# Stochastic evolution equations in weighted $L^p$-spaces and applications to the Heath-Jarrow-Morton model

Diploma Thesis
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# Contents

# Chapter 1

# Introduction

## 1.1 Aims of the diploma thesis

This diploma thesis has got two aims.
These are to confirm the theory introduced by Manthey and Zausinger in [MaZa] for functions defined on $\Theta = \mathbf{R}_+^d$ with $d \in \mathbf{N}$(more detailed than in [MaZa]) and to apply the results received in chapter 2 to the case described by Heath,Jarrow and Morton in [HeJaMo](cf.chapter 7,proposition 4,p.93 there)(cf.chapter 3).

So chapter 2 is concerned about the theory proposed by Manthey and Zausinger in [MaZa].
After repeating some properties of Hilbert spaces from [DPZa92] in section 2.1,the procedure is the same as in Manthey‘s and Zausinger‘s paper,i.e. one defines spaces $L^2_\rho(\Theta)$ with $\rho > d$ for $d \in \mathbf{N}$ and spaces $L^{2\nu}_\rho(\Theta)$ with $\nu \in \mathbf{N}$ analogously to [MaZa](cf. the list of notations in section 1.2),s.t. $L^2_\rho(\Theta)$ is a separable Hilbert space.Then the results from chapter 2 and sections 3.1–3.3 in [MaZa](cf. pp.40–69 there) are shown in the given situation.
First it is shown that the stochastic convolution

$$\int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)$$

is welldefined for a Q-Wienerprocess W on $L^2(\Theta)$ with an operator Q as described in chapter 2,a mapping

$$\Sigma{:}[\,0,T\,] \times L^2_\rho(\Theta) \to \mathcal{L}_2(Q^{\frac{1}{2}}L^2(\Theta), L^2_\rho(\Theta))$$

defined by

$$(\Sigma(t,\varphi)\psi)(x) := \sigma(t,\varphi(x))\psi(x), x \in \Theta$$

for $\sigma{:}[\,0,T\,] \times \mathbf{R} \to \mathbf{R}$ with (L1),(L2) and an almost strong evolution operator U(cf. section 1.3) under certain conditions to X.Theorem 2.2.4 shows the existence of a continuous version of the stochastic convolution under certain restrictions to U,which will be described in chapter 2.

Da Prato's and Zabczyk's theory then leads to the existence of a pathwise unique solution in the sense of [MaZa](cf. section 2.2.,definition 2.2.3 in the following) for an $\omega$-dependent,progressively measurable drift F,s.t. both the drift and the volatility are Lipschitzian.There is even,under additional assumptions on U with initial condition $\xi \in L^{2\kappa}_{\rho}(\Theta)$,a pathwise unique solution,which is P-almost surely in $L^{2\kappa}_{\rho}(\Theta) \subset L^{2}_{\rho}(\Theta)$ for certain $\kappa \in \mathbf{N}$.
Furthermore there is an estimate for the expectation of X depending on the expectation of $\xi$ in $L^{2}_{\rho}(\Theta)$ resp. $L^{2\kappa}_{\rho}(\Theta)$,which Manthey and Zausinger do not show in their paper(cf.theorem 2.2.7).
The comparison theorem 3.3.1(ii) for $\omega$-dependent drifts from [MaZa](cf. section 3.3,p.61 there) holds true in $L^{2}_{\rho}(\Theta)$ as well(cf.theorem 2.2.11).

Having these preparations section 2.3 shows the existence of a solution in case of a non-Lipschitzian function f defining the drift $\omega$-wisely by

$$F(t,\omega,\varphi)(x) := f(t,\omega,\varphi(x)),t \in [\,0,T\,],\varphi \in L^{2}_{\rho}(\Theta),x \in \Theta$$

if f fulfills properties (PG) with exponent $\nu \in \mathbf{N}$ and (LG)(cf.section 1.3) and if the initial condition $\xi$ is appropriate.
Furthermore there is again an estimate for the expectation of the solution depending on the expectation of the initial condition.(cf.theorem 2.3.2).
At the end of chapter 2 existence results in spaces $L^{p}_{\rho}(\Theta)$,which are considered in [AsMa](cf. section 2,theorem 1,p.241 there),are shown with f having the above mentioned properties instead of those mentioned in [AsMa].

Summarizing the results of chapter 2 one has

1. existence of a solution to the SDE
   $X(t) := \int_0^t U(t,s)F(s,\omega,X(s))\,ds + \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s),t \in [\,0,T\,]$
   with f defining F as above being non-Lipschitzian and,in contrast to section 3.4 from [MaZa],$\omega$-dependent.

2. existence of a solution to the above mentioned SDE in weighted $L^p$-spaces with p defined analogously to [AsMa].

3. estimates for the solutions,that Manthey and Zausinger do not make in their paper.

The second aim and content of chapter 3 is the application of the result received in chapter 2 to the model described by Heath,Jarrow and Morton in [HeJaMo](cf.chapter 7,proposition 4,p.93 there).
In order to reach this aim there is first of all an introduction into the theory of markets with bonds and one riskless asset based on [HaPl](cf.chapter 3,pp.232-242 there) in section 3.1 and a repetition of the terms and conditions introduced in [HeJaMo] in section 3.2.
In particular the following holds for $\bar{f}(t,T)$,which is the rate one can receive from $t \in [\,0,T\,]$ up to $T \in [\,0,\tau\,]$ with $\tau > 0$:

$$\bar{f}(t,T) - \bar{f}(0,T) = \int_0^t \alpha(s,T,\omega)\,ds + \sum_{n=1}^{k} \int_0^t \sigma_n(s,T,\omega)\,dw_n(s)$$

for all $T \in [\,0,\tau\,]$,$t \in [\,0,T\,]$ with $\bar{f}(0,\cdot)$,$\alpha$,$\sigma_n$,$n = 1,2,\ldots,k$ as in condition (C1)(cf. section 3.2).
Under certain conditions absence of arbitrage is equivalent to

$$\alpha(t,T) = -\sum_{n=1}^{k} \sigma_n(t,T)\left(\lambda_n(t) - \int_t^T \sigma_n(t,s)\,ds\right)$$

for $\lambda_n$:$[\,0,T\,] \times \Omega \to \mathbf{R}$ given by

$$\lambda_n(t) := \gamma_n(t;S_1,S_2,\ldots,S_k), 0 \le S_1 < S_2 < \ldots < S_k \le \tau \text{ arbitrary}$$

with $\gamma_n$ as in condition (C4)(cf.section 3.2).
In [HeJaMo](cf. chapter 7,p.93 there) there is a result telling that given $\lambda_n$:$[\,0,\tau\,] \times \Omega \to \mathbf{R}$ and bounded,nonnegative $\sigma_n$: $S_T \times \mathbf{R} \to \mathbf{R}$, $n = 1,2,\ldots,k$(with T and k as above,$S_T$ as in section 1.2),all Lipschitzian on $\mathbf{R}$,there is a uniformly measurable family $(\bar{f}(t,T))_{t\in[\,0,T\,]}$ of processes with

$$\bar{f}(t,T) - \bar{f}(0,T) = \int_0^t -\sum_{n=1}^{k} \sigma_n(s,T,\bar{f}(s,T))\left(\lambda_n(s) - \int_s^T \sigma_n(s,y,\bar{f}(s,y))\,dy\right)ds + \sum_{n=1}^{k} \int_0^t \sigma_n(s,T,\bar{f}(s,T))\,dw_n(s)$$

where $w_n$ are real-valued Brownian motions on $[\,0,T\,]$.

Using the time-homogeneity assumption

$$\sigma_n(t,s_1,\cdot) = \sigma_n(t,s_2,\cdot)$$

for all $t \in [\,0,\tau\,]$,$s_1,s_2 \in [\,t,\tau\,]$,$n = 1,2,\ldots,k$,one gets with the shift-semigroup $(S(t))_{t\ge0}$ on $\mathbf{R}$ given by

$$[S(t)h](x) := h(x+t) \quad (1.1)$$

for all $t \ge 0$,$x \in \mathbf{R}$ and all functions h defined on $\mathbf{R}$ and with

$$r_t := \bar{f}(t,t+\cdot) \quad (1.2)$$

the so-called Heath-Jarrow-Morton equation

$$r_t = S(t)r_0 + \int_0^t S(t-s)F(s,\omega,r_s)\,ds + \int_0^t S(t-v)\Sigma(s,r_s)\,dW(s)$$

for all $t \ge 0$,where f belonging to F is defined by

$$\begin{aligned} f(\cdot,\cdot,x) := &\quad (\mathcal{S}(\sigma(\cdot)))(x) + \lambda\sigma(\cdot,x) \\ := &\quad \sigma(\cdot,x)\int_0^x \sigma(\cdot,z)\,dz + \lambda\sigma(\cdot,x) \quad (1.3) \end{aligned}$$

for all $x \in \mathbf{R}_+$.

In order to apply the results of chapter 2 one has to show that the shift-semigroup

$$[S(t)\varphi](x) := \varphi(x+t), \varphi \in L^2_\rho(\Theta), x \in \Theta, t \geq 0$$

leads to an almost strong evolution operator in the sense of [MaZa],which has got the properties needed in chapter 2.

Section 3.3 shows the applicability of the theory in the nuclear case,since there it is shown that the evolution operator defined by U(t,s):=S(t-s) fulfills the conditions needed in chapter 2.
Unfortunately it seems to be impossible to apply the results from chapter 2 in the cylindrical case,s.t. from section 3.3 onwards the results are always restricted to the nuclear case.

The application of chapter 2 in the nuclear case with $d = 1$ shows existence of a solution to the Heath-Jarrow-Morton equation.One can give up the nonnegativity assumption on $\sigma$ from proposition 4 in [HeJaMo] and still get a solution of the Heath-Jarrow-Morton equation.In order to show this one even does not need the full strength of the theory,so it is possible to extend the framework and allow to have a function $\lambda$ defined on $[\,0,T\,] \times \Omega \times \mathbf{R}$, comparably to the situation in definition 4 in [Te](cf.section 2.2,p.4 there).
$\lambda$ being progressively measurable with property (PG) with $\nu = 1$ is enough to have the existence of a solution to the Heath-Jarrow-Morton equation,whereas in case of an exponent $\nu > 1$ nonnegativity of $\sigma$ and a property similar to (LG) for f are needed to have the existence of a solution.

In section 3.5 the differences to the results from [Te] and [AsMa] are described,and it is shown that in case of functions

$$\lambda{:}\mathbf{R}_+ \times \Omega \times \mathbf{R} \to \mathbf{R}$$
$$\sigma{:}\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$$

s.t. the conditions from section 3.4 hold with a constant,which is independent of t and $\omega$,and of a modified semigroup $\bar{S}$ given by

$$\bar{S}(t) := e^{-t}S(t)\ ,\ t \geq 0$$

there exists a solution r of

$$r_t = \bar{S}(t)r_0 + \int_0^t \bar{S}(t-s)F(s,\omega,r_s)\,ds + \int_0^t \bar{S}(t-s)\Sigma(s,r_s)\,dW(s)\ ,\ t \geq 0$$

in the sense that in 2.2.3 condition (ii) holds for arbitrary $T > 0$ and $t \in [\,0,T\,]$ is replaced by $t \geq 0$ in 2.2.3(i) and (iv).The estimates shown before still hold,when

fixing an arbitrary $T > 0$.
Furthermore section 3.5 describes in how far there are restrictions to the rate caused by the definition of spaces in [MaZa] and [AsMa] resp. in [Fi] and [Te] and in how far these restrictions make sense due to the economic interpretation.

Summarizing one can say,that the theory developed in [MaZa] allows for the following improvements of proposition 4 from [HeJaMo]:

1. It is allowed to give up the nonnegativity-assumption on $\sigma$ as in the original Heath-Jarrow-Morton model(cf. condition (C1) in section 3.1)

2. One has existence even with $\lambda$ being of the form

$$\lambda:[\,0,T\,] \times \Omega \times \mathbf{R} \to \mathbf{R}$$

## 1.2 List of notations

The following overview over the notation used in the following chapters is given to finish the introduction:

Let $\Theta$ be a subspace of $\mathbf{R}^d$ for $d \in \mathbf{N}$.

$C_c^1(\Theta) := \{\varphi : (\Theta, \mathcal{B}(\Theta)) \to (\mathbf{R}, \mathcal{B}(\Theta)) \mid \varphi$ is continuously differentiable with compact support$\}$

$$L^2(\Theta) := \left\{\varphi : (\Theta, \mathcal{B}(\Theta)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \mid \int_\Theta \varphi^2(x)\, dx < \infty\right\}$$

$$L^2_\rho(\Theta) := \left\{f : (\Theta, \mathcal{B}(\Theta)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \mid \int_\Theta f^2(x)\, \mu_\rho(dx) < \infty\right\}$$

with $\mathbf{N} \ni \rho > d$,$\mu_\rho(x) := (1 + |x|^2)^{-\frac{\rho}{2}}$,$|\cdot|$ euclidian norm in $\mathbf{R}^d$

$\mathcal{L}(B, K) := \{T : B \to K \mid T$ is bounded and linear$\}$ ,
where B and K are Banach spaces.

$\mathcal{L}(K) := \mathcal{L}(K, K)$

$\mathcal{L}_2(B, K) := \{T : B \to K \mid T$ is a Hilbert-Schmidt operator$\}$ ,
where B and K are Hilbert spaces.

$\mathcal{L}_2(K) := \mathcal{L}_2(K, K)$

$$L^p_\rho(\Theta) := \left\{\varphi : (\Theta, \mathcal{B}(\Theta)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \mid \int_\Theta |\varphi(x)|^p\, \mu_\rho(dx) < \infty\right\}$$

for $p \in \mathbf{N}$ with norm $||\cdot||_{\rho,p}$ given by

$$||\varphi||_{\rho,p} := \left(\int_\Theta |\varphi(x)|^p \mu_\rho(dx)\right)^{\frac{1}{p}}$$

$$L^p([0, T] \times \Omega; B) := \left\{f : [0, T] \times \Omega \to B \mid \mathbf{E}\int_0^T ||f(t)||_B^p\, dt < \infty\right\}$$

for a probability space ($\Omega$,$\mathcal{F}$,P) with expectation $\mathbf{E}$ under P,a Banach space B and fixed $T > 0$ with norm given by

$$||f||_{L^p} := \left(\mathbf{E}\int_0^T ||f(t)||_B^p\, dt\right)^{\frac{1}{p}}, f \in L^p([0, T] \times \Omega; B)$$

$S_T := \{(s, t) \mid 0 \leq s \leq t \leq T\}$ , $T > 0$

Let $C([0, T]; L^p([0, T] \times \Omega; B))$ denote the following set
$\{\varphi : ([0, T] \times \Omega, \mathcal{P}_T) \to (B, \mathcal{B}(B)) \mid \varphi \in L^p([0, T] \times \Omega; B)$is time-continuous$\}$
with $p \in \mathbf{N}$,where $\mathcal{P}_T$ is the $\sigma$-algebra of predictable sets on $[0, T]$ with $T > 0$ and B is a Banach space.Let the norm on this space be given by

$$||X||_{C([0,T];B)(p,q)} := \left(\sup_{t\in[0,T]} \mathbf{E}||X(t)||_B^p\right)^{\frac{q}{p}}, X \in C([0, T]; L^p([0, T] \times \Omega; B);$$

for $p, q \in \mathbf{N}$,write $||\cdot||_{C([0,T];B)(p)} := ||\cdot||_{C([0,T];B)(p,p)}$,$p \in \mathbf{N}$

## 1.3 Notations from [MaZa]

Let B be a Banach space and $T > 0$ fixed:
A family U=$U(t,s)_{(s,t)\in S_T}$ of operators from B onto itself is called an **almost strong evolution operator**,if

(i) U(t,t)=I , $t \in [\,0,T\,]$

(ii) U(t,r)U(r,s)=U(t,s) , $0 \le s \le r \le t \le T$

(iii) U($\cdot$,s) is strongly continuous on $[\,s,T\,]$, U(t,$\cdot$) is strongly continuous on $[\,0,t\,]$ and

$$\sup_{(s,t)\in S_T} ||U(t,s)|| \le c(T) < \infty$$

holds true with the usual operator norm $||\cdot||$ .

(iv) there exists a closed linear operator A(t) on B for almost all $t \in [\,0,T\,]$,s.t. U(t,s):$\mathcal{D}(A(s)) \to \mathcal{D}(A(t))$ holds for all $t > s$ and

$$\int_s^t A(r)U(r,s)\varphi\, dr = (U(t,s) - I)\varphi$$

holds for $\varphi \in \mathcal{D}_{t,s}(A) := \{\varphi \in B \mid U(r,s)\varphi \in D(A(r)), s \le r \le t\}$.

Analogously to the theory of one-parameter semigroups $(A(t))_{t\in[\,0,T\,]}$ is called the **generator** of U.If (iv) even holds for all $t \in [\,0,T\,]$,U is called a **strong evolution operator**.

Let $\Lambda$:$[\,0,T\,] \times \mathbf{R} \to \mathbf{R}$ with a fixed $T > 0$.Take over the following notations from Manthey‘s and Zausinger‘s paper(confer pp.42,44–46,54+69 there):

(L1) There is an $L(T) > 0$,s.t.

$$|\Lambda(t,x) - \Lambda(t,y)| \le L(T)|x-y|$$

holds true for all (t,x,y)$\in [\,0,T\,] \times \mathbf{R} \times \mathbf{R}$.

(L2) There is an $L(T) > 0$,s.t.

$$|\Lambda(t,0)| \le L(T)$$

holds true for all t$\in [0, T]$.

(PG) There exists a constant $c_\Lambda(T) > 0$,s.t.

$|\Lambda(t, x)| \leq c_\Lambda(T)(1 + |x|^\nu)$,$x \in \mathbf{R}$

holds true for all $t \in [0, T]$.

(LG) There exists a nonnegative constant $c_\Lambda(T)$,s.t.

$$\Lambda(t, x) \geq -c_\Lambda(T)(1 - x), x \leq 0$$
$$\Lambda(t, x) \leq c_\Lambda(T)(1 + x), x \geq 0$$

holds true for all t$\in [0, T]$.

# Chapter 2

# Stochastic evolution equations in weighted $L^p$-spaces

For this chapter fix an arbitrary $T > 0$.

## 2.1 Some results on Hilbert spaces

During this section let Y,H be separable Hilbert spaces and let Q be an operator in $\mathcal{L}(Y)$ defined by

$$Qy_n := a_n y_n$$

for an orthonormal basis $(y_n)_{n\in\mathbf{N}}$ of Y and a sequence of nonnegative numbers $a_n$.
Define a Hilbert space $\mathcal{Y} := Q^{\frac{1}{2}}Y$ by

$$Q^{\frac{1}{2}}y_n := \sqrt{a_n}y_n$$

with the inner product given by

$$< \varphi, \psi >_{\mathcal{Y}} := \sum_{\substack{n\in\mathbf{N} \\ a_n\neq 0}} a_n^{-1} < \varphi, y_n >_Y < \psi, y_n >_Y$$

where $< \cdot, \cdot >_Y$ denotes the inner product in Y.
Let $(\Omega,\mathcal{F},P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $(w_n)_{n\in\mathbf{N}}$ be an independent family of real-valued Brownian motions.
Furthermore let W be the corresponding Q-Wienerprocess,i.e. W is given by

$$W(t) := \sum_{n\in\mathbf{N}} \sqrt{a_n}w_n(t)y_n \ , \ t \in [0,T]$$

First of all consider a result from [DPZa92] in order to construct stochastic integrals.

**Definition 2.1.1:**(cf.[DPZa92],section 4.2;pp.90,91;equations (4.6),(4.10))

A $\mathcal{L}_2(\mathcal{Y}, H)$-valued process $(\Phi(t))_{t\in[0,T]}$ is called an **elementary process**,if given a partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_k = T$$

of $[0, T]$,$\mathcal{L}(\mathcal{Y}, H)$-valued random variables there exist

$$\Phi_0 , \Phi_1 , \cdot , \Phi_{k-1}$$

with finitely many values,s.t. each $\Phi_m$ is measurable w.r.t. the $\sigma$-algebra $\mathcal{F}_{t_m}$ and

$$\Phi(t) := \sum_{m=0}^{k-1} \Phi_m \mathbf{1}_{(t_m,t_{m+1}]}(t)$$

is fulfilled.
Given such a process the stochastic integral is defined by

$$\int_0^t \Phi(s)\, dW(s) = \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t))$$

and the so-called Ito-isometry

$$\mathbf{E}\left|\left|\int_0^t \Phi(s)\, dW(s)\right|\right|^2 = \mathbf{E}\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds$$

where $||\cdot||_{\mathcal{L}_2(\mathcal{Y},H)}$ denotes the Hilbert-Schmidt norm,holds for all elementary $\mathcal{L}_2(\mathcal{Y}, H)$-valued processes $(\Phi(t))_{t\in[0,T]}$ and all $t \in [0, T]$.

**Lemma 2.1.2:**(cf. [DPZa92],section 4.2,proposition 4.7(ii),p.93)

If $\Phi$ is a $\mathcal{L}_2(\mathcal{Y}, H)$-valued,predictable process with

$$\mathbf{E}\int_0^T ||\Phi(s)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds < \infty$$

there exists a sequence of elementary processes $(\Phi_n)_{n\in\mathbf{N}}$,s.t.

$$\lim_{n\to\infty} \mathbf{E}\int_0^T ||\Phi(s) - \Phi_n(s)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds = 0$$

holds true.Then one defines the stochastic integral belonging to $\Phi$ as the H-limit of the stochastic integrals belonging to $\Phi_n$ constructed by 2.1.1.
Obviously Ito‘s isometry holds true for this stochastic integral as well.

In their paper Manthey and Zausinger consider a special kind of two-parameter semigroup,which they call almost strong evolution operator(For a definition of this term consider section 1.3).Note,that in his paper [Se],which Manthey and

Zausinger refer to later,Seidler uses the term evolution system,which differs from the term strong evolution operator used in the sense of Manthey and Zausinger. According to [Se](confer section 0,condition (E),pp.68,69 there) this term is defined on separable Hilbert spaces H differing from the definition in section 1.3(with B:=H) in (iii) and (iv).These items are replaced by

(iii)‘
$$U(\cdot,\cdot)\varphi{:}S_T \to H$$
is continuous for each $\varphi \in H$

(iv)‘
$$\frac{d}{dt}U(t,s)\varphi = A(t)U(t,s)\varphi$$
$$\frac{d}{ds}U(t,s)\varphi = -U(t,s)A(t)\varphi$$
for all $\varphi \in \mathcal{D}_{t,s}(A)$

Thus given a Hilbert space H each evolution system in the sense of [Se] is a strong evolution operator in the sense of [MaZa],which leads to the question whether it suffices to assume properties (iii) and (iv) from the definition given here,in order to be able to apply the theory derived in [Se],i.e. one has to check whether one of the conditions (iii)‘,(iv)‘ is needed for one of the results in question from [Se].
First of all one makes use of the following factorization formula cited in [Se]:

**Lemma 2.1.3:**

Let $\psi$ be a predictable $\mathcal{L}_2(\mathcal{Y},H)$-valued process and let $q>2$ with

$$(\Psi) \qquad \mathbf{E}\int_0^T ||\psi(s)||^q_{\mathcal{L}_2(\mathcal{Y},H)}\,ds < \infty$$

Having the properties from section 1.3 and

$$U(\cdot,\cdot)\phi{:}S_T \to \mathcal{L}_2(\mathcal{Y},H),\varphi \in H$$

for U,where $\phi$ denotes the multiplication operator $\mathcal{Y}\to H$ belonging to $\varphi\in H$,the following holds for all $t\in[\,0,T\,]$,$0<\alpha<\frac{1}{2}$:

$$\int_0^t U(t,s)\psi(s)\,dW(s) = \frac{\sin\pi\alpha}{\pi}(R_\alpha Z_{\alpha,U})(t)$$

where W is a Q-Wienerprocess and R,Z are given by

$$Z_{\alpha,U}(t) := \int_0^t (t-s)^{-\alpha}U(t,s)\psi(s)\,dW(s)$$

and

$$R_\alpha f(t) := \int_0^t (t-s)^{\alpha-1}U(t,s)f(s)\,ds$$

for each process f $\in L^q([\,0,T\,]\times\Omega;H)$ and each $t\in[\,0,T\,]$.

**Proof:**
In [Se] Seidler only gives a hint,how this lemma can be shown.He suggests to

apply the stochastic Fubini theorem to a function h defined by

$$h(r,s) := (t-s)^{\alpha-1}(s-r)^{-\alpha}\mathbf{1}_{[0,s]}(r)U(t,r)\psi(r)$$

One has $h(\cdot,s) \in \mathcal{F}_s$ for each predictable process $\psi$ and arbitrary $s \in [0,t]$,since $\psi(r) \in \mathcal{F}_r \subset \mathcal{F}_s$,$r \leq s$ and $h(r,s) = 0_{\mathcal{L}_2(\mathcal{Y},H)} \in \mathcal{F}_0 \subset \mathcal{F}_s$ hold for $r > s$ due to the predictability of $\psi$.
Show that the following property is fulfilled:

$$\int_0^t \left( \int_0^T \mathbf{E}||h(s,x)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{2}} dx < \infty$$

So:

$$\int_0^t \left( \int_0^T \mathbf{E}||h(s,x)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{2}} dx$$
$$= \int_0^t \left( \int_0^T \mathbf{E}||\, (t-x)^{\alpha-1}(x-s)^{-\alpha}\mathbf{1}_{[0,x]}(s)U(t,s)\psi(s)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{2}} dx$$
$$= \int_0^t \left( \int_0^x \mathbf{E}||\, (t-x)^{\alpha-1}(x-s)^{-\alpha}U(t,s)\psi(s)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{2}} dx$$
$$= \int_0^t (t-x)^{\alpha-1} \left( \int_0^x (x-s)^{-2\alpha}\mathbf{E}||\, U(t,s)\psi(s)\, ||^2_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{2}} dx$$
$$\leq c(T) \left( \int_0^T t^{\alpha-1}\, dt \right) \left( \int_0^T s^{-2\alpha}\, ds \right)^{\frac{1}{2}} \left( \int_0^T \mathbf{E}||\psi(s)||^q_{\mathcal{L}_2(\mathcal{Y},H)}\, ds \right)^{\frac{1}{q}} < \infty$$

where property (iii) from the definition of almost strong evolution operators (cf. section 1.3) and Young‘s inequality for convolutions as well as $2\alpha < 1$ and $(\Psi)$ were used in the last step.Thus the stochastic Fubini theorem from [DPZa92](cf. section 4.6,theorem 4.18,p.109(with $\mu$ = Lebesgue measure) there) is applicable and leads with the help of

$$\frac{\pi}{\sin \pi\alpha} = \int_s^t (t-x)^{\alpha-1}(x-s)^{-\alpha}\, dx$$

for $\alpha \in [0,1)$(confer f.e. [DPZa92],section 5.3,proof of theorem 5.9,p.128) to the following:

$$
\begin{aligned}
\int_0^t U(t,s)\psi(s)\,dW(s) = &\ \frac{\sin\pi\alpha}{\pi}\int_0^t \left[\int_s^t (t-x)^{\alpha-1}(x-s)^{-\alpha}\,dx\right] U(t,s)\psi(s)\,dW(s)\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t \left[\int_s^t (t-x)^{\alpha-1}(x-s)^{-\alpha}U(t,s)\psi(s)\,dx\right] dW(s)\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t \left[\int_0^t (t-x)^{\alpha-1}(x-s)^{-\alpha}\mathbf{1}_{[0,x]}(s)U(t,s)\psi(s)\,dx\right] dW(s)\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t \left[\int_0^t (t-x)^{\alpha-1}(x-s)^{-\alpha}\mathbf{1}_{[0,x]}(s)U(t,s)\psi(s)\,dW(s)\right] dx\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t \left[\int_0^x (t-x)^{\alpha-1}(x-s)^{-\alpha}U(t,x)U(x,s)\psi(s)\,dW(s)\right] dx\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t (t-x)^{\alpha-1}U(t,x)\left(\int_0^x (x-s)^{-\alpha}U(x,s)\psi(s)\,dW(s)\right) dx\\
= &\ \frac{\sin\pi\alpha}{\pi}\int_0^t (t-x)^{\alpha-1}U(t,x)Y(x)\,dx\\
= &\ \frac{\sin\pi\alpha}{\pi}(R_\alpha Y)(t)
\end{aligned}
$$

In this chain of equations the stochastic Fubini theorem was used in the fourth step,whereas the semigroup property (ii) from the definition of almost strong evolution operators(cf. section 1.3) was used in the fifth step.Thus the claim holds true for almost strong evolution operators as they were used in [MaZa].
q.e.d.

**Remark 2.1.4:**

The factorization formula for strongly continuous one-parameter semigroups can f.e. be found in [DPZa96](cf.chapter 5,section 2,theorem 5.2.5 there).

Additionally to the factorization formula the following results concerning estimates of stochastic integrals in Hilbert spaces are also needed:

**Theorem 2.1.5:**(cf. [DPZa92],chapter 7,lemma 7.2,p.182)

Let Q be as above with the additional assumption

$$\sum_{n\in\mathbf{N}} a_n < \infty$$

(the so-called nuclear case) and let W be a Q-Wienerprocess on Y.

Given an arbitrary $r \geq 1$ and an arbitrary $\mathcal{L}_2(\mathcal{Y}, H)$-valued,predictable process $(\Phi(t))_{t\in[0,T]}$,one has

$$
\begin{aligned}
\mathbf{E}\left(\sup_{s\in[0,t]}\left|\left|\int_0^s \Phi(\varepsilon)\,dW(\varepsilon)\right|\right|^{2r}\right) \leq &\ c_r \sup_{s\in[0,t]}\mathbf{E}\left(\left|\left|\int_0^s \Phi(\varepsilon)\,dW(\varepsilon)\right|\right|^{2r}\right)\\
\leq &\ C_r\mathbf{E}\left(\int_0^t ||\Phi(\varepsilon)||^2_{\mathcal{L}_2(\mathcal{Y},H)}\,d\varepsilon\right)^r \quad t\in[0,T]
\end{aligned}
$$

with $c_r := \left(\frac{2r}{2r-1}\right)^{2r}$,$C_r := (r(2r-1))^r\left(\frac{2r}{2r-1}\right)^{2r^2}$,where $||\cdot||$ denotes the norm belonging to H.

**Proof:**
Confer [DPZa92],chapter 7,section 1,p.183.

**Theorem 2.1.6:**(cf. [DPZa92],chapter 7,lemma 7.7,p.194)

Let Q be as above with $a_n = 1$ for all $n \in \mathbf{N}$(the so-called cylindrical case) and let W be a Q-Wienerprocess on Y(i.e. one has $\mathcal{Y} = Y$).

For each $r \geq 1$ and each arbitrary $\mathcal{L}_2(Y,H)$-valued,predictable process $(\Phi(t))_{t\in[\,0,T\,]}$ the following holds:

$$\sup_{s\in[\,0,t\,]} \mathbf{E}\left|\left|\int_0^s \Phi(\varepsilon)\,dW(\varepsilon)\right|\right|^{2r} \leq (r(2r{-}1))^r \left(\int_0^t (\mathbf{E}||\Phi(\varepsilon)||^{2r}_{\mathcal{L}_2(Y,H)})^{\frac{1}{r}}\,d\varepsilon\right)^r \;,\; t \in [\,0,t\,]$$

**Proof:**
Confer [DPZa92],chapter 7,section 1,pp.194,195.

In order to finish the section a short repetition concerning integrability of random variables:

**Definition 2.1.7:**(cf. [DPZa92],section 1.1,p.19)

Let B be a separable Banach space.A B-valued random variable X on a probability space $(\Omega, \mathcal{F}$,P) is called **Bochner-integrable**,if

$$\int_\Omega ||X(\omega)||\,P(d\omega) < \infty$$

holds true.Then the Bochner-integral is defined by

$$\mathbf{E}||X|| := \int_\Omega ||X(\omega)||\,P(d\omega)$$

**Lemma 2.1.8:**

Let B be a separable Banach space,let $(X(t))_{t\in[\,0,T\,]}$ be a B-valued process with

$$\mathbf{E}\int_0^T ||X(s)||\,ds < \infty$$

Then

$$\int_0^t X(s)\,ds$$

is Bochner-integrable for all $t \in [\,0,T\,]$.

**Proof:**
According to 2.1.7 the following is to show for arbitrary $t \in [\,0,T\,]$:

$$\int_\Omega \left|\left|\int_0^t X(s)\,ds\right|\right|\,P(d\omega) < \infty$$

With the help of the assumption one gets

$$\begin{aligned}
\int_\Omega \left\| \int_0^t X(s,\omega)\,ds \right\| P(d\omega) &\leq \int_\Omega \int_0^T ||X(s,\omega)||\,ds P(d\omega) \\
&= \mathbf{E} \int_0^T ||X(s)||\,ds \\
&< \infty
\end{aligned}$$

s.t. the Bochner-integral exists.
q.e.d.

## 2.2 Preparing results on $L^2_\rho(R^d_+)$

Let $d \in \mathbf{N}$ be arbitrary but fixed,let $\Theta := \mathbf{R}^d_+$ and $Y := L^2(\Theta)$, $H := L^2_\rho(\Theta)$ with a fixed $\rho > d$.
Then the measure $\mu_\rho$ defined as in section 1.2(cf. the definition of $L^2_\rho(\Theta)$ there) is finite on $\Theta$.In the following leave out $\Theta$,when dealing with spaces of functions defined on $\Theta$,i.e. write $L^2$ instead of $L^2(\Theta)$ resp. $L^2_\rho$ instead of $L^2_\rho(\Theta)$ and so on.

**Remark 2.2.1:**

Note that the separability of $L^2_\rho(\mathbf{R}^d)$ mentioned by Manthey and Zausinger implies the separability of $L^2_\rho$,since one can embed this space into $L^2_\rho(\mathbf{R}^d)$ by

$$\varphi(x) := 0; \varphi \in L^2_\rho, x \in \mathbf{R}^d \setminus \Theta$$

For the rest of this chapter let U be an almost strong evolution operator in the sense of section 1.3 on $L^2_\rho$.

In this section the results of chapter 2 and sections 3.1–3.3 from [MaZa](cf.pp.40–56 there) are shown in the situation of $\Theta = \mathbf{R}^d_+$.Especially the existence of the stochastic integral both in $L^2_\rho$ and $L^{2\kappa}_\rho$ with $\kappa \in \mathbf{N}$ and fixed natural numbers $\rho > d$ is ensured.

First of all let U be an almost strong evolution operator on $L^2_\rho$ with generator $(A(t))_{t\in[0,T]}$.First assume the following properties,which correspond to (A0),(A1) from [MaZa](cf. [MaZa],chapter 2,p.42):

(CD) For each $t \in [0, T]$ A(t):$\mathcal{D}(A(t)) \to L^2_\rho$ is linear and closed with $\mathcal{D}(A(t)) \subset L^2_\rho$.Furthermore

$$\mathcal{D}(A) := \bigcap_{0\leq t\leq T} \mathcal{D}(A(t))$$

is dense in $L^2_\rho$.

(PP) The almost strong evolution operator U on $L^2_\rho$ is positivity preserving in the sense,that

$$\varphi \geq 0 \Rightarrow U(t, s)\varphi \geq 0, (s, t) \in S_T\ , \ \varphi \in L^2_\rho$$

holds true.

**(i) The nuclear case**

Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of nonnegative real numbers with

$$\sum_{n\in\mathbf{N}} a_n < \infty$$

and let $(e_n)_{n\in\mathbf{N}}$ be an orthonormal basis of $L^2$ with $e_n \in L^\infty$ and

$$\sup_{n\in\mathbf{N}} ||e_n||_\infty < \infty$$

Such an orthonormal basis exists according to [MaZa](cf. chapter 2,p.40 there) due to [OsPe].
In complete analogy to section 2.1 one gets a Hilbert space $\mathcal{Y} := Q^{\frac{1}{2}}(L^2)$ and defines the multiplication operator $\phi$ belonging to $\varphi \in L^2_\rho$ in $L^2$ by

$$\phi(\psi)(x) := \varphi(x)\psi(x), \psi \in L^2, x \in \Theta$$

While regarding the nuclear case denote $\mathcal{L}_2(\mathcal{Y}, L^2_\rho)$ by $\mathcal{L}_2$. Thus the multiplication operator fulfills

$$\begin{aligned} ||\phi||_{\mathcal{L}_2} = & \sum_{n\in\mathbf{N}} ||\phi(Q^{\frac{1}{2}}(e_n))||^2_{\rho,2} \\ \leq & \sup_{n\in\mathbf{N}} ||e_n||^2_\infty \, TrQ \, ||\varphi||^2_{\rho,2} \end{aligned}$$

where $TrQ := \sum_{n\in\mathbf{N}} a_n$ denotes the trace of Q.
Since $\mathcal{Y}$ is a Hilbert space

$$\int_0^T \mathbf{E}||\varphi(s)||^2_{\rho,2} \, ds < \infty$$

implies

$$\begin{aligned} \mathbf{E}\int_0^T ||\phi(s)||^2_{\mathcal{L}_2} \, ds \leq & \sup_{n\in\mathbf{N}} ||e_n||^2_\infty \, TrQ \int_0^T \mathbf{E}||\varphi(s)||^2_{\rho,2} \, ds \\ < & \infty \end{aligned}$$

for a $L^2_\rho$-valued,predictable process $\varphi$,as a consequence of which the stochastic integral

$$\int_0^t \phi(s) \, dW(s)$$

exists for all $t \in [\,0, T\,]$ due to 2.1.2,2.1.3.Having $\sigma$ with properties (L1),(L2) define an operator

$$\Sigma : [\,0, T\,] \times L^2_\rho \to \mathcal{L}_2$$

by

$$(\Sigma(t, \varphi)\psi)(x) := \sigma(t, \varphi(x))\psi(x) \ , \ \psi \in \mathcal{Y} \ , \ \varphi \in L^2_\rho \ , \ x \in \Theta \quad (2.1)$$

and define for $t \in [\,0, T\,]$

$$U(t, \cdot)\Sigma : [\,0, t\,] \times L^2_\rho \to \mathcal{L}_2$$

by setting

$$((U(t,\cdot)\Sigma)(s,\varphi)\psi)(x) := (U(t,s)(\Sigma(s,\varphi)\psi))(x), \psi \in L^2, x \in \Theta$$

Then one has for all $L^2_\rho$-valued,predictable processes X with

$$\int_0^T \mathbf{E}||X(s)||^2_{\rho,2}\, ds < \infty$$

and for all $t \in [\,0, T\,]$:

$$\begin{aligned} \mathbf{E}\int_0^t ||U(t,s)\Sigma(s,X(s))||^2_{\mathcal{L}_2}\, ds \leq &\ \sup_{n\in\mathbf{N}} ||e_n||^2_\infty TrQ \int_0^t \mathbf{E}||U(t,s)\sigma(s,X(s))||^2_{\rho,2}\, ds \\ \leq &\ c(T, c(T), c_\sigma(T)) \left(1 + \int_0^T \mathbf{E}||X(s)||^2_{\rho,2}\, ds\right) \\ < &\ \infty \end{aligned}$$

as a consequence of which

$$\int_0^t U(t,s)\Sigma(s,X(s))\, dW(s)$$

is welldefined and the Ito-isometry implies

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,X(s))\, dW(s)\right|\right|^2_{\rho,2} = \mathbf{E}\int_0^t ||U(t,s)\Sigma(s,X(s))||^2_{\mathcal{L}_2}\, ds$$

### (ii) The cylindrical case

Let $a_n = 1$ for all $n \in \mathbf{N}$ and $Qe_n := a_n e_n$ for all $n \in \mathbf{N}$,where $(e_n)_{n\in\mathbf{N}}$ is an orthonormal basis as in case (i).In this case denote $\mathcal{L}_2(L^2, L^2_\rho)$ by $\mathcal{L}_2$.
With (2.1) and (L1),(L2) for $\sigma$

$$\Sigma(t,\varphi)\psi \in \mathcal{M} := \{h \in L^1_\rho \mid h = \phi(\psi)\ ,\ \varphi \in L^2_\rho\ ,\ \psi \in L^2\}$$

holds for arbitrary $\varphi \in L^2_\rho$,$\psi \in L^2$.

Define $U(t,\cdot)\Sigma$ for $t \in [\,0,T\,]$ as in the nuclear case and make the following assumption only for the cylindrical case.This assumption is just (A2) from [MaZa](cf. chapter 2,p.45 there):

(CC) For $(s,t) \in S_T$ there exists an extension of U(t,s) to $\mathcal{M}$(again denoted by U(t,s)). Furthermore there exists a $\gamma \in [\,0,1\,)$,s.t. $U(t,s)\phi \in \mathcal{L}_2$ and

$$||U(t,s)\phi||^2_{\mathcal{L}_2} \leq c(T)(t-s)^{-\gamma}||\varphi||^2_{\rho,2}$$

hold for arbitrary $\varphi \in L^2_\rho$.

