A Gibbsian Approach to Market Demand with Locally Interacting Agents

Diplomarbeit

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The individual is foolish;
the multitude, for the moment is foolish,
when they act without deliberation;
but the species is wise, and, when time is given to it,
as a species it always acts right.
Edmund Burke, 1729-1797

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Bielefeld, August 2005 *Daniel Engelage*

Contents

Chapter 1

Introduction

Background & Objective

A commonly posed question in demand theory asks for properties of individual demand that are inherited by aggregate demand. A discussion of this question may for example be found in [Mas-Colell et al., 95].

However, as declared in the preface to [Hildenbrand, 94], one should rather ask for properties that are satisfied by aggregate demand but not necessarily by individual demands, i.e. properties that are created by aggregation. The idea that is mainly followed is to obtain "well behaved" aggregate demand for "sufficiently heterogeneous" agents in large economies, as "creation" of demand structure is an effect of infinite populations. This perception is conducted for example in [Hildenbrand, 94], [Trockel, 92] or [Trockel, 84]. One may ask for the purpose of "nicely behaved" market demand: Several structural properties are needed to show existence, uniqueness or stability of market equilibrium. Although the properties of market demand that are achieved differ from those that we want to obtain within this diploma thesis, a justification for seeking after structural properties of market demand may be found in [Trockel, 84], pp. 3-5.

For our analysis of market demand, we assume an economic agent to be characterized by a demand-income-pair. In [Hildenbrand, 83], heterogeneity of agents is obtained by dispersion of income. However, the approach that we follow here originates from [Grandmont, 92], where the analysis of aggregate demand relies on the diversification of demand: For this purpose, Grandmont introduces transforms of the commodity space, so called α-transforms. In this context, the "better behaved" market demand becomes, the more spread out the distribution of demand; more precisely of α-transforms. Nevertheless, both approaches show that market demand "is close to" satisfy the *uncompensated law of demand* and thus the *weak axiom of revealed preference*, the "more heterogeneous" the agents are. However, in both approaches, the distributions of income or demand, respectively, are given exogenously.

In this thesis, we obtain an economic model where the distribution of individual states, i.e. demand functions, is obtained endogenously by virtue of a local interaction structure. We then apply the analysis conducted in [Grandmont, 92] to this economy and encounter the parameters relevant for aggregate demand to satisfy the weak axiom of revealed preference. We see that interaction has to be "sufficiently small" for the results in [Grandmont, 92] to hold.

Outline of the Proceeding

As we will see, there is a close relation of socioeconomics with statistical mechanics. This motivates the use of Gibbsian theory. Therefore, we refer to our approach to model interacting agents as Gibbsian, although it traces back to Dobrushin, Lanford and Ruelle and is hence called the DLR approach in statistical mechanics: We are given some graph $\mathbf{S}$ with agents "sitting" at each site. The edges represent possible social interconnections. The local interaction structure is generated by a family of conditional distributions of states for finite subpopulations given the states of the remainders. We then obtain a global distribution of individual states just as a distribution that is consistent with local distributions. Such consistent global distribution is then called a Gibbs measure.

When conducting this approach to aggregate demand analysis, we have to be very specific with the notion of economy under consideration: In [Grandmont, 92], we are given a distribution economy, i.e. a probability measure on the space of individual states $\mathbf{E}$. However, the model introduced in this thesis is a global random economy, i.e. a family $(\sigma_s)_{s\in\mathbf{S}}$ of random variables or random agents with values in $\mathbf{E}$. Hence, a global random economy is given by a probability measure on $\mathbf{E}^{\mathbf{S}}$.

We introduce the *local unbounded spin Ising economy* as an application of the *harmonic oscillator* that we conceive as a generalization of the *Ising ferromagnet*. In this sense, the economy that we consider here is a generalization of the famous Ising economy in [Föllmer, 74], when individual states consist of all possible demand functions; more precisely, of the space of α-transforms.

The economic rationale for the local interaction structure in the harmonic oscillator is the assumption that agents prefer to be similar to their peers. In other words, we assume that agents exhibit what we call a *preference for conformity*. Nevertheless, the Gibbsian approach is much more general. We could even assume random agents to have preference for antagonism in demand for some goods; or we could just assume agents to behave independent as done in [Hildenbrand, 71]. In this sense, the model discussed there is a special case of our model.

The local unbounded spin Ising economy leads to the *global unbounded spin Ising economy*, a special type of Gibbsian random economies, i.e. global random economies rules by a distribution of states that is Gibbsian with respect to an appropriately chosen interaction structure. In case of the global unbounded spin Ising economy, the local interaction structure is given by a "product" of harmonic oscillators. We furthermore assume the Gibbs measure that rules the economy to be ergodic. From this ergodic global unbounded spin Ising economy we obtain the so called *unbounded spin Ising distribution economy* by a proposition from [Hohnisch, 03]. We thus have eventually achieved a distribution economy where the distribution of agents demand is obtained endogenously and we may apply Grandmont's analysis to this distribution economy. We obtain that coupling should be weak for aggregate demand to satisfy the weak axiom of revealed preference.

Incentives for the Application of Gibbsian Theory

An integral part of this thesis is the application of random economies with a local interaction structure. This generates insights in economic behavior by enriching models in general equilibrium theory with social aspects. We hence obtain equilibrium models with agents not only interacting via the price system. Beyond doubt there are more interdependencies going on in society than generated by the market. An elaborate discussion on the *social multiplier* as an indicator of and a measure for the extent of local interactions is given in [Glaeser & Scheinkman, 01]. In [Glaeser et al., 96] the authors present a study of measuring social interactions for commitment of crimes.

Our approach stems from the belief in a formalizable social or local structure. Thus, we are enabled to savagely formalize concepts as for example preference for conformity. Social or local interactions[1] are mediated by a social structure, not by the market.

Here, we do not consider a strategic approach to model interactions. Instead, we apply the probabilistic Gibbsian framework utilizing concepts from statistical mechanics. In light of this approach, we explain situations in large socioeconomic systems where global behavior is not uniquely determined by local data. This is the case when the set of Gibbs measures is not singleton or, as denoted in statistical mechanics, when the system exhibits a phase transition.[2] In Chapter 3, we consider Föllmer's Ising economy as an example of phase transition.

However, there is a rich variety of socioeconomic models based on strategic interaction. A fundamental example in sociology was Schelling's neighborhood model. Here, individually optimal behavior leads to a globally suboptimal outcome. The model explains segregation of neighborhoods in ethnic and social classes. A formal strategic setup of this model can be found in [Young, 98]. It is shown that segregated neighborhoods are most likely to

[1] Henceforth, we use these notions synonymous.

[2] A short but very explanatory compendium of notions from statistical mechanics can be found in [Durlauf, 01].

emerge even if every agent prefers to live in a mixed neighborhood. Other examples of such social pathologies are smoking behavior or commitment of crimes.

There are several ways to motivate the use of Gibbsian theory for socioeconomic systems. One is presented in [Kindermann & Snell, 80] and is predicated on the study of social networks by Holland, Leinhardt and Wassermann. Kindermann and Snell show that the probability measures describing aggregate behavior in Holland's, Leinhardt's and Wassermann's model are Gibbs measures induced by a nearest neighbor potential. A second fundamental model that leads to Gibbs measures as an appropriate notion of aggregate states in an economy with local interactions is the *binary choice model* as discussed in [Brock & Durlauf, 00a].

The objective of interactions-based models is best formulated as follows:

> The object of a typical exercise using these models is to understand the behavior of a population of economic actors rather than that of a single agent. [...] Interactions models typically specify an explicit probability measure characterizing individual behavior conditional on exogenous (to the individual) characteristics [...] and an interaction structure that specifies who affects whom. [...] The goal of the analysis is to characterize a joint probability measure over all agents in the population that is compatible with the conditional probability measures. [...] The reasoning from probabilistic descriptions of individual behavior to population wide aggregates provides a way to explicitly understand the emergence of collective properties in an economy. ([Brock & Durlauf, 01], pp. 18-20)

We have already stated that Gibbsian theory was first encountered in statistical mechanics. Hence, if we can show analogies of socioeconomic models and models in statistical mechanics, we have a motivation for the use of Gibbsian theory: In statistical mechanics, the question considered is for the aggregate behavior of a collection of correlated particles. One famous example is the Ising ferromagnet: For a piece of iron to be a magnet we need the alignment of a great majority of atoms' spins. An atom can either exhibit spin "up" or "down". However, there does not seem to be any tendency for spinning in one direction when considering one atom on its own. Assuming independence, the law of large numbers would imply that the existence of magnets in nature is very unlikely. The fundamental assumption in statistical mechanics is that there exist interdependencies for spins; more concretely, the likelihood of an atom to exhibit a specific spin is increasing in the number of neighboring atoms with the same spin. We elaborately discuss on the Ising ferromagnet in Section 2.7.

Thus, we see that the considerations in both, socioeconomics and statistical mechanics are the same. Both analyze the aggregate behavior of a system with interdependent particles or agents. This motivates the application of the same mathematical tools: the theory of Gibbs measures. In both disciplines, the modeler conjectures an interaction structure, here in form of conditional probabilities, and then obtains a (global) probabilistic description of the system consistent with the local distributions.

Notation & Structure

This thesis constitutes an interdisciplinary approach to market demand, applying concepts form economics, sociology and statistical mechanics. Although identical, some concepts are denoted different contingent on the scientific field. This of course has implications on the notation used here. Thus, we do not distinguish between the notions "microscopic", "microeconomic" or "local"; analogously for the terms "macroscopic", "macroeconomic" or "global". Aggregate demand or market demand is a macroeconomic concept. However, from the perspective of statistical mechanics, it should be a macroscopic observable. We will just use the notation that seems appropriate in the respective context.

When analyzing aggregate demand, we are concerned with its structural properties or structure for short. Here, structural properties represent rationality assumptions like the uncompensated law of demand or the weak axiom of revealed preference.

Chapter 2 gives a comprehensive overview of Gibbsian theory. We give basic definitions and results, in particular extreme and ergodic decomposition of Gibbs measures. In most instances, this chapter is adapted from [Georgii, 88]. We also introduce the notion of a product specification and its corresponding Gibbs measures. This concept is indispensable for the economic models considered within this work. A small but nevertheless important paragraph contains a discussion showing that only extremal Gibbs measures should be considered in physical as well as social systems. The last part of the chapter is devoted to the Ising model.

Chapter 3 is a survey of demand theory. Again, we state basic definitions and results. We then consider large economies and appropriate notions for their characterization: We define distribution and random economies and draw their conceptual distinction. However, we discuss a result connecting these concepts when random agents are independent. The last part of the chapter deals with Föllmer's analysis of interacting agents as it can be found in [Föllmer, 74]. In the course of this, we discuss the famous Ising economy.

In Chapter 4 we examine the concepts and results in [Grandmont, 92] in extenso. Grandmont introduces transforms of the commodity space, so called α-transforms. Thus, transformations of demand functions are obtained and it is shown that market demand is "well behaved" if the distributions of α-transforms of demand are spread out; in this sense, agents are called heterogeneous. Besides elaborating technical but nevertheless important proofs from [Grandmont, 92], we generalize a uniqueness result for market exchange equilibria in [Arrow & Hahn, 71] and conduct a further discussion on connections between distinct structural properties of demand. Grandmont states a result that gives necessary and sufficient conditions for a demand function to be invariant with respect to α-transforms; we state a proof here, that follows the idea in [Trockel, 89].

Based on [Georgii, 88], we define a particular kind of local interaction structure in the first part of Chapter 5: Gaussian specifications. We explicitly characterize the set of homogeneous Gibbs measures for homogeneous Gaussian specifications. We then consider the harmonic oscillator as a generalization of the Ising ferromagnet. In the second part, we introduce the economic concepts that are needed for our approach to market demand analysis: We first introduce local random economies, i.e. a configuration space together with a specification, and then consider l-fold local random economies. These give rise to Gibbsian random economies, basically given by a distribution on the configuration space that is Gibbsian with respect to a (product) specification of the corresponding (l-fold) local random economy. As a particular example, we introduce the *global unbounded spin Ising economy.* We interpret this economic notion as a generalization of Föllmer's Ising economy taking into account individual states as assumed in Grandmont's analysis, i.e. the space of all demand-income-pairs.

In Chapter 6 we first discuss the connection of random economies with distribution economies. We apply a result in [Hohnisch, 03] to obtain an identification of ergodic random economies with distribution economies. This gives rise to the unbounded spin Ising distribution economy corresponding to the ergodic global unbounded spin Ising economy. The unbounded spin Ising distribution economy is what we were looking for: For this type of economy, where the distribution of demand behavior is endogenized, we can conduct Grandmont's analysis . We then discuss two different approaches to the aggregation problem: The first is related to the thermodynamic limit as commonly considered in statistical mechanics. The second is a direct application of Grandmont's analysis and in a way the antipode of the first. However, the second approach establishes a stronger result in the sense that the weak axiom for market demand is created, whereas the first approach "only" allows for the weak axiom to be inherited by market demand.

Chapter 2

Theory of Gibbs Measures

2.1 Introduction

We have already mentioned the close relationship between statistical mechanics and socioeconomic models when agents interact locally. In analogy to statistical mechanics this suggests the use of Gibbs measures as an appropriate framework for modeling equilibria in socioeconomic models. Since Gibbsian theory results from considerations in statistical mechanics the examples given in this chapter mainly refer to physics, as the famous Ising model. In the course of this thesis, we motivate economic interpretations of notions introduced here. However, a rigorous definition of several physical concepts would confuse more than help understanding the general theory needed in economics; in these cases we give an intuition and refer the interested reader to [Georgii, 88], [Georgii, 79], [Preston, 74] or [Preston, 76]. This chapter serves as a very brief introduction to Gibbsian theory. Besides a rigorous discussion, the aim is to generate some intuition for Gibbsian theory.

In this chapter basic definitions and properties of Gibbs measures are stated. These will lead us to some remarkable results that are useful for socioeconomic models in explaining apparently puzzling social phenomena. The subsequent survey on Gibbsian theory is based on [Georgii, 88]. An elaborate historical introduction can be found in [Kindermann & Snell, 80] and [Kindermann & Snell, 80b]. Here, we introduce the notion of a product specification and consider Gibbs measures for these product specifications given by product measures, each factor a Gibbs measure with respect to a corresponding factor of the product specification. Product specifications and corresponding Gibbs measures turn out to be vital for the unbounded spin Ising economy considered in Section 5.3.

Basic Ideas

As introduced by Dobrushin, Lanford and Ruelle the basic idea of a Gibbs measure is that of being an appropriate notion of a (macroscopic) equilibrium in a system consisting of a huge number of interacting components. When the first steps were done in developing the theory of Gibbs measures, research was concentrated on models in statistical mechanics. In recent years there seems to be interest in this theory by sociologists thinking of systems where components (agents) interact locally.

> A Gibbs measure is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components. In the language of Probability Theory, a Gibbs measure is simply the distribution of a stochastic process which, instead of indexed by time, is parametrized by the sites of a spatial lattice, and has the special feature of admitting prescribed versions of the canonical distributions with respect to the configurations outside finite regions. ([Georgii, 88], p. 5)

For an example from Physics consider the famous Ising ferromagnet as introduced in a Section 2.7: We are given a piece of ferromagnetic metal consisting of a large number of atoms each showing spin "up" or "down". Adjacent atoms interact in the way that their spins have the tendency to conform. For high enough temperatures the thermal motion of atoms circumvents the system from the state where all spins are parallel since interaction is not strong enough relative to thermal motion. However, as temperature falls below some critical value the spins align and there can be seen the phenomenon of spontaneous magnetization. When there is no external field, we do not know a priori how the magnet is poled. Global distributions of spins that are consistent with the local interaction structure constitute equilibria of the model.

Following our intuition that equilibrium states are described by Gibbs measures:

> We thus expect that the physical phenomenon of phase transition should be reflected in our mathematical model by the non-uniqueness of the Gibbs measures for a prescribed specification. ([Georgii, 88], p. 6)

Here, a specification is the appropriate way of describing local interactions via conditional distributions.

2.2 Specification of Random Fields

Let $\mathbf{S}$ be a countably infinite set and $(\mathbf{E}, \mathcal{E})$ an arbitrary measurable space. Later on, we assume $\mathbf{S}$ to exhibit some graph structure.

Definition 2.1. ([Georgii, 88], Definition 1.1) *A family* $(\sigma_i)_{i\in\mathbf{S}}$ *of random variables which are defined on some probability space* $(\Omega, \mathcal{F}, \mu)$ *and take values in* $(\mathbf{E}, \mathcal{E})$ *is called a* random field. *The index set* $\mathbf{S}$ *is called the* parameter set *and* $(\mathbf{E}, \mathcal{E})$, *or* $\mathbf{E}$ *for short, the* state space *of the random field.*

We will use a canonical version of a random field by assuming

$$\begin{aligned}\Omega := \mathbf{E}^{\mathbf{S}} &:= \{\omega = (\omega_i)_{i\in\mathbf{S}} : \omega_i \in \mathbf{E}\},\\ \mathcal{F} &:= \mathcal{E}^{\mathbf{S}},\\ \sigma_i &: \Omega \to \mathbf{E}, \omega \mapsto \omega_i,\end{aligned}$$

the projection on the i^{th} coordinate. Thus, a random field is equivalently defined as a probability measure μ on $(\Omega, \mathcal{F})$ and the set of random fields is denoted by the set $\mathcal{P}(\Omega, \mathcal{F})$ of probability measures on $(\Omega, \mathcal{F})$.

Furthermore, for $\Lambda \subset \mathbf{S}$ let $\sigma_\Lambda : \Omega \to \mathbf{E}^\Lambda$ be the projection on Λ, i.e. $\sigma_\Lambda(\omega) := \omega_\Lambda := (\omega_i)_{i\in\Lambda}$ for $\omega = (\omega_i)_{i\in\mathbf{S}} \in \Omega$.
Now, let

$$\mathcal{S} := \{\Lambda \subset \mathbf{S} : 0 < |\Lambda| < \infty\},$$

where $|\Lambda|$ denotes the cardinality of Λ. $\Lambda \in \mathcal{S}$ is called a *finite volume.*

For $\Delta \subset \mathbf{S}$ we consider $\mathcal{F}_\Delta$ to be the σ-algebra generated by

$$\{\sigma_\Lambda \in A\} \quad (\Lambda \in \mathcal{S}, \Lambda \subset \Delta, A \in \mathcal{E}^\Lambda),$$

i.e. by the events in Δ. $\mathcal{F}$ is the smallest σ-algebra containing the cylinder sets $\{\sigma_\Lambda \in A\}$, $(\Lambda \in \mathcal{S}, A \in \mathcal{E}^\Lambda)$.

Definition 2.2. ([Georgii, 88], pp. 13,14) (a) *Let* $(X, \mathcal{X}, \mu)$ *be a measure space,* $g : X \to \mathbb{R}$ *measurable. We write*

$$\mu(g) := \int g d\mu$$

and analogously $\mu(f|\mathcal{C})$ *for the conditional expectation, where* $\mathcal{C}$ *is a sub-*σ*-algebra of* $\mathcal{X}$*.*
(b) *Let* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ *be measurable spaces. A function* $\pi : \mathcal{X} \times Y \to [0, \infty]$ *is called a* (measure) kernel *from* $(Y, \mathcal{Y})$ *to* $(X, \mathcal{X})$ *(or simply from* $\mathcal{Y}$ *to* $\mathcal{X}$*) if*

1. $\pi(\cdot|y)$ *is a measure on* $(X, \mathcal{X})$ *for all* $y \in Y$*, and*

2. $\pi(A|\cdot)$ *is* $\mathcal{Y}$*-measurable for each* $A \in \mathcal{X}$

If, in addition, $\pi(X|\cdot) = 1$ *then* π *is called a* probability kernel. *A probability kernel* π *from a sub-*σ*-algebra* $\mathcal{C}$ *of* $\mathcal{X}$ *to* $\mathcal{X}$ *is called* proper *if the condition* $\pi(C|\cdot) = 1_C(\cdot)$ *holds for all* $C \in \mathcal{C}$*. Intuitively, properness means when starting in* $C \in \mathcal{C}$ *the kernel leads back to* C *for sure.*

Remark 2.3. ([Georgii, 88], p. 13) *Such a kernel π "shifts" a measure μ on $(\mathbf{Y}, \mathcal{Y})$ to a measure $\mu\pi$ on $(\mathbf{X}, \mathcal{X})$ when defined by*

$$\mu\pi(A) := \int \pi(A|\cdot)d(\mu) \quad \forall A \in \mathcal{X}.$$

Remark 2.4. ([Georgii, 88], Remark 1.20) *Let $(\mathbf{X}, \mathcal{X})$ be a measurable space, $\mathcal{C}$ a sub-σ-algebra of $\mathcal{X}$, π a proper probability kernel from $\mathcal{C}$ to $\mathcal{X}$, and $\mu \in \mathcal{P}(\mathbf{X}, \mathcal{X})$. Then*

$$\mu(A|\mathcal{C}) := \mu(1_A|\mathcal{C}) = \pi(A|\cdot) \quad \mu\text{-a.s.} \quad \forall A \in \mathcal{X}$$

if and only if $\mu\pi = \mu$.

Proof. [Georgii, 88], p. 15 □

Remark 2.5. ([Georgii, 88], p. 14) *Let $(Z, \mathcal{Z})$ be a third measurable space and let π_1 and π_2 be kernels from $\mathcal{Z}$ to $\mathcal{Y}$ and $\mathcal{Y}$ to $\mathcal{X}$, respectively. Then*

$$\pi_1\pi_2(A|z) := \int \pi_2(A|y)\pi_1(dy|z) \quad (A \in \mathcal{X}, z \in Z)$$

is a kernel from $\mathcal{Z}$ to $\mathcal{X}$

Having in mind our considerations on systems with interacting components our main interest should be in random fields where the spin variables σ_i are not necessarily independent. This dependence is formalized in a probabilistic manner: We consider distributions of finite collections of spins conditional on the values of remainder spins, i.e. we consider a family of distributions conditional on $\mathcal{T}_\Lambda := \mathcal{F}_{\mathbf{S}\setminus\Lambda}$, $\Lambda \in \mathcal{S}$. We furthermore set $\mathcal{T} := \cap_{\Lambda\in\mathcal{S}}\mathcal{T}_\Lambda$ as the *tail-field-σ-algebra*. One may think of $\mathcal{T}_\Lambda$ as the σ-algebra generated by boundary conditions for Λ or in other words by the information contained in configurations outside of Λ.

Definition 2.6. ([Georgii, 88], Definition 1.23) *A* specification *with parameter set $\mathbf{S}$ and state space $(\mathbf{E}, \mathcal{E})$ is a family $\gamma = (\gamma_\Lambda)_{\Lambda\in\mathcal{S}}$ of proper probability kernels γ_Λ from $\mathcal{T}_\Lambda$ to $\mathcal{F}$ which satisfy the consistency condition $\gamma_\Delta\gamma_\Lambda = \gamma_\Delta$ when $\Lambda \subset \Delta$. The random fields in the set*

$$\mathcal{G}(\gamma) = \{\mu \in \mathcal{P}(\Omega, \mathcal{F}) : \mu(A|\mathcal{T}_\Lambda) = \gamma_\Lambda(A|\cdot) \quad \mu\text{-a.s. } \forall\ A \in \mathcal{F} \text{ and } \Lambda \in \mathcal{S}\}$$

are then said to be specified *or to be* admitted *by γ.*[1]

Let $\omega \in \Omega$ such that 1_ω is $\mathcal{T}_\Lambda$ measurable. Then $\gamma_\Lambda(\cdot|\omega)$ is a probability distribution on $(\Omega, \mathcal{F})$ conditional on the boundary condition ω. One may

[1]Often times, these are already called Gibbs Measures.

think of the consistency condition as it is given in [Georgii, 79] in a more special context but generating much more intuition than that above:

$$\gamma_\Delta(\xi|\omega) = \gamma_\Lambda(\xi_\Lambda|\xi\omega_{\mathbf{S}\setminus\Delta})\gamma_\Delta(\xi_{\Delta\setminus\Lambda}|\omega)$$

for $\Lambda \in \Delta$, $\xi \in \Omega_\Delta$, $\omega \in \Omega$.

In the economic model that will be analyzed in Chapter 6 we actually do not consider a system with local interactions given by a specification but a "product" of $l \in \mathbb{N}$ of those. Thus, we have to undertake the following digression. Given an arbitrary set $\mathbf{X}$, $\mathbf{X}^2$ denotes the product set $\mathbf{X} \times \mathbf{X} := \{(x_1, x_2) : x_1, x_2 \in \mathbf{X}\}$. Similarly, for a σ-algebra $\mathcal{X}$, $\mathcal{X}^2$ denotes the product-σ-algebra.

Proposition 2.7. *Let $\gamma_1 := (\gamma_{1\Lambda})_{\Lambda\in\mathcal{S}}$ and $\gamma_2 := (\gamma_{2\Lambda})_{\Lambda\in\mathcal{S}}$ be specifications on $\mathbf{S}$ with state space $(\mathbf{E}, \mathcal{E})$. The product specification $\gamma := \gamma_1 \otimes \gamma_2$ defined by*

$$\gamma_\Lambda(A_1 \times A_2|\omega) = \gamma_{1\Lambda}(A_1|\omega_1)\gamma_{2\Lambda}(A_2|\omega_2), \quad (A_1, A_2 \in \mathcal{F}, \omega_1, \omega_2 \in \Omega),$$

$\omega := (\omega_1, \omega_2) := (\omega_{1i}, \omega_{2i})_{i\in\mathbf{S}} \in \Omega^2$, is a specification on $\mathbf{S}$ with state space $(\mathbf{E}^2, \mathcal{E}^2)$.

Proof. We first have to show that each γ_Λ, $\Lambda \in \mathcal{S}$, is a proper probability kernel from $\mathcal{T}_\Lambda^2$ to $\mathcal{F}^2$. However, the conditions needed are inherited when taking products: $\gamma_\Lambda(\cdot|\omega)$ is a measure on $(\Omega, \mathcal{F})$ for every $\omega \in \Omega$, $\gamma_\Lambda(A, \cdot)$ is $\mathcal{T}_\Lambda^2$-measurable for each $A \in \mathcal{F}$ since each $\gamma_{i\Lambda}(A_i|\cdot)$ is $\mathcal{T}_\Lambda$-measurable. Furthermore, we have $\gamma_\Lambda(\Omega^2|\cdot) = \gamma_{1\Lambda}(\Omega|\cdot)\gamma_{2\Lambda}(\Omega|\cdot) = 1$. At last, for any $B_1, B_2 \in \mathcal{T}_\Lambda$ we have $\gamma_\Lambda(B_1 \times B_2|\cdot,\cdot) = 1_{B_1}(\cdot)1_{B_2}(\cdot) = 1_{B_1\times B_2}(\cdot,\cdot)$. Thus, γ is a proper probability kernel.

Second, we have to verify the consistency condition for γ: Let $A = A_1 \times A_2 \in \mathcal{F}$, $\omega = (\omega_1, \omega_2) \in \Omega^2$, $\Lambda \subset \Delta$. Then

$$\begin{aligned} \gamma_\Delta\gamma_\Lambda(A|\omega) &= \int_{\Omega^2} \gamma_\Lambda(A|\tilde\omega)\gamma_\Delta(d\tilde\omega|\omega) \\ &= \int_\Omega \gamma_{1\Lambda}(A_1|\tilde\omega_1)\gamma_{1\Delta}(d\tilde\omega_1|\omega_1) \int_\Omega \gamma_{2\Lambda}(A_2|\tilde\omega_2)\gamma_{2\Delta}(d\tilde\omega_2|\omega_2) \\ &= \gamma_{1\Delta}(A_1|\omega_1)\gamma_{2\Delta}(A_2|\omega_2) = \gamma_\Delta(A|\omega). \end{aligned}$$

□

Proposition 2.8. *Let $\gamma_1 := (\gamma_{1\Lambda})_{\Lambda\in\mathcal{S}}$ and $\gamma_2 := (\gamma_{2\Lambda})_{\Lambda\in\mathcal{S}}$ be specifications on $\mathbf{S}$ with state space $(\mathbf{E}, \mathcal{E})$. Define $\gamma := \gamma_1 \otimes \gamma_2$ as in Proposition 2.7. Let $\mu_i \in \mathcal{G}(\gamma_i)$, $i = 1, 2$. Then $\mu := \mu_1 \otimes \mu_2 \in \mathcal{G}(\gamma)$, where this product measure is defined as usual by*

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \quad (A_1, A_2 \in \mathcal{F}).$$

Proof. Let $A_1, A_2 \in \mathcal{F}$ and thus $A := A_1 \times A_2 \in \mathcal{F}^2$, $\Lambda \in \mathcal{S}$.

$$\begin{aligned} \mu\gamma_\Lambda(A_1 \times A_2) &= \int_{\Omega^2} \gamma_\Lambda(A|\cdot)d\mu \\ &= \int_\Omega \gamma_{1\Lambda}(A_1|\cdot)d\mu_1 \int_\Omega \gamma_{2\Lambda}(A_2|\cdot)d\mu_2 \\ &= (\mu_1\gamma_{1\Lambda})(A_1)(\mu_2\gamma_{2\Lambda})(A_2) \\ &= \mu_1(A_1)\mu_2(A_2) = \mu(A_1 \times A_2). \end{aligned}$$

Thus, Remark 2.4 yields the assertion. □

Corollary 2.9 (of Remark 2.4). *Given a specification γ, we have*

$$\mu \in \mathcal{G}(\gamma) \quad \Leftrightarrow \quad \mu\gamma_\Lambda = \mu \quad \forall \Lambda \in \mathcal{S}$$

From [Georgii, 88], Notation 1.26, we take the following definitions: For any given a priori or reference measure $\lambda \in \mathcal{M}(\mathbf{E}, \mathcal{E})$, the set of all σ-finite measures on $(\mathbf{E}, \mathcal{E})$ with $\lambda(\mathbf{E}) > 0$, define the family $\lambda. = (\lambda_\Lambda)_{\Lambda \in \mathcal{S}}$ of measure kernels λ_Λ from $\mathcal{T}_\Lambda$ to $\mathcal{F}$ by

$$\lambda_\Lambda(\cdot|\omega) := \lambda^\Lambda \times \delta_{\omega_{\mathbf{S}\setminus\Lambda}}(\cdot) \quad (\Lambda \in \mathcal{S}, \omega \in \Omega), \tag{2.1}$$

where λ^Λ is defined as the product measure of λ on $(\mathbf{E}^\Lambda, \mathcal{E}^\Lambda)$. Obviously, this constitutes a kernel. Let $\rho = (\rho_\Lambda)_{\Lambda \in \mathcal{S}}$ be a family of densities. We then define $\rho_\Lambda \lambda_\Lambda$ by

$$\rho_\Lambda \lambda_\Lambda(A|\cdot) := \int_A \rho_\Lambda(x)\lambda_\Lambda(dx|\cdot) \quad (A \in \mathcal{F}) \tag{2.2}$$

The a priory measure λ reflects the tendency of each particle to exhibit some specific spin when there would be no interaction among particles. Thus, when assuming that components should not exhibit any tendency, one should assume λ to be the counting measure or Lebesgue measure, depending on the state space $\mathbf{E}$.

Definition 2.10. ([Georgii, 88], Definition 1.27) *Let $\lambda \in \mathcal{M}(\mathbf{E}, \mathcal{E})$. A λ-*modification *is a family $\rho = (\rho_\Lambda)_{\Lambda \in \mathcal{S}}$ of measurable functions $\rho_\Lambda : \Omega \to [0, \infty)$ such that the family $\rho\lambda. = (\rho_\Lambda\lambda_\Lambda)_{\Lambda \in \mathcal{S}}$ is a specification. A* λ-specification *is a specification γ of the form $\gamma = \rho\lambda.$ for some λ-modifications ρ.*

Definition 2.11. ([Georgii, 88], Definition 2.2) *An* interaction potential *is a family $\Phi = (\Phi_A)_{A \in \mathcal{S}}$ of functions $\Phi_A : \Omega \to \mathbb{R}$ with the following properties:*

1. *For all $A \in \mathcal{S}, \Phi_A$ is $\mathcal{F}_A$-measurable.*

2. *For all $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$, the series*

$$H_\Lambda^\Phi(\omega) = \sum_{\substack{A \in \mathcal{S} \\ A \cap \Lambda \neq \emptyset}} \Phi_A(\omega)$$

exists.

$H_\Lambda^\Phi(\omega)$ *is called the* (total) energy *of* ω *in* Λ *for* Φ, *and* H_Λ^Φ *the* Hamiltonian *in* Λ *for* Φ. *Furthermore, we write*

$$h_\Lambda^\Phi(\omega) = e^{-H_\Lambda^\Phi(\omega)}.$$

As in [Georgii, 79], one often thinks of Φ_A as a measurable function on $(\mathbf{E}^A, \mathcal{E}^A)$.

We may now use interaction potentials to define specifications: Let $\lambda \in \mathcal{M}(\mathbf{E}, \mathcal{E})$ be an a priori measure. A potential Φ is called λ-admissible if

$$Z_\Lambda^\Phi(\omega) = \int e^{-H_\Lambda^\Phi(\xi\omega_{\mathbf{S}\setminus\Lambda})}\lambda^\Lambda(d\xi)$$

is finite for all $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$. $Z_\Lambda^\Phi(\omega)$ is called *partition function.* Let Φ be a λ-admissible potential. Then define $\rho_\lambda^\Phi := \frac{h_\Lambda^\Phi}{Z_\Lambda^\Phi}$ for all $\Lambda \in \mathcal{S}$. We obtain that $(\rho_\Lambda^\Phi)_{\Lambda\in\mathcal{S}}$ is a λ-modification. For an elaborate discussion, we refer to [Georgii, 88], pp. 18-28.

Definition 2.12. ([Georgii, 88], Definition 2.9) *Suppose* Φ *is a* λ*-admissible potential and* $\omega \in \Omega, \Lambda \in \mathcal{S}$. *Then the probability measure*

$$\begin{aligned}\mathcal{F} \ni A \mapsto \gamma_\Lambda^\Phi(A|\omega) \quad &:= \quad \rho_\Lambda^\Phi \lambda_\Lambda(A|\omega) \\ &\overset{(2.2)}{=} \quad Z_\Lambda^\Phi(\omega)^{-1}\int_{\mathbf{E}^\Lambda} e^{-H_\Lambda^\Phi(\xi\omega_{\mathbf{S}\setminus\Lambda})}1_A(\xi\omega_{\mathbf{S}\setminus\Lambda})\lambda^\Lambda(d\xi)\end{aligned}$$

on $(\Omega, \mathcal{F})$ *is called a* Gibbs distribution *in* Λ *with boundary condition* $\omega_{\mathbf{S}\setminus\Lambda}$, *interaction potential* Φ *and single spin or a priory measure* λ. *The* λ*-specification* $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda\in\mathcal{S}}$ *is called the* Gibbsian specification *for* Φ *and* λ. *Each random field* $\mu \in \mathcal{G}(\Phi) := \mathcal{G}(\gamma^\Phi)$ *is called* Gibbs measure, Gibbs state *or* Gibbs random field *for* Φ *and* λ. *We reconsider this definition is Section 5.2.*

In [Georgii, 79], the author uses a slightly different framework. However, the underlying idea is the same:

> In order to completely formulate the models [...] it is necessary to specify the interaction between the particles. The [...] commonly accepted ansatz of Gibbs yields the probability distributions for local configurations conditioned with respect to a fixed environment. The objects of our study are probability measures on Ω whose local behavior is determined by the Gibbs distributions. ([Georgii, 79], p. 1)

Our considerations on phase transition at the beginning of the current chapter suggests the following definition.

Definition 2.13. ([Georgii, 88], Definition 2.10) *A potential* Φ *will be said to exhibit a* phase transition *if* $|\mathcal{G}(\gamma^\Phi)| > 1$. *We refer to this case as* non-uniqueness *of Gibbs measures.*

When thinking of a Gibbs measure as an appropriate way to describe an equilibrium state of the system, this means that equilibrium is not unique.

Example 2.14 (Nearest-neighbor potentials on $\mathbb{Z}^d, d \geq 1$)**.** ([Georgii, 88], pp. 29,30) *Let* $\mathbf{S} = \mathbb{Z}^d$, *then* Φ *is called a* nearest neighbor potential *or* Markov potential *if* $\Phi_A = 0$ *whenever* $A \notin \{\{i,j\} \subseteq \mathbb{Z}^d : |i-j| = 1\} \cup \{i \in \mathbb{Z}^d\}$. *Thus,* Φ *is the pair-potential of range 1. A potential* Φ *is called a* pair-potential *if* $\Phi_A = 0$ *whenever* $|A| > 2$. Φ *is called a* self-potential *if* $\Phi_A = 0$ *whenever* $|A| \neq 1$. *A potential* Φ *is said to be of* range $r \in \mathbb{R}_+$ *if* $\Phi_A = 0$ *whenever the diameter of* A, *diam*$(A) > r$.

Markov Specifications

When having a look at our approach to Gibbsian theory, one may conjecture that there is a relation to the theory of Markov processes. This intuition is formalized in [Georgii, 88], Theorem 3.5. This theorem shows that a positive homogeneous Markov specification as defined below uniquely determines a Gibbs measure μ that constitutes a Markov chain. For explicit characterizations we refer to this theorem. The considerations on Markov specifications are not necessary for our further analysis. However, we may yield some intuition for Gibbs measures generated by nearest-neighbor potentials. Throughout this paragraph let $\mathbf{S} = \mathbb{Z}$, $\emptyset \neq \mathbf{E}$ be finite, $\mathcal{E} = 2^{\mathbf{E}}$, the power set of $\mathbf{E}$, and let λ be the counting measure.

Definition 2.15. ([Georgii, 88], Definition 3.1) *Let* γ *be a specification with parameter set* $\mathbb{Z}$ *and state space* $\mathbf{E}$. *We say* γ *is a* positive homogeneous Markov specification *if there is a function* $g(\cdot,\cdot,\cdot) > 0$ *on* $\mathbf{E}^3$ *such that*

$$\gamma_{\{i\}}(\sigma_i = y|\omega) = g(\omega_{i-1}, y, \omega_{i+1})$$

for all $i \in \mathbb{Z}$, $y \in \mathbf{E}$ *and* $\omega \in \Omega$.

As shown in [Georgii, 88], Theorem 1.33, by virtue of the consistency condition posed on specifications, g suffices to characterize a specification γ. Thus, g is called the *determining function* of γ.

For the following result we need some further notation.

Definition 2.16. ([Georgii, 88], p. 46) *A nearest-neighbor potential* Φ *is called* homogeneous *if there are two functions* $\varphi_1 : \mathbf{E} \to \mathbb{R}$ *and* $\varphi_2 : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ *such that*

$$\Phi_A = \begin{cases} \varphi_1(\sigma_i) & \text{if } A = \{i\}, \\ \varphi_2(\sigma_i, \sigma_{i+1}) & \text{if } A = \{i, i+1\}. \end{cases}$$

We are now fully equipped to state a proposition that relates Markov chains and Gibbs measures.

Proposition 2.17. ([Georgii, 88], Corollary 3.9) *A specification γ is a positive homogeneous Markov specification if and only if γ is Gibbsian for some homogeneous nearest-neighbor potential Φ, i.e. $\gamma_\Lambda = \gamma_\Lambda^\Phi$ for all $\Lambda \in \mathcal{S}$.*

Proof. [Georgii, 88], pp. 46,47 □

Thus, the notion of a "Markov potential" makes sense. An Application of these results will be given in the section on the Ising economy.

Markov Chains

In this section we are interested in specifications γ such that $\mathcal{G}(\gamma)$ contains a Markov chain. For the rest of this section let $\mathbf{S} = \mathbb{Z}$.

Definition 2.18. ([Georgii, 88], Definition 10.2) *A specification γ on $\mathbb{Z}$ is said to be a* Markov specification *if $\gamma_{]i,k[}(A|\cdot)$ is $\mathcal{F}_{\{i,k\}}$-measurable for all $A \in \mathcal{F}_{]i,k[}$ and all $i, k \in \mathbb{Z}$ with $i+1 < k$. A λ-modification ρ is called* Markovian *if $\rho_{]i,k[}$ is $\mathcal{F}_{\{i,k\}}$-measurable whenever $i, k \in \mathbb{Z}$ are such that $i+1 < k$.*

Of course, a positive homogeneous Markov specification is a Markov specification, too.

More intuitively, γ is called Markovian if for each interval $\Lambda = \{i, i+1, ..., k\}$, $k > i$, and each $A \in \mathcal{F}_\Lambda$, $\gamma_\Lambda(A|\omega)$ only depends on ω through ω_{i-1} and ω_{k+1}. By Definition 2.6 we have that

$$\mu(A|\mathcal{T}_\Lambda) = \mu(A|\mathcal{F}_{\{i-1,k+1\}}) \quad \mu - a.s$$

for all $A \in= \mathcal{F}_\Lambda$, Λ as above, $\mu \in \mathcal{G}(\gamma)$, whenever γ is a Markov specification. We call this property of μ the *two-sided Markov property* and μ a *Markov field.*

From Proposition 2.17, we obtain:

Corollary 2.19. *If γ is Gibbsian for some homogeneous nearest-neighbor potential Φ then each $\mu \in \mathcal{G}(\gamma)$ is a Markov field.*

A similar result can also be found in [Kindermann & Snell, 80].

Every Markov chain μ constitutes a Markov field, where a Markov chain μ is given by

$$\mu(A|\mathcal{F}_{]-\infty,i[}) = \mu(A|\mathcal{F}_{\{i-1\}}) \quad \mu - a.s.$$

This condition is called the *one-sided Markov property.*

The question is whether $\mathcal{G}(\gamma)$ does not only contain Markov fields when γ is Markovian but also Markov chains.

To gain more intuition for the difference between the two-sided and the one-sided Markov property, let us consider the following remark.

Remark 2.20. ([Georgii, 88], Remark 10.9) (a) *The two sided Markov property is equivalent to the local Markov property on* $\mathbb{Z}$ *which asserts that*

$$\mu(A|\mathcal{T}_\Lambda) = \mu(A|\mathcal{F}_{\partial\Lambda}) \quad \mu - a.s. \quad \textit{for all } A \in \mathcal{F}_\Lambda$$

whenever $\Lambda \subset \mathbb{Z}$ *is finite. Here we set*

$$\partial\Lambda := \{i \in \mathbb{Z} \setminus \Lambda : |i - j| = 1 \quad \textit{for some } j \in \Lambda\}.$$

(b) *The one-sided Markov property is equivalent to the global Markov property on* $\mathbb{Z}$ *which is defined by the requirement that the above property holds for all* $\Lambda \subset \mathbb{Z}$.

