Ornstein-Uhlenbeck Equations with time-dependent coefficients and Lévy Noise in finite and infinite dimensions.

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Contents

Chapter 1

Introduction

The subject of this diploma thesis is the study of an infinite dimensional stochastic differential equation of Ornstein-Uhlenbeck type with Levy noise and time-dependent periodic coefficients.

Chapter 2 contains a general introduction into Lévy processes with particular emphasis on the Lévy-Ito decomposition and the Lévy-Khinchine representation.

Taking advantage of the Lévy-Ito decomposition, we establish the necessary theory of integration to give sense to our solution in chapter 3.

Then, in chapter 4, we focus our interest on the associated semigroup and in particular on its asymptotic behaviour, its invariant measures and its generator. The equation being non-autonomous results in a two-parameter semigroup, since we have to keep track not only of the elapsed time but also of the starting time. In this case the concept of an invariant measure has to be generalized to allow for a whole collection of measures - a so-called *evolution system of measures*- which are invariant in an appropriate sense. We will prove the existence of such a system under some stability conditions. Then we turn the problem into an autonomous one by enlarging the state space, allowing for a one-parameter semigroup. Via the evolution system of measures and thanks to the periodicity of the coefficients we are able to establish a unique invariant measure for this semigroup. Thus we can introduce the L^2-space with respect to the invariant measure where the semigroup is strongly continuous. On a domain of uniqueness for the generator we establish the form of its square field operator and prove an estimate for it, that allows us to obtain a Poincaré and a Harnack inequality for our semigroup. This also gives results for the original two-parameter semigroup.

We will now give an overview of the literature on related problems, while we point out our contributions. Our central reference is [daPr/Lun07] which concentrates on the finite dimensional case and Gaussian noise. A number of our arguments are adapted from this paper, although the Lévy setting forces us to work more heavily with Fourier transforms and the infinite dimensional setting requires additional care. The existence result on evolution systems of measures (Theorem 4.10) is stated in [daPr/Lun07] in the finite-dimensional Gaussian framework, but the extension to Hilbert spaces and Lévy noise is fundamentally new. The same applies to the existence result on invariant measures (Theorem 4.19) for the corresponding one-parameter semigroup .
The form of the Fourier transform (Lemma 4.2) and of the generator (Lemma 4.24) are also genuine generalizations from what was done in [daPr/Lun07], though they are very close to results from the theory of generalized Mehler semigroups as for example in [Fuhr/Röck00] or [Lesc/Röck02]. Nevertheless, our framework differs from theirs in that we allow for time-dependent coefficients, making a direct use of the methods developed there impossible.
Our analogue to the integration by parts formula from [daPr/Lun07] is the concrete calculation of the square field operator (Lemma 4.30). As far as we know, there is no such result in our framework, though the general formula in [Lesc/Röck04] is quite similar.
The gradient estimate in [daPr/Lun07] corresponds to our estimate of the square field operator, which is a generalization of the result in [Röck/Wang03] to the time-dependent case. Both our proofs of the Poincaré and the Harnack inequality follow the ones in [Röck/Wang03] very closely. Nevertheless, as far as we know, the extension to our framework is a new result.

The material in chapter 2 and 3 is of course standard, and the references we used can be found at the beginning of each chapter.

I wish to thank Prof. Dr. Michael Röckner for his motivating lectures on stochastic analysis and for his help in connection with this thesis. I would also like to thank Dr. Walter Hoh for his lectures on jump processes and his help concerning negative definite functions. Finally, special thanks are kindly returned to my brother, Kristian Knäble.

Chapter 2

Introduction to Lévy Processes

2.1 Lévy Processes

The reader well acquainted with Brownian motion, will find Lévy processes to share at least two of its desired properties. Stationarity and independence of increments still hold and, together with stochastic continuity, assure that the whole process can be characterized simply by its distribution after a fixed time span. While Brownian motion is characterized by its drift term and covariance operator, it turns out, that, to incorporate the jumps of a Lévy process, it is sufficient to add a third quantity : the Lévy measure. The correspondence between Lévy processes and these triples is made precise in the seminal Lévy-Khinchine formula 2.36. The Lévy measure codes the information about the size and the likelihood of jumps. This measure will in general not be a probability measure, it need not even be finite, however it is only allowed to "explode" around zero, illustrating the possibility of an accumulation point of jumps of vanishing size. Jumps above a fixed size, on the other hand, cannot accumulate, an important observation on the path structure, that will be stated in proposition 2.13, and follows directly from the càdlàg property. The final result in this section will be the Lévy-Ito decomposition, a representation of a Lévy process as an integral with respect to a so called Poisson Random Measure. This decomposition is most important, as it will be the basis for stochastic integration against Lévy processes.

Our exposition follows [App04] and [Alb/Rue05]. Unless stated otherwise, proofs are taken from [App04] and only slightly adapted. Note that the

Lévy-Khinchine formula and the Lévy-Ito decomposition are of course closely related and that there are basically two different approaches to prove them. We follow what might be called the probabilistic approach and what we deem the more intuitive one. It consists in proving the Lévy-Ito decomposition first and to derive the Lévy-Khinchine representation with its help. There is also an analytic approach that bases everything on the Lévy-Khinchine formula as for example in [Sato99].

In the following let be H a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \| := \| \cdot \|_H$.

Definition 2.1 *An H-valued stochastic process L adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ is a Lévy process if and only if*

(L0) $L(0) = 0$ *(a.s)*

(L1) *L has independent increments , i.e.* *$L(t) - L(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$*

(L2) *L has stationary increments, i.e. for all $0 \leq s < t < \infty$* *$L(t) - L(s)$ has the same distribution as $L(t - s)$*

(L3) *L is stochastically continuous, i.e. for all $t \geq 0$ and $\varepsilon > 0$ holds*

$$\lim_{s \to t} P(\|L(s) - L(t)\|_H > \varepsilon) = 0$$

Remark 2.2 *We could have required the Lévy process to have càdlàg paths by definition, but we prefer to emphasize the result(see [Prott90]), that every Lévy process has a modification that is càdlàg(and indeed still satisfies (L0)-L(3)). The proof is based on the fact, that every martingale admits a càdlàg modification. Although not every Lévy process is a martingale (of course centralization would ensure this, but note that the first moment need not exist), this argument can be made to work by considering a related martingale. In the following, when speaking about a Lévy process, we will always mean a càdlàg modification.*

As it will be useful later on (e.g. in the proof of the strong Markov property) we will introduce the martingale mentioned in the remark.

Lemma 2.3 *For any fixed $u \in H$ the process*

$$M_u(t) := \frac{\exp\{i\langle u, L_t\rangle\}}{\mathbb{E}[\exp\{i\langle u, L_t\rangle\}]}$$

is a martingale.

Although, the numerator is certainly integrable, we now have to rule out that the denominator vanishes. This will be done in two additional lemmas which will give us some insight into the behaviour of the Fourier transforms of L_t as t changes and are interesting in themselves.
In the following we will denote by $\Phi_X(u)$ the Fourier transform of a random variable X.

Lemma 2.4 *For a Lévy process L_t, the map $t \mapsto \Phi_{L_t}(u)$ is continuous for each $u \in H$.*

Proof We have to show

$$\lim_{s \to t} \mathbb{E}[\exp(i\langle u, L_s\rangle)] = \mathbb{E}[\exp(i\langle u, L_t\rangle)]$$

Note, simply, that $x \mapsto \exp(i\langle u, x\rangle)$ is bounded and continuous, and that convergence in probability implies convergence in distribution for Hilbert space valued random variables. ∎

Lemma 2.5 (Lévy symbol) *If L is a Lévy process, then for every $u \in H$:*

$$\Phi_{L_t}(u) = \exp(t\lambda(u))$$

where $\lambda : H \to \mathbb{C}$.

Proof We will show that for every $u \in H$, $\quad t \mapsto \Phi_{L_t}(u)$ fulfills the functional equation of the exponential function. We recall that the functional equation is characterising, that is we prove the following claim :

Claim A function $f : \mathbb{R}_+ \to \mathbb{C}$ with the properties:

- $f(0) = 1$
- $f(t+s) = f(t)f(s) \ \forall s, t \in \mathbb{R}_+$

- $t \mapsto f(t)$ is continuous

is of the form $f(t) = e^{zt}$ for some $z \in \mathbb{C}$.

PROOF (OF THE CLAIM) We must have $f \neq 0$ everywhere since $f(t) = 0$ would imply $f(\frac{t}{n})^n = 0$ for every n and hence $f(\frac{t}{n}) = 0$ for every n but this is impossible by continuity. So we may define $g(t) := \log(f(t))$ where log is the branch of the logarithm that assigns 0 to 1. But g fulfills $g(t+s) = g(t)+g(s)$, is continous and $g(0) = 0$ and hence we must have $g(t) = zt$ for some $z \in \mathbb{C}$. Applying the exponential yields the desired result. ∎

So let us check the three conditions in the claim for $t \mapsto \Phi_{L_t}(u)$.
$\Phi_{L_0}(u) = \mathbb{E}[\exp(0)] = 1$ is trivial. By stationarity and independence of increments:

$$\begin{aligned}
\Phi_{L_{t+s}}(u) &= \mathbb{E}[e^{i\langle u, L_{t+s}\rangle}] \\
&= \mathbb{E}[e^{i\langle u, L_{t+s}-L_s\rangle} e^{i\langle u, L_s\rangle}] \\
&= \mathbb{E}[e^{i\langle u, L_{t+s}-L_s\rangle}]\mathbb{E}[e^{i\langle u, L_s\rangle}] \\
&= \mathbb{E}[e^{i\langle u, L_t\rangle}]\mathbb{E}[e^{i\langle u, L_s\rangle}] \\
&= \Phi_{L_t}(u)\Phi_{L_s}(u)
\end{aligned}$$

The continuity follows, of course, from the last lemma.
Thus the claim applies for every fixed u and we have $\Phi_{L_t}(u) = e^{t\lambda(u)}$, where $\lambda(u)$ is the complex parameter that will depend on u. ∎

Remark 2.6 *So we have seen that the characteristic function of a Lévy process vanishes nowhere, since it can be written as an exponential. Moreover the exponent simply scales by t and hence the whole process is determined by a single Fourier transform. The function λ has a very special structure, which will be given by the Lévy-Khinchine formula 2.36 later on. We will call λ the Lévy symbol associated to L.*

Proof (of Lemma 2.3)

Obviously the first moment exists (this being the point of the construction). Furthermore, we have

$$\begin{aligned}\mathbb{E}\left[\frac{e^{i\langle u,L_t\rangle}}{\mathbb{E}[e^{i\langle u,L_t\rangle}]}|\mathcal{F}_s\right] &= \mathbb{E}\left[\frac{e^{i\langle u,L_t-L_s\rangle}e^{i\langle u,L_s\rangle}}{\mathbb{E}[e^{i\langle u,L_t-L_s\rangle}]\mathbb{E}[e^{i\langle u,L_s\rangle}]}|\mathcal{F}_s\right] \\ &= \frac{e^{i\langle u,L_s\rangle}}{\mathbb{E}[e^{i\langle u,L_s\rangle}]}\mathbb{E}\left[\frac{e^{i\langle u,L_t-L_s\rangle}}{\mathbb{E}[e^{i\langle u,L_t-L_s\rangle}]}|\mathcal{F}_s\right] \\ &= \frac{e^{i\langle u,L_s\rangle}}{\mathbb{E}[e^{i\langle u,L_s\rangle}]}\frac{\mathbb{E}[e^{i\langle u,L_t-L_s\rangle}]}{\mathbb{E}[e^{i\langle u,L_t-L_s\rangle}]} = M_u(s)\end{aligned}$$ ■

Lemma 2.7 *For a fixed $x \in H$ $\langle L(t), x\rangle$ is a real-valued Lévy process.*

Proof Note, that we can consider $\langle L(t), x\rangle$ as $F_x(L(t))$ and F_x is linear and continuous.
So let L be a Lévy-process. (L0) for $F_x(L)$ is trivial. For (L1) and (L2) use linearity of F_x and the obvious facts, that if X is independent of a σ-algebra so is $F(X)$ and that if $X \stackrel{d}{=} Y$, then$F(X) \stackrel{d}{=} F(Y)$ for any measurable F. For (L3) note, that F_x is uniformly continuos, since it is linear. So for a fixed $\varepsilon > 0$ there is $\delta > 0$ such that $\|z-y\| < \delta$ implies $|F_x(z)-F_x(y)| < \varepsilon$. Hence, (using $A \subset B \Rightarrow B^c \subset A^c$):
$P(|F_x(L(t)) - F_x(L(s))| > \varepsilon) \leq P(\|L(t) - L(s)\| > \delta) \stackrel{s\to t}{\longrightarrow} 0$ ■

As for Brownian motion the strong Markov property also holds for Lévy processes. Since we will often be interested in stopping times related to jump occurences, this result will be crucial for several further proofs.

Proposition 2.8 (Strong Markov property) *If L is a Lévy process and T is a stopping time, then, on $\{T < \infty\}$, we have for the process $\{L^T_t\}_{t\geq 0} := L_{T+t} - L_T$:*

1. *$\{L^T_t\}$ is a Lévy process that is independent of $\mathcal{F}_T$*

2. *$L^T_t \stackrel{d}{=} L_t$*

3. *L^T has càdlàg paths and is $\mathcal{F}_T$ -adapted*

PROOF We will give an idea of the proof:
Assume, at first, that T is a bounded stopping time, so we can apply optional stopping.
Let $A \in \mathcal{F}_T$ and recall the martingales $M_u(t) = e^{i\langle u, L_t\rangle} e^{-t\lambda(u)}$ from 2.3
Note that we have written the expectation in the form according to 2.5. For $s < t$ consider the following equalities:

$$\begin{aligned}
\mathbb{E}\left[e^{i\langle u, L_{T+t} - L_{T+s}\rangle}\right] &= \mathbb{E}\left[\frac{e^{i\langle u, L_{T+t}\rangle} e^{-(T+t)\lambda(u)}}{e^{i\langle u, L_{T+s}\rangle} e^{-(T+s)\lambda(u)}} \frac{e^{(T+t)\lambda(u)}}{e^{(T+s)\lambda(u)}}\right] \\
&= \mathbb{E}\left[\frac{M_u(T+t)}{M_u(T+s)} e^{(t-s)\lambda(u)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{M_u(T+t)}{M_u(T+s)} e^{(t-s)\lambda(u)} \mid \mathcal{F}_{T+s}\right]\right] \\
&= e^{(t-s)\lambda(u)} \mathbb{E}\left[\frac{1}{M_u(T+s)} \mathbb{E}\left[M_u(T+t) \mid \mathcal{F}_{T+s}\right]\right] \\
&= e^{(t-s)\lambda(u)} = \mathbb{E}\left[e^{i\langle u, L_{t-s}\rangle}\right]
\end{aligned}$$

Thus, setting $s = 0$, we see that $L_{T+t} - L_T$ and L_t have the same Fourier transforms and hence the same distributions and we have proven (2). For a proof of 1. and 3. see [App04] Theorem 2.2.11. ∎

2.1.1 The Path Structure of Lévy Processes

One of the big differences between Brownian motion and general Lévy processes is, that the latter do not admit continuous paths. We have, however, pointed out, that their paths are still right continuous and admit left limits. So the only discontinuities that can occur are of jump type. As we will aim to split a Lévy process into a continuous part and a jump part in the Lévy-Ito decomposition, it is worthwhile to take a closer look on the occurence of jumps. First, we will combine stochastic continuity with the càdlàg property to see, that jumps at a fixed time occur only with probability 0. Then, again by the càdlàg property, we will show that there are only finitely many jumps on bounded intervals.

Definition 2.9 *Let L_t be a Lévy process. Since we always have left limits, let $L_{t-} := \lim_{s \nearrow t} L_s$. We will call $\Delta L_t := L_t - L_{t-}$ the jump of L at time t. Accordingly, we will call the process $\{\Delta L_t\}_{t>0}$ the jump process of L.*

Lemma 2.10 *If L is a Lévy process we have $P(\Delta L_t \neq 0) = 0$ for any fixed t.*

Proof Let t_n be an increasing real sequence with limit t. By the càdlàg property $\lim_{n\to\infty} L_{t_n} = L_{t-}$ exists for every ω. On the other hand, by stochastic continuity, we obtain a subsequence of t_n that converges for almost all ω to L_t. So $L_{t-} = L_t$ by uniqueness of limits. ■

Remark 2.11 *One may be tempted to assume that ΔL is itself a Lévy process, but this is false in general. Let N_t be a Poisson process. Then, we have by the above : $P(\Delta N_t - \Delta N_s = 0) = 1$ since with probability 1 there will be no jumps. On the other hand (and for the same reason) we have $P(\Delta N_t - \Delta N_s = 0 | \Delta N_s = 1) = 0$. So increments are not independent.*

As we pointed out, we want to prove that there are only finitely many jumps on finite intervals. However, this is only true if we exclude arbitrary small jumps. Those might accumulate without contradicting the càdlàg property. Thus, the following definition is crucial:

Definition 2.12 *We say that $A \in \mathcal{B}(H)$ is bounded below if $0 \notin \bar{A}$*
We denote by $N(t, A)$ the (random) number of "jumps of size A" up to time t, that is $N(t, A) := card\{0 \leq s \leq t | \Delta L_s \in A\}$

Proposition 2.13 *If L has càdlàg paths and A is bounded below, then $N(t, A)$ is finite for every fixed t.*

Proof Assume $N(t, A) = \infty$, so that we have infinitely many jumps of size A in finite time, which have to accumulate, say at s.We can then find a sequence $s_n \subset \{0 \leq s \leq t | \Delta L_s \in A\}$ tending to s either from the left or from the right. As A is bounded below we can find $\varepsilon > 0$ such that $\|\Delta L_{s_n}\| > 2\,\varepsilon$. First assume that $s_n \searrow s$. By the càdlàg property there is $\delta > 0$ such that $\|L_s - L_r\| < \varepsilon \;\forall r \quad$ with $r - s < \delta$. Take n large enough to assure $\|L_s - L_{s_n}\| < \varepsilon$. Of course we also have $\|L_s - L_{s_n-}\| < \varepsilon$ then. By the triangle inequality this implies $\|L_{s_n} - L_{s_n-}\| < 2\,\varepsilon$ contradicting our requirement on A.
Note that, in the proof, the value of L_s is arbitrary, we need only that the left limit exists. So the proof for $s_n \nearrow s$ is the same, with L_s replaced by L_{s-}. ■

We stress once more that $N(t, A)$ depends on ω. It is easy to see that $N(t, A)$ is a random variable (it will become clear in the proof of the next proposition) and we can hence ask if we can find out something about its distribution and how it depends on the set A. Moreover, fixing only A (which must be bounded below, for technical reasons) and writing N_t^A for a more suggestive notation we want to examine the process $\{N_t^A\}_{t\geq 0}$. Note that $N(t, A)$ only takes values in $\mathbb{N}$ and that its jumps are always of size 1. Hence one might hope for a Poisson process, and to show that this is indeed true, we will first prove an auxiliary characterization result.

Proposition 2.14 *If L is a Lévy process that takes values in $\mathbb{N}$ only, is almost surely increasing and has only jumps of size 1, then L is a Poisson process.*

Proof The idea is to show that the waiting times between jumps are exponentially distributed, by using the characteristical functional equation of the exponential again (compare the proof of 2.5). Let be T_n the sequence of stopping times, defined by $T_n := \inf\{t > 0 \,|\, L_t = n\}$. T_n is a stopping time because of $\{T_n \leq t\} = \{L_t > 0\}$. From 2.8 (strong Markov property) we get that the random variables $T_1, T_2 - T_1, T_3 - T_2, ...$ are independent and identically distributed. On the other hand by stationarity and independence of the increments of L we have:

$$\begin{aligned} P(T_1 > s+t) &= P(L_s = 0, L_{s+t} - L_s = 0) \\ &= P(L_s = 0)\; P(L_{s+t} - L_s = 0) \\ &= P(L_s = 0)\; P(L_t = 0) \\ &= P(T_1 > s)\; P(T_1 > t) \end{aligned}$$

So $f(t) = P(L_t > 0)$ fulfills $f(t+s) = f(t)f(s)\ \forall\ t, s > 0$. Furthermore, we have:
$f(0) = P(T_1 > 0) = P(L_0 = 0) = 1$ and
$f(t) = P(L_t = 0) = 1 - P(L_t \geq 1) = 1 - P(|L_t - L_0| \geq 1) \to 1$ as $t \to 0$
because of stochastic continuity, so f is continuous in 0 and since
$f(t+s) - f(s) = f(s)(f(t) - 1)$ we have at least right continuity of f, which allows us to deduce that $f(t) = e^{\alpha t}$ for some $\alpha \in \mathbb{R}$. Moreover we have $f(t) < 1$ for some t, otherwise we would have $1 = P(T_1 > t) = P(L_t = 0)\, \forall\, t$ contradicting the assumption that L_t is increasing. So $-\beta := \alpha$ must be

negative. Having $P(T_1 > t) = e^{-\beta t}$, we get $P(T_1 \leq t) = 1 - e^{-\beta t}$ and differentiation yields the density $\rho_{T_1}(t) = \beta e^{-\beta t}$, thus T_1 is exponentially distributed. It is then well-known that L_t must be a Poisson process. ■

Proposition 2.15 *If A is bounded below, then N_t^A is a Poisson process.*

PROOF By the characterization above, we only have to show that N_t^A is a Lévy Process. We can assume that N_t^A is increasing, otherwise N_t^A will just be the zero process, which we will consider as a Poisson process with intensity 0. (L0) is clear, since by 2.10 with probability 1 we have no jump at 0. (L1) and (L2) follow directly from the respective properties of L_t. Stochastic continuity, on the other hand, is astonishingly difficult to prove. Since $N(t, A) = 0$ implies $N(s, A) = 0$ for all $s < t$ we have for $n \in \mathbb{N}$:

$$\begin{aligned}
P[N(t,A)=0] &= P\left[N\left(\frac{t}{n},A\right)=0, N\left(\frac{2t}{n},A\right)=0,\ldots,N(t,A)=0\right] \\
&= P\left[N\left(\frac{t}{n},A\right)=0, N\left(\frac{2t}{n},A\right)-N\left(\frac{t}{n},A\right)=0,\ldots\right. \\
&\qquad \left.\ldots, N(t,A)-N\left(\frac{(n-1)t}{n},A\right)=0\right] \\
&= \left(P\left[N\left(\frac{t}{n},A\right)=0\right]\right)^n \qquad \text{by independent increments}
\end{aligned}$$

Hence we have :

$$\begin{aligned}
\limsup_{t\to 0} P[N(t,A)=0] &= \limsup_{t\to 0}\left(P\left[N\left(\frac{t}{n},A\right)=0\right]\right)^n \\
&= \lim_{n\to\infty}\limsup_{t\to 0}\left(P\left[N\left(\frac{t}{n},A\right)=0\right]\right)^n \\
&= \lim_{n\to\infty}\left(\limsup_{t\to 0} P\left[N\left(\frac{t}{n},A\right)=0\right]\right)^n
\end{aligned}$$

but

$$\limsup_{t\to 0} P\left[N\left(\frac{t}{n},A\right)=0\right] = \limsup_{t\to 0} P\left[N\left(t,A\right)=0\right]$$

is independent of n, so it can be only 0 or 1. Since the same argument applies to the $\liminf$ as well, there are only three possibilities:

1. $0 = \liminf_{t\to 0} P\left[N\left(t,A\right)=0\right] \neq \limsup_{t\to 0} P\left[N\left(t,A\right)=0\right] = 1$

2. $\lim_{t\to 0} P\left[N\left(t,A\right)=0\right] = 0$

3. $\lim_{t\to 0} P\left[N\left(t,A\right)=0\right] = 1$

3. implies stochastic continuity, so we will show that the other two are impossible:
As N is an increasing process $P\left[N\left(t,A\right)=0\right] =: P_t$ is decreasing in t. Assume 1. especially $\limsup_{t\to\infty} P_t = 1$, so that we must have $P_t > \frac{1}{2}$ for some t. But since P_t is decreasing this implies $P_s > \frac{1}{2}\ \forall\ s < t$ so $\liminf_{t\to\infty} P_t = 0$ is impossible.
Assume 2. so that equivalently $\lim_{t\to 0} P\left[N\left(t,A\right)>0\right] = 1$ Choose another B which is bounded below and disjoint from A. Then $P\left[N\left(t,A\cup B\right)>0\right] = P\left[N\left(t,A\right)>0\right] + P\left[N\left(t,B\right)>0\right]$ so that 2. is impossible, too. ■

2.2 The Lévy-Ito Decomposition

The aim of this decomposition is to show that we can write every Lévy process as a sum of a drift term, a continuous Brownian part and a jump part. The proof of the Lévy-Ito Decomposition illustrates very well the technical difficulties posed by the jumps. First of all, a Lévy process need not have any moments and proposition 2.25 will show that this is only due to the big jumps. Subtracting those, we can centralize and the remaining jump part will be a martingale, although this is difficult to prove in the case where there are arbitrary small jumps. It remains then to show, that the rest term is continuous and a Brownian motion.

