R}^d)} \leq \eta$ and $x \in \Lambda$, we have

$$\mathbb{P}\left[\left|\int_{I_{m,k}} \phi(t, W(t) + x)\ \mathrm{d}t\right| > \frac{\varepsilon}{2 \cdot 2^m}\right] \leq \frac{2 \cdot 2^m}{\varepsilon} \mathbb{E}\left|\int_{I_{m,k}} \phi(t, W(t) + x)\ \mathrm{d}t\right|.$$

We use Lemma 1.29 to get the estimate.

$$\overset{1.29}{\leq} \frac{2 \cdot 2^m}{\varepsilon} C(p,d) \|\phi\|_{L^p([0,1]\times\mathbb{R}^d)} \leq \frac{\varepsilon}{2 \cdot 2^m \#(\Lambda)}.$$

Note that we can put the left-hand side in norms since for every ϕ that satisfies the conditions of Lemma 1.29 $-\phi$ also satisfies the conditions. Therefore

$$\mathbb{P}\left[\bigcup_{x\in\Lambda} \bigcup_{k=0}^{2^m-1} \left|\int_{I_{m,k}} \phi(t, W(t) + x)\ \mathrm{d}t\right| > \frac{\varepsilon}{2 \cdot 2^m}\right] \leq \sum_{x\in\Lambda} \sum_{k=0}^{2^m-1} \frac{\varepsilon}{2 \cdot 2^m \#(\Lambda)} = \frac{\varepsilon}{2}.$$

In conclusion the probability that

$$\left|\int_{I_{m,k}} \phi(t, W(t) + x)\ \mathrm{d}t\right| \leq \frac{\varepsilon}{2 \cdot 2^m}, \qquad \forall x \in \Lambda,\ \forall k \in \{0, ..., 2^m - 1\} \tag{2.9.3}$$

holds for ϕ with $\|\phi\|_{L^p} \leq \eta$ is at least $1 - \varepsilon/2$. Let $\delta := \eta^p$ and $U \subseteq [0,1] \times \mathbb{R}^d$ be an open set with $|U| \leq \delta$. We define an increasing sequence of non-negative, continuous functions, which converge pointwise to $\mathbb{1}_U$ by

$$\phi_r(x) := (r \cdot \mathrm{dist}(x, U^c)) \wedge 1.$$

Observe that

$$\|\phi_r\|_{L^p([0,1]\times\mathbb{R}^d)} \leq \|\mathbb{1}_U\|_{L^p([0,1]\times\mathbb{R}^d)} = |U|^{1/p} \leq \delta^{1/p} = \eta.$$

For each $r \in \mathbb{N}$ we define the events

A_r: (2.9.1) holds for ϕ_r instead of ϕ

and

B_r: (2.9.3) holds for ϕ_r instead of ϕ.

Note that $\mathbb{P}[\mathrm{A}_r], \mathbb{P}[\mathrm{B}_r] \geq 1-\varepsilon/2$ and $\mathbb{P}[\mathrm{A}_r \cap \mathrm{B}_r] \geq 1-\varepsilon$. Let $u \in \Phi^*$. For every $n \in \mathbb{N}$ define

$$u_n(x) := \sum_{k=0}^{2^n-1} \mathbb{1}_{I_{n,k}}(x) \frac{\lfloor 2^n u(k2^{-n}) \rfloor}{2^n}, \qquad \forall x \in [0,1],$$

where $\lfloor \cdot \rfloor$ denotes the componentwise floor function. Observe that u_n is Q-valued and if $u \in \Phi_m$ for some $m \in \mathbb{N}$, u_n converges trivially on $[0,1[$ since $u_n = u$ for n sufficiently large. This convergence even holds for $u \in \Phi$ as the following calculation shows

$$\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \frac{\lfloor 2^n u(\lfloor 2^n x \rfloor 2^{-n}) \rfloor}{2^n} = \lim_{n\to\infty} \frac{2^n u(\lfloor 2^n x \rfloor 2^{-n}) - \mu(2^n u(\lfloor 2^n x \rfloor 2^{-n}))}{2^n}$$

where $\mu(y) := y - \lfloor y \rfloor \in [0,1[$ for all $y \in \mathbb{R}$. Since $u \in \Phi$, u is continuous and hence

$$= \lim_{n\to\infty} u(\lfloor 2^n x \rfloor 2^{-n}) \overset{u\in\Phi}{=} u\left(\lim_{n\to\infty} \lfloor 2^n x \rfloor 2^{-n}\right) = u\left(\lim_{n\to\infty} x - \mu(2^n x) 2^{-n}\right) = u(x).$$

Now, assume that A_r and B_r both hold. In this case we have

$$\left| \int_0^1 \phi_r(t, W(t) + u_m(t))\, \mathrm{d}t \right| \leq \sum_{k=0}^{2^m-1} \left| \int_{I_{m,k}} \phi_r(t, W(t) + u_m(t))\, \mathrm{d}t \right| \overset{(\mathrm{B}_r)}{\leq} \sum_{k=0}^{2^m-1} \frac{\varepsilon}{2 \cdot 2^m} = \frac{\varepsilon}{2}.$$

Observe that u_m is $2^{-m}\mathbb{Z}^d$-valued and therefore $u_m(t) \in \Lambda$. Using (A_r) results in

$$\begin{aligned}
&\left| \int_0^1 \phi_r(t, W(t) + u_{n+1}(t)) - \phi_r(t, W(t) + u_n(t))\, \mathrm{d}t \right| \\
&\leq \sum_{k=0}^{2^{n+1}-1} \left| \int_{I_{n+1,k}} \phi_r(t, W(t) + \underbrace{u_{n+1}(t)}_{\in Q}) - \phi_r(t, W(t) + \underbrace{u_n(t)}_{\in Q})\, \mathrm{d}t \right| \\
&\overset{(\mathrm{A}_r)}{\leq} \sum_{k=0}^{2^{n+1}-1} C(\varepsilon)\sqrt{n+1}2^{-n/2} \max\left(|u_{n+1}(k2^{-n-1}) - u_n(k2^{-n-1})|_\infty, 2^{-2^n} \right) \\
&\leq C(\varepsilon)\sqrt{n+1}2^{-n/2} \sum_{k=0}^{2^{n+1}-1} |u_{n+1}(k2^{-n-1}) - u_n((k/2)2^{-n})|_\infty + 2^{-2^n} \\
&\leq C(\varepsilon)\sqrt{n+1}2^{-n/2} \left[1 + \sum_{k=0}^{2^{n+1}-1} |u_{n+1}(k2^{-n-1}) - u_n(\lfloor k/2 \rfloor 2^{-n})|_\infty \right] \\
&= C(\varepsilon)\sqrt{n+1}2^{-n/2} \left[1 + \sum_{k=0}^{2^{n+1}-1} 2^{-n-1} \left| \lfloor 2^{n+1} u(k2^{-n-1}) \rfloor - 2 \left\lfloor 2^n u\left(\lfloor k/2 \rfloor 2^{-n}\right) \right\rfloor \right|_\infty \right]
\end{aligned}$$

$$
\begin{aligned}
&\le C(\varepsilon)\sqrt{n+1}2^{-n/2}\Bigg[1+\sum_{k=0}^{2^{n+1}-1}2^{-n-1}\underbrace{\left|\lfloor 2^{n+1}u(k2^{-n-1})\rfloor - 2^{n+1}u(k2^{-n-1})\right|_\infty}_{\le 1}\\
&\qquad+\sum_{k=0}^{2^{n+1}-1}\left|u(k2^{-n-1})-u\left(\lfloor k/2\rfloor 2^{-n}\right)\right|_\infty\\
&\qquad+\sum_{k=0}^{2^{n+1}-1}2^{-n}\underbrace{\left|2^n u\left(\lfloor k/2\rfloor 2^{-n}\right)-\lfloor 2^n u\left(\lfloor k/2\rfloor 2^{-n}\right)\rfloor\right|_\infty}_{\le 1}\Bigg]\\
&\le C(\varepsilon)\sqrt{n+1}2^{-n/2}\left[4+\sum_{\substack{k=0\\ 2\nmid k}}^{2^{n+1}-1}\left|u(k2^{-n-1})-u\left(2\lfloor k/2\rfloor 2^{-n-1}\right)\right|_\infty\right]\\
&= C(\varepsilon)\sqrt{n+1}2^{-n/2}\left[4+\sum_{\substack{k=0\\ 2\nmid k}}^{2^{n+1}-1}\left|u(k2^{-n-1})-u\left((k-1)2^{-n-1}\right)\right|_\infty\right]\\
&\le C(\varepsilon)\sqrt{n+1}2^{-n/2}\left[4+\sum_{k=0}^{2^{n+1}-1}\left|u((k+1)2^{-n-1})-u\left(k2^{-n-1}\right)\right|_\infty\right].
\end{aligned}
$$

Using Lemma 2.8 we can estimate the sum and obtain

$$
\overset{2.8}{\le} 5C(\varepsilon)\sqrt{n+1}2^{-n/2}.
$$

Therefore as long as A_r and B_r both hold we have by Lebesgue's dominated convergence Theorem and continuity of ϕ_r

$$
\begin{aligned}
&\int_0^1 \phi_r(t, W(t)+u(t))\ \mathrm{d}t = \lim_{n\to\infty}\int_0^1 \phi_r(t,W(t)+u_n(t))\ \mathrm{d}t\\
&= \int_0^1 \phi_r(t,W(t)+u_m(t))\ \mathrm{d}t + \sum_{n=m}^{\infty}\int_0^1 \phi_r(t,W(t)+u_{n+1}(t)) - \phi_r(t,W(t)+u_n(t))\ \mathrm{d}t\\
&\overset{(\mathrm{A}_r),(\mathrm{B}_r)}{\le} \frac{\varepsilon}{2}+5C(\varepsilon)\sum_{n=m}^{\infty}\sqrt{n+1}2^{-n/2} \overset{(2.9.2)}{\le} \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

We now define the event Q_r as

$$
\mathrm{Q}_r\colon \int_0^1 \phi_r(t,W(t)+u(t))\ \mathrm{d}t \le \varepsilon, \qquad\qquad \forall u\in\Phi^*.
$$

Note that Q_r is measurable since Φ^* is separable and

$$
\mathrm{Q}_r = \bigcap_{u\in\Phi^*}\left\{\int_0^1 \phi_r(t,W(t)+u(t))\ \mathrm{d}t\le\varepsilon\right\} = \left\{\sup_{u\in\Phi^*}\int_0^1 \phi_r(t,W(t)+u(t))\ \mathrm{d}t \le \varepsilon\right\}.
$$

The previous arguments already showed that $\mathrm{A}_r \cap \mathrm{B}_r \subseteq \mathrm{Q}_r$ and hence $\mathbb{P}[\mathrm{Q}_r] \geq 1-\varepsilon$. But since $\phi_r \leq \phi_{r+1}$ we have $\mathrm{Q}_{r+1} \subseteq \mathrm{Q}_r$. This implies that

$$\mathbb{P}\left[\bigcap_{r=0}^{\infty} \mathrm{Q}_r\right] \geq 1-\varepsilon.$$

Using Lebesgue's dominated convergence Theorem again, we deduce that

$$\int_0^1 \mathbb{1}_U(t, W(t)+u(t))\ \mathrm{d}t = \lim_{r\to\infty} \int_0^1 \phi_r(t, W(t)+u(t))\ \mathrm{d}t \leq \varepsilon$$

holds with probability at least $1-\varepsilon$, which concludes the proof.

□

We are now ready to prove the approximation lemma.

Lemma 2.10

Let g be a real-valued Borel function on $[0,1]\times\mathbb{R}^d$ bounded by 1 everywhere. There exists $N \subseteq \Omega$ with $\mathbb{P}[N] = 0$ such that for all sequences $(u_n)_{n\in\mathbb{N}}$ in Φ^* converging pointwise to $u \in \Phi^*$

$$\lim_{n\to\infty} \int_0^1 g(t, W(t)+u_n(t))\ \mathrm{d}t = \int_0^1 g(t, W(t)+u(t))\ \mathrm{d}t, \qquad \forall \omega \in N^c$$

holds.

Proof

Let g be a real-valued Borel function on $[0,1]\times\mathbb{R}^d$ bounded by 1 and $\varepsilon > 0$. Let δ and A_ε be as in Lemma 2.9. By Lusin's Theorem (cf. [Tao11] Theorem 1.3.28) there exists an open set $U \subseteq [0,1]\times\mathbb{R}^d$ with $|U| \leq \delta$ such that $g|_{U^c}$ is continuous. By Tietze's extension Theorem (cf. [BvR97] Theorem 15.15) there exists a continuous function $h\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}$ such that $h = g$ on U^c and $|h| \leq |g|$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in Φ^* converging pointwise to some $u \in \Phi^*$. With probability at least $1-\varepsilon$ the conclusion of Lemma 2.9 holds, i.e.

$$\left|\int_0^1 \mathbb{1}_U(t, W(t)+u_n(t))\ \mathrm{d}t\right| \leq \varepsilon, \qquad \forall \omega \in A_\varepsilon^c,\ \forall n \in \mathbb{N}. \tag{2.10.1}$$

And the same inequality is true if we replace u_n by u. Since g as well as h is bounded by 1 we have $|g-h| \leq 2$ everywhere and we therefore conclude

$$\left|\int_0^1 g(t, W(t)+u_n(t)) - h(t, W(t)+u_n(t))\ \mathrm{d}t\right| \leq \int_0^1 \mathbb{1}_U(t, W(t)+u_n(t))|g-h|\ \mathrm{d}t \overset{(2.10.1)}{\leq} 2\varepsilon.$$

Since h is continuous Lebesgue's dominated convergence Theorem implies

$$\lim_{n\to\infty}\int_0^1 h(t, W(t)+u_n(t))\ \mathrm{d}t = \int_0^1 \lim_{n\to\infty} h(t, W(t)+u_n(t))\ \mathrm{d}t = \int_0^1 h(t, W(t)+u(t))\ \mathrm{d}t.$$

Choose $m \in \mathbb{N}$ sufficiently large such that

$$\left|\int_0^1 h(t, W(t)+u_n(t)) - h(t, W(t)+u(t))\ \mathrm{d}t\right| \leq \varepsilon, \qquad \forall n \geq m. \tag{2.10.2}$$

All in all we have for $n \geq m$ and $\omega \in A_\varepsilon^c$

$$\begin{aligned}
&\left|\int_0^1 g(t, W(t)+u_n(t))\ \mathrm{d}t - \int_0^1 g(t, W(t)+u(t))\ \mathrm{d}t\right| \\
&\leq \underbrace{\left|\int_0^1 g(t, W(t)+u_n(t)) - h(t, W(t)+u_n(t))\ \mathrm{d}t\right|}_{\leq 2\varepsilon \text{ by } (2.10.1)} \\
&\quad + \underbrace{\left|\int_0^1 h(t, W(t)+u_n(t)) - h(t, W(t)+u(t))\ \mathrm{d}t\right|}_{\leq \varepsilon \text{ by } (2.10.2)} \\
&\quad + \underbrace{\left|\int_0^1 h(t, W(t)+u(t)) - g(t, W(t)+u(t))\ \mathrm{d}t\right|}_{\leq 2\varepsilon \text{ by } (2.10.1)} \\
&\leq 5\varepsilon.
\end{aligned}$$

We define

$$A_k := \bigcap_{\substack{(u_n)_{n\in\mathbb{N}}\in\Phi^{*\mathbb{N}} \\ \lim\limits_{n\to\infty} u_n = u}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{ \left| \int\limits_0^1 g(t, W(t)+u_n(t)) - g(t, W(t)+u(t)) \, \mathrm{d}t \right| \leq \frac{5}{k} \right\}.$$

Note that A_k is measurable since Φ^* is separable and

$$\Phi^{*\mathbb{N}} \cong \coprod_{n\in\mathbb{N}} \Phi^*.$$

We obviously have

$$A_{k+1} \subseteq A_k$$

which implies

$$\mathbb{P}\left[\bigcap_{k=1}^{\infty} A_k\right] = \lim_{k\to\infty} \mathbb{P}\left[A_k\right] \geq \lim_{k\to\infty} 1 - \frac{1}{k} = 1$$

concluding the proof.

□

Remark 2.11

Lemma 2.10 also immediately implies that $\sigma_{n,k}$ and thus $\rho_{n,k}$ are continuous. This enables us to generalize the estimate in Lemma 2.5 from dyadic points to the entire cube $[-1,1]^d$. By a simple inductive argument we can also prove the estimate for all points in $\mathbb{R}^d$, which will simplify arguments in later proofs. The following corollary is centered on these observations.

Corollary 2.12

For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for all real-valued Borel functions g on $[0,1]\times\mathbb{R}$ bounded by 1 everywhere

$$|\rho_{n,k}(x,y)| \leq C(\varepsilon)\sqrt{n}2^{-n/2}\left(|x-y|_\infty + 2^{-2^n}\right)$$

holds for all $x, y \in \mathbb{R}^d$, $n \geq 1$, $k \in \{0, ..., 2^n - 1\}$ and $\omega \in A_\varepsilon^c$.