Proof. [Georgii, 88], pp. 194,195 □

Definition 2.21. ([Georgii, 88], Definition 4.7) *A measurable space* $(\mathbf{E}, \mathcal{E})$ *is called a* standard Borel space *if there exists a metric* d *on* $\mathbf{E}$ *which turns* $\mathbf{E}$ *into a complete separable metric space and is such that* $\mathcal{E}$ *is the Borel* σ*-algebra with respect to* d.

We eventually obtain the following result.

Proposition 2.22. ([Georgii, 88], Corollary 10.22) *Suppose* $(\mathbf{E}, \mathcal{E})$ *is a standard Borel space, and let* ρ *be a Markovian* λ*-modification and* $\gamma = \rho\lambda$.*. If* $\mathcal{G}(\gamma) \neq \emptyset$ *then* $\mathcal{G}(\gamma)$ *contains a Markov chain.*

2.3 Existence of Gibbs Measures

In [Georgii, 88], we can find a very elegant way to establish existence. Even though the existence problem of Gibbs measures $\mu \in \mathcal{G}(\gamma)$ for a given specification γ is fundamental and generates insights in the nature of Gibbs measures, a closer look at this problem would be beyond the scope of this diploma thesis. Thus, we state the main results without going much into detail. The answer to the question whether a specification γ admits Gibbs measures follows the subsequent pattern:

1. Choose an appropriate topology on $\mathcal{P}(\Omega, \mathcal{F})$. This will turn out to be the *topology of weak convergence* if $\mathbf{E}$ is finite. For infinite state spaces we need the *topology of local convergence*. These topologies coincide for finite $\mathbf{E}$.

2. Specify a boundary condition $\omega \in \Omega$.

3. Show that the net $(\gamma_\Lambda(\cdot|\omega))_{\Lambda \in \mathcal{S}}$ has a cluster point with respect to the chosen topology.

4. Show that each of those cluster points belongs to $\mathcal{G}(\gamma)$.

We will now use the following shortcut suggested in [Georgii, 88]: Let's assume that we are only interested in the case of a finite state space $\mathbf{E}$. In this case (3) holds without any further conditions on γ and ω since $\mathcal{P}(\Omega, \mathcal{F})$ is compact with respect to the topology of weak convergence. The hard part will now be to show (4).

Definition 2.23. (a) *A* net *of a set* $\mathbf{S}$ *is a mapping from a directed set* D *into* $\mathbf{S}$. D *is* directed *if there exists a relation "$\geq$" that is transitive and reflexive and satisfies: for all* $a, b \in \mathbf{S}$*, there exists* $c \in \mathbf{S}$ *such that* $c \geq a$ *and* $c \geq b$.
(b) ([Georgii, 88], Definition 4.6) *A net* $(\mu_\alpha)_{\alpha \in D}$ *in* $\mathcal{P}(\Omega, \mathcal{F})$ *is said to be* locally equicontinuous *if for each* $\Lambda \in \mathcal{S}$ *and each sequence* $(A_m)_{m \geq 1}$ *in* $\mathcal{F}_\Lambda$ *with* $A_m \downarrow \emptyset$

$$\lim_{m \to \infty} \limsup_{\alpha \in D} \mu_\alpha(A_m) = 0.$$

We will now concentrate on a special type of specifications. These distributions are motivated by the physical assumption that each single parameter has a microscopic horizon of interaction. This notion will separate microscopic and macroscopic quantities.

Definition 2.24. *A function* $g : \Omega \to \mathbb{R}$ *describes a* macroscopic observable *if* g *is measurable with respect to the* σ-algebra at infinity $\mathcal{T} = \cap_{\Lambda \in \mathcal{S}} \mathcal{F}_{\mathbf{S} \setminus \Lambda} \cap_{\Lambda \in \mathcal{S}} \mathcal{T}_\Lambda$*, i.e.* g *does not depend on finite sets of spins.*

A function g describes a microscopic quantity if g is arbitrarily close to functions which only depend on finitely many coordinates; these are called *cylinder functions* or *local functions.* To give a rigorous meaning to this intuitive notion of microscopic states, we consider the following definition.

Definition 2.25. ([Georgii, 88], pp. 31,32) (a) *A function* $g : \Omega \to \mathbb{R}$ *is called* local *if there exists* $\Lambda \subset \mathcal{S}$ *such that* g *is* $\mathcal{F}_\Lambda$*-measurable.*
(b) *A function* $g : \Omega \to \mathbb{R}$ *is called* quasi local *if there is a sequence* $(g_n)_{n \geq 1}$ *of local functions* g_n *such that* $\lim_{n \to \infty} ||g - g_n||_{sup} = 0$*, where* $||\cdot||_{sup}$ *denotes the sup-norm.* $\bar{\mathcal{L}}$ *denotes the space of all bounded quasi local functions.*
(c) *A specification* γ *is called* quasi local *if, for each* $\Lambda \in \mathcal{S}$ *and* $g \in \bar{\mathcal{L}}$ *we have that* $\gamma_\Lambda g \in \bar{\mathcal{L}}$*, where* γ_Λ *is a proper probability kernel from* $\mathcal{T}_\Lambda$ *to* $\mathcal{F}$ *and we let*

$$\gamma_\Lambda g := \gamma_\Lambda(g|\cdot) = \int g(x)\gamma_\Lambda(dx|\cdot).$$

Definition 2.26. ([Georgii, 88], pp. 59,97) (a) *A net* $(\mu_\alpha)_{\alpha \in D} \subset \mathcal{P}(\Omega, \mathcal{F})$ *converges to* μ *with respect to the topology of local convergence,* locally *for short, if and only if* $\lim_D \mu_\alpha(A) = \mu(A)$ *for all* $A \in \cup_{\Lambda \in \mathcal{S}} \mathcal{F}_\Lambda$.
(b) *We say, a net of specifications* $(\gamma^\alpha)_{\alpha \in D}$ converges uniformly in the local topology *to a specification* γ*, if*

$$\lim_{\alpha \in D} ||\gamma^\alpha_\Lambda f - \gamma_\Lambda f||_{sup} = 0$$

for all $\Lambda \in \mathcal{S}$ and bounded local functions f. We write $\gamma^\alpha \to \gamma$.

Now, we can state a general existence result for Gibbs measures.

Proposition 2.27. ([Georgii, 88], Theorem 4.22) *Let $(\mathbf{E}, \mathcal{E})$ be a standard Borel space and γ a quasi local specification. Suppose there is a locally equicontinuous net $(\mu_\alpha)_{\alpha \in D}$ of random fields of the form $\mu_\alpha = \nu_\alpha \gamma^\alpha_\Lambda$, $\alpha \in D$, where $(\nu_\alpha)_{\alpha \in D}$ is a net in $\mathcal{P}(\Omega, \mathcal{F})$, $(\gamma^\alpha)_{\alpha \in D}$ a net of specifications with $\gamma^\alpha \to \gamma$, and $(\Lambda_\alpha)_{\alpha \in D}$ a net in $\mathcal{S}$ with $\Lambda_\alpha \to \mathbf{S}$. Then $\mathcal{G}(\gamma)$ contains a cluster point of $(\mu_\alpha)_{\alpha \in D}$ and is therefore non-empty.*

Proof. [Georgii, 88], p. 71 □

2.4 Symmetries

One way to obtain non-uniqueness of Gibbs measures is the so called *symmetry breaking* or *symmetry breakdown*. An example of symmetry breaking will be given in the section on the two-dimensional Ising model.

Throughout this thesis, we consider transformations in the set $T := \{\tau : \Omega \to \Omega \quad | \quad \tau : \omega \mapsto (\tau_i \omega_{\tau_*^{-1} i})_{i \in \mathbf{S}} \quad (\omega \in \Omega)\}$, where $\tau_* : \mathbf{S} \to \mathbf{S}$ is a bijection (spatial transformation), and $\tau_i : \mathbf{E} \to \mathbf{E}$, $i \in \mathbf{S}$, invertible and measurable (spin transformation) with measurable inverses. For $\tau \in T$ we write $\tau = (\tau_*; \tau_i, i \in \mathbf{S})$

Example 2.28. ([Georgii, 88], Example 5.2) *Let $\mathbf{S} = \mathbb{Z}^d$ for some $d \geq 1$. Then for each $j \in \mathbf{S}$ the transformation*

$$\theta_j : \omega \mapsto (\omega_{i-j})_{i \in \mathbf{S}} \quad (\omega \in \Omega)$$

of Ω is called the (lattice) shift *or* (lattice) translation *by j. Furthermore, we write $\Theta := (\theta_j)_{j \in \mathbf{S}}$.*

Definition 2.29. ([Georgii, 88], pp. 82,83) *Let $\tau \in T$. The* τ-image *$\tau(\gamma) = (\tau(\gamma)_\Lambda)_{\Lambda \in \mathcal{S}}$ of a family $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ of measure kernels is defined by*

$$\tau(\gamma)_\Lambda(A|\omega) = \gamma_{\tau_*^{-1}\Lambda}(\tau^{-1}A|\tau^{-1}\omega)$$

for all $\Lambda \in \mathcal{S}$, $A \subset \Omega$ and $\omega \in \Omega$ or, equivalently,

$$\tau(\gamma)_{\tau_*\Lambda}(\tau A|\tau\omega) = \gamma_\Lambda(A|\omega).$$

Of course, if γ is a specification, then $\tau(\gamma)$ is a specification, too.

Definition 2.30. ([Georgii, 88], Definition 5.7) *Let $\tau \in T$.*
(a) *A function φ on Ω is called* τ-invariant *if $\varphi \circ \tau = \varphi$. More generally, a family $\varphi = (\varphi_\Lambda)_{\Lambda \in \mathcal{S}}$ of functions φ_Λ on Ω is said to be* τ-invariant, *τ is called a* symmetry *of φ, if $\tau(\varphi) = \varphi$, i.e. $\varphi_{\tau_*\Lambda} \circ \tau = \varphi_\Lambda$ for all $\Lambda \in \mathcal{S}$.*

(b) *A measure* μ *on* $(\Omega, \mathcal{F})$ *is said to be* τ-invariant, *and* τ *is called* μ-preserving *or a* symmetry *of* μ, *if* $\tau(\mu) := \mu \circ \tau^{-1} = \mu$. *A specification* γ *is called* τ-invariant *if* $\tau(\gamma) = \gamma$, *i.e. for all* $\Lambda \in \mathcal{S}$, $\omega \in \Omega$

$$\gamma_{\tau_* \Lambda}(\cdot|\tau\omega) = \tau(\gamma_\Lambda(\cdot|\omega)).$$

(c) *The set of all symmetries of an object* φ, μ *or* γ *is called the* symmetry group *of this object.*

To verify the consistency of part (b) above, we should have in mind that $\gamma_\Lambda(\cdot|\omega)$ is a measure. Let $A \in \mathcal{F}$, then for any $\omega \in \Omega$, $\Lambda \in \mathcal{S}$

$$\begin{aligned} \gamma &= \tau(\gamma) \\ \Leftrightarrow \gamma_\Lambda(A|\omega) &= \gamma_{\tau_*^{-1}\Lambda}(\tau^{-1}A|\tau^{-1}\omega) \\ \Leftrightarrow \gamma_{\tau_*\Lambda}(A|\tau\omega) &= \gamma_\Lambda(\tau^{-1}A|\omega) = (\gamma_\Lambda(\cdot|\omega) \circ \tau^{-1})(A) \\ &= \tau(\gamma_\Lambda(\cdot|\omega))(A). \end{aligned}$$

Example 2.31. ([Georgii, 88], Example 5.8) *Let* $\mathbf{S} = \mathbb{Z}^d$ *for some* $d \geq 1$. *A specification* γ *is called* shift-invariant *or* translation-invariant *or* (spatially) homogeneous *if* γ *is invariant under the shift-group* Θ, *i.e. if*

$$\gamma_{\Lambda+j}(\theta_j A|\theta_j\omega) = \gamma_\Lambda(A|\omega)$$

for all $\Lambda \in \mathcal{S}$, $j \in \mathbf{S}$, $A \in \mathcal{F}$ *and* $\omega \in \Omega$. *Similarly, a potential* Φ *is called homogeneous if*

$$\Phi_{A+j} \circ \theta_j = \Phi_A$$

for all $j \in \mathbf{S}$ *and* $A \in \mathcal{F}$.

Definition 2.32. *Let* $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$. *A measure* μ *on* $(\Omega, \mathcal{F})$ *is said to be homogeneous if* μ *is invariant with respect to the shift-group* Θ. *Let* $\mathcal{P}_\Theta(\Omega, \mathcal{F}) := \{\mu \in \mathcal{P}(\Omega, \mathcal{F}) : \theta_j(\mu) = \mu, \quad \forall j \in \mathbf{S}\}$ *denote the set of all homogeneous random fields.*

In a subsequent chapter we need the following inspection.

Remark 2.33. ([Georgii, 88], Remark 5.10) *Let* γ *be a specification and* $\tau \in T$. *If* $\mu \in \mathcal{G}(\gamma)$ *then* $\tau(\mu) \in \mathcal{G}(\tau(\gamma))$. *In particular,* $\mathcal{G}(\gamma)$ *is invariant with respect to all symmetries of* γ.

Proof. By definition of $\mathcal{G}(\gamma)$, we have $\mu \in \mathcal{G}(\gamma)$ if and only if $\mu\gamma_\Lambda = \mu$ for all $\Lambda \in \mathcal{S}$. Thus, we obtain

$$\begin{aligned} \tau(\mu)\tau(\gamma)_\Lambda(\cdot) &= \int \tau(\gamma)_\Lambda(\cdot|\omega)\tau(\mu)(d\omega) \\ &= \int \tau(\gamma)_\Lambda(\cdot|\tau\omega)\mu(d\omega) \\ &= \int \gamma_\Lambda(\tau^{-1} \cdot |\omega)\mu(d\omega) \\ &= \tau(\mu\gamma_\Lambda) = \tau(\mu). \end{aligned}$$

If τ is a symmetry for γ, we have

$$\tau(\mu) \in \mathcal{G}(\tau(\gamma)) = \mathcal{G}(\gamma)$$

and the last assertion follows since τ is invertible. $\square$

Corollary 2.34. ([Georgii, 88], Corollary 5.11) *If $\mathcal{G}(\gamma) = \{\mu\}$, then μ is preserved by all symmetries of γ, i.e. μ is invariant with respect to all of γ's symmetries.*

Definition 2.35. ([Georgii, 88], Definition 5.21) *Let γ be a specification. A symmetry τ of γ is said to be broken if there exists some $\mu \in \mathcal{G}(\gamma)$ such that $\tau(\mu) \neq \mu$, i.e. μ is not invariant with respect to τ.*

The following remark is directly implied by Corollary 2.34 and alludes to the importance of symmetry breaking.

Remark 2.36. *If γ has a broken symmetry then $|\mathcal{G}(\gamma)| > 1$.*

2.5 Extreme Decomposition

In this section we encounter the structure of $\mathcal{G}(\gamma)$. The first observation follows immediately from Definition 2.6: For any specification γ, the set $\mathcal{G}(\gamma)$ is convex, i.e. $\mu_1, \mu_2 \in \mathcal{G}(\gamma)$ then $[\varsigma\mu_1 + (1-\varsigma)\mu_2] \in \mathcal{G}(\gamma)$ for any $\varsigma \in [0,1]$. In particular, whenever $|\mathcal{G}(\gamma)| > 1$, then $\mathcal{G}(\gamma)$ is uncountably infinite. We eventually obtain the *extreme decomposition* of Gibbs measures: Every Gibbs measure can be obtained as the barycenter of a unique probability measure on the set of extreme Gibbs measures. Thus, we first have to characterize the set of extreme Gibbs measures. Again, this Section is based on [Georgii, 88]. However, we also refer to [Georgii, 79], pp. 11-15.

Definition 2.37. *$\mu \in \mathcal{G}(\gamma)$ is said to be* extreme *in $\mathcal{G}(\gamma)$ if for any two $\mu_1, \mu_2 \in \mathcal{G}(\gamma)$, $\mu_1 \neq \mu_2$, $\varsigma \in \mathbb{R}$, $\mu = \varsigma\mu_1 + (1-\varsigma)\mu_2$ implies that $\varsigma \in \{0,1\}$. The set of all extreme elements in $\mathcal{G}(\gamma)$ is denoted by $ex\mathcal{G}(\gamma)$*

The following remark will be used in a subsequent chapter.

Remark 2.38. ([Georgii, 88], Remark 7.2) *Let τ be a transformation in T and γ a specification. We then have $\tau(\mu) \in ex\mathcal{G}(\tau(\gamma))$ whenever $\mu \in ex\mathcal{G}(\gamma)$. In particular, each symmetry τ of a specification γ, maps $ex\mathcal{G}(\gamma)$ onto itself, i.e. we have $\tau(\mu) \in ex\mathcal{G}(\gamma)$, whenever $\mu \in ex\mathcal{G}(\gamma)$.*

Proof. By Remark 2.33, we have $\tau(\mu) \in \mathcal{G}(\tau(\gamma))$. Assume $\tau(\mu)$ not to be extremal, i.e. $\tau(\mu) = \varsigma\mu_1 + (1-\varsigma)\mu_2$, $\varsigma \in]0,1[$, $\mu_1, \mu_2 \in \mathcal{G}(\tau(\gamma))$, $\mu_1 \neq \mu_2$. Remark 2.33 shows that $\tau^{-1}(\mu_i) \in \mathcal{G}(\gamma)$, $i = 1, 2$. Thus,

$$\mu = \tau^{-1}(\tau(\mu)) = \varsigma\tau^{-1}(\mu_1) + (1-\varsigma)\tau^{-1}(\mu_2)$$

contradicting the extremality of μ. $\square$

Proposition 2.39. ([Georgii, 88], Theorem 7.7(a)) *Let γ be a specification. A Gibbs measure $\mu \in \mathcal{G}(\gamma)$ is extreme in $\mathcal{G}(\gamma)$ if and only if μ is trivial on the tail-σ-field $\mathcal{T}$, i.e. $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{T}$. This property is also called* tail triviality.

Proof. [Georgii, 88], pp. 116-118, Remark 7.6 and Corollary 7.4 □

At this point, Georgii states a comment that helps to understand the significance of extreme Gibbs measures:

> Suppose we are observing a well-defined state of a real physical system in equilibrium. We will find that the microscopic quantities are subject to rapid fluctuations, whereas the macroscopic quantities remain constant. [...]
>
> We shall try do describe the state by a probability measure μ. Of course, μ should be consistent with the observed empirical distributions of the microscopic variables. According to the basic principles of Statistical Mechanics, this can be achieved by assuming that μ is a Gibbs measure for a suitably chosen Gibbs specification γ. [...] **[Moreover,] μ should be such that the macroscopic quantities are not random. [...] We have argued that the macroscopic quantities are just the tail measurable functions. Consequently, the tail measurable functions should be constant μ-almost surely and this means that μ should be trivial on $\mathcal{T}$.** [Proposition 2.39] [...] thus tells us that the system's state will be described by a suitable extreme element of $\mathcal{G}(\gamma)$. [...]
>
> From this we conclude that only extreme Gibbs measures are suitable to describe an equilibrium state of a real system. In more catching terms we may say that a physical system will always pick an extreme Gibbs measure for its equilibrium state. For this reason an extreme Gibbs measure is often called a *phase*. ([Georgii, 88], p. 119)

In this sense, non-extreme Gibbs states correspond to some uncertainty about the true underlying state of the system.

Let $\mathscr{V}(ex\mathcal{G}(\gamma))$ denote the *evaluation-σ-algebra* on $ex\mathcal{G}(\gamma)$: Let $e_A : \mu \mapsto \mu(A)$, $A \in \mathcal{F}$, $\mu \in ex\mathcal{G}(\gamma)$, be the evaluation mapping on $ex\mathcal{G}(\gamma)$. Then $\mathscr{V}(ex\mathcal{G}(\gamma))$ is the σ-algebra generated by the sets $\{e_A \leq c\}$, $A \in \mathcal{F}$, $c \in [0, 1]$.

We now state the main result of this section known as *extreme decomposition* of Gibbs measures.

Proposition 2.40. ([Georgii, 88], Theorem 7.26) *Given a standard Borel space $(\mathbf{E}, \mathcal{E})$ and a specification γ, such that $\mathcal{G}(\gamma) \neq \emptyset$. Then $ex\mathcal{G}(\gamma) \neq$*

$\emptyset$, and for each $\mu \in \mathcal{G}(\gamma)$ there exists a unique probability measure ψ_μ on $(ex\mathcal{G}(\gamma), \mathscr{V}(ex\mathcal{G}(\gamma)))$ such that

$$\mu = \int_{ex\mathcal{G}(\gamma)} \nu \psi_\mu(d\nu).$$

Proof. [Georgii, 88], p. 133 □

As argued above, an equilibrium of a physical state will always be given by some extremal element in $\mathcal{G}(\gamma)$. However, there may be situations, when we are not aware of the actual state of the system and we have to guess. This uncertainty is reflected by choosing a weight ψ_μ on extreme Gibbs measures and saying that the system's state is just the barycenter of extreme Gibbs measures with respect to ψ_μ. In this sense, Proposition 2.40 says that the state of a system will always be given by a phase (an extreme Gibbs measure) but an observer may not be sure which phase has emerged and thus "mixes".

The following Corollary of Proposition 2.40 is needed in a subsequent chapter.

Corollary 2.41. ([Georgii, 88], Corollary 7.28) *Let $(\mathbf{E}, \mathcal{E})$ be a standard Borel space and γ a specification with $\mathcal{G}(\gamma) \neq \emptyset$. If $\tau \in T$ is a symmetry of γ, then $\psi_{\tau(\mu)} = \tau(\psi_\mu)$ for all $\mu \in \mathcal{G}(\gamma)$. In particular, $\mu \in \mathcal{G}(\gamma)$ is τ-invariant if and only if ψ_μ is τ-invariant.*

Proof. [Georgii, 88], p. 134 □

2.6 Ergodic Decomposition

In this section we let $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$. Similar to $\mathcal{P}_\Theta(\Omega, \mathcal{F})$, $\mathcal{G}_\Theta(\gamma)$ denotes the set of all homogeneous Gibbs measures. Let extreme elements in $\mathcal{P}_\Theta(\Omega, \mathcal{F})$ and $\mathcal{G}_\Theta(\gamma)$ be defined as in Definition 2.37, i.e. elements are extreme if they can only be obtained as trivial convex combinations.

Definition 2.42. *An extreme homogeneous random field $\mu \in ex\mathcal{P}_\Theta(\Omega, \mathcal{F})$ is called ergodic (with respect to the shift-group Θ). Similarly, $\mu \in ex\mathcal{G}_\Theta(\gamma)$ is called an ergodic Gibbs measure.*

In this section the aim is to encounter the structure of $ex\mathcal{G}_\Theta(\gamma)$ and show the *ergodic decomposition* of an element $\mu \in \mathcal{G}_\Theta(\gamma)$ that coincides with its extreme decomposition.

We first obtain a characterization of ergodic random fields.

Definition 2.43. ([Georgii, 88], p. 291) *The* σ-algebra of shift invariant *or* homogeneous events *is defined by*

$$\mathcal{I} := \{A \in \mathcal{F} : \theta_i(A) = A \quad \forall i \in \mathbf{S}\}$$

Proposition 2.44. ([Georgii, 88], Theorem 14.5(a)) *A probability measure $\mu \in \mathcal{P}_\Theta(\Omega, \mathcal{F})$ is extreme in $\mathcal{P}_\Theta(\Omega, \mathcal{F})$ if and only if μ is trivial on the σ-algebra $\mathcal{I}$ of invariant events.*

Proof. [Georgii, 88], p. 291 $\square$

Remark 2.45. *The proposition above gives rise to an equivalent definition of ergodic random fields as it is done in [Georgii, 88], Definition 14.6: A probability measure $\mu \in \mathcal{P}_\Theta(\Omega, \mathcal{F})$ is said to be ergodic (with respect to the lattice shift-group Θ) if μ is trivial on $\mathcal{I}$. In the language of mathematical physics, any such μ is often called a* pure state*.*

Proposition 2.46. *Let a random field μ be homogeneous. If μ is extreme, then μ is ergodic.*

Proof. Let $\mu \in \mathcal{P}_\Theta(\Omega, \mathcal{F})$. In Proposition 14.9 in [Georgii, 88], it is shown that $\mathcal{I} \subset \mathcal{T}$ μ-a.s. Thus, if μ is trivial on $\mathcal{T}$, then so on $\mathcal{I}$. $\square$

The following proposition is a version of Theorem 14.10 in [Georgii, 88]. It shows that any homogeneous random field μ can uniquely be decomposed in ergodic random fields.

Proposition 2.47. *Let $(\mathbf{E}, \mathcal{E})$ be a standard Borel space and $\mu \in \mathcal{P}_\Theta(\Omega, \mathcal{F})$. Then there exists a unique probability measure ψ_μ on $(ex\mathcal{P}_\Theta(\Omega, \mathcal{F}), \mathscr{V}(ex\mathcal{P}_\Theta(\Omega, \mathcal{F})))$ such that*

$$\mu = \int_{ex\mathcal{P}_\Theta(\Omega,\mathcal{F})} \nu \psi_\mu(d\nu).$$

Proof. [Georgii, 88], p. 294 $\square$

Definition 2.48. *The set $\mathcal{G}_\Theta(\gamma)$ of all homogeneous Gibbs measures with respect to a homogeneous specification γ is given by*

$$\mathcal{G}_\Theta(\gamma) := \mathcal{G}(\gamma) \cap \mathcal{P}_\Theta(\Omega, \mathcal{F}).$$

The following proposition will be needed later on. It furthermore shows that the second part in Definition 2.42 is well defined.

Proposition 2.49. ([Georgii, 88], Theorem 14.15(a)) *Let γ be a homogeneous specification. A Gibbs measure $\mu \in \mathcal{G}_\Theta(\gamma)$ is extreme in $\mathcal{G}_\Theta(\gamma)$ if and only if μ is ergodic, i.e. trivial on $\mathcal{I}$. Thus*

$$ex\mathcal{G}_\Theta(\gamma) = \mathcal{G}_\Theta(\gamma) \cap ex\mathcal{P}_\Theta(\Omega, \mathcal{F}).$$

Proof. [Georgii, 88], p. 297 $\square$

It should be stated that in general there are ergodic Gibbs measures which are not trivial on $\mathcal{T}$, i.e. $\mu \in ex\mathcal{G}_\Theta(\gamma)$ but $\mu \notin ex\mathcal{G}(\gamma)$. In other words, $\mu \in ex\mathcal{G}_\Theta(\gamma)$ is a pure state but not necessarily a phase.

We now obtain that the ergodic decomposition of $\mu \in \mathcal{G}_\Theta(\gamma)$ coincides with its extreme decomposition.

Proposition 2.50. ([Georgii, 88], Theorem 14.17) *Suppose $(\mathbf{E}, \mathcal{E})$ is a standard Borel space and γ a homogeneous specification with $\mathcal{G}_\Theta(\gamma) \neq \emptyset$. Let $\mu \in \mathcal{G}_\Theta(\gamma)$ and ψ_μ the unique probability measure on $ex\mathcal{P}_\Theta(\Omega, \mathcal{F})$ which represents μ as in Proposition 2.47. Then ψ_μ is supported on $ex\mathcal{G}_\Theta(\gamma)$. Consequently, μ has a unique extreme decomposition within $\mathcal{G}_\Theta(\gamma)$, i.e. an ergodic decomposition, namely*

$$\mu = \int_{ex\mathcal{G}_\Theta(\gamma)} \nu \psi_\mu(d\nu).$$

Proof. [Georgii, 88], p. 298 □

The following remark summarizes the several notions introduced and thus helps us keeping track.

Remark 2.51. *Elements in $\mathcal{G}(\gamma)$ are called Gibbs states. We refer to elements in $ex\mathcal{G}(\gamma)$ as phases and to those in $ex\mathcal{G}_\Theta(\gamma)$ as pure states or ergodic Gibbs states. These are just the extreme homogeneous Gibbs states. It is important to note, that a homogeneous phase is a pure state but in general not vice versa; more precisely, any homogeneous extreme Gibbs state is ergodic, i.e. extreme homogeneous. However an extreme homogeneous Gibbs state does not have to be homogeneous extreme. Differing from the notation in [Georgii, 88], we call homogeneous extreme Gibbs states pure phases.*

2.7 The Ising Model

Let us now consider a fundamental example from statistical mechanics: the so called *Ising model*. A more elaborate and in particular historical survey on the Ising model can be found in [Kindermann & Snell, 80b].

After defining the baseline model we state some fundamental results on the properties of Gibbs measures emerging within this model. This section is considered as a comprehensive introduction to the Ising model. A rigorous approach to the Ising model can be found in [Georgii, 88], [Georgii, 79] or [Preston, 76]. Here, the section on the one-dimensional Ising model is taken from [Georgii, 88], Section 3.2, the section on the two-dimensional Ising model from [Georgii, 88], Section 6.2.

As already mentioned, in a physical context, the Ising model may help us to understand the emergence of magnets in nature. However, the Ising model is general enough to be applied not only to problems in physics but also to those in sociology and economics:

> [Ising] tried to explain, using this model, certain empirical facts about ferromagnetic materials. [...]
> While Ising discussed only the magnetic interpretation, the same model has since been found applicable to a number of other physical and biological systems such as gases, binary alloys, and cell structures. A sociologically orientated application has been suggested by Weidlich [...]. Here one considers a group of people, each of whom at a given moment is a 'conservative' ('up') or a liberal ('down'). The energy [...] might better be called 'tension'. The first term [...] [in the interaction potential] is the tension caused by people interacting. [The second term represents the effect of an external magnetic field of intensity h.] The external field represents, for example, the current state of the government, liberal or conservative. Minimum tension (maximum boredom) occurs if all people agree and agree with the government. ([Kindermann & Snell, 80b], p. 5)

In Section 3.5, we encounter an economic interpretation of this model. Moreover, the analysis of market demand in Chapter 6 is based on a model that generalizes the Ising model discussed here with respect to the state space. We refer to that model as the *unbounded spin Ising economy*. Thus, it seems worthwhile to obtain some "feeling" for the Ising model to have a clue what is going on there.

The one-dimensional Ising Model

To formalize the model we let $S = \mathbb{Z}$, $\mathbf{E} = \{-1, 1\}$ and the a priory measure be the counting measure. The specification that describes the interaction is given by the *Ising potential* $\Phi^{J,h}$: with coupling constant $J \in \mathbb{R}$ and external field $h \in \mathbb{R}$, the Ising potential is given by the following homogeneous nearest neighbor potential

$$\Phi_A^{J,h} = \begin{cases} -J\sigma_i\sigma_{i+1} & \text{if } A = \{i, i+1\}, \\ -h\sigma_i & \text{if } A = \{i\}, \\ 0 & \textit{else}. \end{cases}$$

The idea of this one-dimensional Ising model is that we are given a chain of spins on $\mathbb{Z}$ with two possible orientations: "up" (1) or "down" (-1). Given a *coupling constant* J, the interaction energy of two adjacent spins σ_i and σ_j is given by $-J\sigma_i\sigma_j$. We distinguish two cases: The *attractive* case, i.e. $J > 0$. In this case two adjacent spins generate minimal energy if they are parallel. This is often called the *ferromagnetic* case. In socioeconomic terms this assumption means that local interaction is ruled by the agents' preference for conformity in the sense that agents obtain "positive utility" from behaving similar to their neighbors. The *repulsive* case is characterized by $J < 0$. In this case adjacent spins exhibit minimal energy when pointing in opposite directions. This case is often called the *anti ferromagnetic* case. Furthermore there exists an external field given by h. The energy caused by the external field is minimized when spins align with the external field, i.e. spins are "+1" if $h > 0$, "-1" if $h < 0$.

By Proposition 2.17, $\gamma^{\Phi^{J,h}} := \gamma^{J,h}$ is a positive homogeneous Markov specification. Theorem 3.5 in [Georgii, 88] implies the uniqueness of a Gibbs measure for $\gamma^{J,h}$, i.e. $\mathcal{G}(\gamma^{J,h}) = \{\mu_{J,h}\}$ Furthermore, it endows us with an explicit characterization of $\mu_{J,h}$ as the distribution of a homogeneous Markov chain. For this characterization, I refer to [Georgii, 88], pp. 50,51.

An interesting question in statistical mechanics is the behavior of the system for "low temperatures" or in the "thermodynamic limit". Here, we only consider the ferromagnetic case $J > 0$. We introduce the *inverse absolute temperature* β to our model by just multiplying $\Phi^{J,h}$ by β and obtain $\beta\Phi^{J,h} = \Phi^{\beta J,\beta h}$. Furthermore, we let $J = 1$.

Then, the question is how $\mu_{\beta,\beta h}$ behaves as β tends to ∞. We will see, that the thermodynamic limit is closely related to the set of *ground states*: A configuration $\omega \in \Omega$ is called a ground state for a potential Φ if ω exhibits minimal Φ-energy. More rigorously:

Definition 2.52. ([Georgii, 88], Definition 6.18) *Let Ψ be a potential. A configuration $\omega \in \Omega$ is called a* ground state *of Ψ if*

$$H_\Lambda^\Psi(\xi) \geq H_\Lambda^\Psi(\omega)$$

for all $\Lambda \in \mathcal{S}$ and $\xi \in \Omega$ such that $\xi_{\mathbf{S}\setminus\Lambda} = \omega_{\mathbf{S}\setminus\Lambda}$. We then call ξ a local perturbation *of ω.*

For rigorous derivations of the following limiting results, we refer to [Georgii, 88], pp. 51,52. Whenever $h > 0$

$$\mu_{\beta,\beta h} \stackrel{\beta\to\infty}{\longrightarrow} \delta_+ \quad \text{weakly},$$

where δ_+ denotes the Dirac measure on $\omega^+ := (\omega_i^+)_{i\in\mathbb{Z}} \in \Omega$, $\omega_i^+ := 1$ for all i. However, we should also notice, that ω^+ is a ground state for $\Phi^{\beta,\beta h}$ whenever $h > 0$.

In the case $h < 0$, we obtain that $\mu_{\beta,\beta h}$ weakly converges to δ_-, the Dirac measure on $\omega^- := -\omega^+$, as $\beta \to \infty$ Again, in this case ω^- is the unique ground state for $\Phi^{\beta,\beta h}$.

For the case $h = 0$ it turns out that the limiting distribution is given by the uniform distribution on $\{\omega^+, \omega^-\}$, i.e.

$$\mu_{\beta,0} \stackrel{\beta\to\infty}{\longrightarrow} \frac{1}{2}\delta_+ + \frac{1}{2}\delta_- \quad \text{weakly}$$

and in this case $\{\omega^+, \omega^-\}$ is the set of ground states for $\Phi^{\beta,0}$.

The intuition behind these results is immediate: Given a positive directed external field h in a ferromagnetic model. The lower the temperature and thus the stronger the coupling βJ among agents and the stronger the external field βh, a great majority of spins will be positive directed, approaching the case where all spins are positive directed as $\beta \to \infty$. When there is no external field present, i.e. $h = 0$, even for strong coupling there is no significant majority of spins that align.

When discussing the one-dimensional Ising model, Georgii gives a prospect on what is different in multi-dimensional Ising model:

> We will see later that the loss of tail triviality[2] is far more dramatic for the Ising model in two or more dimensions. In higher dimensions the set $\mathcal{G}(\Phi^{\beta,0})$ is so strongly attracted by the two ground states ω^+ and ω^- that for sufficiently large (but finite!) β there exist two distinct Gibbs measures for $\Phi^{\beta,0}$ which are close to δ_+ resp. δ_-. Thus for these β a phase transition occurs [...]. ([Georgii, 88], p. 52)

The two-dimensional Ising Ferromagnet

In this section we introduce the ferromagnetic Ising model on a square lattice. The Ising specification exhibits phase transition by virtue of symmetry breaking. In [Georgii, 88], Section 6.2, phase transition is shown by applying a device introduced by Peierls. For a rigorous application of Peierls' device, we refer to [Georgii, 88], Proof of Theorem 6.9. However, the heuristics are quite intuitive: The device can be applied to potentials with multiple distinct ground states also for more general state spaces than we consider here. To recall, a *ground state* of a potential is a configuration of minimal energy. Consider any ground state $\omega \in \Omega$ and any local perturbation η of ω, i.e. any configuration η, such that $|\{i \in \mathbf{S} : \eta_i \neq \omega_i\}| < \infty$. Let a *contour* of η be given by the outer boundaries of connected subsets of $\{i \in \mathbf{S} : \eta_i \neq \omega_i\}$. One now has to verify some kind of *stability* of the ground state ω: The amount of additional energy needed to obtain configuration η from configuration ω shall be at least proportional to the length of the contour of η. By convexity, $\mathcal{G}(\beta\Phi)$ is uncountably infinite. Now, for sufficiently low temperatures (large β) $\mathcal{G}(\beta\Phi)$ contains a measure μ_ω^β that satisfies $\mu_\omega^\beta \overset{\beta\to\infty}{\longrightarrow} \delta_\omega$ locally. Thus, if there are at least two distinct stable ground states ω for Φ and if β is sufficiently large, we have that $|\mathcal{G}(\beta\Phi)| > 1$.

For the Ising ferromagnet, we let $\mathbf{E} = \{-1, 1\}$, $S = \mathbb{Z}^2$, the a priory measure λ be the counting measure on $\mathbf{E}$ and the interaction potential given

[2] Since $\mu_{\beta,0}$ is the unique Gibbs measure for $\gamma^{\beta,0}$ it is extremal and thus, by Proposition 2.39, it exhibits tail triviality. However, $\lim_{\beta\to\infty} \mu_{\beta,0}$ puts weight $\frac{1}{2}$ on the tail events $\{\sigma = \omega^+\} \in \mathcal{T}$ and $\{\sigma = \omega^-\} \in \mathcal{T}$. Thus, it is not trivial on $\mathcal{T}$.

by

$$\Phi_A = \begin{cases} -\sigma_i\sigma_j & \text{if } A = \{i,j\}, |i-j| = 1, \\ 0 & \textit{else.} \end{cases}$$

For simplicity, we have already assumed coupling $J = 1$ and to obtain the interesting case exhibiting phase transition the external field h to vanish.

The Ising potential Φ is *equivalent* to the potential

$$\Psi_A = \begin{cases} (\sigma_i - \sigma_j)^2 & \text{if } A = \{i,j\}, |i-j| = 1, \\ 0 & \textit{else,} \end{cases}$$

in that $\gamma^\Phi = \gamma^\Psi$ (cf. Definition 2.33 and Theorem 2.34 in [Georgii, 88]) since $(\sigma_i - \sigma_j)^2 = -2\sigma_i\sigma_j + 2$. In particular, we see that ground states for Φ and Ψ coincide. Stating the equivalence of these two potentials is not crucial for our further analysis but for the sake of completion: The form of the Ising potential depends on the literature. Moreover, the Ising potential Ψ directly shows that the local interaction structure implies preference for conformity among agents in an economic context: The interaction energy H^Ψ gets smaller, the more agents agree on a specific spin, i.e. $(\sigma_i(\omega) - \sigma_j(\omega))^2 = 0$.

Φ is homogeneous, i.e. lattice shift invariant, and spin flip invariant[3] and again, ω^+ and ω^- are ground states for Φ. In the two-dimensional Ising ferromagnet, the ground states ω^+ and ω^- are stable in the sense introduced above, i.e. a minimal amount of additional energy is needed to achieve a local perturbation of ground states. We thus obtain, by following Peierls' device, that $\mathcal{G}(\beta\Phi)$ contains elements weakly converging to δ^+ and δ^- as β approaches ∞:

Proposition 2.53. ([Georgii, 88], Theorem 6.9) *Let τ denote the spin flip transformation. In the ferromagnetic Ising model on $\mathbb{Z}^2$ as described above,*

$$\lim_{\beta\to\infty} d(\mathcal{G}_\Theta(\beta\Phi), \delta_+) = \lim_{\beta\to\infty} d(\mathcal{G}_\Theta(\beta\Phi), \delta_-) = 0, \tag{2.3}$$

where $\mathcal{G}_\Theta$ is the set of homogeneous Gibbs measures and d denotes any metric induced by the weak topology on $\mathcal{P}(\Omega, \mathcal{F})$.[4] *In particular, for all sufficiently large β there exist two homogeneous Gibbs measures μ_-^β, $\mu_+^\beta \in \mathcal{G}_\Theta(\beta\Phi)$ with $\tau(\mu_+^\beta) = \mu_-^\beta$ and for all $i \in \mathbf{S}$*

$$\mu_-^\beta(\sigma_i) = \mu_-^\beta(\sigma_0) < 0 < \mu_+^\beta(\sigma_0) = \mu_+^\beta(\sigma_i),$$

where again $\mu_+^\beta(\sigma_i) := \int_\Omega \sigma_i d\mu_+^\beta$ and analogously for μ_-^β.

Proof. [Georgii, 88], pp. 101-105 □

[3] The spin flip transformation τ is given by $\tau : \omega \mapsto (-\omega_i)_{i\in\mathbf{S}}$ for all $\omega \in \Omega$.

[4] In [Georgii, 88] the theorem is stated for the topology of local convergence. However, as already mentioned, the local and the weak topology coincide for finite $\mathbf{E}$.

Remark 2.54. *Since $\mu_+^\beta \neq \mu_-^\beta$, Proposition 2.53 shows in particular that the spin flip transformation is the broken symmetry in the two-dimensional Ising ferromagnet.*

Outright, Georgii interprets this result:

> In physical terms, $\mu_+^\beta(\sigma_0)$ is the magnetization of the Ising spin system when μ_+^β is its state. The last sentence of the theorem above can thus be rephrased as follows: At sufficiently low temperatures, the two-dimensional Ising ferromagnet admits an equilibrium state of positive magnetization,[5] although there is no action of an external field. This phenomenon is called *spontaneous magnetization.* ([Georgii, 88], p.100)

At last, I mention two well known results for the two-dimensional Ising ferromagnet from [Georgii, 88], pp. 100,101, just to give the whole picture: At first, there exists a critical inverse temperature $\beta_c \in \mathbb{R}_{++}$ such that $|\mathcal{G}(\beta\Phi)| = 1$ for $\beta < \beta_c$ and $|\mathcal{G}(\beta\Phi)| > 1$ for $\beta > \beta_c$. In more catching terms, if $\beta < \beta_c$, thermal motion of particles is "stronger" than local interaction in that it prevents the system from exhibiting phase transition. Georgii states that $\mu_+^\beta(\sigma_0) \geq 0$ for all $\beta \geq 0$ and $\mu_+^\beta(\sigma_0) > 0$ if and only if $|\mathcal{G}(\beta\Phi)| > 1$. Thus, we obtain the following remark.