In order to find a convenient representation for the jump part we have to introduce Poisson integrals. In principle we want to give meaning to something like $\sum_{s\le t} f(\Delta L_s)$, but this sum might not be finite, even for bounded f because of an infite number of small jumps , so we have to come up with another approach in the general case. If we consider only jumps above a certain threshold specified by a set A,which is bounded below, we know that the sum above will be finite, as is $N(t,A)$ But then, of course, we will want to make this threshold arbitrarily small. So we need to investigate the influence of A on $N(t,A)$. First of all we note, that jumps of different size cannot influence each other:

Lemma 2.16 *If A and B are bounded below and we have $A \cap B = \emptyset$ then $N(t, A)$ and $N(t, B)$ are independent.*

PROOF see [App04] Theorem 2.3.5 ■

It is fairly obvious that for fixed ω and t $N_t(A)$ is additive for disjoint sets. We prove now that it is even a pre-measure on a properly chosen ring. Moreover, as $N_t(A)$ depends on ω we may take expectation, for t fixed and it turns out, that we still have a pre-measure.

Definition 2.17 *For A bounded below, set $\nu(A) = \mathbb{E}[N(1, A)]$.*
Set $\mathcal{R} := \{A \subset H | A$ is bounded below and Borel measurable$\}$

Lemma 2.18 *$\mathcal{R}$ is a ring in $H \backslash \{0\}$, and ν and $N(t, \cdot)$ are pre-measures on $\mathcal{R}$.*

PROOF To see that $\mathcal{R}$ is a ring in $H \backslash \{0\}$ we note that: because of $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ we have, that $0 \in \overline{A \cup B}$ implies $0 \in \bar{A}$ or $0 \in \bar{B}$ so that $A \cup B$ must still be bounded below, if A and B are. To see that $N(t, \cdot)$ is a pre-measure on $\mathcal{R}$, recall that we have to check σ-additivity only for unions that still belong to the ring. So let $A := \bigcup_n A_n$ where A and all the A_n are bounded below. Then we have:

$$N_t(\bigcup_n A_n) = \sum_{s \leq t} \chi_{\left(\bigcup_n A_n\right)}(\Delta X_s)$$

$$= \sum_{s \leq t} \sum_n \chi_{A_n}(\Delta X_s) = \sum_n \sum_{s \leq t} \chi_{A_n}(\Delta X_s) = \sum_n N_t(A_n)$$

Note that we could interchange sums as all terms are positive, but that if A was not bounded below, the second expression might not even make sense. To see that $\nu(A)$ is finite for A bounded below, we have to refer to proposition 2.25. As $N_t(A)$ is a Lévy process with bounded jumps (of size 1) the first moment must exist. The proof for ν then follows the same lines, we just have to interchange sum and expectation in the last step, which is justified by monotone convergence. ■

Proposition 2.19 *We can extend ν and $N(t, \cdot)$ uniquely to σ-finite measures on the σ-algebra $\mathcal{S} := \{A \subset H | A$ is Borel and $0 \notin A\}$ in $H \backslash \{0\}$*

PROOF To make use of Caratheodory we have to show, that ν and N_t are σ-finite on $\mathcal{R}$ and that $\mathcal{R}$ generates $\mathcal{S}$. But both follows easily on considering the sets $A_n := \{x \in H| \ \|x\| \geq \frac{1}{n}\}$ which are in $\mathcal{R}$. ■

Remark 2.20 *The measure ν is indeed the third member of the characterising triple, mentioned in the introduction. Heuristically, we can already understand that it contains all the information necessary to describe the jump structure. Imagine, we wanted to roughly simulate the jump process of a Lévy process. We would try to cover the space H with sufficiently small and disjoint sets. Then we would simulate a collection of independent (recall proposition 2.16) Poisson processes, one for each set, giving us the respective jump times. Each Poisson process is characterized by its intensity parameter and this is precisely provided by ν, as we state in the next simple lemma.*

Lemma 2.21 *If A is bounded below, $N_t(A) \stackrel{d}{=} \pi(t\nu(A))$, where $\pi(c)$ is the Poisson distribution with parameter c. Moreover the process $N_t(A) - t\nu(A)$ is a martingale.*

PROOF The intensity of a Poisson process P_t is given by $\mathbb{E}[P_1]$. Hence ν gives the intensity by construction. The second result follows by noting that any adapted independent-increment process that is centralized is a martingale. ■

The most important properties of $N_t(\cdot)$ are summarized in the next definition.

Definition 2.22 *Let $(\mathcal{S}, \mathcal{A})$ be a measurable space. A Poisson random measure is a collection of random variables $\{N(A)\}_{A \in \mathcal{A}}$ on a common probability space Ω such that*

- *for almost all $\omega \quad N(\cdot)$ is a measure on $(\mathcal{S}, \mathcal{A})$*
- *if $A_1, ..., A_n$ are mutually disjoint, the random variables $N(A_1), ..., N(A_n)$ are independent*
- *whenever $\mathbb{E}[N(A)] < \infty \quad N(A)$ has a Poisson distribution*

Remark 2.23 *As ν is somehow compensating the drift of N_t, we will call the random measure $N_t - t\nu$ the compensated Poisson random measure.*

As $N_t(\cdot)$ is a measure on $H \backslash \{0\}$ for ω fixed we may define, for $f : H \to H$ measurable $\int_A f(x) N_t(dx)$ as a random Bochner integral. For a short account on Bochner integrals see appendix A of [Spde07].
The following proposition explores the properties of this integral as a random variable in terms of its Fourier transform and its first two moments. In [App04] this is Theorem 2.3.8 , that we have adapted to the Hilbert space setting and where we made the proof of 2. and 3. rigorous.

Proposition 2.24 *Let A be bounded below and $\int_A \|f(x)\| \, \nu(dx) < \infty$, then:*

1.

$$\mathbb{E}\left[\exp\left(i\left\langle u, \int_A f(x) N_t(dx)\right\rangle\right)\right] = \exp\left(t \int_A \left(e^{i\langle u,x\rangle} - 1\right) \nu \circ f^{-1}(dx)\right)$$

2.

$$\mathbb{E}\left[\int_A f(x) N_t(dx)\right] = t \int_A f(x) \nu(dx)$$

3. moreover, if $\int_A \|f(x)\|^2 \, \nu(dx) < \infty$:

$$Var\left[\| \int_A f(x) N_t(dx) \|\right] = t \int_A \|f(x)\|^2 \nu(dx)$$

Proof 1. First, let f be a simple function, $f = \sum_{j=1}^n h_j \chi_{A_j}$ where $h_j \in H$ and the A_j are disjoint measurable subsets of A. Then by 2.16 $N(t, A_j)$

and $N(t, A_i)$ are independent and so we have:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\left\langle u, \int_A f(x)N_t(dx)\right\rangle\right)\right] \\
=&\mathbb{E}\left[\exp\left(i\left\langle u, \sum_{j=1}^n h_j N_t(A_j)\right\rangle\right)\right] \\
=&\mathbb{E}\left[\prod_{j=1}^n \exp\left(i\left\langle u, h_j N_t(A_j)\right\rangle\right)\right] \\
=&\prod_{j=1}^n \mathbb{E}\left[\exp\left(i\left\langle u, h_j N_t(A_j)\right\rangle\right)\right] \\
=&\prod_{j=1}^n \mathbb{E}\left[\exp\left(i\left\langle u, h_j\right\rangle N_t(A_j)\right)\right] \\
=&\prod_{j=1}^n \exp\left(t\left[e^{i\langle u, h_j\rangle} - 1\right]\nu(A_j)\right) \\
=&\exp\left(\sum_{j=1}^n t\left[e^{i\langle u, h_j\rangle} - 1\right]\nu(A_j)\right) \\
=&\exp\left(t\int_A \left[e^{i\langle u, f(x)\rangle} - 1\right]\nu(dx)\right)
\end{aligned}$$

So we have the result for simple functions. For every integrable f we can find a sequence of simple functions f_n, that converge pointwise to f, so we have:

$$\lim_{n\to\infty} \exp\left(t\int_A \left[e^{i\langle u, f_n(x)\rangle} - 1\right]\nu(dx)\right) = \exp\left(t\int_A \left[e^{i\langle u, f(x)\rangle} - 1\right]\nu(dx)\right)$$

by dominated convergence. On the other hand we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\left\langle u, \int_A f_n(x)N_t(dx)\right\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\left\langle u, \int_A f(x)N_t(dx)\right\rangle\right)\right]$$

as follows: If we can show that for $P-$almost all ω :

$$\lim_{n\to\infty}\int_A f_n(x)N_t(dx)(\omega)=\int_A f(x)N_t(dx)(\omega) \tag{2.1}$$

then dominated convergence gives again the result. But (2.1) will follow if:

$$\lim_{n\to\infty}\int_A \|f_n(x)-f(x)\|N_t(dx)(\omega)=0 \tag{2.2}$$

and we will show this by dominated convergence. By assumption we have:

$$\infty>t\int_A\|f(x)\|\nu(dx)=\int_A \mathbb{E}[\|f(x)\|N_t(dx)]=\mathbb{E}\left[\int_A\|f(x)\|N_t(dx)\right]$$

where we used Fubini-Tonelli.
Hence we must have $\int_A\|f(x)\|N_t(dx)<\infty$ almost surely, and since N_t is for any fixed ω a finite measure, we can take $2\|f(x)\|+\text{const}$ as a uniform bound in (2.2) for almost all ω.
Note that the technical problems arise from the fact, that the measure for the Bochner integral does depend on ω, but our sequence of simple functions must not.

2. First assume that f is bounded, that is $\sup_{x\in H}\|f(x)\|=M<\infty$.
For $\lambda\in\mathbb{R}$ we will consider the first identity for λf, then differentiate with respect to λ and set $\lambda=0$:

$$\frac{d}{d\lambda}\mathbb{E}\left[\exp\left(\lambda i\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right)\right]= \\ \frac{d}{d\lambda}\exp\left(t\int_A\left(e^{\lambda i\langle u,f(x)\rangle}-1\right)\nu(dx)\right)$$

Starting with the right hand side, we get by formally interchanging derivation and integration:

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=0}\exp\left(t\int_A\left(e^{\lambda i\langle u,f(x)\rangle}-1\right)\nu(dx)\right)&=t\int_A i\langle u,f(x)\rangle\nu(dx)\\&=i\langle u,t\int_A f(x)\nu(dx)\rangle\end{aligned}$$

where we have also used that the Bochner integral commutes with the scalar product, as it is linear and continuous for fixed u and because f is integrable. Derivation under the integral is justified because the derived integrand is uniformly integrable in λ:

$$\begin{aligned}\sup_\lambda \int_A \left|\left(e^{\lambda i\langle u,f(x)\rangle}-1\right) i\langle u,f(x)\rangle\right| \nu(dx) &\leq 2\int_A |\langle u,f(x)\rangle|\,\nu(dx)\\ &\leq 2\|u\|\int_A \|f(x)\|\nu(dx) < \infty\end{aligned}$$

For the left hand side we can interchange derivation and expectation likewise, since:

$$\begin{aligned}&\sup_\lambda \mathbb{E}\left|\exp\left(\lambda i\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right) i\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right|\\ &\leq \mathbb{E}\left|\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right| \leq \mathbb{E}\left[\|u\|M\int_A N_t(dx)\right]\\ &= \|u\|M\,\mathbb{E}[N_t(A)] = \|u\|Mt\nu(A) < \infty\end{aligned}$$

Thus we obtain:

$$\begin{aligned}&\left.\frac{d}{d\lambda}\right|_{\lambda=0}\mathbb{E}\left[\exp\left(\lambda i\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right)\right]\\ &= \mathbb{E}\left[i\left\langle u,\int_A f(x)N_t(dx)\right\rangle\right] = i\left\langle u,\mathbb{E}\left[\int_A f(x)N_t(dx)\right]\right\rangle\end{aligned}$$

So we have

$$i\left\langle u,\mathbb{E}\left[\int_A f(x)N_t(dx)\right]\right\rangle = i\langle u, t\int_A f(x)\nu(dx)\rangle$$

for arbitrary $u \in H$ and the result follows for bounded f. For merely integrable f we set $f_n := f\chi_{\{\|f\|\leq n\}}$ so that f_n is bounded and $f_n \nearrow f$ pointwise. To complete the proof, we have to show that:

$$\begin{aligned}\mathbb{E}\left[\int_A f(x)N_t(dx)\right] &= \mathbb{E}\left[\int_A \lim_{n\to\infty} f_n(x)N_t(dx)\right] \stackrel{!}{=} \lim_{n\to\infty}\mathbb{E}\left[\int_A f_n(x)N_t(dx)\right]\\ &= \lim_{n\to\infty} t\int_A f_n(x)\nu(dx) \stackrel{!}{=} t\int_A \lim_{n\to\infty} f_n(x)\nu(dx) = t\int_A f(x)\nu(dx)\end{aligned}$$

To see that we may interchange limit and integrals we show

$$\mathbb{E}\left[\int_A \|f_n(x) - f(x)\| N_t(dx)\right] \overset{n\to\infty}{\longrightarrow} 0$$

by dominated convergence. As $\|f_n\| \leq \|f\|$, $2\|f\|$ is an upper bound, and we have by Fatou:

$$\begin{aligned}
&\mathbb{E}\left[\int_A \|f(x)\| N_t(dx)\right] = \mathbb{E}\left[\int_A \lim_{n\to\infty} \|f_n(x)\| N_t(dx)\right] \\
&\leq \mathbb{E}\left[\liminf_{n\to\infty} \int_A \|f_n(x)\| N_t(dx)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_A \|f_n(x)\| N_t(dx)\right] \\
&= \liminf_{n\to\infty} t \int_A \|f_n(x)\| \nu(dx) = \int_A \|f(x)\| \nu(dx) < \infty
\end{aligned}$$

since $\|f\|$ is ν-integrable by assumption. This also implies:

$$\int_A \|f_n(x) - f(x)\| \nu(dx) \overset{n\to\infty}{\longrightarrow} 0$$

and thus the last interchange is also justified.

3. We make the same approach differentiating twice with respect to λ and setting $\lambda = 0$. Note that since $\nu(A) < \infty$ for A bounded below, the assumption from 2. is now valid as well. Unlike in 2. we directly consider a non-bounded f and we obtain along the same lines:

$$\begin{aligned}
&\left.\frac{d^2}{d\lambda^2}\right|_{\lambda=0} \exp\left(t \int_A \left(e^{\lambda i\langle u, f(x)\rangle} - 1\right) \nu(dx)\right) \\
&= \left(ti\langle u, \int_A f(x)\nu(dx)\rangle\right)^2 + \int_A (ti\langle u, f\rangle)^2 \nu(dx)
\end{aligned}$$

where we used the assumption to justify derivation under the integral. So we have seen that the characteristic function is twice differentiable in 0. Borrowing a trick from [Chung68] we will show now that this implies the existence of second moments for $X := \langle u, \int_A f(x) N_t(dx)\rangle$.

$$\begin{aligned}
&\mathbb{E}\left[|\langle u, X\rangle|^2\right] \\
&= 2\mathbb{E}\left[\lim_{h\to 0} \frac{1-\cos(h\langle u, X\rangle)}{h^2}\right] && \text{by Taylor expansion of the cosine} \\
&\leq \liminf_{h\to 0} \mathbb{E}\left[\frac{2-2\cos(h\langle u, X\rangle)}{h^2}\right] && \text{by Fatou} \\
&= \liminf_{h\to 0} \mathbb{E}\left[\frac{2-e^{ih\langle u,X\rangle}-e^{-ih\langle u,X\rangle}}{h^2}\right] && \text{by Euler's formula} \\
&= -\left.\frac{d^2}{d\lambda^2}\right|_{\lambda=0} \mathbb{E}[e^{\lambda i\langle u,X\rangle}] && \text{by central approximation of} \frac{d^2}{d\lambda^2} \\
&< \infty && \text{as we pointed out above}
\end{aligned}$$

Thus the next interchange is also in order:

$$\frac{d^2}{d\lambda^2}\mathbb{E}\left[\exp\left(\lambda i\left\langle u, \int_A f(x)N_t(dx)\right\rangle\right)\right] = \mathbb{E}\left[\left(i\left\langle u, \int_A f(x)N_t(dx)\right\rangle\right)^2\right]$$

As u is arbitrary we now take $u = e_n$ for an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and sum over n, using Parseval's identity:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(t\left\langle e_n, \int_A f(x)\nu(dx)\right\rangle\right)^2 \\
&\qquad = \sum_{n=1}^{\infty}\left(\left\langle e_n, \mathbb{E}\left[\int_A f(x)N_t(dx)\right]\right\rangle\right)^2 = \left\|\mathbb{E}\left[\int_A f(x)N_t(dx)\right]\right\|^2
\end{aligned}$$

where we have employed 2.

$$\sum_{n=1}^{\infty}\int_A t(\langle e_n, f\rangle)^2\nu(dx) = \int_A \sum_{n=1}^{\infty} t(\langle e_n, f\rangle)^2\nu(dx) = \int_A t\|f\|^2\nu(dx)$$

where we have interchanged summation and integration by monotone convergence.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{E}\left[\left(\left\langle e_n, \int_A f(x)N_t(dx)\right\rangle\right)^2\right] = \\
&\qquad \mathbb{E}\left[\sum_{n=1}^{\infty}\left(\left\langle e_n, \int_A f(x)N_t(dx)\right\rangle\right)^2\right] = \mathbb{E}\left[\left\|\int_A f(x)N_t(dx)\right\|^2\right]
\end{aligned}$$

where we have interchanged summation and expectation by Fubini-Tonelli. So we have:

$$\mathbb{E}\left[\left\|\int_A f(x)N_t(dx)\right\|^2\right] = \left\|\mathbb{E}\left[\int_A f(x)N_t(dx)\right]\right\|^2 + \int_A t\|f(x)\|^2\nu(dx)$$

and recalling that $\mathrm{Var}[\|X\|] = \mathbb{E}[\|X\|^2] - \|\mathbb{E}[X]\|^2$ we obtain 3. ■

Proposition 2.25 *If L is a Lévy process with bounded jumps, then it has moments of all orders.*

Proof Let J be the bound for the jumps of L. Define stopping times by : $T_1 := \inf\{t \geq 0 : \|L_t\| > J\}$ and $T_n := \inf\{t \geq T_{n-1} : \|L_t - L_{T_{n-1}}\| > J\}$. So we have $T_n - T_{n-1} = \inf\{t > 0 : \|L_{T_{n-1}+t} - L_{T_{n-1}}\| > J\}$.
By the strong Markov property 2.8 we see that $T_n - T_{n-1}$ has the same distribution as T_1, and that $T_n - T_{n-1}$ is independent of $\mathcal{F}_{T_{n-1}}$. So by iterated optional stopping:

$$\mathbb{E}[e^{-T_n}] = \mathbb{E}[e^{-T_1}e^{-(T_2-T_1)}\cdots e^{-(T_n-T_{n-1})}] = (\mathbb{E}[e^{-T_1}])^n = q^n$$

for some $0 \leq q < 1$ since $0 \leq e^{-T_1} \leq 1$ but $P(T_1 = 0) = 0$ since there are no jumps at 0 with probability 1. Furthermore, note that if we have $\|L_t\| > 2nJ$ this can only happen on $T_n < t$. This is clear, since the process can grow only by a maximum of $2J$ between two stopping times, as it is stopped when the difference exceeds J and a possible jump shortly before the critical point can win not more than another J by the assumption of bounded jumps. Hence, we get by the Markov inequality:

$$P(\|L_t\| \geq 2nJ) \leq P(T_n < t) = P(e^{-T_n} > e^{-t}) \leq \mathbb{E}[e^{-T_n}]e^t = q^n e^t$$

So the tail of $P_{\|L_t\|}$ is so thin, that we may use a rather rough estimate for the moments $m = 1, 2\ldots$

$$\int_{\mathbb{R}_+} y^m \, P_{\|L_t\|}(dy) = \sum_{k=0}^{\infty} \int_{k2J}^{(k+1)2J} y^m P_{\|L_t\|}(dy)$$

$$\leq \sum_{k=0}^{\infty}(k+1)2J \; P(\|L_t\| \geq k2J) \;\leq\; 2J\sum_{k=0}^{\infty}(k+1)e^t q^k < \infty$$

■

Given a general Lévy process, we define a new process with bounded jumps just by subtracting the big jumps. That this process is still a Lévy process is stated in the next lemma.