Proof

Step 1:

Let $\varepsilon > 0$, g a real-valued Borel function with $|g| \leq 1$ everywhere and $x \in \mathbb{R}^d$ such that $x = \lfloor x \rfloor + y$ with $y \in Q$, where $\lfloor \cdot \rfloor$ again denotes the componentwise floor function. Set $r := \lfloor |x|_\infty \rfloor + 1$, $x^{(i)} := ((i \wedge |x_1|)\,\mathrm{sign}(x_1), ..., (i \wedge |x_d|)\,\mathrm{sign}(x_d))^\top$ for $i \in \{0, ..., r\}$ where $x = (x_1, ..., x_d)^\top$. Additionally, we define

$$\tau_z \colon \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad (t,u) \longmapsto (t, u+z).$$

We are now able to write

$$\begin{aligned}
\sigma_{n,k}(x) &= \int\limits_{I_{n,k}} g(t, W(t)+x) - g(t, W(t))\ \mathrm{d}t \\
&= \sum_{i=0}^{r-1} \int\limits_{I_{n,k}} g(t, W(t) + x^{(i+1)}) - g(t, W(t)+x^{(i)})\ \mathrm{d}t \\
&= \sum_{i=0}^{r-1} \int\limits_{I_{n,k}} g \circ \tau_{x^{(i)}}(t, W(t) + x^{(i+1)} - x^{(i)}) - g \circ \tau_{x^{(i)}}(t, W(t))\ \mathrm{d}t.
\end{aligned}$$

Note that $x^{(i+1)} - x^{(i)} \in Q$. We can therefore apply Lemma 2.5

$$\begin{aligned}
|\sigma_{n,k}(x)| &\leq \sum_{i=0}^{r-1} |\sigma_{n,k}(x^{(i+1)} - x^{(i)}; g \circ \tau_{x^{(i)}}, W)| \\
&\overset{2.5}{\leq} \sum_{i=0}^{r-1} C(\varepsilon)\sqrt{n} 2^{-n/2} \max\left(|x^{(i+1)} - x^{(i)}|_\infty, 2^{-2^n}\right) \\
&\overset{n\geq 1}{=} C(\varepsilon)\sqrt{n}2^{-n/2} \sum_{i=0}^{r-2} \underbrace{|x^{(i+1)} - x^{(i)}|_\infty}_{=1} + C(\varepsilon)\sqrt{n}2^{-n/2} \max\Big(\,|\underbrace{x^{(r)} - x^{(r-1)}}_{=y}|_\infty, 2^{-2^n}\,\Big) \\
&\leq C(\varepsilon)\sqrt{n}2^{-n/2}(r-1) + C(\varepsilon)\sqrt{n}2^{-n/2}\left(|y|_\infty + 2^{-2^n}\right) \\
&= C(\varepsilon)\sqrt{n}2^{-n/2}\left(r - 1 + |y|_\infty + 2^{-2^n}\right) = C(\varepsilon)\sqrt{n}2^{-n/2}\left(|x|_\infty + 2^{-2^n}\right).
\end{aligned}$$

Step 2:

Let $x \in \mathbb{R}^d$. Choose a sequence $x_m \in Q$ such that $\lim\limits_{m\to\infty} \lfloor x \rfloor + x_m = x$. We write

$$\begin{aligned}
\lim_{m\to\infty} \sigma_{n,k}(\lfloor x \rfloor + x_m) &= \lim_{m\to\infty} \int_0^1 \mathbb{1}_{I_{n,k}}(t) g(t, W(t) + \lfloor x \rfloor + x_m) - \mathbb{1}_{I_{n,k}}(t) g(t, W(t))\ \mathrm{d}t \\
&\overset{2.10}{=} \int_0^1 \mathbb{1}_{I_{n,k}}(t) g(t, W(t)+x) - \mathbb{1}_{I_{n,k}}(t) g(t, W(t))\ \mathrm{d}t = \sigma_{n,k}(x).
\end{aligned}$$

So, by step 1 we have

$$\begin{aligned}|\sigma_{n,k}(x)| &= \lim_{m\to\infty} |\sigma_{n,k}(\lfloor x\rfloor + x_m)| \le \lim_{m\to\infty} C(\varepsilon)\sqrt{n}2^{-n/2}\left(|\lfloor x\rfloor + x_m|_\infty + 2^{-2^n}\right)\\ &= C(\varepsilon)\sqrt{n}2^{-n/2}\left(|x|_\infty + 2^{-2^n}\right).\end{aligned}$$

Step 3:

Let $x, y \in \mathbb{R}^d$. With the help of the map τ_y we obtain

$$\begin{aligned}\rho_{n,k}(x,y) &= \int_{I_{n,k}} g(t, W(t)+x) - g(t, W(t)+y)\ \mathrm{d}t\\ &= \int_{I_{n,k}} g\circ\tau_y(t, W(t)+x-y) - g\circ\tau_y(t, W(t))\ \mathrm{d}t = \sigma_{n,k}(x-y; g\circ\tau_y, W).\end{aligned}$$

By step 2 we have

$$|\rho_{n,k}(x,y)| = |\sigma_{n,k}(x-y; g\circ\tau_y, W)| \le C(\varepsilon)\sqrt{n}2^{-n/2}\left(|x-y|_\infty + 2^{-2^n}\right)$$

which concludes the proof.

□

2.3 Main result

We will now prove an estimate for the term

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q)|_\infty$$

where $y_q \in Q$. First, we obtain an estimate for the Euler approximation $x_{q+1} := x_q + \sigma_{n,k+q}(x_q)$ (Lemma 2.14). By comparing y_q with the Euler approximation we get the required bound for the above sum (Lemma 2.15), which is the last estimate that is necessary to prove the essential lemma (Lemma 2.16) of the main theorem.

Remark 2.13

Let f be a bounded, $\mathbb{R}^d$-valued Borel function on $[0,1]\times\mathbb{R}^d$ as in (1.0). We write $f = (g_1, ..., g_d)^\top$ and note that

$$|\sigma_{n,k}(x; f, W)|_\infty = \max_{1\le i\le d} |\sigma_{n,k}(x; g_i, W)|.$$

Henceforth, the conclusions of Lemma 2.5 and 2.10 also hold for functions f which are $\mathbb{R}^d$-valued as long as the norm $|\cdot|$ on the left-hand side of the equations is replaced by $|\cdot|_\infty$, i.e.

$$|\sigma_{n,k}(x; f, W)|_\infty \le C(\varepsilon)\sqrt{n}2^{-n/2}\max\left(|x|_\infty, 2^{-2^n}\right),$$

From now on we take f instead of g in the definition of $\sigma_{n,k}$ and $\rho_{n,k}$.

Lemma 2.14

Let $p \in \mathbb{N}$ be an even integer. There exists $C(p) \in \mathbb{R}$ such that for all $\mathbb{R}^d$-valued Borel functions f on $[0,1] \times \mathbb{R}^d$, which are bounded by 1 everywhere, $x_0 \in Q$ and $n, r \in \mathbb{N}$ satisfying $r \leq 2^{n/2}$ we have

$$\mathbb{P}\left[\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1}, x_q)|_\infty > 2^{-n}\left(\lambda\sqrt{r}|x_0|_\infty + C(p)\sum_{q=0}^{r-1}|x_q|_\infty\right)\right] \leq C(p)\lambda^{-p}$$

for all $k \in \{0, ..., 2^n - r - 1\}$ and for any $\lambda > 0$. Where $x_{q+1} = x_q + \sigma_{n,k+q}(x_q)$ for $q \in \{0, ..., r-1\}$.

Proof

Observe that x_q is $\mathcal{F}_{(k+q)2^{-n}}$-measurable, since $\sigma_{n,k+q-1}$ is $\mathcal{F}_{(k+q)2^{-n}}$-measurable. We want to use Corollary 1.28.(ii) for x_q instead of some $x \in \mathbb{R}^d$. Due to this we do a measure theoretic induction. Let $F, A \in \mathcal{F}_{(k+q)2^{-n}}$ and $\alpha, \beta \in \mathbb{R}^d$. Let p be an even integer or 1. We then have

$$\begin{aligned}
&\int_F |\rho_{n,k+q}(\alpha\mathbb{1}_A, \beta\mathbb{1}_A)|_\infty^p \,\mathrm{d}\mathbb{P} = \int_{F\cap A} |\rho_{n,k+q}(\alpha,\beta)|_\infty^p \,\mathrm{d}\mathbb{P} + \int_{F\cap A^c} \underbrace{|\rho_{n,k+q}(0,0)|_\infty^p}_{=0} \,\mathrm{d}\mathbb{P} \\
&= \int_{F\cap A} \mathbb{E}[|\rho_{n,k+q}(\alpha,\beta)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] \,\mathrm{d}\mathbb{P} \overset{1.28.(\mathrm{ii})}{\leq} \int_{F\cap A} C^p\Gamma\left(\frac{p}{2}+1\right)2^{-np/2}|\alpha-\beta|_2^p \,\mathrm{d}\mathbb{P} \\
&\leq \int_F C_1(p)2^{-np/2}|\alpha-\beta|_\infty^p \mathbb{1}_A \,\mathrm{d}\mathbb{P}
\end{aligned}$$

with $C_1(p) := C^p\Gamma(p/2+1)d^{p/2}$, where C is the constant from Corollary 1.28.(ii). Let us now consider step functions. Let $A_i \in \mathcal{F}_{(k+q)2^{-n}}$ be pairwise disjoint sets for $i \in \{1, ..., m\}$. We then have

$$\begin{aligned}
&\int_F \left|\rho_{n,k+q}\left(\sum_{i=1}^m \alpha_i\mathbb{1}_{A_i}, \sum_{j=1}^m \beta_j\mathbb{1}_{A_j}\right)\right|_\infty^p \,\mathrm{d}\mathbb{P} = \sum_{i=1}^m \int_F |\rho_{n,k+q}(\alpha_i\mathbb{1}_{A_i}, \beta_i\mathbb{1}_{A_i})|_\infty^p \,\mathrm{d}\mathbb{P} \\
&\leq \sum_{i=1}^m \int_F C_1(p)2^{-np/2}|\alpha_i-\beta_i|_\infty^p\mathbb{1}_{A_i} \,\mathrm{d}\mathbb{P} = \int_F C_1(p)2^{-np/2}\left|\sum_{i=1}^m(\alpha_i-\beta_i)\mathbb{1}_{A_i}\right|_\infty^p \,\mathrm{d}\mathbb{P}.
\end{aligned}$$

Let ϕ, ψ be non-negative $\mathcal{F}_{(k+q)2^{-n}}$-measurable functions. We approximate ϕ and ψ by increasing sequences of $\mathcal{F}_{(k+q)2^{-n}}$-measurable step functions ϕ_i and ψ_i, respectively. By continuity of $\rho_{n,k+q}$ (Remark 2.11) we obtain

$$\begin{aligned}
&\mathbb{E}[|\rho_{n,k+q}(\phi,\psi)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] = \mathbb{E}[|\rho_{n,k+q}(\lim_{i\to\infty}\phi_i, \lim_{i\to\infty}\psi_i)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] \\
&\overset{2.11}{=} \mathbb{E}[\lim_{i\to\infty}|\rho_{n,k+q}(\phi_i,\psi_i)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] = \lim_{i\to\infty}\mathbb{E}[|\rho_{n,k+q}(\phi_i,\psi_i)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] \\
&\leq \lim_{i\to\infty} C_1(p)2^{-np/2}|\phi_i-\psi_i|_\infty^p = C_1(p)2^{-np/2}|\phi-\psi|_\infty^p.
\end{aligned}$$

As noted above, x_q is $\mathcal{F}_{(k+q)2^{-n}}$-measurable, so we have

$$\mathbb{E}[|\sigma_{n,k+q}(x_q)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] = \mathbb{E}[|\rho_{n,k+q}(x_q,0)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] \le C_1(p)2^{-np/2}|x_q|_\infty^p \tag{2.14.1}$$

$$\mathbb{E}[|\rho_{n,k+q}(x_{q-1},x_q)|_\infty^p|\mathcal{F}_{(k+q)2^{-n}}] \le C_1(p)2^{-np/2}|x_{q-1}-x_q|_\infty^p. \tag{2.14.2}$$

Taking expectation yields

$$\mathbb{E}|\sigma_{n,k+q}(x_q)|_\infty^p \le C_1(p)2^{-np/2}\mathbb{E}|x_q|_\infty^p \tag{2.14.3}$$

$$\mathbb{E}|\rho_{n,k+q}(x_{q-1},x_q)|_\infty^p \le C_1(p)2^{-np/2}\mathbb{E}|x_{q-1}-x_q|_\infty^p. \tag{2.14.4}$$

As of now, let p be an even integer as stated in the assertion. Using the Minkowski inequality we deduce that

$$\begin{aligned}(\mathbb{E}|x_{q+1}|_\infty^p)^{1/p} &= (\mathbb{E}|x_q+\sigma_{n,k+q}(x_q)|_\infty^p)^{1/p} \le (\mathbb{E}|x_q|_\infty^p)^{1/p} + (\mathbb{E}|\sigma_{n,k+q}(x_q)|_\infty^p)^{1/p} \\ &\overset{(2.14.3)}{\le} (\mathbb{E}|x_q|_\infty^p)^{1/p} + C_1(p)^{1/p}2^{-n/2}\,(\mathbb{E}|x_q|_\infty^p)^{1/p} = (1+C_1(p)^{1/p}2^{-n/2})\,(\mathbb{E}|x_q|_\infty^p)^{1/p}.\end{aligned}$$

Taking the p-th power results in

$$\mathbb{E}|x_{q+1}|_\infty^p \le (1+C_1(p)^{1/p}2^{-n/2})^p\mathbb{E}|x_q|_\infty^p.$$

By induction over $q\in\{0,...,r-1\}$ we obtain

$$\begin{aligned}\mathbb{E}|x_q|_\infty^p &\le (1+C_1(p)^{1/p}2^{-n/2})^{pq}|x_0|_\infty^p, \qquad\qquad \forall q\in\{0,...,r\} \\ &\overset{q\le r\le 2^{n/2}}{\le} \left((1+C_1(p)^{1/p}2^{-n/2})^{2^{n/2}}\right)^p|x_0|_\infty^p \\ &\le \exp(C_1(p)^{1/p})^p|x_0|_\infty^p = \exp(pC_1(p)^{1/p})|x_0|_\infty^p.\end{aligned}$$

Finally we have

$$\mathbb{E}|x_q|_\infty^p \le \exp(pC_1(p)^{1/p})|x_0|_\infty^p \tag{2.14.5}$$

for all $q\in\{0,...,r\}$. Define for $q\in\{1,...,r\}$

$$\begin{aligned}Y_q &:= |\rho_{n,k+q}(x_{q-1},x_q)|_\infty, \\ Z_q &:= \mathbb{E}[Y_q|\mathcal{F}_{(k+q)2^{-n}}], \\ X_q &:= Y_q - Z_q.\end{aligned}$$

Observe that X_q is $\mathcal{F}_{(k+q+1)2^{-n}}$-measurable and we have

$$\mathbb{E}[X_q|\mathcal{F}_{(k+q)2^{-n}}] = \mathbb{E}[Y_q-\mathbb{E}[Y_q|\mathcal{F}_{(k+q)2^{-n}}]|\mathcal{F}_{(k+q)2^{-n}}] = 0.$$

We define for $\ell\in\mathbb{N}$

$$M_\ell := \sum_{q=1}^{r\wedge\ell} X_q.$$

And hence we have $M_0=0$. Moreover, M_ℓ is a $\mathcal{G}_\ell := \mathcal{F}_{(k+\ell+1)2^{-n}}$-martingale since

$$\mathbb{E}[M_\ell|\mathcal{G}_m] = \sum_{q=1}^{r\wedge\ell}\mathbb{E}[X_q|\mathcal{G}_m] = \sum_{q=1}^{r\wedge m}\underbrace{\mathbb{E}[X_q|\mathcal{G}_m]}_{=X_q} + \sum_{q=(r\wedge m)+1}^{r\wedge\ell}\underbrace{\mathbb{E}[X_q|\mathcal{G}_m]}_{=0} = \sum_{q=1}^{r\wedge m}X_q = M_m, \qquad \forall \ell\ge m$$

and the quadratic variation process of M is

$$\langle M\rangle_\ell = \sum_{q=1}^{r\wedge\ell}(M_q - M_{q-1})^2 = \sum_{q=1}^{r\wedge\ell} X_q^2.$$

So, by the discrete Burkholder inequality (cf. [Bur66], Theorem 9) we conclude

$$\mathbb{E}\left|\sum_{q=1}^{r} X_q\right|^p = \mathbb{E}\,|M_r|^p \leq C_2(p)\mathbb{E}\,\langle M\rangle_r^{p/2} = C_2(p)\mathbb{E}\left(\sum_{q=1}^{r} X_q^2\right)^{p/2}$$

where $C_2(p) \in \mathbb{R}$ is independent of M. By Proposition A.6 we have

$$\begin{aligned}
&\overset{A.6}{\leq} C_2(p)r^{p/2-1}\mathbb{E}\sum_{q=1}^{r} X_q^p = C_2(p)r^{p/2-1}\sum_{q=1}^{r}\mathbb{E}|Y_q - \mathbb{E}[Y_q|\mathcal{F}_{(k+q)2^{-n}}]|^p \\
&\overset{Y_q\geq 0}{\leq} C_2(p)r^{p/2-1}\sum_{q=1}^{r}\mathbb{E}Y_q^p = C_2(p)r^{p/2-1}\sum_{q=1}^{r}\mathbb{E}|\rho_{n,k+q}(x_{q-1},x_q)|_\infty^p \\
&\overset{(2.14.4)}{\leq} C_1(p)C_2(p)r^{p/2-1}2^{-np/2}\sum_{q=1}^{r}\mathbb{E}|x_{q-1}-x_q|_\infty^p \\
&= C_1(p)C_2(p)r^{p/2-1}2^{-np/2}\sum_{q=1}^{r}\mathbb{E}|\sigma_{n,k+q-1}(x_{q-1})|_\infty^p \\
&\overset{(2.14.3)}{\leq} C_1(p)^2C_2(p)r^{p/2-1}2^{-np}\sum_{q=1}^{r}\mathbb{E}|x_{q-1}|_\infty^p \\
&\overset{(2.14.5)}{\leq} C_1(p)^2C_2(p)\exp(pC_1(p)^{1/p})r^{p/2-1}2^{-np}\sum_{q=1}^{r}|x_0|_\infty^p \\
&\leq C_1(p)^2C_2(p)\exp(pC_1(p)^{1/p})r^{p/2}2^{-np}|x_0|_\infty^p.
\end{aligned}$$