Remark 2.55. *For all $i \in \mathbf{S}$, we have $\mu_+^\beta(\sigma_i) = \mu_+^\beta(\sigma_0) = 0$ whenever $\beta < \beta_c$: There is no magnetic orientation of the Ising ferromagnet when the system is sufficiently "hot", as the law of large numbers would have told us in case of independent particles, i.e. no interaction among spins. In other words: Thermal motion is too strong for particles to couple.*

The existence of the critical temperature β_c is rigorously proved in [Georgii, 72], 4.1.

Second, it can be shown that $ex\mathcal{G}(\beta\Phi) = \{\mu_+^\beta, \mu_-^\beta\}$ whenever $\beta > \beta_c$. In particular, $\mathcal{G}(\beta\Phi) = [\mu_+^\beta, \mu_-^\beta]$.

[5] As well as an equilibrium state μ_-^β of negative magnetization.

Chapter 3

Large Economies & Local Interactions

3.1 Introduction

We are now enabled to rigorously introduce an economy with locally non-strategically interacting agents. In Chapter 1, we have suggested the existence of another form of interaction among agents besides indirect interaction via market prices.

In this chapter, we first have a look at Hildenbrand's approach to large economies: roughly speaking, economies are given by a probability measure on the space of agents' characteristics. After having introduced these *distribution economies*, we motivate a distinct economic notion: the *random economy.* We then reconsider Föllmer's approach to random economies, where agents interact locally in a Gibbsian manner. This generalizes the approach in [Hildenbrand, 71] where agents are thought to act independently in the sense that the random variables describing the agents behavior are stochastically independent. The idea in [Föllmer, 74] is that each agent's distribution of preferences depends on the actual preferences in her peer group. Föllmer's intuition was that the correlation of preferences may represent social interactions. This immediately leads us to Föllmer's fundamental Ising economy and thus to a non-physical interpretation of the Ising model, where the local interaction structure is ruled by preference for conformity among agents.

In this chapter, we have to be very precise in emphasizing the (conceptual) distinction of distribution and random economies. The chapter is structured as follows: We first recall fundamental notions from demand and general equilibrium theory. Thereafter, distribution economies are introduced. The third part is devoted to a brief introduction of random economies and a first approach to link distribution and random economies. The last part of this chapter is a survey on Föllmer's approach to locally interacting agents.

It is important to note that the economic concepts in this chapter are preference-based: The primitives of individual agent's behavior are given by preference relations on the commodity space. We may then generate individual demand functions by virtue of preference optimization given budget restrictions. However, the subsequent chapters turn out to be demand-based in that the primitives of individual behavior are demand functions. These do not necessarily originate from preferences.

3.2 Basic Definitions

Let us first state some fundamental notions from demand theory. The standard definitions below are recalled here since several definitions, as those of preferences and their properties, vary in standard literature. Moreover, it may be helpful to have a look at these fundamental concepts in a basic context that is purely deterministic.

Definition 3.1. (a) *A* binary relation $\succsim$ *on an arbitrary set* $\mathbf{X}$ *is a subset of* $\mathbf{X} \times \mathbf{X}$. *When saying that two elements* $x, y \in \mathbf{X}$ *are in relation, we write* $x \succsim y$ *for* $(x, y) \in \succsim$.
(b) *A binary relation* $\succsim$ *on* $\mathbf{X}$ *is said to be*

- total *or* complete *if*
$$x \succsim y \quad \textit{or} \quad y \succsim x \quad \textit{for all } x, y \in \mathbf{X};$$
- reflexive *if*
$$x \succsim x \quad \textit{for all } x \in \mathbf{X};$$
- transitive *if*
$$x \succsim y \quad \textit{and} \quad y \succsim z \quad \textit{imply} \quad x \succsim z \quad \textit{for all } x, y, z \in \mathbf{X};$$
- asymmetric *if*
$$x \succsim y \quad \Rightarrow \quad \neg y \succsim x \quad \textit{for all } x, y \in \mathbf{X}.$$

(c) *The* asymmetric part $\succ$ *of the relation* $\succsim$ *is defined as*
$$x \succ y \quad \Leftrightarrow \quad [x \succsim y \quad \textit{and} \quad \neg y \succsim x],$$
$x, y \in \mathbf{X}$.

We now consider a binary relation for each agent i on her set of possible consumption vectors. The idea is that specific binary relations reflect an agent's "taste". When saying that a consumption bundle x is *in relation to* or *relates to* a consumption bundle y, or formally $x \succsim_i y$, it is meant that agent i considers consumption bundle x at least as good as consumption bundle y; stated in another way, she weakly prefers x to y. We say that an agent i strictly prefers x to y if $x \succ_i y$. The subsequent definitions basically follow the lines in [Hildenbrand, 74], pp. 83-93. However, later on, we have to confine ourselves to special cases for several results to hold. In this chapter, an agent is entirely characterized by a *consumption set*, a *preference* and a commodity *endowment*. Elaborate discussions on these primitives of an economic model may be found in [Hildenbrand, 74] or [Hildenbrand & Kirman, 76].

Definition 3.2. (a) *Let* $\mathbf{M}$ *be a finite set of agents. A* consumption set $\mathbf{X}_i$ *of agent* $i \in \mathbf{M}$ *is a non-empty subset of the* commodity space $\mathbb{R}^l$ *that is furthermore closed, convex and bounded below. For convenience, we assume the consumption space to be common among agents, i.e.* $\mathbf{X}_i = \mathbf{X}$ *for all* $i \in \mathbf{M}$.
(b) *We define the following relations on* $\mathbb{R}^l$*: For* $x, y \in \mathbb{R}^l$ *let*

- $x \geq y$ *if* $x_k \geq y_k$ *for* $k = 1, ..., l$*;* x *is* greater than or equal to y.
- $x > y$ *if* $x \geq y$ *but* $x \neq y$*;* x *is* semi-greater than y.
- $x \gg y$ *if* $x_k > y_k$ *for* $k = 1, ..., l$*;* x *is* strictly grater than y.

(c) *A binary relation* $\succsim$ *on a consumption set* $\mathbf{X} \subset \mathbb{R}^l$ *is*

- *a* preordering *if it is a transitive binary relation,*
- continuous *if it is upper- and lower-hemi-continuous, i.e. the* upper contour set $\succsim x := \{y \in \mathbf{X} : y \succsim x\}$ *and the* lower contour set $x \succsim := \{y \in \mathbf{X} : x \succsim y\}$ *are closed in* $\mathbf{X}$ *for all* $x \in \mathbf{X}$*. Equivalently: For all sequences* $(x_n)_n \subset \mathbf{X}$ *and* $(y_n)_n \subset \mathbf{X}$ *such that* $x_n \to x \in \mathbf{X}$ *and* $y_n \to y \in \mathbf{X}$*, it holds:* $x_n \succsim y_n \quad \forall n \quad \Rightarrow \quad x \succsim y$.
- strongly monotone *if, for all* $x, y \in \mathbf{X} = \mathbb{R}^l_+$*,* $x > y$ *implies* $x \succ y$*,*
- monotone *if, for all* $x, y \in \mathbf{X} = \mathbb{R}^l_+$*,* $x \gg y$ *implies* $x \succ y$*,*
- weakly convex *if, for all* $x \in \mathbf{X}$*, the upper contour set* $\succsim x$ *is convex,*
- strongly convex *if, for all* $x, y, z \in \mathbf{X}$ *such that* $y \succsim x$*,* $z \succsim x$ *and* $y \neq z$*, it holds: for all* $\theta \in]0, 1[$ *we have* $(\theta y + (1 - \theta)z) \succ x$.
- locally non-satiated *if, for every* $x \in \mathbf{X}$ *and every neighborhood* $V \subset \mathbf{X}$ *of* x*, there exists* $y \in V$ *such that* $y \succ x$.

Definition 3.3. (a) *A* preference $\succsim_i$ *of agent* $i \in \mathbf{M}$ *is a continuous, complete preordering on her* consumption set $X_i \subset \mathbb{R}^l$*. Let* $\mathscr{P}$ *denote the set of all preferences.*
(b) *The* strict preference relation $\succ_i$ *for agent* $i \in \mathbf{M}$ *is the asymmetric part of* $\succsim_i$.

Remark 3.4. (a) *In most introductory textbooks, a preference is defined as a reflexive, complete and transitive binary relation. Our definition assumes a preference to be a continuous, complete and transitive binary relation. However, completeness implies reflexivity.*
(b) *Of course, we could have defined preferences the other way around: Let $x, y \in \mathbf{X}$. Instead of saying x is at least as good as y and writing $x \succsim y$, we could have defined preferences by saying x is at most as good as y and writing $x \precsim y$. However, now we define the preference $\precsim$ by*

$$x \precsim y \quad :\Leftrightarrow \quad \neg x \succ y.$$

All properties of a preference are satisfied.

Definition 3.5. *Agent i's* initial (commodity) endowment *is given by a vector $w_i \in \mathbb{R}^l$. Let $\mathbf{w} = \sum_{i \in \mathbf{M}} w_i$ denote the* aggregate endowment*.*

Definition 3.6. *A* simple (pure exchange) economy *is a tuple $\mathscr{E}^s = ((X_i, \succsim_i, \mathbf{w}_i)_{i \in \mathbf{M}})$, where $\mathbf{M}$ denotes the finite set of agents.*

Definition 3.7. (a) *Given a simple economy $\mathscr{E}^s$, a family $(x_i)_{i \in \mathbf{M}}$ is an* allocation *for $\mathscr{E}^s$ if $x_i \in X_i$ for all $i \in \mathbf{M}$. An allocation $(x_i)_{i \in \mathbf{M}}$ is called* feasible *for $\mathscr{E}^s$ if $\sum_{i \in \mathbf{M}} x_i = \mathbf{w}$.*
(b) *Given a price system $p \in \mathbb{R}^l$ and an endowment $w_i \in \mathbb{R}^l$, the i^{th} agent's Walrasian* demand correspondence *$\chi_i : \mathscr{P} \times \mathbb{R}^l \times \mathbb{R}^l \to 2^{\mathbb{R}^l}$, the power set on $\mathbb{R}^l$, is given by*

$$\chi_i(\succsim_i, w_i, p) = \{x_i \in X_i | \forall \hat{x}_i \in X_i : (\hat{x}_i \succ_i x_i \Rightarrow p \cdot \hat{x}_i > p \cdot w_i)\},$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^l$. Of course, the functional form is the same for all agents and thus, we set $\chi_i = \chi$ for all $i \in \mathbf{M}$.
(c) *Given an exchange economy $\mathscr{E}^s$, a tuple $((x_i)_{i \in \mathbf{M}}, p) \in \mathbb{R}^{l|\mathbf{M}|} \times \mathbb{R}^l$ is a* competitive equilibrium *if*

1. *$x_i \in \chi(\succsim_i, w_i, p)$ for $i \in \mathbf{M}$,*
2. *$(x_i)_{i \in \mathbf{M}}$ is a feasible allocation for $\mathscr{E}^s$.*

When specifying an agent's preference, we have also determined her consumption set. Letting $\mathscr{P}$ denote the set of preferences, an economic agent is given by an element in $\mathscr{P} \times \mathbb{R}^l$. Thus, we may rephrase Definition 3.6:

Definition 3.8. ([Hildenbrand & Kirman, 76], Definition 2.6) *A simple pure exchange economy $\mathscr{E}^s$ is given by a mapping*

$$\mathscr{E}^s : \mathbf{M} \to \mathscr{P} \times \mathbb{R}^l.$$

In this sense, preferences are given by a mapping $\succsim: \mathbf{M} \to \mathscr{P}$, $i \mapsto \succsim_i$, initial endowments by $w : \mathbf{M} \to \mathbb{R}^l$, $i \mapsto w_i$ and analogously for allocations.

To define measures on $\mathscr{P} \times \mathbb{R}^l$, we are faced with the problem of obtaining a suitable σ-algebra on $\mathscr{P}$. We therefore consider the "topology of closed convergence" that turns $\mathscr{P}$ into a separable metric space and use the Borel-σ-algebra with respect to that metric.

Remark 3.9. ([Hildenbrand, 70], p. 163 and [Hildenbrand, 71], p. 415) *Let $\mathscr{P}$ denote the set of all preferences, i.e. of all continuous, complete, transitive (and thus reflexive) binary relations on the consumption set $\mathbf{X}$. The graph $\mathbf{P} = \{(x, y) \in \mathbf{X} \times \mathbf{X} : x \succsim y\}$ of $\succsim$ is called the* preference set *associated with the consumption set $\mathbf{X}$ and preference relation $\succsim$. Then $\mathscr{P}$ is the set of all closed preference sets in $\mathbb{R}^l \times \mathbb{R}^l$. In this sense $\mathscr{P}$ is a collection of closed subsets in $\mathbb{R}^l \times \mathbb{R}^l$.*

Let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{P}$. We say that the sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ converges to some $\mathbf{P} \in \mathscr{P}$ with respect to the topology of closed convergence, if

$$LimInf\mathbf{P}_n = \mathbf{P} = LimSup\mathbf{P}_n,$$

where

$$\begin{aligned} LimInf\mathbf{P}_n := \{x \in \mathbb{R}^l \times \mathbb{R}^l : \quad & \exists n_0 \quad \text{such that for every neighborhood} \\ & U(x) \text{ of } x \text{ and every } n \geq n_0 \text{ it holds} \\ & U(x) \cap \mathbf{P}_n \neq \emptyset\} \quad \text{and} \\ LimSup\mathbf{P}_n := \{x \in \mathbb{R}^l \times \mathbb{R}^l : \quad & \text{for every neighborhood } U(x) \text{ of } x \\ & \text{it holds } U(x) \cap \mathbf{P}_n \neq \emptyset \text{ infinitely often}\}. \end{aligned}$$

It is now stated in [Hildenbrand, 71] that there exists a metric on $\mathscr{P}$ such that $\mathscr{P}$ is a separable metric space and convergence with respect to that metric coincides with the closed convergence above. A more elaborate discussion can be found in [Hildenbrand, 70], pp. 164,165. We are now enabled to consider distributions on the measurable space $(\mathscr{P} \times \mathbb{R}^l, \mathcal{B}(\mathscr{P}) \times \mathcal{B}(\mathbb{R}^l))$, where $\mathcal{B}$ denotes the Borel-σ-algebra for the respective topology chosen.

The economic rationale of this metric is that agents with similar preferences with respect to that metric behave similar in the sense that the demand correspondence χ is upper hemi-continuous. For a rigorous statement, we refer to [Hildenbrand, 71], Appendix 2.

We have already seen how to generate a demand correspondence χ for given preference $\succsim$. In general, the demand correspondence is not a function. Thus, we ask for properties of preferences such that the demand sets $\chi(\succsim, w, p)$ are singleton. Let $\mathscr{P}_{smo,sco}$ denote the set of all strongly monotone and strongly convex preferences.

Henceforth, we consider $\mathbf{X} \subset \mathbb{R}^l_+$ closed and convex, $\mathbf{W} \subset \mathbb{R}^l_{++}$ closed and bounded. We furthermore assume price systems $p \in \mathbb{R}^l_{++}$.[1]

[1] We assume prices to be strictly positive since demand might be unbounded otherwise.

Proposition 3.10. ([Hildenbrand, 74], p. 170) *Let* $\succsim \in \mathscr{P}_{smo,sco}$*, price system* $p \in \mathbb{R}^l_{++}$ *and initial endowment* $w \in \mathbf{W}$*. The demand correspondence* $\chi : \mathscr{P}_{smo,sco} \times \mathbf{W} \times \mathbb{R}^l_{++} \to 2^{\mathbb{R}^l}$ *is a demand function.*

Proof. Theorem 2, Chapter 1.2, in [Hildenbrand, 74] ensures that $\chi(\succsim, p, w)$ is non-empty for every triplet $(\succsim, p, w) \in \mathscr{P}_{smo,sco} \times \mathbb{R}^l_{++} \times \mathbb{R}^l_{++}$. Assume $x, y \in \chi(\succsim, p, w)$, $x \neq y$, for some $(\succsim, p, w)$. Then $x \succsim y$ and $y \succsim y$. By convexity of the consumption set $\mathbf{X}$, we have $\frac{1}{2}x + \frac{1}{2}y \in \mathbf{X}$. Moreover, $\frac{1}{2}x + \frac{1}{2}y \leq p \cdot w$. However, the strong convexity of $\succsim$ yields

$$\frac{1}{2}x + \frac{1}{2}y \succ y$$

and thus a contradiction. □

One major assumption in Walrasian theory is the price-taking behavior of agents, none exhibits market power. If we were to justify this behavior, we would tend to say that the economy is large, i.e. it consists of infinitely many agents. A convenient way to describe such an economy is given in the next chapter.

3.3 Distribution Economies

The purpose of this section is to introduce *distribution economies* that are tailor-made for modeling large economies, i.e. economies with (uncountably) infinitely many agents. Moreover, we consider sequences of simple, i.e. finite, economies converging to economies with an infinite set of agents. Of course, we have to define an appropriate notion of convergence: Since we conceive economies as distributions of agents' characteristics, convergence of economies is just given by weak convergence of corresponding distributions.

We may again identify an agent characterized by $(\mathbf{X}, \succsim, w)$ with an element $(\succsim, w) \in \mathbf{C} := \mathscr{P} \times \mathbf{W}$. As in [Hildenbrand, 70], p. 166, a simple pure exchange economy $\mathscr{E}^s$ is given by a finite family of elements in $\mathbf{C}$.[2] This already motivates to identify an economy with its distribution in $\mathbf{C}$. Consequently, we will now consider a probability measure ν on $(\mathbf{C}, \mathcal{B}(\mathbf{C})) := (\mathscr{P} \times \mathbb{R}^l, \mathcal{B}(\mathscr{P}) \times \mathcal{B}(\mathbf{W}))$ and thus also allow for economies with (uncountably) infinitely many agents.

Nevertheless, if we want to achieve economies as distributions on the set of agents' characteristics, abstracting from the set of agents, we should do so in a way that is motivated by the basic definitions in section 3.2. However, the next definition is not necessary for our models and one may go ahead to Definition 3.12. We generalize Definition 3.6:

[2]The primitives in [Hildenbrand, 70] are more general. However, later on we would have to confine ourselves to this case.

Definition 3.11. ([Hildenbrand, 74], p. 125) *Let $\bar{\mathbf{M}}$ be a not necessarily finite set of agents.*
(a) *A* (pure exchange) economy *$\mathscr{E}$ is a random variable on a probability space $(\bar{\mathbf{M}}, \bar{\mathcal{M}}, Q)$ with values in $\mathbf{C}$ of agent's characteristics such that the mean endowment $\int proj_w \circ \mathscr{E} dQ$ is finite. Here, $proj_w$ denotes the projection mapping on the space of initial endowments $\mathbf{W}$. In this sense, $proj_w \circ \mathscr{E}$ is the initial endowment.*[3]
(b) *A* (per capita) allocation *for the economy $\mathscr{E}$ is an integrable function f of $\bar{\mathbf{M}}$ into $\mathbb{R}^l$ such that a.e. in $\bar{\mathbf{M}}$ the consumption vector $f(i)$ belongs to the consumption set of agent $i \in \bar{\mathbf{M}}$. An allocation is called* attainable *for $\mathscr{E}$ if*

$$\int f dQ = \int proj_w \circ \mathscr{E} dQ.$$

Thus, we see that the initial endowment $proj_w \circ \mathscr{E}$ is well defined as an allocation.
(c) *An economy $\mathscr{E}$ is called* simple *if $\bar{\mathbf{M}}$ is a finite set, $\bar{\mathcal{M}}$ is the power set and $Q(E) = \frac{|E|}{|\bar{\mathbf{M}}|}$ for all $E \subset \bar{\mathbf{M}}$. $\mathscr{E}$ is called* atom-less *if $(\bar{\mathbf{M}}, \bar{\mathcal{M}}, Q)$ is atom-less, i.e. for every $E \in \bar{\mathcal{M}}$ with $Q(E) > 0$ there exists $E \supset \tilde{E} \in \bar{\mathcal{M}}$ such that $0 < Q(\tilde{E}) < Q(E)$.*

Note, that the notion of a simple economy as defined in Definition 3.6 coincides with the notion in (c) above. Motivated by Definition 3.11, a measure ν on $\mathbf{C}, \mathcal{B}(\mathbf{C})$ is called *simple* if ν is the uniform distribution on a finite subset of $\mathbf{C}$.

Due to measurability, $\mathscr{E}$ is "well behaved", so that we may equivalently consider the image distribution $\nu := Q \circ \mathscr{E}^{-1}$ of Q under the mapping $\mathscr{E}$. This motivates the following equivalent definition, where we abstract from the set $\bar{\mathbf{M}}$ of agents.

Definition 3.12. *A* distribution (pure exchange) economy *$\mathscr{E}^d$ is a probability distribution ν on the space of individual characteristics $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$, such that initial endowment $w : \mathbf{C} \to \mathbf{W}$ is ν-integrable. In analogy to Definition 3.11, $\mathscr{E}^d$ is called simple or atom-less if ν is simple or atom-less, respectively. An allocation f for $\mathscr{E}^d$ is given by a ν-integrable function $f : supp(\nu) \to \mathbb{R}^l$ such that $f(i) \in \mathbf{X}_i$ ν-a.e. in $supp(\nu)$. An allocation f is called attainable if $\int_{\mathbf{C}} (f - w) d\nu = 0$.*

In case of a simple economy, per capita or mean initial endowment is given by

$$\int_{\bar{\mathbf{M}}} proj_w \circ \mathscr{E} dQ = \int_{\mathbf{C}} w d\nu = \frac{1}{|supp(\nu)|} \sum_{a \in \mathbf{M}} w_a.$$

[3]In [Hildenbrand, 74], the definition is posed for measurable spaces; however, we confine ourselves to probability spaces. Although, we use the term "random variable" for the economy, there is no stochastics involved so far! Thus, we should just think of a measurable mapping.

Let $\mathcal{M}_0(\mathbf{C})$ denote the set of all probability measures on $\mathbf{C}$ with compact support.

Definition 3.13. ([Hildenbrand, 74], Definition 3, p. 129) *An allocation f for the economy $\mathscr{E}^d$ and a price vector $p \in \mathbb{R}^l$ is called a* Walras equilibrium *for $\mathscr{E}^d$ if $f(i) \in \chi_i(\succ_i, p, w)$ ν-a.e. and $\int f d\nu = \int w d\nu$. A price vector $p \in \mathbb{R}^l$ is called an* equilibrium price *for the economy $\mathscr{E}^d$ if there exists an allocation f for $\mathscr{E}^d$ such that (f, p) is a Walras equilibrium.*

In terms of the *market excess demand* $X(p) := \int_{\mathbf{C}} (\chi - w) d\nu$,[4] p is an equilibrium price if and only if $0 \in X(p)$.

The price-taking behavior of agents can now be described by considering atom-less measures $\nu \in \mathcal{M}_0(\mathbf{C})$: Here, $\nu(i) = 0$ for all $i \in supp(\nu)$. This may be interpreted as the fact that no individual agent exhibits "market power".

When we have introduced economies in the first section of this chapter, we were thinking of a finite population of agents. One may object that economies with a continuum of agents, i.e. given by an atom-less measure, are heuristically not well motivated by economic considerations. However, if we could show that atom-less economies are just limits of "growing" simple economies, the notion of a distribution economy would make sense at all. This is stated in the next theorem.

Theorem 3.14. ([Hildenbrand, 70], Theorem 1) *Let $\nu \in \mathcal{M}_0(\mathbf{C})$ be atom-less. There exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of simple measures on $\mathbf{C}$ converging weakly to ν and such that $supp(\nu_n) \subset supp(\nu_{n+1}) \subset supp(\nu)$*

Proof. [Hildenbrand, 70], pp. 169,170 □

This theorem shows that every atom-less economy $\mathscr{E}^d$ can be obtained as a limit of a sequence $(\mathscr{E}^s_n)_{n\in\mathbb{N}}$ of simple economies. In particular, any distribution economy can be obtained in this manner.

We now have motivated to consider economies with a continuum of agents since we have shown that each of those, at least as long as $\nu \in \mathcal{M}_0(\mathbf{C})$, can be obtained by a "growing" finite population of economic agents.

3.4 Random Economies

So far, no stochastics was involved in our economic model. This section is based on [Hildenbrand, 71], wherein the underlying idea is that certain consistency requirements that are assumed in general equilibrium theory, e.g. budget balancedness, cannot be posed in absolute terms but in a stochastic manner: Despite choosing a commodity vector in $\mathbf{X}$, the individual agent

[4] The introduction of an integral for correspondences is straightforward. For a rigorous definition we refer to [Ellickson, 93], p. 350.

determines a probability to choose a vector from a subset of $\mathbf{X}$. This idea has come into account when experiments have shown that agents do not necessarily make the same choices when all parameters are held constant. The question why non-deterministic behavior arises is a philosophical one and beyond the scope of this diploma thesis. We just assume that such stochastic behavior arises. Nevertheless, as mentioned for example in [Hohnisch, 03], the question is whether stochastic behavior is an intrinsic property of human behavior or if it just arises because of hidden parameters the experimenter is not aware of. However, these considerations suggest to conceive preferences as random and thus, we obtain random choices.

Hildenbrand introduces the notion of random preferences, i.e. preference valued random variables. This approach will eventually lead to the notion of a random economy. Of course, in this context individual demand and total excess demand will be random, too. With the notation from the preceding section, an agent is completely described by an element in $\mathscr{P} \times \mathbf{W}$. Again, we endow $\mathscr{P}$ with the topology of closed convergence. We moreover consider an arbitrary underlying probability space $(\Omega, \mathcal{F}, P)$.

Definition 3.15. ([Hildenbrand, 71], p. 416) (a) *Given* $(\Omega, \mathcal{F}, P)$*. A* random preference ϑ_i *of agent* $i \in \bar{\mathbf{M}}$ *is a random variable, i.e. a measurable mapping,* $\vartheta_i : \Omega \to \mathscr{P}$*. Equivalently, a random preference for agent i is described by the image measure* $P \circ \vartheta_i^{-1}$ *on* $(\mathscr{P}, \mathcal{B}(\mathscr{P}))$*.*
(b) *A* random agent σ_i*,* $i \in \bar{\mathbf{M}}$*, is a random variable* $\sigma_i : \Omega \to \mathscr{P} \times \mathbf{W}$*. Equivalently, random agent i is described by the image measure* $\mu_i := P \circ \sigma_i^{-1}$*. In this sense, a random preference is just the projection of a random agent on the space of preferences.*
(c) (Random) demand *of agent* σ_i*,* $i \in \bar{\mathbf{M}}$*, is given by the random variable* $\chi(\sigma_i(\cdot), p)$ *for the prevailing price system* $p \in \mathbb{R}^l$*, where* χ *is given in Definition 3.7.*

This setup calls for a new type of economy: The *random economy.* Later, we have to distinguish between local and global random economies. As long as random agents are assumed to be independently distributed, local data uniquely specifies global data. Thus, no global data is needed to specify the global distribution of the economy. In this sense, we do not have to distinguish between local and global random economies at this point. However, in terms of Section 5.3, the following definition is of macroeconomic character and we will thus refer to it as a global random economy:

Definition 3.16. *A* global random (pure exchange) economy $\mathscr{E}^{gr} := \{\sigma_i\}_{i \in \bar{\mathbf{M}}}$ *is a family of random agents* σ_i *on* $(\Omega, \mathcal{F}, P)$*; equivalently, a probability measure* μ *on the measurable space* $(\mathscr{P} \times \mathbb{R}^l, \mathcal{B}(\mathscr{P}) \times \mathcal{B}((R)^l))^{\bar{\mathbf{M}}}$*. A random economy is called* simple *if* $\bar{\mathbf{M}}$ *is finite. Using the equivalent characterization,* $\mathscr{E}^{gr}$ *is called simple if its distribution* μ *is the uniform distribution with finite support.*

What follows is not necessarily needed for our further analysis. However, for the sake of completion, it is stated here.

Definition 3.17. *Given a price system $p \in \mathbb{R}^l$ the* (random) total excess demand X *for the simple global random economy $\mathscr{E}^{gr}$ is defined by $\zeta(\cdot, p) := \sum_{i \in \bar{\mathbf{M}}}[\chi(\sigma_i(\cdot), p) - proj_w(\sigma_i(\cdot))]$, where $proj_w(\cdot)$ denotes the projection of $\mathbf{C}$ on $\mathbf{W}$. Again, a price system $\tilde{p}$ is an equilibrium price system if $0 \in \int X(\cdot, \tilde{p})dP$.*

Let now $\mathbf{X} = \mathbb{R}^l_+$. We then consider random agents σ_i with distribution μ_i that has support in $\mathscr{P}_{smo,sco} \times \mathbf{W}$.

Proposition 3.18. ([Hildenbrand, 71], Theorem 1) *Let $\mathscr{E}_n^{gr} = \{\sigma_i^n\}_{i \in M_n}$, $n \in \mathbb{N}$, be a sequence of simple global random economies such that*

- *$|\mathscr{E}_n^{gr}| := |M_n| \to \infty$ as $n \to \infty$,*
- *for every n, the family $\{\sigma_i^n\}_{i \in M_n}$ is stochastically independent,*
- *there is a weakly compact set in $\mathcal{P}(\mathscr{P}_{smo,sco} \times \mathbf{W}, \mathcal{B}(\mathscr{P}_{smo,sco}) \times \mathcal{B}(\mathbf{W}))$ which contains every distribution of σ_i^n, $i \in M_n$, $n \in \mathbb{N}$.*

Then for every economy $\mathscr{E}_n^{gr}$ there exists a price system $\bar{p}_n \in \mathbb{R}^l_{++}$ such that (1) *the expected total excess demand of $\mathscr{E}_n$ with respect to $\bar{p}_n$ is zero, i.e.*

$$\int \zeta_{\mathscr{E}_n^r}(\cdot, \bar{p}_n)dP = 0,$$

(2) *the total excess demand per capita of $\mathscr{E}_n^{gr}$ with respect to $\bar{p}_n$ converges in probability to zero, i.e. for every $\delta > 0$,*

$$P\left\{\omega \in \Omega : \left|\frac{\zeta_{\mathscr{E}_n^{gr}}(\omega, \bar{p}_n)}{|\mathscr{E}_n^{gr}|}\right| \le \delta\right\} \to 1 \quad as \quad n \to \infty$$

Proof. [Hildenbrand, 71], pp. 421,422 □

Part (1) states that there exists an equilibrium for $\mathscr{E}_n^{gr}$. Part (2) gives us some kind of limiting equilibrium state of the economy in probability. We obtain stronger results when considering a particular class of sequences of economies.

However, for our purposes it is not only the result of the above theorem that is remarkable but the independence assumption that is necessary to establish this result. In the random economy with independent random agents interaction only comes into account as global (market) interaction via price systems. In other words, there is a (global) market structure, but a (local) social structure is not introduced to the model so far.

Connecting Random & Distribution Economies for Independent Agents

In this section, we obtain a result that links random and distribution economies if the random agents are independent and identically distributed. In Chapter 6 we obtain a stronger result that combines distribution and random economies when the random economy is ruled by an ergodic distribution.

Given $\omega \in \Omega$ we consider the family of realizations $\{\sigma_i(\omega)\}_{i\in\bar{\mathbf{M}}}$. The *sample or empirical distribution* $\nu_{\mathscr{E}^{gr}}(\omega)$ of the simple global random economy $\mathscr{E}^{gr}$ for the sample ω is defined by

$$\nu_{\mathscr{E}^{gr}}(\omega) = \frac{1}{|\bar{\mathbf{M}}|}\sum_{i\in\bar{\mathbf{M}}}\delta_{\sigma_i(\omega)}. \tag{3.1}$$

Note that $\nu_{\mathscr{E}^{gr}}(\omega)$ is a probability distribution on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$. Thus, $\nu_{\mathscr{E}^{gr}}$ is a distribution valued random variable and $\nu_{\mathscr{E}^{gr}}(\omega)$ is actually a simple distribution economy.

We now try to obtain a result showing some kind of "stability" of $\nu_{\mathscr{E}^{gr}}(\omega)$ in the sense that for economies with "growing populations" $\nu_{\mathscr{E}^{gr}}(\omega)$ gets more and more independent of the particular chosen ω and thus weakly converges to some distribution ν on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$, the limiting empirical distribution as given in Definition 6.2.

Consider a sequence $(\mathscr{E}_n^{gr})_{n\in\mathbb{N}}$ of simple random economies. Two agents i and j are said to be of the same *type* $k \in \mathbb{N}$ if the random variables σ_i^n and σ_j^n are equally distributed on $\mathbf{C}$, i.e. $\mu_i = \mu_j = \varsigma_k$ for some ς_k. Let C_k^n be the number of all agents in $\mathscr{E}_n^{gr}$ that are distributed by ς_k on $\mathbf{C}$, i.e. the number of type k agents in economy $\mathscr{E}_n^{gr}$. The sequence $(\mathscr{E}_n^{gr})_{n\in\mathbb{N}}$ is said to be *steadily increasing* if

1. $|\mathscr{E}_n^{gr}| := |\bar{\mathbf{M}}_n|$ is strictly increasing in n,
2. C_k^n is non-decreasing in n, $\lim_{n\to\infty}\frac{C_k^n}{|\mathscr{E}_n^{gr}|} = c_k$ exists and $\sum_{k=1}^{\infty} c_k = 1$, i.e. c_k is the fraction of type k agents in the limiting economy.

The measure ν on $\mathbf{C}$, given by $\nu = \sum_{k=1}^{\infty} c_k\varsigma_k$, independent of $\omega \in \Omega$, is called the *asymptotic distribution* of $(\mathscr{E}_n^{gr})_{n\in\mathbb{N}}$.

Proposition 3.19. ([Hildenbrand, 71], Theorem 2) *Let $(\mathscr{E}_n^{gr})_{n\in\mathbb{N}}$ be a steadily increasing sequence of simple global random economies. If in every economy $\mathscr{E}_n^{gr}$, the random agents of the same probabilistic type are stochastically independent, then the sequence $(\nu_{\mathscr{E}_n^{gr}}(\omega))_{n\in\mathbb{N}}$ of sample distributions converges in probability to the asymptotic distribution ν. Formally, for all $\epsilon > 0$*

$$P(\{\omega \in \Omega : d(\nu_{\mathscr{E}_n^{gr}}(\omega), \nu) > \epsilon\}) \to 0 \tag{3.2}$$

as $n \to \infty$, where d denotes the metric of weak convergence on $\mathcal{P}(\mathbf{C}, \mathcal{B}(\mathbf{C}))$.

Proof. [Hildenbrand, 71], pp. 424,425 □

As already mentioned, this theorem establishes a relation between random and distribution economies for the independent case. At this point, we are not going to apply Proposition 3.19. However, we should have in mind this result since we discuss a generalization for the ergodic case in Chapter 6.

Proposition 3.20. ([Hildenbrand, 71], Theorem 3) *Let* $(\mathscr{E}_n^{gr})_{n\in\mathbb{N}}$ *be a steadily increasing sequence of simple global random economies where every distribution is concentrated on the set* $\mathscr{P}_{smo,sco} \times \mathbf{W}$. *Assume that in every economy* $\mathscr{E}_n^{gr}$ *the agents of the same type are stochastically independent. Then for every economy* $\mathscr{E}_n^{gr}$ *there is a price system* $\bar{p}_n$ *such that expected total excess demand with respect to* $\bar{p}_n$ *is zero, every adherent point* p *of the sequence* $\bar{p}_n$ *is a price-equilibrium for the asymptotic distribution* ν, *and total excess demand per capita of* $\mathscr{E}_n^{gr}$ *with respect to* $\bar{p}_n$ *converges in probability to zero.*

Proof. [Hildenbrand, 71], pp. 425-427 □

A special case of the above theorems is of course a simple random economy with independent and identically distributed random agents. Proposition 3.20 gives us the adherent points of the sequence of price equilibria as price equilibria of the limiting economy.

As mentioned at the beginning of this chapter, the approaches so far are preference-based in the sense that the state space consists of preferences. However, we may consider more general models when generalizing the state space to any economic or social characteristic we can think of.

3.5 Föllmer's Interacting Agents

In the preceding section it is assumed that agent's preferences are random but independent (within each type). Föllmer generalizes this approach by allowing the probability distributions of an agent's preferences to depend on her environment. Relaxing from independence is one approach taking into account locally interacting agents. We thus obtain another kind of interaction besides market interaction via the price system. The primitives of the model, the conditional distributions that govern the randomness of individual preferences with respect to an agent's environment, are microscopic and given by specifications as defined in Chapter 2. In this sense, the models in this section are of microeconomic type. However, we are interested in the emerging macroeconomic state of an infinite economy with interacting agents. In particular, Föllmer poses the question if equilibrium prices may be derived solely on the knowledge of local data. Due to spontaneous emergence of macroscopic states, we will see that the answer to this question is negative.

Stated another way: "*The microeconomic characteristics may no longer determine the macroeconomic phase, that is, the global probability law which governs the joint behavior of all economic agents.*" ([Föllmer, 74], p.52)

3.5.1 Primitives of the Model

The notation that we use differs from that in [Föllmer, 74] and may seem a little odd to economists. However, the rationale for this notation is the conformity with statistical mechanics. Let $\emptyset \neq \mathbf{S}$ be a countably infinite set of economic agents.[5] Individual consumption sets are closed convex subsets $\mathbf{X}_s \subset \mathbb{R}^l_+$, $s \in \mathbf{S}$. Assume identical consumption sets $\mathbf{X}_s = \mathbf{X}$. An agent $s \in \mathbf{S}$ is characterized by an element in the state space $\mathbf{E} \subset \mathscr{P}_{smo,sco} \times \mathbf{W}$ with appropriate σ-algebra $\mathcal{E}$. For technical convenience, Föllmer assumes $\mathbf{E}$ to be finite.

Remark 3.21. (a) *Assume throughout some underlying probability space $(\Omega, \mathcal{F}, P)$. A random agent at site $s \in \mathbf{S}$, called s, is again given by a random variable $\sigma_s : \Omega \to \mathbf{E}$.*
(b) *As in Chapter 2, we canonically set $\Omega := \mathbf{E}^{\mathbf{S}}$ and $\mathcal{F} := \mathcal{E}^{\mathbf{S}}$. We furthermore define $\sigma_s : \Omega \to \mathbf{E}$ by the projection mapping $\Omega \ni \omega \mapsto \sigma_s(\omega) := \omega_s$. Thus, random agent $s \in \mathbf{S}$ is given by some $\omega_s \in \mathbf{E}$.*

In terms of Chapter 2, $\mathbf{S}$ is the parameter set, $\mathbf{E}$ is the state space and Ω the configuration space, with $\omega \in \Omega$ a configuration.

In a deterministic context, an economy is given by a configuration ω, whereas in our context, an economy may be thought to be given by a probability distribution μ on Ω as seen in the last section. However, if we would define a random exchange economy by a probability space $(\Omega, \mathcal{F}, \mu)$, the model would not be microeconomic any longer: Under the independence assumption of the last section, μ would be given by the product measure $\mu = \otimes_{s \in \mathbf{S}} \mu \circ \sigma_s^{-1}$. However, when relaxing the independence assumption, we have to know the distributions of states for subeconomies in order to determine μ and hence aggregate data.

Thus, to keep the model microeconomic, we define the economy via specifications on Ω. In this setup, we are not given specifications in the general form of chapter 2 but as probability distributions $\gamma_s(\cdot|\cdot)$, $s \in \mathbf{S}$, on $\mathbf{E}$ conditional on the states of all other agents, the "singleton parts" $(\gamma_{\{s\}})_{s \in \mathbf{S}}$ of a specification $(\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ as given in Definition 2.6. However, Theorem 1.33 in [Georgii, 88] justifies this approach for the Ising model. The distributions μ on $(\Omega, \mathcal{F})$ that specify the macroeconomic state of the economy will then

[5] We do not use $\bar{\mathbf{M}}$ for the set of agents here, since we want to emphasize that now the graph structure of the set of agents is crucial for the model.

be given by the corresponding Gibbs measures, i.e. by measures that are consistent with specifications in the sense that the finite volume projections of μ coincide with the conditional distributions.

Definition 3.22. ([Föllmer, 74], Definitions 2.1 & 2.2) (a) *An* environment *of an economic agent $s \in \mathbf{S}$ is a mapping $\eta : \mathbf{S} \setminus \{s\} \to \mathbf{E}$ which specifies the states of the other agents; $\eta \in \Omega_{\{s\}^c}$.*
(b) *Let $\mathcal{E}$ be an appropriate σ-algebra on $\mathbf{E}$. The* characteristic *γ_s of agent $s \in \mathbf{S}$ is given by a probability kernel γ_s from $\mathcal{T}_{\{s\}}$ to $\mathcal{E}$, such that $\gamma_s(\cdot|\eta)$ is a probability distribution on the state space $\mathbf{E}$ conditional on agent s' environment η.*
(c) *A* local random (pure exchange) economy *is a triple $\mathscr{E}^{lr} = (\mathbf{S}, \mathbf{E}, \gamma)$ where $\gamma = (\gamma_s)_{s \in \mathbf{S}}$ is a collection of characteristics. γ is called* (microeconomic, microscopic or local) characteristics *of $\mathscr{E}^{lr}$.*
(c) *A probability measure μ on $(\Omega, \mathcal{F})$ is called a* (macroeconomic, macroscopic or global) Gibbs state *for the economy, if μ is consistent with γ in the following sense:*

$$\mu[\omega_s = e|\eta] = \gamma_s(e|\eta) \quad \mu - \textit{almost surely.}$$

for all $s \in \mathbf{S}$ and $e \in \mathbf{E}$. Local characteristics γ is called consistent *if it admits at least one Gibbs state. Let us denote the set of all Gibbs states of $\mathscr{E}^{lr}$ by $\mathcal{G}(\mathscr{E}^{lr})$.*[6]

Again, by Theorem 1.33 in [Georgii, 88], $\mathcal{G}(\mathscr{E})$ is the set of Gibbs measures with respect to specification γ, $\mathcal{G}(\gamma)$.

In Hildenbrand's context, where preferences are random but independent, the unique Gibbs state μ is given by the product measure $\mu(\cdot) = \otimes_{s \in \mathbf{S}} \gamma_s(\cdot|\eta)$ on $\mathbf{E}^{\mathbf{S}}$. In particular, the Gibbs state is uniquely determined by microeconomic characteristics. However, as we have already seen in chapter 2, there may be consistent characteristics that admit more than one Gibbs state.