Defining $U(t,\cdot)\Sigma$ for $t \in [\,0,T\,]$ as in the nuclear case,(CC) and (L1),(L2) lead to

$$||(U(t,s)\Sigma)(s,\varphi)||^2_{\mathcal{L}_2} \leq c(T)(t-s)^{-\gamma}||\sigma(s,\varphi)||^2_{\rho,2}$$
$$\leq c(c(T),c_\sigma(T))(t-s)^{-\gamma}(1+||\varphi||^2_{\rho,2}) \quad (2.2)$$

for $\varphi \in L^2_\rho$,$(s,t) \in S_T$.Thus with $\gamma \in [0,1)$ one has the following inequation for $t \in [0,T]$ and predictable processes X in $L^2_\rho$ with

$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||^2_{\rho,2}\,ds < \infty$$

$$\mathbf{E}\int_0^t ||U(t,s)\Sigma(s,X(s))||^2_{\mathcal{L}_2}\,ds \leq c(c(T),c_\sigma(T))\int_0^t s^{-\gamma}\,ds$$
$$+c(c(T),c_\sigma(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||^2_{\rho,2}\,ds$$
$$< \infty$$

So as in the nuclear case the stochastic integral

$$\int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)$$

is welldefined for all $t \in [0,T]$.Ito‘s isometry and (2.2) even imply:

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)\right|\right|^2_{\rho,2} = \int_0^t \mathbf{E}||(U(t,s)\Sigma)(s,X(s))||^2_{\mathcal{L}_2}\,ds$$
$$\leq c(c(T),c_\sigma(T))\int_0^t (t-s)^{-\gamma}(1+\mathbf{E}||X(s)||^2_{\rho,2})\,ds \quad (2.3)$$

Define F by

$$F(t,\omega,\varphi)(x) := f(t,\omega,\varphi(x)) \quad (2.4)$$

for all $\varphi \in \mathcal{D}(F)$,$x \in \Theta$,$(t,\omega) \in [0,T]\times\Omega$ with $\mathcal{D}(F) := L^{2\nu}_\rho$ for a function f fulfilling (PG) with exponent $\nu$.Thus it is necessary to have the existence of the stochastic integral in spaces $L^{2\kappa}_\rho$ with $\kappa \in \mathbf{N}$.

In the following the results hold,if the contrary is not explicitly mentioned,both in the nuclear and in the cylindrical case.
(So again $\mathcal{L}_2$ denotes $\mathcal{L}_2(Q^{\frac{1}{2}}L^2, L^2_\rho)$.)

There are further assumptions needed on U,which correspond to (A3),(A4) and (A5) from [MaZa](cf. chapter 2,p.46 resp. pp.54,55 there):

(E1) For each $\kappa \in \mathbf{N}$ there exists a constant $c(\kappa,T) > 0$,s.t.

$$(U(t,s)|\psi|)^\kappa \leq c(\kappa,T)U(t,s)|\psi|^\kappa$$

holds in $L^2_\rho$ for each $\psi \in L^{2\kappa}_\rho$ and each $(s,t) \in S_T$.

(E2) For each $\kappa \in \mathbf{N}$ there exists a constant $c(\kappa,T) > 0$,s.t.

$$\mathbf{E}\int_{\Theta}\left(\int_0^t \sum_{n\in\mathbf{N}}(U(t,s)\phi(s)Q^{\frac{1}{2}}e_n)^2\right)^{\kappa} d\mu_\rho$$
$$\leq c(\kappa,c(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||\varphi(s)||_{\rho,2\kappa}^{2\kappa}\,ds$$

holds for each $L_\rho^{2\kappa}$-valued predictable process $\varphi=(\varphi(t))_{t\in[\,0,T\,]}$ considered as a process of multiplication operators.

(BA) There exists a sequence $(A_N(t))_{t\in[\,0,T\,]}$ of operators with the following properties
(i) $A_N(t)\in\mathcal{L}(L_\rho^2)$,$t\in[\,0,T\,]$,and

$$\sup_{t\in[\,0,T\,]}||A_N(t)||\leq c(N), N\in\mathbf{N}$$

(ii) For each $N\in\mathbf{N}$ the family $(A_N(t))_{t\in[\,0,T\,]}$ generates an almost strong evolution operator $U_N$,which is positivity preserving and fulfills

$$\sup_{(s,t)\in S_T}||(U_N(t,s)-U(t,s))\varphi||_{\rho,2}^2\to 0$$

for $N\to\infty$ and $\varphi\in L_\rho^2$.

(E1),(E2) for U lead to the following lemma,which is an extension of remark 2.3.(ii) from [MaZa](cf. chapter 2,p.48 there):

**Lemma 2.2.2:**

Let $\kappa\in\mathbf{N}$,let $\varphi=(\varphi(t))_{t\in[\,0,T\,]}$ be a $L_\rho^{2\kappa}$-valued,predictable process with the property

$$\sup_{t\in[\,0,T\,]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||\varphi(s)||_{\rho,2\kappa}^{2\kappa}\,ds<\infty$$

Then one has

$$\int_0^t U(t,s)\phi(s)\,dW(s)\in L_\rho^{2\kappa}$$

P-almost surely for $t\in[\,0,T\,]$,where $\phi(s)$ denotes the multiplication operator belonging to $\varphi(s)$.
Furthermore there exists a positive constant c($\kappa$,c(T)) depending on $\kappa$ and U with

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|\right|_{\rho,2\kappa}^{2\kappa}\leq c(\kappa,c(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||\varphi(s)||_{\rho,2\kappa}^{2\kappa}\,ds<\infty \quad (2.5)$$

and there exists a positive constant c($\kappa$,$\gamma$,q,T,c(T)) for all

$$q>\tfrac{2\kappa}{1-\gamma}$$

s.t.

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|\right|_{\rho,2\kappa}^q \leq c(\kappa,\gamma,q,T,c(T))\int_0^t \mathbf{E}||\varphi(s)||_{\rho,2\kappa}^q\,ds \quad (2.6)$$

holds.

**Proof:**
Fix an arbitrary $t \in [0,T]$.Define a predictable process $\chi$ by $\chi(s) := U(t,s)\phi(s)$.Then

$$\chi Q^{\frac{1}{2}} e_n : \Omega \times [0,t] \to L_\rho^2$$

is predictable for each $n \in \mathbf{N}$ as well.Approximating $\chi$ by elementary processes $(\chi_m)_{m\in\mathbf{N}}$ as in section 2.1(cf. theorem 2.1.2 there),one gets

$$\chi_m Q^{\frac{1}{2}} e_n : \Omega \times [0,t] \to L_\rho^2$$

with only finitely many values in $L_\rho^2$.Thus for all n,m$\in \mathbf{N}$ there exists a representative $\psi_m^{(n)}:\Omega \times [0,t] \times \Theta \to \mathbf{R}$ with limiting process

$$\psi^{(n)}:\Omega \times [0,t] \times \Theta \to \mathbf{R}$$

s.t. the following equations hold with the help of the definition of $\chi$:

$$\mathbf{E}\int_0^t \int_\Theta \sum_{n\in\mathbf{N}} [(U(t,s)\phi(s)Q^{\frac{1}{2}}e_n)(x) - \psi^{(n)}(s,x)]^2\,\mu_\rho(dx)\,ds = 0 \quad (2.7)$$

$$\mathbf{E}\int_\Theta \left[\left(\int_0^t U(t,s)\phi(s)\,dW(s)\right)(x) - \left(\sum_{n\in\mathbf{N}}\int_0^t \psi^{(n)}(s,x)\,dw_n(s)\right)\right]^2 \mu_\rho(dx) = 0 \quad (2.8)$$

First of all application of the Burkholder-Gundy equation,(2.7) and (E2) implies

$$\begin{aligned}
&\int_\Theta \mathbf{E}\left(\sum_{n\in\mathbf{N}}\int_0^t \psi^{(n)}(s,x)\,dw_n(s)\right)^{2\kappa} \mu_\rho(dx) \\
&\leq c(\kappa)\mathbf{E}\int_\Theta \left(\int_0^t \sum_{n\in\mathbf{N}} \left(\psi^{(n)}(s,x)\right)^2\,ds\right)^\kappa \mu_\rho(dx) \\
&= c(\kappa)\mathbf{E}\int_\Theta \left(\int_0^t \sum_{n\in\mathbf{N}} (U(t,s)\phi(s)Q^{\frac{1}{2}}e_n)^2(x)\,ds\right)^\kappa \mu_\rho(dx) \\
&\leq c(\kappa,c(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||\varphi(s)||_{\rho,2\kappa}^{2\kappa}\,ds < \infty
\end{aligned}$$

Using (2.8) implies

$$\int_0^t U(t,s)\phi(s)\,dW(s) = \sum_{n\in\mathbf{N}}\int_0^t \psi^{(n)}(s,\cdot)\,dw_n(s)$$

$P \otimes \mu_\rho$-almost surely for arbitrary $t \in [0,T]$. The last two equations together imply (2.5).
In case of $q > \frac{2\kappa}{1-\gamma}$

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|\right|_{\rho,2\kappa}^q = \mathbf{E}\left|\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|^\kappa\right|\right|_{\rho,2}^{2r}$$

with $r := \frac{q}{2\kappa} > \frac{1}{1-\gamma} \geq 1$ for arbitrary $t \in [0,T]$.

Consider the case $\kappa = 1$ first.2.1.5 resp. 2.1.6 immediately lead to

$$\mathbf{E}\left(\sup_{s\in[0,t]}\left|\left|\int_0^s U(t,\epsilon)\phi(\epsilon)\,dW(\epsilon)\right|\right|_{\rho,2}^{2r}\right)$$
$$\leq c(q)\mathbf{E}\left(\int_0^t ||U(t,\epsilon)\phi(\epsilon)||_{\mathcal{L}_2}^2\,d\epsilon\right)^r$$
$$\leq c(q,c(T))\mathbf{E}\left(\int_0^t (t-\epsilon)^{-\gamma}||\varphi(\epsilon)||_{\rho,2}^2\,d\epsilon\right)^r$$
$$\leq c(q,c(T))\mathbf{E}\left(\left[\int_0^T \epsilon^{-\frac{\gamma q}{q-2}}\,d\epsilon\right]^{\frac{q-2}{q}}\left[\int_0^t ||\varphi(\epsilon)||_{\rho,2}^q\,d\epsilon\right]^{\frac{2}{q}}\right)^r$$
$$= c(\gamma,q,T,c(T))\int_0^t \mathbf{E}||\varphi(\epsilon)||_{\rho,2}^q\,d\epsilon$$

for all $t \in [0,T]$,i.e. (2.6) in this case.
Coming to the case of $\kappa > 1$ show that $|U(t,s)\phi(s)|^\kappa$ is a Hilbert-Schmidt operator for all $(s,t) \in S_T$.
(E1) and (E2) lead to

$$\begin{aligned}\sum_{n\in\mathbf{N}}\mathbf{E}\int_\Theta(|U(t,s)\phi(s)|^\kappa Q^{\frac{1}{2}}e_n)^2(x)\mu_\rho(dx) \leq\ & c(\kappa,T)\mathbf{E}\sum_{n\in\mathbf{N}}\int_\Theta(U(t,s)|\phi(s)|^\kappa Q^{\frac{1}{2}}e_n)^2(x)\mu_\rho(dx)\\ \leq\ & c(\kappa,T,c(T))(t-s)^{-\gamma}\mathbf{E}||\,|\varphi(s)|^\kappa||_{\rho,2}^2\\ =\ & c(\kappa,T,c(T))(t-s)^{-\gamma}\mathbf{E}||\varphi(s)||_{\rho,2\kappa}^{2\kappa} < \infty \quad (2.9)\end{aligned}$$

which is just the Hilbert-Schmidt property.
In the case $q > \frac{2\kappa}{1-\gamma}$ one has

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|\right|_{\rho,2\kappa}^q = \mathbf{E}\left|\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|^\kappa\right|\right|_{\rho,2}^{2r}$$

with $r := \frac{q}{2\kappa} > \frac{1}{1-\gamma} \geq 1$ for arbitrary $t \in [0,T]$.
Fix t and consider a sequence $(\tau_N)_{N\in\mathbf{N}}$ of partitions of $[0,t]$ with

$$|\tau_N| := \sup_{t_i\in\tau_N}(t_{i+1}-t_i) \to 0 \text{ for } N \to \infty$$

Then:

$$\mathbf{E}\left|\left|\left|\int_0^t U(t,s)\phi(s)\,dW(s)\right|^\kappa\right|\right|_{\rho,2}^2$$
$$= \mathbf{E}\int_\Theta\left(\left|\lim_{N\to\infty}\sum_{t_i\in\tau_N}(U(t,t_i)\phi(t_i))(W(t_{i+1}\wedge t)-W(t_i\wedge t))\right|^\kappa\right)^2 d\mu_\rho$$
$$= \int_\Theta\mathbf{E}\left(\left|\lim_{N\to\infty}\sum_{t_i\in\tau_N}\sum_{n\in\mathbf{N}}(U(t,t_i)\phi(t_i))(Q^{\frac{1}{2}}e_n)(w_n(t_{i+1}\wedge t)-w_n(t_i\wedge t))\right|^\kappa(x)\right)^2\mu_\rho(dx)$$
$$= \int_\Theta\mathbf{E}\left(\lim_{N\to\infty}\sum_{t_i\in\tau_N}\sum_{n\in\mathbf{N}}(U(t,t_i)\phi(t_i))(Q^{\frac{1}{2}}e_n)(w_n(t_{i+1}\wedge t)-w_n(t_i\wedge t))\right)^{2\kappa}(x)\mu_\rho(dx)$$

$$= c(\kappa)\int\limits_{\Theta}\mathbf{E}\left(\left|\lim_{N\to\infty}\sum_{t_i\in\tau_N}\sum_{n\in\mathbf{N}}(|U(t,t_i)\phi(t_i)|(Q^{\frac{1}{2}}e_n))^2((t_{i+1}\wedge t)-(t_i\wedge t))\right|^{\kappa}\right)(x)\mu_\rho(dx)$$

$$= c(\kappa)\mathbf{E}\int\limits_{\Theta}\sum_{n\in\mathbf{N}}\left(\int\limits_0^t((U(t,s)\phi(s))Q^{\frac{1}{2}}e_n)^2\,ds\right)^{\kappa}(x)\mu_\rho(dx)$$

$$\leq c(\kappa)\mathbf{E}\int\limits_{\Theta}\left(\sum_{n\in\mathbf{N}}\int\limits_0^t(|U(t,s)\phi(s)|^{\kappa}Q^{\frac{1}{2}}e_n\right)^2 ds(x)\mu_\rho(dx)$$

$$= c(\kappa)\mathbf{E}\int\limits_0^t||\,|U(t,s)\phi(s)|^{\kappa}\,||^2_{\mathcal{L}_2}\,ds$$

where in the fourth step the fact that

$$\mathbf{E}(|X_t-X_s|^a)\leq C_a(t-s)^{\frac{a}{2}}$$

holds true for each Brownian motion $(X_t)_{t\geq 0}$,all $0\leq s<t$ and all even natural numbers a by lemma 40.2 from [Ba] was used.
2.1.5 resp. 2.1.6 imply for each $t\in[0,T]$

$$\mathbf{E}\left(\sup_{s\in[0,t]}\left|\left|\left|\int\limits_0^s U(t,\epsilon)\phi(\epsilon)\,dW(\epsilon)\right|^{\kappa}\right|\right|^{2r}_{\rho,2}\right)$$

$$\leq c(\kappa,q)\mathbf{E}\left(\int\limits_0^t||\,|U(t,\epsilon)\phi(\epsilon)|^{\kappa}||^2_{\mathcal{L}_2}\,d\epsilon\right)^r$$

$$\leq c(\kappa,q,T,c(T))\mathbf{E}\left(\int\limits_0^t(t-\epsilon)^{-\gamma}||\varphi(\epsilon)||^{2\kappa}_{\rho,2\kappa}\,d\epsilon\right)^r$$

$$\leq c(\kappa,q,T,c(T))\mathbf{E}\left(\left[\int\limits_0^t\epsilon^{-\frac{\gamma q}{q-2\kappa}}\,d\epsilon\right]^{\frac{q-2\kappa}{q}}\left[\int\limits_0^t||\varphi(\epsilon)||^q_{\rho,2\kappa}\,d\epsilon\right]^{\frac{2\kappa}{q}}\right)^r$$

$$\leq c(\kappa,q,T,c(T))\left[\int\limits_0^T\epsilon^{-\frac{\gamma q}{q-2\kappa}}\,ds\right]^{\frac{(q-2\kappa)r}{2\kappa}}\int\limits_0^t\mathbf{E}||\varphi(\epsilon)||^q_{\rho,2\kappa}\,d\epsilon$$

$$= c(\kappa,\gamma,q,T,c(T))\int\limits_0^t\mathbf{E}||\varphi(\epsilon)||^q_{\rho,2\kappa}\,d\epsilon$$

with r as above,using (2.9) in the second,Hoelder‘s inequality in the third and the fact that

$$q>\frac{2\kappa}{1-\gamma}\Rightarrow\frac{\gamma q}{q-2\kappa}<1$$

holds in the last step.So one especially has for each $t\in[0,T]$:

$$\mathbf{E}\left|\left|\int\limits_0^t U(t,s)\phi(s)\,dW(s)\right|\right|^q_{\rho,2\kappa}\leq c(\kappa,\gamma,q,T,c(T))\int\limits_0^t\mathbf{E}||\varphi(s)||^q_{\rho,2\kappa}\,ds$$

q.e.d.

Define the term solution in the same way as Manthey and Zausinger did:

**Definition 2.2.3**(cf. [MaZa],chapter 2,definition 2.7,p.55)

Let $\Sigma$ and F be defined from $\sigma$ and f as in (2.1) resp. (2.4).Let f be s.t.(PG) is fulfilled with an exponent $\nu \in \mathbf{N}$.

A $L^2_\rho$-valued,predictable process X is called a **solution of Eq($\xi$,F,$\Sigma$)** for

$\xi \in L^2_\rho$, F:$[\,0;T\,] \times \Omega \times L^{2\nu}_\rho \to L^2_\rho$ and $\Sigma : [\,0;T\,] \times L^2_\rho \to \mathcal{L}_2$

if it has the following properties:

(i) $X(t) \in L^{2\nu}_\rho$ P-almost surely for each $t \in [\,0,T\,]$

(ii) $\sup_{t\in[\,0,T\,]} \int_0^t (t-s)^{-\gamma} \mathbf{E}||X(s)||^2_{\rho,2}\, ds < \infty$
where $\gamma = 0$ in the nuclear case and $\gamma$ as in (CC) in the cylindrical case.

(iii) X is pathwise continuous.

(iv) X solves the equation

$$X(t) = \; U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))\, ds + \int_0^t U(t,s)\Sigma(s,X(s))\, dW(s)$$

P-almost surely for each $t \in [\,0,T\,]$.

Following Manthey and Zausinger one first shows:

**Theorem 2.2.4**

Suppose (CD),(PP) and in the cylindrical case additionally (CC) hold.
Given a predictable process $\varphi : [\,0,T\,] \times \Omega \to L^2_\rho$,suppose there exists a q with

$$q > \tfrac{2}{1-\gamma}$$

and

$$\mathbf{E}\int_0^T ||\varphi(s)||^q_{\rho,2}\, ds < \infty$$

s.t. $\gamma$=0 in the nuclear case and $\gamma$ as in (CC) in the cylindrical case.
Then there exists a continuous modification of

$$\int_0^t U(t,s)\phi(s)\, dW(s)$$

in $L^2_\rho$.

**Proof:**
Let $\alpha \in (\,\frac{1}{q}, \frac{1-\gamma}{2}\,)$ with $\gamma$ as in (CC) in the cylindrical case and $\gamma = 0$ in the nuclear case,so especially $\alpha < \frac{1}{2}$,as a consequence of which 2.1.3 is applicable.Thus:

$\int\limits_0^t U(t,s)\phi(s)\,dW(s) = \frac{\sin\pi\alpha}{\pi}(R_\alpha Z_{\alpha,U})(t)$
with $Z_{\alpha,U}(t) := \int_0^t (t-s)^{-\alpha} U(t,s)\phi(s)\,dW(s)$ and $R_\alpha f(t) := \int_0^t (t-s)^{\alpha-1} U(t,s) f(s)\,ds$.

**Claim 1**:$Z_{\alpha,U} \in L^q([\,0,T\,]\times\Omega; L^2_\rho)$
**Proof:** One needs to show:

$$\mathbf{E}\int\limits_0^T ||Z_{\alpha,U}(t)||^q_{\rho,2}\,dt < \infty$$

Changing expectation and integration and applying 2.1.5 resp. 2.1.6 with $r := \frac{q}{2}$ and

$$\Phi(s) := (t-s)^{-\alpha} U(t,s)\phi(s)\,,\ s \in [\,0,t\,]$$

for fixed $t \in [\,0,T\,]$,which is Hilbert-Schmidt-valued by (CC),leads to

$$\begin{aligned}
&\int\limits_0^T \mathbf{E}\left|\left|\int\limits_0^t (t-s)^{-\alpha} U(t,s)\phi(s)\,dW(s)\right|\right|^q_{\rho,2} dt \\
&\le \int\limits_0^T c(q)\mathbf{E}\left(\int\limits_0^t ||(t-s)^{-\alpha} U(t,s)\phi(s)||^2_{\mathcal{L}_2}\,ds\right)^{\frac{q}{2}} dt \\
&= c(q)\int\limits_0^T \mathbf{E}\left(\int\limits_0^t (t-s)^{-2\alpha} ||U(t,s)\phi(s)||^2_{\mathcal{L}_2}\,ds\right)^{\frac{q}{2}} dt \\
&\le c(q,c(T))\mathbf{E}\int\limits_0^T \left(\int\limits_0^t (t-s)^{-(2\alpha+\gamma)} ||\varphi(s)||^2_{\rho,2}\,ds\right)^{\frac{q}{2}} dt \\
&\le T\,c(q,c(T))\left(\int\limits_0^T s^{-(2\alpha+\gamma)}\,ds\right)^{\frac{q}{2}} \mathbf{E}\int\limits_0^T ||\varphi(s)||^q_{\rho,2}\,ds \\
&=: c(\alpha,\gamma,q,T,c(T))\mathbf{E}\int\limits_0^T ||\varphi(s)||^q_{\rho,2}\,ds < \infty
\end{aligned}$$

where Young‘s inequation for convolutions and Jensen‘s inequality were used in the second last step and the fact that

$$\alpha \in \left(\frac{1}{q}, \frac{1-\gamma}{2}\right) \Rightarrow 2\alpha + \gamma < 1$$

holds,which ensures the existence of the integral in brackets,was used in the last step.
Thus claim 1 is proven.

**Claim 2:** $R_\alpha \in \mathcal{L}(L^q([\,0,T\,]\times\Omega; L^2_\rho), C([\,0,T\,]; L^2([\,0,T\,]\times\Omega; L^2_\rho)))$ (cf. section 1.2 with $B := L^2_\rho$)
**Proof:** In a first step show continuity in time under the assumption,that $(R_\alpha Y(t))_{t\in[\,0,T\,]} \subset L^2_\rho$ was already proven for processes Y as in the assumption.
Note that the following holds for arbitrary $t \in [\,0,T\,]$,$\varepsilon > 0$ and fixed $\omega \in \Omega$:

$$
\begin{aligned}
& R_\alpha Y(t+\varepsilon,\omega) - R_\alpha Y(t,\omega) \\
= & \int\limits_0^{t+\varepsilon} (t+\varepsilon-s)^{\alpha-1} U(t+\varepsilon,s) Y(s,\omega)\, ds - \int\limits_0^{t} (t-s)^{\alpha-1} U(t,s) Y(s,\omega)\, ds \\
= & \int\limits_t^{t+\varepsilon} (t+\varepsilon-s)^{\alpha-1} U(t+\varepsilon,s) Y(s,\omega)\, ds \\
& + \int\limits_0^{t} [(t+\varepsilon-s)^{\alpha-1} U(t+\varepsilon,s) - (t-s)^{\alpha-1} U(t,s)] Y(s,\omega)\, ds
\end{aligned}
$$

Obviously the first term tends to 0 in $L^2_\rho$ as $\varepsilon$ tends to 0.By the definition of almost strong evolution operators $U(\cdot,s)$ is strongly continuous on $[\,s,\infty\,)$,which implies continuity of $U(\cdot,s)$ on $[\,s,\infty\,)$,s.t. the second term tends to 0 in $L^2_\rho$ as $\varepsilon$ tends to 0 as well,as a consequence of which,continuity of $R_\alpha$ is shown,since $t \in [\,0,T\,]$ was chosen arbitrarily.

Now consider an arbitrary process Y from $L^q([\,0,T\,]\times\Omega; L^2_\rho)$ and show $R_\alpha Y(t) \in L^2_\rho$ for arbitrary $t \in [\,0,T\,]$.
Young's inequality for convolutions and $1-\alpha < 1$ imply:

$$
\begin{aligned}
\mathbf{E} \left\| \int\limits_0^t (t-s)^{\alpha-1} U(t,s) Y(s)\, ds \right\|^2_{\rho,2} & \leq c^2(T) \mathbf{E} \left\| \int\limits_0^t (t-s)^{-(1-\alpha)} Y(s)\, ds \right\|^2_{\rho,2} \\
& \leq c^2(T) \left( \int\limits_0^t s^{-(1-\alpha)}\, ds \right)^2 \mathbf{E} \int\limits_0^t ||Y(s)||^2_{\rho,2}\, ds \\
& \leq c^2(T) \left( \int\limits_0^T s^{-(1-\alpha)}\, ds \right)^2 \mathbf{E} \int\limits_0^T ||Y(s)||^2_{\rho,2}\, ds \\
& < \infty
\end{aligned}
$$

Thus one has at least $R_\alpha Y(t) \in L^2_\rho$ P-almost surely.So there exists a version of $R_\alpha Y(t)$ in $L^2_\rho$. As the estimate holds true for arbitrary $t \in [\,0,T\,]$,

$$
\sup_{t\in[\,0,T\,]} \mathbf{E} ||R_\alpha Y(t)||^2_{\rho,2} \leq c(\alpha,T,c(T)) \left( \mathbf{E} \int\limits_0^T ||Y(s)||^q_{\rho,2}\, ds \right)^{\frac{2}{q}} < \infty
$$

follows and one has,as continuity was already shown
$R_\alpha Y \in C([\,0,T\,]; L^2([\,0,T\,]\times\Omega; L^2_\rho))$.By the last estimate one gets

$$
||R_\alpha Y||^2_{C([\,0,T\,];L^2_\rho)(2)} \leq c(\alpha,T,c(T)) ||Y||^2_{L^q}
$$

for all $Y \in L^q([\,0,T\,]\times\Omega; L^2_\rho)$.So $R_\alpha$ is a bounded,linear operator,which finishes the proof of claim 2.
Claims 1 and 2 imply the existence of a version of

$$
\int\limits_0^t U(t,s)\phi(s)\, dW(s)
$$

which is continuous in time.
q.e.d.

Then one can transfer theorem 3.2.1 from [MaZa] into the given situation with the help of the following two results:

**Lemma 2.2.5:**

Let $(g_n)_{n\in\mathbf{N}}$ be a sequence of measurable functions $g_n : \mathbf{R}_+ \to \mathbf{R}_+$ with

$$g_n(t) \leq q + b\int_0^t (t-s)^{-\delta} g_{n-1}(s)\,ds$$

for $n \in \mathbf{N}$,$\delta \in [\,0,1\,)$,$b > 0$,$q \geq 0$,$t \in [\,0,T\,]$.Then

$$g_n(t) \leq q\sum_{k=0}^{n-1} q_k\, t^{k(1-\delta)} + q_n\, t^{n(1-\delta)} \sup_{r\in[\,0,T\,]} g_0(r)$$

holds with $q_0 = 1$,$q_1 = \frac{b}{1-\delta}$,$q_k = \frac{c^k(b,\delta)}{\Gamma(k(1-\delta)+1)}$ for $k > 1$,where $\Gamma(\cdot)$ is the gamma-function given by

$$\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}\,dx\ ,\ t > 0$$

Furthermore one has the following property

$$\sum_{k=0}^{\infty} q_k\, T^{k(1-\delta)} < \infty$$

**Proof:**
Cf. [MaSt],Appendix,Lemma A1,pp.158,159

**Remark 2.2.6:**

Having $g_n = g$ for all $n \in \mathbf{N}$ with a bounded g,2.2.5 implies

$$\begin{aligned} g(t) \leq &\quad q\lim_{n\to\infty}\left(\sum_{k=0}^{n-1} q_k\, T^{k(1-\delta)}\right) + \left(\lim_{n\to\infty} q_n\, T^{n(1-\delta)}\right) \sup_{r\in[\,0,T\,]} g(r) \\ = &\quad q\sum_{k=0}^{\infty} q_k T^{k(1-\delta)} =: q\, c(T,b,\delta) \end{aligned}$$

Now one can prove a version of 3.2.1 from [MaZa] :

**Theorem 2.2.7**

Suppose f($\cdot$,$\omega$,$\cdot$),$\omega \in \Omega$,and $\sigma$ fulfill (L1),(L2) with constants $c_f(T)$,$c_\sigma(T)$,with a constant $c_f(T)$ independent of $\omega$. Suppose furthermore that f is progressively measurable and U fulfills (CD),(PP) and (CC).
Let $\gamma$=0 in the nuclear case and let $\gamma$ be as in (CC) in the cylindrical case.Then:

**(i)** For $q > \frac{2}{1-\gamma}$ with

$$\mathbf{E}||\xi||_{\rho,2}^q < \infty$$

there exist a pathwise unique solution X of Eq($\xi$,F,$\Sigma$) and a constant
$c(q,T,\gamma,c(T),c_f(T),c_\sigma(T)) > 0$ depending on q,T,U,f and $\sigma$,s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{\rho,2} \leq c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^q_{\rho,2})$$

**(ii):** Suppose one additionally has (E1),(E2).
Then there exist,given $\kappa > \frac{1}{1-\gamma}$ with

$$\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa} < \infty$$

a pathwise unique solution X of Eq($\xi$,F,$\Sigma$) and a constant $c(\kappa,T,\gamma,c(T),c_f(T),c_\sigma(T)) > 0$ depending on $\kappa$,T,U,f and $\sigma$ with

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\kappa}_{\rho,2\kappa} \leq c(\kappa,T,\gamma,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa})$$

**Remark 2.2.8:**

In [MaZa] the authors only show the finiteness of the sup-terms.As estimates of the above type are shown f.e. in [AsMa] they are proven here as well.

Before it is possible to prove 2.2.7 another lemma is needed:

**Lemma 2.2.9:**

As in Manthey‘s and Zausinger‘s paper (cf. [MaZa],chapter 3,section 3,theorem 3.3.1,p.56 there) it is assumed that f is progressively measurable.
Let f be as in 2.2.7.Then

$$\left(\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds\right)_{t\in[0,T]}$$

is continuous and adapted,i.e especially predictable,for each predictable,continuous process $Y = (Y(t))_{t\in[0,T]}$ with

$$\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^2_{\rho,2} < \infty$$

**Proof of 2.2.9:**
First of all note that the Bochner-integral

$$\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds$$

is welldefined according to 2.1.8,as the $\omega$-independence of $c_f(T)$ implies:

$$\begin{aligned}
\mathbf{E}\left|\left|\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds\right|\right|_{\rho,2} &\leq c(T)\mathbf{E}\left|\left|\int_0^t F(s,\cdot,Y(s))\,ds\right|\right|_{\rho,2} \\
&\leq c(c(T),c_f(T))\int_0^t (1+\mathbf{E}||Y(s)||_{\rho,2})\,ds \\
&\leq c(T,c(T),c_f(T))\left(1+\sup_{r\in[0,T]}\mathbf{E}||Y(r)||_{\rho,2}\right) \\
&< \infty
\end{aligned}$$

where (ii) from 1.3 and (L1),(L2) were used for f in the second step.

Let $t \in [\,0, T\,]$ be arbitrary.For all $s \in [\,0, t\,]$,$\omega \in \Omega$

$$\int_0^t U(t,r)F(r,\omega,Y(r,\omega))\,dr - \int_0^s U(s,r)F(r,\omega,Y(r,\omega))\,dr$$
$$= \int_s^t U(t,r)F(r,\omega,Y(r,\omega))\,dr + \int_0^s [U(t,r)-U(s,r)]F(r,\omega,Y(r,\omega))\,dr$$

holds.As already mentioned in the proof of 2.2.4 $U(\cdot, r)$ is continuous,s.t. the second term in the upper equation tends to 0 in $L^2_\rho$ for $s \to t$,since the continuity of f in $\mathbf{R}$ resulting from (L1) and the time-continuity of Y imply

$$\lim_{s \to t} U(s,r)F(r,\omega,Y(r,\omega)) = U(t,r)F(r,\omega,Y(r,\omega))$$

in $L^2_\rho$ for fixed $r \in [\,0, s\,]$.
As the first term obviously tends to 0 for $s \to t$ and $t \in [\,0, T\,]$ and $\omega \in \Omega$ were chosen arbitrarily,the proof of continuity is finished.

As Y is predictable by assumption it is in particular adapted,so Y(t) is $\mathcal{F}_t$-measurable for each $t \in [\,0, T\,]$.As $(\mathcal{F}_t)_{t \in [\,0,T\,]}$ is a filtration,Y(s) is $\mathcal{F}_t$-measurable for all $s \in [\,0, t\,]$ and fixed $t \in [\,0, T\,]$.
Progressive measurability of f then implies $\mathcal{F}_t$-measurability of $U(t,\cdot)F(\cdot,\cdot,Y(\cdot))$ on $[\,0, t\,]$.Thus

$$\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds$$

is $\mathcal{F}_t$-measurable for the fixed t.As this t was chosen arbitrarily,the process is adapted as well,which finishes the proof.
q.e.d.

**Proof of 2.2.7:**
**(i):** First of all consider $C([\,0, T\,]; L^2([\,0, T\,] \times \Omega; L^2_\rho))$ with norm $||\cdot||_{C([\,0,T\,];L^2_\rho)(2)}$,as it was done in the proof of claim 2 in 2.2.4.This forms a Banach space.
Since f defining F by (2.4) is progressively measurable,

$$\left( \int_0^t U(t,s)F(s,\cdot,Z(s))\,ds \right)_{t \in [\,0,T\,]}$$

is predictable for processes Z from $C([\,0, T\,]; L^2([\,0, T\,] \times \Omega; L^2_\rho))$ by 2.2.9.
As it was suggested in the proof of 3.2.1 in [MaZa],follow [DPZa92].
Define a mapping $\mathcal{K}_1$ for processes Z from $C([\,0, T\,]; L^2([\,0, T\,] \times \Omega; L^2_\rho))$ by

$$\mathcal{K}_1(Z)(t) := \int_0^t U(t,s)F(s,\cdot,Z(s))\,ds$$

(iii) from section 1.3 and (L1),(L2) for f imply

$$
\begin{aligned}
||\mathcal{K}_1(Z)||^2_{C([\,0,T\,];L^2_\rho)(2)} &= \sup_{t\in[\,0,T\,]} \mathbf{E}||\mathcal{K}_1(Z)(t)||^2_{\rho,2} \\
&= \sup_{t\in[\,0,T\,]} \mathbf{E}\left|\left|\int_0^t U(t,s)F(s,\cdot,Z(s))\,ds\right|\right|^2_{\rho,2} \\
&\leq c(c(T),c_f(T))\,\mathbf{E}\int_0^T (1+||Z(s)||^2_{\rho,2})\,ds \\
&\leq T\,c(c(T),c_f(T))\left(1+\sup_{t\in[\,0,T\,]}\mathbf{E}||Z(t)||^2_{\rho,2}\right) \\
&= T\,c(c(T),c_f(T))\left(1+||Z||^2_{C([\,0,T\,];L^2_\rho)(2)}\right)
\end{aligned}
$$

So $\mathcal{K}_1$ is a mapping from $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^2_\rho))$ onto itself.
Define a mapping $\mathcal{K}_2$ for processes Z as above by

$$
\mathcal{K}_2(Z)(t) := \int_0^t U(t,s)\Sigma(s,Z(s))\,dW(s)
$$

2.1.5 resp. 2.1.6 (case $r := 1$),(CC) and (L1),(L2) for $\sigma$ lead to

$$
\begin{aligned}
&||\mathcal{K}_2(Z)||^2_{C([\,0,T\,];L^2_\rho)(2)} \\
&\sup_{t\in[\,0,T\,]} \mathbf{E}\int_0^t ||U(t,s)\Sigma(s,Z(s))||^2_{\mathcal{L}_2}\,ds \\
&\leq C\max\left(\tfrac{1}{1-\gamma}c(c(T),c_\sigma(T))T^{1-\gamma},1\right)\left(1+\sup_{t\in[\,0,T\,]}\int_0^t \mathbf{E}||Z(s)||^2_{\rho,2}\,ds\right) \\
&\leq c(\gamma,T,c(T),c_\sigma(T))\left(1+\int_0^T \mathbf{E}||Z(s)||^2_{\rho,2}\,ds\right) \\
&\leq Tc(\gamma,T,c(T),c_\sigma(T))\left(1+\sup_{t\in[\,0,T\,]}\mathbf{E}||Z(t)||^2_{\rho,2}\right) \\
&= Tc(\gamma,T,c(T),c_\sigma(T))\left(1+||Z||^2_{C([\,0,T\,];L^2_\rho)(2)}\right)
\end{aligned}
$$

where the fact that $\gamma\in[0,1)$ implies

$$
\int_0^t s^{-\gamma}\,ds < \infty
$$

for all $t\in[\,0,T\,]$ was used in the second step.
Thus $\mathcal{K}_2$ also maps each process Z from $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^2_\rho))$ onto $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^2_\rho))$,if Z has got property 2.2.3(ii),which one needs in order to do the second step.

Let X and Y be processes in $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^2_\rho))$ with property 2.2.3(ii).

$$
\mathcal{K}(Z)(t) := U(t,0)\xi + \mathcal{K}_1(Z)(t) + \mathcal{K}_2(Z)(t)\ ,\ Z\in C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^2_\rho))
$$

leads to

$$
\begin{aligned}
||\mathcal{K}(X)-\mathcal{K}(Y)||^2_{C([0,\bar{T}];L^2_\rho)(2)} \leq\ & 2\Big(||\mathcal{K}_1(X)-\mathcal{K}_1(Y)||^2_{C([0,\bar{T}];L^2_\rho)(2)} \\
& +||\mathcal{K}_2(X)-\mathcal{K}_2(Y)||^2_{C([0,\bar{T}];L^2_\rho)(2)}\Big) \\
=:\ & 2(I_1+I_2)
\end{aligned}
$$

for $\bar{T}\in[0,T]$ with the following estimates,which hold analogously to the cases $||\mathcal{K}_1(Z)||$ and $||\mathcal{K}_2(Z)||$:

$$
\begin{aligned}
I_1 \leq\ & c(c(T),c_f(T)) \sup_{t\in[0,\bar{T}]} \mathbf{E}\int_0^t ||X(s)-Y(s)||^2_{\rho,2}\,ds \\
\leq\ & \bar{T}c(c(T),c_f(T)) \sup_{t\in[0,\bar{T}]} \mathbf{E}||X(t)-Y(t)||^2_{\rho,2} \\
=\ & \bar{T}\,c(c(T),c_f(T))||X-Y||^2_{C([0,\bar{T}];L^2_\rho)(2)} \\
I_2 \leq\ & c(c(T),c_\sigma(T))\left(\sup_{t\in[0,\bar{T}]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)-Y(s)||^2_{\rho,2}\,ds\right) \\
\leq\ & \left(\int_0^{\bar{T}} s^{-\gamma}\,ds\right) c(c(T),c_\sigma(T)) \sup_{t\in[0,\bar{T}]} \mathbf{E}||X(t)-Y(t)||^2_{\rho,2} \\
=\ & \tfrac{1}{1-\gamma}\bar{T}^{1-\gamma}\,c(c(T),c_\sigma(T))||X-Y||^2_{C([0,\bar{T}];L^2_\rho)(2)}
\end{aligned}
$$

Thus given $\bar{T}>0$,s.t.

$$
\bar{T}\,c(c(T),c_f(T))+\tfrac{1}{1-\gamma}\bar{T}^{1-\gamma}c(c(T),c_\sigma(T))<1
$$

holds,$\mathcal{K}$ has a unique fixpoint $\bar{X}$,which is a solution to the wanted equation on $[0,\bar{T}]$ by construction and fulfills 2.2.3(ii).Setting $\xi:=\bar{X}(\bar{T})$ leads to a solution on $[\bar{T},2\bar{T}]$ and by finite iteration of this procedure one gets a solution on $[0,T]$,which is unique in $C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ up to modifications and has property 2.2.3(ii).Consider modifications,which are pathwise continuous,that is,which fulfill 2.2.3(iii).
Let X,Y be two such modifications.2.1.5 resp. 2.1.6 with $r:=\frac{q}{2}$,(2.6)(with $\kappa=1$ and $\Phi(s):=U(t,s)[\Sigma(s,X(s))-\Sigma(s,Y(s))],s\in[0,t]$) and (L1) lead to:

$$
\begin{aligned}
\mathbf{E}||X(t)-Y(t)||^q_{\rho,2} \leq\ & c(q)\Bigg(\mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,\cdot,X(s))-F(s,\cdot,Y(s))]\,ds\right|\right|^q_{\rho,2} \\
& +\mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X(s))-\Sigma(s,Y(s))]\,dW(s)\right|\right|^q_{\rho,2}\Bigg) \\
\leq\ & c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))\int_0^t \mathbf{E}||X(s)-Y(s)||^q_{\rho,2}\,ds
\end{aligned}
$$

Due to the continuity of X and Y Gronwall‘s lemma is applicable and shows

$$
||X(t)-Y(t)||_{\rho,2}=0 \ \text{ P-a.s.},t\in[0,T]
$$

i.e. for each $t\in[0,T]$ there exists a P-zeroset $N_t$ with

$$
||X(t,\omega)-Y(t,\omega)||_{\rho,2}=0
$$

for all $\omega\in N_t^C$.
Thus one has

$$||X(t,\omega) - Y(t,\omega)||_{\rho,2} = 0$$

for all $t \in \mathbf{Q}$ and all

$$\omega \in \left( \bigcup_{t \in [\,0,T\,] \cap \mathbf{Q}} N_t \right)^C =: N_Q$$

Since the $N_t$ are P-zerosets,$N_Q$ has P-measure 1,i.e. $X(t) = Y(t)$ for all $t \in [\,0,T\,] \cap \mathbf{Q}$ P-almost surely.Since $\mathbf{Q}$ is dense in $\mathbf{R}$ this implies

$$||X(t) - Y(t)||_{\rho,2} = 0$$

for all $t \in [\,0,T\,]$ P-almost surely due to the continuity of X and Y.
Assume there is an $\omega$,s.t. there exists a $\bar{t} \in [\,0,T\,]$,s.t.

$$||X(\bar{t},\omega) - Y(\bar{t},\omega)||_{\rho,2} \neq 0$$

holds true.But then pathwise continuity of X and Y implies

$$||X(\cdot,\omega) - Y(\cdot,\omega)||_{\rho,2} \neq 0$$

first on an open subset of $[\,0,T\,]$ containing $\bar{t}$ and than inductively on $[\,0,T\,]$,which is a contradiction to

$$X(0,\omega) = \xi(\omega) = Y(0,\omega)$$

Thus one has not only a pathwise continuous solution but even a pathwise unique solution.
Due to its construction the solution fulfills at least 2.2.3(ii)–(iv).As (L1),(L2) obviously imply (PG) with exponent $\nu = 1$ for f,property 2.2.3(i) is trivially fulfilled.
Thus one has existence of a solution in the sense of 2.2.3.
The wanted estimate follows with the help of Gronwall‘s lemma from the following estimate:

$$\begin{aligned}
\mathbf{E}||X(t)||_{\rho,2}^q \leq\ & c(q) \left( \mathbf{E}||U(t,0)\xi||_{\rho,2}^q + \mathbf{E}\left|\left| \int_0^t U(t,s)F(s,\cdot,X(s))\,ds \right|\right|_{\rho,2}^q \right. \\
& \left. + \mathbf{E}\left|\left| \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s) \right|\right|_{\rho,2}^q \right) \\
\leq\ & c(q,c(T))\mathbf{E}||\xi||_{\rho,2}^q \\
& + c(q,T,\gamma,c(T),c_f(T),c_\sigma(T)) \left( 1 + \int_0^t \mathbf{E}||X(s)||_{\rho,2}^q\,ds \right) \\
\leq\ & c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))(1 + \mathbf{E}||\xi||_{\rho,2}^q) \\
& + c(q,T,\gamma,c(T),c_f(T),c_\sigma(T)) \int_0^t \mathbf{E}||X(s)||_{\rho,2}^q\,ds
\end{aligned}$$

for all $t \in [\,0,T\,]$,where the procedure in the second step was analogous to that in the estimate of $\mathbf{E}||X(t) - Y(t)||_{\rho,2}^q$.