This calculation implies that

$$\mathbb{E}\left|\sum_{q=1}^{r} X_q\right|^p \leq C_2(p)r^{p/2-1}\sum_{q=1}^{r}\mathbb{E}Y_q^p \leq C_1(p)^2C_2(p)\exp(pC_1(p)^{1/p})r^{p/2}2^{-np}|x_0|_\infty^p. \tag{2.14.6}$$

Let

$$\begin{aligned}
V_q &:= \mathbb{E}[Z_q|\mathcal{F}_{(k+q-1)2^{-n}}], \\
W_q &:= Z_q - V_q.
\end{aligned}$$

Analog to above we define

$$M'_\ell := \sum_{q=1}^{r\wedge\ell} W_q.$$

Observe that W_q is $\mathcal{F}_{(k+q)2^{-n}}$-measurable and we have

$$\mathbb{E}[W_q|\mathcal{F}_{(k+q-1)2^{-n}}] = \mathbb{E}[Z_q - \mathbb{E}[Z_q|\mathcal{F}_{(k+q-1)2^{-n}}]|\mathcal{F}_{(k+q-1)2^{-n}}] = 0.$$

Note that $M_0' = 0$. Moreover M_ℓ' is a $\mathcal{G}_\ell' := \mathcal{F}_{(k+\ell)2^{-n}}$-martingale since

$$\mathbb{E}[M_\ell'|\mathcal{G}_m'] = \sum_{q=1}^{r\wedge\ell} \mathbb{E}[W_q|\mathcal{G}_m'] = \sum_{q=1}^{r\wedge m} \underbrace{\mathbb{E}[W_q|\mathcal{G}_m']}_{=W_q} + \sum_{q=(r\wedge m)+1}^{r\wedge\ell} \underbrace{\mathbb{E}[W_q|\mathcal{G}_m']}_{=0} = \sum_{q=1}^{r\wedge m} W_q = M_m', \qquad \forall \ell \geq m$$

and the quadratic variation process of M' is

$$\langle M' \rangle_\ell = \sum_{q=1}^{r\wedge\ell} (M_q' - M_{q-1}')^2 = \sum_{q=1}^{r\wedge\ell} W_q^2.$$

We again use the Burkholder inequality (cf. [Bur66], Theorem 9) to establish

$$\mathbb{E}\left|\sum_{q=1}^{r} W_q\right|^p = \mathbb{E}\left|M_r'\right|^p \leq C_2(p)\mathbb{E}\left\langle M'\right\rangle_r^{p/2} = C_2(p)\mathbb{E}\left(\sum_{q=1}^{r} W_q^2\right)^{p/2}$$

with $C_2(p) \in \mathbb{R}$ as before. By Proposition A.6 we have

$$\begin{aligned}
&\overset{A.6}{\leq} C_2(p) r^{p/2-1} \mathbb{E} \sum_{q=1}^{r} W_q^p = C_2(p) r^{p/2-1} \sum_{q=1}^{r} \mathbb{E}|Z_q - \mathbb{E}[Z_q|\mathcal{F}_{(k+q-1)2^{-n}}]|^p \\
&\overset{Z_q\geq 0}{\leq} C_2(p) r^{p/2-1} \sum_{q=1}^{r} \mathbb{E} Z_q^p \leq C_2(p) r^{p/2-1} \sum_{q=1}^{r} \mathbb{E} Y_q^p.
\end{aligned}$$

So by inequality (2.14.6) we deduce that

$$\mathbb{E}\left|\sum_{q=1}^{r} W_q\right|^p \leq C_1(p)^2 C_2(p) \exp(pC_1(p)^{1/p}) r^{p/2} 2^{-np} |x_0|_\infty^p. \tag{2.14.7}$$

Let us now consider the term V_q.

$$\begin{aligned}
V_q &= \mathbb{E}[Z_q|\mathcal{F}_{(k+q-1)2^{-n}}] = \mathbb{E}[\mathbb{E}[Y_q|\mathcal{F}_{(k+q)2^{-n}}]|\mathcal{F}_{(k+q-1)2^{-n}}] \\
&= \mathbb{E}[\mathbb{E}[|\rho_{n,k+q}(x_{q-1}, x_q)|_\infty|\mathcal{F}_{(k+q)2^{-n}}|\mathcal{F}_{(k+q-1)2^{-n}}] \\
&\overset{(2.14.2)}{\leq} C_1(1) 2^{-n/2} \mathbb{E}[|x_{q-1} - x_q|_\infty|\mathcal{F}_{(k+q-1)2^{-n}}] \\
&= C_1(1) 2^{-n/2} \mathbb{E}[|\sigma_{n,k+q-1}(x_{q-1})|_\infty|\mathcal{F}_{(k+q-1)2^{-n}}] \\
&\overset{(2.14.1)}{\leq} C_1(1)^2 2^{-n} |x_{q-1}|_\infty.
\end{aligned}$$

This leads us to

$$\sum_{q=1}^{r} V_q \leq C_1(1)^2 2^{-n} \sum_{q=1}^{r} |x_{q-1}|_\infty = C_1(1)^2 2^{-n} \sum_{q=0}^{r-1} |x_q|_\infty. \tag{2.14.8}$$

Note that $Y_q = X_q + W_q + V_q$. Define $C(p) := \max\left(2^{p+1} C_1(p)^2 C_2(p) \exp(pC_1(p)^{1/p}), C_1(1)^2\right)$. Using the already established estimates we deduce that

$$\begin{aligned}
&\mathbb{P}\left[\sum_{q=1}^{r} Y_q > 2^{-n}\left[\lambda\sqrt{r}|x_0|_\infty + C(p)\sum_{q=0}^{r-1}|x_q|_\infty\right]\right] \\
&= \mathbb{P}\left[\sum_{q=1}^{r} X_q + W_q + V_q > 2^{-n}\left[\lambda\sqrt{r}|x_0|_\infty + C(p)\sum_{q=0}^{r-1}|x_q|_\infty\right]\right] \\
&\leq \underbrace{\mathbb{P}\left[\sum_{q=1}^{r} V_q > 2^{-n}C(p)\sum_{q=0}^{r-1}|x_q|_\infty\right]}_{=0 \text{ by } (2.14.8)} + \mathbb{P}\left[\sum_{q=1}^{r} X_q > \frac{\lambda\sqrt{r}|x_0|_\infty}{2^{n+1}}\right] + \mathbb{P}\left[\sum_{q=1}^{r} W_q > \frac{\lambda\sqrt{r}|x_0|_\infty}{2^{n+1}}\right].
\end{aligned}$$

Applying the Chebychev inequality yields

$$\leq \left(\frac{2^{n+1}}{\lambda\sqrt{r}|x_0|_\infty}\right)^p \mathbb{E}\left[\left|\sum_{q=1}^{r} X_q\right|^p + \left|\sum_{q=1}^{r} W_q\right|^p\right].$$

With the help of (2.14.6) and (2.14.7) we obtain

$$\begin{aligned}
&\leq \left(\frac{2^{n+1}}{\lambda\sqrt{r}|x_0|_\infty}\right)^p 2C_1(p)^2 C_2(p)\exp(pC_1(p)^{1/p}) r^{p/2} 2^{-np}|x_0|_\infty^p \\
&= \lambda^{-p} 2^{p+1} C_1(p)^2 C_2(p) \exp(pC_1(p)^{1/p}) \leq C(p)\lambda^{-p}.
\end{aligned}$$

This concludes the proof.

□

Lemma 2.15

For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for all $\mathbb{R}^d$-valued Borel functions f on $[0,1] \times \mathbb{R}^d$ which are bounded by 1, all $n, r \in \mathbb{N}$ with $r \leq \lfloor 2^{n/4} \rfloor$, every $k \in \{0, ..., 2^n - r - 1\}$ and every $y_0, ..., y_r \in [-1,1]^d$ we have

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q)|_\infty \leq C(\varepsilon)\left[2^{-3n/4}|y_0|_\infty + 2^{-n/8}\sum_{q=0}^{r-1}|\gamma_q|_\infty + 2^{-2^{n/2}}\right], \qquad \forall \omega \in A_\varepsilon^c.$$

Where $\gamma_q := y_{q+1} - y_q - \sigma_{n,k+q}(y_q)$ for $q \in \{0, ..., r-1\}$.

Proof

Let $\varepsilon > 0$. Set $\delta_n := 2^{-n/4}2^{-2^{n/2}}$. By Corollary 2.12, Remark 2.13 and Proposition A.7 with probability $1-\varepsilon/2$ there exists $C_1(\varepsilon) > 0$ such that

$$\begin{aligned} |\rho_{n,k+q}(x,y)|_\infty &\overset{2.12}{\leq} 2^{-1}C_1(\varepsilon)\sqrt{n}2^{-n/2}\left(|x-y|_\infty + 2^{-2^n}\right) \\ &\underset{A.7}{\leq} C_1(\varepsilon)\left(2^{-3n/8}|x-y|_\infty + \delta_n\right) \end{aligned} \tag{2.15.1}$$

holds for all $x,\ y \in \mathbb{R}^d$. Let $\tilde{Q}_s := \{x \in \mathbb{R}^d : |x|_\infty \leq 2^{-s}\}$. Then, for integers s with $0 \leq s \leq 2^{n/2}$ we define

$$\begin{aligned} Q_{n,s} &:= \{x \in \tilde{Q}_s | x = (x_1, ..., x_d)^\top,\ \forall i \in \{1,...,d\}\colon \exists k \in \{-2^n+1, ..., 2^n-1\}\colon x_i = k2^{-s-n}\}, \\ Q_n &:= \bigcup_{s=0}^{\lfloor 2^{n/2}\rfloor} Q_{n,s}. \end{aligned}$$

Furthermore, let $p := 8(d+3)$. Then by Lemma 2.14 with $\lambda := \tilde{\lambda}2^{n/8}$ there is $C > 0$ such that

$$\mathbb{P}\left[\sum_{q=1}^{r} |\rho_{n,k+q}(x_{q-1},x_q)|_\infty > 2^{-n}\left(\tilde{\lambda}2^{n/8}\sqrt{r}|x_0|_\infty + C\sum_{q=0}^{r-1}|x_q|_\infty\right)\right] \leq C\tilde{\lambda}^{-p}2^{-pn/8}$$

holds for some n, r, k and $x_0 \in Q_n$ as in the statement of Lemma 2.14. We deduce that

$$\begin{aligned} &\mathbb{P}\left[\bigcup_{n=0}^{\infty}\bigcup_{r=0}^{\lfloor 2^{n/4}\rfloor}\bigcup_{k=0}^{2^n-r-1}\bigcup_{x_0\in Q_n}\sum_{q=1}^{r}|\rho_{n,k+q}(x_{q-1},x_q)|_\infty > 2^{-n}\left(\tilde{\lambda}2^{n/8}\sqrt{r}|x_0|_\infty + C\sum_{q=0}^{r-1}|x_q|_\infty\right)\right] \\ &\leq C\sum_{n=0}^{\infty}\sum_{r=0}^{\lfloor 2^{n/4}\rfloor}\sum_{k=0}^{2^n-r}\sum_{x_0\in Q_n}\tilde{\lambda}^{-p}2^{-pn/8} \leq C\sum_{n=0}^{\infty}2^{n/4}(2^n-r)\#(Q_n)\tilde{\lambda}^{-p}2^{-pn/8} \\ &\leq C\sum_{n=0}^{\infty}2^{n/4}2^n\sum_{s=0}^{\lfloor 2^{n/2}\rfloor}\underbrace{\#(Q_{n,s})}_{\leq 2^d\cdot 2^{nd}}\tilde{\lambda}^{-p}2^{-pn/8} \leq C\sum_{n=0}^{\infty}2^{n/4}2^n\sum_{s=0}^{2^{n/2}}2^d2^{nd}\tilde{\lambda}^{-p}2^{-pn/8} \\ &= C2^d\tilde{\lambda}^{-p}\sum_{n=0}^{\infty}2^{n+n/2+n/4}2^{nd}2^{-(d+3)n} \\ &\leq C2^d\tilde{\lambda}^{-p}\sum_{n=0}^{\infty}2^{2n}2^{nd}2^{-(d+3)n} \leq C2^d\tilde{\lambda}^{-p}\sum_{n=0}^{\infty}2^{-n} = C2^{d+1}\tilde{\lambda}^{-p}. \end{aligned}$$

Which converges to 0 as $\tilde{\lambda} \to \infty$. Hence with probability $1-\varepsilon/2$ there is $C_2(\varepsilon) > 1$ such that

$$\sum_{q=1}^{r}|\rho_{n,k+q}(x_{q-1},x_q)|_\infty \leq C_2(\varepsilon)2^{-n}\left(2^{n/8}\sqrt{r}|x_0|_\infty + \sum_{q=0}^{r-1}|x_q|_\infty\right) \tag{2.15.2}$$

holds for all n, k, r and $x_0 \in Q_n$ as above. So with probability $1-\varepsilon$ we can assume that both (2.15.1) and (2.15.2) hold with the same constant $C(\varepsilon)$. Fix $n, k, r, y_0, ..., y_r$ as in the statement of this lemma. Take the largest integer $s \in \{0, ..., \lfloor 2^{n/2}\rfloor\}$ such that $y_0 \in \tilde{Q}_s$. Note that

$$|y_0|_\infty \leq 2^{-s}.$$

Since s is maximal with this property we have

$$2^{-s-1} < |y_0|_\infty \qquad \text{or} \qquad |y_0|_\infty \leq 2^{-s} = 2^{-\lfloor 2^{n/2} \rfloor} \leq 2 \cdot 2^{-2^{n/2}}$$

and hence

$$2^{-s} \leq \max\left(2|y_0|_\infty, 2 \cdot 2^{-2^{n/2}}\right).$$

By definition of $Q_{n,s}$ we can find $z_0 \in Q_{n,s}$ such that

$$|z_0 - y_0|_\infty \leq 2^{-s-n} \leq 2^{1-n}|y_0|_\infty + 2^{-2^{n/2}}. \tag{2.15.3}$$

We define $z_1, ..., z_r$ by the recurrence relation

$$z_{q+1} := z_q + \sigma_{n,k+q}(z_q). \tag{2.15.4}$$

Using (2.15.1) we have

$$\begin{aligned} |z_{q+1}|_\infty &\overset{(2.15.4)}{=} |z_q + \sigma_{n,k+q}(z_q)|_\infty \leq |z_q|_\infty + |\rho_{n,k+q}(z_q, 0)|_\infty \\ &\overset{(2.15.1)}{\leq} |z_q|_\infty + C(\varepsilon)\left(2^{-3n/8}|z_q|_\infty + \delta_n\right) \leq (1 + C(\varepsilon)2^{-n/4})(|z_q|_\infty + C(\varepsilon)\delta_n). \end{aligned}$$

By induction on $q \in \{1, ..., r-1\}$ we have

$$|z_q|_\infty \leq (1 + C(\varepsilon)2^{-n/4})^r(|z_0|_\infty + C(\varepsilon)r\delta_n) \overset{r \leq 2^{n/4}}{\leq} \exp(C(\varepsilon))C(\varepsilon)(|z_0|_\infty + 2^{n/4}\delta_n) \tag{2.15.5}$$

for all $q \in \{0, ..., r\}$. Since $z_0 \in Q_{n,s} \subseteq Q_n$ we can apply (2.15.2) to obtain

$$\begin{aligned} \sum_{q=1}^{r} |\rho_{n,k+q}(z_{q-1}, z_q)|_\infty &\overset{(2.15.2)}{\leq} C(\varepsilon)2^{-n}\left(2^{n/8}\sqrt{r}|z_0|_\infty + \sum_{q=0}^{r-1}|z_q|_\infty\right) \\ &\leq C(\varepsilon)2^{-n}\left(2^{n/4}|z_0|_\infty + \sum_{q=0}^{r-1}|z_q|_\infty\right) \\ &\overset{(2.15.5)}{\leq} C(\varepsilon)2^{-n}\left(2^{n/4}|z_0|_\infty + \sum_{q=0}^{r-1} C(\varepsilon)\exp(C(\varepsilon))(|z_0|_\infty + 2^{n/4}\delta_n)\right) \\ &= C(\varepsilon)2^{-n}\left(2^{n/4}|z_0|_\infty + C(\varepsilon)\exp(C(\varepsilon))r(|z_0|_\infty + 2^{n/4}\delta_n)\right) \\ &\leq 2C(\varepsilon)^2\exp(C(\varepsilon))2^{-n}\left(2^{n/4}|z_0|_\infty + 2^{n/4}2^{n/4}\delta_n\right) = C_3 2^{-3n/4}\left(|z_0|_\infty + 2^{n/4}\delta_n\right) \end{aligned}$$

where $C_3 := 2C(\varepsilon)^2\exp(C(\varepsilon))$. So, we have

$$\sum_{q=1}^{r} |\rho_{n,k+q}(z_{q-1}, z_q)|_\infty \leq C_3 2^{-3n/4}\left(|z_0|_\infty + 2^{n/4}\delta_n\right). \tag{2.15.6}$$