As in the last section, we assume $\mathbf{E} \subset \mathscr{P}_{sco,smo} \times \mathbf{W}$. Thus, demand $\chi(\omega(s), p)$ of agent s facing price system p given by preference maximization with respect to budget constraint is a function.

Definition 3.23. ([Föllmer, 74], p. 55) *Let $\mathbf{S}_0$ be a finite subpopulation of $\mathbf{S}$. The* per capita excess demand for subpopulation $\mathbf{S}_0$ *for $\omega \in \Omega$ and price system $p \in \mathbb{R}^l_{++}$ is given by*

$$\frac{1}{|\mathbf{S_0}|} \sum_{s \in \mathbf{S_0}} \zeta(\omega_s, p),$$

where $\zeta : \mathbf{E} \times \mathbb{R}^l_{++} \to \mathbb{R}^l$, $\zeta(\omega_s, p) := \chi(\omega_s, p) - w_s$, is defined as the individual excess demand and w_s denotes the initial endowment of s.[7]

[6] Note that $\mu \in \mathcal{G}(\mathscr{E}^{lr})$ defines a global random economy $\mathscr{E}^{gr}$.

[7] Unlike Föllmer, I assume $p \in \mathbb{R}^l_{++}$. In light of our assumptions on preferences, demand might be unbounded otherwise.

In keeping an eye out for an equilibrium price system, the question arises if we can find a price system such that the global per capita excess demand is approximately zero for sequences of subpopulations approaching the countably infinite set $\mathbf{S}$.

We consider finite subeconomies $\mathbf{S}_n \subset \mathbf{S}$. A sequence $(\mathbf{S}_n)_{n\in\mathbb{N}}$ *exhausts* $\mathbf{S}$ if $\cup_{n\in\mathbb{N}}\mathbf{S}_n = \mathbf{S}$ and $\mathbf{S}_n$ represent $\mathbf{S}$ in a sense that depends on the specific structure of $\mathbf{S}$. In case $\mathbf{S} := \mathbb{Z}^d$ let $B_n := \{s \in \mathbf{S} : |s| \le n\}$. A sequence of finite subpopulations $(\mathbf{S}_n)_{n\in\mathbb{N}}$ exhausts $\mathbf{S}$ if $\mathbf{S}_n \subset B_n$ and there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $\mathbf{S}_n = \dot\cup_{i\le m}\tilde{B}_i$, where $m \le N$, $\tilde{B}_n$ is a "box" parallel to the axes of $\mathbf{S}$ and $|\mathbf{S}_n||B_n|^{-1} \ge \delta$. Intuitively, this means that $(\mathbf{S}_n)_{n\in\mathbb{N}}$ exhausts $\mathbf{S}$ in approximately the same way as $(B_n)_{n\in\mathbb{N}}$.

Definition 3.24. ([Föllmer, 74], Definition 2.6) *Given $\mathscr{E}^{lr}$, we say that a price $p \in \mathbb{R}^l_{++}$* equilibrates, *or* stabilizes, *the Gibbs state μ of the economy $\mathscr{E}^{lr}$ if*

$$\lim_{n\to\infty} \frac{1}{|\mathbf{S}_n|} \sum_{s\in\mathbf{S}_n} \zeta(\omega_s, p) = 0 \quad \mu\text{-almost surely} \tag{3.3}$$

whenever $(\mathbf{S}_n)_{n\in\mathbb{N}}$ is an increasing sequence of coalitions which exhausts $\mathbf{S}$. p equilibrates, or stabilizes, the economy $\mathscr{E}^{lr}$ if p equilibrates each Gibbs state of $\mathscr{E}^{lr}$.

Föllmer now poses the following two questions: Can we stabilize any given Gibbs state? Can we stabilize a local random economy $\mathscr{E}^{lr}$ when we only know the microscopic characteristics?

3.5.2 Markov Economies

In Chapter 2 we have already stated the close relation between the Markov property and a nearest-neighbor potential. We again follow this idea.

Definition 3.25. ([Föllmer, 74], p. 56) *Let the finite set $N(s) \subset \mathbf{S}$ denote the set of* neighbors *or* peers *of agent s.*

Assumption 3.26. (a) *We assume that agent $s \in \mathbf{S}$ directly interacts only with agents $b \in N(s)$, i.e.*

$$\gamma_s(\cdot|\eta) = \gamma_s(\cdot|\tilde{\eta}_s)$$

if $\eta|_{N(s)} = \tilde{\eta}_s|_{N(s)}$ $(s \in \mathbf{S})$.
(b) *Let $\mathbf{S} = \mathbb{Z}^d$ for some $d \ge 1$ and assume $N(s) = \{b \in \mathbf{S} : |b - s| = 1\}$, i.e. $N(s)$ is the set of nearest neighbors in $\mathbb{Z}^d$.*

Definition 3.27. (a) ([Föllmer, 74], Definition 3.4) *A local random economy $\mathscr{E}^{lr} = (\mathbf{S}, \mathbf{E}, \gamma)$ where $\mathbf{S}$ and $N(s)$ are defined as in Assumption 3.26(b) and which local characteristics are consistent and satisfy Assumption 3.26(a) is*

called a Markov economy*. We say that* $\mathscr{E}^{lr}$ *is* homogeneous *if* γ *is homogeneous.*
(b) *Given a homogeneous Markov economy* $\mathscr{E}^{lr}$*, let* $\mathcal{G}_{\boldsymbol{\Theta}}(\mathscr{E}^{lr})$ *denote the set of all homogeneous Gibbs states of* $\mathscr{E}^{lr}$*. In accordance to chapter 2 the extreme elements in* $\mathcal{G}_{\boldsymbol{\Theta}}(\mathscr{E}^{lr})$*,* $ex\mathcal{G}_{\boldsymbol{\Theta}}(\mathscr{E}^{lr})$*, are called* pure states*. The case* $|\mathcal{G}(\mathscr{E}^{lr})| > |\mathcal{G}_{\Theta}(\mathscr{E}^{lr})|$ *is called* symmetry breakdown*, the case* $|\mathcal{G}(\mathscr{E})^{lr}| > 1$ phase transition.

Recall that the set of pure states is just the set of ergodic Gibbs states. From now on, let us consider $\mathscr{E}^{lr}$ to be a homogeneous Markov economy. In this case, γ is consistent by definition and we have $|\mathcal{G}(\mathscr{E}^{lr})| \geq |\mathcal{G}_{\Theta}(\mathscr{E}^{lr})| \geq 1$. Symmetry breakdown "*means that, although the individual agents are all governed by the same conditional probability law, the global phase may be inhomogeneous, and in particular the individual distributions* μ_s *may vary*" ([Föllmer, 74], p. 57) among agents. The existence of phase transition may be interpreted as the possibility of spontaneous changes of phases even though the local characteristics remain fixed. Recall that Remark 2.36 shows that symmetry breaking implies phase transition.

Proposition 3.28. ([Föllmer, 74], Theorem 3.6) *Let* $\mathscr{E}^{lr}$ *be a homogeneous Markov economy. Then, any pure phase can be stabilized.*

Proof. By Proposition 2.49, a Gibbs state $\mu \in \mathcal{G}_{\Theta}(\mathscr{E}^{lr})$ is extreme in $\mathcal{G}_{\Theta}(\mathscr{E}^{lr})$ if and only if μ is ergodic. Thus, a pure state is ergodic and we may apply an ergodic theorem. Let θ_a denote the spatial shift by a, then the ergodic theorem in [Georgii, 72], pp. 125,126, yields for a sequence $(\mathbf{S}_n)_{n\in\mathbb{N}}$ exhausting $\mathbf{S}$ in the sense defined for $\mathbb{Z}^d$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|\mathbf{S}_n|}\sum_{s\in\mathbf{S}_n}\zeta(\omega_s,p) &= \lim_{n\to\infty}\frac{1}{|\mathbf{S}_n|}\sum_{s\in\mathbf{S}_n}(\zeta\circ\theta_s)(\omega_0,p)\\ &= \mu(\zeta(\omega_0,p)) \quad \mu\text{-a.s.}\end{aligned}$$

Existence of a price system p such that $\mu(\zeta(\omega_0,p)) = \int\zeta(\omega_0,p)d\mu = 0$ directly follows from Proposition 3.18(1) for $|\mathscr{E}_n^{lr}| = 1$. □

Corollary 3.29. *Let* $\mathscr{E}^{lr}$ *be a homogeneous Markov economy. If* $|\mathcal{G}(\mathscr{E}^{lr})| = 1$*, then* $\mathscr{E}^{lr}$ *can be stabilized.*

Proof. In this case $\mathcal{G}(\mathscr{E}^{lr}) = ex\mathcal{G}_{\Theta}(\mathscr{E}^{lr}) = \{\mu\}$. Proposition 3.28 shows the assertion. □

Thus, if there is no phase transition, i.e. $|\mathcal{G}(\mathscr{E}^{lr})| = 1$, we can stabilize a homogeneous Markov economy knowing only the microscopic data. In light of this result, it may be interesting to encounter conditions ensuring absence of phase transition:

Assumption 3.30. *For all agents $s \in \mathbf{S}$ and environments $\eta \in \mathbf{E}^{\mathbf{S}\setminus\{s\}}$, $\gamma_s(\cdot|\eta) > 0$.*

Proposition 3.31. ([Föllmer, 74], p. 58) *Let $\mathscr{E}^{lr}$ be a homogeneous Markov economy and Assumption 3.30 hold. Then local characteristics γ can be written as*

$$\gamma_s(e|\eta) = Z(s,\eta)^{-1} e^{\psi(s,e)+\sum_{b\in N(s)} U(s,b,e,\eta(b))}, \tag{3.4}$$

where $Z(\cdot,\cdot)$ is a partition function. U satisfies $U(s,b,\cdot,\cdot) = 0$ whenever $|s-b| \neq 1$. Moreover, $U(s+b,\tilde{s}+b,\cdot,\cdot) = U(s,\tilde{s},\cdot,\cdot)$ and $\psi(s+b,\cdot) = \psi(s,\cdot)$.

Proof. 2.17 Changing slightly our notation in Chapter 2, the assertion follows directly from Definition 2.15 and Corollary 2.17 together with Definition 2.12. However, the latter definition is more general than needed for this proof and alternatively, the assertion can be seen directly in [Preston, 74], Theorem 1.1 or Theorem 4.1 if one is used to the notation therein. □

Let us gain more intuition for this result: The condition $U(s,b,\cdot,\cdot) = 0$ whenever $|s-b| \neq 1$ takes account of the Markov property, whereas $U(s+b,\tilde{s}+b,\cdot,\cdot) = U(s,\tilde{s},\cdot,\cdot)$ and $\psi(s+b,\cdot) = \psi(s,\cdot)$ reflects homogeneity of local characteristics.

Having a look at Definition 2.12 and equation (5.1) we may reinterpret ψ and U in terms of a nearest neighbor potential in the following way: ψ represents the one body potential and the external field, whereas U represents the two body potential, i.e. the coupling among two distinct nearest neighbors. In this sense $\psi(s,\cdot)$ is the *inner direction* of agent s and $U(s,\cdot,\cdot,\cdot)$ the *outer direction*, i.e. the intensity of correlation with her neighbors.

Föllmer now states a theorem due to Spitzer and Dobrushin

Theorem 3.32. ([Föllmer, 74], Theorem 3.12) $|\mathcal{G}(\mathscr{E}^{lr})| = 1$ *if*

$$max|U(\cdot,\cdot,\cdot,\cdot)| \quad \textit{is small enough (relative to } \psi\textit{)},$$

i.e. if the economic agents are sufficiently inner directed, or

$$d = 1,$$

i.e. if the structure of interaction is one-dimensional.

Proof. [Spitzer, 71a], Example 5 □

Intuitively, the theorem says that microscopic data may not be enough to determine the macroscopic state when interaction is both, complex and strong: In this case there may be more than one pure state and thus, due to convexity of $\mathcal{G}(\mathscr{E}^{lr})$, uncountably many Gibbs states.

In Section 2.7, we have already seen this result for the Ising model. We are now able to answer the second question posed at the end of the foregoing section in a limited context: Considering a homogeneous Markov economy that satisfies at least one of the properties in the theorem above, we can stabilize the economy since it exhibits a unique Gibbs state.

3.5.3 The Ising Economy

The type of economy introduced here is basically an economic reinterpretation of the Ising model from Section 2.7. However, the approach is different. Recall that the Ising ferromagnet was introduced by its potential motivated by the preference of agents to conform. We again fix a homogeneous Markov economy $\mathscr{E}^{lr} = (\mathbf{S}, \mathbf{E}, \gamma)$. To be conform with Section 2.7, we assume $\mathbf{S} = \mathbb{Z}^2$. This is not assumed in [Föllmer, 74], but allows us to apply results already shown for the two-dimensional Ising ferromagnet. However, some assertions here are stated for the general case $d \geq 2$.

Definition 3.33. (a) ([Föllmer, 74], p. 59) *$\mathscr{E}^{lr}$ is called* egalitarian *if for all $s \in \mathbf{S}$, $w_s = w \in \mathbf{W}$ and γ_s is rotation invariant, i.e. $\gamma_s(\cdot|\eta) = \gamma_s(\cdot|\tilde{\eta})$ if $\tilde{\eta} = \eta \circ \phi$ for some permutation ϕ on the set of neighbors.*
(b) ([Föllmer, 74], Definition 4.3) *An* Ising economy *is an egalitarian homogeneous Markov economy with two goods and two exclusive preferences.*

By definition of an Ising economy, we may set $\mathbf{E} = \{+1, -1\}$. Thus, the configuration space becomes $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$. In case of $\omega_s = +1$, agent s exhibits an exclusive preference for commodity 1 and for commodity 2 in case $\omega_s = -1$.

Using constants ψ and J, we rephrase representation (3.4) of local characteristics for the Ising economy as

$$\gamma_s(\pm 1|\eta) = Z(\eta)^{-1} e^{\pm(\psi + J \sum_{b \in N(s)} \eta(b))}, \tag{3.5}$$

where $\pm J\eta(b) := U(\cdot, \cdot, \pm 1, \eta(b))$ and $\pm\psi := \psi(\cdot, \pm 1)$.

We have already seen that local characteristics in a homogeneous Markov economy are generated by a nearest neighbor potential. Having a look at the local characteristics (3.5), we note that these are generated by an Ising potential as defined in Section 2.7.[8] In this sense, the approach here and the one in Section 2.7 coincide.

Definition 3.34. ([Föllmer, 74], Definition 4.5) *An Ising economy is called* cyclic *if $J > 0$ and* anti cyclic *if $J < 0$. It is called* outer directed *if $\psi = 0$.*

In economic terms, this means:

- $J > 0$: preference for conformity among neighbors,
- $J < 0$: preference for antagonism among neighbors,
- $\psi = 0$: no individual tendency in an agent's behavior.

Proposition 3.35. *Let $d \geq 2$ and $J > 0$. Then $|\mathcal{G}(\mathscr{E}^{lr})| = 1$ if $\psi \neq 0$*

[8] However, when introducing the two-dimensional Ising ferromagnet, we have assumed the external field to vanish, i.e. $\psi = 0$.

Proof. [Spitzer, 71b] □

Corollary 3.36. ([Föllmer, 74], Proposition 4.8) *Let $d \geq 2$ and $J > 0$. Then $\mathscr{E}^{lr}$ can be stabilized as long as $\psi \neq 0$.*

Proof. The proof follows directly from Proposition 3.35 and Corollary 3.29. □

Thus, to construct an interesting (ferromagnetic) case with phase transition, we assume $d \geq 2$, $J > 0$ and $\psi = 0$ from now on. Henceforth, we let $d = 2$.

As it was mentioned at the end of Section 2.7, there exists a $J_c \in \mathbb{R}_{++}$ such that for $J > J_c$ we have $|ex\mathcal{G}_\Theta(\mathscr{E}^{lr})| = 2$.[9]

Let $\{\mu^1, \mu^2\} = ex\mathcal{G}_\Theta(\mathscr{E}^{lr})$. Proposition 2.53 then implies

$$\frac{\mu_1^1}{\mu_2^1} = \frac{\mu_2^2}{\mu_1^2} > 1, \tag{3.6}$$

where $\mu_1^i := \mu^i[\omega_0 = +1]$ and $\mu_2^i := \mu^i[\omega_0 = -1]$, $i = 1, 2$. Using the notation introduced in Section 2.7, μ^1 is the Gibbs state posing more mass on the ground state ω^+ and μ^2 the distribution posing more mass on ω^-.

For a given price system $p \in \mathbb{R}^2_{++}$ and individual state in $\mathbf{E}$, the derivation of an agent's excess demand is straightforward since an agent with state "+1" spends her whole income for the first commodity and an agent with state "-1" for the second:

$$\begin{aligned}
\zeta_s(+1, p) &= \left(\frac{p_1}{p_1} w_1 + \frac{p_2}{p_1} w_2, 0\right) - (w_1, w_2) \\
&= \left(\frac{p_2}{p_1} w_2, -w_2\right), \\
\zeta_s(-1, p) &= \left(-w_1, \frac{p_1}{p_2} w_1\right).
\end{aligned}$$

Given a Gibbs state $\mu \in \mathcal{G}_\Theta(\mathscr{E}^{lr})$ we would like to find a price system $p = (p_1, p_2) \in \mathbb{R}^2_{++}$ such that the expected excess demand of an agent $s \in \mathbf{S}$ is zero, i.e.

$$\begin{aligned}
& \mu(\zeta(\omega_s, p)) = (0, 0) \qquad\qquad (3.7) \\
\Leftrightarrow \quad & \underbrace{\mu[\omega_s = +1]}_{=:\mu_1} \left(\frac{p_2}{p_1} w_2, -w_2\right) + \underbrace{\mu[\omega_s = -1]}_{=:\mu_2} \left(-w_1, \frac{p_1}{p_2} w_1\right) = (0, 0),
\end{aligned}$$

[9] In Section 2.7 we have done the analysis in terms of the inverse temperature β. Here, the coupling constant J is defined in a way that takes account of β. In the general case, i.e. $d \geq 2$, J_c depends on dimension d.

where μ_1 and μ_2 do not depend on s due to homogeneity. We obtain

$$(3.7) \quad \Leftrightarrow \quad \frac{p_2}{p_1} = \frac{w_1}{w_2}\frac{\mu_2}{\mu_1} \tag{3.8}$$

As in [Hildenbrand, 71] we are now interested in per capita excess demand as the number of agents tends to infinity: Since extremal homogeneous probability distributions are ergodic, we may again apply the ergodic theorem from the proof of Proposition 3.28. Let $(\mathbf{S}_n)_{n\in\mathbb{N}}$ be a sequence of subpopulations that exhausts $\mathbf{S}$, then

$$\frac{1}{|\mathbf{S}_n|}\sum_{s\in\mathbf{S}_n}\zeta(\omega_s,p) \stackrel{n\to\infty}{\longrightarrow} \mu^i(\zeta(\omega_0,p)) \quad \mu^i - \text{a.e. for } i=1,2. \tag{3.9}$$

Assume that there exists some price system $\tilde{p}\in\mathbb{R}^l_{++}$ such that the right hand side in (3.9) is zero for both, μ^2 and μ^1, i.e. $\tilde{p}$ stabilizes μ^2 and μ^1. Such a price system would necessarily imply (3.8) to hold for μ^2 and μ^1 and would thus generate a contradiction in light of (3.6). Hence, we cannot stabilize the economy $\mathscr{E}^{lr}$.

It is even more concussive. Ergodic decomposition, see Proposition 2.50, allows us to rewrite each homogeneous Gibbs state $\mu\in\mathcal{G}_\Theta(\mathscr{E}^{lr})$ as the barycenter of pure states.

Thus, we can write any $\mu\in\mathcal{G}_\Theta(\mathscr{E}^{lr})$ as

$$\mu = \varsigma\mu^1 + (1-\varsigma)\mu^2 \tag{3.10}$$

for an appropriately chosen $\varsigma\in[0,1]$. Since $\mu^1\neq\mu^2$ by Proposition 2.53 and μ^1 and μ^2 are ergodic, it follows that $\mu^1\perp\mu^2$, i.e. there exist $\Omega^1\subset\Omega$ and $\Omega^2\subset\Omega$ such that $\Omega^1\cap\Omega^2=\emptyset$, $\Omega^1\cup\Omega^2=\Omega$ and $\mu^1(\Omega^1)=1$ and $\mu^2(\Omega^2)=1$.

Hence, extreme ergodic decomposition (3.10) implies that

$$\frac{1}{|\mathbf{S}_n|}\sum_{s\in\mathbf{S}_n}\zeta(\omega_s,p) \stackrel{n\to\infty}{\longrightarrow} \mu^i(\zeta(\omega_0,p)) \tag{3.11}$$

μ-almost surely on Ω^i, $i=1,2$.

This yields the following proposition.

Proposition 3.37. ([Föllmer, 74], Proposition 4.13) *Let $d\geq 2$, $J>J_0(d)$ and $\psi=0$. $\mathscr{E}^{lr}$ can "almost never" be stabilized, i.e.*

$$\forall\mu\in\mathcal{G}_\Theta(\mathscr{E}^{lr})\setminus ex\mathcal{G}_\Theta(\mathscr{E}^{lr}) \quad \nexists p\in\mathbb{R}^2_{++}: \quad \mu(\zeta(\omega_0,p))=0.$$

Proof. Let $\{\mu^1,\mu^2\}=ex\mathcal{G}_\Theta(\mathscr{E}^{lr})$. Then, for every $\mu\in\mathcal{G}_\Theta(\mathscr{E}^{lr})$, there exists $\varsigma\in[0,1]$ such that $\mu=\varsigma\mu^1+(1-\varsigma)\mu^2$. For $\mu\in\mathcal{G}_\Theta(\mathscr{E}^{lr})\setminus ex\mathcal{G}_\Theta(\mathscr{E}^{lr})$ we have $\varsigma\in]0,1[$.

Now assume that there actually exists some price vector $p\in\mathbb{R}^2_{++}$ that equilibrates μ. We have $\mu|_{\Omega^1}=\varsigma\mu^1$ and $\mu|_{\Omega^2}=(1-\varsigma)\mu^2$. Again, combining equations (3.8) and (3.6) with (3.11) we see that whenever p is such that $\mu(\zeta(\omega_0, p))$ vanishes μ-a.s. on Ω^1, it cannot vanish μ-a.s. on Ω^2, vice versa, and we obtain a contradiction. Hence, whenever $\varsigma \in]0, 1[$, we cannot stabilize μ. $\square$

3.5.4 Remarks

Let us reconsider the questions posed. We have seen that we can stabilize a given Gibbs state as long as it is a pure state. However, we cannot equilibrate a Gibbs state that is not ergodic. In particular, we cannot stabilize a local random economy $\mathscr{E}^{lr}$ on the basis of microscopic knowledge when phase transition occurs.

It remains the question whether the results obtained in Föllmer's model are as bad news as indicated. Consider an economy with $|ex\mathcal{G}_\Theta(\mathscr{E}^{lr})| \geq 2$. In particular, there exist uncountably many Gibbs states.

We have seen that we can stabilize any pure state. Thus, the problematic states are those not in $ex\mathcal{G}_\Theta(\mathscr{E}^{lr})$. The (philosophical) question is whether these will emerge at all even though they are consistent with local characteristics in the sense of Definition 3.22. We have already argued in Chapter 2 that the only relevant Gibbs states are phases, i.e. extreme Gibbs states. In [Hohnisch, 03] it is furthermore mentioned that it is commonly assumed in statistical mechanics that systems are ruled by pure states, i.e. ergodic Gibbs measures. Again, by ergodic decomposition we obtain any non-pure (homogeneous) state as a "mixture" of pure states. In this sense, we may think of a non-pure state as black box: The system actually is in a pure state but we cannot observe for sure in which one. Gibbs states that are not pure reflect some kind of uncertainty about the true state of the system.

Assuming now that the only Gibbs states to emerge are pure, one could object that even in this case it is not possible to equilibrate the economy $\mathscr{E}^{lr}$ whenever the set of pure states is not singleton. A priori, i.e. by the knowledge of the local characteristics, we are not able to determine the pure state that eventually will emerge. Thus, it is not possible to a priori determine some price system p that equilibrates the resulting pure phase although we have seen that any pure phase can be equilibrated.

Nevertheless, in general equilibrium theory one is often faced with the problem of multiple competitive equilibria and thus multiple equilibrium price systems. In this case, the modeler does not know which of these equilibria will actually be chosen, since every equilibrium exhibits some kind of stability. In this sense we are faced with the same problem: Given the pure state, we can find an equilibrium price system but we do not know which pure state will emerge.

3.6 A Digression: Heterogeneity

We have already mentioned in the Introduction that structural properties of aggregate demand will be obtained when assuming agents to be "sufficiently heterogeneous". So far, we have not rigorously introduced this notion. Intuitively, we call a population of economic agents heterogeneous if the distribution of characterizing variables, or of at least one variable, is dispersed in that the variance is high. In terms of the next chapter, where the distribution of a specific characteristic is given by a density, this means that the density shall be "flat". In that case, curvature of the density is a measure for heterogeneity in the sense that the population is more heterogeneous the less curved the density.

However, a rigorous way to introduce the notion of "heterogeneity" can for example be found in [Kneip, 99].

Another approach is followed in [Hildenbrand & Kneip, 05]: Here, an agent's individual characteristics are given by a demand function and an income. It is then savagely defined the *degree of behavioral heterogeneity of the population*. It is beyond the scope of this diploma thesis to mimic this introduction of a rigorous concept. However, it should be mentioned that this degree of behavioral heterogeneity is zero if all agents have the same demand function and income or if individual demand is of Cobb-Douglas-type, a concept that we introduce in Definition 4.12. One should bear this in mind: Within the next chapter, heterogeneity comes into account through the distribution of so called α-transforms: We say that heterogeneity among agents increases if the variance of the distribution of α-transforms increases. We show that Cobb-Douglas demand functions are invariant with respect to α-transforms, i.e. we cannot increase behavioral heterogeneity via α-transforms, when the underlying demand is of Cobb-Douglas-type. Thus, the degree of behavioral heterogeneity in the sense of [Hildenbrand & Kneip, 05] stays zero when applying α-transforms. This indicates that the concept of behavioral heterogeneity in [Hildenbrand & Kneip, 05] is not entirely consistent with our intuitive notion of heterogeneity. Moreover it shows a shortcoming of our intuition of heterogeneity. Given Cobb-Douglas demand, the behavior of agents does not change when variance of the distribution of α-transforms increases, however, we would say that heterogeneity increases.

However, what we want to "create" by aggregation is the uncompensated law of demand and the weak axiom of revealed preference. Both properties are already satisfied by Cobb-Douglas type demand functions and due to homotheticity of those by aggregate demand when individual demand is of Cobb-Douglas type. Hence, we may discount this case and only consider increasing heterogeneity via α-transforms for demand functions not of Cobb-Douglas type; in the "Cobb-Douglas-case", we do not need heterogeneity for our purpose. This may help us reconciling with our intuition.

Chapter 4

The Aggregation Problem

4.1 Introduction

Within the present chapter we turn to the analysis of structural properties of aggregate demand in context of an economy, where the distribution of agents' characteristics is given exogenously.

In the foregoing chapter we have assumed that an individual agent's demand is generated by preference maximizing behavior. In this sense, we may say that demand is rational as Walrasian demand satisfies properties that indicate rational behavior. Thus, one primitive of the models in the last chapter were preferences. In terms of [Mas-Colell et al., 95], a preference as given in Definition 3.3 is referred to as rational.

If demand is generated by preference maximization, it satisfies the *weak axiom of revealed preference* as a rationality property. Posing some stronger assumptions, as for example homotheticity of preferences, the *(uncompensated) law of demand property* is satisfied. For an exhaustive discussion on these attributes of demand we refer to [Mas-Colell et al., 95] or [Shafer & Sonnenschein, 82] among others. However, we savagely introduce concepts and results concerning the structure of individual demand in the subsequent section.

In this approach, the primitives of agents demand behavior are not preferences but demand functions that do not necessarily have to originate in preference maximizing behavior. A major problem arises when individual demand shall be aggregated, i.e. when analyzing market demand. It is shown for simple economies in [Mas-Colell et al., 95], Example 4.C.1, that even if all agents satisfy the weak axiom of revealed preference as a minimal assumption of rationality, aggregate demand does not necessarily have to do so. Thus, the weak axiom is in general not preserved by aggregation. However, a property that aggregates is the law of demand and moreover, the law of demand implies the weak axiom. Again, a discussion of market demand for simple economies can be found in [Mas-Colell et al., 95], pp. 105-116.

One may now conjecture that an appropriate analysis of market demand structure asks for structural properties of individual demand that are "inherited" by aggregate demand. However, when considering large economies, the question that is tackled is whether aggregation of demand may enhance its structure. In the preface to [Hildenbrand, 94], we read:

> I believe that the relevant question is not to ask which properties of the individual demand behavior are preserved by going from individual to market demand, but rather to analyze which new properties are created by the aggregation procedure.

More concretely, the crucial question is the following: Even if we do not pose any structural assumptions on individual demand, is it possible to obtain such properties for market demand. Here, we try to obtain the weak axiom of revealed preference for market demand without assuming for individual demand. This would be a modern economic interpretation of Edmund Burke's cognition: *The individual is foolish, but the species is wise.*

There are several answers to the question above: We obtain convexifying effects of aggregation or market demand being continuous or even differentiable if, for a continuum of consumers, types are sufficiently dispersed. For a thorough discussion, we refer to [Trockel, 84]. In this chapter we follow an approach by Jean-Michel Grandmont: In [Grandmont, 92] the author considers (atom-less) distributions on the space of agents' types. Here, we discuss the model in detail. However, the idea is given as follows: For each type there corresponds a demand function ξ and an initial income w. Heterogeneity in a type's demand is basically formalized by a distribution on a family of demand functions $(\xi^\alpha)_{\alpha\in\mathbb{R}^l}$ generated from ξ by a specific transformation. The space of transformations may be identified with $\mathbb{R}^l$, where l denotes the number of distinct commodities. The fundamental result in [Grandmont, 92] is the emergence of the weak axiom of revealed preference for aggregate demand whenever individual demand is sufficiently dispersed, i.e. the density of the distribution of α-transforms for all types is "flat enough". We may then use this structural property of market demand to obtain uniqueness of equilibrium. Moreover, stability of equilibrium could be shown. However, this would be beyond the scope of this diploma thesis. In particular, the results show that individual rationality, as for example preference maximizing behavior, is not crucial for the aggregate to behave rational. This model is a special case of the framework discussed in [Hildenbrand, 94], as seen in Example 2 of Chapter 2. Another model considering heterogeneity in income can be found in [Hildenbrand, 83].

In Grandmont's distribution economy, the distribution of demand is given exogenously. The further question of this diploma thesis is whether the same results can be obtained when the distribution of demand is determined endogenously by virtue of Gibbsian interaction. For this purpose we introduce a random exchange economy that generalizes Föllmer's Ising economy with respect to the state space.

Besides elaborating the proofs in [Grandmont, 92], we generalize a uniqueness result for market exchange equilibrium and discuss further connections of distinct notions of market demand structure.

4.2 Grandmont's Approach to Market Demand

As already mentioned in the introduction to this chapter, the fundamental question is which properties of (per capita) market demand can be generated by aggregation. We see that, without severe rationality assumptions on individual demand, market demand may satisfy the weak axiom of revealed preference when basically the distribution of individual demand approaches the uniform distribution.

Even if we would suppose individual demand to arise by preference maximization, one could show that this would not allow for particular properties of market demand, as e.g. the weak axiom of revealed preference. This result can be found in [Shafer & Sonnenschein, 82], Theorem 5 and 6, p. 680: It is shown that any demand function that is homogeneous of degree zero and satisfies Walras' law can be achieved as a market demand function generated by preference maximizing agents. Thus, market demand does not have to satisfy the weak axiom of revealed preference. However, in the proof of this theorem a major assumption is that of an arbitrary distribution of agents' preferences. The idea followed in [Grandmont, 92] is to place restrictions on the shape of the distribution of demand. In [Hildenbrand, 83] the weak axiom is obtained for the aggregate by posing restrictions on the distribution of income.

4.2.1 Transformations of Demand Functions

Before rigorously introducing Grandmont's concept of "α-transforms" on the space of demand functions let us have a look at the intuition of this approach. Thus, we first motivate α-transforms when demand is generated by preference maximizing behavior. Having done so, we generalize the approach to arbitrary demand functions. Assumptions on those may take account of distinct degrees of rationality: The "least rational" case would be to assume a demand function to satisfy Walras' law and homogeneity. The "most rational" would be assuming demand generated by preference maximization.

Let $\succsim \subset \mathbb{R}^l_+ \times \mathbb{R}^l_+$ be a preference relation on the commodity space $\mathbb{R}^l_+$ with $l \geq 2$ distinct commodities. Let $\alpha \in \mathbb{R}^l$ and consider a fixed income $w \in \mathbb{R}_{++}$.[1] We now consider the family $(\succsim_\alpha, w)_{\alpha \in \mathbb{R}^l}$ of preference-income-pairs.

[1] In the last chapter an agent was given an initial commodity endowment. Now we may

The preference $\succsim_\alpha$ is obtained from $\succsim$ by the following procedure: Given $\alpha \in \mathbb{R}^l$. For every commodity $h \leq l$, we expand the commodity-h-axis by e^{α_h}. A similar transformation can be found in [Dierker, Dierker & Trockel, 84]. This yields for every preference $\succsim$ an equivalence class of preferences indexed by $\alpha \in \mathbb{R}^l$.

Given a distribution on $\mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{R}_{++}$, the set of all preference-income-pairs, we define a distribution on the set of all α-transforms $\{(\succsim_\alpha, w)_\alpha : \alpha \in \mathbb{R}^l\}$ conditional on preference-income-pair $(\succsim, w)$. However, this is equivalent to defining a distribution on the space of parameters $\mathbb{R}^l$. In this approach heterogeneity among agents is basically given by the variance of these conditional distributions on $\mathbb{R}^l$.

Let us now turn to a rigorous introduction:

Definition 4.1. ([Grandmont, 92], p. 8) *Given $x \in \mathbb{R}^l_+$ and $\alpha \in \mathbb{R}^l$, we set*

$$e^\alpha \otimes x := (e^{\alpha_1} x_1, ..., e^{\alpha_l} x_l),$$

the α-transform *of x.*

For a given $x \in \mathbb{R}^l$ we consider the equivalence class $(e^\alpha \otimes x)_{\alpha \in \mathbb{R}^l}$ of α-transforms $e^\alpha \otimes x$ of x. It holds $(x_{\alpha^1})_{\alpha^2} = x_{\alpha^1+\alpha^2}$ for the composition of α-transforms.

We start by considering strongly convex and locally non-satiated preferences $\succsim \subset \mathbb{R}^l_+ \times \mathbb{R}^l_+$ on the non-negative orthant. Let the set of those be denoted by $\mathscr{P}_{sco,lns}$.

Definition 4.2. ([Grandmont, 92], p. 9) *Given a preference $\succsim \in \mathscr{P}_{sco,lns}$ and $\alpha \in \mathbb{R}^l$, the* α-transform *$\succsim_\alpha$ of $\succsim$ is defined by*

$$x \succsim_\alpha y :\Leftrightarrow (e^{-\alpha} \otimes x) \succsim (e^{-\alpha} \otimes y)$$

for all $x, y \in \mathbb{R}^l_+$. An α-transform is called homothetic *if α is a multiple of the unit vector, i.e. $\alpha_i = \alpha_j$ for all $i, j \leq l$.*

The α-transform $\succsim_\alpha$ of the preference $\succsim$ is the preference $\succsim_\alpha$ that coincides with $\succsim$ if one unit of commodity h is multiplied by e^{α_h}.

Definition 4.3. *A preference $\succsim$ on the consumption set $\mathbb{R}^l_+$ is called* homothetic *if for every $x, y \in \mathbb{R}^l_+$ and $\theta \in \mathbb{R}_{++}$ we have*

$$x \succsim y \quad \Leftrightarrow \quad \theta x \succsim \theta y.$$

We immediately see that homothetic preferences are invariant with respect to homothetic α-transforms: Let $\succsim$ be homothetic, $\alpha = a(1, ..., 1)$, $a \in \mathbb{R}$. Then we have for all commodity bundles x and y: $x \succsim_\alpha y \quad \Leftrightarrow \quad (e^{-\alpha} x) \succsim (e^{-\alpha} y) \quad \Leftrightarrow \quad x \succsim y$.

Having discussed α-transforms of preferences we now have a closer look on α-transforms of demand functions.

think of initial income w in money metric terms.

Definition 4.4. (a) *Let $m, n \in \mathbf{N}$. A function $g : \mathbb{R}^n_{++} \to \mathbb{R}^m$ is called* homogeneous of degree $k \in \mathbb{Z}$ *if, for all $x \in \mathbb{R}^n_{++}$ and $\lambda \in \mathbb{R}_{++}$, it holds*

$$g(\lambda x) = \lambda^k g(x).$$

(b) ([Grandmont, 92], p. 10) *A function $\xi : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_{+}$ assigning a commodity bundle $\xi(p, w) \in \mathbb{R}^l_{+}$ to each price-income pair $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ is called* demand function *if it is homogeneous of degree* 0 *in (p, w) and satisfies Walras' law, i.e. for all $p \in \mathbb{R}^l_{++}$ and $w \in \mathbb{R}_{++}$ we have*

$$p \cdot \xi(p, w) = w,$$

where "$\cdot$" denotes the scalar product on $\mathbb{R}^l$. Walras' law states that an agent's chosen consumption bundle $\xi(p, w)$ lies on the budget line $p \cdot x = w$ or that the budget balance condition is satisfied.[2]

We start by having a look at Walrasian demand χ that originates from a preference $\succsim \in \mathscr{P}_{sco,lns}$ as given in Definition 3.7.

Remark 4.5. *$\chi(\succsim, \cdot, \cdot)$ is a demand function in the sense of Definition 4.4 on $\mathbb{R}^l_{++} \times \mathbb{R}_{++}$.*

Proof. By definition $\chi(\succsim, \cdot, \cdot) : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to 2^{\mathbb{R}^l}$ is a correspondence. First, due to positive income and strong monotonicity, $\chi(\succsim, p, w) \subset \mathbb{R}^l_{+}$. If a preference relation is locally non-satiated, then it is strictly convex. Thus, we may apply Proposition 3.10 and obtain that the demand correspondence $\chi(\succsim, \cdot, \cdot)$ is actually a function.

$\chi(\succsim, \cdot, \cdot)$ satisfies Walras' law: Assume $x \in \chi(\succsim, p, w)$ for some $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $p \cdot x < w$. Then there exists an $\eta > 0$ such that $p \cdot y < w$ for all $y \in B_\eta(x)$, the η-Ball around x with respect to the Eucledian norm. Since $\succsim$ is assumed to be locally non-satiated there exists $z \in B_\eta(x)$ such that $z \succ x$. But this contradicts the fact that $x \in \chi(\succsim, p, w)$.

It leaves to show that $\chi(\succsim, \cdot, \cdot)$ is homogeneous of degree 0: Let $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $\lambda \in \mathbb{R}_{++}$, then

$$\begin{aligned} \chi(\succsim, \lambda p, \lambda w) &= \{x \in \mathbb{R}^l_{+} | \forall y : (y \succ x \Rightarrow \lambda p y > \lambda w)\} \\ &= \{x \in \mathbb{R}^l_{+} | \forall y : (y \succ x \Rightarrow p y > w)\} \\ &= \chi(\succsim, p, w). \end{aligned}$$

□

[2]We have to be a careful when using this scalar product here: Rigorously defined, we are not talking about a price vector $p \in \mathbb{R}^l$ but about a linear price functional p on the commodity space $\mathbb{R}^l$. That is, p is actually an element of $\mathbb{R}^{l\star}$, the dual space of $\mathbb{R}^l$. However, a well known result from functional analysis allows us to identify the Hilbert space $\mathbb{R}^l$ with its dual space $\mathbb{R}^{l\star}$ via the Riesz isomorphism. Thus, instead of evaluating the linear price functional $p \in \mathbb{R}^{l\star}$ at some commodity bundle $x \in \mathbb{R}^l_{+}$, we may just consider the scalar product $p \cdot x$. However, one should be aware of this technical problem when talking about the price of a commodity bundle.

Let $\chi^\alpha(\cdot,\cdot) := \chi(\succsim,\cdot,\cdot)$ denote the Walrasian demand function generated by the preference relation $\succsim_\alpha$, when $\chi(\cdot,\cdot) := \chi(\succsim,\cdot,\cdot)$ is generated by $\succsim$. Now, $\chi^\alpha : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ is a demand function as in Definition 4.4. We then get the following relation of χ and χ^α for $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$:

$$\begin{aligned}\chi^\alpha(p,w) \quad &= \{x \in \mathbb{R}^l_+ | \forall y \in \mathbb{R}^l_+ : \quad (y \succ_\alpha x \Rightarrow py > w)\} \\ &= \{x \in \mathbb{R}^l_+ | \forall y \in \mathbb{R}^l_+ : \quad (e^{-\alpha} \otimes y \succ e^{-\alpha} \otimes x \\ &\qquad \Rightarrow (e^\alpha \otimes p)(e^{-\alpha} \otimes y) > w)\} \\ &= e^\alpha \otimes \chi(e^\alpha \otimes p, w).\end{aligned}$$

Using these heuristics for χ^α, we pose the following definition for general demand functions as given in Definition 4.4:

Definition 4.6. ([Grandmont, 92], p. 10) *Let $\alpha \in \mathbb{R}^l$. Given a demand function ξ, we define the α-*transform *ξ^α of ξ by*

$$\xi^\alpha(p,w) := e^\alpha \otimes \xi(e^\alpha \otimes p, w) \tag{4.1}$$

for all $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$.