Definition 2.26 *Given a Lévy process L define $L^1(t) := L - \int_{\|x\|>1} xN(t,dx)$*

Lemma 2.27 *L_1 is a Lévy process.*

Proof see [App04] Theorem 2.4.8 ■

Since without large jumps, we can centralize, define:

Definition 2.28 $\tilde{L}^1 := L^1 - \mathbb{E}[L^1]$

Since we now have second moments we can use the powerful theory of square integrable martingales:

Theorem 2.29 *There is a decomposition : $\tilde{L}^1 = L^1_c + L^1_j$ such that L^1_c and L^1_j are independent Lévy processes. L^1_c has continuous sample paths and*

$$L^1_j = \int_{\|x\|\leq 1} x\,[N(t,dx) - t\nu(dx)] := L^2 - \lim_{n\to\infty} \int_{\frac{1}{n}<\|x\|<1} x\,[N(t,dx) - t\nu(dx)]$$

Proof see [App04] Theorem 2.4.11

Corollary 2.30 *For the measure ν we obtain: $\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) < \infty$*

Proof

$$\begin{aligned}\int_{\|x\|\leq 1} \|x\|^2\nu(dx) &= \lim_{n\to\infty} \int_{\frac{1}{n}<\|x\|\leq 1} \|x\|^2\nu(dx) \\ &= \lim_{n\to\infty} \mathrm{Var}\left[\int_{\frac{1}{n}<\|x\|\leq 1} x\,N(1,dx)\right] = \mathrm{Var}[L^1_j] < \infty \end{aligned}$$ ■

We already know that $\nu(\{\|x\| > 1\}) < \infty$ since it is the intensity of the respective Poisson process (note that $\{\|x\| > 1\}$ is bounded below) and hence finite. Thus, we have motivated the following definition:

Definition 2.31 *A measure M on H with :*

$$\int_{H\setminus\{0\}} \min(1, \|x\|^2) M(dx) < \infty$$

is called a Lévy measure.

Proposition 2.32 *A real valued, centred Lévy process B_t with continuous sample paths is a Brownian motion.*

PROOF We will show that $\mathbb{E}[e^{iuB_t}] = e^{-\frac{1}{2}t^2}$ for some $a \geq 0$. As the characteristic function uniquely determines a Lévy process, B_t then must be a Brownian motion.
Since B_t has no jumps, all moments exist by proposition 2.25, thus the characteristic function $\Phi_{B_t}(u) =: \Phi_t(u)$ is infinitely differentiable. By lemma 2.5 it has the form $\Phi_t(u) = e^{t\lambda(u)}$ for some $\lambda : \mathbb{R} \to \mathbb{C}$ which hence must be also smooth. Moreover we have $0 = i\mathbb{E}[B_t] = t\lambda'(0)$ and this easily implies

$$E[B_t^k] = a_1 t + a_2 t^2 + ... + a_{k-1} t^{k-1} \tag{2.3}$$

for the other moments by repeated differentiation, where the a_k are real constants. Note that we can already see: $\mathbb{E}[B_t^2] = a_1 t$ and thus, we must have $a_1 \geq 0$. Note also that since the quadratic variation of B_t is almost surely finite, cubic variation will vanish almost surely. Actually this is a good hint why the remainder in the following expansion should vanish.
Let $\mathcal{P}$ be a partition $\{0 = t_0 < t_1 < ... < t_n = t\}$ and write $\Delta B_j := B_{t_{j+1}} - B_{t_j}$. We employ Taylor expansion up to second order for the function $f(x) = e^{iux}$ to get:

$$\mathbb{E}[e^{iuB_t} - 1] = \mathbb{E}\left[\sum_{j=0}^{n-1} (e^{iuB_{j+1}} - e^{iuB_j})\right] =: \mathbb{E}[T_1 + T_2 + T_R]$$

where

$$T_1 = iu \sum_{j=0}^{n-1} e^{iuB_j} \Delta B_j$$

$$T_2 = -\frac{u^2}{2} \sum_{j=0}^{n-1} e^{iuB_j} (\Delta B_j)^2$$

$$T_R = -\frac{u^2}{2}\sum_{j=0}^{n-1}(e^{iuB_j+\Theta_j\Delta B_j} - e^{iuB_j})(\Delta B_j)^2$$

with $0 < \Theta_j < 1,\ j = 0...n-1$.

T_1 and T_2 yield easily by idependent increments:

$$\mathbb{E}[T_1] = iu\sum_{j=0}^{n-1}\mathbb{E}[e^{iuB_j}]\mathbb{E}[\Delta B_j] = 0$$

and

$$\mathbb{E}[T_2] = -\frac{u^2}{2}\sum_{j=0}^{n-1}\mathbb{E}[e^{iuB_j}]\mathbb{E}[(\Delta B_j)^2] = -a_1\frac{u^2}{2}\sum_{j=0}^{n-1}\Phi_{t_j}(u)(t_{j+1}-t_j)$$

where we have used (2.3) for $k = 2$. Now refine the partition $\mathcal{P}$, that is consider a sequence $\mathcal{P}_n$ of partitions with $\lim_{n\to\infty}\sup_{j_n}|t_{j_n+1} - t_{j_j}| = 0$. We write T_2^n for the term with respect to the n-th partition and we have

$$\lim_{n\to\infty}\mathbb{E}[T_2^n] = -a_1\frac{u^2}{2}\int_0^t \Phi_s(u)ds$$

So that if $\lim_{n\to\infty}\mathbb{E}[T_R^n] = 0$ we will have:

$$\Phi_t(u) - 1 = -a_1\frac{u^2}{2}\int_0^t \Phi_s(u)ds$$

which yields the result by solving the differential equation:

$$\frac{d}{dt}\Phi_t(u) = -a_1\frac{u^2}{2}\Phi_t(u)ds \qquad \Phi_0(u) = 1$$

To show $\lim_{n\to\infty}\mathbb{E}[T_R^n] = 0$ one could try to use the mean value theorem to get by $|e^{iu(x+y)} - e^{iux}| \le |uy|$:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[T_R^n] &\le \limsup_{n\to\infty}\frac{|u|^3}{2}\mathbb{E}\left[\sum_{j_n=0}^{n-1}|\Delta B_j|^3\right]\\ &\le \limsup_{n\to\infty}\frac{|u|^3}{2}\sup_{j_n,\omega}|\Delta B_{j_n}|\mathbb{E}\left[\sum_{j_n=0}^{n-1}|\Delta B_j|^2\right]\end{aligned}$$

Indeed $\mathbb{E}\left[\sum_{j_n=0}^{n-1}|\Delta B_j|^2\right] = a_1 t$ independent of n by (2.3). For $\sup_{j_n,\omega}|\Delta B_{j_n}|$ we get $\lim_{n\to\infty}\sup_{j_n}|\Delta B_{j_n}| = 0$ for ω fixed by uniform continuity. Unfortunately, the modulus of continuity is path-dependent, so the approach does not work that fast. Thus, we introduce the set of "good" paths:

$$G_\varepsilon^n := \{\omega \mid \sup_{j_n\in\mathcal{P}_n}|\Delta B_{j_n}| < \varepsilon\}$$

So splitting $\mathbb{E}[T_R^n] = \mathbb{E}[T_R^n\, 1_{G_\varepsilon^n}] + \mathbb{E}[T_R^n\, 1_{(G_\varepsilon^n)^c}]$ we obtain as above:

$$\mathbb{E}[T_R^n\, 1_{G_\varepsilon^n}] \leq \varepsilon a_1 t\frac{|u|^3}{2} \tag{2.4}$$

On the other hand, using $|e^{ix}| \leq 1$ for a different estimate of T_R we get:

$$\begin{aligned}
\mathbb{E}[T_R^n\, 1_{(G_\varepsilon^n)^c}] &\leq \mathbb{E}[1_{(G_\varepsilon^n)^c} u^2 \sum_{j_n=0}^{n-1}|\Delta B_j|^2] \\
&\leq u^2\sqrt{P(1_{(G_\varepsilon^n)^c})}\left(\mathbb{E}\left[\left(\sum_{j_n=0}^{n-1}|\Delta B_j|^2\right)^2\right]\right)^{\frac{1}{2}}
\end{aligned}$$

by Cauchy-Schwarz. But again by (2.3) it is easy to see that we can estimate the expectation of the squared sum independent of n. So if we can show $\lim_{n\to\infty}P((G_\varepsilon^n)^c) = 0$ for every fixed ε we are done, since by making ε arbitrarily small the right hand side in (2.4) will vanish as well.
But as we noted above, for every fixed ω, uniform continuity will assure that eventually $\omega \in G_\varepsilon^n$ for n large enough. Hence $1_{(G_\varepsilon^n)^c} \to 0$ for $n \to \infty$, so that $P(G_\varepsilon^n)^c) \to 0$ by dominated convergence. ■

Corollary 2.33 *L_c^1 is a Brownian motion.*

PROOF Since L_c^1 has continuous sample paths, so has $\langle L_c^1, h\rangle$ for any $h \in H$. Moreover by 2.7 $\langle L_c^1, h\rangle$ is a Lévy process. Hence $\langle L_c^1, h\rangle$ is a real-valued Brownian motion for any $h \in H$ and thus by Proposition 5.2.3 in [Linde83] L_c^1 is a Wiener process on H. ■

Theorem 2.34 (Lévy-Ito Decomposition) *If L is an H-valued Lévy process, there is a drift vector $b \in H$, a Q-Wiener process W_Q on H, such that W_Q is independent of $N_t(A)$ for any A that is bounded below and we have:*

$$L_t = bt + W_Q(t) + \int_{\|x\|<1} x(N_t(dx) - t\nu(dx)) + \int_{\|x\|\geq 1} xN_t(dx)$$

where N_t is the Poisson random measure associated to L, and ν the corresponding Lévy measure.

Proof Lemma 2.27 allows to separate the big jumps. Then we can centralize as in definition 2.28 with :

$$b = \mathbb{E}[L(1) - \int_{\|x\|<1} x\ N(1, dx)]$$

and theorem 2.29 allows to separate the small jumps. Then by corollary 2.33 the remainder is a Brownian motion. ∎

Remark 2.35 *If $\int_{\|x\|\geq 1} xN_t(dx)$ has first moment we can directly centralize and the form of b would change.*
However, in general the drift only contains the expectation of the small jumps and we will retrieve this asymmetry in the Lévy-Khinchine formula in the form of a cut-off function.

Theorem 2.36 (Lévy-Khinchine Representation) *If L is an H-valued Lévy process with Lévy-Ito decomposition as in 2.34, then its Lévy symbol takes the following form:*

$$\lambda(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Qu\rangle + \int_{H/\{0\}} \left[e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\chi_{\{\|x\|\leq 1\}}\right] \nu(dx) \quad (2.5)$$

Proof Since the four summands in the Lévy-Ito decomposition are independent we have:

$$\mathbb{E}\left[e^{i\langle u,L(1)\rangle}\right] = \mathbb{E}\left[e^{i\langle u,b\rangle}\right] \mathbb{E}\left[e^{i\langle u,W_Q(1)\rangle}\right] \\ \times \mathbb{E}\left[e^{i\langle u,\int_{\|x\|<1} x(N_1(dx)-\nu(dx))\rangle}\right] \mathbb{E}\left[e^{i\langle u,\int_{\|x\|>1} xN_1(dx)\rangle}\right]$$

By 2.24 we have:

$$\mathbb{E}[e^{i\langle u,\int_{\|x\|>1} xN_1(dx)\rangle}] = \exp\left(\int_{\|x\|>1} \left(e^{i\langle u,x\rangle} - 1\right) \nu(dx)\right) \quad (1)$$

and for A bounded below:

$$\mathbb{E}[e^{i\langle u,\int_A x(N_1(dx)-\nu(dx))\rangle}] = \exp\left(\int_A \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\right)\nu(dx)\right)$$

Hence:

$$\begin{aligned}
&\mathbb{E}[\exp\{i\langle u, \int_{\|x\|\le 1} x(N_1(dx)-\nu(dx))\rangle\}] \\
=&\mathbb{E}[\exp\{i\langle u, \lim_{n\to\infty}\int_{\frac{1}{n}<\|x\|\le 1} x(N_1(dx)-\nu(dx))\rangle\}] \\
=&\lim_{n\to\infty}\mathbb{E}[\exp\{i\langle u, \int_{\frac{1}{n}<\|x\|\le 1} x(N_1(dx)-\nu(dx))\rangle\}] \qquad \text{by} L^2 \text{convergence} \\
=&\lim_{n\to\infty}\exp\left(\int_{\frac{1}{n}<\|x\|\le 1}\left(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle\right)\nu(dx)\right) \\
=&\exp\left(\int_{\|x\|\le 1}\left(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle\right)\nu(dx)\right) \qquad (2)
\end{aligned}$$

Writing (1) and (2) under a single integral und recalling the Fourier tranform of Gaussian random variables, the result follows. ■

Definition 2.37 *Since a measure is characterized by its Fourier transform we will say that a measure μ is associated to a triple $[b, Q, \nu]$ if its characteristic exponent has the form (2.5).*

Remark 2.38 *To better memorise the formula recall the Taylor expansion of the exponential function. In the integral we basically subtract the first two terms of the expansion so that the remainder is of second order. Note how this relates to the fact that $\|y\|^2$ is locally ν-integrable.*
We also emphasize that the cut-off function may be replaced by other functions changing the form of the drift b e.g. $g(x) = \frac{1}{1+\|x\|^2}$. Note that g behaves like $\chi_{\{\|x\|\le 1\}}$ near the origin.

Remark 2.39 *Actually the Lévy-Khinchine representation holds not only for Lévy processes but for any infinitely divisible random variable. (See [Sato99] for an account of infinite divisibility) Moreover, Lévy processes and infinite divisible measures can be brought in a one to one correspondence. In particular the converse of 2.36 is true: any function of the form*

$$\exp\left\{i\langle b,u\rangle - \frac{1}{2}\langle u,Qu\rangle + \int_{H/\{0\}} \left[e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\chi_{\{\|x\|\le 1\}}\right]\nu(dx)\right\}$$

is the characteristic function of a measure.

Chapter 3

Generalised Ornstein-Uhlenbeck Processes

3.1 Stochastic Integration with respect to Lévy martingale measures

In this section we take advantage of the Lévy-Ito decomposition to define stochastic integrals with respect to Lévy processes. The only term posing any problems is the integral with respect to the compensated Poisson measure. Quite similarly to the case of Brownian motion, we make strong use of its martingale properties, but the situation is a little more difficult here. We have to introduce the notion of a martingale measure. To get the basic intuition it might be helpful to consider a Brownian motion as a degenerate martingale valued measure and to see how the theory applies to it.

We follow here the approach of [App06], which has been carried out in detail by [Stolze05] and [Knab06]. However, instead of taking the most general setting we adapt the following definition to our situation. Especially we fix the distinction between big and small jumps to be made at size 1, and hence let $\mathcal{R}_1$ be the ring of all Borel subsets of the unit ball of H which are bounded below. It can be immediately seen that this is a ring by reconsidering 2.18. Also define $S_n := \{x \in H | \frac{1}{n} \leq \|x\| \leq 1\}$ and note that $S_n \in \mathcal{R}_1$ for every n.

Definition 3.1 *A Lévy martingale measure on a Hilbert space H is a set function $M : \mathbb{R}_+ \times \mathcal{R}_1 \times \Omega \to H$ satisfying:*

- $M(0, A) = 0$ *almost surely for all* $A \in \mathcal{R}_1$
- $M(t, \emptyset) = 0$ *almost surely*
- *almost surely we have:*$M(t, A \cup B) = M(t, A) + M(t, B)$ *for all t and all disjoint A, $B \in \mathcal{R}_1$*
- $M(t, A)_{\{t \geq 0\}}$ *is a square-integrable martingale for each* $A \in \mathcal{R}_1$
- *if $A \cap B = \emptyset$ $M(t, A)_{\{t \geq 0\}}$ and $M(t, B)_{\{t \geq 0\}}$ are orthogonal, that is: $\langle M(t, A), M(t, B) \rangle$ is a real-valued martingale for every $A, B \in \mathcal{R}_1$*
- $\sup\{\mathbb{E}[\|M(t, A)\|^2] \mid A \in \mathcal{B}(S_n)\} < \infty \qquad$ *for every* $\quad n \in \mathbb{N}$
- *for every sequence A_j decreasing to the empty set such that $A_j \subset \mathcal{B}(S_n)$ for all j we have:* $\qquad \lim_{j \to \infty} \mathbb{E}[\|M(t, A_j)\|^2] < \infty$
- *for every $s < t$ and every $A \in \mathcal{R}_1$ we have that $M(t, A) - M(s, A)$ is independent of $\mathcal{F}_s$*

Proposition 3.2 *$M(t, A) = \int_A x \tilde{N}_t(dx)$ is a Lévy martingale measure on H for every $A \in \mathcal{R}_1$.*

Proof see [Stolze05] Theorem 2.5.2 ∎

Similarly as a Wiener process is characterized by its covariance operator, we can describe the covariance structure of a Lévy martingale measure by a family of operators parametrized by our ring $\mathcal{R}_1$.

Proposition 3.3

$$\mathbb{E}[|\langle M(t, A), v \rangle|^2] = t \langle v, T_A v \rangle$$

for all $t \geq 0$, $v \in H$ $A \in \mathcal{R}_1$, where the operators T_A are given by $T_A v := \int_A T_x v \nu(dx)$ and $T_x v := \langle x, v \rangle x$.

Proof see [Stolze05] Theorem 2.5.4 ∎

We will establish only a limited theory of integration, as for our purposes it will be sufficient to integrate deterministic operator valued functions. We do not even need them to depend on the jump size. The procedure is the same as for Brownian motion, so let us introduce the space of our integrands, the approximating simple functions, and state how the integral is defined for them. For convenience, we set $M([s,t],A) := M(t,A) - M(s,A)$.

Definition 3.4 *Let H' be another real separable Hilbert space. Let $\mathcal{H}^2 := \mathcal{H}^2(T_-,T_+)$ be the space of all $R : [T_-,T_+] \to \mathcal{L}(H,U)$ such that R is strongly measurable and we have:*

$$\|R\|_{\mathcal{H}^2} := \left(\int_{T_-}^{T_+} \int_{\|x\|<1} tr(R(t)T_x R^*(t))\nu(dx)ds\right)^{\frac{1}{2}} < \infty$$

Let $\mathcal{S}$ be the space of all $R \in \mathcal{H}^2$ such that

$$R = \sum_{i=0}^{n} R_i \; \chi_{(t_i,t_{i+1}]}\chi_A$$

where $T_- = t_0 < t_1 < ... < t_{n+1} = T_+$ for some $n \in \mathbb{N}$, where each $R_i \in \mathcal{L}(H,H')$ and where $A \in \mathcal{R}$

For each $R \in \mathcal{S}$, $\quad t \in [T_+,T_-]$ define the stochastic integral as follows:

$$I_t(R) := \sum_{i=0}^{n} R_i M([t_i \wedge t, t_{i+1} \wedge t], A)$$

Proposition 3.5 *The space $\mathcal{H}^2$ with inner product*

$$\langle R, U\rangle := \int_{T_-}^{T_+} \int_{\|x\|<1} tr(R(t)T_x U^*(t))\nu(dx)ds$$

is a Hilbert space.

Proof see [Knab06] Lemma 1.2 ∎

Proposition 3.6 *The space $\mathcal{S}$ is dense in $\mathcal{H}^2$.*

PROOF see [Knab06] Lemma 1.3 ∎

Proposition 3.7 *We have for any $R \in \mathcal{S} : \mathbb{E}[I_t(R)] = 0$ and*

$$\mathbb{E}[\|I_t(R)\|^2] = \int_{T_-}^{t} \int_A tr(R(s)T_x R^*(s))\nu(dx)ds = \|\chi_{[T_-,t]}R(t)\|^2_{\mathcal{H}^2}$$

So for t fixed, $I_t : \mathcal{S} \to L^2(\Omega, \mathcal{F}, P; H)$ is an isometry.

PROOF Let $\{e_k\}_{K\in\mathbb{N}}$be an orthonormal basis of H'.

$$\begin{aligned}
\mathbb{E}[I_t(R)] &= \sum_{i=0}^{n} \mathbb{E}[R_i M([t_i \wedge t, t_{i+1} \wedge t], A)] \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{n} \langle \mathbb{E}[R_i M([t_i \wedge t, t_{i+1} \wedge t], A)], e_k \rangle \, e_k \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{n} \mathbb{E}[\langle R_i M([t_i \wedge t, t_{i+1} \wedge t], A), e_k \rangle] e_k \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{n} \mathbb{E}[\langle M([t_i \wedge t, t_{i+1} \wedge t], A), R_i^* e_k \rangle] e_k = 0
\end{aligned}$$

since $\langle M(t, A), v\rangle$ is a real-valued martingale for any $v \in H$.

For the second part, we first show that :

$$\begin{aligned}
\mathbb{E}[\|I_t(R)\|^2] &= \mathbb{E}[\| \sum_{i=0}^{n} R_i M([t_i \wedge t, t_{i+1} \wedge t], A)\|^2] \\
&= \sum_{i=0}^{n} \mathbb{E}[\|R_i M([t_i \wedge t, t_{i+1} \wedge t], A)\|^2]
\end{aligned}$$

since for $i < j$:

$$\begin{aligned}
&\mathbb{E}[\langle R_i M([t_i \wedge t, t_{i+1} \wedge t], A), R_j M([t_j \wedge t, t_{j+1} \wedge t], A)\rangle] \\
&= \mathbb{E}[\mathbb{E}[\langle R_j^* R_i M([t_i \wedge t, t_{i+1} \wedge t], A), M([t_j \wedge t, t_{j+1} \wedge t], A)\rangle | \mathcal{F}_{t_i}]] \\
&= \mathbb{E}[\langle R_j^* R_i M([t_i \wedge t, t_{i+1} \wedge t], A), \mathbb{E}[M([t_j \wedge t, t_{j+1} \wedge t], A) | \mathcal{F}_{t_i}]\rangle] = 0
\end{aligned}$$

because of independent increments, and where we could introduce the expectation into the scalar product because of $\mathbb{E}[\langle X, Y\rangle|\mathcal{F}] = \langle X, \mathbb{E}[Y|\mathcal{F}]\rangle$ whenever Y is $\mathcal{F}$ measurable.
For $i = j$ we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\|R_i M([t_i \wedge t, t_{i+1} \wedge t], A)\|^2\right] \\
=&\mathbb{E}\left[\sum_{k=0}^{\infty} |\langle R_i M([t_i \wedge t, t_{i+1} \wedge t], A), e_k\rangle|^2\right] && \text{by Parseval's identity} \\
=&\sum_{k=0}^{\infty} \mathbb{E}\left[|\langle M([t_i \wedge t, t_{i+1} \wedge t], A), R_i^* e_k\rangle|^2\right] && \text{by Fubini-Tonelli} \\
=&\sum_{k=0}^{\infty} \langle R_i^* e_k, T_A R_i^* e_k\rangle (t_{i+1} \wedge t - t_i \wedge t) && \text{by proposition 3.3} \\
=&\sum_{k=0}^{\infty} \langle R_i^* e_k, \int_A T_x R_i^* e_k \nu(dx)\rangle (t_{i+1} \wedge t - t_i \wedge t) && \text{by definition of } T_A \\
=&\sum_{k=0}^{\infty} \int_A \langle R_i^* e_k, T_x R_i^* e_k\rangle \nu(dx) (t_{i+1} \wedge t - t_i \wedge t) && \text{as } \langle R_i^* e_k, \cdot\rangle \text{ is continuous} \\
=&\int_A \sum_{k=0}^{\infty} \langle R_i^* e_k, T_x R_i^* e_k\rangle \nu(dx) (t_{i+1} \wedge t - t_i \wedge t) && \text{by Fubini-Tonelli} \\
=&\int_A \operatorname{tr}(R_i T_x R_i^*) \nu(dx) (t_{i+1} \wedge t - t_i \wedge t)
\end{aligned}$$

and the application of Fubini-Tonelli is justified, since:

$$\langle R_i^* e_k, T_x R_i^* e_k\rangle = \langle R_i^* e_k, \langle x, R_i^* e_k\rangle x\rangle = |\langle x, R_i^* e_k\rangle|^2 \geq 0$$

Now, the assertion follows by taking the sum over i. ■

So we can isometrically extend the operator I_t from $\mathcal{S}$ to its closure $\mathcal{H}^2$.