**(ii):** Proceed as in [MaZa],i.e. make use of Picard‘s iteration given by

$$\begin{aligned} X_0(t) :=&\quad U(t,0)\xi, t \in [0,T] \\ X_n(t) :=&\quad X_0(t) + \int_0^t U(t,s)F(s,\cdot,X_{n-1}(s))\,ds \\ &\quad + \int_0^t U(t,s)\Sigma(s,X_{n-1}(s))\,dW(s), t \in [0,T], n \in \mathbf{N} \end{aligned}$$

Then the following holds for all $t \in [0,T]$:

$$\begin{aligned} &\mathbf{E}||X_{n+1}(t) - X_n(t)||^{2\kappa}_{\rho,2\kappa} \\ &\le c(\kappa)\left(\mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,\cdot,X_n(s)) - F(s,\cdot,X_{n-1}(s))]\,ds\right|\right|^{2\kappa}_{\rho,2\kappa}\right. \\ &\quad \left. +\mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X_n(s)) - \Sigma(s,X_{n-1}(s))]\,dW(s)\right|\right|^{2\kappa}_{\rho,2\kappa}\right) \\ &\le c(\kappa,c(T),c_f(T),c_\sigma(T))\left(\int_0^t \mathbf{E}||X_n(s) - X_{n-1}(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right. \\ &\quad \left. + \int_0^t (t-s)^{-\gamma}\mathbf{E}||X_n(s) - X_{n-1}(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right) \\ &\le c(\kappa,\gamma,T,c(T),c_f(T),c_\sigma(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||X_n(s) - X_{n-1}(s)||^{2\kappa}_{\rho,2\kappa}\,ds \end{aligned}$$

where (L1),(E1),(E2) and (2.5) were used in the second step,whereas in the third step the fact,that $(t-s)^\gamma$ can be estimated by $T^\gamma$ for $(s,t) \in S_T$,was used. 2.2.5(set $g_n := \mathbf{E}||X_{n+1} - X_n||^{2\kappa}_{\rho,2\kappa}$) implies

$$\sup_{t\in[0,T]} \mathbf{E}||X_{n+1}(t) - X_n(t)||^{2\kappa}_{\rho,2\kappa} \le q_n T^{n(1-\gamma)} \sup_{t\in[0,T]} \mathbf{E}||X_1(t) - X_0(t)||^{2\kappa}_{\rho,2\kappa}$$

for all $n \in \mathbf{N}$ with $q_0$=1,$q_1 = \frac{c(\kappa,\gamma,T,c_f(T),c_\sigma(T))}{1-\gamma}$ and $q_k = \frac{c(\kappa,\gamma,T,c_f(T),c_\sigma(T))^k}{\Gamma(k(1-\gamma)+1)}$ for $k > 1$.One has:

$$\begin{aligned} \sup_{t\in[0,T]} \mathbf{E}||X_1(t) - X_0(t)||^{2\kappa}_{\rho,2\kappa} \le&\quad c(\kappa)\left(\sup_{t\in[0,T]} \mathbf{E}\left|\left|\int_0^t U(t,s)F(s,U(s,0)\xi)\,ds\right|\right|^{2\kappa}_{\rho,2\kappa}\right. \\ &\quad \left. + \sup_{t\in[0,T]} \mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,U(s,0)\xi)\,dW(s)\right|\right|^{2\kappa}_{\rho,2\kappa}\right) \\ =:&\quad c(\kappa)\left(\sup_{t\in[0,T]} I_F(t) + \sup_{t\in[0,T]} I_\Sigma(t)\right) \end{aligned}$$

Concerning $I_F$

$$I_F(t) \le c(\kappa,T,c(T),c_f(T))(1 + \mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa})\ ,\ t \in [0,T]$$

(2.5) and (L1),(L2) for $\sigma$ lead to

$$\begin{aligned} I_\Sigma(t) \le&\quad c(\kappa,c(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||\sigma(s,U(s,0)\xi)||^{2\kappa}_{\rho,2\kappa}\,ds \\ \le&\quad c(\kappa,\gamma,T,c(T),c_\sigma(T))(1 + \mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa}) \end{aligned}$$

Putting the estimates together one gets for all $n \in \mathbf{N}$:

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||X_{n+1}(t) - X_n(t)||^{2\kappa}_{\rho,2\kappa} \leq q_n T^{n(1-\gamma)} c(\kappa,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa}) < \infty$$

Consider $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;B))$ from section 1.2 with $B := L^{2\kappa}_\rho$.Due to the above estimate and

$$q_n T^{n(1-\gamma)} \to 0\,,\; n\to\infty$$

following from 2.2.5,$(X_n)_{n\in\mathbf{N}}$ is a Cauchy-sequence in the Banach space $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L^{2\kappa}_\rho))$ with norm $||\cdot||_{C([\,0,T\,];L^{2\kappa}_\rho)(2\kappa)}$,s.t. there is a limit process X,which is predictable and continuous by the definition of the Banach space and lies in $L^2_\rho$ due to the fact that $L^{2\kappa}_\rho \subset L^2_\rho$ holds.
Thus at least property 2.2.3(i) is fulfilled.
Hoelder's inequality immediately implies

$$\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||^2_{\rho,2}\,ds \leq \left(\int_0^t s^{-\frac{\gamma\kappa}{\kappa-1}}\,ds\right)^{\frac{\kappa-1}{\kappa}} \left(\int_0^t \mathbf{E}||X(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right)^{\frac{1}{\kappa}}$$

Then X fulfills 2.2.3(ii) with the help of the definition of $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;B))$ and the fact that $\kappa > \frac{1}{1-\gamma}$ holds true,since

$$\kappa > \tfrac{1}{1-\gamma} \iff \tfrac{\gamma\kappa}{\kappa-1} < 1$$

implies the existence of the left integral in the above estimate.
2.2.3(iv) holds due to the construction via Picard's iteration.
The existence of a pathwise unique,continuous solution follows analogously to (i) .
Thus X is a solution of Eq($\xi$,F,$\Sigma$) in the sense of 2.2.3.
Concerning the estimate note that with (2.5) the following holds true for $t \in [\,0,T\,]$:

$$\begin{aligned}
\mathbf{E}||X(t)||^{2\kappa}_{\rho,2\kappa} \leq\; & c(\kappa)\Bigg(\mathbf{E}||U(t,0)\xi||^{2\kappa}_{\rho,2\kappa} + \mathbf{E}\left|\left|\int_0^t U(t,s)F(s,\cdot,X(s))\,ds\right|\right|^{2\kappa}_{\rho,2\kappa} \\
& + \mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)\right|\right|^{2\kappa}_{\rho,2\kappa}\Bigg) \\
\leq\; & c(\kappa)\Bigg(c(\kappa,T)\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa} + c(\kappa,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||X(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right) \\
& + c(\kappa,T,c(T),c_\sigma(T))\left(1+\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right)\Bigg) \\
\leq\; & c(\kappa,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa}) \\
& + c(\kappa,\gamma,T,c(T),c_f(T),c_\sigma(T))\left(1+\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||^{2\kappa}_{\rho,2\kappa}\,ds\right)
\end{aligned}$$

Then 2.2.5,2.2.6($g_n := \mathbf{E}||X||^{2\kappa}_{\rho,2\kappa}$ for all $n \in \mathbf{N}$) imply the wanted estimate.
q.e.d.

By 2.2.7 one immediately gets the following extension of 3.2.2 from [MaZa]:

**Corollary 2.2.10:**

**(i):** Under the assumptions of part(i) of 2.2.7 there exists a pathwise unique solution V to Eq(0,0,Σ),s.t. given arbitrary q with $q > \frac{2}{1-\gamma}$ there exists a positive constant c(q,T,γ,c(T),$c_\sigma(T)$) depending on q,T,U and $\sigma$ with

$$\sup_{t \in [\,0,T\,]} \mathbf{E}||V(t)||^q_{\rho,2} \leq c(q,T,\gamma,c(T),c_\sigma(T))$$

**(ii):** Under the assumptions of part(ii) of 2.2.7 there exists a pathwise unique solution V to Eq(0,0,Σ),s.t. given arbitrary $\kappa$ with $\kappa > \frac{1}{1-\gamma}$ there exists a positive constant c($\kappa$,T,γ,c(T),$c_\sigma(T)$) depending on $\kappa$,T,U and $\sigma$ with

$$\sup_{t \in [\,0,T\,]} \mathbf{E}||V(t)||^{2\kappa}_{\rho,2\kappa} \leq c(\kappa,T,\gamma,c(T),c_\sigma(T))$$

The next step will be to show the comparison theorem 3.3.1(ii) from [MaZa] in case $\Theta = \mathbf{R}^d_+$.

**Theorem 2.2.11:**

Let $f^{(i)}$; i=1,2;$\sigma$ be as in theorem 2.2.7(i).
Suppose,there exists $q > \frac{2}{1-\gamma}$,s.t.

$$\mathbf{E}||\xi^{(i)}||^q_{\rho,2} < \infty$$

and (CD),(PP),(CC) and (BA) hold for U.

Then the conditions

$$f^{(1)}(t,\omega,u) \leq f^{(2)}(t,\omega,u); (t,\omega,u) \in [0,T] \times \Omega \times \mathbf{R}$$

and

$$\xi^{(1)} \leq \xi^{(2)}$$

P-almost surely imply

$$X^{(1)}(t) \leq X^{(2)}(t)$$

P-almost surely for all $t \in [\,0,T\,]$.

For a proof define in analogy to [MaZa] a mapping $Q_M$:$L^2 \rightarrow L^2$ and a $Q_M$-Wienerprocess $W_M$ for fixed $M \in \mathbf{N}$ by

$$Q_M(\psi) := \sum_{n=1}^{M} a_n < \psi, e_n >_0 e_n$$

$$W_M(t) := \sum_{n=1}^{M} \sqrt{a_n} w_n(t) e_n$$

where $< \cdot, \cdot >_0$ denotes the inner product in $L^2$.Thus for $M \to \infty$ one has $Q_M$ tending to Q and $W_M$ tending to W in $L^2$.Given $N \in \mathbf{N}$;i=1,2; denote by $X^{(i)}_{N,M}$ the solution of the equation

$$\begin{aligned} X^{(i)}_{N,M}(t) = \quad & U_N(t,0)\xi^{(i)} + \int_0^t U_N(t,s) F^{(i)}(s,\cdot,X^{(i)}_{N,M}(s))\, ds \\ & + \int_0^t U_N(t,s) \Sigma(s, X^{(i)}_{N,M}(s))\, dW_M(s) \end{aligned}$$

with $t \in [\,0,T\,]$ and $U_N$ from (BA).The existence and uniqueness of this solution will be shown in 2.2.13.

Two lemma are needed in order to show 2.2.11.The first one is a version of lemma 3.3.2 from [MaZa]:

**Lemma 2.2.12:**

Defining $Y^{(i)} := X^{(i)}_{N,M}$;i=1,2; for fixed N,M$\in \mathbf{N}$

$$Y^{(1)}(t) \leq Y^{(2)}(t)$$

holds P-almost surely for all $t \in [0,T]$.

**Proof of 2.2.12:**
Follow the proof of 3.3.2 from [MaZa].
Given a fixed $j \in \mathbf{N}$ and k=1,2,...,j define $t_k := \frac{kT}{j}$ and processes $Z^{(i)}_{k,j}$,$V^{(i)}_{k,j}$ by

$$Z^{(i)}_{0,j}(t) := \xi^{(i)} + \int_0^t \Sigma(s, Z^{(i)}_{0,j}(s))\, dW_M(s)$$
$$V^{(i)}_{0,j}(t) := Z^{(i)}_{0,j}(t_1) + \int_0^t (A_N(s) V^{(i)}_{0,j}(s) + F^{(i)}(s,\cdot,V^{(i)}_{0,j}(s)))\, ds$$

for $t \in [\,0, t_1\,]$ and

$$Z^{(i)}_{k,j}(t) := V^{(i)}_{k-1,j}(t_k) + \int_{t_k}^t \Sigma(s, Z^{(i)}_{k,j}(s))\, dW_M(s)$$
$$V^{(i)}_{k,j}(t) := Z^{(i)}_{k,j}(t_{k+1}) + \int_{t_k}^t (A_N(s) V^{(i)}_{k,j}(s) + F^{(i)}(s,\cdot,V^{(i)}_{k,j}(s)))\, ds$$

for $t \in [\,t_k, t_{k+1}\,]$;k=1,2,...,j-1.Manthey and Zausinger claim,that these processes own pathwise continuous modifications.
Given solutions $Z^{(i)}_{0,j}$,$\bar{Z}^{(i)}_{0,j}$ consider

$$\gamma_d(X) := \xi^{(i)} + \int_0^{\cdot} \Sigma(s, X(s))\, dW_M(s)$$

in $C([\,0,T\,]; L^2([\,0,T\,] \times \Omega; L^2_\rho))$.One has:

$$\sup_{t\in[0,T]} \mathbf{E}||\gamma_d(Z_{0,j}^{(i)})(t) - \gamma_d(\bar{Z}_{0,j}^{(i)})(t)||_{\rho,2}^2$$
$$= \sup_{t\in[0,T]} \mathbf{E}\left|\left|\int_0^t (\Sigma(s, Z_{0,j}^{(i)}(s)) - \Sigma(s, \bar{Z}_{0,j}^{(i)}(s)))\, dW_M(s)\right|\right|_{\rho,2}^2$$
$$\leq Tc(M) \sup_{t\in[0,T]} \mathbf{E}||\Sigma(t, Z_{0,j}^{(i)}(t)) - \Sigma(t, \bar{Z}_{0,j}^{(i)}(t))||_{\mathcal{L}_2}^2$$
$$\leq Tc(M)c_\sigma^2(T) \sup_{t\in[0,T]} \mathbf{E}||Z_{0,j}^{(i)}(t) - \bar{Z}_{0,j}^{(i)}(t)||_{\rho,2}^2$$

and thus in $||\cdot||_{C([0,T];L_\rho^2)(2)}$-norm:

$$||\gamma_d(Z_{0,j}^{(i)}) - \gamma_d(\bar{Z}_{0,j}^{(i)})||_{C([0,\bar{T}];L_\rho^2)(2)} \leq \sqrt{Tc(M)}c_\sigma(T)||Z_{0,j}^{(i)} - \bar{Z}_{0,j}^{(i)}||_{C([0,T];L_\rho^2)(2)}$$

Thus there exists a unique pathwise continuous version of the solution analogously to the proof of 2.2.7(i).
So there is a pathwise unique,continuous,$L_\rho^2$-valued solution $Z_{0,j}^{(i)}$.
Let $V_{0,j}^{(i)}$,$\bar{V}_{0,j}^{(i)}$ be two solutions.Defining

$$\gamma_{A,v}(X)(\omega) := Z_{0,j}^{(i)}(t_1)(\omega) + \int_0^{\cdot} A_N(s)X(s,\omega) + F^{(i)}(s,\omega,X(s,\omega))\, ds$$

for arbitrary $\omega \in \Omega$,one gets:

$$\sup_{t\in[0,T]} \mathbf{E}||\gamma_{A,v}(V_{0,j}^{(i)})(t) - \gamma_{A,v}(\bar{V}_{0,j}^{(i)})(t)||_{\rho,2}^2$$
$$= \sup_{t\in[0,T]} \mathbf{E}\left|\left|\int_0^t A_N(s)[V_{0,j}^{(i)}(s) - \bar{V}_{0,j}^{(i)}(s)] + F^{(i)}(s,\cdot,V_{0,j}^{(i)}(s)) - F^{(i)}(s,\cdot,\bar{V}_{0,j}^{(i)}(s))\, ds\right|\right|_{\rho,2}^2$$
$$\leq 2\left(\sup_{t\in[0,T]} ||A_N(t)||^2 + c_{f^{(i)}}^2(T)\right) T \sup_{t\in[0,T]} \mathbf{E}||V_{0,j}^{(i)}(t) - \bar{V}_{0,j}^{(i)}(t)||_{\rho,2}^2$$
$$\leq 2(c^2(N) + c_{f^{(i)}}^2(T))T \sup_{t\in[0,T]} \mathbf{E}||V_{0,j}^{(i)}(t) - \bar{V}_{0,j}^{(i)}(t)||_{\rho,2}^2$$

s.t. the following holds true in $C([0,T];L^2([0,T]\times\Omega;L_\rho^2))$:

$$||\gamma_{A,v}(V_{0,j}^{(i)}) - \gamma_{A,v}(\bar{V}_{0,j}^{(i)})||_{C([0,T];L_\rho^2)(2)} \leq \sqrt{2(c^2(N) + c_{f^{(i)}}^2(T))T}||V_{0,j}^{(i)} - \bar{V}_{0,j}^{(i)}||_{C([0,T];L_\rho^2)(2)}$$

As above pathwise uniqueness and continuity of $V_{0,n}^{(i)}$ follows.
By the structure of the $Z_{k,n}^{(i)}$ and $V_{k,n}^{(i)}$ it is obvious,that pathwise continuity for k-1 implies first pathwise continuity of $Z_{k,n}^{(i)}$ on $[t_k, t_{k+1}]$ and then pathwise continuity of $V_{k,n}^{(i)}$ on $[t_k, t_{k+1}]$ by the same procedure as above.
Considering these pathwise continuous modifications and defining mappings $Z_j^{(i)}$ , $V_j^{(i)}$:$\Omega\times[0,T]\to L_\rho^2$ by

$$Z_j^{(i)}(t) := Z_{k,j}^{(i)}(t)\,;\, t\in[t_k, t_{k+1})\,;\, k = 0,1,2,...,j-1$$

$$V_j^{(i)}(0) := \xi^{(i)}$$
$$V_j^{(i)}(t) := V_{k,j}^{(i)}(t)\,;\, t\in(t_k, t_{k+1}]\,;\, k = 0,1,...,j-1$$

$$Z_j^{(i)}(T) := V_j^{(i)}(T)$$

leads to the following equations:

$$\begin{aligned} Z_j^{(i)}(t) = \quad & \xi^{(i)} + \int_0^{t_k} (A_N(s)V_j^{(i)}(s) + F^{(i)}(s,\cdot,V_j^{(i)}(s)))\,ds \\ & + \int_0^t \Sigma(s, Z_j^{(i)}(s))\,dW_M(s) \end{aligned}$$

for $t \in [\,t_k, t_{k+1}\,)$;$k = 0, 1, ..., j-1$ and

$$\begin{aligned} V_j^{(i)}(t) = \quad & \xi^{(i)} + \int_0^{t} (A_N(s)V_j^{(i)}(s) + F^{(i)}(s,\cdot,V_j^{(i)}(s)))\,ds \\ & + \int_0^{t_{k+1}} \Sigma(s, Z_j^{(i)}(s))\,dW_M(s) \end{aligned}$$

for $t \in (\,t_k, t_{k+1}\,]$;$k = 0, 1, ..., j-1$;$i = 1, 2$.
Thus by [KaSh](cf. chapter 5,proposition 2.1.8 there) the following holds true for $t \in [\,0, t_1\,)$:

$$\begin{aligned} Z_j^{(1)}(t,x) = \quad & \xi^{(1)}(x) + \left(\int_0^t \Sigma(s, Z_j^{(1)}(s))\,dW_M(s)\right)(x) \\ = \quad & \xi^{(1)}(x) + \sum_{n=1}^{M} \int_0^t \sigma(s, Z_j^{(1)}(s,x))(Q^{\frac{1}{2}}e_n)(x)\,dw_n(s) \\ \leq \quad & \xi^{(2)}(x) + \sum_{n=1}^{M} \int_0^t \sigma(s, Z_j^{(2)}(s,x))(Q^{\frac{1}{2}}e_n)(x)\,dw_n(s) \\ = \quad & Z_j^{(2)}(t,x) \end{aligned}$$

for Lebesgue-almost all $x \in \Theta$.Thus

$$Z_j^{(1)}(t) \leq Z_j^{(2)}(t) \text{ in } L_\rho^2$$

P-almost surely on $[\,0, t_1\,)$ and thus

$$Z_j^{(2)}(t_1) \geq Z_j^{(1)}(t_1) \text{ P-a.s.} \qquad (2.10)$$

with $t_1 = \frac{T}{j}$ for all $j \in \mathbf{N}$.
Given $s \in [\,0, T\,]$,define an operator B(s):$L_\rho^2 \to L_\rho^2$ $\omega$-wisely by

$$B(s)\varphi := \frac{F^{(2)}(s,\cdot,V_j^{(2)}(s)) - F^{(2)}(s,\cdot,V_j^{(1)}(s))}{V_j^{(2)}(s) - V_j^{(1)}(s)}\,\varphi$$

in case $V_j^{(2)}(s,\omega) \neq V_j^{(1)}(s,\omega)$ and

$$B(s)\varphi := C(T)\varphi$$

else,where C(T) denotes the common Lipschitz-constant of $f^{(1)}$ and $f^{(2)}$,i.e.

$$C(T) := \max(c_{f^{(1)}}(T), c_{f^{(2)}}(T))$$

Setting $\bar{A}_N(s) := A_N(s) + B(s)$,$s \in [\,0, T\,]$,one obviously gets

$$
\begin{aligned}
V_j^{(2)}(t) - V_j^{(1)}(t) = \quad & Z_{0,j}^{(2)}(t_1) - Z_{0,j}^{(1)}(t_1) + \int_0^t \bar{A}_N(s)[V_j^{(2)}(s) - V_j^{(1)}(s)]\,ds \\
& + \int_0^t [F^{(2)}(s,\cdot,V_j^{(1)}(s)) - F^{(1)}(s,\cdot,V_j^{(1)}(s))]\,ds
\end{aligned}
$$

for $t \in [0,t_1]$.Since $A_N \in \mathcal{L}(L^2_\rho)$ holds by (BA) the definition of B immediately leads to

$$
U_N(t,s)(B(s)+C(T)*I): L^2_\rho \to L^2_\rho \ , \ (s,t) \in S_T
$$

$$
\int_0^t ||A_N(t)U_N(t,s)(B(s)+C(T)*I)\varphi||_{\rho,2}\,ds \le c(T,C(T))||A_N(t)||\,||\varphi||_{\rho,2}
$$

Thus $\mathcal{D}(A_N) := L^2_\rho$(cf. (BA)) for all $N \in \mathbf{N}$ implies,by theorem 9.11 from [CuPr],the existence of an almost strong evolution operator in the sense of section 1.3 with generator $(A_N(t)+B(t)+C(T)*I)_{t\in[0,T]}$ for all $N \in \mathbf{N}$.
Let $N \in \mathbf{N}$ be fixed.
According to section 1.3 $U_N$ is positivity preserving.Since $||B(t)|| \le C(T)$ obviously holds for all t both $Q_N$ given by

$$
Q_N(t,s) := \sum_{a=0}^{\infty} Q_N^{(a)}(t,s) \ , \ (s,t) \in S_T
$$

with $Q_N^{(0)}(t,s) = U_N(t,s)$

$$
Q_N^{(a)}(t,s)\varphi := \int_s^t U_N(t,r)(B(r)+C(T)*I)Q_N^{(a-1)}(r,s)\varphi\,dr
$$

for $\varphi \in L^2_\rho$,$a = 1,2,\ldots$(cf. [CuPr],pp.252–262) and $\bar{U}_N$ given by

$$
\bar{U}_N(t,s) := Q_N(t,s)e^{-C(T)(t-s)} \ , \ (s,t) \in S_T
$$

have got this property.Then for all $\varphi \in L^2_\rho$

$$
\begin{aligned}
\bar{U}_N(t,s)\varphi = \quad & Q_N(t,s)e^{-C(T)(t-s)}\varphi \\
= \quad & Q_N(t,s)\left(e^{-C(T)(s-s)} + \int_s^t -C(T)e^{-C(T)(r-s)}\,dr\right)\varphi \\
= \quad & Q_N(t,s)\varphi + \int_s^t Q_N(t,r)(-C(T)e^{-C(T)(r-s)}*I)Q_N(r,s)\varphi\,dr
\end{aligned}
$$

where the semigroup property of $Q_N$ was used in the second step.
Thus by [CuPr](cf. theorem 9.2 there)

$$
(A_N + B + C(T)*I) - C(T)*I = A_N + B = \bar{A}_N
$$

generates the operator $\bar{U}_N$.So

$$
\begin{aligned}
V_j^{(2)}(t) - V_j^{(1)}(t) = \quad & \bar{U}_N(t,0)(Z_{0,j}^{(2)}(t_1) - Z_{0,j}^{(1)}(t_1)) \\
& + \int_0^t \bar{U}_N(t,s)[F^{(2)}(s,\cdot,V_j^{(1)}(s)) - F^{(1)}(s,\cdot,V_j^{(1)}(s))]\,ds \\
\ge \quad & 0
\end{aligned}
$$

holds true for all $t \in [0,t_1]$ by (2.10) and $f^{(1)} \le f^{(2)}$.
Then $Z_j^{(i)}(t_1) = V_j^{(i)}(t_1)$;i=1,2;implies

$$Z_j^{(1)}(t_1) \leq Z_j^{(2)}(t_1) \text{ in } L^2_\rho \text{ P-a.s.}$$

s.t. the wanted inequations are shown on $[\,0, t_1\,]$.By the same arguments as before one gets

$$\begin{aligned} V_j^{(1)}(t) &\leq V_j^{(2)}(t) \text{ in } L^2_\rho \text{ P-a.s.} \\ Z_j^{(1)}(t) &\leq Z_j^{(2)}(t) \text{ in } L^2_\rho \text{ P-a.s.} \end{aligned}$$

for $t \in [\,t_1, t_2\,]$,and so on,s.t. finally

$$\begin{aligned} V_j^{(1)}(t) &\leq V_j^{(2)}(t) \text{ in } L^2_\rho \text{ P-a.s.} \\ Z_j^{(1)}(t) &\leq Z_j^{(2)}(t) \text{ in } L^2_\rho \text{ P-a.s.} \end{aligned}$$

is shown for all $t \in [\,0, T\,]$.
Considering $V_j^{(i)}$ for arbitrary $j \in \mathbf{N}$ leads

$$\begin{aligned} \sup_{t\in[\,0,T\,]} \mathbf{E}||V_j^{(i)}(t)||^2_{\rho,2} &\leq c\left(\mathbf{E}||\xi^{(i)}||^2_{\rho,2} + \sum_{k=0}^{j-1} \sup_{t\in[\,t_k,t_{k+1}\,]} \mathbf{E}||V_{k,j}^{(i)}(t)||^2_{\rho,2}\right) \\ &\leq c_V(j) \end{aligned}$$

and an analogue estimation holds for $Z_j$ with a positive constant $c_Z(j)$.
By 2.2.7 the above definition of $Y^{(i)}$ implies

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||Y^{(i)}(t)||^2_{\rho,2} \leq c(\xi^{(i)})$$

for a positive constant depending on $\xi^{(i)}$.
So

$$\sup_{t\in[\,0,T\,]} \mathbf{E}[\,||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{\rho,2} + ||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{\rho,2}\,] \leq c_{V,Z}(j, \xi^{(i)}) < \infty$$

By (L1) for f and (L1),(L2) for $\sigma$ one gets

$$\begin{aligned} &\mathbf{E}||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{\rho,2} \\ &= \mathbf{E}\left|\left|\int_0^t (A_N(s)V_j^{(i)}(s) + F^{(i)}(s,\cdot,V_j^{(i)}(s)))\,ds + \int_0^{t_{k+1}} \Sigma(s, Z_j^{(i)}(s))\,dW_M(s)\right.\right. \\ &\left.\left. - \left(\int_0^t U_N(t,s)F^{(i)}(s,\cdot,Y^{(i)}(s))\,ds - \int_0^t U_N(t,s)\Sigma(s,Y^{(i)}(s))\,dW_M(s)\right)\right|\right|^2_{\rho,2} \\ &\leq c(C(T), c(N))\left(\sum_{n=1}^{M}\left[\int_0^{t_{k+1}} \mathbf{E}||(\Sigma(s, Z_j^{(i)}(s)) - \Sigma(s, Y^{(i)}(s)))e_n||^2_{\rho,2}\,ds\right.\right. \\ &\left.\left. + \int_t^{t_{k+1}} \mathbf{E}||\Sigma(s, Y^{(i)}(s))e_n||^2_{\rho,2}\,ds\right] + \int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{\rho,2}\,ds\right) \\ &\leq c(T, M, c(N), C(T), c_\sigma(T))\left[\int_0^{t_{k+1}} \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||^2_{\rho,2}\,ds\right. \\ &\left. +(t_{k+1} - t_k)\left(1 + \sup_{r\in[\,0,T\,]} \mathbf{E}||Y^{(i)}(r)||^2_{\rho,2}\right) + \int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{\rho,2}\,ds\right] \end{aligned}$$

$$=: c(T,M,c(N),C(T),c_\sigma(T))\left[\beta_j(t_{k+1})+\int_0^t \mathbf{E}||V_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\,ds\right]$$

for $t\in(\,t_k,t_{k+1}\,]$,$k\in\{0,1,\ldots,j-1\}$. Consider $\beta_j$:

$$\begin{aligned}
\beta_j(t_{k+1}) :=\ & \int_0^{t_{k+1}} \mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\,ds+(t_{k+1}-t_k)\left(1+\sup_{r\in[\,0,T\,]}\mathbf{E}||Y^{(i)}(r)||_{\rho,2}^2\right)\\
\leq\ & T\sup_{r\in[\,0,T\,]}\mathbf{E}[\,||V_j^{(i)}(r)-Y^{(i)}(r)||_{\rho,2}^2+||Z_j^{(i)}(r)-Y^{(i)}(r)||_{\rho,2}^2\,]\\
& +\tfrac{1}{j}\left(1+\sup_{r\in[\,0,T\,]}\mathbf{E}||Y^{(i)}(r)||_{\rho,2}^2\right)\\
\leq\ & T\,c_{V,Z}(j,\xi)+\tfrac{1}{j}(1+c(\xi^{(i)}))\\
<\ & \infty
\end{aligned}$$

Due to the continuity of $V_j^{(i)}$ and $Y^{(i)}$ Gronwall‘s lemma is applicable and leads to

$$\begin{aligned}
\mathbf{E}||V_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 \leq\ & c(T,M,c(N),C(T),c_\sigma(T))\beta_j(t_{k+1})e^{c(\ldots)t}\\
=:\ & \bar{c}(T,M,c(N),C(T),c_\sigma(T))\beta_j(t_{k+1})
\end{aligned}$$

Given $t\in[\,t_k,t_{k+1}\,)$,$k\in\{0,1,\ldots,j-1\}$ one has

$$\begin{aligned}
\mathbf{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 \leq\ & C\Bigg(\mathbf{E}||V_j^{(i)}(t_k)-Y^{(i)}(t_k)||_{\rho,2}^2\\
& +\mathbf{E}\left|\left|\int_{t_k}^t(\Sigma(s,Y^{(i)}(s))-\Sigma(s,Z_j^{(i)}(s)))\,dW_M(s)\right|\right|_{\rho,2}^2\\
& +\mathbf{E}\left[\int_{t_k}^t ||A_N(s)||^2\,||Y^{(i)}(s)||_{\rho,2}^2+||F^{(i)}(s,\cdot,Y^{(i)}(s))||_{\rho,2}^2\,ds\right]\Bigg)\\
=:\ & C(I_1(j)+I_2(j)+I_3)
\end{aligned}$$

First consider $I_1(j)$:

$$\begin{aligned}
& I_1(j)\\
& \leq \bar{c}(T,M,c(N),C(T),c_\sigma(T))\left[\int_0^{t_k}\mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\,ds+\tfrac{1}{j}\left(1+\sup_{r\in[\,0,T\,]}\mathbf{E}||Y^{(i)}(r)||_{\rho,2}^2\right)\right]\\
& \leq \bar{c}(T,M,c(N),C(T),c_\sigma(T))\left[\int_0^{t_k}\mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\,ds+\tfrac{1}{j}(1+c(\xi^{(i)}))\right]
\end{aligned}$$

Considering $I_3$,$t\in[\,t_k,t_{k+1}\,)$ implies $t-t_k\leq\frac{1}{j}$,s.t.

$$\begin{aligned}
0\leq I_3 \leq\ & \tfrac{1}{j}c(T,N,C(T))\left(1+\sup_{r\in[\,0,T\,]}\mathbf{E}||Y^{(i)}(r)||_{\rho,2}^2\right)\\
\leq\ & \tfrac{1}{j}c(T,N,C(T))(1+c(\xi^{(i)}))
\end{aligned}$$

holds true using (BA)(ii) and (L1),(L2) for $f^{(i)}$.Considering $I_2(j)$ condition (L1) for $\sigma$ leads to

$$0\leq I_2(j)\leq c(M,c_\sigma(T))\int_{t_k}^t\mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\,ds$$

Thus

$$\mathbf{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 \leq \frac{1}{j}(1+\bar{c}(T,M,c(N),C(T),c_\sigma(T))(1+c(\xi^{(i)})) + c(T,M,c(N),C(T),c_\sigma(T))\int_0^t \mathbf{E}||Z_j^{(i)}(s)-Y^{(i)}(s)||_{\rho,2}^2\, ds$$

s.t. Gronwall‘s lemma implies

$$\mathbf{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 \leq \frac{1}{j}(1+\bar{C})(1+c(\xi^{(i)}))e^{\bar{C}t} < \infty$$

where $\bar{C}$ denotes the maximum of the two constants from the equations before.This term obviously tends to 0 in case of j tending to $\infty$,i.e.

$$\lim_{j\to\infty} \mathbf{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 = 0$$

holds true for $t \in [\,t_k, t_{k+1}\,)$,$k \in \{0,1,\ldots,j-1\}$.Thus by the construction of the $t_k$ this holds true for all $t \in [\,0,T\,]$.

What has been shown so far is:

$$Z_j^{(1)}(t) \leq Z_j^{(2)}(t)\ ,V_j^{(1)}(t) \leq V_j^{(2)}(t)$$

holds true P-almost surely for all $t \in [\,0,T\,]$ and one has

$$\lim_{j\to\infty} \sup_{t\in[\,0,T\,]} \mathbf{E}||Z_j^{(i)}(t)-Y^{(i)}(t)||_{\rho,2}^2 = 0$$

The second property leads to the existence of a subsequence $(Z_{j(l)}^{(i)}(t))_{l\in\mathbf{N}}$,which converges to $Y^{(i)}(t)$ P-almost surely in $L_\rho^2$.With the help of the first property this leads to

$$Y^{(1)}(t) \leq Y^{(2)}(t) \ \text{ P-a.s., } t \in [\,0,T\,]$$

q.e.d.

The second lemma is a version of the claim,Manthey and Zausinger make in the second step of their proof.

**Lemma 2.2.13:**

Under the assumptions of 2.2.11 one has

$$\lim_{N\to\infty} \mathbf{E}||X_{N,M}^{(i)}(t)-X_M^{(i)}(t)||_{\rho,2}^2 = 0$$

and

$$\lim_{M\to\infty} \mathbf{E}||X_M^{(i)}(t)-X^{(i)}(t)||_{\rho,2}^2 = 0$$

where $X_M^{(i)}$ solves

$$dX_M^{(i)}(t) = (\ A(t)X_M^{(i)}(t) + F^{(i)}(t,\omega,X_M^{(i)}(t))\ )\,dt + \Sigma(t,X_M^{(i)}(t))dW_M(t)$$

$X_M^{(i)}(0) = \xi^{(i)}$.

in the sense of 2.2.3.

**Proof:**
For all N,M$\in \mathbf{N}$;$i = 1,2$;$t \in [\,0,T\,]$;$\omega \in \Omega$

$$X_{N,M}^{(i)}(t,\omega) = U_N(t,0)\xi(\omega) + \gamma_{A,v}^{N}(X_{N,M}^{(i)}(\omega))(t) + \gamma_d^{N,M}(X_{N,M}^{(i)}(\omega))(t)$$

holds with the $\gamma$-terms given by
$\gamma_{A,v}^{N}(X(\omega)) := \int_0^{\cdot} U_N(\cdot,s)F^{(i)}(s,\omega,X(s,\omega))\,ds$
$\gamma_d^{N,M}(X(\omega)) := \int_0^{\cdot} U_N(\cdot,s)\Sigma(s,X(s,\omega))\,dW_M(s)$

for a solution.Then one has for any two solutions $X_{N,M}^{(i)}$,$X_{N,M,1}^{(i)}$ and arbitrary $t \in [\,0,\bar{T}\,]$ with $\bar{T} \in [\,0,T\,]$:

- $\mathbf{E}\left|\left|\int_0^t U_N(t,s)[F^{(i)}(s,\cdot,X_{N,M}^{(i)}(s)) - F^{(i)}(s,\cdot,X_{N,M,1}^{(i)}(s))]\,ds\right|\right|_{\rho,2}^2$
  $\leq c(c(N),c_f(T))\int_0^t \mathbf{E}||X_{N,M}^{(i)}(s) - X_{N,M,1}^{(i)}(s)||_{\rho,2}^2\,ds$
  $\leq \bar{T}\,c(c(N),c_f(T)) \sup_{r\in[\,0,\bar{T}\,]} \mathbf{E}||X_{N,M}^{(i)}(r) - X_{N,M,1}^{(i)}(r)||_{\rho,2}^2$
  and thus in $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L_\rho^2))$

  $||\gamma_{A,v}^{N}(X_{N,M}^{(i)}) - \gamma_{A,v}^{N}(X_{N,M,1}^{(i)})||_{C([\,0,\bar{T}\,];L_\rho^2)(2)}$
  $\leq \sqrt{\bar{T}c(c(N),c_f(T))}||X_{N,M}^{(i)} - X_{N,M,1}^{(i)}||_{C([\,0,\bar{T}\,];L_\rho^2)(2)}$

- $\mathbf{E}\left|\left|\int_0^t U_N(t,s)[\Sigma(s,X_{N,M}^{(i)}(s)) - \Sigma(s,X_{N,M,1}^{(i)}(s))]\,dW_M(s)\right|\right|_{\rho,2}^2$
  $\leq c(M)\int_0^t \mathbf{E}||U_N(t,s)[\Sigma(s,X_{N,M}^{(i)}(s)) - \Sigma(s,X_{N,M,1}(s))]\,||_{\mathcal{L}_2}^2\,ds$
  $\leq c(M,c(N),c_\sigma(T))\int_0^t \mathbf{E}||X_{N,M}^{(i)}(s) - X_{N,M,1}^{(i)}(s)||_{\rho,2}^2\,ds$
  $\leq \bar{T}\,c(M,c(N),c_\sigma(T)) \sup_{r\in[\,0,\bar{T}\,]} \mathbf{E}||X_{N,M}^{(i)}(r) - X_{N,M,1}^{(i)}(r)||_{\rho,2}^2$
  and thus in $C([\,0,T\,];L^2([\,0,T\,]\times\Omega;L_\rho^2))$:

  $||\gamma_d^{N,M}(X_{N,M}^{(i)}) - \gamma_d^{N,M,1}(X_{N,M,1}^{(i)})||_{C([\,0,\bar{T}\,];L_\rho^2)(2)}$
  $\leq \sqrt{\bar{T}\,c(M,c(N),c_\sigma(T))}||X_{N,M}^{(i)} - X_{N,M,1}^{(i)}||_{C([\,0,\bar{T}\,];L_\rho^2)(2)}$

Completely analogous to the proof of 2.2.7(i) the above estimates ensure the existence of a pathwise unique and continuous solution $X_{N,M}^{(i)}$.In the same way one gets the existence of pathwise unique,continuous solutions $X^{(i)}_M$ and $X^{(i)}$.