Remark 4.7. *In Definition 4.6, $\alpha \in \mathbb{R}^l$ is a parameter. Thus, when the demand at $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ is given by $\xi(p,w)$, the transformed demand is denoted by $\xi^\alpha(p,w)$ and $\xi^\alpha : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ is again a demand function for every $\alpha \in \mathbb{R}^l$ in the sense of Definition 4.4 as shown below. However, formally we now consider the function $\xi : \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, $(\alpha,p,w) \mapsto \xi(\alpha,p,w) := \xi^\alpha(p,w)$ and the "original" demand $\xi(\cdot,\cdot)$ is given by $\xi(0,\cdot,\cdot)$. Moreover, we consider partial derivatives with respect to α_h. Thus, α is a variable and not a parameter. Nevertheless, for notational convenience, we write $\xi^\alpha(\cdot,\cdot)$ for $\xi(\alpha,\cdot,\cdot)$ and consider partial derivatives*

$$\frac{\partial \xi^\alpha}{\partial \alpha_h}(p,w) := \frac{\partial \xi}{\partial \alpha_h}(a,p,w).$$

Given $\alpha \in \mathbb{R}^l$, the α-transform ξ^α of ξ is again a demand function: Let $(a,p,w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, $\lambda \in \mathbb{R}_{++}$. Then

$$\begin{aligned}\xi^\alpha(\lambda p, \lambda w) &= e^\alpha \otimes \xi(\lambda e^\alpha \otimes p, \lambda w) \\ &= e^\alpha \otimes \xi(e^\alpha \otimes p, w) = \xi^\alpha(p,w). \\ p \cdot \xi^\alpha(p,w) &= p \cdot (e^\alpha \otimes \xi(e^\alpha \otimes p, w)) \\ &= (e^\alpha \otimes p) \cdot \xi(e^\alpha \otimes p, w) = w\end{aligned}$$

Thus, $\xi^\alpha : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_{++}$ is homogeneous of degree zero and satisfies Walras law. In the last chapter we will see that ξ^α satisfies the weak axiom of revealed preference whenever ξ does.

For fixed ξ the collection $(\xi^\alpha)_{\alpha\in\mathbb{R}^l}$ defines an equivalence class of demand functions. When considering distributions on this equivalence class of α-transforms of ξ, we may equivalently consider distributions on $\mathbb{R}^l$.

For the proof of Lemma 4.8 below, we need the following identity named after Euler: Let $g : \mathbb{R}^l_{++} \to \mathbb{R}$ be homogeneous of degree $k \in \mathbb{N}$ and differentiable at $x \in \mathbb{R}^l_{++}$, then the following identity holds:

$$kg(x) = \sum_{i=1}^{l} \frac{\partial g}{\partial x_i}(x)x_i. \tag{4.2}$$

This equation is obtained by differentiating $\lambda^k g(x) = g(\lambda x)$ with respect to λ applying the chain rule and evaluating the derivative at $\lambda = 1$. Note, that ξ_h, $h \leq l$, is homogeneous of degree zero.

In [Grandmont, 92] the following lemma is stated. Here, we provide a proof by straightforward calculations:

Lemma 4.8. ([Grandmont, 92], Lemma 1.2) *The α-transform ξ^α of a demand function ξ is (continuously) partially differentiable with respect to p if and only if it is (continuously) partially differentiable with respect to α. In that case we have for every $h \leq l$, $(\alpha, p, w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$*

$$\frac{\partial \xi^\alpha_h}{\partial \alpha_h}(p, w) = \xi^\alpha_h(p, w) + p_h \frac{\partial \xi^\alpha_h}{\partial p_h}(p, w), \tag{4.3}$$

while for every $k \neq h$,

$$\frac{\partial \xi^\alpha_h}{\partial \alpha_k}(p, w) = p_k \frac{\partial \xi^\alpha_h}{\partial p_k}(p, w). \tag{4.4}$$

Moreover,

$$w \frac{\partial \xi^\alpha_h}{\partial w}(p, w) = \xi^\alpha_h(p, w) - \sum_{k=1}^{l} \frac{\partial \xi^\alpha_h}{\partial \alpha_k}(p, w). \tag{4.5}$$

Proof. Rewriting equation (4.1) as

$$\xi^\alpha_h(p, w) = e^{\alpha_h} \xi_h(e^\alpha \otimes p, w) \quad \forall h \leq l$$

and then differentiating with respect to α_k for $k \leq l$ yields

$$\begin{aligned}
\frac{\partial \xi^\alpha_h}{\partial \alpha_k}(p, w) &= \delta_{hk} e^{\alpha_h} \xi_h(e^\alpha \otimes p, w) + e^{\alpha_h} \underbrace{\frac{\partial \xi_h}{\partial \alpha_k}(e^\alpha \otimes p, w)}_{= \frac{\partial \xi_h}{\partial p_k}(e^\alpha \otimes p, w) e^{\alpha_k} p_k} \\
&= \delta_{hk} \xi^\alpha_h(p, w) + e^{\alpha_h + \alpha_k} p_k \frac{\partial \xi_h}{\partial p_k}(e^\alpha \otimes p, w) \\
&= \delta_{hk} \xi^\alpha_h(p, w) + p_k \frac{\partial \xi^\alpha_h}{\partial p_k}(p, w), \quad \text{since} \\
\frac{\partial \xi^\alpha_h}{\partial p_k}(p, w) &= e^{\alpha_h} \frac{\partial \xi_h}{\partial p_k}(e^\alpha \otimes p, w) e^{\alpha_k},
\end{aligned}$$

by product and chain rule, where δ_{hk} denotes the Kronecker symbol, i.e.

$$\delta_{hk} = \begin{cases} 1 & \text{if } h = k, \\ 0 & \text{if } h \neq k. \end{cases}$$

This verifies equations (4.3) and (4.4). Now, equation (4.5) can be shown by applying Euler's identity: Homogeneity of degree zero yields

$$0 = \sum_{k=1}^{l} \frac{\partial \xi_h^\alpha}{p_k}(p,w)p_k + \frac{\partial \xi_h^\alpha}{\partial w}(p,w)w.$$

Thus, we obtain using equations (4.3) and (4.4)

$$\begin{aligned} w\frac{\partial \xi_h^\alpha}{\partial w}(p,w) &= -\sum_{\substack{k=1 \\ k\neq h}}^{l} \frac{\partial \xi_h^\alpha}{\partial p_k}(p,w)p_k - \frac{\partial \xi_h^\alpha}{\partial p_h}(p,w)p_h \\ &= -\sum_{\substack{k=1 \\ k\neq h}}^{l} \frac{\partial \xi_h^\alpha}{\partial \alpha_k}(p,w) - \frac{\partial \xi_h^\alpha}{\partial \alpha_h}(p,w) + \xi_h^\alpha(p,w) \\ &= \xi_h^\alpha(p,w) - \sum_{k=1}^{l} \frac{\partial \xi_h^\alpha}{\partial \alpha_k}(p,w). \end{aligned}$$

In particular, we have shown the equivalence of continuously partial derivatives with respect to α_k and p_k. $\square$

Definition 4.9. ([Grandmont, 92], p. 11) *Given a demand function ξ and a price-income pair $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, we define the* expenditure function ϵ_h for commodity h *by* $\epsilon_h(p,w) := p_h \xi_h(p,w)$. *The* expenditure function ϵ *is given by the scalar product $\epsilon(p,w) := p \cdot \xi(p,w)$. Analogously, we define the expenditure function ϵ_h^α for commodity h and α-transform by $\epsilon_h^\alpha(p,w) := p_h \xi_h^\alpha(p,w)$.*

Equivalently, we could have defined the α-transform ϵ_h^α of ϵ_h by

$$\epsilon_h^\alpha(p,w) = \epsilon_h(e^\alpha \otimes p, w). \tag{4.6}$$

This is well defined since

$$\begin{aligned} \epsilon_h^\alpha(p,w) &= p_h \xi_h^\alpha(p,w) \\ &= p_h e^{\alpha_h} \xi_h(e^\alpha \otimes p, w) \\ &= (e^\alpha \otimes p)_h \xi_h(e^\alpha \otimes p, w) \\ &= \epsilon_h(e^\alpha \otimes p, w). \end{aligned}$$

Definition 4.10. ([Grandmont, 92], p. 12) *A demand function ξ is called* α-transform invariant *if*

$$\xi^\alpha(p,w) = \xi(p,w) \tag{4.7}$$

for all $(\alpha, p, w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$.

The invariance property of a demand function can equivalently be defined via the expenditure function: For all $(\alpha, p, w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$

$$\begin{aligned} (4.7) \quad &\overset{p_h>0}{\Leftrightarrow} \quad p_h\xi^\alpha_h(p,w) = p_h\xi_h(p,w) \quad \forall h \le l \\ &\Leftrightarrow \quad \epsilon_h(e^\alpha \otimes p, w) = \epsilon^\alpha_h(p,w) = \epsilon_h(p,w) \quad \forall h \le l. \end{aligned} \tag{4.8}$$

In particular, for α-transform invariant demand functions the expenditures for each commodity are independent of prices: For every $p \in \mathbb{R}^l_{++}$ the set $\{e^\alpha \otimes p : \alpha \in \mathbb{R}^l\}$ is isomorphic to $\mathbb{R}^l_{++}$, in that for every $p, \tilde{p} \in \mathbb{R}^l_{++}$ there exists exactly one $\alpha \in \mathbb{R}^l$ such that $e^\alpha \otimes p = \tilde{p}$. Thus, for any $p \in \mathbb{R}^l_{++}$, as α passes through $\mathbb{R}^l$, $\{e^\alpha \otimes p : \alpha \in \mathbb{R}^l\}$ exhausts $\mathbb{R}^l_{++}$ and, vice versa, for fixed $\alpha \in \mathbb{R}^l$ as p passes through $\mathbb{R}^l_{++}$.

Since we have assumed a demand function to be homogeneous of degree 0 in (p,w), we get for all $(\alpha, p, w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $\lambda \in \mathbb{R}_{++}$

$$\begin{aligned} \epsilon^\alpha_h(p, \lambda w) &\overset{\text{independence of prices}}{=} \epsilon^\alpha_h(\lambda p, \lambda w) \\ &= \lambda p_h \xi^\alpha_h(\lambda p, \lambda w) \\ &= \lambda p_h \xi^\alpha_h(p, w) \\ &= \lambda \epsilon^\alpha_h(p,w), \end{aligned} \tag{4.9}$$

whenever ξ is α-transform invariant. We can thus state the following remark:

Remark 4.11. *Let ξ be an α-transform invariant demand function. Then the corresponding expenditure functions ϵ_h are homogeneous of degree* 1 *in income and independent of prices.*

Due to these properties of the expenditure functions ϵ_h, the α-transform invariant demand function ξ belongs to an important class of demand functions:

Definition 4.12. ([Grandmont, 92], p. 12) *A demand function ξ is of* Cobb-Douglas type *if there exists a family $(r_h)_{h \le l}$, $r_h \ge 0$, with $\sum_{h=1}^l r_h = 1$ such that*

$$\xi_h(p,w) = r_h \frac{w}{p_h} \quad \forall (p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}.$$

In particular, for Cobb-Douglas type demand functions budget shares $p_h\xi_h(p,w)$ are independent of prices. There may arise the question for the relevance of Cobb-Douglas type demand functions:

Remark 4.13. *Let us assume that an agent's demand is generated by preference maximizing behavior. Thus, the primitive is a preference* $\succsim$*. We call this a* Cobb-Douglas preference *if it can be represented by a utility function* $u : \mathbb{R}^l_+ \to \mathbb{R}$*, i.e. for all* $x, y \in \mathbb{R}^l_+$ *we have*

$$x \succsim y \quad \Leftrightarrow \quad u(x) \geq u(y),$$

of Cobb-Douglas type, i.e. u *is defined as follows: For all* $x \in \mathbb{R}^l_+$ *and* $(r_h)_{h \leq l}$*,* $r_h > 0$*, let*

$$u(x) = \prod_{h=1}^{l} x_h^{r_h}.$$

Note that a Cobb-Douglas preference relation is homothetic. To generate the Walrasian demand χ*, we have to maximize utility* u *with respect to the budget constraint* $p \cdot x \leq w$*. Since* $\chi_h(p, w) > 0$ *for all* h[3] *we do not have to worry about non-negativity constraints. Representing utility functions are unique up to strictly increasing transformations. Thus, we may rewrite* u *as*

$$u(x) = \sum_{h=1}^{l} r_h \log x_h.$$

Using the Lagrangeian technique with multiplier $\lambda \geq 0$*, we get the Lagrangeian term*

$$L(x, \lambda) = \sum_{h=1}^{l} r_h \log x_h - \lambda(p \cdot x - w)$$

This yields the first order conditions

$$\begin{aligned} \frac{r_h}{x_h} - \lambda p_h = 0 \quad \forall h \leq l \\ w - p \cdot x \geq 0, \quad \lambda \geq 0, \quad \lambda(w - p \cdot x) = 0. \end{aligned}$$

Since $r_h > 0$*, we have* $\lambda > 0$ *and thus* $w - p \cdot x = 0$*. Solving the first order conditions for* x_h *yields* $x_h = \frac{r_h}{\lambda p_h}$*. Then we get* $w - \sum_{h=1}^{l} \frac{r_h}{\lambda} = 0$*. Since the second order conditions hold we get*

$$\chi_h(p, w) = r_h \frac{w}{p_h \sum_{i=1}^{l} r_i}$$

for $h = 1, ..., l$*.*

The proof of the following lemma follows the idea in [Trockel, 89].

Lemma 4.14. ([Grandmont, 92], Lemma 1.3) *A demand function* ξ *is* α*-transform invariant if and only if it is of Cobb-Douglas type.*

[3]Otherwise utility would be zero and could be increased.

In light of this lemma, the remarks in Section 3.6 show an appreciation.

Proof. Let $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$. Assume that the demand function ξ is α-transform invariant. Then we know by Remark 4.11 that the corresponding expenditure functions ϵ_h, $h \leq l$, are independent of prices and homogeneous of degree 1 in income, i.e.

$$\epsilon(p,w) = \epsilon(\tilde{p},w) \quad \forall p,\tilde{p} \in \mathbb{R}^l_{++}, \quad w \in \mathbb{R}_{++}, \tag{4.10}$$
$$\epsilon(p,\lambda w) = \lambda\epsilon(p,w) \quad \forall p \in \mathbb{R}^l_{++}, \quad w,\lambda \in \mathbb{R}_{++}. \tag{4.11}$$

By Definition 4.9 and equations (4.10) and (4.11), the demand function ξ_h for commodity h is given by

$$\xi_h(p,w) = \frac{\epsilon_h(p,w)}{p_h} \overset{(4.11)}{=} \epsilon_h(p,1)\frac{w}{p_h} \overset{(4.10)}{=} \epsilon_h(1,1)\frac{w}{p_h}.$$

Thus, we can see that ξ is of Cobb-Douglas type if we can show that $\sum_{h=1}^l \epsilon_h(p,1) = 1$. However, this follows immediately from Walras' Law:

$$\begin{aligned} p \cdot \xi(p,1) = 1 \quad &\Leftrightarrow \quad \sum_{h=1}^{l} p_h \xi_h(p,1) = 1 \\ &\Leftrightarrow \quad \sum_{h=1}^{l} p_h \frac{\epsilon_h(p,1)}{p_h} = 1 \\ &\Leftrightarrow \quad \sum_{h=1}^{l} \epsilon_h(p,1) = 1. \end{aligned}$$

On the contrary, let ξ be of Cobb-Douglas type. Then for all $h \leq l$

$$\xi_h(p,w) = r_h \frac{w}{p_h} \quad \Leftrightarrow \quad \epsilon_h(p,w) = r_h w.$$

Thus, expenditure ϵ_h for commodity h is independent of prices and for all $(\alpha,p,w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $h \leq l$, we obtain

$$\epsilon^\alpha_h(p,w) = \epsilon_h(e^\alpha \otimes p, w) = \epsilon_h(p,w).$$

Equation (4.8) implies the α-transform invariance of ξ. □

For a better understanding of α-transforms, we have sketched a homothetic α-transform of a general demand function (Figure 4.1) and a Cobb-Douglas demand function (Figure 4.2). The first figure is taken from [Grandmont, 92], p. 14. The bold line DE is the Engel curve, i.e. the set of all demands $\xi(p,w)$ with fixed price system p and varying income w. Furthermore, we set $m = e^{\alpha_2}$ and $n = e^{\alpha_1}$. We start at point $A = \xi(p,w)$. First, we go from point A to point $B = \xi(e^\alpha \otimes p, w)$. This is done by following the

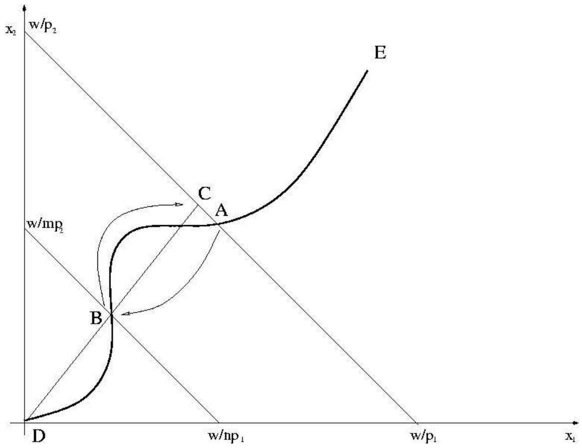

Figure 4.1: α-transform with $\alpha_1 = \alpha_2 > 0$

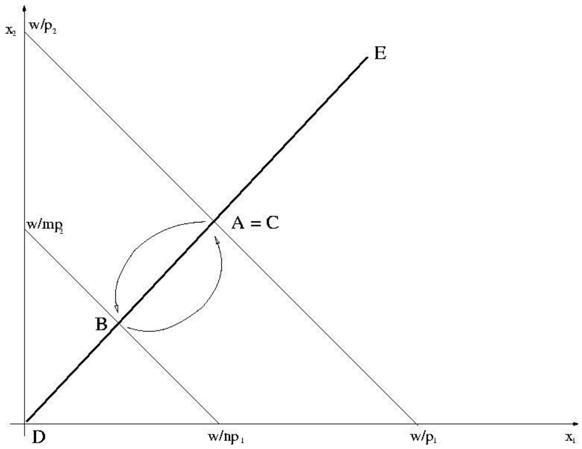

Figure 4.2: α-transform with $\alpha_1 = \alpha_2 > 0$ for Cobb-Douglas demand

Engel curve since the price ratio does not change when applying homothetic transformations but the relative income. Then we have to rescale demand at price system $e^\alpha \otimes p$ to get back to the original budget line. Thus, we obtain $C = e^\alpha \otimes \xi(e^\alpha \otimes p, w) = \xi^\alpha(p, w)$. Figure 4.2 depicts the case of a Cobb-Douglas demand function. Here the Engel curve is a straight line through the origin since the underlying preferences are homothetic. In this case, the α-transform C coincides with the original demand A. Thus demand is α-transform invariant as already shown in Lemma 4.14.

4.2.2 Analysis of Market Demand

> The purpose of this section is to analyze how the shape of the distribution of demand functions [...] can influence the manner

> in which price and/or income changes affect aggregate demand. ([Grandmont, 92], p. 13)

As already mentioned, if we consider a distribution on the equivalence class of α-transforms of some demand ξ, we may equivalently consider a distribution on $\mathbb{R}^l$. The question that we pose is the following: Are there any restrictions on the distributions within each equivalence class that induce some structural properties of market demand? The analysis will lead to the following result: If there is "enough heterogeneity" among agents, we can obtain strong structural properties of aggregate demand. The concept of α-transforms from the previous section equips us with a measure of heterogeneity by considering the variance of distributions within the equivalence classes of demand functions. These distributions are given by densities. Thus, "enough heterogeneity" means that densities are "sufficiently flat". The main insight turns out to be that the structure of aggregate demand does not rely on strong structural assumptions on individual demand. We obtain market demand satisfying the law of demand and the weak axiom of revealed preference due to assumptions on the distribution of agents' demand, whereas individual demand is "only" assumed to satisfy homogeneity and Walras' law.

In [Hildenbrand, 70] an agent is completely characterized by a preference $\succsim$ and a commodity endowment. Now, the primitives have changed: an agent is characterized by a demand function ξ and an income $w \in \mathbb{R}_{++}$ that is, at least in this subsection, independent of the revealing price system.

Grandmont considers distribution economies. Thus, we have to come up with a probability distribution on the space of individual characteristics $\{(\xi, w) | \xi : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+, w \in \mathbb{R}_{++}\}$. The construction will be done in two steps: First, let $\mathbf{A}$ be a separable metric space of types $a \in \mathbf{A}$, $\mathcal{B}(\mathbf{A})$ the Borel-σ-algebra on $\mathbf{A}$ and $\mathcal{P}(\mathbf{A}, \mathcal{B}(\mathbf{A}))$ be the set of all probability distributions on $\mathbf{A}$. We specify a distribution $\mu \in \mathcal{P}(\mathbf{A}, \mathcal{B}(\mathbf{A}))$ that is absolutely continuous with respect to the Lebesgue measure. For each type $a \in \mathbf{A}$ there exists a corresponding demand-income-pair (ξ_a, w_a). Second, for each type $a \in \mathbf{A}$, we specify a conditional probability distribution $\nu(\cdot|a)$ on the space of α-transforms $\{\xi_a^\alpha | \alpha \in \mathbb{R}^l\}$ of ξ_a or equivalently a probability distribution on $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$. We assume that this distribution is given by a density $f(\cdot|a) : \mathbb{R}^l \to \mathbb{R}_+$. Thus, we have for all $B \in \mathcal{B}(\mathbb{R}^l)$ and $a \in \mathbf{A}$

$$\nu(B|a) = \int_B f(\alpha|a) d\alpha =: f(B|a).$$

We now define the probability distribution $P := \mu \otimes f$ on the set of agents' characteristics in the following way: Let $C \in \mathcal{B}(\mathbf{A})$ and $B \in \mathcal{B}(\mathbb{R}^l)$, then we set

$$\begin{aligned} P(C \times B) &:= \int_C f(B|a)\mu(da) \\ &= \int_{B\times C} f(\alpha|a)d\alpha\mu(da). \end{aligned}$$

We may then think of a *Grandmont type distribution economy* as a tuple $((f(\cdot|a))_{a\in\mathbf{A}}, \mu)$, where $\mu \in \mathcal{P}(\mathbf{A}, \mathcal{B}(\mathbf{A}))$ and $f(\cdot|a)$ is a density conditional on $\mathbb{R}^l$ for every $a \in \mathbf{A}$. Let g be $\mathcal{B}(\mathbf{A}) \otimes \mathcal{B}(\mathbb{R}^l) := \sigma(\{A_1 \times A_2 | A_1 \in \mathcal{B}(\mathbf{A}), A_2 \in \mathcal{B}(\mathbb{R}^l)\})$-measurable. Then we set

$$P(g) := \int_{\mathbf{A}\times\mathbb{R}^l} g dP := \int_{\mathbf{A}} \int_{\mathbb{R}^l} g(a,\alpha) f(\alpha|a) d\alpha\mu(da).$$

Assumption 4.15. ([Grandmont, 92], pp. 15,17) *Let $a \in \mathbf{A}$, $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $\alpha \in \mathbb{R}^l$.*

A 1 The demand function $\xi_a(p,w)$ is continuous in (a,p,w).[4]

A 2 The conditional density $f(\alpha|a)$ is continuous in (α, a). Moreover, it is partially differentiable with partial derivatives $\frac{\partial f}{\partial \alpha_k}(\alpha|a)$; these are continuous in (a, α).

A 3 For each type $a \in \mathbf{A}$, partial derivatives are in $\mathscr{L}^1(\mathbb{R}^l, d\alpha)$, i.e.

$$\mathfrak{v}_k(a) := \int_{\mathbb{R}^l} \left| \frac{\partial f}{\partial \alpha_k}(\alpha|a) \right| d\alpha < \infty$$

for all $k = 1, ..., l$.

A 4 For every commodity k, $\mathfrak{v}_k(a)$ is bounded above by $\mathfrak{v}_k$ for μ-almost all $a \in \mathbf{A}$.[5]

A 5 Income level $w_a > 0$ is continuous in a. Per capita income $\bar{w}$ is finite. Stated in functional terms when conceiving income as a mapping $w : \mathbf{A} \to \mathbb{R}_{++}$, $a \mapsto w_a$, $w \in \mathscr{L}^1(\mathbf{A}, \mu)$ and thus

$$0 < \bar{w} = \int_{\mathbf{A}} w_a \mu(da) < \infty.$$

$\mathfrak{v}_k(a)$ is a measure for the variance of $f(\cdot|a)$ “in direction k”: The smaller $\mathfrak{v}_k(a)$, the less curved $f(\cdot|a)$ in direction k and thus, the higher the variance.

For some results on continuity and differentiability of conditional and total market demand we need Lebesgue's dominated convergence theorem. For our purposes, we may directly apply a version in [Dieudonné, 69].

[4] Here, we again consider $\xi_\cdot(\cdot,\cdot) : \mathbf{A} \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$. By continuity and Walras' law, we can show ξ, as a function of a, p, w, to be integrable applying Lebesgue's theorem in the respective contexts. Hereby, we should not forget to mention that integration is conceived component-wise.

[5] From proposition 4.16 we can infer that $\mathfrak{v}_k(a)$ is continuous.

Proposition 4.16. ([Dieudonné, 69], 13.8.6) (a) *Let $(\Omega, \mathcal{F}, \psi)$ be an arbitrary measure space, $\mathbf{E}$ a metric space and let $z_0 \in \mathbf{E}$. Consider $(\omega, z) \mapsto g(\omega, z)$ a mapping of $\Omega \times \mathbf{E}$ into $\bar{\mathbb{R}}$. Suppose that*

1. *for each $z \in \mathbf{E}$, the function $\omega \mapsto g(\omega, z)$ is integrable;*
2. *for almost all $\omega \in \Omega$, the function $z \mapsto g(\omega, z)$ is continuous at z_0;*
3. *there exists an integrable function $\tilde{g} \geq 0$ such that, for all $z \in \mathbf{E}$, we have $|g(\omega, z)| \leq \tilde{g}(\omega)$ for ψ-almost all $\omega \in \Omega$.*

Then $h(z) := \int_\Omega g(\omega, z)\psi(d\omega)$ is continuous at z_0.
(b) *Suppose furthermore that $\mathbf{E}$ is an open interval in $\mathbb{R}$ and that g satisfies the following conditions:*

4. *for ψ-almost all $\omega \in \Omega$, the function $z \mapsto g(\omega, z)$ is finite and admits a partial derivative with respect to the second component, i.e. $\frac{\partial g}{\partial z}(\omega, z)$;*
5. *there exists an integrable function $\tilde{g}_1 \geq 0$ on Ω such that, for all $z \in \mathbf{E}$, we have $|\frac{\partial g}{\partial z}(\omega, z)| \leq \tilde{g}_1(\omega)$ ψ-almost everywhere in Ω.*

Then h is differentiable at every point of $\mathbf{E}$, and we have

$$\frac{dh}{dz}(z) = \int_\Omega \frac{\partial g}{\partial z}(\omega, z)\psi(d\omega).$$

Proof. [Dieudonné, 69], p. 125 □

Definition 4.17. ([Grandmont, 92], p. 15) (a) *For each type $a \in \mathbf{A}$ we define* market demand conditional on type a, *conditional market demand for short, by*

$$X(a, p, w) := \int_{\mathbb{R}^l} \xi_a^\alpha(p, w) f(\alpha|a) d\alpha. \tag{4.12}$$

(b) *Then we define* (total) market demand *by*

$$X(p) := \int_\mathbf{A} X(a, p, w_a)\mu(da) = \int_\mathbf{A} \int_{\mathbb{R}^l} \xi_a^\alpha(p, w_a) f(\alpha|a) d\alpha \mu(da), \tag{4.13}$$

where w_a denotes the income corresponding to type $a \in \mathbf{A}$.

The terms in (4.12) and (4.13) are well-defined in the following sense: Since $0 \leq p_h \xi_{ah}^\alpha(p, w) \leq w$ for all h by Walras' law, we have

$$\begin{aligned}
0 \leq p_h X_h(a, p, w) &= p_h \int_{\mathbb{R}^l} \xi_{ah}^\alpha(p, w) f(\alpha|a) d\alpha \\
&\leq \int_{\mathbb{R}^l} w f(\alpha|a) d\alpha = w < \infty, \\
0 \leq p_h X_h(p) &= p_h \int_\mathbf{A} X_h(a, p, w_a)\mu(da) \\
&\leq \int_\mathbf{A} w_a \mu(da) = \bar{w} < \infty.
\end{aligned}$$

Since $p_h > 0$ for all h we have $X(a,p,w)$ and $X(p)$ finite. Both, conditional market demand $X(a,p,w)$ and total market demand $X(p)$ are non-negative, continuous and satisfy Walras' law: Continuity follows directly from Proposition 4.16. Walras' law can be obtained as follows:[6]

$$\begin{aligned} p\cdot X(a,p,w) &= p\cdot\int_{\mathbb{R}^l}\xi_a(\alpha,p,w)f(\alpha|a)d\alpha \\ &= \int_{\mathbb{R}^l} wf(\alpha|a)d\alpha = w, \\ p\cdot X(p) &= p\cdot\int_{\mathbf{A}} X(a,p,w_a)\mu(da) \\ &= \int_{\mathbf{A}} w_a\mu(da) = \bar{w}, \end{aligned}$$

One should note that total market demand X is not a demand function in the sense of Definition 4.4.

Some Heuristics

To gain some intuition for the approach to market demand structure by increasing variance of α-transforms, let us consider the following heuristic inspection: Let $a \in \mathbf{A}$ and $(p,w) \in \mathbb{R}^l_{++}\times\mathbb{R}_{++}$. Given conditional market demand $X(a,p,w)$ and $\bar{\alpha}\in\mathbb{R}^l$, the $\bar{\alpha}$-transform $X^{\bar{\alpha}}(a,p,w)$ of $X(a,p,w)$ is given by

$$\begin{aligned} X^{\bar{\alpha}}(a,p,w) &= e^{\bar{\alpha}}\otimes X(a,e^{\bar{\alpha}}\otimes p,w) \\ &= e^{\bar{\alpha}}\otimes\int_{\mathbb{R}^l}\xi_a^{\alpha}(e^{\bar{\alpha}}\otimes p,w)f(\alpha|a)d\alpha \\ &\overset{(\xi^\alpha)^{\bar{\alpha}}=\xi^{\alpha+\bar{\alpha}}}{=} \int_{\mathbb{R}^l}\xi_a^{\alpha+\bar{\alpha}}(p,w)f(\alpha|a)d\alpha \\ &\overset{\beta:=\alpha+\bar{\alpha}}{=} \int_{\mathbb{R}^l}\xi_a^{\beta}(p,w)f(\beta-\bar{\alpha}|a)d\beta \end{aligned} \tag{4.14}$$

Thus, we obtain the α-transform $X^{\bar{\alpha}}(a,\cdot,\cdot)$ of the conditional market demand $X(a,\cdot,\cdot)$ by shifting the conditional density $f(\cdot|a)$ within the equivalence class of α-transforms by $\bar{\alpha}$. In this token, conditional demand $X(a,\cdot,\cdot)$ is "arbitrarily" α-transform invariant in the sense of Definition 4.10 if the conditional density $f(\cdot|a)$ is "arbitrarily" invariant with respect to shifts. In this case, Lemma 4.14 would imply that conditional market demand is "arbitrarily close" to being of Cobb-Douglas type.

Heuristics may be stated as follows: The more heterogeneous agents in the sense of an increasing variance, i.e. the "closer" the distribution of

[6] We have to be a little cautious when using the term "Walras' law": We have only defined this notion for demand functions as in Definition 4.4. However, we refer to the analogous concepts in the respective contexts.

agents' characteristics within each equivalence class $\{\xi_a^\alpha : \alpha \in \mathbb{R}^l\}$, $a \in \mathbf{A}$, to uniform distribution, and thus the "flatter" the conditional densities $f(\cdot|a)$ for each $a \in \mathbf{A}$, the "closer" $f(\cdot|a)$ to being invariant. As seen above, conditional market demand would then be "close" to being of Cobb-Douglas type

Assuming conditional market demand $X(a,\cdot,\cdot)$ to be of Cobb-Douglas type for all $a \in \mathbf{A}$, i.e. $X_h(a,p,w) = r_{ha}\frac{w}{p_h}$, where $r_{ha} > 0$ such that $\sum_{h=1}^l r_{ha} = 1$ for all $a \in \mathbf{A}$, we obtain total market demand as follows: Assume $r_{ha} = r_h$ for all $a \in \mathbf{A}$.

$$\begin{aligned} X_h(p) &= \int_{\mathbf{A}} X_h(a,p,w_a)\mu(da) \\ &= \int_{\mathbf{A}} r_h \frac{w_a}{p_h}\mu(da) \\ &= r_h \frac{\bar{w}}{p_h} \end{aligned}$$

for $h = 1,...,l$. Thus, total market demand is of Cobb-Douglas type. Hence, total market demand is close to being of Cobb-Douglas type if densities for α-transforms are sufficiently flat. We should note that Cobb-Douglas type demand has the properties that we are seeking for: uncompensated law of demand and weak axiom of revealed preference. Thus, total market demand is close to satisfying the weak axiom when agents are sufficiently heterogeneous.

Price Effects

In this section we state the main theorem in [Grandmont, 92]: We encounter that total market demand X satisfies structural properties as the weak axiom of revealed preference when agents are sufficiently heterogeneous but before approaching the "Cobb-Douglas limit" above.

It should be noted that given type $a \in \mathbf{A}$, the conditional market demand $X(a,\cdot,\cdot) : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ is actually a demand function: It was already shown above that $X(a,\cdot,\cdot)$ satisfies Walras' law. Thus it suffices to show homogeneity of degree 0 in (p,w):

$$\begin{aligned} X(a,\lambda p,\lambda w) &= \int_{\mathbb{R}^l} \xi_a^\alpha(\lambda p,\lambda w) f(\alpha|a)d\alpha \\ &= \int_{\mathbb{R}^l} \xi_a^\alpha(p,w) f(\alpha|a)d\alpha = X(a,p,w) \end{aligned}$$

for all $\lambda \in \mathbb{R}_{++}$ and $(\alpha,p,w) \in \mathbb{R}^l \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$.

Next, we see that conditional market demand has continuous partial derivatives and obtain bounds on those.

Proposition 4.18. ([Grandmont, 92], p. 19) *Let (A1) to (A3) hold. For every $a \in \mathbf{A}$ conditional market demand $X(a,\cdot,\cdot)$ is continuously partially differentiable with respect to prices. For every $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ we have*

$$\frac{\partial X_h}{\partial p_k}(a,p,w) = -\frac{1}{p_k}\left(\delta_{hk} X_h(a,p,w) + \int_{\mathbb{R}^l} \xi^\alpha_{ah}(p,w)\frac{\partial f}{\partial \alpha_k}(\alpha|a)d\alpha\right)$$

for all $h,k = 1,...,l$, where δ_{hk} again denotes the Kronecker symbol.

Proof. We will first show that the $\bar{\alpha}$-transform $X^{\bar{\alpha}}(a,p,w)$ is continuously partially differentiable with respect to $\bar{\alpha}$. Using Proposition 4.16(2) in connection with (*A2*) and (*A3*) we obtain

$$\begin{aligned}
\frac{\partial X^{\bar{\alpha}}_h}{\partial \bar{\alpha}_k}(a,p,w) &= \frac{\partial}{\partial \bar{\alpha}_k}\int_{\mathbb{R}^l} \xi^\beta_{ah}(p,w) f(\beta - \bar{\alpha}|a)d\beta \\
&= \int_{\mathbb{R}^l} \xi^\beta_{ah}(p,w)\frac{\partial f}{\partial \bar{\alpha}_k}(\beta - \bar{\alpha}|a)d\beta \\
&= -\int_{\mathbb{R}^l} \xi^\beta_{ah}(p,w)\frac{\partial f}{\partial \beta_k}(\beta - \bar{\alpha}|a)d\beta.
\end{aligned}$$

In light of (*A1*) to (*A3*) we may again apply Proposition 4.16(1) to show the continuity of these partial derivatives in $(\bar{\alpha},a,p,w)$ with respect to $\bar{\alpha}_k$. Thus, Lemma 4.8 implies that conditional market demand $X^{\bar{\alpha}}(a,\cdot,\cdot)$ is continuously partially differentiable with respect to prices p_h for all $h = 1,...,l$.

Evaluating the above expression at $\bar{\alpha} = 0$, we get

$$\left.\frac{\partial X^{\bar{\alpha}}_h}{\partial \bar{\alpha}_k}(a,p,w)\right|_{\bar{\alpha}=0} = -\int_{\mathbb{R}^l} \xi^\alpha_{ah}(p,w)\frac{\partial f}{\partial \alpha_k}(\alpha|a)d\alpha.$$

By Definition 4.6 we know that $X^{\bar{\alpha}}_h(a,p,w)|_{\bar{\alpha}=0} = X_h(a,p,w)$. Hence, we have $\left.\frac{\partial X^{\bar{\alpha}}_h}{\partial p_k}(a,p,w)\right|_{\bar{\alpha}=0} = \frac{\partial X_h}{\partial p_k}(a,p,w)$. and equations (4.3) and (4.4) in Lemma 4.8 can be rephrased as

$$p_k\frac{\partial X_h}{\partial p_k}(a,p,w) + \delta_{hk}X_k(a,p,w) = -\int_{\mathbb{R}^l} \xi^\alpha_{ah}(p,w)\frac{\partial f}{\partial \alpha_k}(\alpha|a)d\alpha. \quad (4.15)$$

Since $p_h > 0$ for all $h = 1,...,l$, the assertion follows. □

Proposition 4.19. ([Grandmont, 92], p. 19) *Let (A1) to (A3) hold. For all $(a,p,w) \in \mathbf{A} \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ the following inequality holds:*

$$\left|p_k\frac{\partial X_h}{\partial p_k}(a,p,w) + \delta_{hk}X_h(a,p,w)\right| \le w\frac{\mathfrak{v}_k(a)}{p_h}. \quad (4.16)$$

Proof. Demand ξ is non-negative and satisfies Walras' law. We thus obtain

$$0 \le \xi^\alpha_{ah}(p,w) \le \frac{w}{p_h}$$

for all h and equation (4.15) yields

$$\begin{aligned}\left|p_k \frac{\partial X_h}{\partial p_k}(a,p,w) + \delta_{hk} X_h(a,p,w)\right| &= \left|\int_{\mathbb{R}^l} \xi^{\alpha}_{ah}(p,w) \frac{\partial f}{\partial \alpha_k}(\alpha|a) d\alpha\right| \\ &\leq \int_{\mathbb{R}^l} \xi^{\alpha}_{ah}(p,w) \left|\frac{\partial f}{\partial \alpha_k}(\alpha|a)\right| d\alpha \\ &\leq \frac{w}{p_h} \int_{\mathbb{R}^l} \left|\frac{\partial f}{\partial \alpha_k}(\alpha|a)\right| d\alpha \\ &= \frac{w}{p_h} \mathfrak{v}_k(a).\end{aligned}$$

□

Analogous results hold for total market demand:

Proposition 4.20. ([Grandmont, 92], p. 20) *Suppose assumptions (A1) to (A5) hold. Then total market demand $X(p)$ is continuously partially differentiable with*

$$\frac{\partial X_h}{\partial p_k}(p) = \int_{\mathbf{A}} \frac{\partial X_h}{\partial p_k}(a,p,w_a)\mu(da),$$

and these partial derivatives satisfy

$$\left|p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p)\right| \leq \bar{w} \frac{\mathfrak{v}_k}{p_h}. \tag{4.17}$$

Proof. In light of (*A1*), (*A2*) and (A*3*) we have shown that $X(a,\cdot,\cdot)$ has continuous partial derivatives. Using this result and Proposition 4.16(2) together with (*A4*) and (*A5*), we may change the order of integration and differentiation in equation (4.13) and obtain

$$\frac{\partial X_h}{\partial p_k}(p) = \int_{\mathbf{A}} \frac{\partial X_h}{\partial p_k}(a,p,w_a)\mu(da)$$

for all $h,k = 1,...,l$, $a \in \mathbf{A}$ and $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$. Again by Proposition 4.16 this expression is continuous. Using Lemma 4.8 we obtain, as in the derivation of equation (4.15),

$$\begin{aligned}
p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p) &= \left. \frac{\partial X_h^{\bar{\alpha}}}{\partial \alpha_k}(p) \right|_{\bar{\alpha}=0} \\
&= \int_{\mathbf{A}} \left. \frac{\partial X_h^{\bar{\alpha}}}{\partial \alpha_k}(a, p, w_a) \right|_{\bar{\alpha}=0} \mu(da) \\
&= -\int_{\mathbf{A}} \int_{\mathbb{R}^l} \xi_{ah}^{\alpha}(p, w_a) \frac{\partial f}{\partial \alpha_k}(\alpha|a) d\alpha \mu(da). \\
\left| p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p) \right| &= \left| \int_{\mathbf{A}} \int_{\mathbb{R}^l} \xi_{ah}^{\alpha}(p, w_a) \frac{\partial f}{\partial \alpha_k}(\alpha|a) d\alpha \mu(da) \right| \\
&\leq \int_{\mathbf{A}} \int_{\mathbb{R}^l} \xi_{ah}^{\alpha}(p, w_a) \left| \frac{\partial f}{\partial \alpha_k}(\alpha|a) \right| d\alpha \mu(da) \\
&\leq \frac{1}{p_h} \int_{\mathbf{A}} w_a \int_{\mathbb{R}^l} \left| \frac{\partial f}{\partial \alpha_k}(\alpha|a) \right| d\alpha \mu(da) \\
&\overset{(A3)}{=} \frac{1}{p_h} \int_{\mathbf{A}} w_a \mathfrak{v}_k(a) \mu(da) \\
&\overset{(A4)}{\leq} \frac{\mathfrak{v}_k}{p_h} \int_{\mathbf{A}} w_a \mu(da) \\
&\overset{(A5)}{=} \frac{\bar{w}}{p_h} \mathfrak{v}_k.
\end{aligned}$$

□

The Law of Demand in the Aggregate

Recall from (*A4*) that

$$\mathfrak{v}_k := \sup_{a \in \mathbf{A}} \left\{ \mathfrak{v}_k(a) := \int_{\mathbb{R}^l} \left| \frac{\partial f}{\partial \alpha_k}(\alpha|a) \right| d\alpha < \infty \right\}$$

is well defined. The smaller $\mathfrak{v}_k$ the less curved f uniformly in a in direction k; thus the "flatter" f in direction k and the more heterogeneous agents. Since we want to obtain increasing heterogeneity among agents, the question is whether we can get $\mathfrak{v}_k$ arbitrarily small for all $k = 1, ..., l$. The following procedure based on density transformations can be found in [Grandmont, 92], p. 42:

Note 4.21. ([Grandmont, 92], p. 42) *Let X be a random variable with density $g : \mathbb{R}^l \to \mathbb{R}_+$ that has partial derivatives in $\mathscr{L}^1(\mathbb{R}^l, dx)$. Consider the transformation $h : \mathbb{R}^l \to \mathbb{R}^l$ given by $y = h(x) = \sigma x$, where $\sigma in \mathbb{R}_{++}$. By density transformation, the density g_σ of $h(X)$ is given by $g_\sigma(y) = \frac{g(\frac{y}{\sigma})}{\sigma}$. Thus varying the value of σ, we can get*

$$\int_{\mathbb{R}^l} \left| \frac{\partial g_\sigma}{\partial y_k}(y) \right| dy = \frac{1}{\sigma^2} \int_{\mathbb{R}^l} \left| \frac{\partial g}{\partial x_k}(x) \right| dx$$

arbitrarily small.