3.2 Stochastic Convolution

We want to give meaning to the integral

$$X_{U,B} := \int_s^t U(t,r)B(r)dL(r)$$

which we will call a stochastic convolution. Here L is a H-valued Lévy process and we have $U(t,r) \in \mathcal{L}(H), B(r) \in \mathcal{L}(H)\ \forall\ s \leq r \leq t$.

Remark 3.8

In anticipation of the assumptions in chapter 4 we will pose the following conditions:

- $\sup_{r\in\mathbb{R}} \|B(r)\|_{\mathcal{L}(H)} < \infty$
- there is $M > 0, \omega > 0$ such that : $\|U(t,r)\|_{\mathcal{L}(H)} < Me^{-\omega(t-r)}$
- $r \mapsto B(r)$ is measurable and $r \mapsto U(t,r)$ is measurable for any fixed t

Under these conditions, which are by no means the most general, we can define the stochastic convolution:

Proposition 3.9 *$\int_s^t U(t,r)B(r)dL(r)$ exists, if U and B are as above.*

Proof We write, according to the Lévy-Ito decomposition 2.34:

$$\begin{aligned}\int_s^t U(t,r)B(r)dL(r) = & \int_s^t U(t,r)B(r)b\ dr \\ & + \int_s^t \int_{\|x\|\geq 1} U(t,r)B(r)x\ N_r(dx) \\ & + \int_s^t U(t,r)B(r)dW_Q(r) \\ & + \int_s^t \int_{\|x\|<1} U(t,r)B(r)x\ \tilde{N}_r(dx)\end{aligned} \tag{3.1}$$

The first term in 3.1 is a simple Bochner integral, and by the assumptions on U and B it is obviously finite. The second term is well defined as a finite random sum. The third term is defined as in [Spde07], we just have to make sure that the integrand belongs to the space of integrable processes, that is we have to check if:

$$\int_s^t \|U(t,r)B(r)Q^{\frac{1}{2}}\|_{L^2}^2 dr < \infty$$

where $\|\cdot\|_{L^2}$ is the Hilbert-Schmidt norm.(see e.g. [Spde07]) Since $L^2(H)$, the space of Hilbert-Schmidt operators, is an $\mathcal{L}(H)$-ideal, such that for $A \in \mathcal{L}(H)$ and $C \in L^2(H)$ we have $\|AC\|_{L^2} \leq \|A\|\|C\|_{L^2}$ and we have $\|Q^{\frac{1}{2}}\|_{L^2}^2 < \infty$ it suffices to see that

$$\int_s^t \|U(t,r)B(r)\|^2 dr < \infty$$

and this is clear because U and B are uniformly bounded by assumption. The last term in 3.1 is defined according to the theory of integration against martingale valued measures, established above. We have to check if the integrand is in $\mathcal{H}^2$ that is we have to show:

$$\int_s^t \int_{\|x\|\leq 1} \|U(t,r)B(r)T_x^{\frac{1}{2}}\|_{L^2}^2 dr < \infty$$

But this follows as above, since we have:

$$\int_s^t \int_{\|x\|\leq 1} \|U(t,r)B(r)T_x^{\frac{1}{2}}\|_{L^2}^2 \nu(dx)dr$$
$$\leq \int_s^t \|U(t,r)B(r)\|^2 dr \int_{\|x\|\leq 1} \|T_x^{\frac{1}{2}}\|_{L^2}^2 \nu(dx)$$

where the left integral is finite and for the right integral we calculate:

$$\|T_x^{\frac{1}{2}}\|_{L_2(G)}^2 = tr(T_x) = \sum_{n\in\mathbb{N}} (T_x e_n, e_n) = \sum_{n\in\mathbb{N}} ((x,e_n)x, e_n) = \sum_{n\in\mathbb{N}} (x,e_n)^2 = \|x\|_G^2$$

where (e_n), $n \in \mathbb{N}$, is an orthonormal basis of H.
Since ν is a Lévy measure, we have $\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) < \infty$ and the claim is proved. ∎

3.3 Existence of the Mild Solution

In the following we will have to deal with a non-autonomous abstract Cauchy problem - non-autonomous means we are not in the framework of strongly continuous semigroups anymore. This implies in particular, that we have no easy characterization of well-posedness in the sense of the Hille-Yosida theorem available. There are different, yet technical, approaches (see [Nei/Zag07] for a recent overview), but since this subject is not in the primary interest of our thesis, we content ourselves with assuming that the problem is well posed. This is closely related to the notion of *evolution semigroups*. Our definition is taken from [Chi/Lat99]

We consider the following non-autonomous generalisation of the Langevin equation:

$$\begin{cases} dX_t &= (A(t)X_t + f(t))dt + B(t)dL_t \\ X(s) &= x \end{cases} \tag{3.2}$$

where $B : \mathbb{R} \to \mathcal{L}(H)$ is strongly continuous and bounded in operator norm, $f : \mathbb{R} \to H$ is continuous, $L(t)$ is an H-valued Levy-process and where the $A(t)$ are linear operators on H with common domain $D(A)$ and $A : \mathbb{R} \times D(A) \to H$ is such that we can solve the associated non-autonomous abstract Cauchy problem

$$\begin{cases} dX_t &= (A(t)X_t + f(t))dt \\ X(s) &= x \end{cases} \tag{3.3}$$

according to the following definitions:

Definition 3.10 *An exponentially bounded evolution family on H is a two parameter family $\{U(t,s)\}_{t \geq s}$ of bounded linear operators on H such that we have:*

(i) $U(s,s) = Id$ *and* $U(t,s)U(s,r) = U(t,r)$ *whenever* $r \leq s \leq t$

(ii) for each $x \in H$, $(t,s) \mapsto U(t,s)x$ *is continuous on* $s \leq t$

(iii) there is $M > 0$ *and* $\omega > 0$ *such that :* $\|U(t,s)\| \leq Me^{-\omega(t-s)}$, $s \leq t$

Assumption 3.11 *There is a unique solution to* (3.3) *given by an exponentially bounded evolution family $U(t,s)$ so that the solution takes the form:*

$$X_t = U(t,s)x + \int_s^t U(t,r)f(r)dr$$

Moreover, we assume that :

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x$$

Remark 3.12 *Note that in the finite dimensional case, where each A_t is automatically bounded we get the existence of an evolution family that solves* (3.3)*, under the reasonable assumption that $t \mapsto A_t$ is continuous and bounded in the operator norm, by solving the following matrix-valued ODE:*

$$\begin{cases} \frac{\partial}{\partial t}U(t,s) = A(t)U(t,s) \\ U(s,s) = Id \end{cases}$$

Existence and uniqueness are assured since $(t, M) \mapsto A(t)M$ is globally Lipschitz in M. This result even holds in infinite dimensions, see [Dal/Krei74].

Definition 3.13 *Given assumption 3.11 we call the process:*

$$X(t,s,x) = U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)dL_r$$

a mild solution for (3.2).

3.4 Existence of the Weak Solution

We have called the above expression a mild solution, though there is no obvious relation to the equation yet. Now, we will show that our candidate solution actually solves our equation in a weak sense. The following definition makes this precise, but first we need to strengthen our assumption concerning the common domain of the $A(t)$ a little:

Assumption 3.14 *We require that the adjoint operators $A^*(t)$ also have a common domain independent of t which we will denote by $D(A^*)$. Furthermore, we assume that $D(A^*)$ is dense in H and that we have:*

$$\frac{d}{dt}U^*(t,s)y = U^*(t,s)A^*_t y$$

for every $y \in D(A^*)$.

Definition 3.15 *An H-valued process X_t is called a weak solution for* (3.2) *if for every $y \in D(A^*)$ we have:*

$$\langle X_t, y\rangle = \langle x, y\rangle + \int_s^t \langle X_r, A^*_r y\rangle dr + \int_s^t \langle f(r), y\rangle dr + \int_s^t B^*(r)y dL_r \qquad (3.4)$$

Here $(B^*(r)y)(h) := \langle B^*(r)y, h\rangle$ so that $B^*(r)y \in \mathcal{L}(H, \mathbb{R})$ and the integral is well defined, since
$\|B^*y\|^2_{\mathcal{H}^2} \leq (t-s)\sup_r \|B(r)\|^2\|y\|^2 \sum_k \int_{\|x\|<1} \|T_x^{\frac{1}{2}} e_k\|^2 \nu(dx) < \infty$.

Theorem 3.16 *The mild solution X_t from definition 3.13 is also a weak solution for* (3.2).

PROOF By the expression for X_t (that already contains an integral) we will have to establish the following equality:

$$\left\langle U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)dL_r, y \right\rangle$$
$$= \langle x,y\rangle + \int_s^t \left\langle U(r,s)x + \int_s^r U(r,u)f(u)du + \int_s^r U(r,u)B(u)dL_u, A_r^* y \right\rangle dr$$
$$+ \int_s^t \langle f(u), y\rangle du + \int_s^t B^*(u) y dL_u \quad (3.5)$$

Therefore, we calculate :

$$\int_s^t \langle U(r,s)x, A_r^* y\rangle dr = \int_s^t \langle x, U^*(r,s)A_r^* y\rangle dr = \langle x, \int_s^t U^*(r,s)A_r^* y dr\rangle$$
$$= \left\langle x, \int_s^t \frac{d}{dr} U^*(r,s) y dr \right\rangle = \langle x, [U^*(t,s)U^*(s,s)]y\rangle = \langle U(t,s)x - x, y\rangle$$

and furthermore:

$$\int_s^t \left\langle \int_s^r U(r,u)f(u)du, A_r^* y \right\rangle dr = \int_s^t \int_s^r \langle f(u), U^*(r,u)A_r^* y\rangle \, dudr$$
$$= \int_s^t \int_u^t \langle f(u), U^*(r,u)A_r^* y\rangle \, drdu = \int_s^t \int_u^t \left\langle f(u), \frac{d}{dr} U^*(r,u) y \right\rangle drdu$$
$$= \int_s^t \langle f(u), [U^*(t,u) - Id]y\rangle \, du = \int_s^t \langle U(t,u)f(u), y\rangle \, du - \int_s^t \langle f(u), y\rangle \, du$$

Hence, (3.5) reduces to:

$$\left\langle \int_s^t U(t,r)B(r)dL_r, y \right\rangle =$$
$$\int_s^t \left\langle \int_s^r U(r,u)B(u)dL_u, A_r^* y \right\rangle dr + \int_s^t B^*(u) y dL_u \quad (3.6)$$

and we will prove this equality by carefully transforming the double integral with the help of a stochastic Fubini theorem. We also need to interchange scalar product and the stochastic integral in some respect, requiring a lemma introduced in [App06]. As the Hilbert space valued integral there was called strong stochastic integral and the real valued version was called weak stochastic integral, the ability to interchange scalar product and integral was refered to as *weak-strong-compatibility.* We shortly cite both results giving only the idea of the proof.

Proposition 3.17 (stochastic Fubini) *Let be $(M, \mathcal{M}, \mu)$ a measure space with μ finite. By $G^2(M)$ denote the space of all $\mathcal{L}(H, H')$- valued mappings R on $[s,t] \times M$ such that $(r, m) \mapsto R(r, m)y$ is measurable for each $y \in H$ and $\|R\|^2_{G^2(M)} := \int_s^t \int_M \|R(r,m)T_x^{\frac{1}{2}}\|^2 \nu(dx)\mu(dm)dr < \infty$ Then we have:*

$$\int_M \left(\int_s^t R(u,m) dL_u \right) \mu(dm) = \int_s^t \left(\int_M R(u,m)\mu(dm) \right) dL_u$$

Proof see [Stolze05] Theorem 3.3.4
The idea is simply (after verifying that both sides make sense) to check the equality on simple functions dense in $G^2(M)$ and then to extend it to the whole space. ■

Lemma 3.18 (weak-strong-compatibility) *Let be $R \in \mathcal{H}^2$ and $y \in H$. Then we have:*

$$\left\langle \int_s^t R(r) dL_r, y \right\rangle = \int_s^t R^*(r) y dL_r$$

Proof see [Stolze05] Again, the idea is to assure that both sides are well defined, then to check the equality on simple functions and to extend it to all of $\mathcal{H}^2$. ■

Remark 3.19 *In [Stolze05] the last two results are formulated only for the integral with respect to the martingale measure. Of course the general results involve nothing more than the stochastic Fubini theorem for Wiener integrals, the ordinary Fubini theorem and, for the weak-strong compatibility, the ability to interchange bounded linear operators with Bochner integrals.*

Now we are able to finish our proof of 3.16:

$$
\begin{aligned}
&\int_s^t \left\langle \int_s^r U(r,u)B(u)dL_u, A_r^* y \right\rangle dr \\
&= \int_s^t \left(\int_s^r B^*(u)U^*(r,u)A_r^* y \; dL_u \right) dr && \text{by weak-strong-compatibility} \\
&= \int_s^t \left(\int_u^t B^*(u)U^*(r,u)A_r^* y \; dr \right) dL_u && \text{by stochastic Fubini} \\
&= \int_s^t \left(B^*(u) \int_u^t \frac{d}{dr} U^*(r,u) y \; dr \right) dL_u \\
&= \int_s^t B^*(u)[U^*(t,u) - Id] y \; dL_u \\
&= \left\langle \int_s^t [U(t,u) - Id] B(u) dL_u, y \right\rangle && \text{by weak-strong-compatibility} \\
&= \left\langle \int_s^t U(t,u)B(u)dL_u, y \right\rangle \\
&\qquad - \left\langle \int_s^t B(u)dL_u, y \right\rangle
\end{aligned}
$$

and that is precisely what we had to show. ∎

Chapter 4

Semigroup and Invariant Measure

Now that we have solved our equation, we turn our interest to the associated semigroup. Since the coefficients are time-dependent, this semigroup will depend on two parameters. This leads us to a generalization of invariant measures for two-parameter semigroups - an evolution system of measures, defined in section 4.2.

The proof of our first main theorem - concerning existence and uniqueness of such an evolution system of measures - is already quite demanding, so we have collected some necessary prerequisites in section 4.1. Here we start by deriving the characteristic function of our solution, using our results on stochastic convolution from the preceding chapter. The Fourier transform of the solution will be a valuable tool throughout the rest of this thesis. As another helpful lemma, we provide a slightly extended monotone class theorem. With its help, we establish the required flow property to assure that we are dealing indeed with a semigroup.

Then in section 4.3 we investigate the respective autonomous equation, obtained by artificially enlarging the state space. The reason for this lies in our interest in generators of semigroups. Alas, for two-parameter semigroups, no sensible concept for generators is known, thus we have to introduce a one-parameter semigroup that incorporates all the information of the two-parameter semigroup.

Having done so, we are able to establish existence and uniqueness of an invariant measure for this new semigroup, where the invariant measure is basically given by $dt \otimes \nu_t$ where ν_t is the evolution system of measures from section 4.2. In order to obtain a probability measure, it is crucial to have T-periodicity of the ν_t, so that instead of dt we may consider dt restricted to $[0, T]$ which we can normalize. Moreover, we prove that the semigroup on the respective L^2-space is a contraction.

Next we turn to the infinitesimal generator of our semigroup. Our first result states that the semigroup is indeed strongly continuous on the L^2-space, that is, it fulfills a condition assuring the existence of a generator on a dense subspace. Next we identify a dense subspace of test functions that contains all the information about the generator. In semigroup theory, such a subspace is called a core. On this core, the generator is seen to be the sum of a differential operator and a nonlocal operator given by a superposition of difference operators. Moreover, we obtain some information on the spectrum of the generator.

In section 4.4 we prove two functional inequalities related to our semigroup. We will phrase the results both in terms of the one-parameter and of the two-parameter semigroup. A central tool - the so-called square field operator - is introduced in subsection 4.4.1. We calculate its precise form and establish an important inequality, relating the square field operator and the semigroup. Then we deduce a Poincaré and a Harnack inequality.

Beware of a change in notation, in this chapter the Lévy triple associated to L will be denoted by $[b, R, m]$. Moreover, as a technical assumption that will be necessary for proposition 4.19, we will require the coefficients in (3.2) to be T-periodic for some $T > 0$.

Assumption 4.1 *From now on, we assume that there exists $T>0$ such that the coefficients A, f and B in* (3.2) *are T-periodic.*

4.1 Preliminaries

Recall that the weak solution for (3.2) takes the following form:

$$X(t, s, x) = U(t, s)x + \int_s^t U(t, r)f(r)dr + \int_s^t U(t, r)B(r)dL_r$$

As opposed to the Gaussian case we are no longer able to give an easy representation of the law of $X(t, s, x)$, but we can calculate its Fourier transform:

Lemma 4.2 (characteristic function)

$$\mathbb{E}\left[\exp\left(i\left\langle h, X(t,s,x)\right\rangle\right)\right] =$$
$$\exp\left\{i\left\langle h, U(t,s)x + \int_s^t U(t,r)f(r)dr\right\rangle\right\}\exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\}$$

where λ is the Lévy symbol of L.

Proof : Knowing how the Fourier transform acts on translations, it will be enough to show, that:

$$\mathbb{E}\left[\exp\left(i\left\langle h, \int_s^t U(t,r)B(r)dL_r\right\rangle\right)\right] = \exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\}$$

Using the results from the last chapter, the continuity of U and B allows us to approximate the Lévy stochstic integral by a sequence of sums. More precisely, we want to prove the **claim**:

$$\int_s^t U(t,r)B(r)dL_r = P - \lim_{n\to\infty}\sum_{s_i\in\mathcal{P}_n} U(t,s_i)B(s_i)(L_{s_i} - L_{s_{(i-1)\vee 0}})$$

where the limit is taken in probability and $\mathcal{P}_n$ is a sequence of partitions $s = s_0 < ... < s_N = t$ of $[s,t]$ such that the mesh width tends to zero. As the Lévy stochastic integral is composed of four different terms (see (3.1)) we will show this equality separately for each of them.

We will need $r \mapsto U(t,r)\circ B(r)$ to be strongly continuous and this is the case, since the composition of strongly continuous bounded operators is again strongly continuous:

$$\begin{aligned}
&\|U_t(r)B(r)x - U_t(s)B(s)x\| \\
&\leq \|U_t(r)B(r)x - U_t(s)B(r)x\| + \|U_t(s)B(r)x - U_t(s)B(s)x\| \\
&\leq \|[U_t(r) - U_t(s)]B(r)x\| + \|U_t(s)\|_{\mathcal{L}(H)}\|B(r) - B(s)x\| \xrightarrow{s\to t} 0
\end{aligned}$$

where both terms tend to zero because of strong continuity.

For the drift term which is a Bochner integral we have to show that:

$$\lim_{n\to\infty}\sum_{s_i\in\mathcal{P}_n}\int_{s_{i-1}}^{s_i}\|U(t,s_i)B(s_i)b - U(t,r)B(r)b\|\,dr = 0$$

but since $r \mapsto U_t(r)B(r)b$ is even uniformly continuous on $[s,t]$ we may find $\delta > 0$ such that $\|U_t(r)B(r)b - U_t(r')B(r')b\| < \frac{\varepsilon}{t-s}$ whenever $|r-r'| < \delta$, so that if we choose n such that the mesh width of $\mathcal{P}_n$ is smaller than δ we have

$$\sum_{s_i \in \mathcal{P}_n} \int_{s_{i-1}}^{s_i} \|U(t,s_i)B(s_i)b - U(t,r)B(r)b\| \, dr < \sum_{s_i \in \mathcal{P}_n} \int_{s_{i-1}}^{s_i} \frac{\varepsilon}{t-s} dr < \varepsilon$$

For the small jumps we make use of the isometry from 3.7, so we have to show that our piecewise approximation converges in the $\mathcal{H}^2$ norm, that is we need:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \int_{\|x\|<1} \left(\sum_{s_i \in \mathcal{P}_n} \int_{s_{i-1}}^{s_i} \|[U_t(r)B(r) - U_t(s_i)B(s_i)]T_x^{\frac{1}{2}} e_k\|^2 dr \right) M(dx) = 0$$

For each k and x fixed the expression in round brackets converges to zero, for the same reasons as used for the drift term. So we only have to show that we may take the limit into the sum and the integral, but this follows by dominated convergence on considering the uniform integrable bound :

$$\|[U_t(r)B(r) - U_t(s_i)B(s_i)]T_x^{\frac{1}{2}} e_k\|^2 \leq 2 \sup_{s\leq r\leq t} \|U_t(r)\|_{\mathcal{L}(H)} \sup_{s\leq r\leq t} \|B(r)\|_{\mathcal{L}(H)} \|T_x^{\frac{1}{2}} e_k\|^2$$

Thus we have convergence in L^2 of the approximating sums towards the integral.
The same argument works for the Brownian part, where there is even no dependence on x.
The big jumps, finally are quite simple to treat. Since the expression makes sense pointwise, we consider the approximation for ω fixed and we obtain:

$$\lim_{n\to\infty} \sum_{s_i \in \mathcal{P}_n} \sum_{s_{i-1} \leq r \leq s_i} [U_t(s_i)B(s_i) - U_t(r)B(r)]\Delta L_r(\omega)\chi_{\|\Delta L_r(\omega)\|>1} = 0$$

again because of strong continuity.

So in any of the four cases we have at least convergence in probability and the claim is proved.

Since convergence in probability implies convergence in distribution and $x \mapsto e^{i\langle h,x\rangle}$ is bounded and continuous, we may interchange limit and expectation:

$$\mathbb{E}\left[\exp\left(i\left\langle h, \int_s^t U(t,r)B(r)dL_r\right\rangle\right)\right]$$

$$= \lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\left\langle h, \sum_{k\in\mathcal{P}_n} U(t,s_k)B(s_k)(L_{s_k} - L_{s_{k-1}\vee 0})\right\rangle\right)\right]$$

Using the functional equation of the exponential and the independence of increments of L:

$$\begin{aligned}
&= \lim_{n\to\infty} \prod_{k\in\mathcal{P}_n} \mathbb{E}\left[\exp\left(i\left\langle h, U(t,s_k)B(s_k)(L_{s_k} - L_{s_{k-1}\vee 0})\right\rangle\right)\right] \\
&= \lim_{n\to\infty} \prod_{k\in\mathcal{P}_n} \exp\left\{\lambda(B^*(s_k)U^*(t,s_k)h)(s_k - (s_{k-1}\vee 0))\right\} \\
&= \exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\}
\end{aligned}$$

where we employed the Lévy-Khinchine formula and the functional equation again. Note that the Riemannian sums converge to the integral because of strong continuity. ∎

The following lemma will be of great technical help.