We set $u_q := z_q - y_q$ for $q \in \{0, ..., r\}$. Then we get the following estimate for the difference

$$\begin{aligned}|u_{q+1} - u_q|_\infty &= |z_{q+1} - y_{q+1} - z_q + y_q|_\infty \overset{(2.15.4)}{=} |\sigma_{n,k+q}(z_q) - y_{q+1} + y_q|_\infty \\ &\le |\sigma_{n,k+q}(z_q) - y_{q+1} + y_q + \gamma_q|_\infty + |\gamma_q|_\infty \\ &= |\sigma_{n,k+q}(z_q) - y_{q+1} + y_q + \underbrace{y_{q+1} - y_q - \sigma_{n,k+q}(y_q)}_{=\gamma_q}|_\infty + |\gamma_q|_\infty \\ &= |\rho_{n,k+q}(z_q, y_q)|_\infty + |\gamma_q|_\infty.\end{aligned}$$

We therefore deduce that

$$|u_{q+1}|_\infty \le |u_{q+1} - u_q|_\infty + |u_q|_\infty \le |\rho_{n,k+q}(z_q, y_q)|_\infty + |\gamma_q|_\infty + |u_q|_\infty.$$

Using (2.15.1) we get

$$\begin{aligned}&\overset{(2.15.1)}{\le} C(\varepsilon)2^{-3n/8}|z_q - y_q|_\infty + C(\varepsilon)\delta_n + |\gamma_q|_\infty + |u_q|_\infty \\ &\le \ (1 + C(\varepsilon)2^{-n/4})(|u_q|_\infty + C(\varepsilon)\delta_n + |\gamma_q|_\infty).\end{aligned}$$

Again by induction on q we deduce

$$|u_q|_\infty \le (1 + C(\varepsilon)2^{-n/4})^r \left(|u_0|_\infty + C(\varepsilon)r\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right)$$

for all $q \in \{0, ..., r\}$. Since $|u_0|_\infty = |z_0 - y_0|_\infty \overset{(2.15.3)}{\le} 2^{1-n}|y_0|_\infty + 2^{-2^{n/2}}$ we have

$$\begin{aligned}|u_q|_\infty &\le (1 + C(\varepsilon)2^{-n/4})^r \left(2^{1-n}|y_0|_\infty + 2^{-2^{n/2}} + C(\varepsilon)r\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right) \\ &\le C(\varepsilon)\exp(C(\varepsilon)) \left(2^{1-n}|y_0|_\infty + 2^{-2^{n/2}} + r\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right).\end{aligned}$$

And therefore

$$\begin{aligned}&|\rho_{n,k+q}(z_q, y_q)|_\infty \overset{(2.15.1)}{\le} C(\varepsilon)2^{-3n/8}|z_q - y_q|_\infty + C(\varepsilon)\delta_n = C(\varepsilon)2^{-3n/8}|u_q|_\infty + C(\varepsilon)\delta_n \\ &\le C(\varepsilon)^2\exp(C(\varepsilon))2^{-3n/8} \left(2^{1-n}|y_0|_\infty + 2^{-2^{n/2}} + r\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right) + C(\varepsilon)\delta_n \\ &\le C(\varepsilon)^2\exp(C(\varepsilon))2^{-3n/8} \left(2^{1-n}|y_0|_\infty + 2^{-2^{n/2}} + \underbrace{r}_{\le 2^{3n/8}}\delta_n + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right) \\ &\le C_3 2^{-3n/8} \left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right).\end{aligned}$$

In conclusion we obtain

$$|\rho_{n,k+q}(z_q, y_q)|_\infty \le C_3 2^{-3n/8} \left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1} |\gamma_q|_\infty\right). \tag{2.15.7}$$

Observe that a similar calculation implies that we have the same estimate for z_q, y_q replaced by z_{q-1} and y_{q-1}, respectively. I.e.

$$|\rho_{n,k+q}(z_{q-1}, y_{q-1})|_\infty \le C_3 2^{-3n/8}\left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1}|\gamma_q|_\infty\right). \qquad (2.15.8)$$

In order to complete the proof consider the following equality.

$$\rho_{n,k+q}(y_{q-1}, y_q) = \rho_{n,k+q}(z_{q-1}, z_q) + \rho_{n,k+q}(y_{q-1}, z_{q-1}) + \rho_{n,k+q}(z_q, y_q).$$

Using the above identity for all $q \in \{1, ..., r\}$ and the triangle inequality, we obtain

$$\sum_{q=1}^{r}|\rho_{n,k+q}(y_{q-1}, y_q)|_\infty \le \sum_{q=1}^{r}|\rho_{n,k+q}(z_{q-1}, z_q)|_\infty + |\rho_{n,k+q}(y_{q-1}, z_{q-1})|_\infty + |\rho_{n,k+q}(z_q, y_q)|_\infty.$$

Applying the estimates (2.15.6), (2.15.7) and (2.15.8) yields

$$\begin{aligned}
&\le C_3 2^{-3n/4}\left(|z_0|_\infty + 2^{n/4}\delta_n\right) + 2C_3\sum_{q=1}^{r}2^{-3n/8}\left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q'=0}^{r-1}|\gamma_{q'}|_\infty\right)\\
&= C_3 2^{-3n/4}\left(|z_0|_\infty + 2^{n/4}\delta_n\right) + 2C_3 r 2^{-3n/8}\left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1}|\gamma_q|_\infty\right)\\
&\overset{r\le 2^{n/4}}{\le} C_3 2^{-3n/4}\left(|z_0|_\infty + 2^{n/4}\delta_n\right) + 2C_3 2^{-n/8}\left(2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{3n/8}\delta_n + \sum_{q=0}^{r-1}|\gamma_q|_\infty\right)\\
&\le 2C_3\left(2^{-3n/4}|z_0|_\infty + 2^{-n/2}\delta_n + 2^{-n}|y_0|_\infty + 2^{-2^{n/2}} + 2^{n/4}\delta_n + 2^{-n/8}\sum_{q=0}^{r-1}|\gamma_q|_\infty\right).
\end{aligned}$$

Since $|z_0|_\infty \le |y_0|_\infty + |z_0 - y_0|_\infty \overset{(2.15.3)}{\le} |y_0|_\infty + 2^{1-n}|y_0|_\infty + 2^{-2^{n/2}}$ we have

$$\begin{aligned}
&\le 8C_3\left(2^{-3n/4}|y_0|_\infty + \underbrace{2^{-3n/4}}_{\le 1}2^{-2^{n/2}} + 2^{-2^{n/2}} + \underbrace{2^{-n/2}}_{\le 1}\delta_n + \underbrace{2^{n/4}\delta_n}_{=2^{-2^{n/2}}} + 2^{-n/8}\sum_{q=0}^{r-1}|\gamma_q|_\infty\right)\\
&\le 32C_3\left(2^{-3n/4}|y_0|_\infty + 2^{-2^{n/2}} + 2^{-n/8}\sum_{q=0}^{r-1}|\gamma_q|_\infty\right).
\end{aligned}$$

Which concludes the proof.

□

Using the previous estimates we are finally ready to prove the crucial lemma from which we deduce that the only solution of equation (1.8) is the trivial solution $u = 0$.

Lemma 2.16

Let $\varepsilon > 0$. Let u be a solution of equation (1.8) where f is bounded by 1 and $W(\omega) := \omega$. Assume that $u(\omega) \in \Phi$ for all $\omega \in \Omega$. Then there exist $A_\varepsilon \subseteq \Omega$, $K > 0$ and $m_0 \in \mathbb{N}$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for all integers m with $m > m_0$, $j \in \{0, ..., 2^m - 1\}$ and $\beta \in [2^{-2^{3m/4}}, 2^{-2^{2m/3}}]$ satisfying $|u(j2^{-m})|_\infty \leq \beta$

$$|u((j+1)2^{-m})|_\infty \leq \beta \left(1 + K2^{-m}\log_2(1/\beta)\right), \qquad \forall \omega \in A_\varepsilon^c$$

holds.

Proof

Let $\varepsilon > 0$. Choose $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that the conclusion of Lemma 2.5 and Corollary 2.6 hold with constant $C_2 \geq 1$ and the conclusion of Lemma 2.15 holds with constant $C_3 \geq 1$ for all $\omega \in A_\varepsilon^c$. Let $\omega \in A_\varepsilon^c$. Fix m, j and β as in the statement and suppose $|u(j2^{-m})|_\infty \leq \beta$ as well as $m_0 \geq 2$. We set $N := 4\lfloor \log_2(1/\beta) \rfloor$. Observe that

$$2^{2m/3+2} - 4 \leq N \leq 2^{3m/4+2}. \tag{2.16.1}$$

Suppose $u(\omega) \in \Phi$ satisfies equation (1.8) as stated in the assertion. Define for $n \in \mathbb{N}$ and $x \in [0, 1]$

$$u_n(x) := \sum_{k=0}^{2^n-1} \mathbb{1}_{I_{n,k}}(x) u(k2^{-n}).$$

Note that u_n converges pointwise to u on $[0, 1[$ and $u_n \in \Phi^*$ since $u \in \Phi$. Let α be the smallest real number such that

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_\infty \leq \alpha 2^{-m}\left(\sqrt{n}2^{n/2} + N\right), \quad \forall n \in \{m, ..., N\}. \tag{2.16.2}$$

holds.
For $n \geq m$ define

$$\psi_n := \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty.$$

By splitting the sum in two sums, one where k is even and one where k is odd, we can estimate ψ_n by ψ_{n-1}. For this let $n \in \{m+1, ..., N\}$. We then have

$$\begin{aligned}
\psi_n &= \sum_{\substack{k=j2^{n-m} \\ 2|k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty + \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty \\
&\leq \sum_{\substack{k=j2^{n-m} \\ 2|k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty \\
&\quad + \sum_{\substack{k=j2^{n-m} \\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_\infty + |u((k-1)2^{-n})|_\infty + |u((k+1)2^{-n}) - u(k2^{-n})|_\infty.
\end{aligned}$$

Since $k-1$ is even whenever k is odd, rewriting the term $|u((k-1)2^{-n})|_\infty$ yields

$$\begin{aligned}
&= \sum_{\substack{k=j2^{n-m}\\ 2|k}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty + |u(k2^{-n})|_\infty \\
&\quad + \sum_{\substack{k=j2^{n-m}\\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_\infty + |u((k+1)2^{-n}) - u(k2^{-n})|_\infty \\
&= 2 \sum_{k=j2^{n-m-1}}^{(j+1)2^{n-m-1}-1} |u(k2^{-n+1})|_\infty \\
&\quad + \sum_{\substack{k=j2^{n-m}\\ 2\nmid k}}^{(j+1)2^{n-m}-1} |u(k2^{-n}) - u((k-1)2^{-n})|_\infty + |u((k+1)2^{-n}) - u(k2^{-n})|_\infty \\
&= 2 \sum_{k=j2^{n-1-m}}^{(j+1)2^{n-1-m}-1} |u(k2^{-(n-1)})|_\infty + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_\infty.
\end{aligned}$$

And by the choice of n, we have

$$\overset{(2.16.2)}{\leq} 2\psi_{n-1} + \alpha 2^{-m}\left(\sqrt{n}2^{n/2} + N\right).$$

By induction we deduce

$$\begin{aligned}
\psi_n &\leq 2^{n-m}\psi_m + \sum_{\ell=m+1}^{n} \alpha 2^{n-\ell-m}\left(\sqrt{\ell}2^{\ell/2} + N\right) \\
&= 2^{n-m}|u(j2^{-m})|_\infty + \alpha 2^{n-m} \sum_{\ell=m+1}^{n} 2^{-\ell}\left(\sqrt{\ell}2^{\ell/2} + N\right), \qquad \forall n \in \{m+1, ..., N\}.
\end{aligned}$$

We use that $|u(j2^{-m})|_\infty \leq \beta$ and $\sqrt{\ell}2^{\ell/2} \leq 2 \cdot 2^{2\ell/3}$ to get

$$\begin{aligned}
&\leq 2^{n-m}\left[\beta + \alpha \sum_{\ell=m+1}^{n} 2^{-\ell}\sqrt{\ell}2^{\ell/2} + \alpha N \sum_{\ell=m+1}^{n} 2^{-\ell}\right] \leq 2^{n-m}\left[\beta + 2\alpha \sum_{\ell=m+1}^{n} 2^{-\ell}2^{2\ell/3} + \alpha N 2^{-m}\right] \\
&= 2^{n-m}\left[\beta + 2\alpha \sum_{\ell=1}^{n-m} 2^{-(\ell+m)}2^{2(\ell+m)/3} + \alpha 2^{-m}N\right] \\
&= 2^{n-m}\left[\beta + 2\alpha 2^{-m}2^{2m/3} \sum_{\ell=1}^{n-m} 2^{-\ell/3} + \alpha 2^{-m}N\right] \\
&\overset{(2.16.1)}{\leq} 2^{n-m}\left[\beta + \alpha 2^{-m}N \sum_{\ell=1}^{n-m} 2^{-\ell/3} + \alpha 2^{-m}N\right] \leq 2^{n-m}\left[\beta + \alpha 2^{-m}N\frac{1}{1-2^{-1/3}} + \alpha 2^{-m}N\right].
\end{aligned}$$

By setting $C_1 := \frac{1}{1-2^{-1/3}} + 1$ we have

$$= 2^{n-m}\left[\beta + (C_1-1)\alpha 2^{-m}N + \alpha 2^{-m}N\right] \leq C_1 2^{n-m}\left(\beta + \alpha 2^{-m}N\right).$$

In conclusion we obtain

$$\psi_n \le C_1 2^{n-m}\left(\beta + \alpha 2^{-m} N\right), \qquad \forall n \in \{m+1, ..., N\}. \tag{2.16.3}$$

Since u solves equation (1.8) and by Lemma 2.10 we have

$$u((k+1)2^{-n}) - u(k2^{-n}) \overset{(1.8)}{=} \int\limits_{I_{n,k}} f(t, W(t)+u(t)) - f(t, W(t))\, \mathrm{d}t$$

$$\overset{2.10}{=} \lim_{\ell\to\infty} \int\limits_{I_{n,k}} f(t, W(t)+u_\ell(t)) - f(t, W(t))\, \mathrm{d}t$$

$$= \int\limits_{I_{n,k}} f(t, W(t)+u_n(t)) - f(t, W(t))\, \mathrm{d}t + \sum_{\ell=n}^{\infty} \int\limits_{I_{n,k}} f(t, W(t)+u_{\ell+1}(t)) - f(t, W(t)+u_\ell(t))\, \mathrm{d}t.$$

Since u_n is constant on $I_{n,k}$ we can rewrite this as

$$= \sigma_{n,k}(u(k2^{-n})) + \sum_{\ell=n}^{\infty} \int\limits_{I_{n,k}} f(t, W(t)+u_{\ell+1}(t)) - f(t, W(t)+u_\ell(t))\, \mathrm{d}t$$

$$= \sigma_{n,k}(u(k2^{-n})) + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \int\limits_{2r2^{-\ell-1}}^{(2r+2)2^{-\ell-1}} f(t, W(t)+u_{\ell+1}(t)) - f(t, W(t)+u_\ell(t))\, \mathrm{d}t$$

$$= \sigma_{n,k}(u(k2^{-n})) + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \int\limits_{2r2^{-\ell-1}}^{(2r+1)2^{-\ell-1}} \underbrace{f(t, W(t)+u(2r2^{-\ell-1})) - f(t, W(t)+u(r2^{-\ell}))}_{=0}\, \mathrm{d}t$$

$$+ \int\limits_{(2r+1)2^{-\ell-1}}^{(2r+2)2^{-\ell-1}} f(t, W(t)+u((2r+1)2^{-\ell-1})) - f(t, W(t)+u(r2^{-\ell}))\, \mathrm{d}t$$

$$= \sigma_{n,k}(u(k2^{-n})) + \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \rho_{\ell+1,2r+1}\left(u\left((2r+1)2^{-\ell-1}\right), u\left(r2^{-\ell}\right)\right).$$

This leads us to

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n}) - \sigma_{n,k}(u(k2^{-n}))|_\infty$$

$$\le \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} \sum_{\ell=n}^{\infty} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \left|\rho_{\ell+1,2r+1}\left(u\left((2r+1)2^{-\ell-1}\right), u\left(r2^{-\ell}\right)\right)\right|_\infty.$$

By Fubini's Theorem we have

$$= \sum_{\ell=n}^{\infty} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} \sum_{r=k2^{\ell-n}}^{(k+1)2^{\ell-n}-1} \left|\rho_{\ell+1,2r+1}\left(u\left((2r+1)2^{-\ell-1}\right), u\left(r2^{-\ell}\right)\right)\right|_\infty.$$

We set

$$\Omega_\ell := \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} \left|\rho_{\ell+1,2r+1}\left(u\left((2r+1)2^{-\ell-1}\right), u\left(r2^{-\ell}\right)\right)\right|_\infty$$

and have

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} \left|u((k+1)2^{-n}) - u(k2^{-n}) - \sigma_{n,k}(u(k2^{-n}))\right|_\infty \leq \sum_{\ell=n}^{\infty} \Omega_\ell. \tag{2.16.4}$$

From the reversed triangle inequality we deduce

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_\infty \leq \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\sigma_{n,k}(u(k2^{-n}))|_\infty + \sum_{\ell=n}^{\infty} \Omega_\ell. \tag{2.16.5}$$

The idea of the proof is the following: We will obtain estimates for the two sums on the right-hand side of the above inequality. For the first sum we simply use Lemma 2.5 to obtain the estimate (2.16.6). We will split the second sum in the cases $n \leq \ell < N$ and $N \leq \ell < \infty$. In the first case we use Corollary 2.6, which will lead us to inequality (2.16.7). In the second case we have to do a more direct computation which heavily relies on the fact that u is Lipschitz continuous (Inequality (2.16.8)).
Unfortunately, the final bound for the second sum is not strong enough to prove the assertion. For n in the range $N^{1/6} < n < N$ we will use Lemma 2.15 to get the more sophisticated estimate (2.16.13) for the second sum. Our old estimate estimate will be recycled (see (2.16.9)) to estimate the error term $\gamma_{n,k}$ of the new estimate.
Combining the new with the old estimate will result the final bound (2.16.14).
Using the knowledge of the already established estimate (2.16.2) for the left-hand side and the minimality of α we will finally complete the proof.