From Proposition 4.20 we may infer that ceteris paribus the cross partial derivatives $\frac{\partial X_h}{\partial p_k}(p)$ for all $h \neq k$ tend to zero as $\mathfrak{v}_k$ tends to zero. Moreover, own-price-elasticity of market demand $\frac{p_h}{X_h(p)}\frac{\partial X_h}{\partial p_h}(p)$ tends to -1 as $\mathfrak{v}_k$ tends to zero for all k; of course, we have to ensure market demand $X_h(p)$ to be positive for every commodity h at any price system p. This again suggests the emergence of Cobb-Douglas type demand in the aggregate as $\mathfrak{v}_k$ gets arbitrarily small since for Cobb-Douglas demand cross partial derivatives with respect to prices are zero and own-price-elasticity is -1, too. In this sense, market demand would be well behaved in the "high heterogeneity limit".[7]

We now pose assumptions that ensure market demand $X_h(\cdot)$ for every commodity to be strictly positive.

Assumption 4.22. ([Grandmont, 92], pp. 22) *Let* $a \in \mathbf{A}$, $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ *and* $\alpha \in \mathbb{R}^l$.

A 6 Independence*: For* μ*-a.e.* $a \in \mathbf{A}$ *the conditional density* $f(\alpha|a)$ *does not depend on* a*. Thus, we may set* $f(\alpha) = f(\alpha|a)$ μ*-a.s.*

A 7 Non-vanishing Expenditure*: For every* $h = 1, ..., l$*, there exists* $\varpi_h > 0$*, with* $\sum_{h=1}^l \varpi_h \leq 1$*, such that for all price systems* $p \in \mathbb{R}^l_{++}$ *we have*

$$p_h \int_{\mathbf{A}} \xi_{ah}(p, w_a)\mu(da) \geq \varpi_h \bar{w} > 0.$$

Assumptions (*A6*) and (*A7*) may be applied to show aggregate desirability for every commodity $h \leq l$:

$$\begin{aligned}
p_h X_h(p) &= \int_{\mathbf{A}} \int_{\mathbb{R}^l} p_h \xi^{\alpha}_{ah}(p, w_a) f(\alpha|a) d\alpha \mu(da) \\
&\overset{(A6)}{=} \int_{\mathbb{R}^l} f(\alpha|a) \int_{\mathbf{A}} p_h \xi^{\alpha}_{ah}(p, w_a)\mu(da) d\alpha \\
&\overset{(A7)}{\geq} \int_{\mathbb{R}^l} \varpi_h \bar{w} f(\alpha|a) d\alpha = \varpi_h \bar{w} > 0 \quad \forall h \\
\overset{p_h > 0}{\Rightarrow} X_h(p) &> 0 \quad \forall h.
\end{aligned} \tag{4.18}$$

(*A7*) allows for types of agents that do not consume any amount of commodity h at price system p but in the aggregate, there is positive consumption.

So far, we have mentioned the weak axiom of revealed preference several times but have not rigorously introduced this concept. This will be done within the next paragraphs.

[7] In In this token, the model that is introduced in the next chapter this is just the opposite of the "low temperature limit" considered in Section 2.7.

Definition 4.23. ([Ellickson, 93], p. 63) *A subset $D \in \mathbb{R}^l$ is called a* cone *if $x \in D$ and $\lambda > 0$ imply $\lambda x \in D$. A cone D is a* proper cone *if $D \cap (-D) \in \{\{0\}, \emptyset\}$. A proper cone D is called a* pointed cone *if $0 \in D$. We say that a cone is open, convex, ... if the set D is open, convex, ... in $\mathbb{R}^l$.*

The second part of the following definition calls for further explanation: We define definiteness of a matrix. In standard textbooks in linear algebra this notion is defined for symmetric matrices. However, in economic applications this concept is of interest in more general contexts. Thus we introduce definiteness for not necessarily symmetric matrices and refer to this as quasi-definiteness, although in many economics textbooks as for example in [Mas-Colell et al., 95], it is called definiteness.

Definition 4.24. *Let $A \in M(l \times l, \mathbb{R})$ be an $l \times l$-matrix with entries $a_{ij} \in \mathbb{R}$.*
(a) ([Mas-Colell et al., 95], Definition M.D.2) *We say that A has a* dominant diagonal *if there exists $q \in \mathbb{R}^l_{++}$ such that for every $i = 1, ..., l$ we have*

$$|q_i a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{l} |q_j a_{ij}|. \tag{4.19}$$

(b) ([Mas-Colell et al., 95], Definition M.D.1) *We call A* negative quasi-semidefinite *if for all $x \in \mathbb{R}^l$*

$$^t x A x := \sum_{i,j=1}^{l} x_i a_{ij} x_j \leq 0. \tag{4.20}$$

If this inequality is strict for all $x \neq 0$ we call A negative quasi-definite. *For reversed inequalities we call the matrix A* positive quasi-semidefinite *and* positive quasi-definite, *respectively. As already mentioned above, for symmetric matrices we just drop the term "quasi".*

Definition 4.25. *Let ξ be a demand function, $(p^i, w^i) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$.*
(a) ([Mas-Colell et al., 95], Definition 2.F.1) *ξ satisfies the* weak axiom of revealed preference (WARP) *if the following property holds for any to price-income pairs (p^1, w^1) and (p^2, w^2):*

$$[p^1 \cdot \xi(p^2, w^2) \leq w^1 \quad \textit{and} \quad \xi(p^1, w^1) \neq \xi(p^2, w^2)] \Rightarrow p^2 \cdot \xi(p^1, w^1) > w^2. \tag{4.21}$$

(b) ([Mas-Colell et al., 95], Definition 4.C.2) *ξ satisfies the* uncompensated law of demand (ULD) *property if*

$$(p^1 - p^2) \cdot [\xi(p^1, w) - \xi(p^2, w)] \leq 0 \tag{4.22}$$

for any $p^1, p^2 \in \mathbb{R}^l_{++}$ and $w \in \mathbb{R}_{++}$ with strict inequality if $\xi(p^1, w) \neq \xi(p^2, w)$.
(c) ([Grandmont, 92], p. 23) *ξ is called* strictly monotone *if*

$$(p^1 - p^2) \cdot [\xi(p^1, w) - \xi(p^2, w)] < 0 \tag{4.23}$$

for $p^1 \neq p^2 \in \mathbb{R}^l_{++}$ *and* $w \in \mathbb{R}_{++}$.

Remark 4.26. (a) *Due to fixed aggregate per capita income* $\bar{w}$*, the notions introduced in Definition 4.25 are rephrased for market demand* X *as follows: The weak axiom of revealed preference holds in the aggregate if, for all* $p, q \in \mathbb{R}^l_{++}$

$$[p \cdot X(q) \leq \bar{w} \quad \textit{and} \quad X(p) \neq X(q)] \Rightarrow q \cdot X(p) > \bar{w}.$$

Market demand satisfies the uncompensated law of demand property if

$$(p - q) \cdot [X(p) - X(q)] \leq 0$$

for all $p, q \in \mathbb{R}^l_{++}$ *with strict inequality if* $X(p) \neq X(q)$*. Strict monotonicity is satisfied in the aggregate if, for all* $p \neq q \in \mathbb{R}^l_{++}$,

$$(p - q) \cdot [X(p) - X(q)] < 0.$$

(b) *Having a look at the definition above, we immediately note that strict monotonicity of demand (4.23) implies the uncompensated law of demand property (4.22), but in general not vice versa. As a counterexample we may consider a demand function that is generated by an exclusive preference as in [Föllmer, 74].*

The uncompensated law of demand property states that, without any income compensation or wealth adjustment, price- and demand-changes point in opposite directions.

The weak axiom of revealed preference is a rationality assumption on demand in the following sense: Assume that an agent's demand satisfies WARP. Whenever the choice $\xi(p^2, w^2)$ at price-income pair (p^2, w^2) is also affordable at price-income pair (p^1, w^1), $p^1 \cdot \xi(p^2, w^2) \leq w^1$, but is not chosen, i.e. the choices at those different price-wealth pairs are not identical, $\xi(p^1, w^1) \neq \xi(p^2, w^2)$, then we will necessarily have that the choice $\xi(p^1, w^1)$ at price-income pair (p^1, w^1) is not affordable at price-income pair (p^2, w^2), $p^2 \cdot \xi(p^1, w^1) > w^2$. If the choice $\xi(p^1, w^1)$ at price-income pair (p^1, w^1) would also be affordable at price-income pair (p^2, w^2) we would have to pose the question why the agent prefers $\xi(p^1, w^1)$ over $\xi(p^2, w^2)$ when the budget is given by (p^1, w^1) and after a budget change to (p^2, w^2) even though $\xi(p^1, w^1)$ is still affordable the agent chooses the less preferred bundle $\xi(p^2, w^2)$. This would violate what we would call "rational behavior".

A good intuition for the term "weak axiom of revealed preference" is given in [Mas-Colell et al., 95], p. 29:

> In the consumer demand setting, the idea behind the weak axiom can be put as follows: If $p^1 \cdot \xi(p^2, w^2) \leq w^1$ and $\xi(p^1, w^1) \neq \xi(p^2, w^2)$, then we know that when facing prices p^1 and wealth w^1, the consumer choses consumption bundle

$\xi(p^1, w^1)$ even though bundle $\xi(p^2, w^2)$ was also affordable. We can interpret this choice as "revealing" a preference for $\xi(p^1, w^1)$ over $\xi(p^2, w^2)$. Now, we might reasonably expect the consumer to display some consistency in his demand behavior. In particular, given his revealed preference, we expect that he would chose $\xi(p^1, w^1)$ over $\xi(p^2, w^2)$ whenever they are both affordable. If so, bundle $\xi(p^1, w^1)$ must not be affordable at price-wealth combination (p^2, w^2) at which the consumer chooses bundle $\xi(p^2, w^2)$. That is, as required by the weak axiom, we must have $p^2 \cdot \xi(p^1, w^1) > w^2$.

Let us now state a fundamental result.

Proposition 4.27. ([Hildenbrand, 94], p. 170) *Let ξ be partially differentiable. If the Jacobian matrix $(\frac{\partial \xi_h}{\partial p_k}(p, w))_{h,k=1,\dots,l}$ of demand ξ is negative quasi-definite for all $p \in \mathbb{R}^l_{++}$, $w \in \mathbb{R}_{++}$, then ξ satisfies strict monotonicity. Analogously for market demand X.*[8]

Proof. Let $p, q \in \mathbb{R}^l_{++}$, $p \neq q$ and $w \in \mathbb{R}_{++}$. Furthermore define $x := p - q$, $p(\alpha) := \alpha p - (1 - \alpha)q$ for $\alpha \in [0, 1]$ and $g : [0, 1] \to \mathbb{R}$ by

$$g(\alpha) = x \cdot (\xi(p(\alpha), w) - \xi(q, w)).$$

Then we have

$$\begin{aligned} g(0) &= 0, \\ g(1) &= (p - q) \cdot [\xi(p, w) - \xi(q, w)], \\ \frac{dg}{d\alpha}(\alpha) &= x \cdot \left(\frac{\partial \xi_h}{\partial p_k}(p(\alpha), w) \right)_{h,k=1,\dots,l} x = \sum_{h,k=1}^{l} x_h \frac{\partial \xi_h}{\partial p_k}(p(\alpha), w) x_k, \end{aligned}$$

where the last equation follows by chain rule. By assumption, we have $\frac{dg}{d\alpha}(\alpha) < 0$ for all $\alpha \in [0, 1]$. Since $g(0) = 0$ this yields $g(1) < 0$ and we have obtained the strict monotonicity of ξ. The proof for market demand X is identical when substituting $\xi(p, w)$ by $X(p)$. □

Proposition 4.28. ([Mas-Colell et al., 95], p. 111) *Let ξ be a demand function and $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$. If the Jacobian matrix $(\frac{\partial \xi_h}{\partial p_k}(p, w))_{h,k=1,\dots,l}$ is negative quasi-definite for all p, w, then ξ satisfies the uncompensated law of demand property. Analogously for market demand X.*

Proof. We have seen in proposition 4.27 that negative quasi-definiteness of the Jacobian matrix of demand implies strict monotonicity of demand. On the other hand, strict monotonicity of demand implies the uncompensated law of demand property. □

[8] In [Hildenbrand, 94] the result is shown for market demand; here, we apply the proof to our context of demand functions ξ where income w is fixed.

Remark 4.29. *It is noted in [Mas-Colell et al., 95], 4.C.2, that homothetic preferences imply Walrasian demand χ to satisfy the uncompensated law of demand property. It is well known that the uncompensated law of demand property is inherited by market demand. For simple economies this is shown in [Mas-Colell et al., 95], 4.C.1. The following proposition then shows that aggregate demand satisfies the weak axiom of revealed preference if it is generated by homothetic preference maximizing agents.*

In the proposition that follows, we encounter the connection between the uncompensated law of demand and the weak axiom.

Proposition 4.30. ([Mas-Colell et al., 95], p. 112) *Let ξ be a demand function. If ξ satisfies the uncompensated law of demand property, then ξ satisfies the weak axiom of revealed preference.*

Proof. Let $(p^1, w^1), (p^2, w^2) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ such that $\xi(p^1, w^1) \neq \xi(p^2, w^2)$ and $p^1 \cdot \xi(p^2, w^2) \leq w^1$. Set $p^3 := \frac{w^1}{w^2} p^2$. Due to homogeneity of ξ we have

$$\xi(p^3, w^1) = \xi(\frac{w^1}{w^2} p^2, w^1) = \xi(p^2, w^2).$$

Furthermore, we have

$$\begin{aligned}
&(p^3 - p^1) \cdot [\xi(p^3, w^1) - \xi(p^1, w^1)] < 0 \quad \text{by (ULD)},\\
&p^1 \cdot \xi(p^3, w^1) = p^1 \cdot \xi(p^2, w^2) \leq w^1 \quad \text{and}\\
&p^3 \cdot \xi(p^3, w^1) = w^1, \quad p^1 \cdot \xi(p^1, w^1) = w^1 \quad \text{by Walras' law.}
\end{aligned}$$

Using the above (in-)equalities we may establish the weak axiom:

$$\begin{aligned}
& p^3 \cdot \xi(p^3, w^1) - p^3 \cdot \xi(p^1, w^1) - p^1 \cdot \xi(p^3, w^1) + p^1 \cdot \xi(p^1, w^1) < 0\\
\Leftrightarrow\ & p^3 \cdot \xi(p^1, w^1) > 2w^1 - p^1 \cdot \xi(p^3, w^1)\\
\Leftrightarrow\ & \frac{w^1}{w^2} p^2 \cdot \xi(p^1, w^1) = p^3 \cdot \xi(p^1, w^1) > 2w^1 - p^1 \cdot \xi(p^3, w^1) \geq w^1\\
\Leftrightarrow\ & p^2 \cdot \xi(p^1, w^1) > w^2.
\end{aligned}$$

Thus, the weak axiom of revealed preference holds. □

Remark 4.31. *To summarize, we have shown the following chain of implications: Negative quasi-definiteness implies strict monotonicity (Proposition 4.27) which in turn implies the uncompensated law of demand property that implies the weak axiom of revealed preference (Proposition 4.30).*

Definition 4.32. ([Jehle & Reny, 01], Definition 1.6) *Let ξ be a demand function, strictly positive at $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, and $h, k = 1, \ldots, l$.*
(a) *The* price elasticity *η_{hk} for commodity h and price p_k of commodity k at price-wealth pair $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ is given by*

$$\eta_{hk}(p, w) := \frac{p_k}{\xi_h(p, w)} \frac{\partial \xi_h}{\partial p_k}(p, w) = \frac{\partial log \xi_h}{\partial log p_k}(p, w).$$

If $h \neq k$ *we call* η_{hk} cross-price elasticity*; otherwise* own-price elasticity*.*
(b) *The* income elasticity η_{hw} *for commodity* h *is given by*

$$\eta_{hw}(p,w) := \frac{w}{\xi_h(p,w)}\frac{\partial \xi_h}{\partial w}(p,w) = \frac{\partial log \xi_h}{\partial log w}(p,w).$$

Analogous for total market demand X.[9]

Remark 4.33. *Intuitively, price elasticity measures the percentage change of the quantity of commodity* h *demanded when the price of commodity* k *changes by one percent; income elasticity measures the change in percent when income changes by one percent.*

The next theorem shows that, ceteris paribus, market demand has the desired properties when conditional densities $f(\alpha|a)$ are "sufficiently flat" and X is strictly positive. This constitutes the main result in this chapter and serves as a corner stone of this diploma thesis. We have worked out the proof here elaborately. Since some assertions are stated in a slightly different manner, not all assumptions in [Grandmont, 92] are needed.

Theorem 4.34. ([Grandmont, 92], Theorem 2.3) *Let* $(p,w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, $h,k = 1,...,l$, *and* $\varpi_h > 0$ *be chosen as in (A7). Assume that (A1) to (A7) hold.*[10] *Then* $p_h X_h(p) \geq \varpi_h \bar{w}$ *for every* h *and* p*. Price elasticities* $\frac{\partial \log X_h}{\partial \log p_k}(p)$ *of market demand* X_h *satisfy*

$$\left|\frac{\partial \log X_h}{\partial \log p_k}(p) + \delta_{hk}\right| \leq \frac{\mathfrak{v}_k}{\varpi_h}. \tag{4.24}$$

for all h *and* k*. In particular, we get:*
(a) *For each commodity* h*, total market demand* X_h *is strictly decreasing in its own price, i. e.*

$$\frac{\partial X_h}{\partial p_h}(p) < 0$$

if $\mathfrak{v}_h < \varpi_h$*.*
(b) *Define the proper open cone*

$$\mathscr{D}(m,\varpi) := \{p \in \mathbb{R}^l_{++} | \sum_{k=1}^{l} \frac{\mathfrak{v}_k}{p_k} < \frac{\varpi_h}{p_h} \quad \textit{for every} \quad h\}.$$

Then the Jacobian matrix $\left(\frac{\partial X_h}{\partial p_k}(p)\right)_{h,k=1,...,l}$ *has dominant diagonal on* $\mathscr{D}(m,\varpi)$*.*

[9] Of course, income elasticity for total market demand is not relevant here but in the next section.
[10] We need *(A1)* to *(A5)* in order to apply equation (4.17) and *(A6)* and *(A7)* to obtain non-vanishing demand.

(c) *Assume* $\mathfrak{v}_k l < \varpi_h$ *for all commodities* h, k. *Then total market demand* X *has a negative quasi-definite Jacobian matrix for every price system* $p \in \mathbb{R}^l_{++}$. *Thus total market demand is strictly monotone and in particular it satisfies the weak axiom of revealed preference.*

The theorem shows that market demand may satisfy the weak axiom of revealed preference even though we have not assumed the weak axiom to hold for individual demand. As already argued, even if we would do so, the weak axiom would not necessarily hold in the aggregate. Thus, we have created some structural properties of market demand by assumptions on the distribution of individual demand, or more precisely on the distribution of α-transforms, but without assuming individuals to behave rational except satisfying homogeneity of degree zero and Walras' law.

Proof of Theorem 4.34. After stating Assumption 4.22, we have already shown aggregate desirability, i.e. $p_h X_h(p) \geq \varpi_h \bar{w} > 0$ for all $p \in \mathbb{R}^l_{++}$, $h \leq l$.

Using equation (4.17), we get

$$\begin{aligned}
\left|\frac{\partial log X_h}{\partial \log p_k}(p) + \delta_{hk}\right| &= \left|p_k \frac{\partial \log X_h}{\partial p_k}(p) + \delta_{hk}\right| \\
&= \left|\frac{p_k}{X_h(p)} \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk}\right| \\
&= \frac{1}{X_h(p)} \left|p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p)\right| \\
&\leq \bar{w} \frac{\mathfrak{v}_k}{p_h X_h(p)} \leq \frac{\mathfrak{v}_k}{\varpi_h}
\end{aligned}$$

and have thus established equation (4.24).

Let $0 < \mathfrak{v}_h < \varpi_h$ for all $h = 1, ..., l$, then

$$\begin{aligned}
& \left|\frac{\partial \log X_h}{\partial \log p_h}(p) + 1\right| \leq \frac{\mathfrak{v}_h}{\varpi_h} < 1 \\
\Leftrightarrow \quad & -1 < \frac{\partial \log X_h}{\partial \log p_h}(p) + 1 < 1 \\
\Leftrightarrow \quad & -2 < \frac{\partial \log X_h}{\partial \log p_h}(p) < 0 \\
\Leftrightarrow \quad & -2 \frac{X_h(p)}{p_h} < \frac{\partial X_h}{\partial p_h}(p) < 0
\end{aligned}$$

for all $h = 1, ..., l$ and $p \in \mathbb{R}^l_{++}$. This completes the proof of (a).

Let us now turn to (b): For $k \neq h$, equation (4.24) yields

$$\begin{aligned}
& \left|\frac{p_k}{X_h(p)} \frac{\partial X_h}{\partial p_k}(p)\right| \leq \frac{\mathfrak{v}_k}{\varpi_h} \\
\Leftrightarrow \quad & \frac{1}{X_h(p)} \left|\frac{\partial X_h}{\partial p_k}(p)\right| \leq \frac{\mathfrak{v}_k}{\varpi_h p_k}.
\end{aligned}$$

If $h = k$, we obtain from equation (4.24)

$$\begin{aligned}
\frac{1}{X_h(p)}\left|\frac{\partial X_h}{\partial p_h}(p)\right| &\geq \frac{1}{p_h}\left(1 - \left|\frac{p_h}{X_h(p)}\frac{\partial X_h}{\partial p_h}(p) + 1\right|\right) \\
&\overset{(4.24)}{\geq} \frac{1}{p_h}\left(1 - \frac{\mathfrak{v}_h}{\varpi_h}\right) = \frac{\varpi_h - \mathfrak{v}_h}{p_h \varpi_h}.
\end{aligned}$$

To summarize, we have

$$\frac{1}{X_h(p)}\left|\frac{\partial X_h}{\partial p_k}(p)\right| \quad \begin{cases} \geq \frac{\varpi_h - \mathfrak{v}_h}{p_h \varpi_h} & \text{if } h = k, \\ \leq \frac{\mathfrak{v}_k}{p_k \varpi_h} & \text{if } h \neq k. \end{cases} \tag{4.25}$$

We may now proceed and show that the Jacobian matrix $(\frac{\partial X_h}{\partial p_k}(p))_{h,k=1,\ldots,l}$ has dominant diagonal on $\mathscr{D}(m, \varpi)$:[11] Let $h \leq l$. Then

$$\begin{aligned}
\sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{1}{X_h(p)}\left|\frac{\partial X_h}{\partial p_k}(p)\right| &\overset{(4.25)}{\leq} \sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{\mathfrak{v}_k}{p_k \varpi_h} \\
&= \frac{1}{\varpi_h}\sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{\mathfrak{v}_k}{p_k} \\
&< \frac{1}{\varpi_h}\frac{\varpi_h - \mathfrak{v}_h}{p_h} \\
&\overset{(4.25)}{\leq} \frac{1}{X_h(p)}\left|\frac{\partial X_h}{\partial p_h}(p)\right| \\
\overset{X_h(p)>0}{\Leftrightarrow} \left|\frac{\partial X_h}{\partial p_h}(p)\right| &> \sum_{\substack{k=1 \\ k \neq h}}^{l} \left|\frac{\partial X_h}{\partial p_k}(p)\right|.
\end{aligned}$$

This shows the assertion in (b).

We are now enabled to prove the main result within this theorem: Let $p \in \mathbb{R}^l_{++}$. Assume that for all $h, k = 1, \ldots, l$ we have $\mathfrak{v}_k l < \varpi_h$. In particular, we have $\mathfrak{v}_h l < \varpi_h$ for all h and thus $\mathfrak{v}_h < \varpi_h$. Hence by part (a) we have $\frac{\partial X_h}{\partial p_h}(p) < 0$. Let $v \in \mathbb{R}^l \setminus \{0\}$ and $p \in \mathbb{R}^l_{++}$. Assuming

$$\left|\sum_{h=1}^{l} v_h^2 \frac{\partial X_h}{\partial p_h}(p)\right| > \sum_{\substack{h,k=1 \\ k \neq h}}^{l} \left|v_h v_k \frac{\partial X_h}{\partial p_k}(p)\right|, \tag{4.26}$$

[11] Note that $\sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{\mathfrak{v}_k}{p_k} < \frac{\varpi_h - \mathfrak{v}_h}{p_h}$ for $p \in \mathscr{D}(m, \varpi)$ since $\sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{\mathfrak{v}_k}{p_k} + \frac{\mathfrak{v}_h}{p_h} = \sum_{k=1}^{l} \frac{\mathfrak{v}_k}{p_k} < \frac{\varpi_h}{p_h} \Leftrightarrow \sum_{\substack{k=1 \\ k \neq h}}^{l} \frac{\mathfrak{v}_k}{p_k} < \frac{\varpi_h - \mathfrak{v}_h}{p_h}$.

we obtain

$$\begin{aligned}
\sum_{h,k=1}^{l} v_h \frac{\partial X_h}{\partial p_k}(p) v_k &= \underbrace{\sum_{h=1}^{l} \underbrace{v_h^2}_{\geq 0} \underbrace{\frac{\partial X_h}{\partial p_h}(p)}_{<0}}_{\leq 0} + \sum_{\substack{h,k=1\\ h\neq k}}^{l} v_h \frac{\partial X_h}{\partial p_k}(p) v_k \\
&< 0 \quad \text{by inequality (4.26).}
\end{aligned}$$

Thus, if we can show inequality (4.26) to hold, we have obtained the negative quasi-definiteness of $(\frac{\partial X_h}{\partial p_k}(p))_{h,k=1,\dots,l}$. For the terms in inequality (4.26) equation (4.17) implies

$$\sum_{\substack{h,k=1\\ h\neq k}}^{l} |v_h||v_k| \left|\frac{\partial X_h}{\partial p_k}(p)\right| \leq \bar{w} \sum_{\substack{h,k=1\\ h\neq k}}^{l} |v_h||v_k| \frac{\mathfrak{v}_k}{p_h p_k} \quad \text{and} \tag{4.27}$$

$$\left|\sum_{h=1}^{l} v_h^2 \frac{\partial X_h}{\partial p_h}(p)\right| \geq \bar{w} \sum_{h=1}^{l} v_h^2 \frac{\varpi_h - \mathfrak{v}_h}{p_h^2} > 0 \quad \text{since } v \neq 0. \tag{4.28}$$

These equations are obtained as follows:

$$\begin{aligned}
\sum_{\substack{h,k=1\\ h\neq k}}^{l} |v_h||v_k| \left|\frac{\partial X_h}{\partial p_k}(p)\right| &= \sum_{\substack{h,k=1\\ h\neq k}}^{l} \frac{1}{p_k} |v_h||v_k| \left| p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p)\right| \\
&\leq \bar{w} \sum_{\substack{h,k=1\\ h\neq k}}^{l} \frac{1}{p_k} |v_h||v_k| \frac{\mathfrak{v}_k}{p_h}.
\end{aligned}$$

Thus, (4.27) is shown and we deduce (4.28): Since the summands on the left-hand side of (4.28) are all non-positive and the summands on the right-hand side non-negative, it suffices to show

$$\begin{aligned}
\left|\frac{\partial X_h}{\partial p_h}(p)\right| &= \left| -\frac{X_h}{p_h}(p) + \frac{\partial X_h}{\partial p_h}(p) + \frac{X_h}{p_h}(p)\right| \\
&\overset{|x|-|y|\leq|x+y|}{\geq} \left|\frac{X_h}{p_h}(p)\right| - \left|\frac{\partial X_h}{\partial p_h}(p) + \frac{X_h}{p_h}(p)\right| \\
&\overset{X_h(p)p_h>0}{=} \frac{p_h X_h(p)}{p_h^2} - \left|\frac{\partial X_h}{\partial p_h}(p) + \frac{X_h}{p_h}(p)\right| \\
&\overset{p_h X_h(p)\geq \varpi_h \bar{w}}{\geq} \bar{w}\frac{\varpi_h}{p_h^2} - \left|\frac{\partial X_h}{\partial p_h}(p) + \frac{X_h}{p_h}(p)\right| \\
&\overset{(4.17)}{\geq} \bar{w}\frac{\varpi_h}{p_h^2} - \bar{w}\frac{\mathfrak{v}_h}{p_h^2} = \bar{w}\frac{\varpi_h - \mathfrak{v}_h}{p_h^2}.
\end{aligned}$$

Multiplying by $v_h^2 \geq 0$ and summing up yields (4.28).

Considering inequalities (4.27) and (4.28), we have shown negative quasi-definiteness of $(\frac{\partial X_h}{\partial p_k}(p))_{h,k=1,\dots,l}$ if we have

$$\bar{w}\sum_{h=1}^{l} v_h^2 \frac{\varpi_h - \mathfrak{v}_h}{p_h^2} > \bar{w}\sum_{\substack{h,k=1\\h\neq k}}^{l} |v_h||v_k|\frac{\mathfrak{v}_k}{p_h p_k}$$

Defining $v_h := p_h u_h$ this is equivalent to

$$\sum_{h=1}^{l} u_h^2(\varpi_h - \mathfrak{v}_h) = \sum_{h=1}^{l} u_h^2\varpi_h - \sum_{h=1}^{l} u_h^2\mathfrak{v}_h > \sum_{\substack{h,k=1\\h\neq k}}^{l} |u_h||u_k|\mathfrak{v}_k$$

$$\Leftrightarrow \quad \sum_{h=1}^{l} u_h^2\varpi_h > \sum_{h,k=1}^{l} |u_h||u_k|\mathfrak{v}_k \tag{4.29}$$

A fundamental inequality in linear algebra tells us that for all $u \in \mathbb{R}^l$, we have $l\sum_{h=1}^{l} u_h^2 \geq \sum_{h,k=1}^{l}|u_h||u_k|$. Assuming $\mathfrak{v}_k l < \varpi_h$ for all h and k and defining $\underline{\varpi} := \min_i \varpi_i$ and $m := \max_j \mathfrak{v}_j$, we get

$$\frac{\varpi_i}{\mathfrak{v}_j}\sum_{h=1}^{l} u_h^2 > \sum_{h,k=1}^{l}|u_h||u_k| \quad \frac{\varpi_i}{\mathfrak{v}_j} > l \text{ for all } i,j \leq l$$

$$\Rightarrow \quad \varpi_i\sum_{h=1}^{l} u_h^2 > \mathfrak{v}_j\sum_{h,k=1}^{l}|u_h||u_k| \quad \text{for all } i,j \leq l$$

$$\Rightarrow \quad \sum_{h=1}^{l} u_h^2\varpi_h \geq \sum_{h=1}^{l} u_h^2\underline{\varpi} > \sum_{h,k=1}^{l}|u_h||u_k|m \geq \sum_{h,k=1}^{l}|u_h||u_k|\mathfrak{v}_k.$$

Thus, equation (4.29) holds.

Having established negative quasi-definiteness of the Jacobian matrix $(\frac{\partial X_h}{\partial p_k}(p))_{h,k=1,\dots,l}$ of market demand for all price systems $p \in \mathbb{R}^l_{++}$, we obtain strict monotonicity of market demand and the weak axiom of revealed preference by Proposition 4.27 and Remark 4.31. □

A closer look at part (c) of the foregoing theorem reveals the effect of increasing dispersion of α-transforms: Increasing individual heterogeneity for every type $a \in \mathbf{A}$ means decreasing the $\mathfrak{v}_k$'s for all k. Thus, we obtain the uncompensated law of demand property and the weak axiom of revealed preference for market demand.

Income Effects

Up to now we have not considered income effects, neither for individual nor for market demand. Analysis did not rely on income effects so far because income was fixed and independent of prices. However, what if income is not independent of the price system? One might for example think of an initial commodity endowment as done in Chapter 3. Thus, we now have a look at the effect of income changes on market demand. In more catching terms: "*... how [does] dispersion of the conditional densities $f(a|\alpha)$ influence the partial derivatives of aggregate demand with respect to per capita income* $\bar{w}$." ([Grandmont, 92], p. 25)

In the following discussion we allow per capita income $\bar{w}$ to vary but we assume the distribution of incomes to be fixed, i.e. we assume $\theta : \mathbf{A} \to \mathbb{R}_{++}$, $a \mapsto \theta_a$, such that $\int_{\mathbf{A}} \theta_a \mu(da) = 1$ and $w_a = \theta_a \bar{w}$, the income of type a agent. Thus, an agent of type a always gets a fraction θ_a of aggregate per capita income. We still have as in (*A5*)

$$\bar{w} = \int_{\mathbf{A}} w_a \mu(da) = \int_{\mathbf{A}} \theta_a \bar{w} \mu(da)$$

Definition 4.35. ([Grandmont, 92], p. 25) *Given $a \in \mathbf{A}$, $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $\alpha \in \mathbb{R}^l$. Again,* conditional market demand *is defined as in Definition 4.17:*

$$X(a, p, w) := \int_{\mathbb{R}^l} \xi^\alpha_a(p, w) f(\alpha|a) d\alpha.$$

But now, total market demand $X : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ *is given by*

$$X(p, \bar{w}) := \int_{\mathbf{A}} X(a, p, \theta_a \bar{w}) \mu(da), \tag{4.30}$$

where the distribution of income $(\theta_a)_{a \in \mathbf{A}}$ is a fixed parameter of the system but per capita income $\bar{w}$ may vary.

Of course, the (in-)equalities shown so far, as for example (4.17) and (4.24), still hold. Again by Proposition 4.16, (*A1*) to (*A3*) imply that conditional market demand $X(a, \cdot, \cdot)$ has continuous partial derivatives with respect to p_h, $h = 1 \ldots l$, and w.

Lemma 4.36. *For all $(a, p, w) \in \mathbf{A} \times \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ and $h = 1, \ldots l$, we have*

$$\left| w \frac{\partial X_h}{\partial w}(a, p, w) - X_h(a, p, w) \right| \leq w \sum_{k=1}^{l} \frac{\mathfrak{v}_k(a)}{p_h}. \tag{4.31}$$

Proof. As we have already seen in the last section, conditional market demand is homogeneous of degree 0 in (p, w). Thus, using Euler's identity (4.2), we obtain

$$0 = \sum_{k=1}^{l} p_k \frac{\partial X_h}{\partial p_k}(a, p, w) + w \frac{\partial X_h}{\partial w}(a, p, w).$$

This yields

$$
\begin{aligned}
w\sum_{k=1}^{l}\frac{\mathfrak{v}_k(a)}{p_h} &\overset{(4.16)}{\geq} \sum_{k=1}^{l}\left|p_k\frac{\partial X_h}{\partial p_k}(a,p,w)+\delta_{hk}X_h(a,p,w)\right| \\
&\geq \left|\sum_{k=1}^{l}p_k\frac{\partial X_h}{\partial p_k}(a,p,w)+X_h(a,p,w)\right| \\
&= \left|w\frac{\partial X_h}{\partial w}(a,p,w)-X_h(a,p,w)\right|.
\end{aligned}
$$

□

Analogously to equations (4.17) and (4.24), we obtain the following proposition when considering income effects.

Proposition 4.37. ([Grandmont, 92], Proposition 2.4) *Assume (A1) to (A5) and let $(p,\bar{w}) \in \mathbb{R}^l_{++}\times\mathbb{R}_{++}$. Then total market demand X as defined in equation (4.30) is continuously partially differentiable. Upper and lower bounds for derivatives with respect to per capita income $\bar{w}$ are obtained by*

$$\left|\bar{w}\frac{\partial X_h}{\partial \bar{w}}(p,\bar{w})-X_h(p,\bar{w})\right| \leq \bar{w}\sum_{k=1}^{l}\frac{\mathfrak{v}_k}{p_h} \tag{4.32}$$

for all $h=1,\dots,l$. Assuming (A1) to (A7), then income elasticity $\frac{\partial \log X_h}{\partial \log \bar{w}}(p,\bar{w})$ of total market demand satisfies

$$\left|\frac{\partial \log X_h}{\partial \log \bar{w}}(p,\bar{w})-1\right| \leq \sum_{k=1}^{l}\frac{\mathfrak{v}_k}{\epsilon_h} \tag{4.33}$$

for all $h=1,\dots,l$. In particular,

$$\frac{\partial X_h}{\partial \bar{w}}(p,\bar{w})>0,$$

if $\sum_{k=1}^{l}\mathfrak{v}_k<\epsilon_h$. In this case, commodity h is called a normal good.

Proof. In light of (*A1*), (*A2*) and (*A3*) we have already shown by Proposition 4.16(1) that $X(a,\cdot,\cdot)$ has continuous partial derivatives. Using this result and again Proposition 4.16 together with (*A4*) and(*A5*), we may change the order of integration and differentiation and thus obtain continuous partial derivatives

$$\frac{\partial X_h}{\partial \bar{w}}(p,\bar{w})=\int_{\mathbf{A}}\theta_a\frac{\partial X_h}{\partial w}(a,p,\theta_a\bar{w})\mu(da)$$

for all $h = 1, \ldots, l$. Posing (*A1*) to (*A7*), we have shown equations (4.17) and (4.24). For all $h = 1, \ldots, l$, we have

$$\begin{aligned}
\bar{w} \sum_{k=1}^{l} \frac{\mathfrak{v}_k}{p_h} &\overset{(4.17)}{\geq} \sum_{k=1}^{l} \left| p_k \frac{\partial X_h}{\partial p_k}(p, \bar{w}) + \delta_{hk} X_h(p, \bar{w}) \right| \\
&\geq \left| \sum_{k=1}^{l} p_k \frac{\partial X_h}{\partial p_k}(p, \bar{w}) + X_h(p, \bar{w}) \right| \\
&= \left| \frac{\partial X_h}{\partial \bar{w}}(p, \bar{w}) \bar{w} - X_h(p, \bar{w}) \right|,
\end{aligned}$$

where the last equation follows by Euler's theorem. Thus, we have shown equation (4.32). The following (in-)equalities yield equation (4.33):

$$\begin{aligned}
\sum_{k=1}^{l} \frac{\mathfrak{v}_k}{\epsilon_h} &\overset{(4.24)}{\geq} \sum_{k=1}^{l} \left| \frac{\partial \log X_h}{\partial \log p_k}(p, \bar{w}) + \delta_{hk} \right| \\
&\geq \left| \sum_{k=1}^{l} \frac{\partial \log X_h}{\partial \log p_k}(p, \bar{w}) + 1 \right| \\
&= \left| \sum_{k=1}^{l} \frac{p_k}{X_h(p, \bar{w})} \frac{\partial X_h}{\partial p_k}(p, \bar{w}) + 1 \right| \\
&\overset{\text{Euler}}{=} \left| - \frac{\bar{w}}{X_h(p, \bar{w})} \frac{\partial X_h}{\partial \bar{w}}(p, \bar{w}) + 1 \right| \\
&= \left| \frac{\partial \log X_h}{\partial \log \bar{w}}(p, \bar{w}) - 1 \right|.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\sum_{k=1}^{l} \frac{\mathfrak{v}_k}{\epsilon_h} < 1 \quad &\Rightarrow \quad 0 < \frac{\bar{w}}{X_h(p, \bar{w})} \frac{\partial X_h}{\partial \bar{w}}(p, \bar{w}) < 2 \\
&\Rightarrow \quad \frac{\partial X_h}{\partial \bar{w}}(p, \bar{w}) > 0
\end{aligned}$$

□

4.2.3 Existence & Uniqueness of Equilibrium

We now consider a distribution economy $\mathscr{E}^d$ as in Definition 3.12. In this context an individual agent's income does actually depend on the price system as it is given by an initial commodity endowment. Agent's characteristics are now given by a demand function ξ and an initial endowment $\tau \in \mathbb{R}^l_+ \setminus \{0\}$. An individual agent's income can be obtained by $w = p \cdot \tau$.

The aim of this section is to show existence and uniqueness of equilibrium in $\mathscr{E}^d$ when individual characteristics are sufficiently dispersed. For this purpose we introduce the *gross substitute property* for market excess demand. Moreover, we obtain the weak axiom of revealed preference between an equilibrium price system and any other price vector when densities are sufficiently flat. The weak axiom may be applied to show stability of equilibrium. However, stability is not discussed in this diploma thesis.

We obtain a probability distribution on the space of agents characteristics in the same way as in Section 4.2.2: First consider a marginal distribution μ on the set $\mathbf{A}$ of agents' types. For each type $a \in \mathbf{A}$ we depict a conditional distribution $f(d\alpha|a)$ on the space of α-transforms $\{\xi_a^\alpha | \alpha \in \mathbb{R}^l\}$, where ξ_a and w_a denote the demand-income pair corresponding to type $a \in \mathbf{A}$.

Again, we suppose (*A1*) to (*A4*) to hold but replace (*A5*) by (*A5'*):

Assumption 4.38. ([Grandmont, 92], p. 28) *Let $a \in \mathbf{A}$.*

A 5' The initial endowment $a \mapsto \tau_a \in \mathbb{R}_+^l \backslash \{0\}$ is continuous in a. Per capita initial endowment $\bar{\tau}$ is finite and strictly positive in all components, i.e.

$$\bar{\tau} := \int_{\mathbf{A}} \tau_a \mu(da) \in \mathbb{R}_{++}^l.$$

We then define per capita income $\bar{w} := p \cdot \bar{\tau}$.