Fix N,M$\in$ **N**:
Taking the difference between solutions $X_{N,M}$ and $X_M$,one gets for fixed $t \in [0,T]$:

$X^{(i)}_{N,M}(t) - X^{(i)}_M(t) = a_N(\xi) + b_N(F) + a_N(F) + b_N(\Sigma) + a_N(\Sigma)$

with the terms defined by

$$a_N(\xi) = [U_N(t,0) - U(t,0)]\xi^{(i)}$$
$$b_N(F) = \int_0^t U_N(t,s)[F^{(i)}(s,\cdot,X^{(i)}_{N,M}(s)) - F^{(i)}(s,\cdot,X_M(s))]\,ds$$
$$a_N(F) = \int_0^t [U_N(t,s) - U(t,s)]F^{(i)}(s,\cdot,X^{(i)}_M(s))\,ds$$
$$b_N(\Sigma) = \int_0^t U_N(t,s)[\Sigma(s,X^{(i)}_{N,M}(s)) - \Sigma(s,X^{(i)}_M(s))]\,dW_M(s)$$
$$a_N(\Sigma) = \int_0^t [U_N(t,s) - U(t,s)]\Sigma(s,X^{(i)}_M(s))\,dW_M(s)$$

Use the following consideration for an estimate of the $a_N$-terms:
By (BA)(ii)

$$\lim_{N\to\infty} \sup_{(s,t)\in S_T} ||[U_N(t,s) - U(t,s)]\varphi||^2_{\rho,2} = 0$$

holds true for all $\varphi \in L^2_\rho$.Let $r \in [0,T]$ be arbitrary.As f and $\sigma$ fulfill (L1),(L2) by assumption,$X^{(i)}_M(r) \in L^2_\rho$ for i=1,2 and $M \in \mathbf{N}$ hold true

$$F(r,\cdot,X^{(i)}_M(r)) \in L^2_\rho\,,\ \sigma(r,X^{(i)}_M(r)) \in L^2_\rho$$

follows obviously with F defined by (2.4) and $\sigma$ defined analogously to (2.4)(without $\omega$-dependence).Thus

$$\lim_{N\to\infty} \sup_{(s,t)\in S_T} \mathbf{E}||[U_N(t,s) - U(t,s)]F(r,\cdot,X^{(i)}_M(r))||^2_{\rho,2} = 0$$

and

$$\lim_{N\to\infty} \sup_{(s,t)\in S_T} \mathbf{E}||[U_N(t,s) - U(t,s)]\sigma(r,X^{(i)}_M(r))||^2_{\rho,2} = 0$$

As $r \in [0,T]$ was arbitrary one has in particular

$$\lim_{N\to\infty} \sup_{(s,t)\in S_T} \mathbf{E}||[U_N(t,s) - U(t,s)]F(s,\cdot,X^{(i)}_M(s))||^2_{\rho,2} = 0$$

and

$$\lim_{N\to\infty} \sup_{(s,t)\in S_T} \mathbf{E}||[U_N(t,s) - U(t,s)]\sigma(s,X^{(i)}_M(s))||^2_{\rho,2} = 0$$

Consider the following estimates for the $a_N$-terms:

$$\mathbf{E}||a_N(\xi)||^2_{\rho,2} \leq \sup_{(s,t)\in S_T} ||[U_N(t,s) - U(t,s)]\xi||^2_{\rho,2}$$

$$\mathbf{E}||a_N(F)||^2_{\rho,2} \leq T \sup_{(s,t)\in S_T} ||[U_N(t,s) - U(t,s)]F^{(i)}(s,\cdot,X^{(i)}_M(s))||^2_{\rho,2}$$

$$\begin{aligned}
\mathbf{E}||a_N(\Sigma)||^2_{\rho,2,+} &= \mathbf{E}\left|\left|\int_0^t [U_N(t,s) - U(t,s)]\Sigma(s, X^{(i)}_M(s))\, dW_M(s)\right|\right|^2_{\rho,2} \\
&= \mathbf{E}\left|\left|\int_0^t [U_N(t,s) - U(t,s)]\Sigma(s, X^{(i)}_M(s))\, d\left(\sum_{n=1}^M \sqrt{a_n} w_n(s) e_n\right)\right|\right|^2_{\rho,2} \\
&= \sum_{n=1}^M a_n\, \mathbf{E}\int_\Theta \left(\int_0^t [U_N(t,s) - U(t,s)]\Sigma(s, X^{(i)}_M(s)) e_n\, ds\right)^2 (x)\mu_\rho(dx) \\
&\leq \sum_{n=1}^M a_n \sup_{n\in\mathbf{N}} ||e_n||^2_\infty\, T \sup_{(s,t)\in S_T} \mathbf{E}||[U_N(t,s) - U(t,s)]\sigma(s, X^{(i)}_M(s))||^2_{\rho,2}
\end{aligned}$$

By (BA)(ii) resp. the above considerations these terms tend to 0 as N tends to $\infty$.

Furthermore the properties of f and $\sigma$ lead to

$$\mathbf{E}||b_N(F)||^2_{\rho,2} \leq c(c(N), c_f(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{\rho,2}\, ds$$

$$\mathbf{E}||b_N(\Sigma)||^2_{\rho,2} \leq c(M, c(N), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{\rho,2}\, ds$$

s.t.

$$\begin{aligned}
\mathbf{E}||X^{(i)}_{N,M}(t) - X^{(i)}_M(t)||^2_{\rho,2} \leq\ & C\Big((\mathbf{E}||a_N(\xi)||^2_{\rho,2} + \mathbf{E}||a_N(F)||^2_{\rho,2} + \mathbf{E}||a_N(\Sigma)||^2_{\rho,2}) \\
&+ c(M, c(N), c_f(T), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{\rho,2}\, ds\Big)
\end{aligned}$$

holds true.As $X^{(i)}_{N,M}$ and $X^{(i)}_M$ are time-continuous, Gronwall‘s lemma is applicable,s.t. the first part of the claim follows by the fact that the $a_N$-terms tend to 0 for $N \to \infty$.

Consider $X^{(i)}_M$ with an arbitrary M$\in \mathbf{N}$:

$$\begin{aligned}
X^{(i)}_M(t) - X^{(i)}(t) =\ & \int_0^t U(t,s)[F^{(i)}(s,\omega,X^{(i)}_M(s)) - F^{(i)}(s,\omega,X^{(i)}(s))]\, ds \\
&+ \int_0^t U(t,s)[\Sigma(s, X^{(i)}_M(s)) - \Sigma(s, X^{(i)}(s))]\, dW_M(s) \\
&- \sum_{n=M+1}^\infty \int_0^t \sqrt{a_n}[U(t,s)\Sigma(s, X^{(i)}(s))](e_n)\, dw_n(s)
\end{aligned}$$

for all $t \in [0,T]$,$\omega \in \Omega$,s.t. analogously to the $b_N$-terms above the following holds:

$$\begin{aligned}
\mathbf{E}||X_M^{(i)}(t) - X^{(i)}(t)||_{\rho,2}^2 \leq\ & c(M, c(T), c_{f^{(i)}}(T), c_\sigma(T)) \int_0^t (t-s)^{-\gamma} \mathbf{E}||X_M^{(i)}(s) - X^{(i)}(s)||_{\rho,2}^2 \, ds \\
& + \mathbf{E} \left|\left| \sum_{n=M+1}^{\infty} \int_0^t \sqrt{a_n} [U(t,s)\Sigma(s, X^{(i)}(s))](e_n) \, dw_n(s) \right|\right|_{\rho,2}^2 \\
\leq\ & c(M, c(T), c_f(T), c_\sigma(T)) \int_0^t (t-s)^{-\gamma} \mathbf{E}||X_M^{(i)}(s) - X^{(i)}(s)||_{\rho,2} \, ds \\
& + \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}|| \, [U(t,s)\Sigma(s, X^{(i)}(s))](e_n)||_{\rho,2}^2 \, ds
\end{aligned}$$

Since the numbers $a_n$ are nonnegative and fulfill

$$\sum_{n=1}^{\infty} a_n < \infty$$

in the nuclear case,one gets

$$\begin{aligned}
& \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}|| \, [U(t,s)\Sigma(s, X^{(i)}(s))](e_n)||_{\rho,2}^2 \, ds \\
& \leq T \, c(c(T), c_\sigma(T)) \left( \sup_{n \in \mathbf{N}} ||e_n||_\infty \right)^2 \left( \sum_{n=1}^{\infty} a_n \right) \left( 1 + \sup_{t \in [\,0,T\,]} \mathbf{E}||X^{(i)}(t)||_{\rho,2}^2 \right) \\
& < \infty
\end{aligned}$$

This sum obviously tends to 0 as M tends to $\infty$,s.t. the nuclear case is finished.
For the cylindrical case(i.e. $a_n = 1$ for all n) apply (2.3) for $\varphi \in L_\rho^2$, $(s,t) \in S_T$.Thus the summand belonging to the stochastic rest is again finite and converges to 0 for $M \to \infty$.
Having this property both in the nuclear and in the cylindrical case,applying 2.2.5/2.2.6 (with $g_n := \mathbf{E}||X_M^{(i)} - X^{(i)}||_{\rho,2}^2$) finishes the proof.
q.e.d.

**Proof of 2.2.11:**
First by 2.2.12

$$X_{N,M}^{(1)}(t) \leq X_{N,M}^{(2)}(t), t \in [\,0,T\,]$$

P-almost surely in $L_\rho^2$.Then 2.2.13 implies the claim by first taking $N \to \infty$ and then taking $M \to \infty$.

## 2.3 The case with a non-Lipschitzian f

The aim of this section is to show the main result of this chapter,which is a result like 3.4.1 from [MaZa],in case $\Theta = \mathbf{R}_+^d$ with an $\omega$-dependent f.In order to do so,there is still one lemma necessary.

**Lemma 2.3.1**

Consider an arbitrary,real-valued,progressively measurable function f defined on

$$[0,T] \times \Omega \times \mathbf{R}$$

and define

$$\begin{aligned} f_N(t,\omega,x) &:= f(t,\omega,x) \vee -N \\ f_{N,M}(t,\omega,x) &:= \inf_{u \in \mathbf{R}} f_N(t,\omega,u) + M|u-x| \end{aligned}$$

for all $t \in [0,T]$,$\omega \in \Omega$,$x \in \mathbf{R}$,N,M$\in \mathbf{N}$.
Then $f_N$ and $f_{N,M}$ are progressively measurable as well.

**Proof:**
Consider $f_N$ first.
Fix $x \in \mathbf{R}$.What has to be shown is

$$\mathbf{1}_{[0,t]} f_N(\cdot,\cdot,x)^{-1}(A) \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$

for all Borel-sets A on $\mathbf{R}$.
Let $b \in \mathbf{R}$ and $t \in [0,T]$ be arbitrary. For all $N \in \mathbf{N}$ with $b \geq -N$ one gets

$$\begin{aligned} \{(s,\omega) \in [0,t] \times \Omega \mid f_N(s,\omega,x) < b\} &= \{(s,\omega) \mid f(s,\omega,x) \vee -N < b\} \\ &= \{(s,\omega) \mid f(s,\omega,x) < b\} \\ &\in \mathcal{B}([0,t]) \times \mathcal{F}_t \end{aligned}$$

due to the progressive measurability of f.But for $b < -N$:

$$\begin{aligned} \{(s,\omega) \in [0,t] \times \Omega \mid f_N(s,\omega,x) < b\} &= \{(s,\omega) \mid f(s,\omega,x) \vee -N < b\} \\ &= \emptyset \\ &\in \mathcal{B}([0,t]) \times \mathcal{F}_t \end{aligned}$$

Thus progressive measurability for each $f_N(\cdot,\cdot,x)$ with $x \in \mathbf{R}$,$N \in \mathbf{N}$ is shown,which finishes the consideration of $f_N$.

Finally consider $f_{N,M}$ with N,M $\in \mathbf{N}$. As above one gets for arbitrary u,x $\in \mathbf{R}$

$$\begin{aligned} \{(s,\omega) \in [0,t] \times \Omega \mid f_N(s,\omega,u) + M|u-x| < b\} &= \{(s,\omega) \mid f_N(s,\omega,u) < b - M|u-x|\} \\ &\in \mathcal{B}([0,t]) \times \mathcal{F}_t \end{aligned}$$

for fixed M$\in \mathbf{N}$ due to the progressive measurability of $f_N$.
Thus $f_N(\cdot,\cdot,u) + M|u-x|$ is progressively measurable for fixed u,x$\in \mathbf{R}$,M$\in \mathbf{N}$, which implies the progressive measurability of

$$\hat{f}_{N,M}(\cdot,\cdot,x) := \inf_{u\in\mathbf{Q}}(f_N(\cdot,\cdot,u) + M|u-x|)$$

for fixed x$\in$ **R**,M$\in$ **N** as countable infimum of progressively measurable mappings.
Let u$\in$ $\mathbf{R}\setminus\mathbf{Q}$.
As **Q** is dense in **R**,there exists a sequence $(u_n)_{n\in\mathbf{N}}$ of rational numbers converging to u for $n\to\infty$.
The continuity of f obviously implies the continuity of $f_N$ and

$$f_N + M|\cdot - x|$$

for fixed $x\in\mathbf{R}$.Thus by $u_n\to u$

$$f_N(\cdot,\cdot,u_n) + M|u_n - x| \to f_N(\cdot,\cdot,u) + M|u-x|$$

As such a sequence exists for every u$\in$ **R**,one has for arbitrary x$\in$ **R**:

$$\begin{aligned} f_{N,M}(\cdot,\cdot,x) = & \inf_{u\in\mathbf{R}} f_N(\cdot,\cdot,u) + M|u-x| \\ = & \inf_{u\in\mathbf{Q}} f_N(\cdot,\cdot,u) + M|u-x| \\ = & \hat{f}_{N,M}(\cdot,\cdot,x) \end{aligned}$$

Thus one gets progressive measurability of $f_{N,M}(\cdot,\cdot,x)$ for each x$\in$ **R**,which finishes the proof.
q.e.d.

Now one can finally show the wanted result:

**Theorem 2.3.2:**

Let (CD),(PP),(CC) and (BA) be fulfilled,let f:$[0,T]\times\Omega\times\mathbf{R}\to\mathbf{R}$ defining F be progressively measurable and continuous on **R** with properties (PG) with exponent $\nu\in\mathbf{N}$ and (LG) with $\omega$-independent constant and let $\sigma$:$[0,T]\times\mathbf{R}\to\mathbf{R}$ defining $\Sigma$ fulfill (L1),(L2). Let $\gamma$ be as in (CC) in the cylindrical case and let it be 0 in the nuclear case.

**(i):**In case $\nu=1$,given a natural number $q>\frac{2}{1-\gamma}$ with

$$\mathbf{E}||\xi||_{\rho,2}^q < \infty$$

there exists a solution X to Eq($\xi$,F,$\Sigma$),s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||_{\rho,2}^q \le c(q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

holds with a positive constant c(...) depending on U,f,$\sigma$,q,T.

**(ii):** If in addition (E1) and (E2) hold and there is a natural number $\nu>\frac{1}{1-\gamma}$ with

$$\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu} < \infty$$

there exists a solution X to Eq(ξ,F,Σ),s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

holds for a positive constant c(...) only depending on U,f,$\sigma$,$\nu$ and T.

**Proof:**
Following the proof of Manthey and Zausinger show (ii) first,and then show (i),which was not shown in [MaZa].

**Proof of (ii):**
**Step 1:** Define mappings $\bar{g}$,$\bar{h}$:$\mathbf{R}\to\mathbf{R}$ by

$$\bar{g}(v) := \min\left(\inf_{\substack{0\leq u\leq v\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u)\mathbf{1}_{[0,\infty)}(v),0\right) - c_f(T)(1-v)\mathbf{1}_{(-\infty,0)}(v) \quad (2.11)$$

$$\bar{h}(v) := \max\left(\sup_{\substack{v\leq u\leq 0\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u)\mathbf{1}_{(-\infty,0)}(v),0\right) + c_f(T)(1+v)\mathbf{1}_{(0,\infty)}(v) \quad (2.12)$$

Since f fulfills (LG),$\bar{g}$ and $\bar{h}$ fulfill the conditions

$$\bar{g}\leq 0, \bar{g}(v)\leq f(t,\omega,v), (t,\omega,v)\in[0,T]\times\Omega\times\mathbf{R} \quad (2.13)$$

$$\bar{h}\geq 0, \bar{h}(v)\geq f(t,\omega,v), (t,\omega,v)\in[0,T]\times\Omega\times\mathbf{R} \quad (2.14)$$

Then define mappings $f_{N,M}$ for $N,M\in\mathbf{N}$ as in 2.3.1.Then:

$$-N\leq f_{N,M}(t,\omega,0)\leq f_N(t,\omega,0)\leq c_f(T)$$

i.e. $f_N$ and $f_{N,M}$ fulfill (L2) with $\omega$-independent constant
$c(N) := \max\{N,c_f(T)\}$.

**Claim:** For arbitrary N,M$\in\mathbf{N}$

$$|f_{N,M}(t,\omega,x)-f_{N,M}(t,\omega,y)|\leq M|x-y|; t\in[0,T],\omega\in\Omega, x,y\in\mathbf{R}$$

holds true,s.t. the $f_{N,M}$ also fulfill (L1).

**Proof:** Fix arbitrary $t\in[0,T]$, $\omega\in\Omega$ and x,y$\in\mathbf{R}$ and define

$$z_x := \arg\inf_{z\in\mathbf{R}}(f_N(t,\omega,z)+M|x-z|), z_y := \arg\inf_{z\in\mathbf{R}}(f_N(t,\omega,z)+M|y-z|)$$

First consider the case $z_x = z_y =: \bar{z}$.Then the definition of $f_{N,M}$ implies

$$\begin{aligned} f_{N,M}(t,\omega,x)-f_{N,M}(t,\omega,y) &= M(|x-\bar{z}|-|y-\bar{z}|)\\ &\leq M(|x-\bar{z}-(y-\bar{z})|) = M|x-y| \end{aligned}$$

and analogously

$$-(f_{N,M}(t,\omega,x) - f_{N,M}(t,\omega,y)) = f_{N,M}(t,\omega,y) - f_{N,M}(t,\omega,x) \leq M|x-y|$$

which proves the claim.

So let $z_x \neq z_y$ in what follows.
Assuming $f_{N,M}(t,\omega,x) - f_{N,M}(t,\omega,y) < 0$ the claim follows from

$$f_{N,M}(t,\omega,x) - (f_N(t,\omega,z_x) + M|y-z_x|) < f_{N,M}(t,\omega,x) - f_{N,M}(t,\omega,y)$$

and the above case.Otherwise the claim follows from the following chain of inequations:

$$\begin{aligned} M|x-y| \geq \quad & M(|x-z_y| - |y-z_y|) \\ = \quad & f_N(t,\omega,z_y) + M|x-z_y| - (f_N(t,\omega,z_y) + M|y-z_y|) \\ \geq \quad & f_{N,M}(t,\omega,x) - f_{N,M}(t,\omega,y) \\ > \quad & 0 \end{aligned}$$

**Step 2:** Defining $F_{N,M}$ by (2.4),theorems 2.3.1, 2.2.7 and 2.2.11 imply the existence of pathwise unique solutions $X_{N,M}$ of Eq$(\xi, F_{N,M}, \Sigma)$ with

$$X_{N,M} \leq X_{N,M+1} \quad (2.15)$$

Denoting solutions of Eq($\xi^+$,$F_{0,M}$,$\Sigma$),Eq.($\xi^-$, $F^-_{N,M}$, $\Sigma$) resp. Eq.(0,0,$\Sigma$) by $\bar{X}_{0,M}$,$\underline{X}_{N,M}$ resp. V,one gets the following relations:

$$\begin{gathered} \underline{X}_{N,M}(t) \leq X_{N,M}(t) \leq \bar{X}_{0,M}(t) \quad (2.16) \\ \underline{X}_{N,M}(t) \leq V(t) \leq \bar{X}_{0,M}(t) \quad (2.17) \end{gathered}$$

P-almost surely for each $t \in [\,0,T\,]$ and arbitrary $N, M \in \mathbf{N}$.Theorem 2.2.7(ii) implies

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{\rho,2\nu} \leq c(\nu,M,T)(1 + \mathbf{E}||\xi^+||^{2\nu}_{\rho,2\nu})$$

for each $M \in \mathbf{N}$.Furthermore:

$$\begin{aligned} \mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{\rho,2\nu} \leq \quad & c(\nu)\left(\mathbf{E}||U(t,0)\xi^+||^{2\nu}_{\rho,2\nu} + \mathbf{E}\left|\left|\int_0^t U(t,s)F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))\,ds\right|\right|^{2\nu}_{\rho,2\nu}\right. \\ & \left. + \mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\bar{X}_{0,M}(s))\,dW(s)\right|\right|^{2\nu}_{\rho,2\nu}\right) \\ =: \quad & \bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}_M(t) \end{aligned}$$

First (E1) leads to

$$\begin{aligned} \bar{I}^{(1)}(t) = \quad & c(\nu)\mathbf{E}||U(t,0)\xi^+||^{2\nu}_{\rho,2\nu} \leq c(\nu,T)\mathbf{E}||\xi^+||^{2\nu}_{\rho,2\nu} \\ \leq \quad & c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{\rho,2\nu} \end{aligned}$$

By (PP) (E1) additionally implies

$$\bar{I}_M^{(2)}(t) \leq c(\nu,T)\mathbf{E}\left|\left|\int\limits_0^t U(t,s)|F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))|^\nu \, ds\right|\right|_{\rho,2}^2$$

and thus

$$\begin{aligned}
\bar{I}_M^{(2)}(t) \leq &\quad c(\nu,T,c(T))\mathbf{E}\int\limits_0^t ||\bar{h}(\bar{X}_{0,M}(s))||_{\rho,2\nu}^{2\nu} \, ds \\
\leq &\quad c(\nu,T,c(T),c_f(T))\mathbf{E}\int\limits_0^t \int\limits_{\bar{\Theta}} \Big[(1+\bar{X}_{0,M}^{2\nu}(s,y))\mathbf{1}_{\{\bar{X}_{0,M}(s,y)>0\}}(s,y) \\
&\quad +(1+V^{2\nu^2}(s,y))\mathbf{1}_{\{\bar{X}_{0,M}(s,y)<0\}}(s,y)\Big] \; \mu_\rho(dy)\, ds \\
\leq &\quad c(\nu,T,c(T),c_f(T))\left(1+\int\limits_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds + \int\limits_0^T \mathbf{E}||V(s)||_{\rho,2\nu^2}^{2\nu^2} \, ds\right) \\
\leq &\quad c(\nu,T,c(T),c_f(T),c_\sigma(T))\left(1+\int\limits_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds\right) \\
\leq &\quad c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))\left(1+\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds\right)
\end{aligned}$$

where (2.14) was used in the first,(2.12) and (2.17) were used in the second,2.2.10(ii) was used in the fourth and the fact,that $(t-s)^\gamma$ can be estimated by $T^\gamma$,was used in the fifth step.
Considering the process $\bar{\varphi}(t) := \sigma(t,\bar{X}_{0,M}(t))$ , $t \in [\,0,T\,]$, in $L_\rho^{2\nu}$, (E2);(L1),(L2) for $\sigma$ and $\gamma \in [\,0,1\,)$ imply

$$\begin{aligned}
\bar{I}_M^{(3)}(t) \leq &\quad c(\nu,T)\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||\sigma(s,\bar{X}_{0,M}(s))||_{\rho,2\nu}^{2\nu} \, ds \\
\leq &\quad c(\nu,\gamma,T,c_\sigma(T))\left(1+\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds\right)
\end{aligned}$$

Thus one has for arbitrary $t \in [\,0,T\,]$:

$$\begin{aligned}
\mathbf{E}||\bar{X}_{0,M}(t)||_{\rho,2\nu}^{2\nu} \leq &\quad c(\nu,T)\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu} \\
&\quad +c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))\left(1+\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds\right) \\
\leq &\quad c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu}) \\
&\quad +c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))\int\limits_0^t (t-s)^{-\gamma}\,\mathbf{E}||\bar{X}_{0,M}(s)||_{\rho,2\nu}^{2\nu} \, ds
\end{aligned}$$

Setting $g := \mathbf{E}||\bar{X}_{0,M}||_{\rho,2\nu}^{2\nu}$ one gets

$$g(t) \leq C(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu}) + C\int\limits_0^t (t-s)^{-\gamma} g(s)\, ds$$

for $\gamma \in [\,0,1\,)$,C:=c($\nu$,$\gamma$,T,c(T),$c_f(T)$,$c_\sigma(T)$),i.e. the conditions of 2.2.5 are fulfilled with $g_n := g$ for all $n \in \mathbf{N}$.Since g is bounded for all $M \in \mathbf{N}$,2.2.6 leads to

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

for arbitrary $M \in \mathbf{N}$,s.t. due to the M-independence of the constant

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

It was essential for the M-independence of the constant,that (2.14) is applicable for all $M \in \mathbf{N}$.
An analogous estimate holds true for $\underline{X}_{N,M}$(with a constant possibly dependent on N) and thus by (2.16) also for $X_{N,M}$.

**Step 3:** As Manthey and Zausinger did,show the convergence of $(X_{N,M})_{M\in\mathbf{N}}$ in $L^2_\rho$ to a process $X_N$ solving Eq.$(\xi, F_N, \Sigma)$.
Manthey and Zausinger define

$$Z_{N,M}(t) := X_{N,M}(t) - X_{N,1}(t); N, M \in \mathbf{N}; t \in [\,0,T\,]$$

(2.15) implies

$$0 \leq Z_{N,M}(t) \leq Z_{N,M+1}(t)$$

and

$$\begin{aligned} \sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||Z_{N,M}(t)||^{2\nu}_{\rho,2\nu} \leq \quad & c(\nu)\left[\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||X_{N,M}(t)||^{2\nu}_{\rho,2\nu} + \sup_{t\in[\,0,T\,]} \mathbf{E}||X_{N,1}(t)||^{2\nu}_{\rho,2\nu}\right] \\ < \quad & \infty \end{aligned}$$

**Claim 1:**By the definition

$$Z_N(t) := \sup_{M\in\mathbf{N}} Z_{N,M}(t), N \in \mathbf{N}, t \in [\,0,T\,]$$

from [MaZa] one gets,for each $N \in \mathbf{N}$,a pathwise continuous process $Z_N$,which is unique up to a zeroset in $\Omega \times \Theta$.
**Proof:** Consider continuity first:
$X_{N,M}$ is pathwise continuous for all N,M $\in \mathbf{N}$ by 2.2.7,as a consequence of which $Z_{N,M}(\omega)$ is continuous for arbitrary $\omega \in \Omega$ as the difference of $X_{N,M}(\omega)$ and $X_{N,1}(\omega)$ and $Z_N(\omega)$ is continuous as the supremum of the continuous $Z_{N,M}(\omega)$.As $\omega$ was chosen arbitrarily,pathwise continuity is shown.
By 2.2.7 $X_{N,M}(t)$ is pathwise unique for all N,M$\in \mathbf{N}$ and arbitrary $t \in [\,0,T\,]$,s.t. $Z_{N,M}(t)$ is pathwise unique as well for all N,M$\in \mathbf{N}$ and arbitrary $t \in [\,0,T\,]$.Thus

$$Z_N(t,\omega) := \sup_{M\in\mathbf{N}} Z_{N,M}(t,\omega)$$

is unique in $L^2_\rho$ except for $\omega \in \Omega$,s.t.

$$||Z_{N,\tilde{M}}(t,\omega)||_{\rho,2} = ||Z_{N,\bar{M}}(t,\omega)||_{\rho,2} = \sup_{M\in\mathbf{N}} ||Z_{N,M}(t,\omega)||_{\rho,2}$$

with natural numbers $\tilde{M} \neq \bar{M}$ and fixed N,t holds true.But then,according to the definition of the norm,the definition of $\mu_\rho$ from section 1.2 implies

$$Z_{N,\tilde{M}}(t,\omega,x) = Z_{N,\bar{M}}(t,\omega,x) \text{ for } \mu_\rho\text{-a.a. } x \in \Theta$$

s.t. $Z_N(t)$ is unique $P \otimes \mu_\rho$-almost everywhere for all N,t,which finishes the proof.

Due to the construction of the process one gets:

$$\sup_{t\in[0,T]} \mathbf{E}||Z_N(t)||_{\rho,2\nu}^{2\nu} < \infty$$

**Claim 2:** By setting

$$X_N(t) := Z_N(t) + X_{N,1}(t), N \in \mathbf{N}, t \in [0,T]$$

one gets a pathwise continuous stochastic process $X_N$,s.t. $X_N(t,\omega) \in L_\rho^{2\nu}$ holds true for every fixed $N \in \mathbf{N}$ and every $\omega \in \Omega$.
**Proof:**From the pathwise continuity of $Z_N$ shown in claim 1 and the pathwise continuity of $X_{N,1}$ one gets pathwise continuity of $X_N$.
Furthermore one already knows

$$\sup_{t\in[0,T]} \mathbf{E}||Z_N(t)||_{\rho,2\nu}^{2\nu} < \infty$$
$$\sup_{t\in[0,T]} \mathbf{E}||X_{N,1}(t)||_{\rho,2\nu}^{2\nu} < \infty$$

for each fixed N$\in \mathbf{N}$,which implies

$$\sup_{t\in[0,T]} \mathbf{E}||Z_N(t) + X_{N,1}(t)||_{\rho,2}^{2\nu} < \infty$$

Thus there exists a process $X_N \in \{\varphi : [0,T] \times \Omega \to L_\rho^{2\nu} \mid \sup_t \mathbf{E}||\varphi(t)||_{\rho,2\nu}^{2\nu} < \infty\}$,s.t.

$$X_N(t) = Z_N(t) + X_{N,1}(t), t \in [0,T]$$

P-almost surely in $L_\rho^{2\nu}$,i.e. for fixed N there exists a modification $X_N$,s.t. $X_N(t,\omega) \in L_\rho^{2\nu}$ holds true for all $t \in [0,T]$,$\omega \in \Omega$,which was the claim.

Fix an arbitrary $t \in [0,T]$.By (2.15) one gets $X_{N,M}(t) \uparrow X_N(t)$ P-almost surely.Since

$$\sup_{M\in\mathbf{N}} \mathbf{E}||X_{N,M}(t)||_{\rho,2\nu}^{2\nu} < \infty$$

dominated convergence implies

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||_{\rho,2\nu}^{2\nu} = 0$$

As $t \in [0,T]$ was arbitrarily chosen,one gets

$$\begin{aligned}\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||_{\rho,2\nu}^{2\nu}\, dt \leq& \quad T \sup_{t\in[0,T]} \lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||_{\rho,2\nu}^{2\nu} \\ =& \quad 0 \quad (2.18)\end{aligned}$$

Furthermore one has the following estimate for all $t \in [0,T]$:

$$
\begin{aligned}
\mathbf{E}||X_N(t)||^{2\nu}_{\rho,2\nu} = & \quad \mathbf{E}\left|\left|\sup_M(X_{N,M}(t)-X_{N,1}(t))+X_{N,1}(t)\right|\right|^{2\nu}_{\rho,2\nu} \\
\leq & \quad c(\nu)\left[\mathbf{E}\left|\left|\sup_M X_{N,M}(t)\right|\right|^{2\nu}_{\rho,2\nu}+2\mathbf{E}||X_{N,1}||^{2\nu}_{\rho,2\nu}\right] \\
\leq & \quad 3c(\nu)\sup_{M\in\mathbf{N}}\mathbf{E}||X_{N,M}(t)||^{2\nu}_{\rho,2\nu}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\sup_{t\in[\,0,T\,]}\mathbf{E}||X_N(t)||^{2\nu}_{\rho,2\nu} \leq & \quad 3c(\nu)\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}}\mathbf{E}||X_{N,M}(t)||^{2\nu}_{\rho,2\nu} \\
\leq & \quad c(N,\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^{2\nu}_{\rho,2\nu})
\end{aligned}
$$

In the same manner one gets processes $\underline{X}_N$,$\bar{X}$ in $L^{2\nu}_\rho$,s.t.

$$
\lim_{M\to\infty}\int_0^T\mathbf{E}||\underline{X}_{N,M}(t)-\underline{X}_N(t)||^{2\nu}_{\rho,2\nu}\,ds=0
$$

$$
\lim_{M\to\infty}\int_0^T\mathbf{E}||\bar{X}_{0,M}(t)-\bar{X}(t)||^{2\nu}_{\rho,2\nu}\,ds=0
$$

and

$$
\begin{gathered}
\underline{X}_N(t)\leq X_N(t)\leq\bar{X}(t) \\
\underline{X}_N(t)\leq V(t)\leq\bar{X}(t)
\end{gathered}
$$

P-almost surely for all $t\in[\,0,T\,]$.

**Step 4:** (2.18) is just convergence in probability of $(X_{N,M})_{M\in\mathbf{N}}$ to $X_N$,as a consequence of which there must be a subsequence of $(X_{N,M})_{M\in\mathbf{N}}$,which converges to $X_N$ P-almost surely.As in the fourth step of the proof of 3.4.1(ii) in [MaZa] let w.l.o.g. $(X_{N,M})_{M\in\mathbf{N}}$ itself be this sequence.

$$
\begin{aligned}
&\mathbf{E}\left|\left|X_N(t)-U(t,0)\xi-\int_0^t U(t,s)F_N(s,\cdot,X_N(s))\,ds-\int_0^t U(t,s)\Sigma(s,X_N(s))\,dW(s)\right|\right|^2_{\rho,2,+} \\
&\leq 3(I^{(1)}_{N,M}(t)+I^{(2)}_{N,M}(t)+I^{(3)}_{N,M}(t))
\end{aligned}
$$

where the terms are given by:

$$
\begin{aligned}
&I^{(1)}_{N,M}(t):=\mathbf{E}||X_N(t)-X_{N,M}(t)||^2_{\rho,2,+}\leq c(\nu,\rho)(\mathbf{E}||X_n(t)-X_{N,M}(t)||^{2\nu}_{\rho,2\nu})^{\frac{1}{\nu}} \\
&I^{(2)}_{N,M}(t):=c^2(T)\mathbf{E}\left|\left|\int_0^t F_N(s,\cdot,X_N(s))-F_{N,M}(s,\cdot,X_{N,M}(s))\,ds\right|\right|^2_{\rho,2} \\
&I^{(3)}_{N,M}(t):=\mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X_N(s))-\Sigma(s,X_{N,M}(s))]\,dW(s)\right|\right|^2_{\rho,2}
\end{aligned}
$$

s.t. at least the first term tends to 0 for $M\to\infty$ by (2.18).Consider the second term:

$$
\begin{aligned}
I^{(2)}_{N,M}(t) \le \quad & c(c(T))\left(\mathbf{E}\int_0^T ||F_N(s,\cdot,X_N(s)) - F_N(s,\cdot,X_{N,M}(s))||^2_{\rho,2}\,ds\right.\\
& \left.+\mathbf{E}\int_0^T ||F_N(s,\cdot,X_{N,M}(s)) - F_{N,M}(s,\cdot,X_{N,M}(s))||^2_{\rho,2}\,ds\right)\\
=: \quad & c(c(T))(I^{(21)}_{N,M}(T) + I^{(22)}_{N,M}(T))
\end{aligned}
$$

By (PG):

$$
\begin{aligned}
I^{(21)}_{N,M}(T) = \quad & \mathbf{E}\int_0^T\int_{\mathbf{\Theta}}(f_N(s,\cdot,X_N(s,x)) - f_N(s,\cdot,X_{N,M}(s,x)))^2\mu_\rho(dx)\,ds\\
\le \quad & \mathbf{E}\int_0^T\int_{\mathbf{\Theta}}\left(c(N,c_f(T))(1+|X_N(s,x)|^\nu + |X_{N,M}(s,x)|^\nu)\right)^2\mu_\rho(dx)\,ds\\
\le \quad & c(N,T,c(T),\nu,c_f(T))\left(1+\int_0^T(\mathbf{E}||X_N(s)||^{2\nu}_{\rho,2\nu} + \mathbf{E}||X_{N,M}(s)||^{2\nu}_{\rho,2\nu})\,ds\right)
\end{aligned}
$$

s.t.

$$
X_{N,1}(t) \le X_{N,M}(t) \le X_N(t),\text{P-f.s},t\in[0,T]
$$

implies

$$
I^{(21)}_{N,M}(T) \le c(N,T,\nu,c(T),c_f(T),c_\sigma(T))\left(1+2\int_0^T\mathbf{E}||X_N(s)||^{2\nu}_{\rho,2\nu}\,ds\right)
$$

So $(I^{(21)}_{N,M})_{M\in\mathbf{N}}$ is a bounded sequence in $L^{2\nu}_\rho$.Then (2.18) and the continuity of $F_N$ ensure

$$
\lim_{M\to\infty} I^{(21)}_{N,M}(T) = 0
$$

Fix $L \le M$;L,M$\in$ $\mathbf{N}$;and get in analogy to the consideration of $I^{(21)}_{N,M}$ with $f_{N,M}\uparrow f_N$

$$
\begin{aligned}
\lim_{M\to\infty} I^{(22)}_{N,M} \le \quad & \lim_{M\to\infty}\int_0^T\mathbf{E}||F_N(s,\omega,X_{N,M}(s)) - F_{N,L}(s,\omega,X_{N,M}(s))||^2_{\rho,2}\,ds\\
= \quad & \int_0^T\mathbf{E}||F_N(s,\omega,X_N(s)) - F_{N,L}(s,\omega,X_N(s))||^2_{\rho,2}\,ds
\end{aligned}
$$

which tends to 0 for $L\to\infty$ since $f_{N,L}\uparrow f_N$.Thus

$$
\lim_{M\to\infty} I^{(2)}_{N,M}(T) = 0
$$

Finally one gets by (E2),(L1),Hoelder‘s inequality and

$$
\nu > \tfrac{1}{1-\gamma} \Rightarrow \tfrac{\gamma\nu}{\nu-1} < 1 \quad (2.19)
$$

the following estimate for the third term:

$$
\begin{aligned}
I^{(3)}_{N,M}(T) \leq\ & c^2(T)\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||\sigma(s,X_N(s))-\sigma(s,X_{N,M}(s))||^2_{\rho,2}\,ds \\
\leq\ & c(c(T),c_\sigma(T))\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)-X_{N,M}(s)||^2_{\rho,2}\,ds \\
\leq\ & c(c(T),c_\sigma(T))\left(\int\limits_0^T s^{\frac{-\gamma\nu}{\nu-1}}\,ds\right)^{\frac{\nu-1}{\nu}}\left(\int\limits_0^T \mathbf{E}||X_N(s)-X_{N,M}(s)||^{2\nu}_{\rho,2\nu}\,ds\right)^{\frac{1}{\nu}} \\
\leq\ & c(\gamma,T,c(T),c_\sigma(T))\left(\int\limits_0^T \mathbf{E}||X_N(t)-X_{N,M}(t)||^{2\nu}_{\rho,2\nu}\,ds\right)^{\frac{1}{\nu}}
\end{aligned}
$$

which tends to 0 for $M\to\infty$ by (2.18).Thus (iv) from 2.2.3 holds true for $X_N$.Consider the other properties from 2.2.3:
Since $X_N$ is predictable by its construction and fulfills $X_N \in L^{2\nu}_\rho$ according to claim 2 from step 3,property 2.2.3(i) is fulfilled.By (2.18),(2.19) and property 2.2.3(ii) for $X_{N,M}$ one gets

$$
\begin{aligned}
\sup_{t\in[0,T]}\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)||^2_{\rho,2}\,ds \leq\ & 2\Bigg[\sup_{t\in[0,T]}\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)-X_{N,M}(s)||^2_{\rho,2}\,ds \\
& +\sup_{t\in[0,T]}\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||X_{N,M}(s)||^2_{\rho,2}\,ds\Bigg] \\
\leq\ & 2\Bigg[\left(\int\limits_0^T s^{-\frac{\gamma\nu}{\nu-1}}\,ds\right)^{\frac{\nu-1}{\nu}}\left(\int\limits_0^T \mathbf{E}||X_N(s)-X_{N,M}(s)||^{2\nu}_{\rho,2\nu}\,ds\right)^{\frac{1}{\nu}} \\
& +\sup_{t\in[0,T]}\int\limits_0^t (t-s)^{-\gamma}\mathbf{E}||X_{N,M}(s)||^2_{\rho,2}\,ds\Bigg] \\
<\ & \infty
\end{aligned}
$$

for all $t\in[0,T]$ and arbitrary N,M$\in\mathbf{N}$.
Thus $X_N$ has property 2.2.3(ii) as well.
By claim 2 from step 3 $X_N$ is pathwise continuous,which is just 2.2.3(iii),s.t. all properties from 2.2.3 are fulfilled,which means that $X_N$ is a solution of Eq.($\xi$,$F_N$,$\Sigma$) for all $N\in\mathbf{N}$.