We will now estimate the two sums on the right-hand side starting with the $\sigma_{n,k}$ sum. We apply Lemma 2.5 to obtain

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\sigma_{n,k}(u(k2^{-n}))|_\infty \overset{2.5}{\leq} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} C_2\sqrt{n}2^{-n/2}\left(\left|u(k2^{-n})\right|_\infty + 2^{-2^n}\right)$$

and since $n > m$ and $N \leq 2^m$ we get

$$\begin{aligned}
&\leq \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} C_2\sqrt{n}2^{-n/2}\left(|u(k2^{-n})|_\infty + 2^{-N}\right)\\
&= C_2\sqrt{n}2^{-n/2}\left(2^{n-m}2^{-N} + \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty\right).
\end{aligned}$$

Again, using that $n \in \{m+1, ..., N\}$ implies

$$\begin{aligned}
&= C_2\sqrt{n}2^{-n/2}\left(2^{n-m}2^{-N} + \psi_n\right) \overset{(2.16.3)}{\leq} C_2\sqrt{n}2^{-n/2}\left(2^{n-m}2^{-N} + C_1 2^{n-m}\left(\beta + \alpha 2^{-m}N\right)\right)\\
&= C_2\sqrt{n}2^{n/2-m}\left(2^{-N} + C_1\beta + C_1\alpha 2^{-m}N\right) \overset{2^{-N}\leq\beta}{\leq} 2C_1C_2\sqrt{n}2^{n/2-m}\left(\beta + \alpha 2^{-m}N\right)
\end{aligned}$$

and hence for the first sum we obtain

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\sigma_{n,k}(u(k2^{-n}))|_\infty \le 2C_1C_2\sqrt{n}2^{n/2-m}\left(\beta+\alpha 2^{-m}N\right), \quad \forall n\in\{m+1,...,N\}. \tag{2.16.6}$$

Next we bound Ω_ℓ. By Corollary 2.6 we have

$$|\rho_{\ell,k}(x,y)|_\infty \le C_2 2^{-\ell/2}\sqrt{N}(2^{-N}+|x-y|_\infty), \qquad \forall \ell\le N\le 2^\ell.$$

So, for $m\le\ell$ and $\ell+1\le N\le 2^{\ell+1}$ this leads to the following estimate

$$\begin{aligned}
\Omega_\ell &= \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} \left|\rho_{\ell+1,2r+1}\left(u((2r+1)2^{-\ell-1}),u(r2^{-\ell})\right)\right|_\infty \\
&\le \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}+|u((2r+1)2^{-\ell-1})-u(r2^{-\ell})|_\infty\right) \\
&\le C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m}+\sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1}|u((2r+1)2^{-\ell-1})-u(r2^{-\ell})|_\infty\right) \\
&\le C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m}+\sum_{r=j2^{\ell+1-m}}^{(j+1)2^{\ell+1-m}-1}|u((r+1)2^{-\ell-1})-u(r2^{-\ell-1})|_\infty\right) \\
&\overset{(2.16.2)}{\le} C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m}+\alpha 2^{-m}\left(\sqrt{\ell+1}2^{(\ell+1)/2}+N\right)\right) \\
&\le C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m}+\alpha 2^{-m}\left(2\sqrt{\ell}2^{\ell/2}+N\right)\right).
\end{aligned}$$

Hence

$$\Omega_\ell \le C_2 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m}+\alpha 2^{-m}\left(2\sqrt{\ell}2^{\ell/2}+N\right)\right), \quad \forall m\le\ell,\ \ell+1\le N\le 2^{\ell+1}. \tag{2.16.7}$$

This implies that

$$\begin{aligned}
\sum_{\ell=m}^{N-1}\Omega_\ell &\le C_2\sqrt{N}\sum_{\ell=m}^{N-1}2^{-\ell/2}\left(2^{-N}2^{\ell-m}+\alpha 2^{-m}\left(2\sqrt{\ell}2^{\ell/2}+N\right)\right) \\
&= C_2\sqrt{N}2^{-m}\left[2^{-N}\sum_{\ell=m}^{N-1}2^{\ell/2}+2\alpha\sum_{\ell=m}^{N-1}\sqrt{\ell}+\alpha N\sum_{\ell=m}^{N-1}2^{-\ell/2}\right] \\
&\le C_2\sqrt{N}2^{-m}\left[2^{-N}4\cdot 2^{N/2}+2\alpha N\sqrt{N}+4\alpha N\right].
\end{aligned}$$

Estimating N by $N\sqrt{N}$ yields

$$\le 6C_2\sqrt{N}2^{-m}\left[2^{-N/2}+\alpha N\sqrt{N}\right]\le 6C_2 2^{-m}\left[\sqrt{N}2^{-N/2}+\alpha N^2\right].$$

Now consider the case $\ell \geq N$

$$\sum_{\ell=N}^{\infty} \Omega_\ell = \sum_{\ell=N}^{\infty} \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} \left|\rho_{\ell+1,2r+1}\left(u((2r+1)2^{-\ell-1}), u(r2^{-\ell})\right)\right|_\infty$$
$$\overset{2.6}{\leq} \sum_{\ell=N}^{\infty} \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} C_2 2^{-\ell/2}\sqrt{\ell+1}\left(2^{-\ell} + |u((2r+1)2^{-\ell-1}) - u(r2^{-\ell})|_\infty\right).$$

By the Lipschitz continuity of u we get

$$\overset{u\in\Phi}{\leq} \sum_{\ell=N}^{\infty} \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} 2C_2 2^{-\ell/2}\sqrt{\ell}\left(2^{-\ell} + |(2r+1)2^{-\ell-1} - 2r2^{-\ell-1}|\right)$$
$$= \sum_{\ell=N}^{\infty} \sum_{r=j2^{\ell-m}}^{(j+1)2^{\ell-m}-1} 2C_2 2^{-\ell/2}\sqrt{\ell}(2^{-\ell} + 2^{-\ell-1})$$
$$= \sum_{\ell=N}^{\infty} 3C_2 2^{\ell-m} 2^{-\ell/2}\sqrt{\ell}2^{-\ell} = 3C_2 2^{-m}\sum_{\ell=N}^{\infty}\sqrt{\ell}2^{-\ell/2}$$
$$= 3C_2 2^{-m}\sum_{\ell=0}^{\infty}\sqrt{\ell+N}2^{-(\ell+N)/2} = 3C_2 2^{-m}2^{-N/2}\sum_{\ell=0}^{\infty}\sqrt{\ell+N}2^{-\ell/2}$$
$$\leq 3C_2 2^{-m}2^{-N/2}\sum_{\ell=0}^{\infty}(\sqrt{\ell}+\sqrt{N})2^{-\ell/2} = 3C_2 2^{-m}2^{-N/2}\left[\sum_{\ell=1}^{\infty}\sqrt{\ell}2^{-\ell/2} + \sqrt{N}\sum_{\ell=0}^{\infty}2^{-\ell/2}\right].$$

Using $\sqrt{\ell}2^{-\ell/2} \leq 2 \cdot 2^{-\ell/3}$ results in

$$\leq 3C_2 2^{-m}2^{-N/2}\left[2\sum_{\ell=1}^{\infty}2^{-\ell/3} + 4\sqrt{N}\right]$$
$$\leq 3C_2 2^{-m}2^{-N/2}\left[8 + 4\sqrt{N}\right] \leq 24C_2\sqrt{N}2^{-m-N/2}.$$

And hence

$$\sum_{\ell=N}^{\infty}\Omega_\ell \leq 24C_2\sqrt{N}2^{-m-N/2}. \tag{2.16.8}$$

Combing these two estimates results in

$$\sum_{\ell=m}^{\infty}\Omega_\ell \leq 6C_2 2^{-m}\left[\sqrt{N}2^{-N/2} + \alpha N^2\right] + 24C_2\sqrt{N}2^{-m-N/2}$$
$$\leq 24C_2 2^{-m}\left[\sqrt{N}2^{-N/2} + \alpha N^2 + \sqrt{N}2^{-N/2}\right] = 48C_2 2^{-m}\left[\sqrt{N}2^{-N/2} + \alpha N^2\right].$$

Our next aim is to improve the estimate for large n using Lemma 2.15. Let $N^{1/6} \leq n \leq N$. We define

$$\gamma_{n,k} := u((k+1)2^{-n}) - u(k2^{-n}) - \sigma_{n,k}(u(k2^{-n})), \qquad \forall k \in \{0, ..., 2^n - 1\}.$$

Therefore, we deduce

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\gamma_{n,k}|_\infty \overset{(2.16.4)}{\leq} \sum_{\ell=n}^{\infty} \Omega_\ell \overset{n\geq m}{\leq} 48C_2 2^{-m}(\sqrt{N}2^{-N/2} + \alpha N^2). \tag{2.16.9}$$

We also set

$$\Lambda_n := \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-2} |\rho_{n,k+1}(u((k+1)2^{-n}), u(k2^{-n}))|_\infty.$$

Comparing Ω with Λ results in

$$\begin{aligned}\Omega_n &= \sum_{r=j2^{n-m}}^{(j+1)2^{n-m}-1} \left|\rho_{n+1,2r+1}\left(u\left((2r+1)2^{-n-1}\right), u\left(2r2^{-n-1}\right)\right)\right|_\infty \\ &\leq \sum_{k=j2^{n+1-m}}^{(j+1)2^{n+1-m}-2} \left|\rho_{n+1,k+1}\left(u\left((k+1)2^{-n-1}\right), u\left(k2^{-n-1}\right)\right)\right|_\infty = \Lambda_{n+1}.\end{aligned}$$

We set $r := \lfloor 2^{n/4} \rfloor$. In order to estimate Λ_n we will use Lemma 2.15. To this end we split the sum into s r-sized pieces. For reader's convenience we define

$$\hat{u}(x) := \mathbb{1}_{[0,(j+1)2^{n-m}-1]}(x2^n)u(x), \qquad \forall x \in \mathbb{R}.$$

$\hat{u}$ is the trivial extension of u which vanishes outside of $[0, (j+1)2^{-m} - 2^{-n}]$. Choose $i \in \{0, ..., r-1\}$ such that

$$\sum_{t=0}^{\lfloor r^{-1}2^{n-m} \rfloor} |\hat{u}((j2^{n-m} + i + tr)2^{-n})|_\infty \leq \frac{1}{r} \sum_{q=0}^{r-1} \sum_{t=0}^{\lfloor r^{-1}2^{n-m} \rfloor} |\hat{u}((j2^{n-m} + q + tr)2^{-n})|_\infty$$

holds. Since we calculate the mean of $\sum_{t=0}^{\lfloor r^{-1}2^{n-m} \rfloor} |\hat{u}((j2^{n-m}+q+tr)2^{-n})|$ on the right-hand side, it is clear that such an i always exists. Set $s := \lfloor r^{-1}(2^{n-m} - i) \rfloor$ and note that $s \leq \lfloor r^{-1}2^{n-m} \rfloor$. Using that we have

$$\sum_{t=0}^{s} |\hat{u}((j2^{n-m} + i + tr)2^{-n})|_\infty \overset{s \leq \lfloor r^{-1}2^{n-m} \rfloor}{\leq} \frac{1}{r} \sum_{q=0}^{r-1} \sum_{t=0}^{\lfloor r^{-1}2^{n-m} \rfloor} |\hat{u}((j2^{n-m} + q + tr)2^{-n})|_\infty.$$

Since $\hat{u}$ vanishes if $q + tr \geq 2^{n-m}$ this simplifies to

$$= \frac{1}{r} \sum_{k=0}^{2^{n-m}-1} |\hat{u}((j2^{n-m} + k)2^{-n})|_\infty = \frac{1}{r} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u(k2^{-n})|_\infty = r^{-1}\psi_n.$$

So, we obtain

$$\sum_{t=0}^{s} |\hat{u}((j2^{n-m}+i+tr)2^{-n})|_\infty \leq r^{-1}\psi_n. \tag{2.16.10}$$

For $t \in \{0, ..., s\}$ we define $k_t := j2^{n-m}+i+tr$ and $y_q^{(t)} := \hat{u}((k_t+q)2^{-n})$ for $q \in \{0, ..., r\}$. Observe that for $t \leq s-1$ we have

$$\begin{aligned} k_t &= j2^{n-m}+i+tr \leq (2^m-1)2^{n-m}+i+(s-1)r = 2^n - 2^{n-m}+i+r\lfloor r^{-1}(2^{n-m}-i)\rfloor - r \\ &\leq 2^n - 2^{n-m}+i+2^{n-m}-i-r = 2^n - r. \end{aligned}$$

Therefore, we are able to apply Lemma 2.15 for every $t \in \{0, ..., s-1\}$

$$\begin{aligned} \sum_{\substack{q=1 \\ k_t+q<2^n}}^{r} \left|\rho_{n,k_t+q}(y_{q-1}^{(t)}, y_q^{(t)})\right|_\infty \overset{2.15}{\leq} C_3 \Bigg[& 2^{-3n/4}|y_0^{(t)}|_\infty \\ & + 2^{-n/8}\sum_{q=0}^{r-1}|\gamma_{n,j2^{n-m}+i+tr+q}|_\infty + 2^{-2^{n/2}} \Bigg]. \end{aligned} \tag{2.16.11}$$

In the case $t = s$ we also apply Lemma 2.15

$$\begin{aligned} \sum_{q=1}^{2^{n-m}-i-sr-1} \left|\rho_{n,k_s+q}(y_{q-1}^{(s)}, y_q^{(s)})\right|_\infty \overset{2.15}{\leq} C_3 \Bigg[& 2^{-3n/4}|y_0^{(s)}|_\infty \\ & + 2^{-n/8} \sum_{q=0}^{2^{n-m}-i-sr-2} |\gamma_{n,j2^{n-m}+i+sr+q}|_\infty + 2^{-2^{n/2}} \Bigg]. \end{aligned} \tag{2.16.12}$$

Note that the sum on the left-hand side has less than r summands and $\gamma_{n,k_t+q} = \gamma_{n,j2^{n-m}+i+tr+q}$. Summing over t results in

$$\begin{aligned} &\sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-2} |\rho_{n,k+1}(u(k2^{-n}), u((k+1)2^{-n}))|_\infty \\ &= \sum_{t=0}^{s-1} \sum_{\substack{q=0 \\ k_t+q+1<2^n}}^{r-1} |\rho_{n,k_t+q+1}(\underbrace{\hat{u}((j2^{n-m}+i+tr+q)2^{-n})}_{=y_q^{(t)}}, \underbrace{\hat{u}((j2^{n-m}+i+tr+q+1)2^{-n})}_{=y_{q+1}^{(t)}})|_\infty \\ &\quad + \sum_{q=0}^{2^{n-m}-i-sr-2} |\rho_{n,k_s+q+1}(\underbrace{\hat{u}((j2^{n-m}+i+sr+q)2^{-n})}_{=y_q^{(s)}}, \underbrace{\hat{u}((j2^{n-m}+i+sr+q+1)2^{-n})}_{=y_q^{(s)}})|_\infty \\ &= \sum_{t=0}^{s-1} \sum_{\substack{q=1 \\ k_t+q<2^n}}^{r} |\rho_{n,k_t+q}(y_{q-1}^{(t)}, y_q^{(t)})|_\infty + \sum_{q=1}^{2^{n-m}-i-sr-1} |\rho_{n,k_s+q}(y_{q-1}^{(s)}, y_q^{(s)})|_\infty \end{aligned}$$

Applying inequality (2.16.11) and (2.16.12) yields

$$\begin{aligned} &\leq C_3 \sum_{t=0}^{s} \left[2^{-3n/4}|\hat{u}((j2^{n-m}+i+tr)2^{-n})|_\infty + 2^{-n/8} \sum_{\substack{q=0 \\ i+tr+q\leq 2^{n-m}-1}}^{r-1} |\gamma_{n,j2^{n-m}+i+tr+q}|_\infty + 2^{-2^{n/2}} \right] \\ &= C_3 \left[2^{-3n/4} \sum_{t=0}^{s} |\hat{u}((j2^{n-m}+i+tr)2^{-n})|_\infty + 2^{-n/8} \sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-1} |\gamma_{n,k}|_\infty + (s+1)2^{-2^{n/2}} \right] \end{aligned}$$

$$\stackrel{(2.16.10)}{\leq} C_3 \left[2^{-3n/4} r^{-1} \psi_n + 2^{-n/8} \sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-1} |\gamma_{n,k}|_\infty + (s+1) 2^{-2^{n/2}} \right].$$

Since $i \leq r-1$ we can use Lemma 2.15 directly to establish

$$\sum_{k=j2^{n-m}}^{j2^{n-m}+i-1} |\rho_{n,k+1}(u(k2^{-n}), u((k+1)2^{-n}))|_\infty \leq C_3 \left[2^{-3n/4} |u(j2^{-m})|_\infty \right.$$
$$\left. + 2^{-n/8} \sum_{k=j2^{n-m}}^{j2^{n-m}+i-1} |\gamma_{n,k}|_\infty + 2^{-2^{n/2}} \right].$$