Definition 4.39. ([Grandmont, 92], pp. 28,29) (a) *Let $a \in \mathbf{A}$, $(p, \tau) \in \mathbb{R}_{++}^l \times \mathbb{R}_+^l \setminus \{0\}$ and ξ a demand function.* Conditional market demand $X : \mathbf{A} \times \mathbb{R}_{++}^l \times \mathbb{R}_+^l \setminus \{0\}$ *is given by*

$$X(a, p, p \cdot \tau) := X(a, p, w) := \int_{\mathbb{R}^l} \xi_a^\alpha(p, w) f(\alpha|a) d\alpha,$$

where $w := p \cdot \tau$. Note that conditional market demand has all properties that were stated in Section 4.2.2, in particular continuous partial derivatives.
(b) Conditional market excess demand $Z(a, \cdot)$ *given $a \in \mathbf{A}$ is defined by*

$$Z(a, p) := X(a, p, p \cdot \tau_a) - \tau_a.$$

(c) *For any price system p we define* total market excess demand $Z : \mathbb{R}_{++}^l \to \mathbb{R}^l$ *by*

$$Z(p) := \int_{\mathbf{A}} Z(a, p) \mu(da).$$

Remark 4.40. (a) *Conditional market excess demand $Z(a, \cdot)$ obviously inherits the following properties from conditional market demand $X(a, \cdot, \cdot)$: $Z(a, p)$ is*

- *well defined in the sense that $\int_{\mathbb{R}^l} \xi_a^\alpha(p, p \cdot \tau_a) f(\alpha|a) d\alpha - \tau_a < \infty$,*

- *continuously partially differentiable; in particular continuous,*
- *bounded below by* $-\tau_a$ *since* $X_h(a,p,p\cdot\tau_a)\geq 0$ *for all* $h=1,\dots,l$,
- *homogeneous of degree zero in price* $p\in\mathbb{R}^l_{++}$ *since conditional market demand is homogeneous of degree zero in* (p,w)[12] *and*
- *satisfies Walras' law, i.e.* $p\cdot Z(a,p)=0$ *for all* $a\in\mathbf{A}$ *and* $p\in\mathbb{R}^l_{++}$.[13]

(b) *We can rewrite total market excess demand as*

$$\begin{aligned} Z(p) &= \int_{\mathbf{A}} (X(a,p,p\cdot\tau_a)-\tau_a)\mu(da) \\ &\overset{(A5')}{=} \int_{\mathbf{A}} X(a,p,p\cdot\tau_a)\mu(da)-\bar{\tau} \\ &= \int_{\mathbf{A}}\int_{\mathbb{R}^l} \xi^{\alpha}_a(p,p\cdot\tau_a)f(\alpha|a)d\alpha\mu(da)-\bar{\tau}. \end{aligned}$$

Posing (A1) to (A4) and (A5') we note that total market excess demand has the properties given in part (a) *of this remark. Most properties are obvious, the others are shown in subsequent remarks and propositions.*

Definition 4.41. ([Grandmont, 92], p. 29) *A price system* $p^*\in\mathbb{R}^l_{++}$ *such that* $Z(p^*)=0$ *is called an* equilibrium price system *or equilibrium for short.*

Applying the chain rule yields that $Z(a,\cdot)$ is continuously partially differentiable: Let $a\in\mathbf{A}$ and $p\in\mathbb{R}^l_{++}$, then for all $h,k=1,\dots,l$, we have

$$\frac{\partial Z_h}{\partial p_k}(a,p)=\frac{\partial X_h}{\partial p_k}(a,p,p\cdot\tau_a)+\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau_a)\tau_{ak}.$$

Lemma 4.42. ([Grandmont, 92], p. 29) *Let* $a\in\mathbf{A}$ *and* $p\in\mathbb{R}^l_{++}$. *Then for all* $h,k=1,\dots,l$, *we have*

$$p_hp_k\left|\frac{\partial Z_h}{\partial p_k}(a,p)\right|\leq(p\cdot\tau_a)(1+\mathfrak{v}_k)+p_k\tau_{ak}(1+\sum_{j=1}^{l}\mathfrak{v}_j).$$

[12] We have for all $a\in\mathbf{A}$, $p\in\mathbb{R}^l_{++}$ and $\lambda\in\mathbb{R}^l_{++}$

$$Z(a,\lambda p)=X(a,\lambda p,\lambda p\cdot\tau_a)-\tau_a=X(a,p,p\cdot\tau_a)-\tau_a=Z(a,p).$$

[13] For all $p\in\mathbb{R}^l_{++}$ and $a\in\mathbf{A}$

$$p\cdot Z(a,p)=p\cdot X(a,p,p\cdot\tau_a)-p\cdot\tau_a=0.$$

Proof. Using the chain rule, we get

$$p_h p_k \left|\frac{\partial Z_h}{\partial p_k}(a,p)\right| \leq \underbrace{p_h p_k \left|\frac{\partial X_h}{\partial p_k}(a,p,p\cdot\tau_a)\right|}_{=:\mathrm{I}} + \underbrace{p_h p_k \left|\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau_a)\tau_{ak}\right|}_{=:\mathrm{II}}.$$

By equation (4.31), we obtain the following series of inequalities:

$$\begin{aligned}
\frac{p\cdot\tau_a}{p_h}\sum_{k=1}^{l}\mathfrak{v}_k &\overset{(A4)}{\geq} (p\cdot\tau_a)\sum_{k=1}^{l}\frac{\mathfrak{v}_k(a)}{p_h} \\
&\overset{4.31}{\geq} \left|(p\cdot\tau_a)\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau_a) - X_h(a,p,p\cdot\tau_a)\right| \\
\Leftrightarrow \sum_{k=1}^{l}\mathfrak{v}_k &\geq \left|p_h\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau) - \frac{p_h}{p\cdot\tau}X_h(a,p,p\cdot\tau)\right| \\
&\geq \left|p_h\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau)\right| \quad \underbrace{-\quad\frac{p_h}{p\cdot\tau}X_h(a,p,p\cdot\tau)}_{\geq -\frac{p\cdot\tau}{p\cdot\tau}=-1} \\
&\geq \left|p_h\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau)\right| - 1 \\
\Leftrightarrow 1+\sum_{j=1}^{l}\mathfrak{v}_j &\geq \left|p_h\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau)\right| \\
\Leftrightarrow \mathrm{II} &= p_k\tau_{ak}\left|p_h\frac{\partial X_h}{\partial w}(a,p,p\cdot\tau)\right| \leq p_k\tau_{ak}(1+\sum_{j=1}^{l}\mathfrak{v}_j).
\end{aligned}$$

Now, let $h \neq k$, then by equation (4.16) we get

$$\begin{aligned}
\left|p_k\frac{\partial X_h}{\partial p_k}(a,p,p\cdot\tau_a)\right| &\leq (p\cdot\tau_a)\frac{\mathfrak{v}_k}{p_h} \\
\Leftrightarrow \mathrm{I} &\leq (p\cdot\tau_a)\mathfrak{v}_k \leq (p\cdot\tau_a)(1+\mathfrak{v}_k).
\end{aligned}$$

In case that $h = k$, we again apply equation (4.16) and obtain

$$\begin{aligned}
&\left|p_h\frac{\partial X_h}{\partial p_h}(a,p,p\cdot\tau_a) + X_h(a,p,p\cdot\tau_a)\right| \leq (p\cdot\tau_a)\frac{\mathfrak{v}_h}{p_h} \\
\Leftrightarrow\quad &(p\cdot\tau_a)\mathfrak{v}_h \geq \left|p_h^2\frac{\partial X_h}{\partial p_h}(a,p,p\cdot\tau_a) + p_h X_h(a,p,p\cdot\tau_a)\right| \\
&\overset{p_hX_h\geq 0}{\geq} \left|p_h^2\frac{\partial X_h}{\partial p_h}(a,p,p\cdot\tau_a)\right| - \underbrace{p_h X_h(a,p,p\cdot\tau_a)}_{\leq p\cdot\tau_a} \\
&\geq \left|p_h^2\frac{\partial X_h}{\partial p_h}(a,p,p\cdot\tau_a)\right| - p\cdot\tau_a \\
\Leftrightarrow\quad &\mathrm{I} \leq (p\cdot\tau_a)(1+\mathfrak{v}_h).
\end{aligned}$$

This completes the proof. □

The following proposition allows to "differentiate under the integral":

Proposition 4.43. ([Grandmont, 92], Proposition 3.1) *Let $p \in \mathbb{R}^l_{++}$. Assume that (A1) to (A4) and (A5') hold. Then total market excess demand is continuously partially differentiable and for every $h, k = 1, \ldots, l$, it holds*

$$\frac{\partial Z_h}{\partial p_k}(p) = \int_{\mathbf{A}} \frac{\partial Z_h}{\partial p_k}(a,p)\mu(da).$$

Proof. By Lemma 4.42 we see that $\left|\frac{\partial Z_h}{\partial p_k}(a,p)\right| < \infty$ for all a, p, h, k with integrable majorant function. Moreover, we have already stated that the partial derivatives $\frac{\partial Z_h}{\partial p_k}(a,p)$ of conditional market excess demand $Z_h(a,p)$ are continuous in (a,p). Thus, the assertion follows from Proposition 4.16[14] using Definition 4.39. □

We now tackle the problem of existence and uniqueness of equilibrium. However, first we have to think about the term "uniqueness".

Remark 4.44. *We have seen that total market excess demand is homogeneous of degree zero in prices. Hence, when talking about uniqueness of an equilibrium price system p^* we always mean uniqueness up to a scalar multiple: If p^* is an equilibrium then λp^*, $\lambda \in \mathbb{R}_{++}$ is an equilibrium, too. In light of homogeneity, not the absolute price system p is the relevant entity but rather relative prices are important. This allows us to consider normalized prices and we will do so whenever it seems appropriate. Moreover, when considering specific sets $\mathbf{G} \subset \mathbb{R}^l_{++}$ of prices, as for example convex sets, we may without loss of generality consider cones generated by those sets, i.e.* $\{\tilde{p} \in \mathbb{R}^l_{++} : \exists p \in \mathbf{G} \quad \textit{and} \quad \lambda \in \mathbb{R}_{++} \quad \textit{such that} \quad \tilde{p} = \lambda p\}$.

In the foregoing section we have posed assumptions (*A6*) and (*A7*) to ensure aggregate desirability for every commodity. Again, we will pose assumption (*A6*) stating that the conditional density $f(\cdot|a)$ does almost surely not depend on type $a \in \mathbf{A}$. Following Grandmont, we slightly modify assumption (*A7*) in view of the fact that each type $a \in \mathbf{A}$ corresponds to a demand-endowment pair (ξ_a, τ_a). We reconsider the primitives of the model in the following way: Let the type space $\mathbf{A}$ be given by

$$\mathbf{A} \subset \mathbf{B} \times \mathbb{R}^l_+ \setminus \{0\}, \quad \mathbf{A} \ni a = (b, \tau),$$

where $\mathbf{B}$ is the space of types b of demand functions ξ_b and $\mathbb{R}^l_+ \setminus \{0\}$ the space of corresponding initial endowments τ.

We now specify the distribution μ on $\mathbf{A}$ in a way similar to the definition of the distribution on agents characteristics in Section 4.2.2:

[14] Differentiability from part (b), continuity from part (a).

Definition 4.45. *Let $(\Omega, \mathcal{F})$ be a standard Borel space. Let P be a probability distribution on $(\Omega, \mathcal{F})$ and $\mathcal{F}_0 \subset \mathcal{F}$ be a sub-σ-algebra. A probability kernel $K_{\mathcal{F}_0}$ from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F})$ such that for all $F \in \mathcal{F}$ it holds*

$$K_{\mathcal{F}_0}(\omega, F) = P(F|\mathcal{F}_0)(\omega) \quad \text{for } P\text{-almost all } \omega \in \Omega$$

is called a regular conditional probability *given $\mathcal{F}_0$. For existence and uniqueness we refer to [Bauer, 74], Proposition 56.5.*

Let ν be a probability distribution on the space of endowments $(\mathbb{R}^l_+, \mathcal{B}(\mathbb{R}^l_+))$. Given initial endowment τ, we depict a regular conditional probability $\nu(\cdot|\tau)$ on $(\mathbf{B}, \mathcal{B}(\mathbf{B}))$, where $\mathcal{B}(\mathbf{B})$ denotes the Borel σ-algebra induced by the $\mathscr{L}^1$-norm.[15] Now, a *Grandmont type distribution economy* is a tuple $((f(\cdot|(b,\tau)))_{b\in\mathbf{B},\tau\in\mathbb{R}^l_+}, (\nu(\cdot|\tau))_{\tau\in\mathbb{R}^l_+}, \nu)$.

Together with assumption (*A6*) we obtain aggregate desirability of every commodity by assuming a stronger version of (*A7*):

Assumption 4.46. ([Grandmont, 92], p. 30) *The following assumption states that the budget shares for every commodity are strictly positive:*

A 7' For each commodity $h = 1, \ldots, l$, there exists $\varpi_h > 0$, with $\sum_{k=1}^l \varpi_k \leq 1$, such that for ν-almost every initial endowment $\tau \in \mathbb{R}^l_+ \setminus \{0\}$, for all price systems $p \in \mathbb{R}^l_{++}$ and incomes $w = p \cdot \tau \in \mathbb{R}_{++}$ we have

$$p_h \int_{\mathbf{B}} \xi_{bh}(p, w)\nu(db|\tau) \geq \varpi_h w > 0.$$

We can thus show aggregate desirability for each commodity: For all $h = 1, \ldots, l$ and $p \in \mathbb{R}^l_{++}$, we have

$$\begin{aligned}
p_h X_h(p) &= \int_{\mathbf{A}} p_h X_h(a, p, p\cdot\tau_a)\mu(da) \\
&= \int_{\mathbb{R}^l_+} \int_{\mathbf{B}} p_h X_h(a, p, p\cdot\tau)\nu(db|\tau)\nu(d\tau) \\
&= \int_{\mathbb{R}^l_+} \left[\int_{\mathbf{B}} \left[\int_{\mathbb{R}^l} p_h \xi^\alpha_{bh}(p, p\cdot\tau) \underbrace{f(\alpha|(b,\tau))}_{\overset{(A6)}{=}:f(\alpha) \quad \mu\text{-a.s.}} d\alpha\right]\nu(db|\tau)\right]\nu(d\tau) \\
&= \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^l_+} \left[\int_{\mathbf{B}} p_h \xi^\alpha_{bh}(p, p\cdot\tau)\nu(db|\tau)\right]\nu(d\tau)\right] f(\alpha)d\alpha \\
&\geq \varpi_h p \cdot \int_{\mathbb{R}^l} \underbrace{\left[\int_{\mathbb{R}^l_+} \tau\nu(d\tau)\right]}_{=\bar{\tau}<\infty} f(\alpha)d\alpha \\
&= \varpi_h(p\cdot\bar{\tau}) \quad = \quad \varpi_h \bar{w} \quad > \quad 0.
\end{aligned}$$

[15] In terms of definition 4.45, conditioning on τ actually means conditioning on the σ-algebra that is generated by the constant random variable τ. Note that by positivity of individual demand and the satisfaction of Walras' law we may conclude that a demand function ξ as defined in 4.4 is in $\mathscr{L}^1$.

Since $p_h > 0$ for all $h = 1, \dots, l$, this yields

$$X_h(p) \geq \varpi_h \frac{\bar{w}}{p_h} > 0.$$

Using aggregate desirability, we can now show the following existence result for equilibria using a slightly different proof than in [Grandmont, 92]:

Proposition 4.47. ([Grandmont, 92], Proposition 3.2) *Let (A1) to (A3), (A5'), (A6) and (A7') hold.*
(a) *For every $p \in \mathbb{R}^l_{++}$ and $h = 1, \dots, l$, it holds*

$$p_h(Z_h(p) + \bar{\tau}_h) \geq \varpi_h (p \cdot \bar{\tau}).$$

(b) *There exists an equilibrium price system $p^* \in \mathbb{R}^l_{++}$. For all $h = 1, \dots, l$, p^* satisfies*

$$p^*_h \bar{\tau}_h \geq \varpi_h (p^* \cdot \bar{\tau}). \tag{4.34}$$

Proof. Subsequent to Assumption 4.46 we have shown

$$p_h(Z_h(p) + \bar{\tau}_h) = p_h X_h(p) \geq \varpi_h (p \cdot \bar{\tau}).$$

Thus part (a) is shown.

Let $(p^n)_{n \in \mathbb{N}} \in \mathbb{R}^l_{++}$ be a sequence of prices such that $p^n \overset{n \to \infty}{\longrightarrow} \bar{p} \neq 0$ and there exists $k \leq l$ such that $\bar{p}_k = 0$. For convenience we normalize prices in the following way: We only consider price systems $p \in \{p \in \mathbb{R}^l_{++} | p \cdot \bar{\tau} = 1\}$. Using part (a), we obtain for all $n \in \mathbb{N}$

$$\begin{aligned} Z_k(p^n) \quad &\geq \quad \frac{\varpi_k}{p^n_k}(p^n \cdot \bar{\tau}) - \bar{\tau} = \frac{\varpi_k}{p^n_k} - \bar{\tau}_k \\ &\longrightarrow \quad \infty \quad \text{whenever} \quad n \to \infty. \end{aligned}$$

Proposition 4.48 below, shows the existence of an equilibrium price system p^*. Furthermore, we have

$$p^*_h \bar{\tau}_h = p^*_h (Z_h(p^*) + \bar{\tau}_h) \geq \varpi_h (p^* \cdot \bar{\tau}).$$

□

A version of [Mas-Colell et al., 95], Proposition 17.C1.1,[16] for our context may be given as follows:

Proposition 4.48. *Let $Z : \mathbb{R}^l_{++} \to \mathbb{R}^l$ be a total market excess demand function as defined in 4.39.*[17] *Let $(p^n)_{n \in \mathbb{N}} \in \mathbb{R}^l_{++}$ be a sequence of prices such that $p^n \overset{n \to \infty}{\longrightarrow} p \neq 0$ and there exists $k \leq l$ such that $p_k = 0$. If*

$$\max_{k \leq l} Z_k(p^n) \overset{n \to \infty}{\longrightarrow} \infty,$$

then there exists $p^* \in \mathbb{R}^l_{++}$ such that $Z(p^*) = 0$.

[16] A more general version of this existence theorem in case of demand correspondences is given in [Debreu, 82], Theorem 8.

[17] In particular, we have shown that $Z(\cdot)$ is continuous, homogeneous of degree zero, bounded below and satisfies Walras' law.

Proof. [Mas-Colell et al., 95], pp. 586,587 □

Unless otherwise stated, we expect all assumptions in Proposition 4.47 to hold that imply existence of equilibrium. Now that we have established the existence of equilibrium, we may turn our attention to the uniqueness problem. Here, uniqueness will be implied by the *gross substitute property* of market excess demand.

Definition 4.49. (a) ([Moore, 04] Definition 10.11) *Let Z be a total market excess demand function and $p \in \mathbb{R}^l_{++}$. We say that two commodities k and h, $h \neq k \leq l$, are* gross substitutes at price system p *if*

$$\frac{\partial Z_h}{\partial p_k}(p) > 0, \qquad \frac{\partial Z_k}{\partial p_h}(p) > 0 \tag{4.35}$$

and these partial derivatives exist.
(b) *We say that the total market excess demand function Z satisfies the gross substitutes property if all commodities h are gross substitutes at all price systems $p \in \mathbb{R}^l_{++}$. In addition, we say that Z exhibits the gross substitutes property on some subset $\mathbf{G} \subset \mathbb{R}^l_{++}$ if all commodities h are gross substitutes at all price systems $p \in \mathbf{G}$.*

Intuitively, two commodities are gross substitutes whenever an increase in price of one good causes an increase in demand of the other good.

We now show that the gross substitutes property implies uniqueness of equilibrium. The following proposition is a version of [Arrow & Hahn, 71], Theorem 9.7.7: We do not state the result for prices in $\mathbb{R}^l_{++}$ but moreover for prices in any convex subset of $\mathbb{R}^l_{++}$. The proof, that we provide here, follows an idea in [Moore, 04], Proposition 10.13, but in a more general context. It crucially relies on the satisfaction of the gross substitutes property on a convex set of prices.

Proposition 4.50. *Let Z be a total market excess demand function that satisfies the gross substitutes property on some convex set $\mathbf{G} \subset \mathbb{R}^l_{++}$ of prices. Without loss of generality assume $\mathbf{G}$ to be a convex cone. Let $p^* \in \mathbf{G}$ be an equilibrium price vector if any such exists in $\mathbf{G}$. Then p^* is the unique equilibrium price vector in $\mathbf{G}$ up to scalar multiples.*

Proof. Let $\tilde{p} \in \mathbf{G}$ be a further equilibrium price system for Z and

$$m := \min_{1 \leq h \leq l} \frac{p^*_h}{\tilde{p}_h}.$$

Now assume that $p^* \neq m\tilde{p}$, $m\tilde{p} \in \mathbf{G}$. We show that p^* cannot be an equilibrium price system and thus obtain a contradiction. Because of convexity of $\mathbf{G}$ we have that the whole "line" $[p^*, m\tilde{p}]$ is contained in $\mathbf{G}$: Define the map $p(\cdot)$ of convex combinations, i.e.

$$p(\eta) := \eta p^* + (1 - \eta) m\tilde{p} \quad \text{for all } \eta \in [0, 1].$$

Then $p(\eta) \in \mathbf{G}$, $p(1) = p^*$ and $p(0) = m\tilde{p}$.

By definition of m, we have $p^* \geq m\tilde{p}$. Moreover, there exists some $k \leq l$ such that $p^*_k > m\tilde{p}_k$ and some $r \leq l$ such that $p^*_k = m\tilde{p}_k$. Due to the gross substitutes property of total market excess demand Z on $[p^*, m\tilde{p}] \subset \mathbf{G}$ it follows

$$0 = Z_k(m\tilde{p}) < Z_k(p^*) :$$

Consider the h^{th} component of $p(\cdot)$. Sine $p(\cdot)$ is the linear interpolation of p^* and $m\tilde{p}$, we know that $p_h(\eta)$ may not decrease in η, i.e. for all $h = 1, \ldots, l$, we have

$$\frac{dp_h}{d\eta}(\eta) \geq 0 \quad \text{for all } \eta \in [0, 1].$$

But by definition of k, it holds

$$\frac{dp_k}{d\eta}(\eta) > 0 \quad \text{for all } \eta \in [0, 1],$$

i.e. $p_k(\cdot)$ is strictly increasing along $\eta \in [0, 1]$. The chain rule yields

$$\begin{aligned} \frac{dZ_k}{d\eta}(p(\eta)) &= \sum_{h=1}^{l} \frac{\partial Z_k}{\partial p_h}(p(\eta)) \frac{dp_h}{d\eta}(\eta) \\ &\geq \frac{\partial Z_k}{\partial p_k}(p(\eta))(p^*_k - m\tilde{p}_k) > 0 \end{aligned}$$

for all $\eta \in [0, 1]$. Thus it follows

$$Z_k(p^*) = Z_k(p(1)) > Z_k(p(0)) = Z_k(m\tilde{p}) = Z_k(\tilde{p}) = 0$$

and hence, p^* cannot be an equilibrium price system. □

Corollary 4.51. *Posing the standard assumption in Proposition 4.47, let Z be a total market excess demand function that satisfies the gross substitutes property. Then there exists a unique equilibrium price system $p^* \in \mathbb{R}^l_{++}$ up to scalar multiples.*

Proof. The existence follows from Proposition 4.47. Uniqueness is implied by Proposition 4.50 since $\mathbb{R}^l_{++}$ is a convex cone. □

Remark 4.52. *Total market excess demand function Z satisfies the* weak axiom of revealed preference *if, for any $p^1, p^2 \in \mathbb{R}^l_{++}$, it holds*

$$[Z(p^1) \neq Z(p^2) \quad \text{and} \quad p^1 \cdot Z(p^2) \leq 0] \Rightarrow p^2 Z(p^1) > 0.$$

The following proposition is a version of Theorem 9.7.9 in [Arrow & Hahn, 71]. The proof basically follows the lines therein.

Proposition 4.53. *Let Z be a total market excess demand function that satisfies the gross substitutes property and the assumptions in Proposition 4.47. Let $p^* \in \mathbb{R}^l_{++}$ be the unique equilibrium price system. Then for any $p \in \mathbb{R}^l_{++}$ that is not collinear to p^*, i.e. there is no $\lambda \in \mathbb{R}_{++}$ such that $p^* = \lambda p$, we have*

$$p^* \cdot Z(p) > 0.$$

By Remark 4.52 this inequality states that the weak axiom of revealed preference holds for total market excess demand between the equilibrium price system $p^* \in \mathbb{R}^l_{++}$ and any other price system $p \in \mathbb{R}^l_{++}$.

Proof. Existence and uniqueness of equilibrium is given by Corollary 4.51.

Again, we apply a normalizing device for price systems in this proof. We will only consider price systems $p \in \Delta := \{p \in \mathbb{R}^l_{++} | \sum_{k=1}^{l} p_k = 1\}$. Thus, for every price system p, we have eliminated all collinear price systems and now "uniqueness really means uniqueness" on Δ. Note that Δ is a bounded open subset of $\mathbb{R}^{l-1}$.

Let us have a look at the map $\kappa : \Delta \to \mathbb{R}$, $p \mapsto \kappa(p) := p^* \cdot Z(p) = \sum_{j=1}^{l} p_j^* Z_j(p)$. We know that $\kappa(p^*) = 0$. Thus, the assertion follows if we can show that p^* is the unique minimizer of $\kappa(\cdot)$.

Since total market excess demand $Z(\cdot)$ is continuous and bounded below, as the feasible price systems p are, we know that there exists a lower bound for κ and κ is continuous. In a foregoing proof we have already seen that $\max_{k\leq l} Z_k(p^n) \overset{n\to\infty}{\longrightarrow} \infty$, whenever the sequence $(p^n)_{n\in\mathbb{N}} \subset \Delta$ is such that $p_k^n \overset{n\to\infty}{\longrightarrow} 0$ for some $k \leq l$. Since $p^* \in \mathbb{R}^l_{++}$ is fixed we get that $\kappa(p^n) \overset{n\to\infty}{\longrightarrow} \infty$ for sequences $(p^n)_{n\in\mathbb{N}} \subset \Delta$ that approach the boundary of Δ. In this sense, κ "explodes at the boundary". Hence, we see that the minimization problem for $\kappa(\cdot)$ on Δ exhibits an interior solution and a necessary condition for a price system $\tilde{p}$ to be a minimizer is given by

$$\nabla(\kappa)(\tilde{p}) := \left(\frac{\partial\kappa}{\partial p_1}, \ldots, \frac{\partial\kappa}{\partial p_l}\right) = \left(\sum_{j=1}^{l} p_j^* \frac{\partial Z_j}{\partial p_k}(\tilde{p})\right)_{k\leq l} = 0. \tag{4.36}$$

Applying Walras' law, we see that for all $p \in \Delta$ and all $k = 1, \ldots, l$, we have

$$\frac{\partial[p \cdot Z(p)]}{\partial p_k} = 0.$$

For p^* and $k \leq l$ this yields

$$0 = \frac{\partial[p^* \cdot Z(p^*)]}{\partial p_k} = \sum_{j=1}^{l} p_j^* \frac{\partial Z_j}{\partial p_k}(p^*) + \underbrace{Z_k(p^*)}_{=0}.$$

Thus, equation (4.36) holds at the unique equilibrium p^*.

We finish the proof by showing that equation (4.36) does not hold for any price system $p \in \Delta$ other than p^*:[18] Let $\Delta \ni p \neq p^*$ and assume that p satisfies equation (4.36). Set

$$m := \min_{1 \leq h \leq l} \frac{p_h^*}{p_h}$$

and consider $r \leq l$ such that $\frac{p_r^*}{p_r} = m$. We know that $Z_r(p^*) = 0$; this implies

$$Z_r(p) < 0 :$$

Let $p^1 := \frac{p^*}{p_r^*}$ and $p^2 := \frac{p}{p_r}$. By homogeneity of degree zero, we obtain $Z(p^1) = Z(p^*)$ and $Z(p^2) = Z(p)$. This yields

$$0 = Z_r(p^*) = Z_r(p^1) > Z_r(p^2) = Z_r(p),$$

where the inequality follows from the gross substitute property: We have $p^1 = \left(\frac{p_1^*}{p_r^*}, \ldots, 1, \ldots, \frac{p_l^*}{p_r^*}\right)$ and $p^2 = \left(\frac{p_1}{p_r}, \ldots, 1, \ldots, \frac{p_l}{p_r}\right)$. By definition of r, it holds $\frac{p_k^*}{p_k} \geq \frac{p_r^*}{p_r}$ for all $k = 1, \ldots, l$ and since $p^* \neq p$ there exists k_0 such that $\frac{p_{k_0}^*}{p_{k_0}} > \frac{p_r^*}{p_r}$. But these properties are equivalent to:

$$\frac{p_k^*}{p_r^*} \geq \frac{p_k}{p_r} \quad \forall k = 1, \ldots, l \quad \text{and} \quad \exists k_0 \quad \text{such that} \quad \frac{p_{k_0}^*}{p_r^*} > \frac{p_{k_0}}{p_r}.$$

As in the proof of 4.50, this yields $Z_r(p^1) > Z_r(p^2)$ and thus $Z_r(p) < 0$.

Now Walras' law implies

$$0 = \frac{\partial [p \cdot Z(p)]}{\partial p_r} = \sum_{j=1}^{l} p_j \frac{\partial Z_j}{\partial p_r}(p) + \underbrace{Z_r(p)}_{<0}.$$

Thus,

$$\sum_{j=1}^{l} p_j \frac{\partial Z_j}{\partial p_r}(p) > 0.$$

But then we obtain

$$\begin{aligned} 0 &= \frac{1}{m} \sum_{j=1}^{l} p_j^* \frac{\partial Z_j}{\partial p_r}(p) = p_r \frac{\partial Z_r}{\partial p_r}(p) + \frac{1}{m} \sum_{j \neq r} p_j^* \frac{\partial Z_j}{\partial p_r}(p) \\ &> \sum_{j=1}^{l} p_j \frac{\partial Z_j}{\partial p_r}(p) > 0, \end{aligned}$$

[18] In terms of the assertion this means that (4.36) may not hold for any price system p that is not collinear to p^*.

which yields a contradiction.[19] Hence, there is no $p \neq p^*$ in Δ such that equation (4.36) is satisfied. We have a continuous function κ on a convex open bounded subset Δ of $\mathbb{R}^{l-1}$ that is bounded below and tends to infinity when approaching the boundary of Δ. Since there is only one element p^* in Δ satisfying the necessary condition (4.36) for an extreme point, p^* has to be the unique minimizer of $\kappa(\cdot)$. $\square$

Corollary 4.54. *Let Z be a total market excess demand function that satisfies the gross substitutes property on some convex open set $\mathbf{G} \in \mathbb{R}^l_{++}$, without loss of generality a convex cone. Let $\mathbf{G}$ contain an equilibrium price system p^* that is then unique in $\mathbf{G}$ by Proposition 4.50. Then for any $p \in \mathbf{G}$ that is not collinear to p^* we have*

$$p^* \cdot Z(p) > 0,$$

if we furthermore suppose Z to be such that $\kappa : p \mapsto p^* \cdot Z(p)$ is twice continuously differentiable on $\mathbf{G}$ and the Hessian matrix $\left(\frac{\partial \kappa^2}{\partial^2 p_h p_k}(p)\right)_{h,k=1,\ldots,l}$ is positive definite at p^*.

Proof. The proof basically uses the same arguments as the foregoing proof: We can show that there is no price system $p \neq p^*$ in $\mathbf{G}$ such that the necessary condition (4.36) for an extremum is satisfied by any $p \in \mathbf{G}$ not collinear to p^*: In the proof of Proposition 4.53 we have used that the set $\mathbb{R}^l_{++}$ where gross substitutes property holds is a convex cone.

The problem that we face is that κ does not necessarily have to increase when approaching the boundary of $\mathbf{G}$. However, by positive definiteness of the Hessian matrix at p^*, p^* is the unique minimizer of κ on $\mathbf{G}$. $\square$

By equation (4.34) we can see that equilibrium price systems are bounded away from zero. Using (*A7'*), we show that the set where the gross substitutes property holds will eventually contain any compact subset of $\mathbb{R}^l_{++}$ as individual behavior gets more and more heterogeneous. The set of equilibria is thus eventually be contained in such a set and there will be a unique equilibrium by Proposition 4.50.

The theorem that follows originates in [Grandmont, 92], Theorem 3.3. The difference is that we do not pose an assumption to get the weak axiom of revealed preference between an equilibrium price system and any other

[19] The second to last inequality can be obtained straightforward: By definition of r and since $p \neq p^*$, we get

$$\begin{aligned} & \frac{p_k^*}{p_k} \geq \frac{p_r^*}{p_r} \quad \forall k = 1, \ldots, l \quad \text{and} \quad \exists k_0 \quad \text{such that} \quad \frac{p_{k_0}^*}{p_{k_0}} > \frac{p_r^*}{p_r} \\ \Leftrightarrow \quad & \frac{p_k^* p_r}{p_r^*} \geq p_k \quad \forall k = 1, \ldots, l \quad \text{and} \quad \exists k_0 \quad \text{such that} \quad \frac{p_{k_0}^* p_r}{p_r^*} > p_{k_0}. \end{aligned}$$

Applying these inequalities yields the result.

price system in $\mathbb{R}^l_{++}$ but between an equilibrium price system and any other price system in a set that eventually contains any compact set of prices when agents become sufficiently heterogeneous. The proof basically follows the lines in [Grandmont, 92], pp. 34,35.

Theorem 4.55. *Let assumptions (A1) to (A4), (A5'), (A6) and (A7') hold. For $\varpi = (\varpi_1, \ldots, \varpi_l), \mathfrak{v} = (\mathfrak{v}_1, \ldots, \mathfrak{v}_l) \in \mathbb{R}^l_{++}$ as chosen in (A4) and (A7), we set for all h*

$$\mathscr{G}(\mathfrak{v}, \varpi) := \left\{ p \in \mathbb{R}^l_{++} | p_k \bar{\tau}_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k (p \cdot \bar{\tau}) \quad \text{for all} \quad k \neq h \right\}. \tag{4.37}$$

(a) *Total market excess demand Z exhibits the gross substitutes property on $\mathscr{G}(\mathfrak{v}, \varpi)$. In particular, whenever κ, as defined in the proof of Proposition 4.53, is twice continuously differentiable and has a positive definite Hessian matrix at the unique equilibrium price system p^* in $\mathscr{G}(\mathfrak{v}, \varpi)$, given existence, the weak axiom of revealed preference holds for total market excess demand between the unique equilibrium price system $p^* \in \mathscr{G}(\mathfrak{v}, \varpi)$ and any other price system $p \in \mathscr{G}(\mathfrak{v}, \varpi)$ that is not collinear to p^*. That is, we have $p^* \cdot Z(p) > 0$.*
(b) *For all $h = 1, \ldots, k$, assume that*

$$\varpi_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k \tag{4.38}$$

holds for all $k \neq h$. Then $\{p \in \mathbb{R}^l_{++} | p_k \bar{\tau}_k \geq \varpi_k (p \cdot \bar{\tau}) \quad \forall k\} \subset \mathscr{G}(\mathfrak{v}, \varpi)$. By equation (4.34) this implies that the set of equilibrium prices is contained in $\mathscr{G}(\mathfrak{v}, \varpi)$ and thus the uniqueness of equilibrium up to scalar multiples follows by Proposition 4.50.

Proof. First, we note that $\mathscr{G}(\mathfrak{v}, \varpi)$ is a convex cone: Let $\lambda \in \mathbb{R}_{++}$ and $p \in \mathscr{G}(\mathfrak{v}, \varpi)$. Then for all $k = 1, \ldots, l$

$$\begin{aligned} & p_k \bar{\tau}_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k (p \cdot \bar{\tau}) \\ \Leftrightarrow \quad & \lambda p_k \bar{\tau}_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k (\lambda p \cdot \bar{\tau}). \end{aligned}$$

Thus, $\lambda p \in \mathscr{G}(\mathfrak{v}, \varpi)$. Furthermore let $\eta \in [0,1]$ and $p, q \in \mathscr{G}(\mathfrak{v}, \varpi)$. Then for all $k = 1, \ldots, l$

$$(\eta p_k - (1-\eta) q_k) \bar{\tau}_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k ((\eta p + (1-\eta) q) \cdot \bar{\tau}).$$

This shows convexity.

We now turn to the proof of part (a): Equation (4.16) implies for all $k \neq h$

$$
\begin{aligned}
\left| p_h p_k \int_{\mathbf{A}} \frac{\partial X_h}{\partial p_k}(a, p, p \cdot \tau_a)\mu(da) \right| &\leq p_h \int_{\mathbf{A}} \underbrace{\left| p_k \frac{\partial X_h}{\partial p_k}(a, p, p \cdot \tau_a) \right|}_{\stackrel{(4.16)}{\leq} \frac{\mathfrak{v}_k}{p_h}(p \cdot \tau_a)} \mu(da) \\
&\leq \mathfrak{v}_k(p \cdot \bar{\tau})
\end{aligned}
$$

$$
\Leftrightarrow -\mathfrak{v}_k(p \cdot \bar{\tau}) \leq p_h p_k \int_{\mathbf{A}} \frac{\partial X_h}{\partial p_k}(a, p, p \cdot \tau_a)\mu(da) \leq \mathfrak{v}_k(p \cdot \bar{\tau}). \tag{4.39}
$$

By equation (4.31) we obtain

$$
\begin{aligned}
&\int_{\mathbf{A}} p_k \tau_{ak} p_h \left[\frac{\partial X_h}{\partial w}(a, p, p \cdot \tau_a) - \frac{X_h(a, p, p \cdot \tau_a)}{p \cdot \tau_a} \right] \mu(da) \\
\stackrel{4.31}{\geq} &\int_{\mathbf{A}} p_k \tau_{ak} p_h \left[- \frac{(p \cdot \tau_a) \sum_{j=1}^{l} \mathfrak{v}_j}{p_h (p \cdot \tau_a)} \right] \mu(da) \\
= &-p_k \sum_{j=1}^{l} \mathfrak{v}_j \int_{\mathbf{A}} \tau_{ak} \mu(da) = -p_k \bar{\tau}_k \sum_{j=1}^{l} \mathfrak{v}_j.
\end{aligned}
$$

Reordering of this inequality and further estimation yields

$$
\begin{aligned}
& \int_{\mathbf{A}} p_k \tau_{ak} \left[p_h \frac{\partial X_h}{\partial w}(a, p, p \cdot \tau_a) \right] \mu(da) \\
\geq \quad & \int_{\mathbf{A}} p_k \tau_{ak} \left[p_h \frac{X_h(a, p, p \cdot \tau_a)}{p \cdot \tau_a} \right] \mu(da) - p_k \bar{\tau}_k \sum_{j=1}^{l} \mathfrak{v}_j \\
\stackrel{(A6)}{=} \quad & \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^l_+} p_k w_k \left[\int_{\mathbf{B}} p_h \frac{\xi^{\alpha}_{bh}(p, p \cdot \tau)}{p \cdot \tau} \nu(db|\tau) \right] \nu(d\tau) \right] f(\alpha) d\alpha \\
& -p_k \bar{\tau}_k \sum_{j=1}^{l} \mathfrak{v}_j \\
\stackrel{(A7')}{\geq} \quad & \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^l_+} \varpi_h p_k \tau_k \quad \nu(d\tau) \right] f(\alpha) d\alpha - p_k \bar{\tau}_k \sum_{j=1}^{l} \mathfrak{v}_j \\
= \quad & \varpi_h p_k \bar{\tau}_k - p_k \bar{\tau}_k \sum_{j=1}^{l} \mathfrak{v}_j \quad = \quad p_k \bar{\tau}_k (\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j).
\end{aligned} \tag{4.40}
$$

Proposition 4.43 and inequalities (4.39) and (4.40) then imply $\forall k \neq h$

$$\begin{aligned} p_h p_k \frac{\partial Z_h}{\partial p_k}(p) &\overset{4.43}{=} p_h p_k \int_{\mathbf{A}} \frac{\partial Z_h}{\partial p_k}(a,p)\mu(da) \\ &= p_h p_k \int_{\mathbf{A}} \left[\frac{\partial X_h}{\partial p_k}(a,p,p\cdot\tau_a) + \frac{\partial X_h}{\partial w}(a,p,p\cdot\tau_a)\tau_{ak} \right] \mu(da) \\ &\geq p_k \bar{\tau}_k (\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j) - \mathfrak{v}_k (p\cdot\bar{\tau}) > 0 \quad \text{if } p \in \mathscr{G}(\mathfrak{v},\varpi). \end{aligned}$$

Since $p_h p_k > 0$ for all $h, k = 1, \ldots, l$, we obtain for all h:

$$\frac{\partial Z_h}{\partial p_k}(p) > 0$$

for all $k \neq h$ and $p \in \mathscr{G}(\mathfrak{v},\varpi)$. Thus, Z satisfies the gross substitute property on $\mathscr{G}(\mathfrak{v},\varpi)$.

Applying Corollary 4.54 to the convex set $\mathscr{G}(\mathfrak{v},\varpi)$ yields the weak axiom of revealed preference for total market excess demand Z in the special form stated in part (a), whenever there exists an equilibrium $p^* \in \mathscr{G}(\mathfrak{v},\varpi)$. Uniqueness of such a p^*, if it exists, follows from Proposition 4.50.

Let us now turn to the proof of part (b): Assume that inequality (4.38) holds and consider price systems $p \in \mathbb{R}^l_{++}$ such that

$$p_k \bar{\tau}_k \geq \varpi_k (p\cdot\bar{\tau}) \quad \text{for all } k.$$

Merging these inequalities, we obtain for all $h \leq l$

$$\begin{aligned} & p_k \bar{\tau}_k \varpi_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \varpi_k \mathfrak{v}_k (p\cdot\bar{\tau}) \\ \Leftrightarrow \quad & p_k \bar{\tau}_k \left(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j \right) > \mathfrak{v}_k p\cdot\bar{\tau} \end{aligned}$$

for all $k \neq h$. Thus, we have

$$\{p \in \mathbb{R}^l_{++} | p_k \bar{\tau}_k \geq \varpi_k (p\cdot\bar{\tau}) \quad \forall k\} \subset \mathscr{G}(\mathfrak{v},\varpi).$$

Hence, the set of equilibrium price systems is contained in the convex cone $\mathscr{G}(\mathfrak{v},\varpi)$. Proposition 4.47 yields existence and Proposition 4.50 uniqueness of equilibrium. □

So far, we have shown that the weak axiom is satisfied in the form of Theorem 4.55 on a set of prices that eventually contains all equilibrium prices. In [Grandmont, 92] we can find a stronger result in the sense that no assumptions are posed on the functional form of market excess demand Z. However, stronger assumptions are posed on the distributions.