**Step 5:**

Due to the definition of $X_N$ for $N\in\mathbf{N}$ and step 2 one already knows that

$$
\sup_{t\in[0,T]}\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{\rho,2\nu}\leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T),N)(1+\mathbf{E}||\xi||^{2\nu}_{\rho,2\nu})
$$

holds.Show the N-independence of the constant.First of all

$$
\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{\rho,2\nu}\leq c(\nu)\left(\underline{I}^{(1)}(t)+\underline{I}^{(2)}_N(t)+\underline{I}^{(3)}_N(t)\right)
$$

for all $t\in[0,T]$ with

$$
\underline{I}^{(1)}(t):=\mathbf{E}||U(t,0)\xi^-||^{2\nu}_{\rho,2\nu}
$$

$$
\underline{I}^{(2)}_N(t):=\mathbf{E}\left|\left|\int\limits_0^t U(t,s)F^-_N(s,\cdot,\underline{X}_n(s))\,ds\right|\right|^{2\nu}_{\rho,2\nu}
$$

$$\underline{I}_N^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\underline{X}_N(s))\,dW(s)\right|\right|_{\rho,2\nu}^{2\nu}$$

Then

$$\begin{array}{rl} \underline{I}^{(1)}(t) = & \mathbf{E}||\,|U(t,0)\xi^-\,|^\nu\,||_{\rho,2}^2 \\ \leq & c(\nu,T)\mathbf{E}||U(t,0)|\xi^-|^\nu||_{\rho,2}^2 \\ = & c(\nu,T,c(T))\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu} \end{array}$$

by (E1),

$$\begin{array}{rl} \underline{I}_N^{(2)}(t) \leq & c(\nu,c(T))\int_0^t ||F_N^-(s,\cdot,\underline{X}_N(s))||_{\rho,2\nu}^{2\nu}\,ds \\ \leq & c(\nu,c(T))\int_0^t \mathbf{E}||\bar{g}(\underline{X}_N(s))||_{\rho,2\nu}^{2\nu}\,ds \\ \leq & c(\nu,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||_{\rho,2\nu}^{2\nu}\,ds\right) \\ \leq & c(\nu,\gamma,T,c(T),c_f(T))\left(1+\int_0^t (t-s)^{-\gamma}\mathbf{E}||\underline{X}_N(s)||_{\rho,2\nu}^{2\nu}\,ds\right) \end{array}$$

by (2.13) and by (CC),(L1),(L2)

$$\underline{I}_N^{(3)}(t) \leq c(\nu,T,c(T),c_\sigma(T))\left(1+\int_0^t (t-s)^{-\gamma}\mathbf{E}||\underline{X}_N(s)||_{\rho,2\nu}^{2\nu}\,ds\right)$$

Together the three estimates lead to

$$\begin{array}{rl} \mathbf{E}||\underline{X}_N(t)||_{\rho,2\nu}^{2\nu} \leq & c(\nu,T)\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu} \\ & +c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))\left(1+\int_0^t (t-s)^{-\gamma}\mathbf{E}||\underline{X}_N(s)||_{\rho,2\nu}^{2\nu}\,ds\right) \\ \leq & c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu}) \\ & +c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||\underline{X}_N(s)||_{\rho,2\nu}^{2\nu} \end{array}$$

for all $t \in [\,0,T\,]$.Then 2.2.5 and 2.2.6 with $g_n := \mathbf{E}||\underline{X}_N||_{\rho,2\nu}^{2\nu}$ for all $n \in \mathbf{N}$ show

$$\mathbf{E}||\underline{X}_N(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

for all $N \in \mathbf{N}$,s.t.

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbf{N}}} \mathbf{E}||\underline{X}_N(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

holds true.The fact that (2.13) holds true for all $N \in \mathbf{N}$ was essential for the N-independence of the constant. Then

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||_{\rho,2\nu}^{2\nu} \leq c(\nu)\left[\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||\bar{X}(t)-\bar{X}_{0,M}(t)||_{\rho,2\nu}^{2\nu} + \sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||_{\rho,2\nu}^{2\nu}\right]$$

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{\rho,2\nu,+}^{2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})$$

and

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \text{ in } L^{2\nu}_\rho$$

lead to

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{\rho,2\nu} \leq c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^{2\nu}_{\rho,2\nu}) \quad (2.20)$$

for all $t \in [\,0,T\,]$.

By $f_N \downarrow f$ 2.2.11 implies

$$X_{N+1}(t) \leq X_N(t) \text{ P-a.s.,t} \in [\,0,T\,]\,,\ N \in \mathbf{N} \quad (2.21)$$

Show that X given by

$$X(t) := \inf_{N\in\mathbf{N}} X_N(t)\,,\ t \in [\,0,T\,]$$

is a solution in the sense of 2.2.3.
First fix $t \in [\,0,T\,]$.Define

$$Y_N(t) := X_1(t) - X_N(t)\ ,\ N \in \mathbf{N}$$

By (2.21) this is a sequence of random variables,that are positive almost surely with

$$Y_N(t) \leq Y_{N+1}(t) \text{ P-a.s.}$$

and

$$\begin{aligned}\sup_{t\in[\,0,T\,]} \mathbf{E}||Y_N(t)||^{2\nu}_{\rho,2\nu} = &\sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t)-X_N(t)||^{2\nu}_{\rho,2\nu} \\ \leq\ & c(\nu)\left(\sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t)||^{2\nu}_{\rho,2\nu} + \sup_{t\in[\,0,T\,]} \mathbf{E}||X_N(t)||^{2\nu}_{\rho,2\nu}\right)\end{aligned}$$

which is finite by (2.20).Analogously to the procedure in the case of the $X_{N,M}$ one gets

$$\lim_{N\to\infty} \int_0^T \mathbf{E}||Y_N(t)-Y(t)||^{2\nu}_{\rho,2\nu}\,dt = 0$$

for Y given by

$$Y(t) := \sup_{N\in\mathbf{N}} Y_N(t)\ ,\ t \in [\,0,T\,]$$

By the definitions of X and Y

$$\begin{aligned}Y(t) = \sup_{N\in\mathbf{N}} Y_N(t) = &\sup_{N\in\mathbf{N}} (X_1(t) - X_N(t)) \\ = &\ X_1(t) - \inf_{N\in\mathbf{N}} X_N(t) \\ = &\ X_1(t) - X(t)\end{aligned}$$

holds true for all $t \in [0,T]$,s.t.

$$\begin{aligned}\mathbf{E}||X_N(t) - X(t)||^{2\nu}_{\rho,2\nu} &= \mathbf{E}||X_1(t) - Y_N(t) - X_1(t) + Y(t)||^{2\nu}_{\rho,2\nu} \\ &= \mathbf{E}||Y_N(t) - Y(t)||^{2\nu}_{\rho,2\nu}\end{aligned}$$

for all $t \in [0,T]$ implies

$$\lim_{N\to\infty} \int_0^T \mathbf{E}||X_N(t) - X(t)||^{2\nu}_{\rho,2\nu}\, dt = 0 \quad (2.22)$$

With the estimate

$$\begin{aligned}&\mathbf{E}\left|\left|X(t) - U(t,0)\xi - \int_0^t U(t,s)F(s,\cdot,X(s))\,ds - \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)\right|\right|^2_{\rho,2} \\ &\leq 3\left(\mathbf{E}||X(t) - X_N(t)||^2_{\rho,2} + c^2(T)\mathbf{E}\left|\left|\int_0^t F(s,\cdot,X(s)) - F_N(s,\cdot,X_N(s))\,ds\right|\right|^2_{\rho,2}\right. \\ &\left.+\mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X(s)) - \Sigma(s,X_N(s))]\,dW(s)\right|\right|^2_{\rho,2}\right) \\ &=: 3(I^{(1)}_N(t) + I^{(2)}_N(t) + I^{(3)}_N(t))\end{aligned}$$

one gets analogously to the procedure in step 4:

$$I^{(1)}_N(t) \leq c(\nu,\rho)\left(\mathbf{E}||X(t) - X_N(t)||^{2\nu}_{\rho,2\nu}\right)^{\frac{1}{\nu}}$$

$$\begin{aligned}I^{(2)}_N(t) \leq\ & c(\rho,T)\left(\mathbf{E}\int_0^T ||F(s,\cdot,X(s)) - F(s,\cdot,X_N(s))||^2_{\rho,2}\,ds\right. \\ &\left.+\mathbf{E}\int_0^T ||F(s,\cdot,X_N(s)) - F_N(s,\cdot,X_N(s))||^2_{\rho,2}\right)\end{aligned}$$

$$I^{(3)}_N(t) \leq c(T,c(T),c_\sigma(T))\left(\int_0^T s^{\frac{-\gamma\nu}{\nu-1}}\,ds\right)^{\frac{\nu-1}{\nu}}\left(\int_0^T \mathbf{E}||X(s) - X_N(s)||^{2\nu}_{\rho,2\nu}\,ds\right)^{\frac{1}{\nu}}$$

By the definition of $f_N$ for any $x \in \Theta$ $f_N(\cdot,\cdot,x)$ differs from $f(\cdot,\cdot,x)$ if and only if one has $f(\cdot,\cdot,x) < -N$. But then

$$f^2(\cdot,\cdot,x) > f^2_N(\cdot,\cdot,x)$$

s.t. $F(\cdot,\cdot,\varphi) \geq F_N(\cdot,\cdot,\varphi)$ holds true in $L^2_\rho$ for all $\varphi \in L^{2\nu}_\rho$,which means $F_N \uparrow F$ in $L^2_\rho$.
Thus analogously to the procedure in step 4 (2.20) implies

$$\lim_{N\in\mathbf{N}} I^{(j)}_N(T) = 0$$

for $j = 1,2,3$,since f is continuous in $\mathbf{R}$ as well.
Thus X solves the equation from 2.2.3 (iv) P-a.s. for $t \in [0,T]$.
Due to the fact that $X(t) \leq X_N(t)$ holds true P-a.s. for all $t \in [0,T]$, $N \in \mathbf{N}$,one gets

$$\mathbf{E}||X(t)||^{2\nu}_{\rho,2\nu} \leq \inf_{N\in\mathbf{N}} \mathbf{E}||X_N(t)||^{2\nu}_{\rho,2\nu}$$

But one also has

$$D := \inf_{N\in\mathbf{N}} \mathbf{E}||X_N(t)||_{\rho,2\nu}^{2\nu} \le \mathbf{E}||X_{\bar N}(t)||_{\rho,2\nu}^{2\nu}$$

for all $\bar N \in \mathbf{N}$,s.t. (2.18) implies

$$D \le \mathbf{E}\left|\left|\inf_{N\in\mathbf{N}} X_N(t)\right|\right|_{\rho,2\nu}^{2\nu} = \mathbf{E}||X(t)||_{\rho,2\nu}^{2\nu}$$

Thus for all $t \in [\,0,T\,]$:

$$\begin{aligned}\sup_{t\in[\,0,T\,]} \mathbf{E}||X(t)||_{\rho,2\nu}^{2\nu} = &\sup_{t\in[\,0,T\,]}\inf_{N\in\mathbf{N}} \mathbf{E}||X_N(t)||_{\rho,2\nu}^{2\nu} \le \sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||_{\rho,2\nu}^{2\nu}\\ \le\ & c(\nu,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2\nu}^{2\nu})\end{aligned}$$

Thus the wanted estimate holds true and

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||X(t)||_{\rho,2\nu}^{2\nu} < \infty$$

leads to $X(t) \in L_\rho^{2\nu}$ P-almost surely for all $t \in [\,0,T\,]$,s.t. 2.2.3(i) is fulfilled.Completely analogous to the $X_N$-case one gets

$$\begin{aligned}\sup_{t\in[\,0,T\,]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||_{\rho,2}^2\,ds \le\ & 2\left[\left(\int_0^T s^{-\frac{\gamma\nu}{\nu-1}}\,ds\right)^{\frac{\nu-1}{\nu}}\left(\int_0^T \mathbf{E}||X(s)-X_N(s)||_{\rho,2\nu}^{2\nu}\,ds\right)^{\frac{1}{\nu}}\right.\\ &\left.+\sup_{t\in[\,0,T\,]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)||_{\rho,2}^2\,ds\right]\end{aligned}$$

for arbitrary $N \in \mathbf{N}$.As all $X_N$ have property 2.2.3(ii),the last integral is finite,s.t. by (2.22) property 2.2.3(ii) is also fulfilled for X .
Due to $X_N$ being a solution of Eq.($\xi$,$F_N$,$\Sigma$) each path of a process $X_N$ is continuous.But then each path of X is continuous as the infimum of a family of continuous processes.So X owns property 2.2.3(iii) as well,which finishes the proof.

**Proof of (i):**
Consider functions $f_{N,M}^-$,$f_{N,M}$ and $f_{0,M}$ as in the proof of (ii).Then given $q > \frac{2}{1-\gamma}$ and $\xi$ with

$$\mathbf{E}||\xi||_{\rho,2}^q < \infty$$

theorem 2.2.7(i) ensures the existence of pathwise unique continuous solutions $\underline{X}_{N,M}$,$\bar X_{0,M}$,$X_{N,M}$ and V (case $\xi = F = 0$) with properties (2.16) and (2.17).Consider again $\bar X_{0,M}$.For arbitrary $t \in [\,0,T\,]$ one has

$$\begin{aligned}\mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,2} \leq\ & c(q)\left(\mathbf{E}||U(t,0)\xi||^q_{\rho,2} + \left|\left|\int_0^t U(t,s)F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))\,ds\right|\right|^q_{\rho,2}\right.\\ & \left. +\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\bar{X}_{0,M}(s))\,dW(s)\right|\right|^q_{\rho,2}\right)\\ =:\ & \tilde{I}^{(1)}(t)+\tilde{I}^{(2)}_M(t)+\tilde{I}^{(3)}_M(t)\end{aligned}$$

with the following estimates for the $\tilde{I}$-terms:

$$\tilde{I}^{(1)}(t) \leq c(q,c(T))\mathbf{E}||\xi||^q_{\rho,2}$$

$$\begin{aligned}\tilde{I}^{(2)}_M(t) =\ & \mathbf{E}\left[\int_\Theta\left(\int_0^t U(t,s)F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))\,ds\right)^2(x)\mu_\rho(dx)\right]^{\frac{q}{2}}\\ \leq\ & c(c(T))\mathbf{E}\int_0^t ||\bar{h}(\bar{X}_{0,M}(s))||^q_{\rho,2}\,ds\\ \leq\ & c(c(T),c_f(T))\mathbf{E}\int_0^t\left[\int_\Theta\left[(1+\bar{X}^2_{0,M}(s,y))\mathbf{1}_{\{\bar{X}_{0,M}(s,y)>0\}}(s,y)\right.\right.\\ & \left.\left.+\left(1+V^2(s,y)\right)\mathbf{1}_{\{\bar{X}_{0,M}(s,y)<0\}}(s,y)\right]\mu_\rho(dy)\right]^{\frac{q}{2}}ds\\ \leq\ & c(q,T,c(T),c_f(T))\left(1+\int_0^t\mathbf{E}||\bar{X}_{0,M}(s)||^q_{\rho,2}\,ds+\int_0^T\mathbf{E}||V(s)||^q_{\rho,2}\,ds\right)\\ \leq\ & c(q,T,c(T),c_f(T),c_\sigma(T))\left(1+\int_0^t\mathbf{E}||\bar{X}_{0,M}(s)||^q_{\rho,2}\,ds\right)\end{aligned}$$

where (2.14) was used in the second and 2.2.10(i) was used in the last step and

$$\begin{aligned}\tilde{I}^{(3)}_M(t) \leq\ & c(q,T,\gamma,c(T))\int_0^t\mathbf{E}||\sigma(s,\bar{X}_{0,M}(s))||^q_{\rho,2}\,ds\\ \leq\ & c(q,T,\gamma,c(T),c_\sigma(T))\left(1+\int_0^t\mathbf{E}||\bar{X}_{0,M}(s)||^q_{\rho,2}\,ds\right)\end{aligned}$$

where (2.6) with $\kappa=1$ was used for the process $\bar{\varphi}(t):=\sigma(t,\bar{X}_{0,M}(t))$ in the first and (L1),(L2) for $\sigma$ were used in the second step.

Putting the estimates together

$$\begin{aligned}\mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,2} \leq\ & c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^q_{\rho,2})\\ & +c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))\int_0^t\mathbf{E}||\bar{X}_{0,M}(s)||^q_{\rho,2}\,ds\end{aligned}$$

holds true,s.t. Gronwall‘s lemma leads to

$$\sup_{t\in[0,T]}\mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,2} \leq c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^q_{\rho,2})$$

and one gets by the M-independence of the constant

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbf{N}}}\mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,2} < c(q,T,\gamma,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^q_{\rho,2})$$

As in the proof of (ii) the analogue of the last estimate holds true for $\underline{X}_{N,M}$

with a constant possibly depending on N and thus by (2.16) for $X_{N,M}$ as well. Defining $Z_{N,M}$ as in part (ii) one has

$$0 \leq Z_{N,M}(t) \leq Z_{N,M+1}(t) \text{ P-a.s,} t \in [\,0,T\,]$$

and

$$\begin{aligned}\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||Z_{N,M}(t)||_{\rho,2}^q \leq &\quad c(q)\left[\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||X_{N,M}(t)||_{\rho,2}^q + \sup_{t\in[\,0,T\,]} \mathbf{E}||X_{N,1}(t)||_{\rho,2}^q\right] \\ < &\quad \infty\end{aligned}$$

Completely analogous to the proof of claim 1 in step 3 of (ii) $Z_N$ defined by

$$Z_N(t) := \sup_{M\in\mathbf{N}} Z_{N,M}(t), N \in \mathbf{N}, t \in [\,0,T\,]$$

is a pathwise continuous process,which is unique up to a P-zeroset on $\Omega \times \Theta$ and fulfills

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||Z_N(t)||_{\rho,2}^q < \infty$$

Then $X_N$ defined by

$$X_N(t) := Z_N(t) + X_{N,1}(t), N \in \mathbf{N}, t \in [\,0,T\,]$$

is pathwise continuous as well.
Analogously to the proof of (ii) dominated convergence implies

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||_{\rho,2}^q = 0$$

for all $t \in [\,0,T\,]$ and thus

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||_{\rho,2}^q \, dt = 0 \quad (2.23)$$

In the same manner one gets processes $\bar{X}$ and $\underline{X}_N$ with

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||_{\rho,2}^q \, dt = 0$$
$$\lim_{M\to\infty} \int_0^T \mathbf{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||_{\rho,2}^q \, dt = 0$$

By the same method as in the proof of (ii) one gets

$$\mathbf{E}||X_N(t)||_{\rho,2}^q \leq c(N,q,T,c(T),c_f(T),c_\sigma(T))(1 + \mathbf{E}||\xi||_{\rho,2}^q)$$

and again there exists a subsequence of $(X_{N,M})_{M\in\mathbf{N}}$,which converges to $X_N$ P-almost surely.Again let w.l.o.g. the sequence itself be this subsequence.Then

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)F_N(s,\cdot,X_N(s))\,ds - \int_0^t U(t,s)\Sigma(s,X_N(s))\,dW(s)\right|\right|_{\rho,2}^2$$

$\leq 3(I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t))$

with I-terms given by:

$$I^{(1)}_{N,M}(t) := \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{\rho,2}$$
$$I^{(2)}_{N,M}(t) := c^2(T)\int_0^t \mathbf{E}||F_N(s,\cdot,X_N(s)) - F_{N,M}(s,\cdot,X_{N,M}(s))||^2_{\rho,2}\,ds$$
$$I^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X_N(s)) - \Sigma(s,X_{N,M}(s))]\,dW(s)\right|\right|^2_{\rho,2}$$

By the estimate

$$\begin{aligned} I^{(1)}_{N,M}(t) = &\quad \mathbf{E}||X_N(t) - X_{N,M}(t)||^{\frac{2q}{q}}_{\rho,2} \\ \leq &\quad (\mathbf{E}||X_N(t) - X_{N,M}(t)||^q_{\rho,2})^{\frac{2}{q}} \end{aligned}$$

for all $q > \frac{2}{1-\gamma} \geq 2$,$I^{(1)}_{N,M}(t)$ converges to 0 for $M \to \infty$ given arbitrary $t \in [0,T]$.
Dividing $I^{(2)}_{N,M}$ into terms $I^{(2i)}_{N,M}(T)$;i=1,2; as in the first part of the proof leads to

$$I^{(21)}_{N,M}(T) := \mathbf{E}\int_0^T ||F_N(s,\cdot,X_N(s)) - F_N(s,\cdot,X_{N,M}(s))||^2_{\rho,2}\,ds$$
$$I^{22}_{N,M}(T) := \mathbf{E}\int_0^T ||F_N(s,\cdot,X_{N,M}(s)) - F_{N,M}(s,\cdot,X_{N,M}(s))||^2_{\rho,2}\,ds$$

Condition (PG) with exponent $\nu = 1$ implies

$$\begin{aligned} &|f_N(s,\omega,X_N(s,\omega,y)) - f_N(s,\omega,X_{N,M}(s,\omega,y))| \\ &\leq c(N,c_f(T))(1 + |X_N(s,\omega,y)| + |X_{N,M}(s,\omega,y)|) \end{aligned}$$

Analogously to the first part of the proof $(I^{(21)}_{N,M})_{M\in\mathbf{N}}$ is a bounded sequence in $L^2_\rho$ with

$$I^{(21)}_{N,M}(T) \leq \left(\mathbf{E}\int_0^T ||F_N(s,\cdot,X_N(s)) - F_N(s,\cdot,X_{N,M}(s))||^q_{\rho,2}\,ds\right)^{\frac{2}{q}}$$

which converges to 0 for $M \to \infty$ due to the continuity of $F_N$.
Analogously to the procedure in the proof of part (ii)

$$\lim_{M\to\infty} I^{(22)}_{N,M}(T) = 0$$

holds true,s.t. $I^{(2)}_{N,M}$ converges to 0 for $M \to \infty$ for all $t \in [0,T]$.
By Ito's isometry,(CC),(L1) and Hoelder's inequality

$$\begin{aligned} I^{(3)}_{N,M}(t) \leq &\quad c(T)\int_0^t (t-s)^{-\gamma}\mathbf{E}||\sigma(s,X_N(s)) - \sigma(s,X_{N,M}(s))||^2_{\rho,2}\,ds \\ \leq &\quad c(c(T),c_\sigma(T))\int_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s) - X_{N,M}(s)||^2_{\rho,2}\,ds \\ \leq &\quad c(c(T),c_\sigma(T))\left(\int_0^T s^{-\frac{\gamma q}{q-2}}\,ds\right)^{\frac{q-2}{q}}\left(\int_0^T \mathbf{E}||X_N(s) - X_{N,M}(s)||^q_{\rho,2}\,ds\right)^{\frac{2}{q}} \end{aligned}$$

which converges to 0 for $M \to \infty$ by (2.23),since

$$q > \tfrac{2}{1-\gamma} \Rightarrow \tfrac{q\gamma}{q-2} < 1 \quad (2.24)$$

ensures the existence of the first integral on the righthand side.

Then the sum of the $I_{N,M}^{(i)}$ converges to 0 for $M \to \infty$,s.t. $X_N$ fulfills property 2.2.3(iv).
By its construction $X_N$ is predictable.For all $N \in \mathbf{N}$

$$\begin{aligned}
\sup_{t\in[\,0,T\,]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)||_{\rho,2}^2\,ds \leq\ & 2\left[\sup_{t\in[\,0,T\,]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)-X_{N,M}(s)||_{\rho,2}^2\,ds\right.\\
&+ \sup_{t\in[\,0,T\,]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||X_{N,M}(s)||_{\rho,2}^2\,ds\\
\leq\ & 2\left[\left(\int_0^T s^{-\frac{\gamma q}{q-2}}\,ds\right)^{\frac{q-2}{q}}\left(\int_0^T \mathbf{E}||X_N(s)-X_{N,M}(s)||_{\rho,2}^q\,ds\right)^{\frac{2}{q}}\right.\\
&+ \sup_{t\in[\,0,T\,]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||X_{N,M}(s)||_{\rho,2}^2\,ds
\end{aligned}$$

holds true with arbitrary $M \in \mathbf{N}$.The righthand side is finite,since $X_{N,M}$ as a solution of Eq($\xi$,$F_{N,M}$,$\Sigma$) has property 2.2.3(ii) and the first term tends to 0 for $M \to \infty$ as in the estimate of $I_{N,M}^{(3)}$,s.t. 2.2.3(ii) is fulfilled for $X_N$.$\nu = 1$ implies that 2.2.3(i) is trivially fulfilled and the pathwise continuity follows analogously to the proof of part (ii).
Thus for arbitrary $N \in \mathbf{N}$ $X_N$ is a solution of Eq.($\xi$,$F_N$,$\Sigma$).
Defining processes $\underline{X}_N$,$\bar{X}$ analogously to $X_N$,these processes solve Eq.($\xi^-$,$F_N^-$,$\Sigma$) resp. Eq.($\xi^+$,$F^+$,$\Sigma$) with estimates

$$\begin{gathered}
\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t)\\
\underline{X}_N(t) \leq V(t) \leq \bar{X}(t)
\end{gathered}$$

P-almost surely for all $t \in [\,0,T\,]$.
Consider arbitrary $N \in \mathbf{N}$:

$$\mathbf{E}||\underline{X}_N(t)||_{\rho,2}^q \leq c(q)(\underline{I}^{(1)}(t) + \underline{I}_N^{(2)}(t) + \underline{I}_N^{(3)}(t))$$

with

$$\begin{aligned}
\underline{I}^{(1)}(t) &= \mathbf{E}||U(t,0)\xi^-||_{\rho,2}^q\\
\underline{I}_N^{(2)}(t) &= \mathbf{E}\left|\left|\int_0^t U(t,s)F_N^-(s,\cdot,\underline{X}_N(s))\,ds\right|\right|_{\rho,2}^q\\
\underline{I}_N^{(3)}(t) &= \mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\underline{X}_N(s))\,dW(s)\right|\right|_{\rho,2}^q
\end{aligned}$$

For an estimate of $\underline{I}_N^{(3)}$ consider (2.6) (with $\kappa = 1$ and $\varphi := \Sigma(\cdot,\underline{X}_N)$). This leads to

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\underline{X}_N(s))\,dW(s)\right|\right|^q_{\rho,2} \leq c(q,\gamma,T,c(T),c_\sigma(T))\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{\rho,2}\,ds$$

for all $t \in [\,0,T\,]$.
So the $\underline{I}_N$-terms have the following properties:

1. $\sup_{t\in[\,0,T\,]} \underline{I}^{(1)}(t) \leq c(q,c(T))\mathbf{E}||\xi^-||^q_{\rho,2} \leq c(q,c(T))\mathbf{E}||\xi||^q_{\rho,2}$

2. $$\begin{aligned} \underline{I}_N^{(2)}(t) \leq & \quad c(q,c(T))\int_0^t \mathbf{E}||F_N^-(s,\cdot,\underline{X}_N(s))||^q_{\rho,2}\,ds \\ \leq & \quad c(q,c(T))\int_0^t \mathbf{E}||g(\underline{X}_n(s))||^q_{\rho,2}\,ds \\ \leq & \quad c(q,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{\rho,2}\,ds\right) \end{aligned}$$

3. $\underline{I}_N^{(3)}(t) \leq c(q,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{\rho,2}\,ds\right)$

Then applying Gronwall‘s lemma leads to

$$\mathbf{E}||\underline{X}_N(t)||^q_{\rho,2} \leq c(q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||^q_{\rho,2})$$

i.e. $\underline{X}_N$ is bounded in $L^q([\,0,T\,]\times\Omega;L^2_\rho)$ by an N-independent constant.
Defining $Y_N$,Y and X as in part (ii) implies

$$Y_N(t) \leq Y_{N+1}(t) \text{ P-a.s.}$$

and

$$\begin{aligned} \sup_{t\in[\,0,T\,]}\mathbf{E}||Y_N(t)||^q_{\rho,2} = & \quad \sup_{t\in[\,0,T\,]}\mathbf{E}||X_1(t)-X_N(t)||^q_{\rho,2} \\ \leq & \quad c(q)\left(\sup_{t\in[\,0,T\,]}\mathbf{E}||X_1(t)||^q_{\rho,2} + \sup_{t\in[\,0,T\,]}\mathbf{E}||X_N(t)||^q_{\rho,2}\right) \\ < & \quad \infty \end{aligned}$$

Analogously to the procedure in case $X_{N,M}$ one gets

$$\lim_{N\to\infty}\int_0^T \mathbf{E}||Y_N(t)-Y(t)||^q_{\rho,2}\,dt = 0$$

By the obvious chain of equations

$$\begin{aligned} \mathbf{E}||X_N(t)-X(t)||^q_{\rho,2} = & \quad \mathbf{E}||X_1(t)-Y_N(t)-X_1(t)+Y(t)||^q_{\rho,2} \\ = & \quad \mathbf{E}||Y_N(t)-Y(t)||^q_{\rho,2} \end{aligned}$$

for all $t \in [\,0,T\,]$ one has

$$\lim_{N\to\infty}\int_0^T \mathbf{E}||X_N(t)-X(t)||^q_{\rho,2}\,dt = 0 \quad (2.25)$$

Analogously to part (ii) one estimates in the following way:

$$\mathbf{E}\left|\left|X(t)-U(t,0)\xi-\int_0^t U(t,s)F(s,\cdot,X(s))\,ds-\int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)\right|\right|_{\rho,2}^2$$

$$\leq 3\left(\mathbf{E}||X(t)-X_N(t)||_{\rho,2}^2+c^2(T)\mathbf{E}\left|\left|\int_0^t F(s,\cdot,X(s))-F_N(s,\cdot,X_N(s))\,ds\right|\right|_{\rho,2}^2\right.$$

$$\left.+\mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X(s))-\Sigma(s,X_N(s))]\,dW(s)\right|\right|_{\rho,2}^2\right)$$

$$=: 3(\tilde{I}_N^{(1)}(t)+\tilde{I}_N^{(2)}(t)+\tilde{I}_N^{(3)}(t))$$

Consider the $\tilde{I}$-terms:

$$\tilde{I}_N^{(1)}(t)\leq(\mathbf{E}||X(t)-X_N(t)||_{\rho,2}^q)^{\frac{2}{q}}$$

$$\begin{aligned}\tilde{I}_N^{(2)}(t)\leq\; & 2\left(\mathbf{E}\int_0^T ||F(s,\cdot,X(s))-F(s,\cdot,X_N(s))||_{\rho,2}^2\,ds\right.\\ & \left.+\mathbf{E}\int_0^T ||F(s,\cdot,X_N(s))-F_N(s,\cdot,X_N(s))||_{\rho,2}^2\,ds\right)\end{aligned}$$

$$\tilde{I}_N^{(3)}(t)\leq c(c(T),c_\sigma(T))\left(\int_0^T s^{\frac{-\gamma q}{q-2}}\,ds\right)^{\frac{q-2}{q}}\left(\int_0^T \mathbf{E}||X(s)-X_N(s)||_{\rho,2}^q\,ds\right)^{\frac{2}{q}}$$

With the help of (2.24) and (2.25) one gets property 2.2.3(iv) for X with the same arguments as in (ii).
As $\nu=1$ 2.2.3(i) is trivially fulfilled.Concerning 2.2.3(ii) estimate in the following manner:

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||X(s)||_{\rho,2}^2\,ds\leq\; & 2\left[\left(\int_0^T s^{-\frac{\gamma q}{q-2}}\,ds\right)^{\frac{q-2}{q}}\left(\int_0^T \mathbf{E}||X(s)-X_N(s)||_{\rho,2}^q\,ds\right)^{\frac{2}{q}}\right.\\ & \left.+\sup_{t\in[0,T]}\int_0^t (t-s)^{-\gamma}\mathbf{E}||X_N(s)||_{\rho,2}^2\,ds\right]\end{aligned}$$

This estimate holds true for arbitrary $N\in\mathbf{N}$.As $X_N$ fulfills 2.2.3(ii) for arbitrary $N\in\mathbf{N}$ (2.25) then implies the finiteness of the righthand side,s.t. 2.2.3(ii) is fulfilled for X.As in part (ii) pathwise continuity of X follows from that of $X_N$ for arbitrary $N\in\mathbf{N}$.
So X is a solution to Eq.($\xi$,F,$\Sigma$) in the sense of 2.2.3.
Analogously to the proof of (ii) one gets the following chain of inequations for X:

$$\begin{aligned}\sup_{t\in[0,T]}\mathbf{E}||X(t)||_{\rho,2}^q=\; & \sup_{t\in[0,T]}\inf_{N\in\mathbf{N}}\mathbf{E}||X_N(t)||_{\rho,2}^q\leq\sup_{\substack{t\in[0,T]\\ N\in\mathbf{N}}}\mathbf{E}||X_N(t)||_{\rho,2}^q\\ \leq\; & c(q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,2}^q)\end{aligned}$$

Thus one also has the wanted estimate,which finishes the proof.
q.e.d.

As it was already mentioned in the introduction the aim of chapter 3 is to show an existence result for the so-called Heath-Jarrow-Morton model with the help of the theory developed here.In this model,which will be described at the beginning of chapter 3,U will be defined by the shift-semigroup.In [AsMa] the authors claim an existence (and uniqueness) result(confer [AsMa],section 2,theorem 1,p.241) in spaces $L^{\bar{p}}_{\rho}(\mathbf{R}^d)$ with a semigroup,which is not the shift-semigroup,but builds an almost strong evolution operator with properties (CD),(PP),(CC),(E1) and (E2) by setting

$$U(t,s) = S(t-s), (s,t) \in \mathbf{R}^2_+$$

and a function f fulfilling conditions different from those here.In [AsMa] $\bar{p}$ was defined by

$$\bar{p} := \max(2,\nu)$$

In order to be able to emphasize differences between the Heath-Jarrow-Morton model and the model from [AsMa] show that the existence result from [AsMa] can be transferred to the given situation.

**Theorem 2.3.3:**

Let $\sigma$,f and U be as in 2.3.2.Set

$$p := \max(2,\nu)$$

if $\max(2,\nu)$ is an even number and

$$p := \max(2,\nu) + 1$$

else.
Let $h \in L^p_\rho$ be a deterministic initial value.
Then there is a pathwise continuous solution of

$$X(t) = U(t,0)h + \int_0^t U(t,s)F(s,\cdot,X(s))\,ds + \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)$$
$$X(0) = h$$

in $L^p_\rho$ and for each q with $q > \frac{p}{1-\gamma}$ there is a positive constant c(p,q,$\gamma$,T,c(T),$c_f(T)$,$c_\sigma(T)$),s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{\rho,p} \le c(p,q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+||h||^q_{\rho,p})$$

holds true with c(p,q,$\gamma$,T,c(T),$c_f(T)$,$c_\sigma(T)$) depending on p,q,U,f,$\sigma$ and T.

**Proof:**
Show a number of claims in case $\nu > 1$.
Note,that in case $\nu = 1$ one has $p = 2$,s.t. theorem 2.3.2(i) immediately leads to the wanted result.
So let $\nu > 1$.

**Claim 1:**
If Y is a predictable $L^p_\rho$-valued process with

$$\sup_{t\in[0,T]} \mathbf{E}||Y(t)||_{\rho,p}^p < \infty$$

the Bochner-integral

$$\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds$$

exists in $L_\rho^p$ for all $t \in [0,T]$ and the corresponding process is continuous and adapted.

**Proof:** Proceed as in the proof of 2.2.9,i.e. consider for arbitrary $t \in [0,T]$ the following estimate:

$$\begin{aligned}
\mathbf{E}\left|\left|\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds\right|\right|_{\rho,p} &= \mathbf{E}\left[\left(\left|\left|\int_0^t |U(t,s)F(s,\cdot,Y(s))|^{\frac{p}{2}}\,ds\right|\right|_{\rho,2}\right)^{\frac{2}{p}}\right] \\
&\leq c(p,c(T))\mathbf{E}\left[\left(\left|\left|\int_0^t |F(s,\cdot,Y(s))|^{\frac{p}{2}}\,ds\right|\right|_{\rho,2}\right)^{\frac{2}{p}}\right] \\
&\leq c(p,c(T),c_f(T))\mathbf{E}\left(\int_0^t (1+||\,|Y(s)|^{\frac{p}{2}}\,||_{\rho,2})\,ds\right)^{\frac{2}{p}} \\
&\leq c(p,T,c(T),c_f(T))\mathbf{E}\left(1+\sup_{r\in[0,T]} ||Y(r)||_{\rho,p}^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&= c(p,T,c(T),c_f(T))\mathbf{E}\left(1+\left(\sup_{r\in[0,T]} ||Y(r)||_{\rho,p}^{p}\right)^{\frac{1}{2}}\right)^{\frac{2}{p}} \\
&\leq c(p,T,c(T),c_f(T))\left(1+\left(\sup_{r\in[0,T]} \mathbf{E}||Y(r)||_{\rho,p}^{p}\right)^{\frac{1}{2}}\right)^{\frac{2}{p}} \\
&< \infty
\end{aligned}$$

For this estimate (E1),(ii) from section 1.3,(L1),(L2) for f and the fact that one has $\frac{2}{p} < 1$ by the definition of p,which implies the applicability of the reverse of Jensen‘s inequality were needed.
Thus the Bochner-integral

$$\int_0^t U(t,s)F(s,\cdot,Y(s))\,ds \in L_\rho^p$$

exists for processes Y as in the claim.Completely analogous to the proof of 2.2.9 one has continuity and adaptedness.
Having a predictable,$L_\rho^p$-valued process Y with

$$\sup_{t\in[0,T]} \mathbf{E}||Y(t)||_{\rho,p}^p < \infty$$

and thus

$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-\gamma}\mathbf{E}||Y(s)||_{\rho,p}^p\,ds \leq \left(\int_0^T s^{-\gamma}\,ds\right) \sup_{t\in[0,T]} \mathbf{E}||Y(t)||_{\rho,p}^p < \infty$$

2.2.2 applied in case $\kappa := \frac{p}{2}$ leads to

$$\int_0^t U(t,s)\Sigma(s,Y(s))\,dW(s) \in L^p_\rho$$

P-almost surely for all t$\in [0,T]$ and for all $q > \frac{p}{1-\gamma}$ there exists a positive constant c(p,q,$\gamma$,T,$c_\sigma(T)$) s.t.

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,Y(s))\,dW(s)\right|\right|^q_{\rho,p} \leq c(p,q,\gamma,T,c_\sigma(T))\int_0^t \mathbf{E}||Y(s)||^q_{\rho,p}\,ds < \infty$$

Thus by claim 1

$$U(t,0)h + \int_0^t U(t,s)F(s,\cdot,Y(s))\,ds + \int_0^t U(t,s)\Sigma(s,Y(s))\,dW(s) \in L^p_\rho$$

P-almost surely for all $t \in [0,T]$ and each predictable,$L^2_\rho$-valued process Y with

$$Y(t) \in L^p_\rho \text{ P-a.s. f.a. } t \in [0,T], \sup_{t\in[0,T]} \int_0^t \mathbf{E}||Y(s)||^p_{\rho,p}\,ds < \infty$$

and each deterministic $h \in L^p_\rho$.
But the wanted process X should additionally fulfill

$$X(t) = U(t,0)h + \int_0^t U(t,s)F(s,\cdot,X(s))\,ds + \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)$$

for all $t \in [0,T]$ pathwisely,and it should be continuous.
In order to prove existence follow the proof of 2.3.2(ii).

**Claim 2:**
Suppose f fulfills (L1) and (L2),whereas $\sigma$,U and $\gamma$ have the same properties as above.
Then there exists a pathwise unique continuous solution X of the wanted equation and for all $q > \frac{p}{1-\gamma}$ there exists a positive constant c(p,q,$\gamma$,T,c(T),$c_f(T)$,$c_\sigma(T)$),s.t.

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{\rho,p} \leq c(p,q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+||h||^q_{\rho,p})$$

holds true.
**Proof:** Define analogously to the proof of 2.2.7(ii) a sequence $(X_n)_{n\in\mathbf{N}}$ in $C([0,T];L^2([0,T]\times\Omega;L^p_\rho))$ ($B := L^p_\rho$ in section 1.3).By the above considerations

$$\mathbf{E}||X_{n+1}(t)-X_n(t)||^q_{\rho,p} \leq c(\gamma,p,q,T,c_\sigma(T),c_f(T))\int_0^t \mathbf{E}||X_n(s)-X_{n-1}(s)||^q_{\rho,p}\,ds$$

s.t. with the same steps as in the proof of 2.2.7(ii) $(X_n)_{n\in\mathbf{N}}$ is a Cauchy-sequence in $C([0,T];L^2([0,T]\times\Omega;L^p_\rho))$ with norm $||\cdot||_{C([0,T];L^p_\rho)(q)}$ and its limit process has the wanted properties.