Combing the last two estimates yields our desired estimate for Λ_n

$$\begin{aligned}
\Lambda_n =& \sum_{k=j2^{n-m}+i}^{(j+1)2^{n-m}-2} |\rho_{n,k+1}(u(k2^{-n}), u((k+1)2^{-n}))|_\infty + \sum_{k=j2^{n-m}}^{j2^{n-m}+i-1} |\rho_{n,k+1}(u(k2^{-n}), u((k+1)2^{-n}))|_\infty \\
\leq & \; C_3 \left[2^{-3n/4} r^{-1} \psi_n + 2^{-n/8} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\gamma_{n,k}|_\infty + 2^{-3n/4} |u(j2^{-m})|_\infty + (s+2) 2^{-2^{n/2}} \right].
\end{aligned}$$

By (2.16.3) and the assumption $|u(j2^{-m})|_\infty \leq \beta$ we have

$$\begin{aligned}
&\stackrel{(2.16.3)}{\leq} C_3 \left[2^{-3n/4} C_1 r^{-1} 2^{n-m} (\beta + \alpha 2^{-m} N) + 2^{-n/8} \sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |\gamma_{n,k}|_\infty + 2^{-3n/4} \beta + (s+2) 2^{-2^{n/2}} \right] \\
&\stackrel{(2.16.9)}{\leq} C_3 \left[2C_1 2^{-m} (\beta + \alpha 2^{-m} N) + 48 C_2 2^{-m-n/8} (\sqrt{N} 2^{-N/2} + \alpha N^2) + 2^{-3n/4} \beta + 2^{n-m} 2^{-2^{n/2}} \right] \\
&\leq \; 48 C_1 C_2 C_3 \left[2^{-m} (\beta + \alpha 2^{-m} N) + 2^{-m-n/8} (\sqrt{N} 2^{-N/2} + \alpha N^2) + 2^{-m} 2^{n-2^{n/2}} \right].
\end{aligned}$$

The following calculations show that the first term dominates the last expression.

$$\begin{aligned}
&2^{-n/8} \sqrt{N} 2^{-N/2} \leq \sqrt{N} 2^{-N/2} \leq 2^3 \sqrt{\log_2(1/\beta)} 2^{-2\log_2(1/\beta)} \leq 2^3 \beta^{-1/2} \beta^2 \leq 2^3 \beta, \\
&2^{-n/8} N \leq 2^{-N^{1/6}/8} N \stackrel{(2.16.1)}{\leq} 2^{-2^{m/9}/8} N \stackrel{(2.16.1)}{\leq} 2^{-2^{m/9-3}} 2^{3m/4+2} \leq 2^{98} 2^{-m}, \\
&2^{n-2^{n/2}} \leq 2^{-2^{n/3}} \leq 2^{-2^{N^{1/6}/3}} \leq 2^{2^{41}} 2^{-N} \leq 2^{2^{41}} 2^4 2^{-4\log_2(1/\beta)} = 2^{2^{41}+4} \beta^4 \leq 2^{2^{41}+4} \beta.
\end{aligned}$$

Therefore, we get $\Lambda_n \leq 3 \cdot 2^{2^{41}+8} C_1 C_2 C_3 2^{-m} (\beta + \alpha 2^{-m} N)$ and since the same bound holds for Λ_{n+1} we have

$$\Omega_n \leq \Lambda_{n+1} \leq 2^{2^{41}+8} C_1 C_2 C_3 2^{-m} (\beta + \alpha 2^{-m} N), \qquad \forall N^{1/6} \leq n \leq N-1.$$

We deduce

$$\sum_{\ell=N^{1/6}+1}^{N-1} \Omega_\ell \leq C_4 2^{-m} N (\beta + \alpha 2^{-m} N) \tag{2.16.13}$$

where $C_4 := 2^{2^{41}+8} C_1 C_2 C_3$.

Using the old estimate we get

$$\begin{aligned}
\sum_{\ell=m}^{N^{1/6}} \Omega_\ell &\overset{(2.16.7)}{\leq} C_2 \sum_{\ell=m}^{N^{1/6}} 2^{-\ell/2}\sqrt{N}\left(2^{-N}2^{\ell-m} + \alpha 2^{-m}\left(2\sqrt{\ell}2^{\ell/2} + N\right)\right) \\
&= C_2\sqrt{N}\left[2^{-N}2^{-m}\sum_{\ell=m}^{N^{1/6}} 2^{\ell/2} + 2\alpha 2^{-m}\sum_{\ell=m}^{N^{1/6}}\sqrt{\ell} + \alpha 2^{-m}N\sum_{\ell=m}^{N^{1/6}} 2^{-\ell/2}\right] \\
&\leq C_2\sqrt{N}2^{-m}\left[2^{-N}4\cdot 2^{N^{1/6}/2} + 2\alpha N^{1/6}N^{1/12} + \alpha N 4\cdot 2^{-m/2}\right] \\
&\leq 4C_2\sqrt{N}2^{-m}\left[2^{-N}2^{N^{1/6}/2} + \alpha N^{1/4} + \alpha 2^{-m/2}N\right].
\end{aligned}$$

Combining these two estimates with our old estimate yields

$$\begin{aligned}
&\sum_{\ell=m}^{\infty}\Omega_\ell = \sum_{\ell=m}^{N^{1/6}}\Omega_\ell + \sum_{\ell=N^{1/6}+1}^{N-1}\Omega_\ell + \sum_{\ell=N}^{\infty}\Omega_\ell \\
&\overset{(2.16.13),(2.16.8)}{\leq} 4C_2\sqrt{N}2^{-m}\left[2^{-N}2^{N^{1/6}/2} + \alpha N^{1/4} + \alpha 2^{-m/2}N\right] \\
&\qquad + C_4 2^{-m}N(\beta + \alpha 2^{-m}N) + 24C_2\sqrt{N}2^{-m-N/2} \\
&\leq 24C_2C_4 2^{-m}\Big[N(\beta + \alpha 2^{-m}N) + \sqrt{N}(\alpha N^{1/4} + \alpha 2^{-m/2}N) \\
&\qquad + \sqrt{N}2^{-N}2^{N^{1/6}/2} + \sqrt{N}2^{-N/2}\Big] \\
&\leq 24C_2C_4 2^{-m}\Big[N(\beta + \alpha 2^{-m}N) + \alpha N(N^{-1/4} + 2^{-m/2}\sqrt{N}) \\
&\qquad + N2^{-N}2^{N^{1/6}/2} + N2^{-N/2}\Big].
\end{aligned}$$

The last two summands can be estimated by the first one as the following calculations show

$$\begin{aligned}
&2^{-N}2^{N^{1/6}/2} \leq 2^4\cdot 2^{-4\log_2(1/\beta)}2^{4^{1/6}\log_2(1/\beta)/2} = 2^4\cdot 2^{\frac{4^{1/6}-8}{2}\log_2(1/\beta)} \leq 2^4\cdot 2^{-3\log_2(1/\beta)} = 2^4\beta^3 \leq \beta, \\
&2^{-N/2} \leq 4\cdot 2^{-2\log_2(1/\beta)} = 4\beta^2 \leq \beta.
\end{aligned}$$

So, the first two terms dominate. We therefore obtain

$$\sum_{\ell=m}^{\infty}\Omega_\ell \leq 24C_2C_4 2^{-m}\left[N(\beta + \alpha 2^{-m}N) + \alpha N(N^{-1/4} + 2^{-m/2}\sqrt{N})\right]. \tag{2.16.14}$$

To conclude the proof we use both estimates to bound the term

$$\sum_{k=j2^{n-m}}^{(j+1)2^{n-m}-1} |u((k+1)2^{-n}) - u(k2^{-n})|_\infty.$$

With the help of (2.16.5), (2.16.6) and (2.16.14) we estimate the sum by

$$\begin{aligned}
&\leq 2C_1C_2\sqrt{n}2^{n/2-m}\left(\beta+\alpha 2^{-m}N\right)\\
&\quad + 24C_2C_4 2^{-m}\left[N(\beta+\alpha 2^{-m}N)+\alpha N(N^{-1/4}+2^{-m/2}\sqrt{N})\right]\\
&\leq C_5 2^{-m}\left[\sqrt{n}2^{n/2}\left(\beta+\alpha 2^{-m}N\right)+N(\beta+\alpha 2^{-m}N)+\alpha N(N^{-1/4}+2^{-m/2}\sqrt{N})\right]\\
&\leq C_5 2^{-m}\left[\sqrt{n}2^{n/2}+N\right]\cdot\left[\beta+\alpha(2^{-m}N+N^{-1/4}+2^{-m/2}\sqrt{N})\right]
\end{aligned}$$

with $C_5 := 24C_1C_2C_4$. Hence, by the minimality of α and (2.16.2) we have

$$\alpha 2^{-m}\left[\sqrt{n}2^{n/2}+N\right] \leq C_5 2^{-m}\left[\sqrt{n}2^{n/2}+N\right]\cdot\left[\beta+\alpha(2^{-m}N+N^{-1/4}+2^{-m/2}\sqrt{N})\right]$$

for all $n \in \{m+1, ..., N\}$. This implies that

$$\alpha \leq C_5\left[\beta+\alpha(2^{-m}N+N^{-1/4}+2^{-m/2}\sqrt{N})\right].$$

Since $2^{-m}N \leq N^{-1/4} \leq 2^{-m/2}\sqrt{N}$ and

$$\lim_{m\to\infty} 2^{-m/2}\sqrt{N} \overset{(2.16.1)}{\leq} \lim_{m\to\infty} 2^{-m/2}2^{3m/8+1} = \lim_{m\to\infty} 2^{-m/8+1} = 0$$

we can choose m_0 large enough such that

$$C_5\left(2^{-m}N+N^{-1/4}+2^{-m/2}\sqrt{N}\right) \leq \frac{1}{2}$$

holds for all $m \geq m_0$. It now follows

$$\begin{aligned}
&\alpha \leq C_5\beta+\alpha C_5(2^{-m}N+N^{-1/4}+2^{-m/2}\sqrt{N}) \leq C_5\beta+\frac{\alpha}{2}\\
\Rightarrow\;&\alpha \leq 2C_5\beta.
\end{aligned}$$

Setting $n = m$ in (2.16.2) yields

$$\begin{aligned}
|u((j+1)2^{-m})|_\infty - |u(j2^{-m})|_\infty &\leq |u((j+1)2^{-m})-u(j2^{-m})|_\infty \overset{(2.16.2)}{\leq} \alpha 2^{-m}\left(\sqrt{m}2^{m/2}+N\right)\\
\Rightarrow\qquad |u((j+1)2^{-m})|_\infty &\leq |u(j2^{-m})|_\infty+\alpha 2^{-m}\left(\sqrt{m}2^{m/2}+N\right)\\
&\leq \beta+2C_5\beta 2^{-m}\left(\sqrt{m}2^{m/2}+N\right)\\
&= \beta\left(1+2C_5\sqrt{m}2^{-m/2}+2C_5 2^{-m}N\right)\\
&\overset{\sqrt{m}\leq 2^{-m/2}N}{\leq} \beta\left(1+4C_5 2^{-m}N\right)\\
&= \beta\left(1+16C_5 2^{-m}\lfloor\log_2(1/\beta)\rfloor\right)\\
&\leq \beta\left(1+K2^{-m}\log_2(1/\beta)\right),
\end{aligned}$$

where the constant K can be expressed as

$$K = 16C_5 = 3\cdot 2^7 C_1C_2C_4 = 3\cdot 2^{2^{41}+15}C_1^2C_2^2C_3 \leq 2^{2^{42}}C_2^2C_3$$

which completes the proof.

□

With this lemma we can now find the zero set N, which is required in Lemma 1.8 to prove the main result Theorem 1.5.

Theorem 2.17

Let f be a $\mathbb{R}^d$-valued Borel function, which is bounded by 1/2 everywhere. There exists a set $N \subseteq \Omega$ with $\mathbb{P}[N] = 0$ such that

$$u(t) = \int_0^t f(s, \omega(s) + u(s)) - f(s, \omega(s)) \, \mathrm{d}s \;\Rightarrow\; u = 0, \qquad \forall \omega \in N^c.$$

Proof

Step 1:

Let u be a solution to the above equation. For $t_1, t_2 \in \mathbb{R}$ we have

$$|u(t_2) - u(t_1)| = \left| \int_{t_1}^{t_2} f(s, \omega(s) + u(s)) - f(s, \omega(s)) \, \mathrm{d}s \right| \leq |t_2 - t_1| \cdot 2\|f\| = |t_2 - t_1|.$$

Therefore $u \in \Phi$. Let $\varepsilon > 0$. Applying Lemma 2.16 gives us a $K > 0$ and $m_0 \in \mathbb{N}$. For $m \in \mathbb{N}$ and $j \in \{0, ..., 2^m - 1\}$ we define

$$\begin{aligned}
\beta_0 \; &:= 2^{-2^{3m/4}}, \\
\beta_{j+1} &:= \beta_j (1 + K 2^{-m} \log_2(1/\beta_j)), \\
\gamma_j \; &:= \log_2(1/\beta_j).
\end{aligned}$$

Let m be sufficiently large i.e. $\ln(2)^{-1} K 2^{-m} \leq 1$. Note that $\gamma_0 \geq 0$. Assume $\gamma_j \geq 0$ for some $j \in \{0, .., 2^m - 1\}$. We then have

$$\gamma_{j+1} = -\log_2(\beta_{j+1}) = \gamma_j - \log_2(1 + K 2^{-m} \gamma_j) \overset{\gamma_j \geq 0}{\geq} \gamma_j \underbrace{(1 - K' 2^{-m})}_{\in [0,1[}$$

with $K' := K/\ln(2)$. By induction this proves that γ_j is non-negative and decreasing. Again by induction on j we also deduce

$$\gamma_j \geq \gamma_0 (1 - K' 2^{-m})^j \geq \gamma_0 (1 - K' 2^{-m})^{2^m} \geq \gamma_0 e^{-K'-1} = 2^{3m/4} e^{-K'-1} \geq 2^{2m/3},$$

where we again used that m is “sufficiently large”. Since β_j is increasing, we obtain

$$2^{-2^{3m/4}} = \beta_0 \leq \beta_j \leq 2^{-2^{2m/3}}, \qquad \forall j \in \{0, ..., 2^m\}.$$

This and the fact that $u(0) = 0$ implies that β_j fulfills the conditions of Lemma 2.16 for all $j \in \{0, ..., 2^m - 1\}$ as long as m is large enough and hence we have

$$|u(j 2^{-m})|_\infty \leq \beta_j \leq 2^{-2^{2m/3}}, \qquad \forall j \in \{0, ..., 2^m\}.$$

By letting m go to infinity, we deduce that u vanishes at all dyadic points. By continuity of u it follows $u = 0$ on $[0, 1]$ with probability at least $1 - \varepsilon$.

Step 2:

Let $k \in \mathbb{N}$. By setting $\varepsilon := 1/k$ in step 1 we conclude that there is $A_k \subseteq \Omega$ with $\mathbb{P}[A_k] \leq 1/k$ such that $u = 0$ for all $\omega \in A_k^c$. By defining

$$N := \bigcap_{k=1}^{\infty} A_k$$

we have $u = 0$ on $[0, 1]$ for all $\omega \in N^c$ concluding the proof.

□

Proof (of Theorem 1.5)

Let f and σ be as in equation (1.0). By Proposition 1.3 and Remark 1.4 it is enough to consider the case where f is bounded by $1/2$ everywhere. Using Theorem 2.17 there exists $N \subseteq \Omega$ with $\mathbb{P}[N] = 0$ such that N satisfies the conditions of Lemma 1.8. Invoking Lemma 1.8 yields the required result.

□

3 Applications

As an application of the main result of this thesis we focus on Davie's Corollary 4.1 of [Dav07] in this chapter. Furthermore, we establish a corollary which shows the connection between path-by-path solutions and solutions of perturbed IEs.

For partitions

$$\mathcal{P} = \{0 = t_0 < ... < t_N = T\}$$

we define the mesh

$$\text{mesh}(\mathcal{P}) := \max_{1 \le n \le N} |t_n - t_{n-1}|$$

and the Euler approximation

$$x_{n+1} = x_n + W(t_{n+1}) - W(t_n) + (t_{n+1} - t_n) f(t_n, x_n)$$

for $n \in \{0, ..., N-1\}$ with $x_0 := 0$.

Corollary 3.1

Additionally to the previous conditions in equation (1.0) let f be continuous with $|f|_\infty \neq 0$. For almost all Brownian paths W and every sequence of partitions

$$\mathcal{P}_k = \left\{ t_0^{(k)}, ..., t_{N_k}^{(k)} \right\}$$

with $\lim\limits_{k\to\infty} \text{mesh}(\mathcal{P}_k) = 0$ we have

$$\lim_{k\to\infty} \sup_{0 \le n \le N_k} \left| x_n^{(k)} - x(t_n^{(k)}) \right| = 0,$$

where x is the unique solution from Theorem 1.5 and $x_n^{(k)}$ the Euler approximation w.r.t. the partition $\mathcal{P}_k$.