Lemma 4.3 (complex monotone classes) *Let $\mathcal{H}$ be a complex vector space of complex-valued bounded functions, that contains the constants and is closed under componentwise monotone convergence. Let $\mathcal{M} \subset \mathcal{H}$ be closed under multiplication and complex conjugation. Then, all bounded $\sigma(\mathcal{M})$- measurable functions belong to $\mathcal{H}$.*

Proof Without loss of generality, we can assume that $\mathcal{M}$ already is an algebra, by taking its linear hull, which changes neither the multiplicativity, nor the generated σ-algebra. As $\mathcal{M}$ is closed under complex conjugation, we have $\mathcal{M}_\mathbb{R} := \{\Re f, \Im f \mid f \in \mathcal{M}\} \subset \mathcal{M}$. As $\mathcal{M}$ is multiplicative, $\mathcal{M}_\mathbb{R}$ is a real algebra of real-valued functions. Of course, $\mathcal{M}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$, where $\mathcal{H}_\mathbb{R}$ is the monotone real vector space of real-valued elements of $\mathcal{H}$. Now we can apply the standard monotone class theorem, which states, that every bounded $\sigma(\mathcal{M}_\mathbb{R})$-measurable real-valued function belongs to $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$. To obtain our conclusion, we have to show, that $\sigma(\mathcal{M}) \subset \sigma(\mathcal{M}_\mathbb{R})$. The elements of $\sigma(\mathcal{M})$ are of the form $A_f := f^{-1}(A)$ for a Borel-set $A \in \mathbb{C}$ and a $f \in \mathcal{M}$.If $A = B \times C$ is a rectangle, we have $A_f = B_{\Re f} \cap C_{\Im f}$. But since preimages and set operations commute, it is clear, that it is sufficient to have the inclusion for rectangles, as these form a generator for $\mathcal{B}(\mathbb{C})$. It is also obvious, that any complex-valued function in $\mathcal{H}$ can be reconstructed from its real and imaginary parts. ∎

The last and the next result in combination will be particularly useful:

Lemma 4.4 *The functions $\mathcal{M} := \{e^{i\langle h,x\rangle}, h \in H\}$ form a complex multiplicative systems that generates the Borel σ-algebra of H.*

Proof It is obvious that $\mathcal{M}$ is closed under multiplication and complex conjugation.
To show that indeed $\sigma(\mathcal{M}) = \mathcal{B}(H)$ we make use of the following lemma: (see [Schw73] page 108)

Lemma 4.5 *A countable family of real-valued functions on a Polish space X separating the points of X already generates the Borel-sigma-algebra of X.*

Our countable family will be $\{f_{n,k}(x) := \sin(\langle \frac{1}{n}e_k, x\rangle)\}_{k,n\in\mathbb{N}} \subset \mathcal{M}$ where $\{e_k\}$ is an orthonormal basis of H.
Since the sine function is injective in a neighborhood of zero, and the functions $\langle \frac{1}{n}e_k, x\rangle)$ separate the points of H, so do the $f_{n,k}$. As real and imaginary parts of functions in $\mathcal{M}$, it is clear, that the sigma-algebra generated by them is included in $\sigma(\mathcal{M})$. ■

Now we will show that our solution induces a two-parameter semigroup, defined as follows:

Definition 4.6 *Whenever $f : H \to \mathbb{C}$ is measurable and bounded, define*

$$P(s,t)f(x) := \mathbb{E}[f(X(t,s,x))]$$

$P(s,t)$ will be called the two-parameter semigroup (associated to the solution X).

Lemma 4.7 *For f as above, we have the following flow property:*

$$P(r,s)P(s,t)f(x) = P(r,t)f(x)$$

Proof We will show the equality for the functions $f_h(x) = e^{i\langle h,x\rangle}$ and extend it with the help of 4.3. First note, that by 4.2 we have

$$\begin{aligned} P(s,t)f_h(x) = \quad & \exp\left\{i\left\langle h, U(t,s)x\right\rangle\right\} \exp\left\{i\left\langle h \int_s^t U(t,r)f(r)dr\right\rangle\right\} \\ & \times \exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\} \end{aligned}$$

so that:

$$\begin{aligned}
&P(r,s)P(s,t)f_h(x)\\
&= \mathbb{E}[P(s,t)f_h(X(s,r,x)]\\
&= \mathbb{E}\left[\exp\left\{i\left\langle U^*(t,s)h, X(s,r,x)\right\rangle\right\}\right] \times \exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle\right\}\\
&\quad \times \exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\}
\end{aligned}$$

but again 4.2 gives us the Fourier transform of $X(s,r,x)$ this time evaluated at $U^*(t,s)h$:

$$\begin{aligned}
&= \exp\left\{i\left\langle U^*(t,s)h, U(s,r)x\right\rangle\right\}\\
&\times \exp\left\{i\left\langle U^*(t,s)h, \int_r^s U(s,q)f(q)dq\right\rangle\right\} \exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle\right\}\\
&\times \exp\left\{\int_r^s \lambda(B^*(q)U^*(s,q)U^*(t,s)h)dq\right\} \exp\left\{\int_s^t \lambda(B^*(r)U^*(t,r)h)dr\right\}
\end{aligned}$$

Interchanging $U(t,s)$ with the integral, as it is a bounded operator, making use of the semigroup property of U and U^* and combining the integrals yields the result for exponential f. The space of all bounded measurable f for which the flow property holds is a complex monotone vectorspace, since by monotone convergence :

$$\begin{aligned}
P(s,t)\lim_{n\to\infty} f_n(x) &= \mathbb{E}[\lim_{n\to\infty} f_n(X(s,r,x))]\\
= \lim_{n\to\infty} \mathbb{E}[f_n(X(s,r,x))] &= P(s,t)\lim_{n\to\infty} f_n(x)P(s,t)f_n(x)
\end{aligned}$$

and by monotonicity of the integral we have even $P(s,t)f_n(x) \nearrow P(s,t)f(x)$ so that we can apply monotone covergence again to obtain:

$$P(r,s)P(s,t)\lim_{n\to\infty} f_n(x) = P(r,s)\lim_{n\to\infty} P(s,t)f_n(x) = \lim_{n\to\infty} P(r,s)P(s,t)f_n(x)$$

Hence, the proof is complete. ■

4.2 Evolution Systems of Measures

Since our equation is non-autonomous we cannot hope for a single invariant measure. What one can still expect in our setting is a so called *evolution* *system of measures*, a whole family $\{\nu_t\}_{t\in\mathbb{R}}$ of measures such that for all $s < t$ and all bounded measurable f:

$$\int_{\mathbb{R}^n} P(s,t)f(x)\nu_s(dx) = \int_{\mathbb{R}^n} f(x)\nu_t(dx) \tag{4.1}$$

To assure the existence of such a system, besides assumption 3.11 we will henceforth require the following condition to hold:

Assumption 4.8 *We assume a weak regularity for the Levy symbol λ , that is: for the corresponding Lévy measure M holds:*

$$\int_{\|x\|>1} \|x\| M(dx) < \infty$$

Taking a clue from the Gaussian situation, we will consider the distribution of our solutions after an infinite time span. As we have explicit knowledge of the Fourier transforms, we will characterize the measures in that way.

The following lemma will give a useful growth condition for the Lévy symbol that will allow us to construct limit measures.

Lemma 4.9 *Every Lévy symbol λ with a Lévy measure M satisfying 4.8 is Fréchet differentiable. In particular such a λ is locally of linear growth.*

Proof : Let be λ the corresponding Lévy symbol and M the Lévy measure. By the Lévy-Khinchine formula 2.36 we know that:

$$\lambda(u) = i\langle u, b\rangle - \frac{1}{2}\langle u, Au\rangle + \int \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\chi_{\{\|x\|\leq 1\}}\right) M(dx)$$

Clearly, it is enough to show that the integral expression is differentiable. We first show Gâteaux differentiability, hence we will need the directional derivatives to be integrable to obtain the result via dominated convergence. We have:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \left(e^{i\langle u+tv,x\rangle} - 1 - i\langle u+tv, x\rangle\chi_{\{\|x\|\leq 1\}}\right) = i\langle v, x\rangle e^{i\langle u,x\rangle} - i\langle v, x\rangle\chi_{\{\|x\|\leq 1\}}$$

To see the integrability we split the integral in two parts:

$$\begin{aligned}
&\int_{\|x\|\le 1} \left|i\langle v,x\rangle e^{i\langle u,x\rangle} - i\langle v,x\rangle \chi_{\{\|x\|\le 1\}}\right| M(dx)\\
&= \int_{\|x\|\le 1} \left| i\langle v,x\rangle \sum_{k=0}^{\infty} \frac{(i\langle u,x\rangle)^k}{k!} - i\langle v,x\rangle \right| M(dx)\\
&\le \int_{\|x\|\le 1} \left(\|v\|\,\|x\| \sum_{k=1}^{\infty} \frac{|(i\langle u,x\rangle)|^k}{k!} \right) M(dx)\\
&\le \int_{\|x\|\le 1} \left(\|v\|\,\|x\| \sum_{k=1}^{\infty} \frac{\|u\|^k\|x\|^k}{k!} \right) M(dx)\\
&\le \int_{\|x\|\le 1} \left(\|v\|\,\|x\|^2\|u\| \sum_{k=1}^{\infty} \frac{\|u\|^{k-1}\|x\|^{k-1}}{k!} \right) M(dx)\\
&\le \sup_{\|x\|\le 1} \exp\{\|u\|\|x\|\} \int_{\|x\|\le 1} \left(\|v\|\,\|x\|^2\|u\|\right) M(dx)\\
&= \exp\{\|u\|\}\|u\|\,\|v\| \int_{\|x\|\le 1} \|x\|^2 M(dx) < \infty
\end{aligned}$$

for every fixed u, v and s since M is a Lévy measure. On the other hand, we have:

$$\int_{\|x\|>1} \left|i\langle v,x\rangle e^{i\langle u,x\rangle} - i\langle v,x\rangle \chi_{\{\|x\|\le 1\}}\right| M(dx) \le \|v\| \int_{\|x\|>1} \|x\| M(dx) < \infty$$

by assumption.
Moreover, from the above, it is easy to see that the Gâteaux derivative is linear and bounded and depends continuously on u with respect to the operator norm, so λ is Fréchet differentiable and hence locally Lipschitz. ∎

Theorem 4.10 *Assume hypothesis 4.8. Then the functions*

$$\hat{\nu}_t(h) := \exp\left\{ i \left\langle h, \int_{-\infty}^{t} U(t,r)f(r)dr \right\rangle \right\} \exp\left\{ \int_{-\infty}^{t} \lambda\{B^*(r)U^*(t,r)h\}dr \right\}$$

are the Fourier transforms of an evolution system of measures.
This system is T-periodic, that is we have $\nu_{T+t} = \nu_t$ for any t.
Any other T-periodic evolution system of measures, coincides with the above.

PROOF : To establish T-periodicity, first note, that we have:
$U(t,s) = U(t+T, s+T)$ for any $s < t$, which follows easily from its defining differential equation and the assumption that A is T-periodic. Hence, we get

$$\int_{-\infty}^{t+T} U(t+T,r)f(r)dr = \int_{-\infty}^{t} U(t+T,r+T)f(r+T)dr = \int_{-\infty}^{t} U(t,r)f(r)dr$$

and for the other integral the argument is the same.

We have to assure that the integrals above exist. Since U is stable and f is bounded on all of $\mathbb{R}$ (as it is continuous and periodic) we have:

$$\int_{-\infty}^{t} \|U(t,r)f(r)\|dr \leq \int_{-\infty}^{t} Me^{-\omega(t-r)}\|f\|_\infty dr = \frac{M}{\omega}\|f\|_\infty$$

As λ is Fréchet differentiable it has locally linear growth, so that with $\lambda(0) = 0$ we have $\|\lambda(u)\| \leq C\|u\|$ on the bounded range of the argument for some $C > 0$. So with $\|B^*\|$ bounded we can treat the second integral as the first:

$$\begin{aligned}\int_{-\infty}^{t} \|\lambda\{B^*(r)U^*(t,r)h\}\|dr &\leq C\int_{-\infty}^{t} \|B^*(r)U^*(t,r)h\|dr \\ &\leq C\sup_r \|B^*(r)\|\frac{M}{\omega}\|h\| < \infty \qquad (4.2)\end{aligned}$$

where we have used, that $\|U^*\| = \|U\|$.

To show that these functions are indeed Fourier transforms of measures we can make use of Lévy's continuity theorem in the finite dimensional case. We have just proven pointwise convergence of the Fourier transforms of $P \circ [X(t,s,x)]^{-1}$, and that the limit function is continuous in 0 follows easily by dominated convergence. Pointwise convergence under the integral is clear by continuity of λ, U and B and a majorizing function is found by looking at (4.2) again.
In the infinite dimensional case, however, we cannot apply Lévy's continuity theorem for reasons explained in appendix B. For a better readability we postpone the somewhat technical alternative.

In order to see that the respective measures constitute an evolution system

of measures we will check (4.1) for exponential functions and then extend the result via monotone classes.
So if we take $k(x) = e^{i\langle h,x\rangle}$ in (4.1) we get:

$$\int_{\mathbb{R}^n} k(x)\nu_t(dx) = \hat{\nu_t}(h)$$

by the very definition of Fourier transformation.
On the other hand we have by 4.2:

$$P(s,t)k(x) =$$
$$e^{i\langle h,U(t,s)x\rangle} \exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle\right\} \exp\left\{\int_s^t \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

Using the adjoint of U and the fact that Fourier transformation is only with respect to x we obtain by definition of $\hat{\nu_s}$:

$$\int_{\mathbb{R}^n} P(s,t)k(x)\nu_s(dx)$$
$$= \hat{\nu_s}(U^*(t,s)h)\ \exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle + \int_s^t \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

$$= \exp\left\{i\left\langle U^*(t,s)h, \int_{-\infty}^s U(s,r)f(r)dr\right\rangle + \int_{-\infty}^s \lambda\{B^*(r)U^*(s,r)U^*(t,s)h\}dr\right\}$$
$$\exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle + \int_s^t \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

$$= \exp\left\{i\left\langle h, \int_{-\infty}^s U(t,s)U(s,r)f(r)dr\right\rangle + \int_{-\infty}^s \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$
$$\exp\left\{i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle + \int_s^t \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

where we used in the left part, that linear continuous operators commute with the integral, and in the right part the corresponding ("twisted") semigroup property for the adjoints.

$$= \exp\left\{i\left\langle h, \int_{-\infty}^s U(t,r)f(r)dr\right\rangle + i\left\langle h, \int_s^t U(t,r)f(r)dr\right\rangle\right\}$$
$$\exp\left\{\int_{-\infty}^s \lambda\{B^*(r)U^*(t,r)h\}dr + \int_s^t \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

$$= \exp\left\{ i \left\langle h, \int_{-\infty}^{t} U(t,r) f(r) dr \right\rangle \right\} \exp\left\{ \int_{-\infty}^{t} \lambda\{B^*(r) U^*(t,r) h\} dr \right\}$$

but the last line equals $\hat{\nu_t}(h)$ and that is precisely what we had to show.

To prove the full assertion we have to show that (4.1) not only holds for functions of the form $k_h(x) := e^{i\langle h,x\rangle}$, but for any bounded measurable function.
By 4.4 we can apply 4.3, because the bounded and measurable functions for which (4.1) holds, form a complex monotone vector space:
for constant functions the equality is trivial and that (4.1) holds for monotone limits is essentially an iterated application of Levy's theorem about monotone convergence. Hence, all bounded measurable functions satisfy (4.1) and the existence of an evolution system of measures is proved.

To prove uniqueness, let $\{\nu_s\}$ be another T-periodic family satisfying (4.1), then it follows by periodicity:

$$\begin{aligned} \hat{\nu_s}(h) = \quad & \hat{\nu_s}(U^*(s+T,s)h) \\ \times \exp & \left\{ i \left\langle h, \int_{s}^{s+T} U(s+T,r) f(r) dr \right\rangle + \int_{s}^{s+T} \lambda\{B^*(r) U^*(s+T,r) h\} dr \right\} \end{aligned}$$

Using the easy to check relations:

$$\int_{s}^{s+T} U(s+T,r) f(r) dr = \int_{-\infty}^{s} U(s,r) f(r) dr - U(s+T,s) \int_{-\infty}^{s} U(s,r) f(r) dr$$

$$\begin{aligned} & \int_{s}^{s+T} \lambda\{B^*(r) U^*(s+T,r) h\} dr = \\ & \qquad \int_{-\infty}^{s} \lambda\{B^*(r) U^*(s,r) h\} dr - \int_{-\infty}^{s} \lambda\{B^*(r) U^*(s+T,r) h\} dr \end{aligned}$$

we get:

$$\begin{aligned}\hat{\nu_s}(h) = \hat{\nu_s}(U^*(s+T,s)h)\ &\exp\left\{i\left\langle h, \int_{-\infty}^{s} U(s,r)f(r)dr\right\rangle\right\}\\ &\exp\left\{\int_{-\infty}^{s} \lambda\{B^*(r)U^*(s,r)h\}dr\right\}\\ &\exp\left\{i\left\langle h, -U(s+T,s)\int_{-\infty}^{s} U(s,r)f(r)dr\right\rangle\right\}\\ &\exp\left\{-\int_{-\infty}^{s} \lambda\{B^*(r)U^*(s+T,r)h\}dr\right\}\end{aligned}$$

or equivalently:

$$\begin{aligned}&\hat{\nu_s}(h)\left[\exp\left\{i\left\langle h, \int_{-\infty}^{s} U(s,r)f(r)dr\right\rangle\right\}\exp\left\{\int_{-\infty}^{s} \lambda\{B^*(r)U^*(s,r)h\}dr\right\}\right]^{-1}\\ &\quad= \hat{\nu_s}(U^*(s+T,s)h)\left[\exp\left\{i\left\langle U^*(s+T,s)h, \int_{-\infty}^{s} U(s,r)f(r)dr\right\rangle\right\}\right.\\ &\qquad\qquad\left.\exp\left\{\int_{-\infty}^{s} \lambda\{B^*(r)U^*(s+T,r)h\}dr\right\}\right]^{-1} \qquad \blacksquare\end{aligned}$$

Finding, that the second line is the first, with h replaced by $U^*(s+T,r)h$ we can iterate, since the relation was valid for all $h \in H$. As $\|U^*(s+T,r)\| < 1$ must hold by our stability assumption, all the factors in the second equation will tend to 1, so $\hat{\nu_s}$ must indeed have the desired form.

Remark 4.11 *The condition that the Lévy symbol is of linear growth is actually stronger than necessary. To assure the existence of the integral in* (4.2) *it would be even sufficient to have a very weak estimate of the form* $|\lambda(u)| = O(\sqrt{\|u\|})$. *But we were unable to find any other easy to check conditions to control the growth of a Lévy symbol around the origin. Moreover, in the infinite dimensional case, we make full use of our assumption, as can be seen next.*

PROOF ($\hat{\nu_t}$ IS A CHARACTERISTIC FUNCTION - HILBERT SPACE CASE)
The general idea is the following. In the Gaussian case, it is known that the limit distributions are Gaussian again. In the same manner we take advantage of the fact that, in the Lévy case, our limit distribution is infinitely

divisible. We proceed here as in [Fuhr/Röck00] chapter 3. First of all we show that our distributions $P \circ X(t,s,x)$ are infinitely divisible for any $t > s, x$. By the Lévy-Khinchine representation it is sufficient to prove that their characteristic functions have the form (2.5) for some triple $[b, Q, M]$. Therefore, we calculate:

$$
\begin{aligned}
&\exp\left\{\int_s^t \lambda(B_r^* U_{t,r}^* h)dr\right\} \\
&= \exp\left\{\int_s^t i\langle b, B_r^* U_{t,r}^* h\rangle dr - \frac{1}{2}\int_s^t \langle B_r^* U_{t,r}^* h, RB_r^* U_{t,r}^* h\rangle dr\right. \\
&\left. + \int_s^t \left(\int_H e^{i\langle x, B_r^* U_{t,r}^* h\rangle} - 1 - i\langle x, B_r^* U_{t,r}^* h\rangle \chi_{\{\|x\|\le 1\}} M(dx)\right) dr\right\}
\end{aligned}
$$

For the jump part we have:

$$
\begin{aligned}
&\int_H e^{i\langle x, B_r^* U_{t,r}^* h\rangle} - 1 - i\langle x, B_r^* U_{t,r}^* h\rangle \chi_{\{\|x\|\le 1\}} M(dx) \\
&= \int_H e^{i\langle U_{t,r} B_r x, h\rangle} - 1 - i\langle U_{t,r} B_r x, h\rangle \chi_{\{\|x\|\le 1\}} M(dx) \\
&+ \int_H [-\chi_{\{\|U_{t,r} B_r x\|\le 1\}} + \chi_{\{\|U_{t,r} B_r x\|\le 1\}}] M(dx) \\
&= \int_H e^{i\langle x, h\rangle} - 1 - i\langle x, h\rangle \chi_{\{\|x\|\le 1\}} M \circ (U_{t,r} B_r)^{-1}(dx) \qquad (4.3) \\
&- \int_H i\langle U_{t,r} B_r x, h\rangle [\chi_{\{\|x\|\le 1\}} - \chi_{\{\|U_{t,r} B_r x\|\le 1\}}] M(dx) \qquad (4.4)
\end{aligned}
$$

Note that (4.3) is finite because of: (setting $C := \|U_{t,r} B_r\|_{\mathcal{L}(H)}$)

$$
\begin{aligned}
&\int_H (1 \wedge \|x\|^2) M \circ (U_{t,r} B_r)^{-1}(dx) = \int_H (1 \wedge \|U_{t,r} B_r x\|^2) M(dx) \\
&\le \int_H (1 \wedge C^2 \|x\|^2) M(dx) \le C^2 \int_H (\frac{1}{C^2} \wedge \|x\|^2) M(dx) < \infty
\end{aligned}
$$

and only in that way we can argue that (4.4) must be finite as well. Thus, we obtain:

$$
\begin{aligned}
&\exp\left\{\int_s^t \lambda(B_r^* U_{t,r}^* h)dr\right\} \\
&= \exp\left\{ i\left\langle \int_s^t U_{t,r}B_r b\,dr, h\right\rangle - \frac{1}{2}\left\langle h, \int_s^t U_{t,r}B_r R B_r^* U_{t,r}^* h dr\right\rangle \right. \\
&\quad + \int_s^t \left(\int_H e^{i\langle x,h\rangle} - 1 - i\langle x,h\rangle \chi_{\{\|x\|\le 1\}} M\circ (U_{t,r}B_r)^{-1}(dx)\right) dr \\
&\quad \left. - i\left\langle \int_s^t \int_H U_{t,r}B_r x[\chi_{\{\|x\|\le 1\}} - \chi_{\{\|U_{t,r}B_r x\|\le 1\}}]M(dx)dr, h\right\rangle \right\}
\end{aligned}
$$

so that with:

- $b(t,s) := \int_s^t U_{t,r}B_r b dr - \int_s^t \int_H U_{t,r}B_r x[\chi_{\{\|x\|\le 1\}} - \chi_{\{\|U_{t,r}B_r x\|\le 1\}}]M(dx)dr$
- $Q(t,s) := \int_s^t U_{t,r}B_r R B_r^* U_{t,r}^* dr$
- $M_{t,s}(A) := \int_s^t M\circ (U_{t,r}B_r)^{-1}(A)dr$ for $0\notin A$

$\exp\left\{\int_s^t \lambda(B_r^* U_{t,r}^* h)dr\right\}$ is associated to the triple $[b(t,s), Q(t,s), M_{t,s}]$, where $Q(t,s)$ is still symmetric and nonnegative and we have :

$$
\begin{aligned}
\mathrm{tr}Q(t,s) &= \sum_k \langle e_k, Q(t,s)e_k\rangle = \sum_k \int_s^t \|\sqrt{R}B_r^* U_{t,r}^* e_k\|^2 \\
&= \int_s^t \|\sqrt{R}B_r^* U_{t,r}^*\|_2^2 \le \int_s^t \|\sqrt{R}\|_2^2 \|B_r^* U_{t,r}^*\|^2 < \infty
\end{aligned}
$$

and $M_{t,s}$ is a Lévy measure, as we have [since $(1\wedge\|x\|^2) \le (\|x\|\wedge\|x\|^2)$]:

$$
\begin{aligned}
&\int_s^t \int_H (1\wedge\|x\|^2) M\circ(U_{t,r}B_r)^{-1}(dx) \le \int_s^t \int_H (\|x\|\wedge\|x\|^2) M\circ(U_{t,r}B_r)^{-1}(dx) \\
&= \int_s^t \int_H (\|U_{t,r}B_r x\| \wedge \|U_{t,r}B_r x\|^2) M(dx) \\
&\le \int_s^t \|U_{t,r}B_r\| \int_H (\|x\| \wedge \|U_{t,r}B_r\|\,\|x\|^2) M(dx) \\
&\le \int_s^t \max_{r\le t}\|B_r\| M e^{-\omega(t-r)} \underbrace{\int_H (\|x\| \wedge \max_{r\le t}\|U_{t,r}B_r\|\,\|x\|^2) M(dx)}_{<\infty \text{ by assumption } 4.8} < \infty
\end{aligned}
$$

Moreover, we see that we can let $s \to -\infty$ and $Q(t, -\infty)$ will still be trace class as well as $M_{t,-\infty}$ will still be a Lévy measure, because of the exponential stability of U. Since we already know that the Fourier transform as a whole converges, convergence of the first part of $b(t, -\infty)$ (which is obvious) implies convergence of the second part. Hence the limit function is associated to a Lévy triple and thus the characteristic function of an infinitely divisible measure. ■

Remark 4.12 *Note that we did not show weak convergence and that we do not need to do so. The point is only in proving that our limit functions are indeed associated to measures. Extension to functions more general than complex exponentials is then done by monotone class arguments.*

4.3 The Reduced Equation

Since we are interested in generators, we have to reduce our equation to the autonomous case, so that we obtain a one-parameter semigroup, that we can relate to a generator. It will turn out, that we can establish an invariant measure for our new semigroup on the extended state space, using our evolution system of measures.

Reduction of non-autonomous problems is a well-known method in the theory of ordinary differential equations.(see e.g. [Dal/Krei74]) We recall that the basic idea is to enlarge the state space, thus allowing to keep track of the elapsed time. The reduced problem then looks:

$$\begin{cases} dX(t) = \{A(y(t))X(t) + f(y(t))\}dt + B(y(t))dL(t) & X(0) = x \\ dy(t) = dt & y(0) = s \end{cases}$$

The one-parameter semigroup is then defined as follows:

$$P_\tau u(t,x) := P_{t,t+\tau} u(t+\tau, \cdot)(x) := (P_{t,t+\tau} u_{t+\tau})(x)$$

meaning that we apply the two-parameter semigroup to u as a function of x only. That the family $\{P_\tau\}_{\tau \in \mathbb{R}}$ is indeed a semigroup, follows, of course, from the semigroup property of $\{P_{s,t}\}_{s<t}$ and is a simple calculation:

$$\begin{aligned} (P_\sigma(P_\tau u))(t,x) &= P_{t,t+\sigma} P_{t+\sigma,t+\sigma+\tau} u(t+\sigma+\tau, \cdot)(x) \\ &= P_{t,t+\sigma+\tau} u(t+\sigma+\tau, \cdot)(x) = P_{\tau+\sigma} u(t,x) \end{aligned}$$

4.3.1 The Invariant Measure ν and the Space $L^2_*(\nu)$

Starting from our evolution system of measures, we will establish an invariant measure for the one-parameter semigroup. On the respective L^2-space the semigroup will then be a contraction.

From now on we will require the following assumption to hold:

Assumption 4.13 *$D(A^*)$ is dense in H and we have $U^*(t,s)D(A^*) \subset D(A^*)$ for all $s \leq t$. Moreover, we have $\frac{d}{dt}U^*(t,s)x = U^*(t,s)A^*(t)x$ for all $x \in D(A^*)$*

To obtain our invariant measure we need the following lemma:

Lemma 4.14 *The function $F : (t, A) \mapsto \nu_t(A)$ is a transition kernel.*

PROOF Obviously, for fixed t we have a measure.
To show measurability, we will need a monotone class argument:
$\mathcal{H} := \{f : H \to \mathbb{C} \quad | \quad t \mapsto \int f(x)\nu_t(dx)$ is measurable $\}$ is a monotone complex vector space, since pointwise limits of measurable functions are again measurable.
$\mathcal{M} := \{\exp(\langle h, x\rangle)\}_{h\in H}$ is a complex multiplicative system, such that $\sigma(\mathcal{M}) = \mathcal{B}(H)$ (see 4.5 and the following arguments)
We will have proved the lemma, once we know that $\mathcal{M} \subset \mathcal{H}$.
Therefore, we will show that $t \mapsto \int f(x)\nu_t(dx)$ is even continuous for all $f \in \mathcal{M}$
Recalling the form of the Fourier transforms,

$$\hat{\nu_t}(h) = \exp\left\{i\left\langle h, \int_{-\infty}^{t} U(t,r)f(r)dr\right\rangle\right\}\exp\left\{\int_{-\infty}^{t} \lambda\{B^*(r)U^*(t,r)h\}dr\right\}$$

it will be sufficient to show that:

$$\int_{\mathbb{R}} \chi_{\{r\leq s\}}\lambda\{B^*(r)U^*(s,r)h\}dr \xrightarrow{s\to t} \int_{-\infty}^{t} \lambda\{B^*(r)U^*(t,r)h\}dr$$

Pointwise convergence of the integrands is clear, and an integrable upper bound is given by: $C\,\chi_{\{r\leq s_{\max}\}}\,M\,e^{-\omega(s_{\min}-r)}$
where C is composed of the Lipschitz constant of λ and a bound for $\|B(r)\|$ and $s_{\min}, s_{\max}$ are upper and lower bounds for the convergent sequence s_n, while M and ω are the stability constants for U.
The argument for the other integral follows the same lines. ■

Now, we introduce the space, on which the semigroup will be strongly continuous and the subspace that will be the core for the generator of our semigroup. Let be

Definition 4.15 *As F is a kernel from H to $\mathbb{R}$ we can form $\nu := F \otimes \frac{1}{T}dt$, a measure on $H \times \mathbb{R}$.*

$$\begin{aligned} L^2_*(\nu) :=\{&f : \mathbb{R}\times H \to \mathbb{R} \text{ measurable} \,|\, f(t+T,x) = f(t,x) \quad \nu - a.e. \\ &\int_{[0,T]\times H} \|f\|^2(y)\nu(dy) < \infty\} \end{aligned}$$

$$\begin{aligned}\mathcal{M} := span_{\mathbb{C}}\{f : \mathbb{R}\times H \to \mathbb{C} \quad | \quad & f = \Phi(t)e^{i\langle x,h(t)\rangle},\ \textit{where} \\ & \Phi \in C^1(\mathbb{R},\mathbb{R})\ \textit{and T-periodic}, \\ & h \in C^1(\mathbb{R},H)\ \textit{ and T-periodic such that } Im\, h \subset D(A^*)\}\end{aligned}$$

$$K := \{\Re(f) \mid \quad f \in \mathcal{M}\}$$

That is, K comprises the real parts of the functions in $\mathcal{M}$.

Remark 4.16 *L^2_* is a Hilbert space. Because of the periodicity it is clear, that $(\int_{[0,T]\times H} \|f\|^2(y)\nu(dy))^{\frac{1}{2}}$ is a norm(where we introduce ν a.e.-equivalence classes as usual). Given a Cauchy-sequence f_n we consider f_n^z the restriction of f_n to the interval $I_z := [zT,(z+1)T], z \in \mathbb{Z}$. By Riesz-Fischer we obtain a limit f^0 of f_n^0 on $[0,T]$, and because of periodicity it is clear that the other restrictions form the same Cauchy sequences, that is:* $\lim f_n^z = f^z = f^0\ \forall z \in \mathbb{Z}$*. Hence, the limit function is periodic, and the space is complete.*

Lemma 4.17 *K is dense in $L^2_*(\nu)$.*

Proof Note that by periodicity, we can think of our functions to be defined on $[0,T]\times H$ and in the following we will do so without changing notation. We will show density of $\mathcal{M}$ in $L^2_*(\nu;\mathbb{C})$. This implies density of the respective real vector spaces. We will use complex monotone classes again. The space $\mathcal{M}$ is closed under multiplication and conjugation. Consider $\mathcal{H} := \bar{\mathcal{M}}$ as a subspace of $L^2_*(\nu;\mathbb{C})$ where we allow complex-valued integrable and periodic functions. $\mathcal{H}$ is a complex monotone vector space, by monotone convergence, applied separately to real and imaginary parts. So, $\mathcal{H}$ contains all $\sigma(\mathcal{M})$-measurable functons. If we can show that $\sigma(\mathcal{M}) = \mathcal{B}(H\times\mathbb{R})$, then we will have all step functions in $\mathcal{H}$, so density will be obvious. Note that we want to show, that functions of the form $\Phi_i \otimes e_h$ generate a product σ-algebra $\mathcal{B}([0,T])\otimes\mathcal{B}(H)$. Since both families contain the constant function, we can break the problem down, as $(1_{\mathbb{R}}\otimes f)^{-1}(A) = \mathbb{R}\times f^{-1}(A)$ and knowing that Φ_i generates $\mathcal{B}(R)$ and $e_h := e^{i\langle\cdot,h\rangle}$ generates $\mathcal{B}(H)$ (which follows again from 4.5 and the fact that $D(A^*)$ is dense), we have the result. ∎

Remark 4.18 *Note that to prove the density of K, $h \equiv const$ is sufficient, but we will need the t-dependence of h later to show that K is P_τ-invariant.*

Proposition 4.19 *The measure ν is the unique invariant measure for the semigroup P_τ on $L^2_*(\nu)$. The semigroup P_τ is a contraction on $L^2_*(\nu)$.*

PROOF We will prove both invariance and contractivity for functions from K first. Since K is dense, it is then easy to see that they hold on the whole of $L^2_*(\nu)$. Note, however, that we first need invariance to prove contractivity, then extend contractivity by density, and then obtain global invariance by density and contractivity. For invariance we have to show:

$$\int_{[0,T]\times H} P_\tau u(t,x)d\nu = \int_{[0,T]\times H} u(t,x)d\nu \quad \forall \tau > 0,\ u \in K$$

Writing $u_t(x) := u(t,x)$, remember, that $(P_\tau u)(t,x) = (P_{t,t+\tau}u_{t+\tau})(x)$. Taking into account (4.1),which is valid, since the elements from K are bounded, we have:

$$\int_{[0,T]\times H} P_\tau u(t,x)d\nu = \frac{1}{T}\int_{[0,T]}\int_H (P_{t,t+\tau}u_{t+\tau})(x)\nu_t(dx)dt$$

$$= \frac{1}{T}\int_{[0,T]}\int_H u_{t+\tau}(x)\nu_{t+\tau}(dx)dt = \frac{1}{T}\int_{[\tau,T+\tau]}\int_H u_t(x)\nu_t(dx)$$

$$= \frac{1}{T}\int_{[0,T]}\int_H u_t(x)\nu_t(dx) = \int_{[0,T]\times H} u(t,x)d\nu$$

because of translation invariance of dt and T-periodicity of u and ν_t.

For the contraction property we have to show: $\|P_\tau u\|_{L^2_*} \leq \|u\|_{L^2_*}$
Using the Jensen inequality for the expectation and afterwards the invariance property for u^2 (recall that K is closed under multiplication):

$$\|P_\tau u\|_{L^2_*} = \int_{[0,T]\times H} \mathbb{E}[u(t+\tau, X(t+\tau,t,x))]^2\nu(dx,dt)$$

$$\leq \int_{[0,T]\times H} \mathbb{E}[u^2(t+\tau, X(t+\tau,t,x))]\nu(dx,dt) = \int_{[0,T]\times H} (P_\tau u^2)(t,x)\nu(dx,dt)$$

$$= \int_{[0,T]\times H} u^2(t,x)\nu(dx,dt) = \|u\|_{L^2_*}$$

To show uniqueness, let μ be another invariant measure for P_τ, so that we have:

$$\int_{[0,T]\times H} P_\tau u(t,x)\mu(dx,dt) = \int_{[0,T]\times H} u(t,x)\mu(dx,dt) \quad \forall \tau > 0,\ u \in L^2_*(\nu) \tag{4.5}$$

By [Dudley89] : corollary 10.2.8 , we can disintegrate μ as follows:

$$\int u(t,x)\mu(dt,dx) = \int_{[0,T]} \left(\int_H u(t,x)\mu_t(dx) \right) \mu_1(dt) \tag{4.6}$$

for the marginal $\mu_1(dt) = \mu \circ Pr^{-1}$ where Pr is the Projection on the t-component, and $\{\mu_t\}_{t\in\mathbb{R}}$ is a family of probability measures on H. Choosing $u(t,x) = f(t)$ independent of x in (4.5) we have by (4.6):

$$\int_{[0,T]\times H} f(t+\tau)\mu_1(dt) = \int_{[0,T]\times H} f(t)\mu_1(dt)$$

Since f is T-periodic, μ_1 is translation invariant (note, that we need here a similar monotone class argument as in 4.17). So μ_1 must be Lebesgue measure.

To show $\mu_t = \nu_t$, we will of course use the uniqueness property from 4.10. Choosing $u(t,x) = f(t)g(x)$ and $\tau = T$ in (4.5) yields:

$$\int_{[0,T]} f(t) \left(\int_H P_{t,t+T}g(x)\mu_t(dx) \right) \mu_1(dt) = \int_{[0,T]} f(t) \left(\int_H g(x)\mu_t(dx) \right) \mu_1(dt)$$

Clearly, if this holds for a fixed, bounded g and arbitrary bounded f, we must have

$$\int_H P_{t,t+T}g(x)\mu_t(dx) = \int_H g(x)\mu_t(dx)$$

Since this holds for any bounded measurable g we can apply 4.10 to obtain $\nu_t = \mu_t \; \forall\, t \in \mathbb{R}$. ∎

Remark 4.20 *Note that we cannot abandon the T-periodicity, because we need translation invariance in our proof above. But as we want a probability measure, we cannot take Lebesgue measure on the whole of $\mathbb{R}$. The only alternative known to us, is to restrict ourselves to a finite interval by periodicity.*

4.3.2 Generator and Domain of Uniqueness

In this subsection we prove that the generator is given by a pseudo-differential operator. Compared to the Gaussian case, we have an additional nonlocal part. However, we still obtain a result on the spectrum of the generator, exactly as in the Gaussian case.

Proposition 4.21 (strong continuity) *P_τ is strongly continuous on $L^2_*(\nu)$*

PROOF The invariance of ν and the density of K allow us to use proposition 4.3 from [Ma/Röck92]. Hence, it will be sufficient to show, that $P_\tau u \xrightarrow{t\to 0} u$ $\nu - a.e.$ For $u(t,x) = \Phi(t)e^{i\langle x,h(t)\rangle}$ we have by 4.2:

$$\begin{aligned}(P_\tau u)(t,x) &= \exp\left\{\int_t^{t+\tau} \lambda(B^*(r)U^*(t+\tau,r)h(t+\tau))dr\right\} \\ \times \quad & \Phi(t+\tau)\exp\left\{i\left\langle h(t+\tau), U(t+\tau,t)x + \int_t^{t+\tau} U(t+\tau,r)f(r)dr\right\rangle\right\}\end{aligned} \tag{4.7}$$

Recalling that Φ, h and U are continuous and that $U(t,t) = Id$ we obtain the result, since all the integrals vanish. Note, that by linearity, this extends to general $u \in K$. ■

Since P_τ is strongly continuous, it admits a generator, say G. For the concept of cores, that is domains of uniqueness for operators, see appendix A.

Lemma 4.22 *Let assumption 4.8 hold. Then K is a core for G.*

PROOF Looking closely at (4.7) again, one notes that $(P_\tau u)(t,x)$ is again of the form $\Psi(t)e^{i\langle x,k(t)\rangle}$ with Ψ and k as follows:

$$\begin{aligned}\Psi(t) := \Phi(t+\tau)\exp&\left\{i\left\langle h(t+\tau), \textstyle\int_t^{t+\tau} U(t+\tau,r)f(r)dr\right\rangle\right\} \\ \times \quad \exp&\left\{\textstyle\int_t^{t+\tau} \lambda(B^*(r)U^*(t+\tau,r)h(t+\tau))dr\right\}\end{aligned}$$

$k(t) := U^*(t+\tau,t)h(t+\tau)$

Since by assumption 4.13 $k : \mathbb{R} \to D(A^*)$, K is invariant under P_τ, since by linearity it is sufficient to check the invariance for special u. Furthermore, we have again by (4.7):

$$\begin{aligned} Gu = \frac{d}{d\tau} P_\tau u \Big|_{\tau=0} &= [\Phi'(t) + i\Phi(t)\langle x, h'(t)\rangle]\, e^{i\langle x,h(t)\rangle} \\ &\quad + i\langle x + f(t), A^*(t)h(t)\rangle\, \Phi(t)\, e^{i\langle x,h(t)\rangle} \\ &\qquad + \lambda[B^*(t)h(t)]\, \Phi(t)\, e^{i\langle x,h(t)\rangle} \end{aligned} \tag{4.8}$$

Note that we have used the differentiability of λ.

Seeing, that $K \subset D(G)$ we can apply corollary A.3 to prove the assertion. ■

Remember, that (b, R, M) is the triple of our Levy process (see 2.5)
Set $\Sigma = \sqrt{R}$.
To obtain a realization of G let us define the following operator on K:

Definition 4.23 *For $u \in K$ we set:*

$$\begin{aligned} Lu(t,x) := {} & u_t(t,x) + \langle A(t)x + f(t), \nabla_x u(t,x)\rangle \\ & + \langle B(t)b, \nabla_x u(t,x)\rangle + \frac{1}{2} Tr\{\Sigma^* B^* \nabla_{xx} u(t,x) B\Sigma\} \\ & + \int_{\mathbb{R}^d} \{u(t, x + B(t)y) - u(t,x) - \langle B(t)y, \nabla_x u(t,x)\chi_{\|y\|\le 1}\}\nu(dy) \end{aligned}$$

where ∇_x u denotes the gradient of u, ∇_{xx} u denotes the generalized Hessian of u and Tr denotes the trace of an operator.
Note that ∇_{xx} u is Hilbert-Schmidt, (see [daPr04] 1.2.4.) so the trace is well defined.