**Claim 3:**
Let f fulfill conditions (PG) (with an exponent $\nu > 1$),
(LG) and (D).Then there exists a solution X with the properties claimed in the

theorem.
**Proof:** Follow the proof of 2.3.2(ii).
As in step 1 there define mappings $\bar{g}$,$\bar{h}$ and,for N,M$\in$ **N** , $f_{N,M}$ and processes $\underline{X}_{N,M\in\mathbf{N}}$,$X_{N,M}$ and $\bar{X}_{0,M}$(,which exist by claim 2).
Since $||\varphi||^q_{\rho,p} = ||\ |\varphi|^{\frac{p}{2}}||^{\frac{2q}{p}}_{\rho,2}$ for arbitrary $\varphi \in L^p_\rho$ (E2) implies the following:

$$\begin{aligned}\mathbf{E}\left|\left|\int_0^t U(t,s)F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))\,ds\right|\right|^q_{\rho,p} &\leq c(p,q,T)\mathbf{E}\left|\left|\int_0^t U(t,s)|F_{0,M}(s,\cdot,\bar{X}_{0,M}(s))|^{\frac{p}{2}}\,ds\right|\right|^{\frac{2q}{p}}_{\rho,2}\\ &\leq c(p,q,T,c(T))\mathbf{E}\int_0^t ||\bar{h}(\bar{X}_{0,M})||^q_{\rho,p}\,ds\end{aligned}$$

i.e. one gets

$$\mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,p} \leq c(p,q,\gamma,T,c_f(T),c_\sigma(T),h)\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||^q_{\rho,p}\,ds\right)$$

analogously to steps 1 and 2 in Manthey's and Zausinger's proof,since

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\bar{X}_{0,M}(s))\,dW(s)\right|\right|^q_{\rho,p} \leq c(p,q,\gamma,c(T))\int_0^t \mathbf{E}||\sigma(s,\bar{X}_{0,M}(s))||^q_{\rho,p}\,ds \quad (2.25)$$

holds true for arbitrary $M \in \mathbf{N}$.But by claim 2

$$\sup_{t\in[0,T]} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,p} \leq c(p,q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+||h||^q_{\rho,p})$$

for all $M \in \mathbf{N}$,s.t. the M-independence of the constant implies

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbf{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{\rho,p} \leq c(p,q,\gamma,T.c(T),c_f(T),c_\sigma(T))(1+||h||^q_{\rho,p})$$

Analogous estimates hold true for $X_{N,M}$ and $\underline{X}_{N,M}$ with fixed $N \in \mathbf{N}$.
Obviously step 3 from 2.3.2(ii) still works.
Step 4 differs from the considerations in the proof of 2.3.2(ii) in terms $I^{(1)}_{N,M}$ and $I^{(3)}_{N,M}$.

$$\begin{aligned}I^{(1)}_{N,M}(t) :=&\ \mathbf{E}||X_N(t)-X_{N,M}(t)||^2_{\rho,2}\\ \leq&\ c(p,\rho)\mathbf{E}||X_N(t)-X_{N,M}(t)||^2_{\rho,p}\\ \leq&\ c(p,q,\rho)(\mathbf{E}||X_N(t)-X_{N,M}(t)||^q_{\rho,p})^{\frac{2}{q}}\\ I^{(3)}_{N,M}(t) :=&\ \mathbf{E}\left|\left|\int_0^t U(t,s)[\Sigma(s,X_N(s))-\Sigma(s,X_{N,M})]\,dW(s)\right|\right|^2_{\rho,2}\\ \leq&\ c(c(T),c_\sigma(T))\int_0^t \mathbf{E}||X_N(s)-X_{N,M}(s)||^2_{\rho,2}\,ds\\ \leq&\ c(p,q,\rho,c(T),c_\sigma(T))\left(\int_0^T \mathbf{E}||X_N(s)-X_{N,M}(s)||^q_{\rho,p}\,ds\right)^{\frac{2}{q}}\end{aligned}$$

By these estimates the two terms converge to 0 for $M \to \infty$ in this case as well and the results from step 4 in the proof of 2.3.2.(ii) still hold.
Analogously to (2.25) one gets

$$\mathbf{E}\left|\left|\int_0^t U(t,s)\Sigma(s,\underline{X}_N(s))\,dW(s)\right|\right|_{\rho,p}^q \leq c(p,q,\gamma,c(T))\int_0^t \mathbf{E}||\sigma(s,\underline{X}_N(s))||_{\rho,p}^q\,ds$$
$$\leq c(p,q,\gamma,T,c(T),c_\sigma(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||_{\rho,p}^q\,ds\right)$$

and thus analogously to the beginning of the proof

$$\sup_{\substack{t\in[0,T]\\ N\in\mathbf{N}}} \mathbf{E}||\underline{X}_N(s)||_{\rho,p}^q \leq c(p,q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+||h||_{\rho,p}^q)$$

i.e. the estimates are N-independent as they already were in the other proofs.
Define

$$X(t) := \inf_{N\in\mathbf{N}} X_N(t), t\in[0,T]$$

as in step 5 in the proof of 2.3.2(ii).
Analogously to that proof one can show that this defines a solution.
Pathwise uniqueness follows as in the proof of 2.3.3.
q.e.d.

**Remark 2.3.4:**

(i) As it was already mentioned Assing and Manthey chose

$$\bar{p} := \max(2,\nu)$$

in their paper.The way p is chosen in 2.3.3 ensures

$$\frac{p}{2}\in\mathbf{N}$$

which is necessary,since (E2) and 2.2.2 only hold for **natural** numbers $\kappa$ and thus would not be applicable in situations with odd numbers $\nu \geq 3$.

(ii) In [AsMa] the existence result was denoted for a **deterministic** initial condition h,whereas the existence claim in [MaZa] was denoted for random initial conditions.
As already mentioned in the proof of 2.3.3,the case $\nu = 1$ is already given by 2.3.2(i),s.t. in case $\nu = 1$ 2.3.3 also holds true for random initial conditions $\xi$ with

$$\mathbf{E}||\xi||_{\rho,2}^q < \infty$$

for fixed $q > \frac{2}{1-\gamma}$.The proof of 2.3.3 shows that one can keep the claim for random initial conditions with fixed $q > \frac{p}{1-\gamma}$,if one additionally assumes

$$\mathbf{E}||\xi||_{\rho,p}^q < \infty$$

This leads to the following corollary:

**Corollary 2.3.5:**

Let $\sigma$,f and U as in 2.3.2,define p as in theorem 2.3.3 .
Let $q > \frac{p}{1-\gamma}$ be fixed and let $\xi$ be a random initial condition,s.t.

$$\mathbf{E}||\xi||_{\rho,p}^q < \infty$$

holds true.

Then there exists a solution of

$$X(t) = U(t,0)\xi + \int_0^t U(t,s)F(s,\cdot,X(s))\,ds + \int_0^t U(t,s)\Sigma(s,X(s))\,dW(s)$$
$$X(0) = \xi$$

in $L_\rho^2$ with $X(t) \in L_\rho^p$ P-almost surely for all $t \in [\,0,T\,]$ and a positive constant c(p,q,$\gamma$,T,c(T),$c_f(T)$,$c_\sigma(T)$) depending on p,q,U,f,$\sigma$ and T,s.t.

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||X(t)||_{\rho,p}^q \leq c(p,q,\gamma,T,c(T),c_f(T),c_\sigma(T))(1+\mathbf{E}||\xi||_{\rho,p}^q)$$

holds true.

# Chapter 3

# The application to the Heath-Jarrow-Morton model

## 3.1 A market model by Harrison and Pliska

In their paper [HaPl] Harrison and Pliska develop a stochastic model of a so-called frictionless market,i.e. of a market with short selling and without transaction costs.
Consider the model with continuous trading-time,which is described in chapter 3(cf. pp.232–242 there) of [HaPl].

Let $(\Omega,\mathcal{F},\mathrm{P})$ be a probability space,$(\mathcal{F}_t)_{t\in[\,0,\bar{\tau}\,]}$,$\bar{\tau} > 0$ arbitrary but fixed,the augmented right-continuous,complete filtration.Let

$$\bar{P} = \bar{P}(t,\cdot)\,;\,0 \le t \le \bar{\tau}$$

with $\bar{P} = (B, \bar{P}(\cdot, S_1), \dots, \bar{P}(\cdot, S_k))$ for $k \in \mathbf{N}$ arbitrary with adapted,right-continuous $\bar{P}(\cdot, S^n)$;$n = 0, 1, \dots, k$ with left limits denote the $k+1$-dimensional price process belonging to bonds $S_0$–$S_k$,where $S_0$ is locally riskless,i.e. there exists a process $(r^t)_{t\in[\,0,\bar{\tau}\,]}$ with

$$B(t) = \exp\left(\int\limits_0^t r^s\,ds\right),0 \le t \le \bar{\tau}$$

$r^t$ is interpreted as the riskless interest rate at time t.
The condition of adaptedness implies,that at any time each agent knows the current and former prices of the bonds.One defines the so-called intrinsic discount process for $\bar{P}$ by

$$\beta_t := \tfrac{1}{B(t)}$$

and the discounted price process by

$$Z(t, S_n) := \beta_t \bar{P}(t, S_n);t \in [\,0, \bar{\tau}\,],n = 1, 2, \dots, k$$

Suppose that the set **P** of probability-measures under which the family $(Z(t,S_n);t \in [\,0,\bar{\tau}\,])_{n=1,2,\ldots,k}$ is a $(\mathcal{F}_t)_{t\in[\,0,\bar{\tau}\,]}$-martingale is not empty.

**Definition 3.1.1:**

A **trading strategy** is a k+1-dimensional process $\phi$=$(\phi_t)_{t\in[\,0,\bar{\tau}\,]}$ with components $\phi^n$;$n = 0, 1, \ldots, k$;which are locally bounded and predictable.
Here $\phi^n_t$ denotes the number of shares the agent holds at time t.
$\phi_t$ is called the **portfolio** of the agent at time t.
There exist a value-process V($\phi$) and a gain-process G($\phi$) associated to the trading strategy given by

$$V(t,\phi) := \sum_{n=0}^{k} \phi^n_t \bar{P}(t,S_n),0 \leq t \leq \bar{\tau}$$
$$G(t,\phi) := \sum_{n=0}^{k} \int_0^t \phi^n_s \, d\bar{P}(s,S_n),0 \leq t \leq \bar{\tau}$$

A strategy $\phi$ is called **self-financing**,if

$$V(t,\phi) = V(0,\phi) + G(t,\phi),0 \leq t \leq \bar{\tau}$$

holds.The corresponding discounted value- resp. gain-processes are defined by

$$V^*(t,\phi) := \phi^0_t + \sum_{n=1}^{k} \phi^n_t Z(t,S_n),0 \leq t \leq \bar{\tau}$$
$$G^*(t,\phi) := \sum_{n=1}^{k} \int_0^t \phi^n_s \, dZ(s,S_n),0 \leq t \leq \bar{\tau}$$

**Remark 3.1.2:**

A trading strategy $\phi$ is self-financing if and only if

$$V^*(t,\phi) = V^*(0,\phi) + G^*(t,\phi),t \in [\,0,\bar{\tau}\,]$$

holds true.

**Definition 3.1.3:**

(i) A trading strategy $\phi$ is called **admissible**,if $V^*(\phi) = (V^*(t,\phi))_{t\in[\,0,\bar{\tau}\,]}$ is a martingale with respect to P and fulfills both $V^*(\phi) \geq 0$ and

$$V^*(t,\phi) = V^*(0,\phi) + G^*(t,\phi),t \in [\,0,\bar{\tau}\,]$$

(ii) A contingent claim is a random variable X:$\Omega \to \mathbf{R}$.It is called **available**,if there exists an admissible trading strategy $\phi$ with $V^*(\bar{\tau},\phi) = \beta_{\bar{\tau}}X$.
Then one says $\phi$ generates X and one denotes $\pi := V^*(0,\phi)$ as the price associated with X.

(iii) A market is called **complete**,if each contingent claim with

$$\mathbf{E}\beta_{\bar{\tau}}X < \infty$$

is available.

**Theorem 3.1.4:** (cf. [HaPl],chapter 3,section 3.4,theorem 3.35,p.241)

Denote by $\mathcal{M}_P$ the set of all martingales w.r.t the probability measure P and by $\mathcal{M}(Z) \subset \mathcal{M}_P$ the set of all martingales M w.r.t. P with

$$M_t = M_0 + \sum_{n=1}^{k} \int_0^t H_s^n \, dZ(s, S_n)$$

for predictable processes $H^n$;$n = 1, 2, \ldots, k$; with

$$\mathbf{E}|H^n_{t\wedge\tau_m}| < \infty, 0 \leq t \leq \bar{\tau}$$

for all $n = 1, 2, \ldots, k$ with stopping times $(\tau_m)_{m\in\mathbf{N}}$,s.t.

$$\lim_{m\to\infty} P(\tau_m = \tau) = 1$$

Then the model is complete if and only if $\mathcal{M}_P = \mathcal{M}(Z)$ holds true.

**Corollary 3.1.5:** (cf. [HaPl],chapter 3,section 3.4,corollary 3.36,p.241)

If **P** only consists of a single probability measure,the model is complete.

## 3.2 The framework of the Heath-Jarrow-Morton paper

As already mentioned in the introduction,this section‘s aim is to repeat the essential conditions and results from [HeJaMo].
First of all $[\,0,\tau\,]$ with $\tau > 0$ denotes the trading interval of the economy.Furthermore let $(\Omega,\mathcal{F},\mathrm{P})$ be a probability space and $(\mathcal{F}_t)_{t\in[\,0,\tau\,]}$ the augmented,right-continuous and complete filtration generated by $k \geq 1$ independent,real-valued Brownian motions $w_1,w_2,...,w_k$ on $[\,0,\tau\,]$ with $w_n(0) = 0$ for all $n = 1, 2, \ldots, k$.
To each $T \in [\,0,\tau\,]$ there is a bond with payoff-time T,i.e. an option,which gives the owner a known payoff at time T,and a riskless asset,i.e. an asset,which ensures at any time t a certain interest rate for a bond starting in t with payoff an infinitesimal unit later.
Denote by $\bar{P}(t,T)$ the price of a bond with payoff-time T in t.
For arbitrary $T \in [\,0,\tau\,]$ and $t \in [\,0,T\,]$ Heath, Jarrow and Morton assume the following:

1. $\bar{P}(T,T) = 1$
2. $\bar{P}(t,T) > 0$
3. $\frac{d}{dT}\log\bar{P}(t,T)$ exists

Condition 1 means,the payoff of a bond must always be 1 unit of money(=UM), condition 2 means that there is no trivial arbitrage opportunity,i.e. there is no chance to get the payoff of a bond without having payed a positive number of UM before.By condition 3 the following definition is possible:

$$\bar{f}(t,T) := -\tfrac{d}{dT}\log\bar{P}(t,T),T \in [\,0,\tau\,],t \in [\,0,T\,]$$

As the payoff for a bond is always 1 UM at the payoff-time by condition 1,the writer,i.e. the one who offers the bond,must choose the price in t of a bond with payoff-time T in such a way,that he can surely(,i.e. almost surely under the preceding probability measure P) make exactly one UM out of $\bar{P}(t,T)$ UM between t and T.Then $\bar{P}(t,T) > 1$ would mean,that it is P-almost surely impossible to get a positive rate by investing of money in bonds,s.t. it would make sense to keep the money instead of investing it into a bond with payoff-time T.
Suppose for T and T+h with small $h > 0$ there is a safe interest rate r,f.e. via a savings book.Then the following must hold true,since the payoff is equal at any time by condition 1:

$$\bar{P}(t,T)e^{-rh} = \bar{P}(t,T+h) \iff r = -\left(\tfrac{\log\bar{P}(t,T+h)-\log\bar{P}(t,T)}{h}\right)$$

Thus the rate of a riskless asset with start in T and payoff an infinitesimal unit later viewed from time t is

$$\lim_{h\to 0} -\frac{\log \bar{P}(t,T+h)-\log \bar{P}(t,T)}{h} = -\frac{d}{dT}\log \bar{P}(t,T) = \bar{f}(t,T)$$

So $\bar{f}(t,T)$ is the interest rate one can contract for in time t for a riskless asset starting in T with payoff an infinitesimal unit later.In order to ensure this rate,the writer of such a riskless asset must be able to reach this interest rate between t and T for sure.

Besides the trivial arbitrage possibility,which is excluded by the condition 2 for $\bar{P}(t,T)$,it should in general be impossible to have an arbitrage possibility,i.e. in the sense of section 3.1 one has to exclude the existence of an admissible trading strategy $\phi$ with

$$V(0,\phi)=0 \text{ and } \mathbf{E}V(\bar{\tau},\phi)\geq 0$$

for $\bar{\tau}\in[\,0,\tau\,]$.Therefore introduce analogously to section 3.1 the following notations.

By Z(t,T) denote the discounted price in t of a bonds with payoff-time T given by

$$Z(t,T):=\frac{\bar{P}(t,T)}{B(t)}$$

where

$$B(t):=exp\left(\int_0^t \bar{f}(y,y)\,dy\right)$$

is the number of UM one gets in T by investing 1 UM in t,s.t. one buys a bond with payoff an infinitesimal unit later at any time in $[\,t,T\,]$.

In [HeJaMo] the authors assume the following property:

$$\text{(C1)}\qquad \begin{aligned}\bar{f}(t,T)-\bar{f}(0,T) = & \int_0^t \alpha(s,T,\omega)\,ds \\ & +\sum_{n=1}^{k}\int_0^t \sigma_n(s,T,\omega)\,dw_n(s)\end{aligned}$$

for all $T\in[\,0,\tau\,]$,$t\in[\,0,T\,]$

where $(\bar{f}(0,T))_{T\in[\,0,\tau]}$ is deterministic with

$$\bar{f}(0,\cdot){:}([\,0,\tau\,],\mathcal{B}([\,0,\tau\,]))\rightarrow(\mathbf{R},\mathcal{B}(\mathbf{R}))$$

$\alpha{:}S_T\times\Omega\rightarrow\mathbf{R}$ is a uniformly measurable mapping $\mathcal{B}(S_T)\times\mathcal{F}\rightarrow\mathcal{B}(\mathbf{R})$ with

$$\int_0^T|\alpha(t,T,\omega)|\,dt<\infty \text{ P-a.s.}$$

and $\sigma_n{:}S_T\times\Omega\rightarrow\mathbf{R}$ is a uniformly measurable mapping $\mathcal{B}(S_T)\times\mathcal{F}\rightarrow\mathcal{B}(\mathbf{R})$ with

$$\int_0^T\sigma_n^2(t,T,\omega)\,dt<\infty \text{ P-a.s.};n=1,2,\ldots,k$$

The different processes $\sigma_n$;$n = 1, 2, \ldots, k$;underline the sensitivity of the process against changes of any of the Brownian motions $w_n$;$n = 1, 2, \ldots, k$.
In order to assure,that B(t) defined as above is P-almost surely positive and finite,Heath,Jarrow and Morton assume

$$(C2) \quad \int_0^\tau |\bar{f}(0,t)|\, dt < \infty, \int_0^\tau \left( \int_0^t |\alpha(s,t,\omega)|\, ds \right) dt < \infty \text{ P-a.s.}$$

Furthermore they show(cf. [HeJaMo] appendix,proof of equation (8),pp.99,100),that (C2) and assumption (C3) consisting of the following three conditions

(a) $$\int_0^t \left( \int_s^t \sigma_n(s,y,\omega)\, dy \right)^2 ds < \infty \text{ P-a.s.;} t \in [0,\tau]; n = 1, 2, \ldots, k$$

(b) $$\int_0^t \left( \int_t^T \sigma_n(s,y,\omega)\, dy \right)^2 ds < \infty \text{ P-a.s.;} t \in [0,t]; T \in [0,\tau]; n = 1, 2, \ldots, k$$

(c) $t \to \int_t^T \left( \int_0^t \sigma_n(s,y,\omega)\, dw_n(s) \right) dy$ is continuous P-a.s. for $T \in [0,\tau]$;
$n = 1, 2, \ldots, k$

imply the following property of the bond-price process:

$$\begin{aligned} ln\, \bar{P}(t,T) = \quad & ln\, \bar{P}(0,T) + \int_0^t [\bar{f}(s,s) + b(s,T)]\, dv \\ & -\tfrac{1}{2} \sum_{n=1}^k \int_0^t a_n^2(s,T)\, ds + \sum_{n=1}^k \int_0^t a_n(s,T)\, dw_n(s) \quad (3.1) \end{aligned}$$

P-almost surely with t and T as above,$\omega \in \Omega$ and

$$a_n(t,T,\omega) := -\int_t^T \sigma_n(t,s,\omega)\, ds$$

$$b(t,T,\omega) := -\int_t^T \alpha(t,s,\omega)\, ds + \tfrac{1}{2} \sum_{n=1}^k a_n^2(t,T,\omega)$$

for all $n = 1, 2, \ldots, k$,s.t.

$$\begin{aligned} d\bar{P}(t,T) = \quad & [\bar{f}(t,t) + b(t,T)]\bar{P}(t,T)\, dt \\ & + \sum_{n=1}^k a_n(t,T)\bar{P}(t,T)\, dw_n(t) \end{aligned}$$

follows P-almost surely with the help of Ito's formula.Applying Ito's lemma one gets

$$\begin{aligned} \ln Z(t,T) = \quad & \ln Z(0,T) + \int_0^t \left( b(s,T) - \tfrac{1}{2} \sum_{n=1}^k a_n^2(s,T) \right) ds \\ & + \sum_{k=1}^n \int_0^t a_n(s,T)\, dw_n(s) \quad (3.2) \end{aligned}$$

for t,T as above.Then Heath,Jarrow and Morton describe under which assumptions there is a probability measure,s.t. the relative prices of the bonds are martingales w.r.t. $(\mathcal{F}_t)$.
To proceed one needs the following assumptions:

(C4) For arbitrary $S_1, S_2, \ldots, S_k \in [0, \tau]$ with $0 < S_1 < S_2 < \ldots < S_k \leq \tau$ there exist solutions

$$\gamma_n(\cdot, \cdot; S_1, S_2, \ldots, S_k): \Omega \times [0, S_1] \to \mathbf{R}; n = 1, 2, \ldots, k$$

of

$$b(t, S_j) = \sum_{n=1}^{k} a_n(t, S_j)(-\gamma_n(t; S_1, S_2, \ldots, S_k)) \quad (3.3)$$

$P \times ds$-almost everywhere for $j = 1, 2, \ldots, k$,s.t.:

$$\int_0^{S_1} \gamma_n^2(v; S_1, S_2; \ldots, S_k)\, dv < \infty \text{ P-a.s.}; n = 1, 2, \ldots, k$$

$$\mathbf{E}\left( exp\left( \sum_{n=1}^{k} \int_0^{S_1} \gamma_n(s; S_1, S_2, \ldots, S_k)\, dw_n(s) - \frac{1}{2} \sum_{n=1}^{k} \int_0^{S_1} \gamma_n^2(s; S_1, S_2, \ldots, S_k)\, ds \right)\right)$$
$$= 1$$

$$\mathbf{E} \quad \left( exp\left( \sum_{n=1}^{k} \int_0^{S_1} [a_n(s, y) + \gamma_n(s; S_1, S_2, \ldots, S_k)]\, dw_n(s) \right.\right.$$
$$\left.\left. -\frac{1}{2} \sum_{n=1}^{k} \int_0^{S_1} [a_n(s, y) + \gamma_n(s; S_1, S_2, \ldots, S_k)]^2\, ds \right)\right) = 1$$

$y \in \{S_1, S_2, \ldots, S_k\}$

(C5) For arbitrary $S_1, S_2, \ldots, S_k \in [0, \tau]$ with $0 < S_1 < S_2 < \ldots < S_k \leq \tau$ let

$$\begin{matrix} \mathrm{a}_1(t, S_1) & \mathrm{a}_2(t, S_1) & \cdots & \mathrm{a}_k(t, S_1) \\ \mathrm{a}_1(t, S_2) & \mathrm{a}_2(t, S_2) & \cdots & \mathrm{a}_k(t, S_2) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{a}_1(t, S_k) & \mathrm{a}_2(t, S_k) & \cdots & \mathrm{a}_k(t, S_k) \end{matrix}$$

be nonsingular $P \times ds$-almost surely.

One interprets $\gamma_n$ as the market price of risk associated with the factor $w_n$ for any n.
To understand this write (3.3) for a bond with payoff in T,i.e.

$$b(t, T) = \sum_{n=1}^{k} a_n(t, T)(-\gamma_n(t; S_1, S_2, \ldots, S_k))$$

Remember the definitions from the preceding page:

$$a_n(t, T, \omega) := -\int_t^T \sigma_n(t, s, \omega)\, ds; n = 1, 2, \ldots, k$$

$$b(t,T,\omega) := -\int_t^T \alpha(t,s,\omega)\,ds + \tfrac{1}{2}\sum_{n=1}^{k} a_n^2(t,T,\omega)$$

Thus $a_n(t,T)$ describes the covariance between the rate of the bond with payoff-time T and the n-th random factor $w_n$,whereas b(t,T) describes the expected rate above the riskless rate for the bond with payoff-time T.

With $S_1, S_2, \ldots, S_k \in [0,\tau]$, $0 < S_1 < S_2 < \ldots < S_k \leq \tau$,s.t. (C1)–(C3) are fulfilled, (C4) is fulfilled if and only if there is a probability measure P$(S_1, S_2, \ldots, S_k)$ equivalent to P,s.t. Z(t,$S_n$);$n = 1, 2, \ldots, k$; is a $(\mathcal{F}_t)_{t\in[0,S_1]}$-martingale under P$(S_1, S_2, \ldots, S_k)$(cf. [HeJaMo],chapter 4,proposition 1,p.84). The proof works via a Girsanov-argument and one receives the probability measure by

$$\frac{dP(S_1,S_2,\ldots,S_k)}{dP} = \exp\left(\sum_{n=1}^{k}\int_0^{S_1}\gamma_n(s;S_1,S_2,\ldots,S_k)\,dw_n(s) - \frac{1}{2}\sum_{n=1}^{k}\int_0^{S_1}\gamma_n^2(s;S_1,S_2,\ldots,S_k)\,ds\right)$$

By

$$\tilde{w}_n^{S_1,S_2,\ldots,S_k}(t) := w_n(t) - \int_0^t \gamma_n(s;S_1,S_2,\ldots,S_k)\,ds; t \in [0,S_1]; n = 1,2,\ldots,k \quad (3.4)$$

one gets real-valued independent Brownian motions on $((\Omega,\mathcal{F},\mathrm{P}(S_1, S_2, \ldots, S_k)),(\mathcal{F}_t)_{t\in[0,S_1]})$.

Having (C1)–(C4) and thus the existence of a probability measure P$(S_1, S_2, \ldots, S_k)$,(C5) is fulfilled if and only if P$(S_1, S_2, \ldots, S_k)$ is unique(cf. [HeJaMo],chapter 4,proposition 2,p.85).
So (C1)–(C5) for $0 < S_1 < S_2 < \ldots < S_k \leq \tau$ implies the existence of a unique,but $S_n$-depending probability measure under which $Z(t,S_n)$,$t \in [0,S_1]$,is a $(\mathcal{F}_t)_{t\in[0,S_1]}$-martingale for any $n = 1, 2, \ldots, k$.
Furthermore one has the following result(cf. [HeJaMo],chapter 4,proposition 3,p.86)

**Theorem 3.2.1:**

Given $\{\alpha(\cdot,T)\ ;\ T \in [0,\tau]\}$,$\{\sigma_n(\cdot,T)\ ;\ T \in [0,\tau]\}$ for $n = 1, 2, \ldots, k$ with (C1)–(C5).Then the following are equivalent:

(i) There is a unique probability measure,denoted by P again,s.t. Z(t,T) is a martingale under P for all $T \in [0,\tau]$ and $t \in [0,S_1]$ with arbitrary $S_1 \in [0,\tau]$.

(ii) $\gamma_n(t;S_1,S_2,\ldots,S_k) = \gamma_n(t;T_1,T_2,\ldots,T_k)$ for $n = 1, 2, \ldots, k$ for all sequences $(S_n)$ and $(T_n)$ as in (C4),(C5).

(iii) For all $T \in [0,\tau]$,$t \in [0,T]$

$$\alpha(t,T) = -\sum_{n=1}^{k} \sigma_n(t,T)\left(\lambda_n(t) - \int_t^T \sigma_n(t,s)\,ds\right)$$

holds true with $\lambda_n(t) := \gamma_n(t; S_1, S_2, \ldots, S_k)$ for $n = 1, 2, \ldots, k$ and for arbitrary $S_1, S_2, \ldots, S_k$,$t \in [\,0, S_1\,)$.

(ii) is called the standard finance condition for arbitrage free pricing,since this condition is necessary to have absence of arbitrage.
(iii) is called the forward drift restriction,since the restriction to an $\alpha$ of that type is sufficient to have (i).

Thus in case that (C1)–(C5) are fulfilled and the relative prices $(Z(t,T))_{t\in[\,0,T\,]}$ are martingales under P for $T \in [\,0,\tau\,]$,the following equation holds:

$$\begin{aligned}\bar{f}(t,T) - \bar{f}(0,T) = \quad & \int_0^t -\sum_{n=1}^{k} \sigma_n(s,T)\left(\lambda_n(s) - \int_s^T \sigma_n(s,y)\,dy\right) ds \\ & + \sum_{n=1}^{k} \int_0^t \sigma_n(s,T)\,dw_n(s)\end{aligned}$$

Having this result Heath,Jarrow and Morton describe how to price a contingent claim having these assumptions.
As it was already mentioned,having assumptions (C1)–(C5) for fixed

$$0 < S_1 < S_2 < \ldots < S_k \leq \tau$$

there is exactly one probability measure P$(S_1, S_2, \ldots, S_k)$,s.t. all Z(t,$S_n$); $n = 1, 2, \ldots, k$; are martingales.
By corollary 3.1.5 the uniqueness of P$(S_1, S_2, \ldots, S_k)$ as measure under which the Z(t,$S_n$);$n = 1, 2, \ldots, k$; are martingales implies the completeness of the market considered here.
The discounted payoff induced by the admissible,self-financing trading strategy $\phi$ is given by:

$$V^*(t,\phi) := \phi_t^0 + \sum_{n=1}^{k} \phi_t^n Z(t, S_n), 0 \leq t \leq \bar{\tau}$$

By $\bar{\tau} := S_1$ one gets

$$\phi_{S_1}^0 + \sum_{n=1}^{k} \phi_{S_1}^{S_n} Z(S_1, S_n)$$

as the discounted payoff of a contingent claim in $S_1$ and thus

$$\phi_{S_1}^0 B(S_1) + \sum_{n=1}^{k} \phi_{S_1}^{S_n} \bar{P}(S_1, S_n)$$

as the actual payoff,where $\phi_{S_1}^0$ denotes the number of riskless assets and $\phi_{S_1}^{S_n}$ denotes the number of bonds with payoff in $S_n$ held by the agent at time $S_1$.
In [HaPl] Harrison and Pliska show the absence of arbitrage given completeness of the market at least in case of a discrete trading time,i.e. in case $t = 0, 1, \ldots, \bar{\tau}$ instead of $t \in [\,0, \bar{\tau}\,]$ in section 3.1.So there is no admissible,self-financing strategy $\phi$ with

$$V(0,\phi)=0,\mathbf{E}_Q V(\bar{\tau},\phi)\geq 0$$

for at least one Q$\in\mathcal{P}$.Since the market considered here is complete,one has

$$\mathbf{E}\left(\frac{X}{B(S_1)}|\mathcal{F}_t\right)B(t)=\bar{P}(t,S_1;X)$$

where $\mathbf{E}$ denotes the expectation under the probability measure P,under which all Z(t,T) are martingales,and $\bar{P}(t,S_1;X)$ denotes the value of the contingent claim X with payoff in $S_1$ in $t\in[\,0,S_1\,]$.Thus one has

$$\bar{P}(t,S_1;X)=\mathbf{E}\left(\phi^0_{S_1}+\frac{\phi^{S_1}_{S_1}}{B(S_1)}+\sum_{n=2}^{k}\phi^{S_n}_{S_1}Z(S_1,S_n)|\mathcal{F}_t\right)B(t)$$

So,in order to be able to price the contingent claim,it must be possible to determine B and Z.With the help of (C1),(3.4) and the forward drift restriction resp. (3.2) and (3.4)

$$\begin{aligned}\bar{f}(t,t)=\quad&\bar{f}(0,t)+\sum_{n=1}^{k}\int_0^t\sigma_n(s,t)\int_s^t\sigma_n(s,y)\,dy\,ds\\&+\sum_{n=1}^{k}\int_0^t\sigma_n(s,t)d\tilde{w}_n(s)\\Z(t,u)=Z(0,u)\exp\quad&\left(-\frac{1}{2}\sum_{n=1}^{k}\int_0^t a_n(s,u)\,ds\right.\\&\left.+\sum_{n=1}^{k}\int_0^t a_n(s,u)\,d\tilde{w}_n^{S_1,S_2,\ldots,S_k}(s)\right)\end{aligned}$$

hold true P-almost surely for $t\in[\,0,S_1\,]$,$u\in\{S_1,S_2,\ldots,S_k\}$.Because of

$$\mathbf{E}\left(\frac{\bar{P}(S_1,u)}{B(S_1)}\right)<\infty,u\in[\,S_1,\tau\,]$$

and the completeness of the market all the other bonds can be replicated by an admissible,self-financing strategy,applying only the bonds with payoff-times $S_n$;$n=1,2,\ldots,k$;and the riskless asset.

In the following have a closer look at the stochastic differential equation considered in chapter 7 of [HeJaMo].

As it was already mentioned in section 1.1,one finds the following result in [HeJaMo]:

Given trading time $[\,0,\tau\,]$ with $\tau>0$,$k\in\mathbf{N}$ and $T\in[0,\tau]$,s.t.property (i) from 3.2.1 holds.
For n=1,2,...,k let

- $\lambda_n$:$[\,0,\tau\,]\times\Omega\rightarrow\mathbf{R}$ be predictable and bounded

- $\sigma_n$: $S_T\times\mathbf{R}\rightarrow\mathbf{R}$ be bounded,nonnegative and Lipschitzian on $\mathbf{R}$

Then there is a uniformly measurable family $(\bar{f}(t,T))_{t\in[\,0,T\,]}$ of processes with

$$\bar{f}(t,T)-\bar{f}(0,T) = \int_0^t -\sum_{n=1}^k \sigma_n(s,T,\bar{f}(s,T))\left(\lambda_n(s)-\int_s^T \sigma_n(s,y,\bar{f}(s,y))\,dy\right)ds + \sum_{n=1}^k\int_0^t \sigma_n(s,T,\bar{f}(s,T))\,dw_n(s)$$

for all $T\in[0,\tau]$.
Thus one has for all $x\in\mathbf{R}$ with t+x$\leq\tau$ :

$$\bar{f}(t,t+x)-\bar{f}(0,t+x) = \int_0^t -\sum_{n=1}^k \sigma_n(s,t+x,\bar{f}(s,t+x))\left(\lambda_n(s)-\int_s^{t+x} \sigma_n(s,y,\bar{f}(s,y))\,dy\right)ds + \sum_{n=1}^k\int_0^t \sigma_n(s,t+x,\bar{f}(s,t+x))\,dw_n(s)$$

In order to have at least the situation described in [MaZa] one needs the following time-homogeneity assumption on $\sigma$:

$$\sigma_n(t,s_1,\cdot)=\sigma_n(t,s_2,\cdot)$$

for all $t\in[0,\tau]$,$s_1,s_2\in[t,\tau]$,$n=1,2,\ldots,k$. Then the equation becomes:

$$\bar{f}(t,t+x)-\bar{f}(0,t+x) = \int_0^t -\sum_{n=1}^k \sigma_n(s,\bar{f}(s,t+x))\left(\lambda_n(s)-\int_s^{t+x} \sigma_n(s,\bar{f}(s,y))\,dy\right)ds + \sum_{n=1}^k\int_0^t \sigma_n(s,\bar{f}(s,t+x))\,dw_n(s)$$

So (1.1) and (1.2) imply the following equation:

$$r_t(x) = (S(t)r_0)(x)+\int_0^t -\sum_{n=1}^k \sigma_n(s,(S(t-s)r_s)(x))\left(\lambda_n(s)-\int_0^{t-s+x} \sigma_n(s,r_s(y))\,dy\right)ds + \sum_{n=1}^k\int_0^t \sigma_n(s,(S(t-s)r_s)(x))\,dw_n(s)$$

Analogously to (2.4)(without $\omega$-dependence) one gets for the shift-semigroup $(S(t))_{t\geq 0}$

$$[S(t)F(s,\varphi)]=f(s,[S(t)\varphi])$$

and the equation for r becomes

$$\begin{aligned} r_t(x) = \; & (S(t)r_0)(x) \\ & +\int_0^t \sum_{n=1}^k \left((S(t-s)\sigma_n(s,r_s))(x)\left(S(t-s)\int_0^{\cdot} \sigma(s,r_s(y))\,dy\right)(x)\right)ds \\ & -\int_0^t \sum_{n=1}^k (S(t-s)\sigma_n(s,r_s))(x)\lambda_n(s)\,ds \\ & +\int_0^t \sum_{n=1}^k [S(t-s)\sigma_n(s,r_s)](x)\,dw_n(s) \end{aligned}$$

Remember,that $t\in[0,\tau]$ and t+x$\leq\tau$ must hold true,s.t. this is,given a fixed t,only the case for $x\in[0,\tau-t]$.Choose $\tau$ very big,in order to claim that r is defined on $\mathbf{R}_+$.

Now fix $k = 1$.Then one has for all $t \in [\,0, T\,]$ with $T > 0$ and all $x \in \mathbf{R}_+$

$$
\begin{aligned}
r_t(x) = \quad & (S(t)r_0)(x) \\
& + \int_0^t \left( (S(t-s)\sigma(s, r_s))(x) \left( S(t-s) \int_0^{\cdot} \sigma(s, r_s(y))\, dy \right)(x) \right) ds \\
& - \int_0^t (S(t-s)\sigma(s, r_s))(x)\lambda(s)\, ds \\
& + \int_0^t (S(t-s)\sigma(s, r_s))(x)\, dw(s)
\end{aligned}
$$

for an arbitrary real-valued Brownian motion w with $w(0) = 0$.
On the Hilbert space $L^2_\rho(\mathbf{R}_+)$ the equation becomes

$$
r_t = S(t)r_0 + \int_0^t S(t-s)(\mathcal{S}(\sigma(s, r_s)) - \lambda(s)\sigma(s, r_s))\, ds + \int_0^t S(t-s)\Sigma(s, r_s)\, dW(s)
$$

with $\mathcal{S}$ defined by(cf. (1.3))

$$
(\mathcal{S}(e))(x) := e(x) \int_0^x e(z)\, dz
$$

for all functions e defined on $\mathbf{R}_+$,$\Sigma$ defined from $\sigma$ by (2.1),and $\sigma$ defined analogously to (2.4) (without $\omega$-dependence)

Thus given the assumptions

$$
\begin{aligned}
\sigma : \ & [0, T] \times \mathbf{R} \to \mathbf{R} \\
\lambda : \ & [0, T] \times \Omega \to \mathbf{R}
\end{aligned}
$$

one gets a function f:$[\,0, T\,] \times \Omega \times \mathbf{R} \to \mathbf{R}$ defined by (1.3).Then defining F by (2.4) gives the so-called Heath-Jarrow-Morton equation

$$
r_t = S(t)r_0 + \int_0^t S(t-s)F(s, \omega, r_s)\, ds + \int_0^t S(t-s)\Sigma(s, r_s)\, dW(s)
$$

## 3.3 The shift-semigroup

Consider the shift-semigroup as an evolution operator in the framework of chapter 2 with dimension $d = 1$,s.t. $\rho$ is a fixed number with $\rho > 1$.The aim of this section is to show that the shift-semigroup fulfills the conditions imposed on U in chapter 2 with U defined on $L^2_\rho(\mathbf{R}_+)$ resp. $L^{2\nu}_\rho(\mathbf{R}_+)$ by

$$U(t,s) := S(t-s) \quad (3.5)$$

for all $(s,t) \in S_T$.For the rest of the chapter write $L^2$ instead of $L^2(\mathbf{R}_+)$,$L^2_\rho$ instead of $L^2_\rho(\mathbf{R}_+)$,and so on.

As it was already mentioned by Manthey nd Zausinger in their paper(cf.[MaZa],section 2,example 2.6,p.55) each positivity preserving,strong continuous semigroup defines an almost strong evolution operator in the sense of section 1.3 by (3.5). First of all show the strong continuity of the shift-semigroup on $L^2_\rho$.

A shift-semigroup $(T_t)_{t\geq 0}$ defined on a Banach space B is called **strongly continuous**,if

$$\lim_{t\to 0} T_t\varphi = \varphi, \varphi \in B$$

holds true.

First consider functions $\varphi \in C^1_c$.As such functions are continuous and differ from 0 only on compact sets,each $\varphi \in C^1_c$ is uniformly continuous on this compact set,i.e. for each $\varepsilon > 0$ there exists a $t(\varepsilon) > 0$,s.t.

$$|\varphi(x+t) - \varphi(x)| < \varepsilon c_\rho^{-\frac{1}{2}}$$

for all $t \leq t(\varepsilon)$,$x \in Cp(\varphi)$,where $Cp(\varphi)$ denotes the compact set associated with $\varphi$ and $c_\rho$ is given by

$$c_\rho := \int_{\mathbf{R}_+} (1+x^2)^{-\frac{\rho}{2}}\, dx \quad (3.6)$$

which is finite since $\rho > 1$.
As $\varphi$ equals 0 outside the above compact set and is continuous on this compact set

$$||\varphi||_\infty := \sup_{x\in\mathbf{R}_+} |\varphi(x)|$$

exists and for all $t \in [\,0, T\,]$

$$\int_{\mathbf{R}_+} (\varphi(x+t) - \varphi(x))^2\, \mu_\rho(dx) = \int_{\mathbf{R}_+} (\varphi(x+t) - \varphi(x))^2 (1+x^2)^{-\frac{\rho}{2}}\, dx \leq 4\, c_\rho\, ||\varphi||^2_\infty$$

holds true,i.e. $[(S(t) - Id)\varphi] \in L^2_\rho$ for all $t \in [\,0, T\,]$ and all $\varphi \in C^1_c$.In particular one has the following for all $t \leq t(\varepsilon)$:

$$||S(t)\varphi - \varphi||_{\rho,2} = \left( \int\limits_{\mathbf{R}_+} (\varphi(x+t) - \varphi(x))^2 \, \mu_\rho(dx) \right)^{\frac{1}{2}} \quad (3.7)$$

$$\leq \varepsilon \, c_\rho^{-\frac{1}{2}} \left( \int\limits_{\mathbf{R}_+} (1+x^2)^{-\frac{\rho}{2}} \, dx \right)^{\frac{1}{2}} = \varepsilon$$

Thus one has strong continuity on $C_c^1$ in the $|| \cdot ||_{\rho,2}$-norm.
In order to continue the following lemma is needed.