Proof

Let W be a Brownian path for which the conclusion of Theorem 1.5 holds. Suppose there is a sequence of partitions $\mathcal{P}_k$ with $\lim_{k\to\infty} \operatorname{mesh}(\mathcal{P}_k) = 0$ but $\sup_{0\le n\le N_k} \left|x_n^{(k)} - x(t_n^{(k)})\right| \ge \delta > 0$. Set

$$u_n^{(k)} := x_n^{(k)} - W(t_n^{(k)}).$$

We then have

$$\begin{aligned}\left|u_{n+1}^{(k)} - u_n^{(k)}\right| &= \left|x_{n+1}^{(k)} - W(t_{n+1}^{(k)}) - x_n^{(k)} + W(t_n^{(k)})\right| \\ &= |t_{n+1}^{(k)} - t_n^{(k)}||f(t_n^{(k)}, x_n^{(k)})| \le |t_{n+1}^{(k)} - t_n^{(k)}||f|_\infty.\end{aligned}$$

Define $u^{(k)} \in \mathcal{C}([0,T],\mathbb{R})$ as $u_n^{(k)}$ at $t_n^{(k)}$ and interpolate linearly at the other points. Note that $u^{(k)}$ is continuous (even Lipschitz continuous) and uniformly bounded. Next, we will prove that the family $u^{(k)}$ is equicontinuous. To this end let $\varepsilon > 0$ and set $\delta' := \frac{\varepsilon}{|f|_\infty}$. Let $z_1, z_2 \in [0,T]$ with $|z_2 - z_1| \le \delta'$. W.l.o.g. we assume that $z_1 < z_2$. Choose $m, \ell \in \mathbb{N}$ such that $z_1 \le t_m^{(k)} < \ldots < t_\ell^{(k)} \le z_2$. Using the triangle inequality and applying the above estimate immediately yields

$$\begin{aligned}\left|u^{(k)}(z_1) - u^{(k)}(z_2)\right| &\le \sum_{i=m}^{\ell-1}\left|u_{i+1}^{(k)} - u_i^{(k)}\right| + \left|u^{(k)}(z_1) - u^{(k)}(t_m^{(k)})\right| + \left|u^{(k)}(z_2) - u^{(k)}(t_\ell^{(k)})\right| \\ &\le (t_\ell^{(k)} - t_m^{(k)})|f|_\infty + (t_m^{(k)} - z_1)|f|_\infty + (z_2 - t_\ell^{(k)})|f|_\infty = |z_2 - z_1||f|_\infty \le \delta'|f|_\infty = \varepsilon\end{aligned}$$

proving the equicontinuity of the family $u^{(k)}$. So by the Arzelà-Ascoli Theorem the set $\overline{\{u^{(k)}\}}$ is compact in $\mathcal{C}([0,T],\mathbb{R})$. By passing to a subsequence we have $u \in \mathcal{C}([0,T],\mathbb{R})$ with

$$\lim_{k\to\infty}\sup_{0\le n\le N_k}\left|u_n^{(k)} - u(t_n^{(k)})\right| = 0.$$

We define $y(t) := u(t) + W(t)$ for $t \in [0,T]$. Using our assumption we obviously have $x \ne y$

$$\begin{aligned}\lim_{k\to\infty}\sup_{0\le n\le N_k}\left|y(t_n^{(k)}) - x(t_n^{(k)})\right| &= \sup_{0\le n\le N_k}\lim_{k\to\infty}\left|u_n^{(k)} + W(t_n^{(k)}) - x(t_n^{(k)})\right| \\ &= \lim_{k\to\infty}\sup_{0\le n\le N_k}\left|x_n^{(k)} - x(t_n^{(k)})\right| \ge \delta > 0.\end{aligned}$$

Nevertheless, y satisfies (1.1) since

$$u(t) = \lim_{k\to\infty}\sum_{n=1}^{N_k} u(t_n^{(k)}\wedge t) - u(t_{n-1}^{(k)}\wedge t) = \lim_{k\to\infty}\sum_{n=1}^{N_k}(t_n^{(k)}\wedge t - t_{n-1}^{(k)}\wedge t)f(t_{n-1}^{(k)}, x_{n-1}^{(k)}).$$

Using the continuity of f this is the same as

$$= \lim_{k\to\infty}\sum_{n=1}^{N_k}(t_n^{(k)}\wedge t - t_{n-1}^{(k)}\wedge t)f(t_{n-1}^{(k)}, u_{n-1}^{(k)} + W(t_{n-1}^{(k)})) = \int_0^t f(s, u(s) + W(s))\,\mathrm{d}s,$$

which is a contradiction to the conclusion of Theorem 1.5.

□

Remark 3.2

Observe that Corollary 3.1 implies that the partitions in the Euler approximation can be chosen arbitrarily i.e. $t_n^{(k)}$ might depend on ω in a "non-anticipating" way. Usually one is restricted to partition points which are stopping times. From the view point of numeric's this corollary implies that variable step size algorithm converge to the correct solution. This seems to be related to the simplicity of the SDE which we consider. For example the Euler approximation for the SDE

$$\mathrm{d}x(t) = W(t)\mathrm{d}W(t)$$

converges to different functions if the partition points are chosen in an "anticipating" way (cf. [GL97] section 4.1).

Corollary 3.3

Let f be a bounded Borel function, $\sigma \in \mathbb{R} \setminus \{0\}$, $u_0 \in \mathbb{R}^d$. For almost all Brownian paths W the differential equation

$$\begin{cases} \mathrm{d}u(t) = f(t, u(t) + \sigma W(t))\mathrm{d}t \\ u(0) = u_0 \end{cases}$$

has a unique solution in the integral sense.

Proof

Let x be the unique solution of

$$x(t) = u_0 + \int_0^t f(s, x(s))\ \mathrm{d}s + \sigma W(t)$$

from Theorem 1.5. We set $u(t) := x(t) - \sigma W(t)$. Note that $u(0) = u_0$ and

$$u(t) = u_0 + \int_0^t f(s, x(s))\ \mathrm{d}s = u_0 + \int_0^t f(s, u(s) + \sigma W(s))\ \mathrm{d}s.$$

□

Conclusion

In conclusion we have demonstrated that for almost all canonical Brownian paths W the stochastic differential equation

$$\begin{cases} \mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \\ x(0) = x_0 \end{cases}$$

in the finite dimensional space $\mathbb{R}^d$ with bounded Borel measurable drift f, diffusion coefficient $\sigma \in \mathbb{R} \setminus \{0\}$ and deterministic initial condition $x_0 \in \mathbb{R}^d$ admits a unique solution in the path-by-path sense. Hence, we confirmed that the above equation can be solved in the sense of randomly perturbed ordinary differential equations. As a consequence of this, we have shown that the Euler approximation converges to the solution for almost all Brownian paths even if the partition points are chosen randomly.

In this thesis we only considered the finite dimensional case with a constant non-degenerate diffusion term. Recently there has been some development in the case where the diffusion term is not constant and satisfies some mild regularity properties (cf. [Dav11]). To the author's knowledge the question whether the above equation admits a unique solution in the path-by-path sense in infinite dimensions remains open.

A Some basic estimates

This appendix contains a few estimates which are used in Chapter 1. The proofs of these bounds would have disturbed the flow of reading and hence this appendix acts as a collection of estimates that are used in Proposition 1.16 and Proposition 1.19.

Lemma A.1

For all $x \leq 0$ the following estimate holds

$$e^x \leq \frac{1}{1 - x + x^2/2 - x^3/6}.$$

Proof

Define

$$f(x) := e^x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right), \qquad \forall x \leq 0.$$

An easy calculation yields

$$f'(x) = -\frac{x^3}{6} e^x \geq 0, \qquad \forall x \leq 0.$$

Since $f(0) \leq 1$ holds, $f'(x) \geq 0$ implies that $f(x) \leq 1$ for all $x \leq 0$ which shows the assertion.

□

Proposition A.2

There exists a constant $C \in \mathbb{R}$ such that

$$\frac{(k+1)^2 - x}{x^{5/2}} \leq C e^{(k-3/2)/x}, \qquad \forall x \in]0, 1/2], \; k \geq 2.$$

Proof

Let x and k be as specified in the assertion. Since $(k-3/2)/x$ is positive we can use Lemma A.1 to obtain

$$\frac{(k+1)^2 - x}{x^{5/2}} e^{-(k-3/2)/x} \overset{A.1}{\leq} \frac{(k+1)^2 - x}{x^{5/2}} \frac{1}{1 + \frac{k-3/2}{x} + \frac{1}{2}\frac{(k-3/2)^2}{x^2} + \frac{1}{6}\frac{(k-3/2)^3}{x^3}}$$
$$= \frac{(k+1)^2 - x}{x^{5/2} + (k-3/2)x^{3/2} + \frac{1}{2}(k-3/2)^2 x^{1/2} + \frac{1}{6}(k-3/2)^3 x^{-1/2}} \overset{x\downarrow 0}{\longrightarrow} 0.$$

Therefore, there exists $C \in \mathbb{R}$ such that

$$\frac{(k+1)^2 - x}{x^{5/2}} e^{-(k-3/2)/x} \leq C, \qquad \forall x \in]0, 1/2].$$

□

Proposition A.3

There exists a constant $C \in \mathbb{R}$ such that for all $r \geq 0$

$$\sum_{k=0}^{\infty} 2^{-k}(1-2^{-k-1})^r \leq C(1+r)^{-1}$$

holds.

Proof

For $x \geq 0$ we define

$$f(x) := 2^{-x}(1-2^{-x-1})^r$$
$$\Rightarrow f'(x) = \ln(2)2^{-x}\left(1-2^{-x-1}\right)^{r-1}\left[\frac{r+1}{2^{x+1}} - 1\right].$$

An easy calculation shows

$$f'(x) \leq 0 \iff x \geq \log_2(r+1) - 1 =: a(r).$$

Note that $a(r)$ is the global maximum of f. Using this we split the sum into the increasing and decreasing part

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\lfloor a(r)\rfloor - 1} f(k) + f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \sum_{k=\lfloor a(r)\rfloor+2}^{\infty} f(k)$$

and estimate the sum via its integral

$$
\begin{aligned}
&\leq \int_0^{\lfloor a(r)\rfloor} f(x)\ \mathrm{d}x + f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \int_{\lfloor a(r)\rfloor+1}^{\infty} f(x)\ \mathrm{d}x \\
&\leq f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \int_0^{\infty} 2^{-x}(1-2^{-x-1})^r\ \mathrm{d}x \\
&\leq 2f(a(r)) + \int_0^{\infty} e^{-x\ln 2}(1-e^{-(x+1)\ln 2})^r\ \mathrm{d}x \\
&= \frac{4}{r+1}\underbrace{\left(1-\frac{1}{r+1}\right)^r}_{\leq 1} + \frac{1}{\ln 2}\int_0^{\infty} e^{-y}\left(1-\frac{1}{2}e^{-y}\right)^r\ \mathrm{d}y.
\end{aligned}
$$

Since $\frac{2}{r+1}\left(1-\frac{1}{2}e^{-y}\right)^{r+1}$ is an anti-derivative of the integrand, we obtain

$$
\leq \frac{4}{r+1} + \frac{1}{\ln 2}\frac{2}{r+1}\underbrace{\left(1-\frac{1}{2^{r+1}}\right)}_{\leq 1} \leq \frac{4}{r+1} + \frac{3}{r+1} = \frac{7}{r+1}.
$$

□

Lemma A.4

Let $\alpha \geq 0$. For every $r \geq 0$ the following inequality holds

$$
\frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} \leq e^{\alpha+1/12}(r+1)^{-\alpha}.
$$

Proof

We use Stirling's formula for the gamma function

$$
\Gamma(x) = \sqrt{2\pi}x^{x-1/2}e^{-x}e^{\mu(x)}, \qquad \forall x > 0,
$$

where $0 < \mu(x) < \frac{1}{12x}$, to estimate

$$
\begin{aligned}
\frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} &\leq \frac{\sqrt{2\pi}(r+1)^{r+1/2}e^{-r-1}e^{1/(12(r+1))}}{\sqrt{2\pi}(r+1+\alpha)^{r+1/2+\alpha}e^{-r-1-\alpha}} = \frac{(r+1)^{r+1/2}e^{1/(12(r+1))}}{(r+1+\alpha)^{r+1/2+\alpha}e^{-\alpha}} \\
&\leq \frac{e^{1/(12(r+1))}}{(r+1)^{\alpha}e^{-\alpha}} \leq e^{1/12}e^{\alpha}(r+1)^{-\alpha} = e^{\alpha+1/12}(r+1)^{-\alpha}.
\end{aligned}
$$

□

Proposition A.5

There exists a constant $C \in \mathbb{R}$ such that for all $r \geq 0$

$$\sum_{k=0}^{\infty} 2^{-k/2}(1-2^{-k-1})^r \leq C(r+1)^{-1/2}$$

holds.

Proof

For $x \geq 0$ we set

$$f(x) := 2^{-x/2}(1-2^{-x-1})^r$$
$$\Rightarrow f'(x) = \ln(2) 2^{-x/2}\left(1-2^{-x-1}\right)^{r-1}\left[\frac{r+1/2}{2^{x+1}} - \frac{1}{2}\right].$$

An easy calculation shows

$$f'(x) \leq 0 \quad \iff \quad x \geq \log_2(r+1/2) =: a(r).$$

Note that $a(r)$ is the global maximum of f. Using this we split the sum into the increasing and decreasing part

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\lfloor a(r)\rfloor -1} f(k) + f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \sum_{k=\lfloor a(r)\rfloor+2}^{\infty} f(k).$$

and estimate the sum via its integral

$$\leq \int_0^{\lfloor a(r)\rfloor} f(x)\,\mathrm{d}x + f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \int_{\lfloor a(r)\rfloor+1}^{\infty} f(x)\,\mathrm{d}x$$
$$\leq f(\lfloor a(r)\rfloor) + f(\lceil a(r)\rceil) + \int_0^{\infty} 2^{-x/2}\left(1-2^{-x-1}\right)^r\,\mathrm{d}x$$
$$\leq 2f(a(r)) + \int_0^{\infty} e^{-x\ln(2)/2}(1-e^{-(x+1)\ln 2})^r\,\mathrm{d}x$$
$$= \frac{2}{\sqrt{r+1/2}}\underbrace{\left(1-\frac{1}{2r+1}\right)^r}_{\leq 1} + \frac{1}{\ln 2}\int_0^{\infty} e^{-y/2}\left(1-\frac{1}{2}e^{-y}\right)^r\,\mathrm{d}y.$$

Substituting $\frac{1}{2}e^{-y}$ with u results in

$$
\begin{aligned}
&\leq \frac{2}{\sqrt{r+1/2}} - \frac{\sqrt{2}}{\ln 2}\int_{1/2}^{0} u^{-1/2}(1-u)^r \, \mathrm{d}y \\
&= \frac{2}{\sqrt{r+1/2}} + \frac{\sqrt{2}}{\ln 2}\int_{0}^{1/2} u^{-1/2}(1-u)^r \, \mathrm{d}y \\
&\leq \frac{2}{\sqrt{r+1/2}} + \frac{\sqrt{2}}{\ln 2}\int_{0}^{1} u^{-1/2}(1-u)^r \, \mathrm{d}y \\
&= \frac{2}{\sqrt{r+1/2}} + \frac{\sqrt{2}}{\ln 2}\beta(1/2, r+1) \\
&= \frac{2}{\sqrt{r+1/2}} + \frac{\sqrt{2\pi}}{\ln 2}\frac{\Gamma(r+1)}{\Gamma(r+3/2)}.
\end{aligned}
$$

Using Lemma A.4 with $\alpha = \frac{1}{2}$ we can estimate the gamma function.

$$
\begin{aligned}
&\leq \frac{2}{\sqrt{r+1/2}} + \frac{\sqrt{2\pi}e^{7/12}}{\ln 2}(r+1)^{-1/2} \leq \frac{3}{(r+1)^{1/2}} + \underbrace{\frac{\sqrt{2\pi}e^{7/12}}{\ln 2}}_{\leq 7}(r+1)^{-1/2} \\
&\leq 3(r+1)^{-1/2} + 7(r+1)^{-1/2} = 10(r+1)^{-1/2}.
\end{aligned}
$$

□

Proposition A.6

Let r, $n \in \mathbb{N}$ and a_i non-negative numbers for every $i \in \{1, ..., r\}$. Then the following inequality holds

$$\left(\sum_{k=1}^{r} a_k\right)^n \leq r^{n-1}\sum_{k=1}^{r} a_k^n.$$

Proof

We have

$$\left(\sum_{k=1}^{r} a_k\right)^n = \sum_{k_1+...+k_r=n} \binom{n}{k_1, ..., k_r} a_1^{k_1}\cdots a_r^{k_r}$$

where

$$\binom{n}{k_1, ..., k_r} = \frac{n!}{k_1!\cdots k_r!}$$

is the multinomial coefficient. Applying Young's inequality with $p_i = \frac{n}{k_i}$ results in

$$\leq \frac{1}{n} \sum_{k_1+\ldots+k_r=n} \sum_{j=1}^{r} \binom{n}{k_1, \ldots, k_r} k_j a_j^n$$
$$= \sum_{j=1}^{r} \sum_{\substack{k_1+\ldots+k_r=n \\ k_j>0}} \binom{n-1}{k_1, \ldots, k_{j-1}, k_j-1, k_{j+1}, k_r} a_j^n$$
$$= \sum_{j=1}^{r} a_j^n \underbrace{\sum_{k_1+\ldots+k_r=n-1} \binom{n-1}{k_1, \ldots, k_r}}_{=r^{n-1}}$$
$$= r^{n-1} \sum_{j=1}^{r} a_j^n.$$

□

Proposition A.7

For every $n \in \mathbb{N}$ the following inequality holds

$$\sqrt{n} 2^{-n/2} \leq 2 \cdot 2^{-3n/8}.$$

Proof

We have

$$\log_2 \left(n^{1/2} 2^{-n/2}\right) = \underbrace{\log_2 \left(n^{1/2}\right)}_{\leq n/8+1} - \frac{n}{2} \leq -\frac{3n}{8} + 1$$
$$= \log_2 2^{-3n/8} + \log_2 2 = \log_2 \left(2 \cdot 2^{-3n/8}\right),$$

which concludes the proof.

□

B Fourier transform of E and D

In this appendix we calculate the Fourier transform of E and D, which are used in Proposition 1.16 of Chapter 1.

Proposition B.1

Let $t \in \mathbb{R}$ with $t > 0$ then

$$\mathcal{F}[\mathrm{E}(t,\,\cdot\,)](\xi) = e^{-2\pi^2 t\xi^2}$$

holds.