Lemma 4.24 (realization of G) $L = G_{|K}$ *, so that $G = \bar{L}$*

Proof Again, we will only check this for special u, since we deal with linear operators. Note, for the calculations involved, that for $u(t,x) = \Phi(t)e^{i\langle x,h(t)\rangle}$ we have $\nabla_x u(t,x) = ih(t)u(t,x)$ $\quad \nabla_{xx} u(t,x) = -h(t)h^*(t)u(t,x)$ and that

$\mathrm{Tr}(Auu^*A^*) = \langle Au, Au\rangle$. Hence the Levy-Khinchine formula yields for λ:

$$\begin{aligned}
&u(t,x)\ \lambda(B^*(t)h(t)) \\
&= i\langle B(t)b, h(t)\rangle\ u(t,x) - \frac{1}{2}\langle \Sigma^* B^*(t)h(t), \Sigma^* B^*(t)h(t)\rangle\ u(t,x) \\
&\quad + u(t,x)\int_{\mathbb{R}^d}\{e^{i\langle B^*(t)h(t),y\rangle} - 1 - i\langle B^*(t)h(t), y\rangle\chi_{\|y\|\le 1}\}\nu(dy) \\
&= \langle B(t)b, \nabla_x u(t,x)\rangle - \frac{1}{2}\mathrm{Tr}(\Sigma^* B^*(t)h(t)h^*(t)u(t,x)B(t)\Sigma) \\
&\quad + \int_{\mathbb{R}^d}\{\Phi(t)e^{i\langle h(t),x\rangle}e^{i\langle h(t),B(t)y\rangle} - u - i\langle h(t)u, B(t)y\rangle\chi_{\|y\|\le 1}\}\nu(dy) \\
&= \langle B(t)b, \nabla_x u(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\{\Sigma^* B^* \nabla_{xx} u(t,x)B\Sigma\} \\
&\quad + \int_{\mathbb{R}^d}\{u(t,x+B(t)y) - u(t,x) - \langle B(t)y, \nabla_x u(t,x)\chi_{\|y\|\le 1}\}\nu(dy)
\end{aligned}$$

That the first two summands in both expressions also coincide is very easy to see. ∎

Lemma 4.25 *For all $u \in D(L)$ we have*

$$\int_{[0,T]\times\mathbb{R}^d} Lu(t,x)\nu(dt,dx) = 0 \tag{4.9}$$

Proof We will prove this for $u \in K$ first. As ν is P_τ invariant, let us consider the equality:

$$\int_{[0,T]\times\mathbb{R}^d} P_\tau u(t,x)\nu(dt,dx) = \int_{[0,T]\times\mathbb{R}^d} u(t,x)\nu(dt,dx)$$

Differentiating both sides with respect to τ we obtain the result, if we can show that we can interchange integral and differential on the left hand side. We know, that for a fixed $u \in K \quad \tau \mapsto P_\tau u(t,x)$ is differentiable $\nu-$ almost everywhere, since $\lim_{n\to\infty} nP_{\frac{1}{n}}u(t,x) = Lu(t,x)$ in $L^2_*(\nu)$ implies pointwise *a.e.*-convergence along a subsequence, that we can choose without loss of generality. Furthermore, for a fixed u the derivative in τ is given by $P_\tau Lu(t,x)$, but since every P_τ is a contraction with respect to the supremum norm, and every $Gu \in K$ is bounded we have a uniform bound for the derivatives. Hence we can apply Lebesgue's dominated convergence theorem.
Since K is a core, for $u \in D(G)$ we can find a sequence u_n such that $u_n \to u$ in L^2_* and $Gu_n \to Gu$ in L^2_*. Taking the limit, we obtain the equality for u. ∎

Exactly as in [daPr/Lun07] we obtain the following result on the spectrum of G:

Corollary 4.26 *For any $z \in \sigma(G)$ and $k \in \mathbb{Z}$ we have $z + 2\frac{\pi}{T}ki \in \sigma(G)$. Moreover 0 is a simple eigenvalue of G.*

Proof For fixed $k \in \mathbb{Z}$ consider the operator $T_k u(t,x) = e^{2k\frac{\pi}{T}it}u(t,x)$. Since T is unitary the spectrum of G is equal to the spectrum of $T_k^{-1}GT_k = G + (2ki\frac{\pi}{T})Id$ where the equality holds, because the factors cancel out everywhere, except for the derivative with respect to t where the product rule applies. This proves the first statement.
Since every unique invariant measure is ergodic, we also have the equivalent property (see [daPr/zab03]):
If $u \in L^2_*$ fulfills $P_\tau u = u$ for every $\tau > 0$ then u is equal to a constant in L^2_*.
Now assume $Gu = 0$. Then we have:

$$P_\tau u - u = \int_0^\tau P_s G u ds = 0 \quad \text{for all } \tau > 0$$

Hence, the kernel of G is one-dimensional and contains exactly the constants. Let now $u \in \text{Ker}G^2$ that is $Gu \in \text{Ker}G$. Thus, we must have $Gu \equiv c$ for some constant c. But since $\int Gudv = 0$ by 4.25 we can deduce $c = 0$, thus u already was in Ker G and we have Ker G^2 =Ker$G$. ■

4.4 Asymptotic Behaviour of the Semigroup

Having obtained a unique invariant measure for the semigroup of the reduced equation, we may use ergodic theory to ivestigate the asymptotics of the two-parameter semigroup. By a simple application of the ergodic theorem we get:

Proposition 4.27 *Assume that $f : H \to \mathbb{R}$ is such that $\int_0^T \int_H f^2(x)\nu_t(dx)dt < \infty$ Then we have:*

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau P_{t,t+s}f(x)ds = \frac{1}{T}\int_0^T \int_H f(x)\nu_t(dx)dt$$

Proof Since every unique invariant measure is automatically ergodic, the result follows directly from the equality (see [daPr/zab03] 3.2.4):

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau P_s g(t,x)ds = \int_0^T \int_H g(t,x)\nu(dx)$$

which is valid for all $g \in L^2_*(\nu)$. We have only to take $g(t,x) = f(x)$ independent of t and recall, that then $P_s g(t,x) = P_{t+s,t} f(x)$. ∎

Under the condition of weak convergence, we are able to characterize the asymptotic behaviour of $P_{s,t}$ for $s \to -\infty$ with the help of our evolution system of measures.

Proposition 4.28 *Assume that there is $x \in H$ such that for $s \to -\infty$ $P \circ (X(t,s,x))^{-1} \to \nu_t$ weakly. Then we have for $f \in C_b(H)$:*

$$\lim_{s \to -\infty} P_{s,t} f(x) = \int_H f(x) \nu_t(dx)$$

Proof By definition of weak convergence. ∎

4.4.1 The Square Field Operator and an Estimate

In the following we will introduce the square field operator. Its importance lies in the crucial role, that it will play in the proof of the following functional inequalities.

Definition 4.29 $\Gamma(u,u) := Gu^2 - 2uGu$ *will be called the square field operator.*

Lemma 4.30 (square field operator) *On K we have:*

$$\begin{aligned} Gu^2 - 2uGu = \quad & \langle \Sigma^* B^*(t) \nabla_x u, \Sigma^* B^*(t) \nabla_x u \rangle \\ & + \int_H \left[u(x + B(t)y, t) - u(x,t) \right]^2 M(dy) \end{aligned}$$

Proof Let u be given by $u(t,x) = \Phi(t) e^{i\langle x, h(t) \rangle}$.
First note, that $u^2(t,x) = \Phi^2(t) e^{i\langle x, 2h(t) \rangle}$, so that by (4.8):

$$Gu^2(t,x) = \left[2\Phi'(t)\Phi(t) + i\Phi^2(t) \langle x, 2h'(t) \rangle \right] e^{i\langle x, 2h(t) \rangle} \tag{4.10}$$

$$+ \, i \langle A(t)x + f(t), 2h(t) \rangle \, \Phi^2(t) \, e^{i\langle x, 2h(t) \rangle} \tag{4.11}$$

$$+ \, \lambda[B^*(t) 2h(t)] \, \Phi^2(t) \, e^{i\langle x, 2h(t) \rangle}$$

$$2u(t,x)Gu(t,x) = 2[\Phi'(t) + i\Phi(t) \langle x, h'(t) \rangle] \, \Phi(t) \, \left(e^{i\langle x, h(t) \rangle} \right)^2 \tag{4.12}$$

$$+ \, 2i \langle A(t)x + f(t), h(t) \rangle \, \Phi^2(t) \, \left(e^{i\langle x, h(t) \rangle} \right)^2 \tag{4.13}$$

$$+ \, 2\lambda[B^*(t) h(t)] \, \Phi^2(t) \, \left(e^{i\langle x, h(t) \rangle} \right)^2$$

We see immediately, that (4.10)=(4.12) and (4.11)=(4.13), so that:

$$Gu^2 - 2uGu$$

$$
\begin{aligned}
&= \Big(i\langle b, B^*(t)2h(t)\rangle - \frac{1}{2}\langle \Sigma^* B^*(t)2h(t), \Sigma^* B^*(t)2h(t)\rangle \\
&\quad + \int_H \left[e^{i\langle B^*(t)2h(t),y\rangle} - 1 - i\langle B^*(t)2h(t), y\rangle\right] M(dy) \\
&\quad - 2i\langle b, B^*(t)h(t)\rangle + 2\frac{1}{2}\langle \Sigma^* B^*(t)h(t), \Sigma^* B^*(t)h(t)\rangle \\
&\quad -2\int_H \left[e^{i\langle B^*(t)h(t),y\rangle} - 1 - i\langle B^*(t)h(t), y\rangle\right] M(dy)\Big) \\
&\quad \times \Phi^2(t)e^{i\langle x,2h(t)\rangle} \\
&= -\langle \Sigma^* B^*(t)h(t), \Sigma^* B^*(t)h(t)\rangle \Phi^2(t)e^{i\langle x,2h(t)\rangle} \\
&\quad + \int_H \left[e^{2i\langle h(t),B(t)y\rangle} - 2e^{i\langle h(t),B(t)y\rangle} + 1\right] M(dy)\Phi^2(t)e^{i\langle x,2h(t)\rangle} \\
&= \langle \Sigma^* B^*(t)h(t)i\Phi(t)e^{i\langle x,h(t)\rangle}, \Sigma^* B^*(t)h(t)i\Phi(t)e^{i\langle x,h(t)\rangle}\rangle \\
&+ \int_H \left[\Phi^2(t)e^{2i\langle h(t),x+B(t)y\rangle} - 2\Phi^2(t)e^{i\langle h(t),2x\rangle}e^{i\langle h(t),B(t)y\rangle} + \Phi^2(t)e^{i\langle x,2h(t)\rangle}\right] M(dy) \\
&= \langle \Sigma^* B^*(t)h(t)\nabla_x u(t,x), \Sigma^* B^*(t)h(t)\nabla_x u(t,x)\rangle \\
&+ \int_H \left[\left(\Phi(t)e^{i\langle h(t),x+B(t)y\rangle}\right)^2 - 2\Phi(t)^2 e^{i\langle h(t),x+x+B(t)y\rangle} + \left(\Phi(t)e^{i\langle x,h(t)\rangle}\right)^2\right] M(dy) \\
&= \langle \Sigma^* B^*(t)h(t)\nabla_x u(t,x), \Sigma^* B^*(t)h(t)\nabla_x u(t,x)\rangle \\
&+ \int_H \left[\left(\Phi(t)e^{i\langle h(t),x+B(t)y\rangle} - \Phi(t)e^{i\langle x,h(t)\rangle}\right)^2\right] M(dy)
\end{aligned}
$$

To see, that the equality also holds for sums of such u note that:

$$G(u+v)^2 - 2(u+v)G(u+v) - \langle \Sigma^* B^*(t)\nabla_x(u+v), \Sigma^* B^*(t)\nabla_x(u+v)\rangle \\ - \int_H \left[(u+v)(x+B(t)y,t) - (u+v)(x,t)\right]^2 M(dy)$$

$$= Gu^2 - 2uGu - \langle \Sigma^* B^*(t)\nabla_x u, \Sigma^* B^*(t)\nabla_x u\rangle + \int_H \left[u(x+B(t)y,t) - u(x,t)\right]^2 M(dy)$$
$$+ Gv^2 - 2vGv - \langle \Sigma^* B^*(t)\nabla_x v, \Sigma^* B^*(t)\nabla_x v\rangle + \int_H \left[v(x+B(t)y,t) - v(x,t)\right]^2 M(dy)$$
$$+ 2G(uv) - 2vGu - 2uGv - 2\left\langle \Sigma^* B^*(t)\nabla_x u, \Sigma^* B^*(t)\nabla_x v\right\rangle$$
$$- 2\int_H [u(x+B(t)y,t) - u(x,t)][v(x+B(t)y,t) - v(x,t)]\, M(dy) \quad (4.14)$$

so that by our previous result the first two lines in (4.14) vanish. To see that the last two lines vanish as well we compute with $u(t,x) = \Phi(t)e^{i\langle x,h(t)\rangle}$ and $v(t,x) = \Psi(t)e^{i\langle x,k(t)\rangle}$ (recalling, that $\nabla_x u = ih(t)u$ and $\nabla_x v = ik(t)v$):

$$\begin{aligned} G(uv) - vGu - uGv = & \\ & \langle \Sigma^* B^*(t)[h(t)+k(t)], \Sigma^* B^*(t)[h(t)+k(t)]\rangle\, uv \\ & - \langle \Sigma^* B^*(t)h(t), \Sigma^* B^*(t)h(t)\rangle\, uv - \langle \Sigma^* B^*(t)k(t), \Sigma^* B^*(t)k(t)\rangle\, uv \\ & + \int_H [u(x+B(t)y,t)v(x+B(t)y,t) - v(x,t)u(x,t)] M(dy) \\ & - u\int_H [v(x+B(t)y,t) - v(x,t)] M(dy) \\ & - v\int_H [u(x+B(t)y,t) - u(x,t)] M(dy) \end{aligned}$$

where the terms of the generator involving simple differentiation have canceled out, precisely because of the "product rule structure" of $G(uv) - vGu - uGv$.
Cautiously comparing, we obtain the result by using bilinearity of the scalar product and for the integrals an equality of the form:
$(a-b)(c-d) = ac - bd - d(a-b) - b(c-d)$ ■

Assumption 4.31

(i) For every $t, \tau > 0$: $U(t+\tau, t)RH \subset \sqrt{R}H$ and there is a strictly positive $C_1 \in C[0,\infty)$ such that:

$$\|U(t,s)Rx\|_{H_0} \leq \sqrt{C_1(t-s)}\|Rx\|_{H_0} \quad x \in H,\ t > s$$

(ii) There is a strictly positive $C_2 \in C[0,\infty)$ such that:

$$M \circ U(t+\tau, t)^{-1} \leq C_2(\tau)M \quad \tau > 0$$

that is $C_2(\tau)M - M \circ U(t+\tau,t)^{-1}$ is a positive measure.

Lemma 4.32 (estimate of the square field operator) *If $B = Id$, we have for $u \in K$:*

$$\sqrt{\langle D_x P_\tau u(t,x), R D_x P_\tau u(t,x)\rangle} \leq \sqrt{C_1(\tau)}\ P_\tau \left(\|\sqrt{R} D_x u\|\right)(t,x) \qquad (4.15)$$

$$\int_H \left[P_\tau u(x+y,t) - P_\tau u(x,t)\right]^2 M(dy)$$
$$\leq C_2(\tau) P_\tau \left(\int_H [u(\cdot+y) - u(\cdot)]^2 M(dy)\right)(x,t) \quad (4.16)$$

So that combining the two estimates, we have:

$$\Gamma(P_\tau u, P_\tau u) \leq \max(C_1, C_2)(\tau) P_\tau \Gamma(u,u)$$

PROOF Let be $z \in H$ and $u(t,x) = \Phi(t)e^{i\langle x, h(t)\rangle}$, then:

$$
\begin{aligned}
&\langle D_x P_\tau u(t,x), Rz\rangle \\
&\qquad = \langle iU^*(t+\tau,t)h(t+\tau)P_\tau u(t,x), Rz\rangle \\
&\qquad = \left\langle iU^*(t+\tau,t)h(t+\tau)\int_H u(t+\tau,y)P\circ X(t+\tau,t,x)^{-1}(dy), Rz\right\rangle \\
&\qquad = \int_H \langle iU^*(t+\tau,t)h(t+\tau)u(t+\tau,x), Rz\rangle \, P\circ X(t+\tau,t,x)^{-1}(dy) \\
&\qquad = P_\tau \langle iU^*(t,t-\tau)h(t)u(t,x), Rz\rangle \\
&\qquad = P_\tau \langle D_x u(t,x), U(t,t-\tau)Rz\rangle \\
&\qquad = P_\tau \langle D_x u(t,x), \sqrt{R}\sqrt{R}^{-1}U(t,t-\tau)Rz\rangle \\
&\qquad = P_\tau \langle \sqrt{(}R)D_x u(t,x), \sqrt{R}^{-1}U(t,t-\tau)Rz\rangle \qquad (+) \\
&\qquad \le P_\tau \|\sqrt{(}R)D_x u(t,x)\| \|\sqrt{R}^{-1}U(t,t-\tau)Rz\| \\
&\qquad = P_\tau \|\sqrt{(}R)D_x u(t,x)\| \|U(t,t-\tau)Rz\|_{H_0} \\
&\qquad \le P_\tau \|\sqrt{(}R)D_x u(t,x)\| \sqrt{C_1(\tau)}\|Rz\|_{H_0} \\
&\qquad = P_\tau \|\sqrt{(}R)D_x u(t,x)\| \sqrt{C_1(\tau)}\|\sqrt{R}z\|
\end{aligned}
$$

Now, for every pair (t,x) choosing $z = D_x P_\tau u(t,x)$ we obtain:

$$
\begin{aligned}
&\langle D_x P_\tau u(t,x), RD_x P_\tau u(t,x)\rangle \\
&\qquad\qquad \le \sqrt{C_1(\tau)}\ \|\sqrt{R}D_x P_\tau u(t,x)\|\ \ P_\tau\left(\|\sqrt{R}D_x u\|\right)(t,x)
\end{aligned}
$$

or

$$
\sqrt{\langle D_x P_\tau u(t,x), RD_x P_\tau u(t,x)\rangle} \le \sqrt{C_1(\tau)}\ P_\tau\left(\|\sqrt{R}D_x u\|\right)(t,x) \tag{4.17}
$$

Note that we have used the special form of u only up to equation $(+)$, but by linearity of P and D_x it is clear that this also holds for sums. So we obtain (4.17) on all of K

Setting $\tilde{P} := P \circ X(t+\tau,t,0)^{-1}$ and $\tilde{M} := M \circ U(t+\tau,t)^{-1}$ we have for general $u \in K$: (setting $\tilde{\tau} := t+\tau$ for brevity)

$$
\begin{aligned}
&\int_H |P_\tau u(x+y,t) - P_\tau u(x,t)|^2 \, M(dy) \\
=&\int_H \left| \int_H u(U(\tilde{\tau},t)(x+y)+z,\tilde{\tau}) - u(U(\tilde{\tau},t)x+z,\tilde{\tau}) \tilde{P}(dz) \right|^2 M(dy) \\
\leq&\int_H \left(\int_H |u(U(\tilde{\tau},t)(x+y)+z,\tilde{\tau}) - u(U(\tilde{\tau},t)x+z,\tilde{\tau})|^2 \tilde{P}(dz) \right) M(dy) \\
=&\int_H \left(\int_H |u(U(\tilde{\tau},t)(x+y)+z,\tilde{\tau}) - u(U(\tilde{\tau},t)x+z,\tilde{\tau})|^2 M(dy) \right) \tilde{P}(dz) \\
=&\int_H \left(\int_H |u(U(\tilde{\tau},t)x+y+z,\tilde{\tau}) - u(U(\tilde{\tau},t)x+z,\tilde{\tau})|^2 \tilde{M}(dy) \right) \tilde{P}(dz) \\
\leq& C_2(\tau) \int_H \left(\int_H |u(U(\tilde{\tau},t)x+y+z,\tilde{\tau}) - u(U(\tilde{\tau},t)x+z,\tilde{\tau})|^2 M(dy) \right) \tilde{P}(dz) \\
=& C_2(\tau) P_\tau \left(\int_H |u(\cdot + y) - u(\cdot)|^2 M(dy) \right)(x,t)
\end{aligned}
$$

■

Corollary 4.33

$$\sqrt{\langle D_x P_\tau u(t,x), D_x P_\tau u(t,x) \rangle} \leq \ \|U(t+\tau,t)\| P_\tau \left(\|D_x u\| \right)(t,x) \qquad (4.18)$$

PROOF Reconsidering the proof above and setting $R = Id$ yields the result. ■

4.4.2 Functional Inequalities

Following [Röck/Wang03], we will now prove a Poincaré and a Harnack inequality.

Definition 4.34

$$\overline{u}_t := \int_H u(t,x)\nu_t(dx)$$

Proposition 4.35 *Assume that the ν_t have uniformly bounded first moments, that is:* $\sup_t\{\int_H \|x\|\nu_t(dx)\} < \infty$ *Then we have for all $u \in K$:*

$$\lim_{\tau\to\infty}\left(\sup_t |P_\tau u(t,x) - \overline{u}_{t+\tau}|\right) = 0 \quad \textit{for every fixed } x$$

Proof We have, since $\overline{u}_{t+\tau} := \int_H u(t+\tau,y)\nu_{t+\tau}(dy) = P_{t,t+\tau}u(t+\tau,\cdot)(y)\nu_t(dy)$ by the property of the evolution system :

$$\begin{aligned}
&|P_\tau u(t,x) - \overline{u}_{t+\tau}| = \left|\int_H P_{t,t+\tau}u(t+\tau,\cdot)(x) - P_{t,t+\tau}u(t+\tau,\cdot)(y)\nu_t(dy)\right| \\
&\le \|D_x P_{t,t+\tau}u(t+\tau,\cdot)\|_\infty \int_H |x-y|\nu_t(dy) \\
&\le \|U(t+\tau,t)\|\|P_\tau D_x u(t,\cdot)\|_\infty \int_H |x-y|\nu_t(dy) \\
&\le Me^{-\omega\tau}\|D_x u(t+\tau,\cdot)\|_\infty \int_H |x-y|\nu_t(dy) \overset{\tau\to\infty}{\longrightarrow} 0
\end{aligned}$$

since the integral is bounded by assumption and $\|D_x u(t,x)\| = \|h(t)\|$ but $h : \mathbb{R} \to H$ is continuous and periodic, hence bounded. ■

Proposition 4.36 (Poincaré Inequality) *Given assumption 4.31 and $B = Id$ then we have for* $C(\tau) := \max\left(\int_0^\tau C_1(s)ds, \int_0^\tau C_2(s)ds\right)$:

$$P_\tau u^2 - (P_\tau u)^2 \le C(\tau)P_\tau\Gamma(u,u) \qquad \textit{for all} \quad \tau > 0, u \in K \tag{4.19}$$

Proof Set $f(s) := P_{\tau-s}(P_s u)^2$ Then we have by the product rule:

$$\begin{aligned}
\frac{d}{ds}f(s) &= -P_{\tau-s}G(P_s u)^2 + P_{\tau-s}2P_s uGP_s u \\
&= -P_{\tau-s}[G(P_s u)^2 - 2P_s uGP_s u] = -P_{\tau-s}\Gamma(P_s u, P_s u)
\end{aligned}$$

Hence,

$$
\begin{aligned}
-\frac{d}{ds}f(s) &= P_{\tau-s}\Gamma(P_s u, P_s u) \\
&= P_{\tau-s}\left\langle \Sigma^*\nabla_x P_s u, \Sigma^*\nabla_x P_s u\right\rangle \\
&\quad + P_{\tau-s}\int_H \left[P_s u(x+y,t) - P_s u(x,t)\right]^2 M(dy) \\
&\le C_1(s) P_{\tau-s}P_s\left\langle \Sigma^*\nabla_x u, \Sigma^*\nabla_x u\right\rangle && \text{by (4.15)} \\
&\quad + C_2(s) P_{\tau-s}P_s \int_H \left[u(x+y,t) - u(x,t)\right]^2 M(dy) && \text{by (4.16)}
\end{aligned}
$$

Integrating with respect to s and noting that $f(0) = P_\tau f^2$ and $f(t) = (P_\tau f)^2$ we obtain:

$$
\begin{aligned}
P_\tau f^2 - (P_\tau f)^2 \le& \left(\int_0^\tau C_1(s)ds\right) P_\tau \left\langle \Sigma^*\nabla_x u, \Sigma^*\nabla_x u\right\rangle \\
&+ \left(\int_0^\tau C_2(s)ds\right) P_\tau \int_H \left[u(x+y,t) - u(x,t)\right]^2 M(dy)
\end{aligned}
$$

and the result is proved. ■

Corollary 4.37 *Let be C as in 4.36.*
Given that $C(\infty) < \infty$ we also have for all $u \in K$:

$$
\int_{[0,T]\times H} [u(t,x) - \overline{u}_t]^2 \nu(dt,dx) \le C(\infty) \int_{[0,T]\times H} \Gamma(u,u)\nu(dt,dx)
$$

Proof Integrating (4.19) with respect to ν yields, because of invariance:

$$
\int_{[0,T]\times H} u^2 - (P_\tau u)^2 \nu(dt,dx) \le C(\tau) \int_{[0,T]\times H} \Gamma(u,u)\nu(dt,dx)
$$

Letting $\tau \to \infty$ and using 4.35 together with dominated convergence we have:

$$
\int_{[0,T]\times H} u^2 - (\overline{u}_t)^2 \nu(dt,dx) \le C(\infty) \int_{[0,T]\times H} \Gamma(u,u)\nu(dt,dx)
$$

Since $\overline{u}$ does not depend on x anymore, we have:

$$\int_{[0,T]\times H} [u(t,x) - \overline{u}_t]^2 \nu(dt,dx) = \int_{[0,T]\times H} u^2(t,x) - 2u(t,x)\overline{u}_t + \overline{u}_t^2 \nu(dt,dx)$$
$$= \int_{[0,T]\times H} u^2(t,x) + \overline{u}_t^2 \nu(dt,dx) - 2\int_{[0,T]} \overline{u}_t \underbrace{\int_H u(t,x)\nu_t(dx)}_{\overline{u}_t} dt$$
$$= \int_{[0,T]\times H} u^2(t,x) - \overline{u}_t^2 \nu(dt,dx)$$

and the result follows. ∎

For the following Harnack inequality we need a definition:

Definition 4.38

$$\rho(x,y) := \inf\{ \|z\| \ : \ \sqrt{R}z = x - y\}$$

with the usual convention that $\inf\emptyset = \infty$ *, so* ρ *may take the value infinity if* $(x-y) \notin Im\sqrt{R}$*.*

The defintion of ρ is closely related to the notion of the pseudo inverse, for information on that see [Spde07] appendix B. The vague intuition for the term ρ in the Harnack inequality might be explained as follows: the operator $\sqrt{R}$ governs the diffusion of the underlying equation, if there is no diffusion possible from x to y then we have no result.