**Lemma 3.3.1:**

$L^2$ is a dense subset of $L^2_\rho$ for each $\rho \in \mathbf{N}$.

**Proof:**
Let $\varphi \in L^2_\rho$ and define

$$\varphi_n(x) := \min(\varphi(x), n(1+x^2)^{-\frac{\rho}{4}})$$

for all $x \in \mathbf{R}_+$ and all $n \in \mathbf{N}$.Then by (3.6) $\rho > 1$ implies

$$\int\limits_{\mathbf{R}_+} \varphi_n^2(x) \, dx \leq n^2 c_\rho < \infty$$

for all $n \in \mathbf{N}$.By the definition of $\varphi_n$ one obviously has $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ for all $x \in \mathbf{R}_+$. So $(\varphi_n)_{n \in \mathbf{N}}$ is a sequence in $L^2$,s.t. $\varphi_n \to \varphi$ in $L^2_\rho$ for $n \to \infty$ holds true,which finishes the proof,since $\varphi$ was chosen arbitrarily from $L^2_\rho$.
q.e.d.

It is known (f.e. from [Fi],section 5.1,proof of 5.1.1,p.77),that $C_c^1$ is a dense subset from $L^2$.By 3.2.1 this implies that $C_c^1$ is also dense in $L^2_\rho$,i.e. for each $\varphi \in L^2_\rho$ there exists a sequence $(\varphi_n)_{n \in \mathbf{N}} \subset C_c^1$ with

$$\varphi_n \to \varphi \text{ in } L^2_\rho \text{ for } n \to \infty$$

Such a sequence leads to

$$||S(t)\varphi - \varphi||_{\rho,2} \leq ||S(t)(\varphi - \varphi_n)||_{\rho,2} + ||S(t)\varphi_n - \varphi_n||_{\rho,2} + ||\varphi_n - \varphi||_{\rho,2} \quad (3.8)$$

for all $t \in [0, T]$.In order to get the wanted estimate for $S(t)\varphi - \varphi$,one needs another lemma:

**Lemma 3.3.2:**

There is a constant $c(\rho, T) > 0$,s.t. the following holds true for all $t \in [0, T]$ and all $\varphi \in L^2_\rho$:

$$||S(t)\varphi||_{\rho,2} \leq c(\rho, T)||\varphi||_{\rho,2} \quad (3.9)$$

**Proof:**
Let $\varphi$ resp. t be arbitrary from $L^2_\rho$ resp. $[0,T]$.Then:

$$\int\limits_{\mathbf{R}_+} \varphi^2(x+t)(1+x^2)^{-\frac{\rho}{2}}\,dx = \int\limits_0^t \varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}\left(\frac{1+x^2}{1+(x+t)^2}\right)^{-\frac{\rho}{2}}\,dx + \int\limits_t^\infty \varphi^2(x+t)\left(1+(x+t)^2\left(\frac{x}{x+t}\right)^2\right)^{-\frac{\rho}{2}}\,dx$$

The second term is the simpler one.The estimate

$$\left(\frac{x}{x+t}\right)^{-\rho} \leq 2^\rho \ , \ x \geq t$$

and the fact,that $\frac{x}{x+t}$ is at most 1 for all x,t in this framework,lead to:

$$\begin{aligned}\int\limits_t^\infty \varphi^2(x+t)\left(1+(x+t)^2\left(\frac{x}{x+t}\right)^2\right)^{-\frac{\rho}{2}}\,dx \leq & \int\limits_t^\infty \varphi^2(x+t)\left(\frac{x}{x+t}\right)^{-\rho}(1+(x+t)^2)^{-\frac{\rho}{2}}\,dx \\ \leq & \ 2^\rho \int\limits_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

Now the difficult first term.By the estimates

$$(x+t)^2 \leq 4T^2 \ , \ 0 \leq x \leq t \leq T \quad (3.10)$$

$$(1+x^2)^{-\frac{\rho}{2}} \leq 1 \ , \ 0 \leq x \leq t$$

one gets

$$\begin{aligned}&\int\limits_0^t \varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}\left(\frac{1+x^2}{1+(x+t)^2}\right)^{-\frac{\rho}{2}}\,dx \\ &\leq (1+4T^2)^{\frac{\rho}{2}} \int\limits_0^t \varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}(1+x^2)^{-\frac{\rho}{2}}\,dx \\ &\leq (1+4T^2)^{\frac{\rho}{2}} \int\limits_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

These estimates imply:

$$\begin{aligned}||S(t)\varphi||_{\rho,2} = & \left(\int\limits_{\mathbf{R}_+} \varphi^2(x+t)\,\mu_\rho(dx)\right)^{\frac{1}{2}} \\ = & \left(\int\limits_{\mathbf{R}_+} \varphi^2(x+t)(1+x^2)^{-\frac{\rho}{2}}\,dx\right)^{\frac{1}{2}} \\ \leq & \ c(\rho,T)||\varphi||_{\rho,2}\end{aligned}$$

where the constant is given by

$$c(\rho,T) := \sqrt{2^\rho + (1+4T^2)^{\frac{\rho}{2}}}$$

q.e.d.

By 3.3.2(cf. (3.9)) one can simplyfy (3.8) to

$||S(t)\varphi - \varphi||_{\rho,2} \leq (c(\rho,T)+1)||\varphi - \varphi_n||_{\rho,2} + ||S(t)\varphi_n - \varphi_n||_{\rho,2}$

With the help of the preceding consideration of $C_c^1$ for each $n \in \mathbf{N}$ and each $\varepsilon > 0$ there is $t(\varepsilon, n) > 0$ with

$$||S(t)\varphi_n - \varphi_n||_{\rho,2} \leq \frac{\varepsilon}{2}$$

for all $t \leq t(\varepsilon, n)$(cf.(3.7)).
As $\varphi_n$ is an approximating sequence of $\varphi$ in $L^2_\rho$,i.e. for each $\varepsilon > 0$ there is $N(\varepsilon) \in \mathbf{N}$ with

$$||\varphi - \varphi_n||_{\rho,2} \leq \frac{\varepsilon}{2(c(\rho,T)+1)}$$

for all $n \geq N(\varepsilon)$,with fixed $\varepsilon > 0$ one has for arbitrary $n \geq N(\varepsilon)$

$$||S(t)\varphi - \varphi||_{\rho,2} \leq \varepsilon$$

for all $t \leq t(\varepsilon, n)$,s.t. the following result is shown:

**Theorem 3.3.3:**

The shift-semigroup $(S(t))_{t\in[0,T]}$ is strongly continuous on $L^2_\rho$.

**Remark 3.3.4:**

Note,that S(t):$L^2_\rho \rightarrow L^2_\rho$ holds true for arbitrary $t \geq 0$,as one can replace T by t for fixed $t \geq 0$ in (3.9).But as already seen above there is no constant $C > 0$ with

$$||S(t)\varphi||_{\rho,2} \leq C||\varphi||_{\rho,2}$$

for all $t \geq 0, \varphi \in L^2_\rho$,note the T-dependence of the constant in 3.3.2 .
Thus it is important to fix $T > 0$,in order to be able to apply 2.3.2.Thus by (3.5) $(S(t))_{t\in[0,T]}$ generates an almost strong evolution operator U.
So one has to prove that U has the properties (BC),(PP),(CC),(E1),(E2) and (BA).

**(i) The nuclear case**
First consider the nuclear case,since there

- property (CC) is not needed.
- (E1) implies (E2) analogously to [MaZa](cf. chapter 2,remark 2.3(ii),p.47 there),since the shift-semigroup is positivity preserving.

It will be shown that U defined by (3.5) has the necessary properties in order to apply Manthey‘s and Zausinger‘s theory in this case.

**Theorem 3.3.5:**

Consider the shift-semigroup $(S(t))_{t\in[0,T]}$ and define an almost strong evolution operator U by (3.5).
Then U has properties (CD),(PP),(E1),(E2) and (BA).

**Proof:**
Denote by A the generator of the shift-semigroup on $L^2_\rho$.
By the definition of the generator:

$$\textbf{(i):}\quad \begin{array}{rl} \mathcal{D}(A) := & \{\varphi \in L^2_\rho \mid \lim\limits_{t\to 0} \frac{1}{t}(([S(t)\varphi])(x) - \varphi(x)) \in \mathbf{R}, x \in \mathbf{R}_+\} \\ = & \{\varphi \in L^2_\rho \mid \varphi \text{ is differentiable}\} \end{array}$$

**(ii):** $[A\varphi](x) = \varphi'(x)$,$x \in \mathbf{R}_+$,$\varphi \in \mathcal{D}(A)$

Thus A is a linear operator.
Since $C^1_c$ is a dense subset in $L^2_\rho$

$$C^1_c \subset \mathcal{D}(A) \subset L^2_\rho$$

implies the density of $\mathcal{D}(A)$ in $L^2_\rho$.

Consider $\varphi \in C^1_c \subset \mathcal{D}(A)$.As $\varphi$ has compact support,$\varphi'$ has compact support as well.Due to its continuity on this compact set $\varphi'$ has a maximum on this set,s.t. with $c_\rho < \infty$ for $c_\rho$ as in (3.6) $A\varphi \in L^2_\rho$ holds true.
By the density of $C^1_c$ in $\mathcal{D}$(A) there is to each $\varphi \in \mathcal{D}(A)$ a sequence $(\varphi_n)_{n\in\mathbf{N}} \subset C^1_c$,converging to $\varphi$ in $L^2_\rho$-norm,i.e.

$$\lim_{n\to\infty} \varphi_n(x) \to \varphi(x), \mu_\rho\text{-f.a. } x \in \mathbf{R}_+$$

Concerning the derivatives of $\varphi$ and $\varphi_n$ one gets

$$\begin{aligned} |\varphi'(x) - \varphi_n'(x)| = & \left| \lim_{h\to 0} \frac{\varphi(x+h)-\varphi(x)}{h} - \lim_{h\to 0} \frac{\varphi_n(x+h)-\varphi_n(x)}{h} \right| \\ = & \left| \lim_{h\to 0} \frac{\varphi(x+h)-\varphi(x)-(\varphi_n(x+h)-\varphi_n(x))}{h} \right| \\ \leq & \lim_{h\to 0} \frac{|\varphi(x+h)-\varphi_n(x+h)|+|\varphi(x)-\varphi_n(x)|}{|h|} \end{aligned}$$

which implies

$$\lim_{n\to\infty} |\varphi'(x) - \varphi_n'(x)| = 0$$

for $\mu_\rho$-almost all $x \in \mathbf{R}_+$ as the terms in the nominator tend to 0 $\mu_\rho$-almost everywhere for $n \to \infty$,s.t. one gets the existence of

$$\int_{\mathbf{R}_+} \varphi'^2(x)\,\mu_\rho(dx) \leq 2\left( \int_{\mathbf{R}_+} (\varphi'(x) - \varphi_n'(x))^2\,\mu_\rho(dx) + \int_{\mathbf{R}_+} {\varphi_n'}^2(x)\,\mu_\rho(dx) \right)$$

since the righthand side is smaller than $2(\varepsilon^2 c_\rho + ||A\varphi_n||^2_{\rho,2}) < \infty$ for n big

enough.
So $A\varphi \in L^2_\rho$ also holds true for $\varphi \in \mathcal{D}(A)$.

Let $(\varphi_n)_{n\in\mathbf{N}} \subset \mathcal{D}(A)$ be a converging sequence with limit $\varphi \in \mathcal{D}(A)$.
Of course convergence in $||\cdot||_{\rho,2}$-norm is meant,i.e.

$$\lim_{n\to\infty} \int_{\mathbf{R}_+} (\varphi_n(x) - \varphi(x))^2 \, \mu_\rho(dx) = 0$$

which implies $\varphi_n \to \varphi$ $\mu_\rho$-almost everywhere on $\mathbf{R}_+$.In analogy to the above consideration one gets $\varphi_n' \to \varphi'$ $\mu_\rho$-almost everywhere.
This obviously implies

$$A\varphi_n = \varphi_n' \to \varphi' = A\varphi$$

in $L^2_\rho$,i.e. A is closed.So:

A:$\mathcal{D}(A) \to L^2_\rho$ is closed,linear operator with $\mathcal{D}(A)$ being a dense subset of $L^2_\rho$.
This is just property (CD).

As already mentioned the shift-semigroup is positivity preserving,s.t. U defined by (3.5) is positivity preserving as well.
Thus one gets a family $(A(t))_{t\in[0,T]}$ defined by $A(t) := A$ for all t,which generates an almost strong evolution operator,which is positivity preserving.
This is just property (PP).

Furthermore one has for arbitrary $\varphi \in L^2_\rho$ and $\kappa \in \mathbf{N}$:

$(U(t,s)|\varphi|)^\kappa = (|\varphi(\cdot + t - s)|)^\kappa = |\varphi(\cdot + t - s)|^\kappa = U(t,s)|\varphi|^\kappa$

for all $(s,t) \in S_T$,i.e. (E1) holds true with constant

$$c(\kappa, T) = 1, \kappa \in \mathbf{N}$$

As already mentioned above this immediately implies (E2).

As Manthey and Zausinger mention in their paper(cf.[MaZa],example 2.6,p.55) a positivity preserving,strong continuous,one-parameter semigroup fulfills (BA),which finishes the proof.

**(ii) The cylindrical case**

In Manthey‘s and Zausinger‘s theory the cylindrical case differs from the nuclear one in two aspects:

1. Property (CC) is needed.

2. (E1) does not imply (E2),i.e.(E2) must be shown separately.

Remember that (CC) was the following property:

There exists an extension of U(t,s) on $\mathcal{M}$ for $(s,t) \in S_T$,s.t.

$$U(t,s)\phi \in \mathcal{L}_2(L^2, L^2_\rho) =: \mathcal{L}_2$$

holds true for all $\varphi \in L^2_\rho$,(where $\phi$ denotes the multiplication operator associated with $\varphi$,)and there is a $\gamma \in [\,0,1\,)$ with

$$||U(t,s)\phi||^2_{\mathcal{L}_2} \leq c(T)(t-s)^{-\gamma}||\varphi||^2_{\rho,2} \quad (3.11)$$

First consider the part of (CC) concerning the extension.
For each function h of the form $h = \varphi\psi$ the definition of the shift-semigroup and (3.5) lead to:

$$[U(t,s)h](x) = \varphi(x+t-s)\psi(x+t-s)$$

Then one can show $U(t,s)h \in \mathcal{M}$ with the help of Hoelder‘s inequality:

$$\begin{aligned}\int_{\mathbf{R}_+} |[U(t,s)h](x)|\,\mu_\rho(dx) &= \int_{\mathbf{R}_+} |\varphi(x+t-s)||\psi(x+t-s)|\,\mu_\rho(dx)\\ &\leq \left(\int_{\mathbf{R}_+} \varphi^2(x+t-s)\,\mu_\rho(dx)\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}_+} \psi^2(x+t-s)\,dx\right)^{\frac{1}{2}}\\ &\leq c(\rho,T)||\varphi||_{\rho,2}||\psi||_2 < \infty\end{aligned}$$

Thus one has an extension of U(t,s) on $\mathcal{M}$ .

But considering the orthonormal sequence $(e_n)_{n\in\mathbf{N}}$ in $L^2$ from chapter 2 one gets for $\varphi \in L^2_\rho$:

$$\begin{aligned}\sum_{n\in\mathbf{N}} ||U(t,s)\phi(e_n)||^2_{\rho,2} &= \sum_{n\in\mathbf{N}} \int_{\mathbf{R}_+} \varphi^2(x+t-s)e_n^2(x+t-s)(1+x^2)^{-\frac{\rho}{2}}\,dx\\ &\leq c(\rho,T)\sum_{n\in\mathbf{N}} \int_{\mathbf{R}_+} \varphi^2(x)e_n^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

For an estimate of type (3.11) one needs a term of the type

$$c(\ldots)(t-s)^{-\gamma}\sum_{n\in\mathbf{N}} <\varphi, h_n>^2_{\rho,2}$$

where $h_n := e_n(1+x^2)^{\frac{\rho}{4}}$ is an orthonormal-basis of $L^2_\rho$ and $\gamma \in [\,0,1\,)$.With $\varphi^2 e_n^2(1+\cdot^2)^{-\frac{\rho}{2}} \geq 0$ Jensen‘s inequality implies

$$\begin{aligned}<\varphi, h_n>^2_{\rho,2} &= \left(\int_{\mathbf{R}_+} \varphi(x)h_n(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\right)^2\\ &\leq \int_{\mathbf{R}_+} \varphi^2(x)h_n^2(x)(1+x^2)^{-\rho}\,dx\\ &= \int_{\mathbf{R}_+} \varphi^2(x)e_n^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

s.t. it seems impossible to get the wanted estimate.
So one has (at least by this method) **no**

- convergence of the sum
- appropriate estimate of the sum.

## 3.4 Existence of solutions to the HJM equation

As Heath,Jarrow and Morton consider an SDE with the operator generating the shift-semigroup,one has to restrict oneself to the nuclear case,since,as seen above,not all the properties needed in the proof of 2.3.2 hold true in the cylindrical case.

**(i) The Heath-Jarrow-Morton case**
In the following it will be shown,that there is a solution to the Heath-Jarrow-Morton equation

$$r_t = S(t)h + \int_0^t S(t-s)F(s,\cdot,r_s)\,ds + \int_0^t S(t-s)\Sigma(s,r_s)\,dW(s) \quad (3.12)$$

in $L^2_\rho$ with given $\sigma$ and $\lambda$ and f defined by (1.3) under appropriate assumptions.

**Remark 3.4.1:**

Let $\lambda$:$[0,T] \times \Omega \to \mathbf{R}$ be predictable and $\sigma$:$[0,T] \times \mathbf{R} \to \mathbf{R}$ and f defined as in (1.3).
Note that in order to apply the results from chapter 2 one needs a progressively measurable function f,i.e. one needs

$$\mathbf{1}_{[0,t]\times\Omega} f(\cdot,\cdot,x) \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$

for all $t \in [0,T]$,$x \in \mathbf{R}$.To have this it is enough to assume

$$\mathbf{1}_{[0,t]}\sigma(\cdot,x) \in \mathcal{B}([0,t])\,,\ t \in [0,T]\,,\ x \in \mathbf{R} \quad (3.13)$$

**Proof:**
Given $(t,\omega) \in [0,T] \times \Omega$ define

$$\sigma(t,\omega,x) := \sigma(t,x)$$

for all fixed $x \in \mathbf{R}$.As $(\mathcal{F}_t)_{t\in[0,T]}$ is a family of $\sigma$-algebras,(3.13) implies the following:

$$\mathbf{1}_{[0,t]\times\Omega}\sigma(\cdot,\cdot,x) \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$

for all $t \in [0,T]$,$x \in \mathbf{R}$.Then one also has

$$\mathbf{1}_{[0,t]\times\Omega} \int_0^x \sigma(\cdot,\cdot,z)\,dz \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$

and

$$\mathbf{1}_{[0,t]\times\Omega}\,\sigma(\cdot,\cdot,x) \int_0^x \sigma(\cdot,\cdot,z)\,dz \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$

for all $t \in [0,T]$ and all $x \in \mathbf{R}$.As a predictable process $\lambda$ is progressively measurable as well,as a consequence of which

$$\mathbf{1}_{[\,0,t\,]\times\Omega}\lambda\sigma(\cdot,\cdot,x)\in\mathcal{B}([\,0,t\,])\times\mathcal{F}_t$$

holds true.With the help of the definition of f one gets

$$\mathbf{1}_{[\,0,t\,]\times\Omega}f(\cdot,\cdot,x)\in\mathcal{B}([\,0,t\,])\times\mathcal{F}_t$$

for all $t\in[\,0,T\,]$ and all $x\in\mathbf{R}$.
q.e.d.

Consider the situation given by the existence claim from [HeJaMo] (cf. section 7,proposition 4,p.93),i.e. $\sigma$ is a real-valued function on $[\,0,T\,]\times\mathbf{R}$,which is Lipschitzian in $\mathbf{R}$,nonnegative and bounded and $(\lambda(t))_{t\in[\,0,T\,]}$ is a real-valued process,which is predictable and bounded.
(Note that the authors in [HeJaMo] write $\gamma$ instead of $\lambda$.)

Denote the exact assumptions on $\sigma$ and $\lambda$: Let

$\sigma$:$(\,[\,0,T\,]\times\mathbf{R}\,,\,\mathcal{B}([\,0,T\,])\times\mathcal{B}(\mathbf{R})\,)\;\rightarrow\;(\,\mathbf{R},\mathcal{B}(\mathbf{R})\,)$ be nonnegative,s.t. there exists a positive constant $c_\sigma(T)$ with

$$(\sigma 1)\qquad |\sigma(t,x)-\sigma(t,y)|\leq c_\sigma(T)|x-y|$$
$$(\sigma 2)\qquad \sigma(t,x)\leq c_\sigma(T)$$

for all $t\in[\,0,T\,]$;x,y$\in\mathbf{R}$.

Let $\lambda$:$([\,0,T\,]\times\Omega,\mathcal{P}_T)\rightarrow(\mathbf{R},\mathcal{B}(\mathbf{R}))$ be such that there exists a positive constant $M_\lambda(T)$ with

$$(\lambda)\qquad |\lambda(t,\omega)|\leq M_\lambda(T)$$

for all $t\in[\,0,T\,]$,$\omega\in\Omega$.

By remark 3.4.1 these assumptions imply the progressive measurability of f defined as above and the following lemma:

**Lemma 3.4.2:**

**(i)** $\sigma$ fulfills (L1) and (L2).
**(ii)** For arbitrary $\omega\in\Omega$ $f(\cdot,\omega,\cdot)$ fulfills (PG) with exponent $\nu=1$ and (LG) with an $\omega$-independent constant.

**Proof:**
First (i).(L1) holds true obviously(cf.($\sigma$1) above) and (L2) is the special case $x=0$ of ($\sigma$2).
Concerning (ii) the assumptions lead to the following estimates for f:

$$
\begin{aligned}
|f(t,\omega,x)| = & \left| \sigma(t,x) \int_0^x \sigma(t,z)\, dz - \lambda(t,\omega)\sigma(t,x) \right| \\
\leq & |\sigma(t,x)| \left| \int_0^x \sigma(t,z)\, dz \right| + |\lambda(t,\omega)\sigma(t,x)| \\
\leq & c_\sigma^2(T)|x| + M_\lambda(T)c_\sigma(T) \\
\leq & c_f(T)(1+|x|)
\end{aligned}
$$

with $c_f(T) := (\max(c_\sigma(T), M_\lambda(T)))^2 \geq c_\sigma(T) \max(c_\sigma(T), M_\lambda(T))$ for all $(t,\omega) \in [\,0,T\,] \times \Omega$ and all $x \in \mathbf{R}$.
For this $c_f(T)$,all $x \geq 0$ and arbitrary $(t,\omega) \in [\,0,T\,] \times \Omega$ one has

$$
\begin{aligned}
f(t,\omega,x) = & \sigma(t,x) \int_0^x \sigma(t,z)\, dz - \lambda(t,\omega)\sigma(t,x) \\
\leq & c_\sigma^2(T)x + |\lambda(t,\omega)|\sigma(t,x) \\
\leq & c_\sigma^2(T)x + M_\lambda(T)c_\sigma(T) \\
\leq & c_f(T)(1+x)
\end{aligned}
$$

where the nonnegativity of $\sigma$ was used in the second and $x \geq 0$ was used in the last step.But for $x \leq 0$ one gets:

$$
\begin{aligned}
f(t,\omega,x) = & -\sigma(t,x) \int_x^0 \sigma(t,z)\, dz - \lambda(t,\omega)\sigma(t,x) \\
\geq & -\left( \left| \sigma(t,x) \int_x^0 \sigma(t,z)\, dz \right| + |\lambda(t,\omega)||\sigma(t,x)| \right) \\
\geq & -c_\sigma^2(T)(-x) - M_\lambda(T)c_\sigma(T) \\
\geq & -c_f(T)(1-x)
\end{aligned}
$$

for arbitrary $(t,\omega) \in [\,0,T\,] \times \Omega$,where $x \leq 0$ was used in the last step.
q.e.d.

**Remark 3.4.3:**

It seems to be impossible to have (L1),(L2) for f,since one has in case of $(t,\omega) \in [\,0,T\,] \times \Omega$ and x,y$\in \mathbf{R}$,s.t. w.l.o.g. $x > y$,

$$
\begin{aligned}
|f(t,\omega,x) - f(t,\omega,y)| = & \Big| \sigma(t,x) \int_0^x \sigma(t,z)\, dz - \sigma(t,y) \int_0^y \sigma(t,z)\, dz \\
& +\lambda(t,\omega)(\sigma(t,y) - \sigma(t,x))| \\
\leq & \left| (\sigma(t,x) - \sigma(t,y)) \int_0^y \sigma(t,z)\, dz \right| + \left| \sigma(t,x) \int_y^x \sigma(t,z)\, dz \right| \\
& +|\lambda(t,\omega)(\sigma(t,y) - \sigma(t,x))|
\end{aligned}
$$

where it seems impossible to estimate the first term by a term of the form $C|x-y|$ with a positive,y-independent constant C.

Defining $\Sigma$ by (2.1) in the nuclear case and F from f by (2.4),theorems 2.3.2(i) and 3.3.5 imply:

**Theorem 3.4.4:**

Given $\sigma$ with the above formulated assumptions and (3.13) and $\lambda$ with the above formulated assumptions, there exists a solution r of the Heath-Jarrow-

Morton equation

$$r_t = S(t)h + \int_0^t S(t-s)F(s,\cdot,r_s)\,ds + \int_0^t S(t-s)\Sigma(s,r_s)\,dW(s)$$

in the sense of 2.2.3 on $[\,0,T\,]$ for all finite $T>0$ and all deterministic $h \in L^2_\rho$,s.t. for all $q>2$ there exists a positive constant c(q,T,$c(\rho,T)$,$c_\sigma(T)$,$M_\lambda(T)$) depending on q,T,U,$\sigma$ and $\lambda$ with

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||r_t||^q_{\rho,2} \le c(q,T,c(\rho,T),c_\sigma(T),M_\lambda(T))(1+||h||^q_{\rho,2})$$

**Remark 3.4.5:**

If one puts the question,whether all properties of $\sigma$ assumed in 3.4.4 are necessary to apply 2.3.2,one immediately notices,that one cannot give up the Lipschitz-assumption,as this is just (L1).As $\mathcal{S}(\sigma(t,\cdot))$;$t\in[\,0,T\,]$; is a part of the function f(t,·);$t\in[\,0,T\,]$;it is necessary to assume boundedness of $\sigma$ in order to have property (LG) for f.
But it is **not** necessary to have the nonnegativity-assumption from [HeJaMo].It suffices to assume,that $\sigma$ is Lipschitzian in $\mathbf{R}$ and bounded,since then the following holds true for all $(t,\omega)\in[\,0,T\,]\times\Omega$:

$$\begin{aligned} f(t,\omega,x) \le\ & \left|\sigma(t,x)\int_0^x \sigma(t,z)\,dz\right| + |\lambda(t,\omega)\sigma(t,x)| \\ \le\ & c_\sigma^2(T)|x| + M_\lambda(T)c_\sigma(T) \\ \le\ & (\max(c_\sigma(T),M_\lambda(T)))^2(1+x) \end{aligned}$$

for $x\ge 0$ and analogously for $x\le 0$

$$\begin{aligned} f(t,\omega,x) \ge\ & -\left(\left|\sigma(t,x)\int_0^x \sigma(t,z)\,dz\right| + |\lambda(t,\omega)\sigma(t,x)|\right) \\ \ge\ & -(c_\sigma^2(T)|x| + M_\lambda c_\sigma(T)) \\ \ge\ & -(\max(c_\sigma(T),M_\lambda(T)))^2(1-x) \end{aligned}$$

Thus even without assuming nonnegativity of $\sigma$ f fulfills (PG) with exponent $\nu=1$ and (LG) with an $\omega$-independent constant,as a consequence of which one gets the existence of the solution.
So for $\sigma$,$\lambda$ with the assumptions from proposition 4 from [HeJaMo] there exists a solution of the equation

$$\begin{aligned} r_t(x) =\ & r_0(t+x) + \int_0^t f(s,\cdot,r_s(x+t-s))\,ds \\ & + \left(\int_0^t S(t-s)\Sigma(s,r_s)\,dW(s)\right)(x) \quad (3.14) \end{aligned}$$

for $x\in\mathbf{R}_+$ and the solution r has the property

$$(3.15) \qquad \int_{\mathbf{R}_+} r_t^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx < \infty\ ,\ t\in[\,0,T\,]$$

This also holds true without the nonnegativity-assumption for $\sigma$ from [HeJaMo].

Note,that in the model regarded by Heath,Jarrow and Morton there was no nonnegativity-assumption on $\sigma$ (cf.[HeJaMo],section.2,condition C1(iii),p.80),s.t. the result is a real improvement of proposition 4 from [HeJaMo] at least in case $k = 1$.

In this simple case only theorem 2.3.2(i) was applied.So it may be possible to get an existence result in a model extending the one from [HeJaMo] by theorem 2.3.2(ii) resp. theorem 2.3.3/corollary 2.3.5.

**(ii) Extension of the model from [HeJaMo]**

Consider from now on functions $\lambda$ of the form

$$\lambda:[\,0,T\,] \times \Omega \times \mathbf{R} \to \mathbf{R}$$

Thus one gets for f:

$$f(t,\omega,x) := \sigma(t,x) \int_0^x \sigma(t,z)\,dz - \lambda(t,\omega,x)\sigma(t,x)$$

Then:

**Theorem 3.4.6:**

**(i)**Let $\sigma:[\,0,T\,] \times \mathbf{R} \to \mathbf{R}$ be Lipschitzian in $\mathbf{R}$ and bounded with property (3.13),for all $\omega \in \Omega$ let $\lambda(\cdot,\omega,\cdot)$ be continuous in $\mathbf{R}$,predictable and let it fulfill (PG) with exponent $\nu = 1$ for a constant independent of $\omega$.
Having a deterministic $h \in L^2_\rho$,there is a solution of the Heath-Jarrow-Morton equation in $L^2_\rho$ in the sense of 2.2.3 and for all $q > 2$

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||r_t||^q_{\rho,2} \le c(q,T,c(\rho,T),c_\sigma(T),c_\lambda(T))(1+||h||^q_{\rho,2})$$

holds true with a positive constant c(q,T,$c(\rho,T)$,$c_\sigma$,$c_\lambda$) dependent on q,T,U,$\sigma$ and $\lambda$.

**(ii)** Let $\sigma$ be nonnegative,bounded and Lipschitzian in $\mathbf{R}$ with property (3.13),let $\lambda$ be continuous in $\mathbf{R}$,predictable and let it fulfill (PG) with exponent $\nu > 1$ and (LG$\Lambda$) given by the inequations

$$-\lambda(t,\omega,x) \le c_\lambda(T)(1+x)\ ,\ x \ge 0$$
$$\lambda(t,\omega,x) \le c_\lambda(T)(1-x)\ ,\ x \le 0$$

for all $(t,\omega) \in [\,0,T\,] \times \Omega$ and a constant $c_\lambda(T) > 0$.
Having a deterministic initial condition $h \in L^p_\rho$ with p as in 2.3.5,there is a solution of the Heath-Jarrow-Morton equation in $L^2_\rho$ in the sense of 2.2.3.The paths of the solution are almost surely in $L^p_\rho$ and have the property

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||r_t||^q_{\rho,p} \le c(p,q,T,c(\rho,T),c_\sigma(T),c_\lambda(T))(1+||h||^q_{\rho,p})$$

for all $q > p$ with a constant $c(p,q,T,c(\rho,T),c_\sigma(T),c_\lambda(T)) > 0$ dependent on p,q,T,U,$\sigma$ and $\lambda$.

**Proof:**
**(i):** The proof of lemma 3.4.2 shows,that the assumption,that $\lambda$ fulfills (PG) with exponent $\nu = 1$ is sufficient to have (PG) with exponent $\nu = 1$ and (LG) for f.Then the claim follows again by 2.3.2(i).

**(ii):** Show that f fulfills (PG) with exponent $\nu > 1$ and (LG) under the given assumptions.Then 2.3.3 leads to the claim.

So let $(t, \omega) \in [0, T] \times \Omega$ be arbitrary.Then one has for arbitrary $x \in \mathbf{R}$:

$$\begin{aligned} |f(t,\omega,x)| = & \left|\sigma(t,x)\int_0^x \sigma(t,z)\,dz - \lambda(t,\omega,x)\sigma(t,x)\right| \\ \leq & \ |\mathcal{S}(\sigma(t,\cdot))(x)| + |\lambda(t,\omega,x)||\sigma(t,x)| \\ \leq & \ c_\sigma^2(T)|x| + c_\sigma(T)c_\lambda(T)(1+|x|^\nu) \\ \leq & \ c_f(T)(1+|x|^\nu) \end{aligned}$$

with $c_f(T) := c_\sigma^2(T) + 2c_\sigma(T)c_\lambda(T)$,which is obvious,if one considers the term

$$c_\sigma^2(T)|x| + c_\sigma(T)c_\lambda(T)(1+|x|^\nu)$$

first for x with $|x| < 1$ and then for x with $|x| > 1$.
So it remains to show,that (LG) also holds true for f.Fix an arbitrary pair $(t, \omega) \in [0, T] \times \Omega$.
First consider $x \geq 0$:

$$\begin{aligned} f(t,\omega,x) = & \ \sigma(t,x)\int_0^x \sigma(t,z)\,dz + (-\lambda(t,\omega,x))\sigma(t,x) \\ \leq & \ c_\sigma^2(T)x + \sigma(t,x)(-\lambda(t,\omega,x)) \\ \leq & \ c_\sigma^2(T)x + \sigma(t,x)(c_\lambda(T)(1+x)) \\ \leq & \ (c_\sigma^2(T) + c_\sigma(T)c_\lambda(T))(1+x) \end{aligned}$$

where (LGΛ) was used in the third step. For $x \leq 0$ (LGΛ) leads to:

$$\begin{aligned} f(t,\omega,x) = & \ -\sigma(t,x)\left(\int_x^0 \sigma(t,z)\,dz + \lambda(t,\omega,x)\right) \\ \geq & \ -\sigma(t,x)\left(\left|\int_0^x \sigma(t,z)\,dz\right| + \lambda(t,\omega,x)\right) \\ \geq & \ -\sigma(t,x)(c_\sigma(T)|x| + \lambda(t,\omega,x)) \\ \geq & \ -c_\sigma^2(T)|x| + \sigma(t,x)(-\lambda(t,\omega,x)) \\ \geq & \ -c_\sigma^2(T)(-x) + \sigma(t,x)(-c_\lambda(T)(1-x)) \\ \geq & \ -(c_\sigma^2(T) + c_\sigma(T)c_\lambda(T))(1-x) \end{aligned}$$

So property (LG(Λ)) for $\lambda$ ensures property (LG) for f,which finishes the proof.
q.e.d.

**Remark 3.4.7:**

Note,that in contrast to 3.4.4 and 3.4.6(i) one needs nonnegativity of $\sigma$ in 3.4.6(ii),since without this assumption one could only estimate

$$f(t,\omega,x) \leq |\sigma(t,x)| \left( \left| \int_0^x \sigma(t,z)\,dz \right| + |\lambda(t,\omega,x)| \right) \;,\; x \geq 0$$
$$f(t,\omega,x) \geq -|\sigma(t,x)| \left( \left| \int_0^x \sigma(t,z)\,dz \right| + |\lambda(t,\omega,x)| \right) \;,\; x \leq 0$$

s.t.,as $\lambda$ fulfills (PG) with exponent $\nu > 1$,there seems to be no chance to have the estimates from (LG) for x with $|x| > 1$.

So one can conclude:
Extending $\lambda$ to a mapping $\lambda$:$[\,0, T\,] \times \Omega \times \mathbf{R} \to \mathbf{R}$,one gets

- a solution of (3.14) with property (3.15) in case of $\lambda$ fulfilling (PG) with exponent without the need to assume nonnegativity of $\sigma$.

- a solution of (3.14) with

  (3.16) $\qquad \int_{\mathbf{R}_+} |r_t(x)|^p \, \mu_\rho(dx) < \infty$ P-f.s,$t \in [\,0, T\,]$

  for p as in 2.3.5. in case of $\lambda$ fulfilling (PG) with an exponent $\nu > 1$ and (LG$\Lambda$) and $\sigma$ being nonnegative.

So in this model one does not lose the existence result from [HeJaMo] and in case $\nu = 1$ one even does not lose the improvement,that one can give up the nonnegativity-assumption on $\sigma$.

**Remark 3.4.8:**

The extension does not only allow the application of part (ii) from 2.3.2,it also makes it possible to compare this model to the one in [Te],since there the market price of risk is defined not only on $[\,0, T\,] \times \Omega$ as well.

How can this extension of $\lambda$ be interpreted?
Since $\lambda$ represents the so-called market price of risk,the extension means that the market price depends both on the time at which the consideration starts and on how far the times,up to which the rate shall be determined,are away from this time.

So one now has existence of a solution to the Heath-Jarrow-Morton equation (3.12) in different situations(see the assumptions in 3.4.4.,3.4.6(i) resp. 3.4.6(ii)).

## 3.5 Comparison to other models

**(i) Tehranchi's model**

In his paper Tehranchi defines the term Heath-Jarrow-Morton model on spaces

$$H_w := \left\{ f : \mathbf{R}_+ \to \mathbf{R} \mid f \text{ ist absolut stetig} \int\limits_{\mathbf{R}_+} f'^2(x) w(x)\, dx < \infty \right\}$$
$$H_w^0 := \left\{ f \in H_w \mid f(\infty) := \lim_{x\to\infty} f(x) = 0 \right\}$$

with a growing weight-function w:$\mathbf{R}_+ \to [\,1, \infty\,)$ in the following way (cf.[Te],section 2.2,definition 4,p.4 there):

A Heath-Jarrow-Morton model is a pair ($\lambda$,$\sigma$) with

$\lambda$:$(\mathbf{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}(H_w)) \to (G, \mathcal{B}(G))$
$\Sigma$:$(\mathbf{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}(H_w)) \to (\mathcal{L}_2(G, H_w^0), \mathcal{B}(\mathcal{L}_2(G, H_w^0)))$

s.t. there is a non-empty set of initial conditions $h \in H_w$,for which there exists a unique,continuous,$H_w$-valued solution $(f_t)_{t\geq 0}$ of the Heath-Jarrow-Morton equation

$$f_t = S(t)h + \int\limits_0^t S(t-s)a(s,\omega,f_s)\, ds + \int\limits_0^t S(t-s)\Sigma(s,\omega,f_s)\, dW(s)$$

with

$$a(s,\omega,f) := F_{HJM}(\Sigma(t,\omega,f)) - \Sigma(t,\omega,f)\lambda(t,\omega,f)$$

where $F_{HJM}$ is a mapping from $\mathcal{L}_2(G, H_w^0)$ to $H_w$ defined by

$$F_{HJM}(A)(x) := < A^*\delta_x\, ,\, A^* I_x >_G\, ,\ A \in \mathcal{L}(G, H_w^0)$$

with $\delta_x(f) := f(x)$,$I_x(f) := \int_0^x f(z)\, dz$ for $f \in H_w$,$x \in \mathbf{R}_+$.
In this case $\mathcal{P}$ denoted the $\sigma$-algebra of predictable sets on $\mathbf{R}_+ \times \Omega$,G was a Hilbert space and W a cylindrical Brownian motion on G.

The weighted spaces are also used in [Fi] in order to define a Heath-Jarrow-Morton model.Fillipovic describes,why it makes sense to consider this kind of spaces:
Given a fixed time it is unrealistic to expect the rate from this time onward up to a time far away to differ essentially from the rate up to a time,which is only an infinitesimal unit further away,i.e. for fixed $t \geq 0$ and large $x \in \mathbf{R}$ one expects

$$r_t(x) \approx r_t(x+\varepsilon)$$

for small enough $\varepsilon > 0$.This is ensured by the fact that,since $w \geq 1$ is a growing function,a large difference between $r_t(x)$ and $r_t(x+\varepsilon)$ and thus a large $r_t'(x)$ is punished by $w \geq 1$ and the larger x the larger is this punishment.
In Fillipovic's and Tehranchi's Heath-Jarrow-Morton model the processes are

such that it is possible to integrate the weighted terms $r'_t(x)w(x)$ over $\mathbf{R}_+$.