Proof

Define f as

$$f(\xi) := \mathcal{F}[\mathrm{E}(t,\,\cdot\,)](\xi) = \int_{-\infty}^{\infty} e^{-2\pi \mathrm{i}\xi z}(2\pi t)^{-1/2} e^{-z^2/2t}\ \mathrm{d}z.$$

By interchanging differentiation with integration we get

$$f'(\xi) = -\int_{-\infty}^{\infty} 2\pi \mathrm{i} z e^{-2\pi \mathrm{i}\xi z}(2\pi t)^{-1/2} e^{-z^2/2t}\ \mathrm{d}z = 2\pi \mathrm{i}\int_{-\infty}^{\infty} e^{-2\pi \mathrm{i}\xi z}(2\pi t)^{-1/2}\partial_z t e^{-z^2/2t}\ \mathrm{d}z.$$

Using integration by parts yields

$$f'(\xi) = -4\pi^2 t\xi \int_{-\infty}^{\infty} e^{-2\pi \mathrm{i}\xi z}(2\pi t)^{-1/2} e^{-z^2/2t}\ \mathrm{d}z.$$

And since we have $f(0) = 1$ this results in the following initial value problem.

$$\begin{cases} f'(\xi) &= -4\pi^2 t\xi f(\xi) \\ f(0) &= 1 \end{cases}$$

This problem has clearly the unique solution

$$f(\xi) = e^{-2\pi^2 t\xi^2}.$$

□

Proposition B.2

Let $t \in \mathbb{R}$ with $t > 0$ then

$$\mathcal{F}[\mathrm{D}(t,\,\cdot\,)](\xi) = -4\pi^2\xi^2 e^{-2\pi^2 t\xi^2}$$

holds.

Proof

By splitting the integral in two parts we obtain

$$\begin{aligned}
\mathcal{F}[\mathrm{D}(t,\,\cdot\,)](\xi) &= \int\limits_{-\infty}^{\infty} e^{-2\pi\mathrm{i}\xi z}(2\pi t)^{-1/2}\frac{z^2-t}{t^2}e^{-z^2/2t}\;\mathrm{d}z \\
&= \frac{1}{t^2}\int\limits_{-\infty}^{\infty} e^{-2\pi\mathrm{i}\xi z}(2\pi t)^{-1/2}z^2 e^{-z^2/2t}\;\mathrm{d}z - \frac{1}{t}\int\limits_{-\infty}^{\infty} e^{-2\pi\mathrm{i}\xi z}(2\pi t)^{-1/2}e^{-z^2/2t}\;\mathrm{d}z \\
&= \frac{1}{t^2}\frac{-1}{4\pi^2}\int\limits_{-\infty}^{\infty} \partial_\xi^2 e^{-2\pi\mathrm{i}\xi z}(2\pi t)^{-1/2}e^{-z^2/2t}\;\mathrm{d}z - \frac{1}{t}\mathcal{F}[\mathrm{E}(t,\,\cdot\,)](\xi).
\end{aligned}$$

Interchanging differentiation with integration yields

$$= \frac{-1}{4\pi^2 t^2}\partial_\xi^2\mathcal{F}[\mathrm{E}(t,\,\cdot\,)](\xi) - \frac{1}{t}\mathcal{F}[E(t,\,\cdot\,)](\xi).$$

Using Proposition B.1 results in

$$\begin{aligned}
&= \frac{-1}{4\pi^2 t^2}\left[-4\pi^2 t + 16\pi^4 t^2\xi^2\right]e^{-2\pi t\xi^2} - \frac{1}{t}e^{-2\pi^2 t\xi^2} \\
&= \left[\frac{1}{t} - 4\pi^2\xi^2 - \frac{1}{t}\right]e^{-2\pi t\xi^2} = -4\pi^2\xi^2 e^{-2\pi t\xi^2}.
\end{aligned}$$

□

C Davie's estimate for ρ

In this appendix we give a detailed proof of Davie's Lemma 3.1 in [Dav07] which was replaced by Lemma 2.5 in this thesis. We also give a derivation of equation (21) in Davie's paper (Proposition C.3) which was used to prove Lemma 2.14 in the original proof.

Lemma C.1

For all $m \in \mathbb{N}$ with $m \geq 1$

$$\sum_{k=m+1}^{\infty} \sqrt{k} 2^{-k} \leq 6\sqrt{m} 2^{-m}$$

holds.

Proof

Since $\sqrt{k}2^{-k}$ is decreasing on $[1, \infty[$. We can estimate

$$\sum_{k=m+1}^{\infty} \sqrt{k} 2^{-k} \leq \int_{m}^{\infty} \sqrt{k} 2^{-k} \, \mathrm{d}k = \int_{m}^{\infty} k^{1/2} e^{-k \ln 2} \, \mathrm{d}k.$$

Substitution with $u := k \ln 2$ results in

$$= \int_{m \ln 2}^{\infty} \left(\frac{u}{\ln 2}\right)^{1/2} e^{-u} \frac{1}{\ln 2} \, \mathrm{d}u = \left(\frac{1}{\ln 2}\right)^{3/2} \int_{m \ln 2}^{\infty} u^{3/2-1} e^{-u} \, \mathrm{d}u = \left(\frac{1}{\ln 2}\right)^{3/2} \Gamma(3/2, m \ln 2).$$

Where Γ is the complementary incomplete gamma function. Using the estimate (2.14) in [Olv03] page 70 results in

$$\leq \left(\frac{1}{\ln 2}\right)^{3/2} \frac{e^{-m \ln 2} (m \ln 2)^{3/2}}{m \ln 2 - 3/2 + 1} = \frac{e^{-m \ln 2} m^{3/2}}{m \ln 2 - 1/2} \leq 6 \frac{2^{-m} m^{3/2}}{m} \leq 6\sqrt{m} 2^{-m}.$$

□

We now turn to the proof of Davie's Lemma 3.1 in [Dav07].

Lemma C.2

For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that for every real-valued Borel function g on $[0,1] \times \mathbb{R}$ bounded by 1

$$|\rho_{n,k}(x,y)| \leq C(\varepsilon) \left(\sqrt{n} + \sqrt{\log_2^+ \frac{1}{|x-y|_\infty}}\right) 2^{-n/2} |x-y|_\infty, \qquad \forall \omega \in A_\varepsilon^c$$

holds for all dyadic points $x, y \in Q$, $n \geq 1$, $k \in \{0, \dots, 2^n - 1\}$ and $\omega \in A_\varepsilon^c$.

Proof

Step 1: $x \sim y$

Claim: For every $\varepsilon > 0$ there exist $C(\varepsilon) \in \mathbb{R}$ and $A_\varepsilon \subseteq \Omega$ with $\mathbb{P}[A_\varepsilon] \leq \varepsilon$ such that

$$|\rho_{n,k}(x,y)| \leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-n/2}|x-y|_\infty,$$
$$\forall \omega \in A_\varepsilon^c,\ x,y \in Q,\ x \sim_m y,\ n \geq 1,\ m \geq 1,\ k \in \{0,...,2^n-1\}.$$

Let $m \in \mathbb{N}$ and $x, y \in Q$ be dyadic neighbors of order m. Applying Corollary 1.28.(i) with $\lambda = \lambda'(\sqrt{n}+\sqrt{m})$ for some $\lambda' > 0$, $s = 0$ and using that $\sqrt{d}|x-y|_\infty \geq |x-y|_2$ yields

$$\mathbb{P}[|\rho_{n,k}(x,y)| > \lambda'(\sqrt{n}+\sqrt{m})2^{-m-n/2}] \leq 2e^{-\lambda'^2(\sqrt{n}+\sqrt{m})^2/(2C^2)} \leq 2e^{-\lambda'^2(n+m)/(2C^2)}.$$

So, we can estimate the probability that $|\rho_{n,k}(x,y)| > \lambda'(\sqrt{n}+\sqrt{m})2^{-m-n/2}$ holds for any k, n and any pair of dyadic neighbors x, y by

$$\mathbb{P}\left[\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}\bigcup_{\substack{x,y\in Q\\ x\sim_m y}}\bigcup_{k=0}^{2^n-1} |\rho_{n,k}(x,y)| > \lambda'(\sqrt{n}+\sqrt{m})2^{-m-n/2}\right]$$
$$\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{x,y\in Q\\ x\sim_m y}}\sum_{k=0}^{2^n-1} 2e^{-\lambda'^2(n+m)/(2C^2)}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \#\{(x,y)\in Q^2 | x \sim_m y\} \cdot 2^{n+1}e^{-\lambda'^2(n+m)/(2C^2)}$$
$$\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{(m+2)d}3^d 2^{n+1}e^{-\lambda'^2(n+m)/(2C^2)}$$
$$= 2\cdot 3^d 4^d \sum_{n=0}^{\infty} 2^n e^{-\lambda'^2 n/(2C^2)} \sum_{m=0}^{\infty} 2^{md}e^{-\lambda'^2 m/(2C^2)}.$$

By calculating the derivative of $2^{md}e^{-\lambda'^2 m/(2C^2)}$ w.r.t. m we see easily see that the series is decreasing as long as we choose $\lambda' > \sqrt{2\ln(2)dC^2}$. So, we can use the integral criteria to estimate the series and hence we obtain

$$\int_0^\infty 2^{md}e^{-\lambda'^2 m/(2C^2)}\ dm \leq \int_0^\infty e^{md-\lambda'^2 m/(2C^2)}\ dm = \int_0^\infty \alpha^m\ dm.$$

Where $\alpha := e^{d-\lambda'^2/(2C^2)}$

$$= \frac{-1}{\ln\alpha} = \frac{1}{\lambda'^2/(2C^2)-d} \stackrel{\lambda'\to\infty}{\longrightarrow} 0.$$

This implies that both sums convergence for $\lambda' > \sqrt{2\ln(2)dC^2}$. We therefore deduce that

$$\lim_{\lambda'\to\infty}\mathbb{P}\left[\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}\bigcup_{\substack{x,y\in Q\\ x\sim_m y}}\bigcup_{k=0}^{2^n-1} |\rho_{n,k}(x,y)| > \lambda'(\sqrt{n}+\sqrt{m})2^{-m-n/2}\right] = 0.$$

Let $\varepsilon > 0$. Then there exists $C(\varepsilon)$ such that

$$\mathbb{P}\left[\bigcap_{n=0}^{\infty}\bigcap_{m=0}^{\infty}\bigcap_{\substack{x,y\in Q\\ x\sim_m y}}\bigcap_{k=0}^{2^n-1}|\rho_{n,k}(x,y)| \leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2}\right] \geq 1-\varepsilon.$$

So, we have

$$\begin{aligned}&|\rho_{n,k}(x,y)| \leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2},\\ &\forall x,y \in Q,\ x \sim_m y,\ n \geq 1,\ m \geq 1,\ k \in \{0,...,2^n-1\}\end{aligned}$$

with probability greater than $1-\varepsilon$.

Step 2: $|x-y|_\infty < 1/2$

Let x, $y \in Q$ be dyadic points with $|x-y|_\infty < 1/2$. The claim is trivial for $x = y$, so we assume $x \neq y$. We can now use Lemma 2.4. Let m, x_r and y_r be as in Lemma 2.4. This also means that $m \geq 1$ and $2^{-m-1} \leq |x-y|_\infty$ by the maximality of m.

$$\rho_{n,k}(x,y) = \rho_{n,k}(x_m,y_m) + \sum_{r=m}^{\infty}\rho_{n,k}(x_{r+1},x_r) + \sum_{r=m}^{\infty}\rho_{n,k}(y_{r+1},y_r).$$

Observe that both sums converge since we have $x_{r+1} = x_r$ for large r. We know that $x_m \sim_m y_m$ or $x_m = y_m$ and that $x_{r+1} \sim_{r+1} x_r$ or $x_{r+1} = x_r$ and that $y_{r+1} \sim_{r+1} y_r$ or $y_{r+1} = y_r$, so we can use step 1 to deduce that

$$\begin{aligned}|\rho_{n,k}(x,y)| &\leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} + 2C(\varepsilon)\sum_{r=m}^{\infty}(\sqrt{n}+\sqrt{r+1})2^{-r-1-n/2}\\ &= C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} + 2C(\varepsilon)2^{-n/2}\left[\sqrt{n}\sum_{r=m}^{\infty}2^{-r-1} + \sum_{r=m}^{\infty}\sqrt{r+1}2^{-r-1}\right]\\ &\leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} + 2C(\varepsilon)2^{-n/2}\left[\sqrt{n}2^{-m} + \sum_{r=m+1}^{\infty}\sqrt{r}2^{-r}\right].\end{aligned}$$

Since $m \geq 1$ we can use Lemma C.1 to obtain

$$\begin{aligned}&\leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} + 2C(\varepsilon)2^{-n/2}\left[\sqrt{n}2^{-m} + 6\sqrt{m}2^{-m}\right]\\ &\leq C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} + 12C(\varepsilon)2^{-m}2^{-n/2}\left[\sqrt{n}+\sqrt{m}\right]\\ &= 13C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-m-n/2} = 26C(\varepsilon)(\sqrt{n}+\sqrt{m})2^{-n/2}|x-y|_\infty\\ &= 26C(\varepsilon)\left(\sqrt{n}+\sqrt{\log_2^+\frac{1}{|x-y|_\infty}}\right)2^{-n/2}|x-y|_\infty.\end{aligned}$$

Step 3: $|x-y|_\infty \geq 1/2$

Let $x, y \in Q$ be dyadic points with $|x-y|_\infty \geq 1/2$. The claim is trivial in this case since

$$\begin{aligned}
|\rho_{n,k}(x,y)| &= \left| \int_{I_{n,k}} g(t, W(t)+x) - g(t, W(t)+y) \, \mathrm{d}t \right| \\
&\leq \int_{I_{n,k}} 2\|g\| \, \mathrm{d}t \leq 2 \cdot 2^{-n} \leq 4 \cdot 2^{-n} |x-y|_\infty \leq 4\sqrt{n} 2^{-n} |x-y|_\infty \\
&\leq 4\sqrt{n} 2^{-n/2} |x-y|_\infty \leq 4 \left(\sqrt{n} + \sqrt{\log_2^+ \frac{1}{|x-y|_\infty}} \right) 2^{-n/2} |x-y|_\infty .
\end{aligned}$$

□

We are now ready to derive equation (21) in Davie's paper which enables the possibility to use Lemma C.2 instead of Lemma 2.5 in the proof of Lemma 2.14.

Proposition C.3

There exists $C \in \mathbb{R}$ such that for all n, $m \in \mathbb{N}$

$$C(\sqrt{n} + \sqrt{m}) 2^{-n/2} 2^{-m} \leq C 2^{-n/4} 2^{-m} + 2^{-2^{n/2}}.$$

holds.

Proof

Consider the function

$$f(n,m) := 2^{-m} \left[(\sqrt{n} + \sqrt{m}) 2^{-n/2} - 2^{-n/4} \right].$$

We will show that $f(n,m) \leq C 2^{-2^{n/2}}$ holds with some constant $C \in \mathbb{R}$. To this end we calculate the maximum of $f(n,m)$ with respect to m. We calculate the first derivative

$$\partial_m f(n,m) = \frac{\ln 4 \sqrt{m}(2^{n/4} - \sqrt{n}) - m \ln 4 + 1}{2^{m+n/2+1}\sqrt{m}}$$

and calculate the zeros of this function

$$\begin{aligned}
&\partial_m f(n,m) = 0 \\
&\Leftrightarrow \sqrt{m} \underbrace{(2^{n/4} - \sqrt{n})}_{=:\alpha} - m + \underbrace{\frac{1}{\ln 4}}_{=:\beta} = 0 \\
&\Leftrightarrow \alpha\sqrt{m} - m + \beta = 0.
\end{aligned}$$

Note that the above equation has at most one solution. If m_1, m_2 are two distinct solutions we obtain

$$\begin{aligned}&\alpha\sqrt{m_1}-m_1=\alpha\sqrt{m_2}-m_2\\&\Rightarrow\alpha\left(\sqrt{m_1}-\sqrt{m_2}\right)=m_1-m_2=\left(\sqrt{m_1}-\sqrt{m_2}\right)\left(\sqrt{m_1}+\sqrt{m_2}\right)\\&\Rightarrow\alpha=\left(\sqrt{m_1}+\sqrt{m_2}\right).\end{aligned}$$

And hence solving for β yields

$$\beta=m_1-\alpha\sqrt{m_1}=m_1-\left(\sqrt{m_1}+\sqrt{m_2}\right)\sqrt{m_1}=-\sqrt{m_1}\sqrt{m_2}\leq 0,$$

which contradicts $\beta=1/\ln(4)>0$. Therefore, the only solution to the above equation is

$$m_0:=\left(\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}+\beta}\right)^2.$$

An easy calculation shows that $f(n,m_0)$ is the global maximum, because of the fact that $\lim\limits_{m\to\infty}f(n,m)=0$. Concerning the dependency of n we have

$$\alpha\in\Theta\left(2^{n/4}\right)$$

and henceforth

$$m_0\in\Theta\left(2^{n/2}\right).$$

We get

$$f(n,m)\leq f(n,m_0)=\underbrace{2^{-m_0}}_{\in\mathcal{O}\left(2^{-2^{n/2}}\right)}\underbrace{\left[\underbrace{\left(\sqrt{n}+\sqrt{m_0}\right)}_{\in\Theta\left(2^{n/4}\right)}2^{-n/2}-2^{n/4}\right]}_{\in\mathcal{O}(1)}\in\mathcal{O}\left(2^{-2^{n/2}}\right),$$

which concludes the proof.

$\square$

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