Proposition 4.39 (Harnack Inequality)

$$|P_\tau u(y)|^2 \le P_\tau u^2(x) \exp\left[\frac{\rho^2(x,y)}{\int_0^\tau \frac{1}{h(s)}ds}\right] \quad \textit{for all } u \in C_b \tag{4.20}$$

Proof First, let be $u \in K$ such that u is strictly positive. Since $P_{\tau-s}(P_s u)^2(t,x)$ will then also be strictly positive we can define:

$$\Phi(s) := \log[P_{\tau-s}(P_s u)^2(t,x_s)]$$

where x_s is given by:

$$x_s := x + \frac{(y-x)\int_0^s \frac{1}{h(\tau-u)}du}{\int_0^\tau \frac{1}{h(u)}du}$$

Note that we have $x_0 = x$ and $x_\tau = y$.
Differentiating Φ we obtain:

$$\frac{d}{ds}\Phi(s) = \frac{\frac{d}{ds}P_{\tau-s}(P_s u)^2(t,x_s)}{P_{\tau-s}(P_s u)^2(t,x_s)} \tag{4.21}$$

and for the numerator:

$$\begin{aligned}
\frac{d}{ds}[P_{\tau-s}(P_s u)^2(t,x_s)] &= \frac{d}{ds}[P_{\tau-s}(P_s u)^2](t,x_s) + \left\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s), \frac{dx_s}{ds}\right\rangle \\
&= -GP_{\tau-s}(P_s u)^2(t,x_s) + P_{\tau-s}[2P_s u G P_s u](t,x_s) \\
&\quad + \frac{1}{h(\tau-s)\int_0^\tau \frac{1}{h(u)}du}\left\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s),(y-x)\right\rangle \\
&= -P_{\tau-s}\Gamma(P_s u, P_s u) \\
&\quad + \frac{1}{h(\tau-s)\int_0^\tau \frac{1}{h(u)}du}\left\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s),(y-x)\right\rangle
\end{aligned} \tag{4.22}$$

We will now estimate $\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s),(y-x)\rangle$:

$$\begin{aligned}
&\left\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s),(y-x)\right\rangle \\
&= \inf_{\{z:\sqrt{R}z=x-y\}}\left\langle D_x[P_{\tau-s}(P_s u)^2](t,x_s),\sqrt{R}z\right\rangle && (x-y)\in \mathrm{Im}\sqrt{R} \\
&\leq \sqrt{\langle RD_x[P_{\tau-s}(P_s u)^2](t,x_s), D_x[P_{\tau-s}(P_s u)^2](t,x_s)\rangle}\rho(x,y) && \text{Cau.-Schw.} \\
&\leq \rho(x,y)\sqrt{h(\tau-s)}P_{\tau-s}\left(\sqrt{\langle RD_x(P_s u)^2, D_x(P_s u)^2\rangle}\right)(t,x_s) && \text{by (4.15)} \\
&\leq 2\rho(x,y)\sqrt{h(\tau-s)}P_{\tau-s}\left(P_s u\sqrt{\langle RD_x(P_s u), D_x(P_s u)\rangle}\right)(t,x_s) && \text{chain rule}
\end{aligned} \tag{4.23}$$

Combining (4.21),(4.22) and (4.23) we obtain:

$$\frac{d}{ds}\Phi(s) \leq \frac{-P_{\tau-s}\Gamma(P_s u, P_s u)}{P_{\tau-s}(P_s u)^2(t,x_s)}$$

$$+ \frac{\frac{1}{h(\tau-s)\int_0^\tau \frac{1}{h(u)}du} 2\rho(x,y)\sqrt{h(\tau-s)}P_{\tau-s}\left(P_s u\sqrt{\langle RD_x(P_s u), D_x(P_s u)\rangle}\right)(t,x_s)}{P_{\tau-s}(P_s u)^2(t,x_s)}$$

writing out Γ *but omitting the non-local part, we get:*

$$\leq \frac{-P_{\tau-s}\left(\langle RD_x(P_s u), D_x(P_s u)\rangle\right)(t,x_s)}{P_{\tau-s}(P_s u)^2(t,x_s)}$$

$$+ \frac{\frac{1}{\sqrt{h(\tau-s)}\int_0^\tau \frac{1}{h(u)}du} 2\rho(x,y)P_{\tau-s}\left(P_s u\sqrt{\langle RD_x(P_s u), D_x(P_s u)\rangle}\right)(t,x_s)}{P_{\tau-s}(P_s u)^2(t,x_s)}$$

$$= \frac{1}{P_{\tau-s}(P_s u)^2(t,x_s)}$$
$$\times P_{\tau-s}\left((P_s u)^2\left[2H\frac{\sqrt{\langle RD_x(P_s u), D_x(P_s u)\rangle}}{P_s u} - \frac{\langle RD_x(P_s u), D_x(P_s u)\rangle}{(P_s u)^2}\right]\right)(t,x_s)$$

where we have set $H := \frac{\rho(x,y)}{\sqrt{h(\tau-s)}\int_0^\tau \frac{1}{h(u)}du}$ for brevity.

Furthermore, setting $G := \frac{\sqrt{\langle RD_x(P_s u), D_x(P_s u)\rangle}}{P_s u}$:

$$\begin{aligned}
\frac{d}{ds}\Phi(s) &\leq \frac{1}{P_{\tau-s}(P_s u)^2(t,x_s)}P_{\tau-s}\left((P_s u)^2\left[-G^2+2HG\right]\right)(t,x_s) \\
&= \frac{1}{P_{\tau-s}(P_s u)^2(t,x_s)}P_{\tau-s}\left((P_s u)^2\left[-G^2+2HG-H^2+H^2\right]\right)(t,x_s) \\
&\leq \frac{1}{P_{\tau-s}(P_s u)^2(t,x_s)}P_{\tau-s}\left((P_s u)^2\left[H^2\right]\right)(t,x_s) \\
&= H^2
\end{aligned}$$

since H depends neither on x_s nor on t. Integration over s yields:

$$\begin{aligned}
\log[(P_\tau u)^2(t,y)] - \log[(P_\tau u^2)(t,x)] &= \Phi(\tau) - \Phi(0) \\
&\le \int_0^\tau H^2(s)ds \\
&= \int_0^\tau \frac{\rho^2(x,y)}{h(\tau-s)(\int_0^\tau \frac{1}{h(u)}du)^2}ds \\
&= \frac{\rho^2(x,y)}{\int_0^\tau \frac{1}{h(u)}du}
\end{aligned}$$

Hence, applying the exponential yields:

$$(P_\tau u)^2(t,y) \;\le\; P_\tau u^2(t,y) \quad \frac{\rho^2(x,y)}{\int_0^\tau \frac{1}{h(u)}du}$$

and the proof is complete for positive functions.
To obtain the result for general u, note first, that it is sufficient to have it for $|u|$, since we have:

$$|P_\tau u(t,y)|^2 \le [P_\tau |u|(t,y)]^2 \le P_\tau u^2(t,x) \exp\left[\frac{\rho^2(x,y)}{\int_0^\tau \frac{1}{h(s)}ds}\right]$$

Of course, we cannot take modulus without leaving K, but as K is an algebra we may take the square of our functions. Thus, let be $u \in C_b$ and $\varepsilon > 0$. Then $f := \sqrt{|u|} \in C_b$. Now, by 4.40 we can approximate f pointwisely by functions u_n from K. Then, $u_n^2 + \varepsilon$ is strictly positive, it will approach $|u| + \varepsilon$ and since the approximating functions are uniformly bounded, we can take limits in (4.20) and obtain the result via dominated convergence and then letting $\varepsilon \to 0$. ■

Lemma 4.40 *For every $f \in C_b$ we can find a sequence $u_n \in K$ such that:*

- *$u_n \to f$ pointwisely*
- $\sup_{x,n} |u_n(x)| \le 1 + \sup_x |f(x)|$

Proof We try to approximate a given $f \in C_b$ as follows:
Let be $(e_k)_{k\in\mathbb{N}}$ a complete orthonormal system in $D(A^*)$. Let be

$$f_n(h) := g_n(P_n h)$$

where $P_n : H \to \mathbb{R}^n \quad h \mapsto (\langle h, e_1\rangle, \dots, \langle h, e_n\rangle)$
and $g_n : \mathbb{R}^n \to \mathbb{R} \quad (x_1, \dots, x_n) \mapsto f(x_1 e_1 + \cdots + x_n e_n)$

Note that each g_n is continuous and bounded and that $f_n = f$ on $\operatorname{span}\{e_1, \dots, e_n\}$. Moreover, we have $P_n \to Id$ strongly and hence $f_n \to f$ pointwisely.

To obtain an approximation by functions of K we now need to approach every g_n by trigonometric polynomials. So let be $g : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded. For each $n \in \mathbb{N}$ let be be ϕ_n a function that coincides with g on $\{\|x\|_\infty \le n\}$ vanishes on $\{\|x\|_\infty \ge n+1\}$ and is still continuous. Since ϕ_n can be extended to a continuous $(n+1)$-periodic function, we can approximate it with respect to the supremum norm on $\{\|x\|_\infty \le n+1\}$ with trigonometric polynomials, by Fejér's Theorem. Let us call $g^{(n)}$ an approximation of ϕ_n on $\{\|x\|_\infty \le n+1\}$ up to $\frac{1}{n}$, that is we have:

$$\sup_{\|h\|_\infty \le n+1} |g^{(n)}(h) - \phi_n(h)| < \frac{1}{n}$$

This implies, of course:

$$\sup_{\|h\|_\infty \le n} |g^{(n)}(h) - g(h)| < \frac{1}{n}$$

hence, by construction, we have $g^{(n)} \to g$ pointwisely for $n \to \infty$ and by periodicity it is clear that we have $\sup_{x,n} |g^{(n)}(x)| \le 1 + \sup_x |g(x)|$.

Now, applying the above approximation to our functions g_n from above, we denote:

$$u_n(h) := f_n^{(n)}(h) := g_n^{(n)}(P_n h)$$

and an easy calculation shows that $f_n^{(n)}$ is indeed a function from K, since we have:

$$\begin{aligned} e^{i\langle (k_1,\dots,k_n), P_n h\rangle} &= e^{i\langle (k_1,\dots,k_n), (\langle h,e_1\rangle,\dots,\langle h,e_n\rangle)\rangle} \\ &= e^{i(\langle h,k_1e_1\rangle+\cdots+\langle h,k_ne_n\rangle)} = e^{i\langle h,k_1e_1+\cdots+k_ne_n\rangle} \end{aligned}$$

and $(k_1e_1 + \cdots + k_ne_n) \in D(A^*)$.
To see that $f_n^{(n)}(h) \to f(h)$ for each fixed h we calculate:

$$|f(h) - f_n^{(n)}(h)| \le |f(h) - f_n(h)| + |f_n(h) - f_n^{(n)}(h)|$$

We have already stated that the first tern will tend to 0. For the second term, note, that for fixed h there is N independent of n such that $\|P_nh\|_\infty < N$. Thus, we obtain:

$$|f_n(h) - f_n^{(n)}(h)| = |g_n(P_nh) - g_n^{(n)}(P_nh)| \le \sup_{\|x\|_\infty \le N} |g_n(x) - g_n^{(n)}(x)| < \frac{1}{n}$$

whenever n is bigger than N. ■

Appendix A

Cores

As it is often difficult to describe the action of a generator G on the whole of its domain D, one would like to find a subspace, C say, of D which is in some sense fitted to G. For example, a space of smooth functions might be suited to a differential operator. When this is possible without loss of information, in the sense that the whole generator can be reconstructed from its values on the test functions, such a C is called a core. Formally:

Definition A.1 *Let be G a generator of a strongly continuous semigroup on a Banach space $(B, \| \, \|)$ with domain D. A subspace C of D is called a core for G if it is dense in D with respect to the graph norm $\| \, \|_G$ on D, that is $\|x\|_G := \|x\| + \|Gx\|$.*

So the above mentioned reconstruction can be carried out by taking the closure of the graph of the restricted operator. There are other characterizations of a core available. We will cite one and derive another one from it. The material here is taken from [Arendt86], another useful reference for cores is [Eberle97].

Proposition A.2 *Let G be the generator of a strongly continuous semigroup $S(t)$ and let C be a subspace of $D(G)$. If S is the only semigroup, which has a generator that extends $G|_C$, then C is a core.*

We will sketch a proof:
Notice that the contraposition of the statement is: If C is not a core than $G|_C$ admits more than one generating extension. So the idea is to construct a

semigroup $T(t)$ different from S, whose generator coincides with G on C. By a pertubation result, whose proof we omit here, $G+P$ will be a generator for any P that is continuous with respect to the graph norm on $D(G)$. Since we do not want to change the generator on C, we need a P that vanishes on C but not on all of $D(G)$. Since C is not a core, hence not dense in $D(G)$ with respect to the graph norm, we have the existence of a linear functional f on $D(G)$ continuous for the graph norm, that vanishes on C but not everywhere. (Recall that a subspace is dense if and only if every continuous functional vanishing there is trivial). Setting $P(x) = f(x)u$ for a fixed $0 \neq u \in D(G)$ we have the reqired pertubation, and it is clear that $G + P \neq G$

A complete proof can be found in [Arendt86] page 46.

Lemma A.3 *Let $S(t)$ be a strongly continuous semigroup with generator G on a Banach space B. Let C be a dense subspace of $D(G)$ that is invariant under S. Then C is a core.*

Proof We have to prove, that G is the only generating extension of $G|_C$. Assume A is another generating extension, with semigroup $T(t)$. Consider the following Cauchy problem, where $f \in C$:

$$dX(t) = GX(t)dt\ ,\ X(0) = f$$

By definition $S(t)f$ is a solution that stays in C by assumption. So $S(t)f$ is also a solution for the problem:

$$dX(t) = AX(t)dt\ ,\ X(0) = f$$

since G and A coincide on C. Of course $T(t)f$ is also a solution of the second problem. As both problems admit a unique solution (see [Pazy74] Chapter 4) we conclude $S(t)f = T(t)f$ whenever $f \in C$. As the operators $S(t)$ and $T(t)$ are continuous for any t, and C is dense in B (as it is dense in a dense subset)we have $S = T$. ∎

Appendix B

Lévy's Continuity Theorem in Infinite Dimensions

In this section we follow [Vakh81]. [Somm07] was helpful.
Consider a sequence of probability measures μ_n on a real separable Hilbert space H, and the sequence f_n of their respective characteristic functions.
On $\mathbb{R}^d$ Lévy's continuity theorem states, that the pointwise convergence of f_n to a limit function f, which is continuous in 0, implies that f is the characteristic function of some probability measure μ to whom the μ_n converge weakly.

On an infinite dimensional Hilbert space, this need no longer hold: Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H and let ν_n be the standard normal distribution on R^n. Let $T_n : \mathbb{R}^n \to H$ be an isometric embedding with $Im(T_n) = span(e_1, ..., e_n)$
Then the image measure $T_n\nu_n$ on H will have characteristic function:

$$f_n(h) = e^{-\frac{1}{2}\sum_{k=1}^{n}|\langle h,e_k\rangle|^2}$$

It is clear, that $\lim_{n\to\infty} f_n(h) = e^{-\frac{1}{2}\|h\|^2} =: f(h)$ and that f is continuous in the norm topology.
Alas, f is not a characteristic function. To see this, note that every characteristic function must be even weakly continuous, since

$$\mathbb{E}[e^{i\langle h_n,X\rangle}] \quad \overset{h_n\overset{w}{\to}h}{\longrightarrow} \quad \mathbb{E}[e^{i\langle h,X\rangle}]$$

holds by dominated convergence.
But f is not weakly continuous. We have $e_k \overset{w}{\to} 0$ for $k \to \infty$ but $f(e_k) = e^{-\frac{1}{2}}$ for each k.

So, we have to impose stronger conditions on the limit function f to assure that it is itself a characteristic function. It turns out that one has to ask for continuity with respect to a weaker topology, the Sazonov topology.

Definition B.1 *The Sazonov topology on a Hilbert space H is the coarsest topology such that the functions*

$$x \mapsto \langle Sx, x \rangle$$

are continuous for any positive , self-adjoint trace class operator S.

With the help of this definition, we can formulate an extension of Bochner's theorem:

Theorem B.2 (Sazonov) *A functional f on a Hilbert space H is the characteristic function of a measure on H, if and only if:*

- *f is positive definite*
- $f(0) = 1$
- *f is continuous in the Sazonov topology*

PROOF See [Vakh81] 3.1.6 ■

For the proof of 4.10 we have tried to make use of this theorem without success. However, we give the underlying idea.
Recall that our limit function was essentially of the form:

$$f(h) := \exp\left\{\int_{-\infty}^{t} \lambda(L_r h)dr\right\}$$

where the $L_r := B^*(r)U^*(t,r)$ were linear operators.
$f(0) = 1$ is trivial and positive definiteness of the limit is rather easy to show. For the continuity, we know that λ is Sazonov continuous and then it is not hard to show that also $\lambda \circ L$ is Sazonov continuous for any bounded linear operator L. Hence, we have pointwise convergence under the integral, and the result would follow by dominated convergence, if we had an integrable bound uniform in h_n as $h_n \to h$. Regrettably, we were unable to find norm-bounded neighborhoods in the Sazonov topology and without those this approach seems to be hopeless.

Bibliography

[Alb/Rue05] Albeverio, R. and Rüdiger, B. : Stochastic Integrals and the Lévy-Ito Decomposition Theorem on Separable Banach Spaces - Stochastic Analyis and Applications 23, 2005

[App04] Applebaum, David : Lévy Processes and Stochastic Calculus - Cambridge University Press, 2004

[App06] Applebaum, David: Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space - Séminaire de Probabilités 39, 2006

[Arendt86] Arendt, Wolfgang in: One-parameter Semigroups of Positive Operators - Heidelberg : Springer, 1986

[Chi/Lat99] Chicone, Carmen and Latushkin, Yuri : Evolution Semigroups in Dynamical Systems and Differential Equations - Providence : American Mathematical Society , 1999

[Chung68] Chung, Kai Lai : A course in probability theory - Academic Press, 1968

[Dal/Krei74] Daleckii, Ju. L. and Krein, M. G. : Stability of solutions of differential equations in Banach space - Providence : American Mathematical Society , 1974

[daPr/Lun07] DaPrato, Guiseppe and Lunardi Alessandra : Ornstein-Uhlenbeck operators with time periodic coeffients - Journal of Evolution Equations 7, 2007

[daPr04] DaPrato, Guiseppe : Kolmogorov Equations for Stochastic PDEs - Birkhäuser, 2004 [daPr/zab03] DaPrato, Guiseppe and Zabczyk, Jerzy : Ergodicity for infinite dimensional systems - Cambridge University Press, 2003

[Dudley89] Dudley, Richard : Real Analysis and Probability - Wadsworth & Brooks/Cole 1989

[Eberle97] Eberle, Andreas : Uniqueness and Non-Uniqueness of Singular Diffusion Operators - University of Bielefeld : Doctorate Thesis, 1997

[Fuhr/Röck00] Fuhrmann, Marco and Röckner, Michael : Generalized Mehler Semigroups: The Non-Gaussian Case - Potential Analysis 12, 2000

[Knab06] Knäble, Kristian : Stochastic Evolution Equations with Lévy Noise and applications to the Heath-Jarrow-Morton Model - University of Bielefeld : Diploma Thesis, 2006

[Lesc/Röck02] Lescot, Paul and Röckner, Michael : Generators of Mehler-Type Semigroups as Pseudo-Differential Operators - Infinite Dimensional Analysis, Quantum Probability and Related Topics 5, 2002

[Lesc/Röck04] Lescot, Paul and Röckner, Michael : Perturbations of Generalized Mehler Semigroups and Applications to Stochastic Heat Equations with Levy Noise and Singular Drift - Potential Analysis 20, 2004

[Linde83] Linde, Werner : Infinitely Divisible and Stable Measures on Banach Spaces - Leipzig : Teubner, 1983

[Ma/Röck92] Ma,Zhi-Ming and Röckner,Michael : Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. - Heidelberg : Springer, 1992

[Nei/Zag07] Neidhardt, Hagen and Zagrebnov, Valentin : Linear non-autonomous Cauchy problems and evolution semigroups - arXiv:0711.0284v1 [math-ph] 2007

[Pazy74] Pazy, Amnon : Semi-groups of linear operators and applications to partial differential equations . - College Park Maryland, 1974

[Prott90] Protter, Philip : Stochastic Integration and Differential Equations - Springer, 1990

[Röck/Wang03] Röckner, Michael and Wang, Feng-Yu : Harnack and functional inequalities for generalized Mehler semigroups - Journal of Functional Analysis 203, 2003

[Sato99] Sato, Ken-Iti : Lévy processes and infinitely divisible distributions - Cambridge University Press , 1999 .

[Spde07] Röckner, Michael and Prevot, Claudia : A Concise Course on Stochastic Partial Differential Equations - Heidelberg : Springer 2007

[Schw73] Schwartz, Laurent : Radon measures on arbitrary topological spaces and cylindrical measures . - Oxford : Univ. Pr. , 1973

[Somm07] Sommer, Christian : Lévy's continuity Theorem on Banach Spaces - University of Bielefeld : Bachelor Thesis, 2007

[Stolze05] Stolze, Stefan : Stochastic Equations in Hilbert Space with Lévy Noise and their Application in Finance - University of Bielefeld : Diploma Thesis, 2005

[Vakh81] Vakhania, Nikolai : Probability Distributions on Linear Spaces - Elsevier, 1981