In his paper Tehranchi uses corollary 5.1.2 from [Fi] in order to have a local Lipschitz property for $F_{HJM}$ (cf.[Te],section 2.1,proposition 3,p.5) for appropriate weighting functions w.
Taking $G = L^2$ the model Tehranchi uses differs from the one considered here in two aspects.The first one is,that here only the nuclear case is considered,since the shift-semigroup does not fulfill (CC) and (E2) in the cylindrical case.The second one is that $S$ defined as in [Fi] does not allow for an estimate like in 5.1.2 in [Fi],as a consequence of which one does **not** get the existence of a **unique**,continuous solution by assuming $\sigma\lambda$ to be Lipschitzian,which is the case in the basic situation in [HeJaMo](cf. chapter 7,proposition 4,p.93 there).
Furthermore in [Te] one has existence of $(f_t)_{t\geq 0}$ instead of $(r_t)_{t\in[0,T]}$ for fixed $T > 0$,since by the definition of the norm in [Te](cf. section 2.1,definition 1,p.3 there) the following holds true for arbitrary $t \geq 0$,$h \in H_w$:

$$\begin{aligned} ||S(t)f||_{H_w} = &\quad f(\infty + t) + \int_0^\infty f'^2(x+t)w(x)\,dx \\ \leq &\quad \int_0^\infty f'^2(x+t)w(x+t)\,dx \\ \leq &\quad f(\infty) + \int_0^\infty f'^2(x)w(x)\,dx = ||f||_{H_w} \end{aligned}$$

where $f \in H_w^0$ was used in the second and the third and the fact,that w is a growing function,was used in the second step.Thus one gets for the operator norm $||\cdot||_{\mathcal{L},w}$:

$$||S(t)||_{\mathcal{L},w} \leq 1\ ,\ t \geq 0$$

whereas due to (3.9) one has for the operator norm $||\cdot||$ in $L^2_{\rho,+}$

$$||S(t)|| \leq c(\rho, T)\ ,\ t \in [\,0, T\,]$$

for the operator norm $||\cdot||$ in $L^2_\rho$ with the constant having the property

$$c(\rho, T) \to \infty \text{ for } T \to \infty$$

To understand the importance of this difference consider once again the proof of 2.3.2.
In that proof one first considered solutions to Eq($\xi$,$F_{N,M}$,$\Sigma$) in the sense of 2.2.3 with $f_{N,M}$, N,M$\in \mathbf{N}$ corresponding to $F_{N,M}$ given by

$$\begin{aligned} f_N(t,\omega,u) &:= f(t,\omega,u) \vee (-N) \\ f_{N,M}(t,\omega,u) &:= \inf_{v\in\mathbf{R}} (f_N(t,\omega,v) + M|u-v|) \end{aligned}$$

for $(t,\omega,u) \in [\,0,T\,] \times \Omega \times \mathbf{R}$.
The $f_{N,M}$ fulfill (L1),(L2),s.t. given $\sigma$ fulfilling (L1),(L2) as well there exist pathwise unique solutions to Eq($\xi$,$F_{N,M}$,$\Sigma$) for initial conditions $\xi$ with

$$\mathbf{E}||\xi||^q_{\rho,2} < \infty$$

for a $q > \frac{2}{1-\gamma}$ resp.

$$\mathbf{E}||\xi||^{2\kappa}_{\rho,2\kappa} < \infty$$

for a $\kappa > \frac{1}{1-\gamma}$ by theorem 2.2.7.
Looking in the proof of 2.2.7 it strikes,that in the proof of part (i) the solution is found by finding a contraction in $C([0,\bar{T}];L^2([0,T]\times\Omega;L^2_\rho))$ for $\bar{T}>0$,s.t.

$$\bar{T}(c(c(T),c_f(T))+c(c(T),c_\sigma(T)))<1$$

holds true.Thus there is exactly one process X on $[0,\bar{T}]$ being a fixed point of $\mathcal{K}$ given by

$$\mathcal{K}(Y)(t) := U(t,0)\xi + \int_0^t U(t,s)F(s,\cdot,Y(s))\,ds + \int_0^t U(t,s)\Sigma(s,Y(s))\,dW(s)$$

After that by setting $\xi := X(\bar{T})$ one gets a fixpoint
$X \in C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ by induction,which is the wanted solution according to the definition of $\mathcal{K}$.

In order to get a solution $X=(X(t))_{t\geq 0}$ by this contraction argument it would be necessary to have

$$\sup_{(s,t)\in S}||U(t,s)|| \leq C < \infty$$

with $S := \{(s,t) \mid 0\leq s\leq t; s,t\in\mathbf{R}_+\}$ for a positive constant C.
Since this does not hold true in the shift-semigroup case,it is only possible to prove the existence of processes $(r_t)_{t\in[0,T]}$ for fixed $T>0$ with the help of the theory from [MaZa].

But one can solve the problem by considering a modification of the shift-semigroup.
Define

$$[\bar{S}(t)\varphi](x) := e^{-t}\varphi(x+t)\ ;\ t\geq 0\ ,\ x\in\mathbf{R}_+\ ,\ \varphi\in L^2_\rho$$

Then:

**Lemma 3.5.1:**

**(i)** There exists a positive constant c($\rho$),s.t. the following holds true for all $t\geq 0$ and all $\varphi\in L^2_\rho$:

$$||\bar{S}(t)\varphi||_{\rho,2} \leq c(\rho)||\varphi||_{\rho,2} \quad (3.18)$$

**(ii)** $(\bar{S}(t))_{t\geq 0}$ is a strongly continuous semigroup.

**Proof:**
Let $\varphi\in L^2_\rho$ be arbitrary.

**(i):** Let $t \geq 0$ be arbitrary.Then one gets analogously to the proof of 3.3.2:

$$\begin{aligned}\int_{\mathbf{R}_+} e^{-2t}\varphi^2(x+t)(1+x^2)^{-\frac{\rho}{2}}\,dx = \quad & e^{-2t}\int_0^t \varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}\left(\frac{1+x^2}{1+(x+t)^2}\right)^{-\frac{\rho}{2}} dx \\ & +e^{-2t}\int_t^\infty \varphi^2(x+t)\left(1+(x+t)^2\left(\frac{x}{x+t}\right)^2\right)^{-\frac{\rho}{2}} dx\end{aligned}$$

In analogy to the proof of 3.3.2 one gets the following estimates:

$$\begin{aligned}e^{-2t}\int_t^\infty \varphi^2(x+t)\left(1+(x+t)^2\left(\frac{x}{x+t}\right)^2\right)^{-\frac{\rho}{2}} dx \leq \quad & 2^\rho e^{-2t}\int_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx \\ \leq \quad & 2^\rho \int_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

$$\begin{aligned}& e^{-2t}\int_0^t \varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}\left(\frac{1+x^2}{1+(x+t)^2}\right)^{-\frac{\rho}{2}} dx \\ & \leq e^{-2t}(1+4t^2)^{\frac{\rho}{2}}\int_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx\end{aligned}$$

where remark 3.3.4 was applied for the second estimate.
In order to get an estimate independent of t,search for a constant,by which $g(t) := e^{-2t}(1+4t^2)$ is bounded.Since g is nonnegative 0 is a lower bound for g. So look for maximizers of g.The first derivative is of the following form:

$$\begin{aligned}g'(t) = \quad & e^{-2t}(-2-8t^2+8t) = -8e^{-2t}\left(t^2-t+\frac{1}{4}\right) \\ = \quad & -8e^{-2t}\left(t-\frac{1}{2}\right)^2 \leq 0\end{aligned}$$

for all $t \geq 0$,so g is monotonically decreasing in t,i.e. one gets for all $t \geq 0$

$$g(t) \leq g(0) = 1$$

s.t.

$$\begin{aligned}\int_{\mathbf{R}_+} e^{-2t}\varphi^2(x+t)(1+(x+t)^2)^{-\frac{\rho}{2}}\,dx \leq \quad & (2^\rho+1)\int_{\mathbf{R}_+} \varphi^2(x)(1+x^2)^{-\frac{\rho}{2}}\,dx \\ = \quad & (2^\rho+1)||\varphi||_{\rho,2}^2\end{aligned}$$

follows by the estimates above.Thus for all $t \geq 0$

$$\begin{aligned}||\bar{S}(t)\varphi||_{\rho,2} = \quad & \left(\int_{\mathbf{R}_+} e^{-2t}\varphi^2(x+t)\,\mu_\rho(dx)\right)^{\frac{1}{2}} \\ = \quad & \left(\int_{\mathbf{R}_+} e^{-2t}\varphi^2(x+t)(1+x^2)^{-\frac{\rho}{2}}\,dx\right)^{\frac{1}{2}} \\ \leq \quad & c(\rho)||\varphi||_{\rho,2}\end{aligned}$$

with a positive constant c($\rho$) given by

$$c(\rho) := \sqrt{2^\rho+1}$$

**(ii):** Let s,t$\in \mathbf{R}_+$ be arbitrary.One has for all $x \in \mathbf{R}_+$:

$$[\bar{S}(t+s)\varphi] = e^{-(t+s)}\varphi(x+t+s) = [\bar{S}(t)(e^{-s}\varphi(\cdot+s))](x) = [\bar{S}(t)(\bar{S}(s)\varphi)](x)$$

Thus the semigroup property is proven.
It remains to show strong continuity,i.e.

$$\lim_{t\to 0} \bar{S}(t)\varphi = \varphi$$

must hold true.
This property is obviously fulfilled for $\varphi \in C_c^1$ with

$$\lim_{t\to 0} e^{-t} = 1 \ , \ \lim_{t\to 0} \varphi(x+t) = \varphi(x)$$

As already mentioned in section 3.3 $C_c^1$ is a dense subset of $L_\rho^2$,s.t. there exists a sequence $(\varphi_n)_{n\in\mathbf{N}} \subset C_c^1$ with

$$\varphi_n \to \varphi \text{ in } L_\rho^2 \text{ for } n \to \infty$$

for the given $\varphi \in L_\rho^2$.Analogously to the procedure there one gets by part (i)

$$||\bar{S}(t)\varphi - \varphi||_{\rho,2} \leq (c(\rho)+1)||\varphi - \varphi_n||_{\rho,2} + ||\bar{S}(t)\varphi_n - \varphi_n||_{\rho,2}$$

for all $t \geq 0$ and finally for fixed $\varepsilon > 0$

$$||\bar{S}(t)\varphi - \varphi||_{\rho,2,+} \leq \varepsilon$$

for all $t \leq \bar{t}$ with $\bar{t} > 0$ depending on $\varepsilon$,which proves strong continuity.
q.e.d.

Define an almost strong evolution operator $\bar{U}$ from $\bar{S}$ by (3.5).Then:

**Theorem 3.5.2**

$\bar{U}$ has properties (CD),(PP),(E1),(E2) and (BA).

**Proof:**
Denote the generator of $\bar{S}$ by $\bar{A}$.By the definition of the generator one gets

$$\mathcal{D}(\bar{A}) := \{\varphi \in L_\rho^2 \mid \lim_{t\to 0} \tfrac{1}{t}([\bar{S}(t)\varphi](x) - \varphi(x)) \in \mathbf{R}, x \in \mathbf{R}_+\}$$

In this case the following holds true for arbitrary $x \in \mathbf{R}_+$:

$$\begin{aligned}\lim_{t\to 0} \tfrac{1}{t}(e^{-t}\varphi(x+t) - \varphi(x)) &= \lim_{t\to 0} \tfrac{1}{t}(e^{-t}\varphi(x+t) - e^{-0}\varphi(x)) \\ &= \lim_{t\to 0} \tfrac{1}{t}(e^{-t}(\varphi(x+t) - \varphi(x)) + (e^{-t} - e^{-0})\varphi(x)) \\ &= \varphi'(x) - \varphi(x)\end{aligned}$$

Thus $\bar{A} = A - I$ and $\mathcal{D}(\bar{A}) = \mathcal{D}(A)$ with A from section 3.3.
So $\bar{A}$ is a linear operator and analogously to the proof of 3.3.5 $\mathcal{D}(\bar{A})$ is a dense subset of $L_\rho^2$.
Have a closer look at the proof of 3.3.5.In that proof one first considered functions from $C_c^1$.Both these functions and their derivatives reach a maximum,s.t. by $c_\rho < \infty$ $\bar{A}\varphi \in L_\rho^2$ holds true in case $\varphi \in C_c^1 \subset \mathcal{D}(\bar{A})$.By the above mentioned

density of $C_c^1$ in $\mathcal{D}(\bar{A})$ for any $\varphi \in \mathcal{D}(\bar{A})$ there is a sequence $(\varphi_n)_{n\in\mathbf{N}} \subset C_c^1$, s.t. $||\varphi_n - \varphi||_{\rho,2} \to 0$ for $n \to \infty$.
Analogously to the proof of 3.3.5 one shows

$$\lim_{n\to\infty} \varphi_n(x) \to \varphi(x), \mu_\rho\text{-a.a } x \in \mathbf{R}_+$$
$$\lim_{n\to\infty} |\varphi'(x) - \varphi_n'(x)| = 0, \mu_\rho\text{-a.a. } x \in \mathbf{R}_+$$

This implies the finiteness of

$$\int\limits_{\mathbf{R}_+} [\varphi' - \varphi]^2(x)\,\mu_\rho(dx) \leq \quad 4\Bigg(\int\limits_{\mathbf{R}_+} (\varphi'(x) - \varphi_n'(x))\,\mu_\rho(dx) + \int\limits_{\mathbf{R}_+} {\varphi_n'}^2(x)\,\mu_\rho(dx) + \int\limits_{\mathbf{R}_+} (\varphi(x) - \varphi_n(x))^2\,\mu_\rho(dx) + \int\limits_{\mathbf{R}_+} \varphi_n^2(x)\,\mu_\rho(dx)\Bigg)$$

analogously to the proof of 3.3.5,s.t. $\bar{A}\varphi \in L^2_\rho$ for $\varphi \in \mathcal{D}(\bar{A})$.
By the above limit properties $\bar{A}$ is a closed operator analogously to the situation in 3.3.5.Thus one has the following conclusion:
$\bar{A}$:$\mathcal{D}(\bar{A}) \to L^2_\rho$ is a linear,closed operator,with $\mathcal{D}(\bar{A})$ being a dense subset of $L^2_\rho$.
This is exactly property (CD).

Since $\bar{S}$ differs from the shift-semigroup only by the positive factor $e^{-\cdot}$,again analogously to 3.3.5 $\bar{U}$ is a positivity preserving almost strong evolution operator generated by $\bar{A}(t) := \bar{A}$ for $t \in [\,0, T\,]$ with $T > 0$ arbitrary,which is just property (PP).

For arbitrary $\varphi \in L^2_\rho$ and $\kappa \in \mathbf{N}$ one has

$$(\bar{U}(t,s)|\varphi|)^\kappa = e^{-\kappa(t-s)}|\varphi(\cdot + t - s)|^\kappa \leq e^{-(t-s)}|\varphi(\cdot + t + s)|^\kappa = \bar{U}(t,s)|\varphi|^\kappa$$

for all $s \leq t$ from $\mathbf{R}_+$,i.e. (E1) holds true with $c(\kappa) = 1$ for all $\kappa \in \mathbf{N}$.
As already mentioned at the beginning of section 3.3. (E1) implies (E2).
By example 2.6 from [MaZa](cf. p.55 there) $\bar{S}$ has property (BA) as well,since it is a (strongly continuous) one-parameter semigroup.This finishes the proof.

Note,that as in the case of the shift-semigroup property (CC) cannot be shown,s.t. one has to restrict to the nuclear case in this section as well.

Let $\sigma$ and $\lambda$ be analogously to the situation in theorem 3.4.4,i.e. one has:

$\sigma$:$(\,\mathbf{R}_+ \times \mathbf{R}\,, \mathcal{B}(\mathcal{R}_+) \times \mathcal{B}(\mathbf{R})\,) \;\;\to\;\; (\,\mathbf{R}, \mathcal{B}(\mathbf{R})\,)$ is nonnegative,and there is a positive constant $c_\sigma$ with

$(\tilde{\sigma}1)$ $$|\sigma(t,x) - \sigma(t,y)| \leq c_\sigma|x - y|$$
$(\tilde{\sigma}2)$ $$\sigma(t,x) \leq c_\sigma$$

for all $t \in \mathbf{R}_+$;x,y$\in \mathbf{R}$.

Given $\lambda$:$(\mathbf{R}_+ \times \Omega, \mathcal{P}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ there is a positive constant $M_\lambda(T)$ with

$$(\tilde{\lambda}) \qquad |\lambda(t,\omega)| \leq M_\lambda(T)$$

for all $t \in \mathbf{R}_+$,$\omega \in \Omega$.

**Remark 3.5.3:**

Let $\sigma$ be such,that for all $t \geq 0$

$$\mathbf{1}_{[\,0,t\,]}\sigma(\cdot, x) \in \mathcal{B}([\,0,t\,])\;,\; t \in \mathbf{R}_+\;,\; x \in \mathbf{R}$$

holds true.Then f:$\mathbf{R}_+ \times \Omega \times \mathbf{R} \to \mathbf{R}$ given by

$$f(t,\omega,x) := \sigma(t,x)\int_0^x \sigma(t,z)\,dz + \lambda(t,\omega)\sigma(t,x)\;,\;(t,\omega,x) \in \mathbf{R}_+ \times \Omega \times \mathbf{R}$$

is progressively measurable.

**Proof:**
The proof works in the same manner as the one of 3.4.1.

As in section 3.4(cf. 3.4.2) $\sigma$ fulfills (L1),(L2),whereas f fulfills (PG) with exponent 1 and (LG),where one can replace $t \in [\,0,T\,]$ by $t \in \mathbf{R}_+$ and $c_\sigma(T)$ resp. $c_f(T)$ by $c_\sigma$ resp. $c_f$.

**Theorem 3.5.4:**

Given $\sigma$,$\lambda$ with $(\tilde{\sigma} i)$;i=1,2;$(\tilde{\lambda})$ and $\sigma$ having the additional property from 3.5.3,there exists a solution r of the modified Heath-Jarrow-Morton equation

$$r_t = \bar{S}(t)h + \int_0^t \bar{S}(t-s)F(s,\cdot,r_s)\,ds + \int_0^t \bar{S}(t-s)\Sigma(s,r_s)\,dW(s)$$

for $t \geq 0$ and all deterministic $h \in L^2_\rho$,s.t. there is a positive constant c(q,T,$c(\rho)$,$c_\sigma(T)$,$M_\lambda(T)$) for all $q > 2$ and all $T > 0$,which is depending on q,T,$\bar{U}$,$\sigma$ and $\lambda$ with

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||r_t||^q_{\rho,2} \leq c(q,T,c(\rho),c_\sigma(T),M_\lambda(T))(1 + ||h||^q_{\rho,2})$$

**Proof:**
As in the proof of 2.3.2 define for N,M$\in \mathbf{N}$ mappings
$f_N$,$f_{N,M}$:$\mathbf{R}_+ \times \Omega \times \mathbf{R} \to \mathbf{R}$ by

$$f_N(t,\omega,x) := f(t,\omega,x) \vee (-N)\;;\; t \geq 0\;,\; \omega \in \Omega\;,\; x \in \mathbf{R}_+$$

and

$$f_{N,M}(t,\omega,x) := \inf_{u\in\mathbf{R}}(f_N(t,\omega,u) + M|x-u|)\;;\; t \geq 0\;,\; \omega \in \Omega\;,\; x \in \mathbf{R}_+$$

Again the $f_{N,M}$ fulfill (L1),(L2).
Analogously to the proof of 2.2.9 one has continuity and predictability of

$$\int_0^t \bar{S}(t-s)F_{N,M}(s,\cdot,Z(s))\,ds$$

for all $t \geq 0$,if Z is a $L^2_\rho$-valued predictable process with

$$\sup_{r\geq 0}\mathbf{E}||Z(r)||^2_{\rho,2} < \infty$$

since

$$\begin{aligned}
\mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)F_{N,M}(s,\cdot,Z(s))\,ds\right|\right|_{\rho,2} &\leq c(\rho)\mathbf{E}\left|\left|\int_0^t F_{N,M}(s,\cdot,Z(s))\,ds\right|\right|_{\rho,2}\\
&\leq c(c(\rho),c_f(N,M))\int_0^t (1+\mathbf{E}||Z(s)||_{\rho,2})\,ds\\
&\leq c(c(\rho),c_f(N,M))\,t\left(1+\sup_{r\geq 0}\mathbf{E}||Z(r)||_{\rho,2}\right)\\
&< \infty
\end{aligned}$$

holds true,as a consequence of which the Bochner-integral is welldefined.Considering $\mathcal{K}_1$ from the proof of 2.2.7 for processes $Z \in C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ one gets for arbitrary $T > 0$:

$$\begin{aligned}
||\mathcal{K}_1(Z)||^2_{C([0,T];L^2_\rho)(2)} &= \sup_{t\in[0,T]}\mathbf{E}||\mathcal{K}_1(Z)(t)||^2_{\rho,2}\\
&= \sup_{t\in[0,T]}\mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)F_{N,M}(s,\cdot,Z(s))\,ds\right|\right|^2_{\rho,2}\\
&\leq c^2(\rho)\,\mathbf{E}\int_0^T ||F_{N,M}(s,\cdot,Z(s))||^2_{\rho,2}\,ds\\
&\leq c^2(\rho)\,\mathbf{E}\int_0^T c_f(N,M)^2(1+||Z(s)||^2_{\rho,2})\,ds\\
&\leq T\,c(c(\rho),c_f(N,M))\left(1+\sup_{t\in[0,T]}\mathbf{E}||Z(t)||^2_{\rho,2}\right)\\
&= T\,c(c(\rho),c_f(N,M))(1+||Z||^2_{C([0,T];L^2_\rho)(2)})
\end{aligned}$$

So $\mathcal{K}_1$ is a mapping from $C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ onto itself.
$\mathcal{K}_2$ from the proof of 2.2.7(i) maps processes Z
from $C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ to $C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$,if these have property 2.2.3(ii),as one has analogously to the proof there:

$$\begin{aligned}
||\mathcal{K}_2(Z)||^2_{C([0,T];L^2_\rho)(2)} &\leq \mathbf{E}\int_0^t ||U(t,s)\Sigma(s,Z(s))||^2_{\mathcal{L}_2}\,ds\\
&\leq T\,c(c(\rho),c_\sigma)\left(1+\int_0^t \mathbf{E}||Z(s)||^2_{\rho,2}\,ds\right)\\
&\leq T(1+T)\,c(c(\rho),c_\sigma)\left(1+\sup_{t\in[0,T]}\mathbf{E}||Z(t)||^2_{\rho,2}\right)\\
&= T(1+T)c(c(\rho),c_\sigma)(1+||Z||^2_{C([0,T];L^2_\rho)(2)})
\end{aligned}$$

Given processes X,Y with 2.2.3(ii) one gets:

$$
\begin{aligned}
||\mathcal{K}(X)-\mathcal{K}(Y)||^2_{C([0,T];L^2_\rho)(2)} \leq \quad & 2\Big(||\mathcal{K}_1(X)-\mathcal{K}_1(Y)||^2_{C([0,T];L^2_\rho)(2)} \\
& +||\mathcal{K}_2(X)-\mathcal{K}_2(Y)||^2_{C([0,T];L^2_\rho)(2)}\Big) \\
=: \quad & 2(I_1+I_2)
\end{aligned}
$$

with estimates

$$
\begin{aligned}
I_1 \leq \quad & c^2(\rho)\sup_{t\in[0,T]}\mathbf{E}\int_0^t ||F_{N,M}(s,\cdot,X(s))-F_{N,M}(s,\cdot,Y(s))||^2_{\rho,2}\,ds \\
\leq \quad & c^2(\rho)\,c_f^2(N,M)\sup_{t\in[0,T]}\mathbf{E}\int_0^t ||X(s)-Y(s)||^2_{\rho,2}\,ds \\
\leq \quad & Tc(c(\rho),c_f(N,M))\sup_{t\in[0,T]}\mathbf{E}||X(t)-Y(t)||^2_{\rho,2} \\
= \quad & T\,c(c(\rho),c_f(N,M))||X-Y||^2_{C([0,T];L^2_\rho)(2)} \\
I_2 \leq \quad & c(c(\rho),c_\sigma)\left(\int_0^t \mathbf{E}||X(s)-Y(s)||^2_{\rho,2}\,ds\right) \\
\leq \quad & T\,c(c(\rho),c_\sigma)\left(\sup_{t\in[0,T]}\mathbf{E}||X(t)-Y(t)||^2_{\rho,2}\right) \\
= \quad & T\,c(c(\rho),c_\sigma)||X-Y||^2_{C([0,T];L^2_\rho)(2)}
\end{aligned}
$$

Thus one has the existence of a unique fixpoint X of $\mathcal{K}$ with

$$\mathcal{K}(Z)(t)=\bar{S}(t)\xi+\mathcal{K}_1(Z)(t)+\mathcal{K}_2(Z)(t)$$

in $C([0,T];L^2([0,T]\times\Omega;L^2_\rho))$ for stochastic $\xi$ in $L^2_\rho$ with a fixed $T>0$,s.t.

$$T<\frac{1}{c(c(\rho),c_f(N,M))+c(c(\rho),c_\sigma)}$$

holds true.Analogously one gets a unique fixpoint on $[0,T]$ by setting $\xi:=X(T)$,s.t. one has a fixpoint on $[0,2T]$.Due to the T-independence of the constants c(...) in the estimates one can repeat this procedure infinitely many times and thus get a fixpoint on $\mathbf{R}_+$,which is unique up to modifications and fulfills 2.2.3(ii)(with $\gamma=0$ (nuclear case)) for all $T>0$.As it was shown in the proof of 2.2.7(i) there is even a pathwise unique,continuous fixpoint,which corresponds to a solution of the wanted equation.Analogously to the proof of 2.2.7(i) (i) and (iv) from 2.2.3 hold true with $T=\infty$.Denote this solution by $r^{N,M}$.

For fixed $T>0$ and $\xi$ with

$$\mathbf{E}||\xi||^q_{\rho,2}<\infty$$

one gets by the above estimates:

$$
\begin{aligned}
\mathbf{E}||r_t^{N,M}||_{\rho,2}^q \leq \quad & c(q)\left(\mathbf{E}||\bar{S}(t)\xi||_{\rho,2}^q + \mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)F(s,\cdot,r_s^{N,M})\,ds\right|\right|_{\rho,2}^q\right. \\
& \left. +\mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)\Sigma(s,r_s^{N,M})\,dW(s)\right|\right|_{\rho,2}^q\right) \\
\leq \quad & c(q,c(\rho))\mathbf{E}||\xi||_{\rho,2}^q \\
& +c(q,T,C,c_f,c_\sigma)\left(1+\int_0^t \mathbf{E}||r_s^{N;M}||_{\rho,2}^q\,ds\right) \\
\leq \quad & c(q,T,c(\rho),c_f(N,M),c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q) \\
& +c(q,T,c(\rho),c_f(N,M),c_\sigma)\int_0^t \mathbf{E}||r_s^{N,M}||_{\rho,2}^q\,ds
\end{aligned}
$$

for all $t\in[\,0,T\,]$,s.t. Gronwall‘s lemma implies the wanted estimate for $r^{N,M}$.
In the same manner one gets existence of $\underline{r}^{N,M}$ and $\bar{r}^{0,M}$ defined analogously to the proof of 2.3.2.
In order to have a comparison between these processes,one needs a comparison theorem like 2.2.11.
Consider the procedure from section 2.2.
The estimates in the proof of 2.2.12 show,that the corresponding estimates in the new situation are such,that the considered processes are defined on $\mathbf{R}_+$,s.t. completely analogous to the proof of 2.2.12 one first gets a version on $[\,0,T\,]$,then on $[\,0,2T\,]$ and inductively on $\mathbf{R}_+$.Due to the T-independence of the constants belonging to $\bar{U}$,f and $\sigma$ the proof of 2.2.13 also works in the new situation,s.t. one has a comparison result like 2.2.11 with $t\geq 0$ instead of $t\in[\,0,T\,]$.

Defining $\bar{r}^{0,M}$;$M\in\mathbf{N}$;analogously to the proof of 2.3.2 (L1),(L2) for $\sigma$ and $f_{N,M}$ with time-independent constants $c_\sigma$ and $c_f(N,M)$ for all N,M$\in\mathbf{N}$ implies for $t\in[\,0,T\,]$ with $T>0$ arbitrary

$$
\mathbf{E}||\bar{r}_t^{0,M}||_{\rho,2}^q \leq \tilde{I}^{(1)}(t) + \tilde{I}_M^{((2)}(t) + \tilde{I}_M^{(3)}(t)
$$

where the terms are defined as follows:

$$
\tilde{I}^{(1)}(t) := c(q)\mathbf{E}||\bar{S}(t)\xi||_{\rho,2}^q \leq c(q,c(\rho))\mathbf{E}||\xi||_{\rho,2}^q
$$

$$
\begin{aligned}
\tilde{I}_M^{(2)}(t) := \quad & c(q)\mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)F_{0,M}(s,\cdot,\bar{r}_s^{0,M})\,ds\right|\right|_{\rho,2}^q \\
\leq \quad & c(q,c(\rho))\mathbf{E}\int_0^t ||\bar{h}(\bar{r}_s^{0,M})||_{\rho,2}^q\,ds \\
\leq \quad & c(q,c(\rho),c_f)\mathbf{E}\int_0^t \left[\int_{\mathbf{R}_+}\left[(1+(\bar{r}_s^{0,M}(y))^2)\mathbf{1}_{\bar{r}_s^{0,M}(y)>0}(s,y)\right.\right. \\
& \left.\left. +(1+V^2(s,y))\mathbf{1}_{\bar{r}_s^{0,M}(y)<0}(s,y)\right]\mu_\rho(dy)\right]^{\frac{q}{2}}\,ds \\
\leq \quad & c(q,T,c(\rho),c_f,c_\sigma)\left(1+\int_0^t \mathbf{E}||\bar{r}_s^{0,M}||_{\rho,2}^q\,ds\right)
\end{aligned}
$$

$$\begin{aligned}\tilde{I}_M^{(3)}(t) :=\ & c(q)\mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)\Sigma(s,\bar{r}_s^{0,M})\,dW(s)\right|\right|_{\rho,2}^q \\ \leq\ & c(q,c(\rho))\int_0^t \mathbf{E}||\sigma(s,\bar{r}_s^{0,M})||_{\rho,2}^q\,ds \\ \leq\ & c(q,T,c(\rho),c_\sigma)\left(1+\int_0^t \mathbf{E}||\bar{r}_s^{0,M}||_{\rho,2}^q\,ds\right)\end{aligned}$$

The procedure in the estimates is the same as in the proof of 2.3.2(i) except for the fact that there is no $\gamma$-dependence of the constant belonging to $\tilde{I}_M^{(3)}$,since $\gamma = 0$ in the nuclear case.
Thus for arbitrary $T > 0$ and $t \in [\,0,T\,]$

$$\begin{aligned}\mathbf{E}||\bar{r}_t^{0,M}||_{\rho,2}^q \leq\ & c(q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q) \\ & +c(q,T,c(\rho),c_f,c_\sigma)\int_0^t \mathbf{E}||\bar{r}_s^{0,M}||_{\rho,2}^q\,ds\end{aligned}$$

s.t. Gronwall‘s lemma implies

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{r}_t^{0,M}||_{\rho,2}^q \leq c(q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

for arbitrary $T > 0$ and by the M-independence of the constant

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in N}} \mathbf{E}||\bar{r}_t^{0,M}||_{\rho,2}^q \leq c(q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

holds true as well.Defining $\underline{r}^{N,M}$,N,M$\in \mathbf{N}$,analogously to the proof of 2.3.2 one gets

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in \mathbf{N}}} \mathbf{E}||\underline{r}_t^{N,M}||_{\rho,2}^q \leq c(N,q,t,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

and by the comparison theorem one gets an estimate of that kind for $r^{N,M}$ and arbitrary $T > 0$.

As in the proof of 2.3.2(i) there are processes $r^N$,$\bar{r}$ and $\underline{r}^N$ with

$$\underline{r}_t^N \leq r_t^N \leq \bar{r}_t \text{ P-a.s.,} t \in [\,0,T\,]$$

$$\underline{r}_t^N \leq V(t) \leq \bar{r}(t) \text{ P-a.s.,} t \in [\,0,T\,]$$

for all $N \in \mathbf{N}$ with the following properties:

$$\begin{aligned}\lim_{M\to\infty}\int_0^T \mathbf{E}||r_t^{N;M} - r_t^N||_{\rho,2}^q\,dt = 0 \\ \lim_{M\to\infty}\int_0^T \mathbf{E}||\bar{r}_t^{0,M} - \bar{r}_t||_{\rho,2}^q\,dt = 0 \\ \lim_{M\to\infty}\int_0^T \mathbf{E}||\underline{r}_t^{N,M} - \underline{r}_t^N||_{\rho,2}^q\,dt = 0\end{aligned}$$

for arbitrary $T > 0$.For such $T > 0$ one has analogously to the proof of 2.3.2

$$\mathbf{E}||r_t^N||_{\rho,2}^q \leq c(N,q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q), t \in [\,0,T\,]$$

Furthermore $r^N$ solves Eq.(h,$F_N$,$\Sigma$) in the sense of 2.2.3 for this and thus for all $T > 0$ and $(r_t^N)_{t\geq 0}$ exists for all $N \in \mathbf{N}$,since except T all constants are time-independent.
As in the proof of 2.3.2(i) one gets

$$\mathbf{E}||\underline{r}_t^N||_{\rho,2}^q \leq c(q)(\underline{I}^{(1)}(t) + \underline{I}_N^{(2)}(t) + \underline{I}_N^{(3)}(t))$$

for arbitrary $t \in [\,0,T\,]$ with $T > 0$ and

$$\begin{aligned}
\sup_{t\in[\,0,T\,]} \underline{I}^{(1)}(t) &:= \sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{S}(t)\xi^-||_{\rho,2}^q \leq c(q,c(\rho))\mathbf{E}||\xi||_{\rho,2}^q \\
\underline{I}_N^{(2)}(t) &:= \mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)F_N^-(s,\cdot,\underline{r}_s^N)\,ds\right|\right|_{\rho,2}^q \\
&\leq c(q,c(\rho))\int_0^t \mathbf{E}||F_N^-(s,\cdot,\underline{r}_s^N)||_{\rho,2}^q\,ds \\
&\leq c(q,c(\rho))\int_0^t \mathbf{E}||g(\underline{r}_s^N)||_{\rho,2}^q\,ds \\
&\leq c(q,T,c(\rho),c_f)\left(1+\int_0^t \mathbf{E}||\underline{r}_s^N||_{\rho,2}^q\,ds\right) \\
\underline{I}_N^{(3)}(t) &:= \mathbf{E}\left|\left|\int_0^t \bar{S}(t-s)\Sigma(s,\underline{r}_s^N)\,dW(s)\right|\right|_{\rho,2}^q \\
&\leq c(q,c(\rho))\int_0^t \mathbf{E}||\sigma(s,\underline{r}_s^N)||_{\rho,2}^q\,ds \\
&\leq c(q,T,c(\rho),c_\sigma)\left(1+\int_0^t \mathbf{E}||\underline{r}_s^N||_{\rho,2}^q\,ds\right)
\end{aligned}$$

Then Gronwall's lemma leads to

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\underline{r}_t^N||_{\rho,2}^q \leq c(q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

Completely analogous to the proof of 2.3.2(i) one defines a process r solving Eq.($\xi$,F,$\Sigma$) in the sense of 2.2.3 for arbitrary $T > 0$ and,since all constants except T are time-independent,one even has existence of $r = (r_t)_{t\geq 0}$.
Since $\gamma = 0$ in the nuclear case the estimate for the solution process X in the proof of 2.3.2(i) implies

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||r_t||_{\rho,2}^q \leq c(q,T,c(\rho),c_f,c_\sigma)(1+\mathbf{E}||\xi||_{\rho,2}^q)$$

for all $T > 0$.Thus by replacing the stochastic initial condition $\xi$ by the deterministic initial condition $h \in L_\rho^2$ the proof is finished.
q.e.d.

As a conclusion one can say,that the spaces $L_\rho^2$ are not that appropriate for considering existence questions in the Heath-Jarrow-Morton model as the spaces $H_w$ resp. $H_w^0$ from [Te],since the properties of the shift-semigroup are such,that they only allow the existence of a solution $(r_t)_{t\in[\,0,T\,]}$ for fixed $T > 0$ and not for a solution $(r_t)_{t\geq 0}$.

For the modified Heath-Jarrow-Morton equation described in this section there exists a solution $(r_t)_{t\geq 0}$,if the conditions for f and $\sigma$ are T-independent.
Here the shift-semigroup was modified by adding a dampening factor,which becomes larger with growing time.
With $r_t(x)$ denoting the rate one can contract for in t for a bond with start in t+x and payoff an infinitesimal unit later the modification means,that the promised rate is in comparison to the model in [HeJaMo] dampened the stronger the larger the time t,when the contract is made,is,i.e. with raising t the writers of the contracts become more carefully with their offered contracts,which for a single writer could be interpreted as a sinking risk affinity with growing age.

### (ii) The existence and uniqueness claim in [AsMa]

What is interesting about the result in [AsMa] is,that it is an existence (and uniqueness) result with a non-Lipschitzian drift.As it was already mentioned in chapter 2,Assing and Manthey show their result on spaces $L^p_\rho$ with
$p := \max(2, \nu)$.But they show the existence of a solution to the equation

$$u(t) = T(t)\xi + \int_0^t T(t-s)F(u(s))\,ds + \int_0^t T(t-s)\Sigma(u(s))\,dW(s)\ ,\ t \geq 0$$

where the semigroup $(T_t)_{t\geq 0}$ is given by

$$\begin{aligned}[T(t)u](x) :=& \int_{\mathbf{R}^d} G(t, x-y)u(y)\,dy \\ :=& \int_{\mathbf{R}^d} (4\lambda\pi t)^{-\frac{d}{2}} \exp -\tfrac{|x-y|^2}{4\lambda\pi} u(y)\,dy\ ,\ t \geq 0 \quad (3.19)\end{aligned}$$

and f defining F fulfills the conditions

$$(f1) \qquad |f(x) - f(y)| \leq c_f|x-y|(1 + |x|^{\nu-1} + |y|^{\nu-1})\ ,\ x, y \in \mathbf{R}\ ,\ c_f > 0$$

which implies (PG) with exponent $\nu$ (cf.[AsMa],section 2,remark 1(c),p.241),and

$$(f2) \qquad uf(u) \leq \kappa(1 + |u|^2)\ ,\ u \in \mathbf{R}\ ,\ \kappa > 0$$

(cf.section 2,theorem 1,p.240/241 there).

Further more remark 1(c) from [AsMa] tells that condition (f1) goes back to [Ma] and ensures pathwise uniqueness.
But considering the proof of the uniqueness result from [Ma] it strikes,that one estimates by (f1) in the following way there:

$$\begin{aligned}\mathbf{E}\mathbf{1}_{t\leq\tau_N}(u(t,x)-v(t,x))^2 \leq\quad & c(N,K,\nu)\Bigg(\Bigg(\int_0^t\int_{\mathbf{R}^d} G(t-s,x-y)l^{2\nu-2}(y)\,dy\,ds\Bigg)\\ & *\Bigg(\int_0^t\int_{\mathbf{R}^d} G(t-s,x-y)\mathbf{1}_{s\leq\tau_N}|u(s,y)-v(s,y)|\,dy\,ds\Bigg)\\ & +\mathbf{E}\mathbf{1}_{t\leq\tau_N}\left[\int_0^t\int_{\mathbf{R}^d} G(t-s,x-y)(\sigma(u(s,y))-\sigma(v(s,y)))\,dW(s,y)\right]^2\Bigg)\end{aligned}$$

Here l is given by(cf.[Ma],section 4.3,p.32)

$$l(x) = 1+|x|^n\ ,\ x\in\mathbf{R}^d \text{ for a } n\in\mathbf{N}$$

$$\tau_N := \inf_{t\in[0,T]}(|u(t,\cdot)|_l^{\nu-1}+|v(t,\cdot)|_l^{\nu-1} > N),\inf\emptyset := T,N\in\mathbf{N}$$

with $|\cdot|_l$ defined by(cf. [Ma],chapter 1,p.17)

$$|\varphi|_l := \sup_{x\in\mathbf{R}^d} l^{-1}(x)\varphi(x)$$

In order to have this estimate one needs

$$\int_0^t\int_{\mathbf{R}^d} G(t-s,x-y)l^{2\nu-2}(y)\,dy\,ds < \infty$$

for arbitrary $x\in\mathbf{R}^d$.I.e. the proof can only work if the semigroup is of type (3.19).
Since both the shift-semigroup and $\bar{S}$ are not of this type,the uniqueness result from [AsMa] is not applicable to the situation of the general resp. modified Heath-Jarrow-Morton equation.
Concluding the above considerations one can say,that in the case considered in [AsMa] there is a unique solution,but for the uniqueness results one needs a condition for f,which is stronger than (PG),and a dissipativity condition.

Considering the general suitability of the spaces $L^p_\rho$ one can say the following:
By (3.15) resp. (3.16) one gets in some sense restrictions to the absolute value of $r_t(x) := \bar{f}(t,t+x)$,$x\in\mathbf{R}_+$.Noting that $r_t(x)$ denotes the rate from time t onwards to time t+x,this restriction makes sense,as one can not get arbitrarily rich within a fixed time-window,here the one from t to t+x.
Note that in $L^2_\rho$,since the dampening given by $\mu_\rho$ is the stronger the larger x is,functions f with $f(x) < f(y)$ for $x < y$ are preferred,which makes sense,since the interpretation for $r_t(x) > r_t(y)$ for $x < y$ would be,that between t and t+x one can contract for a higher rate than between t and t+y,as a consequence of which there would be negative spot rates between t+x and t+y,which is not part of the Heath-Jarrow-Morton model.
So in general the spaces $L^p_\rho$ are suitable as state space for the Heath-Jarrow-Morton model.

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