Proposition 4.56. ([Grandmont, 92], Theorem 3.3(3)) *Let assumptions (A1) to (A4), (A5'), (A6) and (A7') hold and* $\varpi = (\varpi_1, \ldots, \varpi_l)$, $\mathfrak{v} = (\mathfrak{v}_1, \ldots, \mathfrak{v}_l) \in \mathbb{R}^l_{++}$ *be chosen as in (A4) and (A7). We now pose the following condition: For all* $h \leq l$,

$$\varpi_k^2(\varpi_h - \sum_{j=1}^{l} \mathfrak{v}_j) > \mathfrak{v}_k \tag{4.41}$$

for all $k \neq h$. *Then the weak axiom of revealed preference holds for total market excess demand between the unique equilibrium price system* p^* *and any other price system* $p \in \mathbb{R}^l_{++}$ *not collinear to* p^*, *i.e.* $p^* \cdot Z(p) > 0$ *if there does not exist any* $\lambda \in \mathbb{R}$ *such that* $p^* = \lambda p$.

In our analysis of market demand, when the distribution of demand is endogenized, we do not directly apply this theorem. Thus, we will only sketch the proof here.

Proof. Existence and uniqueness of the equilibrium price system p^* follows by the fact that (4.38) is implied by (4.41) since by definition $0 < \varpi_k < 1$ for all $k \leq l$. As in the proof of Proposition 4.53, the idea is to show that the mapping $\kappa : \mathbb{R}^l_{++} \to \mathbb{R}$ defined by

$$p \mapsto \kappa(p) := p^* \cdot Z(p)$$

is uniquely minimized at p^* up to scalar multiples, where $\kappa(p^*) = 0$. The main difference of this proof and the proof of Proposition 4.53 is that total market excess demand Z satisfies the gross substitute property on $\mathscr{G}(\mathfrak{v}, \varpi)$ but not necessarily on the whole $\mathbb{R}^l_{++}$. Thus, we have to generalize Corollary 4.54 in that $p^* \cdot Z(p) > 0$ not only for $p \in \mathscr{G}(\mathfrak{v}, \varpi)$ but for all $p \in \mathbb{R}^l_{++}$.

For a formal proof we refer to [Grandmont, 92], pp. 35,36. □

This completes our analysis of market demand for Grandmont type distribution economies.

Chapter 5

Economies with Gaussian Interaction

5.1 Introduction

In section 4.2 it is shown that market demand satisfies strong structural properties when individual demand is sufficiently dispersed. However, the distribution of demand is a primitive of that model. In this chapter we introduce a model where the distribution of demand is not given exogenously by the modeler but is generated endogenously by virtue of a local interaction structure. The basic idea is to generalize Föllmer's Ising economy with respect to the spin space: Whereas Föllmer has assumed $\mathbf{E} = \{-1, +1\}$, representing exclusive preferences in a two-commodity exchange economy, we come up with a model where the spin space $\mathbf{E}$ represents the space of all demand-income pairs. The main point of this chapter is to generalize the local interaction structure in the Ising model to some potential that takes account of an unbounded spin space.

We now have to be very specific in distinguishing the different economic concepts introduced in Chapter 3: Grandmont's model is given by a distribution economy, i.e. a probability distribution on $\mathbf{E}$. In this chapter we consider a random economy, i.e. a family $(\sigma_s)_{s\in\mathbf{S}}$ of random agents or random variables on some underlying probability space $(\Omega, \mathcal{F}, P)$, each of which assumes values in $\mathbf{E}$. In this sense the random economy is given by a probability measure μ on $\mathbf{E}^{\mathbf{S}}$. However, as seen in Section 3.5, in context of locally interacting agents, this turns out to be a macroeconomic concept.

When using a Gibbsian approach to model local interactions, the primitives of the model turn out to be

- a graph $\mathbf{S}$, where each site of $\mathbf{S}$ represents an agent and edges are used to define the microscopic interaction structure;
- a spin space $\mathbf{E}_s$ of individual states for every agent $s \in \mathbf{S}$ and a ten-

dency for every agent to exhibit some spin given the agent does not interact;[1]

- a microscopic interaction structure given by a specification, in turn generated by a potential as in Definition 2.12.

As in the Ising economy we assume $\mathbf{S}$ to be the d-dimensional integer lattice $\mathbb{Z}^d$, $d \geq 1$. The choice of the spin space $\mathbf{E} = \{-1, +1\}$ is a major objection to the Ising economy. A spin space that consists of just two elements only admits two different individual states. When assuming the consumption set to be equal to $\mathbb{R}^l_+$ and agents to be specified by demand-income pairs, this assumptions cannot be maintained as the set of preferences or demand functions is uncountably infinite: We have stated that any equivalence class of α-transforms for a given demand-income-pair can be identified with $\mathbb{R}^l$. In the new model that we refer to as *unbounded spin Ising economy*[2], we assume an underlying type, or more concretely a demand-income-pair, and then the spin space for every agent is given by the space of all α-transforms: $\mathbf{E} = \mathbb{R}^l$, where l denotes the number of distinct goods available. The underlying type may be thought of as some sort of consensus within society. However, we will argue that this assumption is not an improper restriction. In this model, we do not want an agent to exhibit some a priori tendency for a specific demand function, i.e. for some specific $\alpha \in \mathbb{R}^l$. Thus, we choose the Lebesgue measure as reference or a priori measure.

The last crucial primitive of the model is the local interaction structure. In Föllmer's Ising economy this structure was generated by the Ising potential taking account of preference for conformity. The objects of study were the pure states. By the same token, the interaction structure in the new model will be given by an interaction potential, too. Since we have changed the spin space drastically from a model exhibiting two spins to a model with unbounded and uncountably infinite spin space, we have to consider a more general interaction structure. In a first step, we consider so called *Gaussian potentials* generating *Gaussian specifications* with state space $\mathbb{R}$. More precisely, we consider the potential that is assumed for the *harmonic oscillator*. Then we take the product specification as introduced in Proposition 2.7 with spin space $\mathbf{E} = \mathbb{R}^l$. In this sense, the unbounded spin Ising economy that we consider is a "product" of harmonic oscillators.

We have chosen this special type of local interaction structure, that we refer to as Gaussian interaction, for several reasons: First of all, it directly generalizes the Ising potential used by Föllmer to state space $\mathbf{E} = \mathbb{R}^l$. A second motivation is mathematical convenience; there are many well known and elegant results for Gibbs states when interaction is specified by homogeneous Gaussian potentials.

[1] We assume a common spin space $\mathbf{E}$ and a common reference or a priori measure λ among agents.

[2] The name originates from the unbounded spin Ising model in statistical mechanics.

Justifiably so, the skeptical reader may now object that such a model is hardly to accommodate with the "real world". However, this objection may be refuted by the fact that the specific Gaussian interaction potential under consideration is equivalent to an interaction potential that accommodates for "preference for conformity". Thus, assuming that an agent wants to be similar to her peers, our choice of interaction structure is warrantable.

The present chapter is divided into two main sections. The first section states fundamental results for Gaussian specifications as the mathematical cornerstone of the models that are introduced in the second section. These models generalize the Ising economy with respect to spin space. The results obtained in the first section are crucial for the analysis that is carried out in the next chapter.

5.2 Gaussian Specifications

This section serves as a comprehensive introduction to Gaussian specifications and thus, provides a basis for further modeling and analysis of market demand. This discussion originates in [Georgii, 88], Chapter 13.

We assume that the local interaction structure is given by a Gaussian potential, i.e. a pair potential $\Phi^{J,h}$ with a quadratic part given by some symmetric and positive definite *coupling function* $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and a linear part given by some *external field* $h : \mathbf{S} \to \mathbb{R}$. We then consider the set $\mathcal{G}(\gamma^{J,h})$ of Gibbs measures with respect to the specification $\gamma^{J,h}$, induced by the interaction potential $\Phi^{J,h}$ as in Definition 2.12. For our economic model, we confine ourselves to homogeneous coupling functions and external field 0. Two questions are of interest:

1. Does there exist a Gibbs measure for the Gaussian potential $\Phi^{J,h}$, i.e. $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$?

2. If so, do we obtain an explicit characterization of ergodic elements in $\mathcal{G}(\gamma^{J,h})$ when J and h are assumed to be homogeneous?

5.2.1 Basic Definitions

The first definition recalls basic concepts from probability theory; parts (b) to (d) are taken from [Röckner, 05]. Let $(\Omega, \mathcal{F}, P)$ be an underlying probability space.

Definition 5.1. (a) ([Bauer, 74], Definition 47.7) *Let μ be a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$. We define the* Fourier transform $\hat{\mu} : \mathbb{R}^n \to$

$\mathbb{C}$ of μ by

$$\begin{aligned}\hat{\mu}(u) &:= \int_{\mathbb{R}^n} e^{\iota u\cdot y}\mu(dy) \\ &= \int_{\mathbb{R}^n} \cos(u\cdot y)\mu(dy) + \iota \int_{\mathbb{R}^n} \sin(u\cdot y)\mu(dy)\end{aligned}$$

for all $u \in \mathbb{R}^n$, where ι denotes the imaginary unit.
(b) *Let $\sigma : \Omega \to \mathbb{R}^n$ be a random variable, i.e. $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable, and $\sigma(P) := \mu_\sigma := P \circ \sigma^{-1}$ denote the distribution of σ with respect to P; particularly, $\mu_\sigma \in \mathcal{P}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. We then define the* characteristic function

$$\varphi_\sigma := \hat{\mu}_\sigma$$

of σ. By the transformation formula for image measures (cf. [Bauer, 74], Proposition 18.1), we may equivalently define the characteristic function by

$$\varphi_\sigma(u) = \int_{\mathbb{R}^n} e^{\iota u\cdot y}\mu_\sigma(dy) = \int_\Omega e^{\iota u\cdot\sigma} dP = P\left(e^{\iota u\cdot\sigma}\right),$$

the expectation with respect to P, for all $u \in \mathbb{R}^n$.
(c) *Let $\sigma : \Omega \to \mathbb{R}^n$ be a random variable. We say that σ is* Gaussian distributed *(with respect) to P if there exists a real, symmetric, positive semi-definite $n \times n$-matrix C and some $m \in \mathbb{R}^n$ such that*

$$\varphi_\sigma(u) = e^{\iota u\cdot m - \frac{1}{2}u\cdot Cu} \quad \forall u \in \mathbb{R}^n.$$

In this case C is called the covariance *and m the* mean *of σ. We call μ_σ a* Gauss measure *or* Gaussian (measure).
(d) *Let $\mathbf{S}$ be an arbitrary index set. A family of $\mathbb{R}$-valued random variables $(\sigma_i)_{i\in\mathbf{S}}$ on Ω is called* (jointly) Gaussian *if for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbf{S}$, the random vector $\sigma := (\sigma_{i_1}, \ldots \sigma_{i_n}) : \Omega \to \mathbb{R}^n$ is Gaussian distributed; equivalently, if all finite dimensional marginal distributions $\mu_{(\sigma_{i_1},\ldots,\sigma_{i_n})} = P \circ (\sigma_{i_1}, \ldots, \sigma_{i_n})^{-1}$ are Gauss measures.*

Remark 5.2. *The Fourier transform uniquely determines the corresponding probability measure. Thus, a Gaussian measure is uniquely determined by its covariance C and mean m.*

Proof. [Bauer, 74], Proposition 48.4 □

From Chapter 2 we know that we first have to specify a parameter set $\mathbf{S}$, a state space $\mathbf{E}$ and a reference or a priori measure λ: We assume $\mathbf{S}$ to be countably infinite. Henceforth, let $\mathbf{E} = \mathbb{R}$ and $\mathcal{E}$ be the Borel-σ-algebra $\mathcal{B}(\mathbb{R})$. The reference measure λ on $\mathbf{E}$ is assumed to be the Lebesgue measure.

As in Chapter 2 we choose the following canonical representation for the underlying measurable space $(\Omega, \mathcal{F})$: $\Omega := \mathbf{E}^{\mathbf{S}}$ and $\mathcal{F} := \mathcal{E}^{\mathbf{S}}$. Thus, we write $(\Omega, \mathcal{F}) := (\mathbf{E}, \mathcal{E})^{\mathbf{S}} := (\mathbb{R}^{\mathbf{S}}, \mathcal{B}(\mathbb{R})^{\mathbf{S}})$. Again, a random field $(\sigma_i)_{i\in\mathbf{S}}$ as introduced in Definition 2.1 is equivalently given by some probability measure $\mu \in \mathcal{P}(\Omega, \mathcal{F})$.

The following definition is consistent with Definition 5.1(d):

Definition 5.3. (a) ([Georgii, 88], Definition 13.1) *A probability measure μ on $(\mathbf{E}, \mathcal{E})^{\mathbf{S}}$ is called a* Gaussian field *if all finite dimensional marginal distributions $\sigma_\Lambda(\mu) := \mu \circ \sigma_\Lambda^{-1}$, $\Lambda \in \mathcal{S}$, are Gaussian. The vector $m := (m_i)_{i\in\mathbf{S}} := (\mu(\sigma_i))_{i\in\mathbf{S}}$, where $\mu(\sigma_i) := \int_{\mathbf{E}^{\mathbf{S}}} \sigma_i d\mu$, is then called the* mean *of μ, and the symmetric function $C : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$,*

$$C(i,j) := \mu((\sigma_i - m_i)(\sigma_j - m_j)) = \mu(\sigma_i\sigma_j) - m_i m_j \quad \forall i,j \in \mathbf{S},$$

is called the covariance function *of μ. μ is said to be centered if $m = 0$.*
(b) *A specification γ as given in Definition 2.6 is called* Gaussian *if, for all $\omega \in \Omega$ and $\Lambda \in \mathcal{S}$, $\gamma_\Lambda(\cdot|\omega)$ is a Gaussian field.*

Remark 5.4. ([Georgii, 88], p. 257) *In terms of Definition 5.1 we can restate Definition 5.3 in the following way: μ on $(\mathbf{E}, \mathcal{E})^{\mathbf{S}}$ is Gaussian with mean m and covariance function C if and only if*

$$\begin{aligned} \varphi_\sigma(u) := \int_{\mathbf{E}^{\mathbf{S}}} e^{\iota \sum_{i\in\mathbf{S}} u_i\sigma_i} \mu(d\omega) &= \mu(e^{\iota \sum_{i\in\mathbf{S}} u_i\sigma_i}) \\ &= e^{\iota \sum_{i\in\mathbf{S}} u_i m_i - \frac{1}{2}\sum_{i,j\in\mathbf{S}} u_i C(i,j) u_j} \end{aligned}$$

for all $u := (u_i)_{i\in\mathbf{S}}$, $u_i \in \mathbb{R}$, such that $|\{i \in \mathbf{S} : u_i \neq 0\}| < \infty$. Moreover, the symmetric covariance function C interpreted as an infinite dimensional matrix can be shown to be positive semi-definite, i.e.

$$\sum_{i,j\in\mathbf{S}} u_i C(i,j) \bar{u}_j \geq 0$$

for all sequences $(u_i)_{i\in\mathbf{S}}$ in $\mathbb{C}$ with $|\{i \in \mathbf{S} : u_i \neq 0\}| < \infty$, where $\bar{u}_i$ denotes the complex conjugate of u_i.

Definition 5.5. ([Georgii, 88], pp. 260-265) (a) *Let $(\Omega, \mathcal{F})$ be given as above, $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and $h \in \Omega$. We set*

$$\Omega_J := \left\{ \omega \in \Omega : \sum_{j\in\mathbf{S}} |J(i,j)\omega_j| < \infty \quad \text{for all } i \in \mathbf{S} \right\}.$$

If J has finite range, *i.e. $\{j \in \mathbf{S} : J(i,j) \neq 0\} \in \mathcal{S}$ for all $i \in \mathbf{S}$, then $\Omega_J = \Omega$.*
(b) *We define the potential*

$$\Phi_A^{J,h} := \begin{cases} \frac{1}{2} J(i,i)\sigma_i^2 + h_i\sigma_i & \text{if } A = \{i\}, \\ J(i,j)\sigma_i\sigma_j & \text{if } A = \{i,j\},\ i \neq j, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

J is called coupling function *and h* external field. *$\Phi^{J,h}$ is called a* Gaussian potential *if J is symmetric and positive definite.*[3] *Let $H_\Lambda^{J,h} := H_\Lambda^{\Phi^{J,h}}$ denote the Hamiltonian and $Z_\Lambda^{J,h}$ the partition function with respect to $\Phi^{J,h}$.*[4]
(c) *Given $\Lambda \in \mathcal{S}$ and $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$, we define the matrix*

$$\mathscr{J}_\Lambda := (J(i,j))_{i,j\in\Lambda}.$$

(d) *For $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and $h \in \Omega$, we set*

$$\mathbf{M}_{J,h} := \left\{ m \in \Omega_J : h_i + \sum_{j\in\mathbf{S}} J(i,j)m_j = 0 \quad \textit{for all } i \in \mathbf{S} \right\}.$$

We now characterize the specification $\gamma^{J,h} := \gamma^{\Phi^{J,h}}$ generated by $\Phi^{J,h}$ whenever J is symmetric and positive definite.

Proposition 5.6. ([Georgii, 88], Proposition 13.13 & Definition 13.18) *Let $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be symmetric and positive definite and $h \in \Omega$. Then, for $\omega \in \Omega_J$, $Z_\Lambda^{J,h}(\omega) < \infty$. $\gamma^{J,h}$, given by*

$$\gamma_\Lambda^{J,h}(A|\omega) := \begin{cases} \frac{1}{Z_\Lambda^{J,h}(\omega)} \lambda_\Lambda(1_A e^{-H_\Lambda^{J,h}}|\omega) & \textit{if } \omega \in \Omega_J, \\ \delta_{0_\Lambda \omega_{\mathbf{S}\setminus\Lambda}}(A) & \textit{else,} \end{cases}$$

for all $\Lambda \in \mathcal{S}$, $A \in \mathcal{F}$ and $\omega \in \Omega$, defines a Gaussian specification. Here, $\delta_{0_\Lambda \omega_{\mathbf{S}\setminus\Lambda}}$ denotes the Dirac measure on the configuration $0_\Lambda \omega_{\mathbf{S}\setminus\Lambda}$ that is equal to zero on Λ and to ω everywhere else. $\gamma_\Lambda^{J,h}(\cdot|\omega)$ is the unique Gaussian field with mean

$$m_i(\Lambda,\omega) = \begin{cases} - \sum_{k\in\Lambda} \mathscr{J}_\Lambda^{-1}(i,k) \left(h_k + \sum_{j\notin\Lambda} J(k,j)\omega_j \right) & \textit{if } i \in \Lambda \textit{ and } \omega \in \Omega_J, \\ 0 & \textit{if } i \in \Lambda \textit{ and } \omega \notin \Omega_J, \\ \omega_i & \textit{if } i \notin \Lambda, \end{cases}$$

and covariance function

$$C_\Lambda(i,j) = \begin{cases} \mathscr{J}_\Lambda^{-1}(i,j) & \textit{if } i,j \in \Lambda \textit{ and } \omega \in \Omega_J, \\ 0 & \textit{else.} \end{cases}$$

[3] We show in Proposition 5.6 that in this case $\gamma^{\Phi^{J,h}}$ is a Gaussian specification.

[4] In fact, $\Phi^{J,h}$ is not a potential as defined in 2.11, where a potential is given by a family of functions $\Phi_A : \Omega \to \mathbb{R}$: The reason is that $H_\Lambda^{J,h}(\omega)$ may not exist for all $\omega \in \Omega$. Nevertheless, the Hamiltonian exists for all $\omega \in \Omega_J$. Thus, $\Phi^{J,h}$ is a potential as in Definition 2.11 if J exhibits finite range. In general, we have to generate a specification in a slightly different way than in Definition 2.12: We obtain $\gamma_\Lambda^{\Phi^{J,h}}(A|\omega)$ as in Definition 2.12 if $\omega \in \Omega_J$ and thus the partition function and the Hamiltonian are well-defined; for $\omega \notin \Omega_J$, we just assume some distribution. However, the procedure can be seen in Proposition 5.6.

Proof. By definition of $\gamma^{J,h}$, $\mu(\Omega_J) = 1$ for all $\mu \in \mathcal{G}(\gamma^{J,h})$. In [Georgii, 88], p. 265, there is an outline for proving that $\gamma^{J,h}$ is indeed a specification The Gaussian property and the explicit form of mean and covariance when $\omega \in \Omega_J$ is shown in the proof of [Georgii, 88], Proposition 13.13. For $\omega \notin \Omega_J$, the assertion is immediate by definition of γ as a specific Dirac measure. □

Assumption 5.7. *Henceforth, if we consider a Gaussian specification, we always refer to a specification $\gamma^{J,h}$ as defined in the foregoing proposition with symmetric and positive definite coupling function J.*

When considering the terms in Proposition 5.6, one should always be aware of the formal definitions as introduced in Chapter 2. To recall: For $\Lambda \subset \mathcal{S}$ let λ^Λ denote the Lebesgue measure on $(\mathbf{E}, \mathcal{E})^\Lambda$, i.e. the $|\Lambda|$-fold product of the Lebesgue measure on $(\mathbf{E}, \mathcal{E})$. In equation (2.1), $\lambda_\Lambda(A|\omega)$ is defined as

$$\begin{aligned}
\lambda_\Lambda(A|\omega) := \lambda_\Lambda(1_A|\omega) &:= \lambda^\Lambda \times \delta_{\omega_{\mathbf{S}\setminus\Lambda}}(A) \\
&= \int_\Omega 1_A(\lambda^\Lambda \otimes \delta_{\omega_{\mathbf{S}\setminus\Lambda}})(d\omega) \\
&= \int_{\mathbf{E}^\Lambda} 1_A(\xi\omega_{\mathbf{S}\setminus\Lambda})\lambda^\Lambda(d\xi)
\end{aligned}$$

for $\Lambda \in \mathcal{S}$, $A \in \mathcal{F}$, $\omega \in \Omega$, $\xi \in \Omega_\Lambda := \mathbf{E}^\Lambda$, where $\delta_{\omega_{\mathbf{S}\setminus\Lambda}}$ denotes the Dirac measure on $(\mathbf{E}, \mathcal{E})^{\mathbf{S}\setminus\Lambda}$ with mass on $\omega_{\mathbf{S}\setminus\Lambda}$. $\omega_\Lambda : \Omega \ni \omega \mapsto (\omega_i)_{i\in\Lambda}$ is the projection mapping on $\mathbf{E}^\Lambda$. Thus, we obtain

$$\lambda_\Lambda(1_A e^{-H_\Lambda^{J,h}}|\omega) = \int_{\mathbf{E}^\Lambda} 1_A(\xi\omega_{\mathbf{S}\setminus\Lambda}) e^{-H_\Lambda^{J,h}(\xi\omega_{\mathbf{S}\setminus\Lambda})}\lambda^\Lambda(d\xi).$$

And thus, Proposition 5.6 and Definition 2.12 are consistent.

Let δ_m denote the Dirac measure with mass in $m \in \Omega$. Since

$$\int_\Omega e^{\iota \sum_{i\in\mathbf{S}} u_i\sigma_i(\omega)}\delta_m(d\omega) = e^{\iota \sum_{i\in\mathbf{S}} u_i m_i},$$

we say that δ_m is Gaussian with mean m and covariance 0.

5.2.2 Characterization of Gibbs States

The following proposition provides us with a characterization of Gaussian Gibbs measures in terms of mean m and “inverse” J of covariance C.

Proposition 5.8. ([Georgii, 88], Theorem 13.22) *Let μ be a Gaussian field with mean m and covariance function C. Also, let $h \in \Omega$ and $J : \mathbf{S}\times\mathbf{S} \to \mathbb{R}$ be a positive definite symmetric function. Then the following conditions are equivalent.*

1. $\mu \in \mathcal{G}(\gamma^{J,h})$,
2. $\mu(\Omega_J) = 1$, $m \in \mathbf{M}_{J,h}$ *and*

$$\sum_{j \in \mathbf{S}} J(i,j)C(j,k) = \delta_{ik} \quad \text{for all } i,k \in \mathbf{S}.$$

Proof. [Georgii, 88], pp. 266,267 □

We now tackle the problem of characterizing the set of Gibbs measures $\mathcal{G}(\gamma^{J,h})$ for the Gaussian specification $\gamma^{J,h}$, J symmetric and positive definite. Since $\mathbf{E}$ is a linear space, the following notion analogous to the spin flip in the Ising model is well defined. Recall: In the Ising model, spin flip is the broken symmetry.

Definition 5.9. *Given $m \in \Omega$ we define the* spin translation $T \ni \tau^m : \Omega \to \Omega$ *by m as follows:*

$$\tau^m \omega := \omega + m := (\omega_i + m_i)_{i \in \mathbf{S}}$$

for all $\omega \in \Omega$.

Remark 5.10. *Let $\mu \in \mathcal{P}((\mathbf{E}, \mathcal{E})^{\mathbf{S}})$ be Gaussian with mean $\tilde{m}$ and covariance C. Consider the transformation τ^m. Then $\tau^m(\mu) := \mu \circ (\tau^m)^{-1} = \mu \circ \tau^{-m}$ is Gaussian with mean $\tilde{m} + m$ and covariance C.*

Proof. By Remark 5.4 we have that μ is Gaussian with mean $\tilde{m}$ and covariance C if and only if

$$\mu\left(e^{\iota \sum_{i \in \mathbf{S}} u_i \sigma_i}\right) = \int e^{\iota \sum_{i \in \mathbf{S}} u_i \sigma_i} d\mu = e^{\iota \sum_{i \in \mathbf{S}} u_i \tilde{m}_i - \frac{1}{2} \sum_{i,j \in \mathbf{S}} u_i C(i,j) u_j}$$

for all $u \in \mathbb{R}^{\mathbf{S}}$, such that $|\{i \in \mathbf{S} : u_i \neq 0\}| < \infty$. Applying the transformation formula for image measures, we obtain

$$\begin{aligned}
\tau^m(\mu)\left(e^{\iota \sum_{i \in \mathbf{S}} u_i \sigma_i}\right) &= \int e^{\iota \sum_{i \in \mathbf{S}} u_i \sigma_i} d(\tau^m(\mu)) \\
&= \int e^{\iota \sum_{i \in \mathbf{S}} u_i \overbrace{(\sigma_i \circ \tau^m)(\omega)}^{=m_i + \sigma_i(\omega)}} \mu(d\omega) \\
&= \int e^{\iota \sum_{i \in \mathbf{S}} u_i m_i + \iota \sum_{i \in \mathbf{S}} u_i \sigma_i} d\mu \\
&= e^{\iota \sum_{i \in \mathbf{S}} u_i (\tilde{m}_i + m_i) - \frac{1}{2} \sum_{i,j \in \mathbf{S}} u_i C(i,j) u_j}.
\end{aligned}$$

□

The proof of the following remark is an elaborate version of that in [Georgii, 88].

Remark 5.11. ([Georgii, 88], Remark 13.23) *Let $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be symmetric and positive definite. Let $h, \tilde{h} \in \Omega$.*
(a) *If $m \in \mathbf{M}_{J,h}$, we have*

$$\gamma_\Lambda^{J,h+\tilde{h}}(\cdot|\tau^m\omega) = \tau^m(\gamma_\Lambda^{J,\tilde{h}}(\cdot|\omega)) \quad \forall \Lambda \in \mathcal{S}, \omega \in \Omega_J.$$

(b) *For each $m \in \mathbf{M}_{J,h}$, we have*

$$\mathcal{G}(\gamma^{J,h}) = \{\tau^m(\mu) : \mu \in \mathcal{G}(\gamma^{J,0})\}.$$

(c) *$\mathcal{G}(\gamma^{J,h})$ is invariant under τ^m, $m \in \mathbf{M}_{J,0}$. In this case, τ^m is a symmetry of $\gamma^{J,h}$.*
(d) *If $\mathbf{M}_{J,0}$ contains an element $m \neq 0$, then either $\mathcal{G}(\gamma^{J,h}) = \emptyset$ or $ex\mathcal{G}(\gamma^{J,h})$ is uncountably infinite.*

Proof. (a) From Proposition 5.6 we infer that $\gamma_\Lambda^{J,h+\tilde{h}}(\cdot|\tau^m\omega)$ and $\gamma_\Lambda^{J,\tilde{h}}(\cdot|\omega)$ are Gaussian fields. Thus, $\tau^m(\gamma_\Lambda^{J,\tilde{h}}(\cdot|\omega))$ is a Gaussian field by Remark 5.10. In light of Remark 5.2, it suffices to show that the left and the right hand side of the assertion have the same mean and covariance. Proposition 5.6 and Remark 5.10 imply the covariances to be identical. The explicit form of the mean in Proposition 5.6 shows consilience of means.
(b) "$\supset$" Let $\mu \in \mathcal{G}(\gamma^{J,0})$ and $m \in \mathbf{M}_{J,h}$. Then Remark 2.33 and part (a) imply

$$\tau^m(\mu) \in \mathcal{G}(\tau^m(\gamma^{J,0})) \overset{(a)}{=} \mathcal{G}(\gamma^{J,h}).$$

"$\subset$" Let $\mu^h \in \mathcal{G}(\gamma^{J,h})$ and $m \in \mathbf{M}_{J,h}$. We then have to find $\mu^0 \in \mathcal{G}(\gamma^{J,0})$ such that $\tau^m(\mu^0) = \mu^h$. But again,

$$\mu^h \in \mathcal{G}(\gamma^{J,h}) \quad :\Leftrightarrow \quad \mu^h\gamma_\Lambda^{J,h} = \mu^h \quad \forall \Lambda \in \mathcal{S}.$$

As in the proof of Remark 2.33, this yields

$$\tau^{-m}(\mu^h)\tau^{-m}(\gamma_\Lambda^{J,h}) = \tau^{-m}(\mu^h) \quad \forall \Lambda \in \mathcal{S}.$$

By part (a), we have $\tau^{-m}(\gamma^{J,h}) = \gamma^{J,0}$ and we set $\mu^0 := \tau^{-m}(\mu^h)$. We then have $\mu^0 = \tau^{-m}(\mu^h) \in \mathcal{G}\gamma^{J,0}$.
(c) Let $m \in \mathbf{M}_{J,0}$. In this case, part (a) yields

$$\tau^m(\gamma^{J,h}) = \gamma^{J,h},$$

i.e. τ^m is a symmetry of $\gamma^{J,h}$:

$$\gamma_\Lambda^{J,0+\tilde{h}}(\cdot|\tau^m\omega) = \gamma_\Lambda^{J,\tilde{h}}(\cdot|\tau^m\omega) \overset{(a)}{=} \tau^m(\gamma_\Lambda^{J,\tilde{h}}(\cdot|\omega)).$$

By Definition 2.30(b), τ^m is a symmetry; Remark 2.33 yields the assertion.
(d) Note that $\mathbf{M}_{J,0}$ is a linear subspace of Ω.[5] Let us now assume there

[5] Having a look at the definition of $\mathbf{M}_{J,0}$, we see that $0 \in \mathbf{M}_{J,0}$, and, for $m_1, m_2 \in \mathbf{M}_{J,0}$, $k \in \mathbb{R}$, we have $(m_1 + m_2) \in \mathbf{M}_{J,0}$ and $km_1 \in \mathbf{M}_{J,0}$.

exists $0 \neq m \in \mathbf{M}_{J,0}$. Then $\mathbf{M}_{J,0}$ is uncountable. We have already shown that in this case $\tau^m(\gamma^{J,h}) = \gamma^{J,h}$, that is $\gamma^{J,h}$ exhibits the symmetries τ^m, $m \in \mathbf{M}_{J,0}$. Since $\mathbf{M}_{J,0}$ is uncountable, the set of symmetries is uncountable (and thus $\mathcal{G}(\gamma^{J,h})$ has to be uncountable whenever $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$ by Remark 2.33). Let now $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$. By Proposition 2.40 we have $ex\mathcal{G}(\gamma^{J,h}) \neq \emptyset$. Remark 2.38 yields that $ex\mathcal{G}(\gamma^{J,h})$ is uncountable: Let $\mu \in ex\mathcal{G}(\gamma^{J,h})$, then $\{\tau^m(\mu) : m \in \mathbf{M}_{J,0}\} \subset ex\mathcal{G}(\gamma^{J,h})$ is already uncountable. □

The next definition is needed for the main result in this section stating that $\mu \in \mathcal{G}(\gamma^{J,h})$ if and only if μ is a random translation of some Gaussian field μ_C.

Definition 5.12. ([Georgii, 88], p. 268) *Let $\mu_1, \mu_2 \in \mathcal{P}(\Omega, \mathcal{F})$. We define the* convolution *$\mu_1 * \mu_2$ by*

$$\mu_1 * \mu_2(f) := \int \int f(\xi + \eta)\mu_1(d\xi)\mu_2(d\eta)$$

for all bounded measurable functions f, where $\xi + \eta := (\xi_i + \eta_i)_{i \in \mathbf{S}} \in \Omega$.

Remark 5.13. *Given a random field $\mu \in \mathcal{P}(\Omega, \mathcal{F})$. Let δ_m denote the Dirac measure with mass in $m \in \Omega$. Then $\mu * \delta_m = \tau^m(\mu)$.*

Proof. Let f be a bounded measurable function. Then

$$\begin{aligned} \mu * \delta_m(f) &= \int \left[\int f(\xi + \eta)\mu(d\xi) \right] \delta_m(d\eta) \\ &= \int f(\xi + m)\mu(d\xi) \\ &= \int (f \circ \tau^m)(\xi)\mu(d\xi) \\ &= \int f(\xi)[\tau^m(\mu)](d\xi) \\ &= \tau^m(\mu)(f). \end{aligned}$$

□

Remark 5.14. *Given random fields $\mu, \nu \in \mathcal{P}(\Omega, \mathcal{F})$. Let δ_m denote the Dirac measure with mass in $m \in \Omega$. Then*

$$\mu * \nu = \int_\Omega \tau^m(\mu)\nu(dm) \stackrel{5.13}{=} \int_\Omega (\mu * \delta_m)\nu(dm).$$

Thus, $\mu * \nu$ is a random translation of μ when randomness is ruled by ν.

Proof. Let f be a bounded measurable function. Then

$$\begin{aligned}
\mu * \nu(f) &= \int \left[\int f(\xi + \eta)\mu(d\xi) \right] \nu(d\eta) \\
&= \int \left[\int (f \circ \tau^{\eta})(\xi)\mu(d\xi) \right] \nu(d\eta) \\
&= \int \left[\int f d(\tau^{\eta}(\mu)) \right] \nu(d\eta) \\
&= \int \tau^{\eta}(\mu)(f)\nu(d\eta) \\
&\overset{5.13}{=} \int (\mu * \delta_{\eta})(f)\nu(d\eta)
\end{aligned}$$

□

The following proposition gives us a characterization for Gibbs states with respect to Gaussian specifications. We elaborately state the proof here. Particularly, the proof of the second part exhibits a good intuition for the set of Gibbs states.

Proposition 5.15. ([Georgii, 88], Theorem 13.24) *Let $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be symmetric and positive definite and let $h \in \Omega$. Suppose that $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$. Then the limits*

$$C(i,j) := \lim_{\Lambda \in \mathcal{S}} \mathscr{J}_{\Lambda}^{-1}(i,j) \tag{5.2}$$

exist for all $i, j \in \mathbf{S}$. A random field μ belongs to $ex\mathcal{G}(\gamma^{J,h})$ if and only if μ is a Gaussian field with covariance C and mean $m \in \mathbf{M}_{J,h}$. Hence,

$$\mathcal{G}(\gamma^{J,h}) = \{\mu_C * \nu : \nu \in \mathscr{P}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1\}, \tag{5.3}$$

where μ_C is the unique centered Gaussian field with covariance function C.

Proof. Proposition 2.40 ensures $ex\mathcal{G}(\gamma^{J,h}) \neq \emptyset$. Let $\mu \in ex\mathcal{G}(\gamma^{J,h})$ and $(\Lambda_n)_{n \geq 1} \subset \mathcal{S}$ be increasing and cofinal in $\mathcal{S}$, i.e. for each $\Delta \in \mathcal{S}$ there exists n_0 with $\Lambda_n \supset \Delta$ for all $n \geq n_0$. Theorem 7.12 in [Georgii, 88] implies that there exists an $\omega \in \Omega_J$ such that the Gaussian field $\gamma_{\Lambda_n}^{J,h}(\cdot|\omega)$ converges to μ locally as $n \to \infty$. Propositions 5.6 and 5.16 below imply that μ is Gaussian with mean m and covariance $C(i,j) = \lim_{\Lambda \in \mathcal{S}} \mathscr{J}_{\Lambda}^{-1}(i,j) \quad \forall i, j \in \mathbf{S}$. In particular, this limit exists. By Proposition 5.8 we obtain $m \in \mathbf{M}_{J,h}$.

To keep track of the results obtained so far: $\mu \in ex\mathcal{G}(\gamma^{J,h})$ is Gaussian with mean $m \in \mathbf{M}_{J,h}$ and covariance C given by equation (5.2).

Let μ_C be the centered Gaussian field with this covariance C. Remark 5.13 implies $\mu_C * \delta_m = \tau^m(\mu_C)$. By Remark 5.10 we obtain $\mu_C * \delta_m$ to be a Gaussian field with mean m and covariance C. Thus, for μ as above, $\mu = \mu_C * \delta_m$ by Remark 5.2.

So far, we have shown $\emptyset \neq ex\mathcal{G}(\gamma^{J,h}) \subset \{\mu_C * \delta_m : m \in \mathbf{M}_{J,h}\}$, i.e. for each $\mu \in ex\mathcal{G}(\gamma^{J,h})$ we can find $m \in \mathbf{M}_{J,h}$ such that $\mu = \tau^m(\mu_C)$. Thus, we have contrariwise $\mu_C = \tau^{-m}(\mu)$. Remark 2.38 then implies

$$\mu_C \in ex\mathcal{G}(\tau^{-m}(\gamma^{J,h})) = ex\mathcal{G}(\gamma^{J,0}),$$

where the equality of sets follows from Remark 5.11(b):

$$\begin{aligned} & ex\mathcal{G}(\gamma^{J,h}) = ex\{\tau^m(\mu) : \mu \in \mathcal{G}(\gamma^{J,0})\} \\ \Leftrightarrow \quad & ex\mathcal{G}(\tau^{-m}(\gamma^{J,h})) = ex\{\mu : \mu \in \mathcal{G}(\gamma^{J,0})\} = ex\mathcal{G}(\gamma^{J,0}) \end{aligned}$$

And again by Remarks 2.38 and 5.11(b), we have

$$\mu_C * \delta_m = \tau^m(\mu_C) \in ex\mathcal{G}(\tau^m(\gamma^{J,0})) = ex\mathcal{G}(\gamma^{J,h}) \quad \forall m \in \mathbf{M}_{J,h}.$$

Thus, we have $\{\mu_C * \delta_m : m \in \mathbf{M}_{J,h}\} \subset ex\mathcal{G}(\gamma^{J,h})$.

Hence, we have shown

$$ex\mathcal{G}(\gamma^{J,h}) = \{\mu_C * \delta_m : m \in \mathbf{M}_{J,h}\}$$

with μ_C the unique centered Gaussian field with covariance C.

Having characterized the extreme Gibbs states for $\gamma^{J,h}$ explicitly, we may now use Proposition 2.40 to obtain the set of Gibbs measures:

$$\begin{aligned} \mathcal{G}(\gamma^{J,h}) \quad &= \quad \left\{ \int_{ex\mathcal{G}(\gamma^{J,h})} \nu\psi(d\nu) : \psi \in \mathcal{P}(ex\mathcal{G}(\gamma^{J,h}), \mathscr{V}(ex\mathcal{G}(\gamma))) \right\} \\ &\overset{Note 5.17}{=} \quad \left\{ \int_{\Omega} (\mu_C * \delta_m)\nu(dm) : \nu \in \mathcal{P}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1 \right\} \\ &\overset{5.14}{=} \quad \{\mu_C * \nu : \nu \in \mathcal{P}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1\}. \end{aligned}$$

$\square$

The notation "$\int \tau^m(\mu_C)\nu(dm)$" for "$\mu_C * \nu$" exhibits a precious intuition: We see that a measure μ is Gibbsian if and only if it is a random translation of μ_C. In the foregoing proof we have used the following result:

Proposition 5.16. ([Georgii, 88], Proposition 13.A5) *Let $(U^{(k)})_{k\geq 1}$ be a sequence of $\mathbb{R}^n$-valued Gaussian random vectors $U^{(k)}$ with mean $m^{(k)}$ and covariance $C^{(k)}$. Suppose $U^{(k)}$ converges in distribution to a random vector U. Then the limits $m := \lim_{k\to\infty} m^{(k)}$ and $C := lim_{k\to\infty} C^{(k)}$ exist, and U is Gaussian with mean m and covariance C.*

Proof. [Georgii, 88], p. 285 $\square$

Note 5.17. *Considering $\mu_C * \nu \in ex\mathcal{G}(\gamma^{J,h})$, Remark 5.14 and the explicit characterization of $ex\mathcal{G}(\gamma^{J,h})$ allow us to identify ν with the probability distribution ψ on $ex\mathcal{G}(\gamma^{J,h})$ that represents $\mu_C * \nu$ in Proposition 2.40:*

$$\int_{ex\mathcal{G}(\gamma^{J,h})} \eta\psi(d\eta) \overset{2.40}{=} \mu_C * \nu \overset{5.14}{=} \int_{\mathbf{M}_{J,h}} (\mu_C * \delta_m)\nu(dm)$$

Assuming $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$, Proposition 5.15 provides us with a complete description of $ex\mathcal{G}(\gamma^{J,h})$. Thus, we have to find conditions ensuring the existence of Gibbs measures. In [Georgii, 88], p. 269, the following conditions are given. We state a proof here.

Proposition 5.18. *Let $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be symmetric and positive definite. $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$ if and only if $\mathbf{M}_{J,h} \neq \emptyset$ and J has inverse C given by equation (5.2) such that $\mu_C(\Omega_J) = 1$.*

Proof. "$\Rightarrow$" This implication follows directly from Proposition 5.15 since $\mu_C(\Omega_J) = 1$ by Proposition 5.8 as $\mu_C \in \mathcal{G}(\gamma^{J,h})$.
"$\Leftarrow$" Since $0 \in \mathbf{M}_{J,0}$, $\mu_C(\Omega_J) = 1$ and J has inverse C, Proposition 5.8 implies $\mu_C \in \mathcal{G}(\gamma^{J,0})$. Let now $m \in \mathbf{M}_{J,h}$, then by Remark 5.11(b), we obtain

$$\mathcal{G}(\gamma^{J,h}) = \{\tau^m(\mu) : \mu \in \mathcal{G}(\gamma^{J,0})\} \ni \tau^m(\mu_C).$$

□

If J has finite range, we have $\Omega_J = \Omega$ and thus $\mu_C(\Omega_J) = 1$. In this case, Proposition 5.18 reduces to

$$\mathcal{G}(\gamma^{J,h}) \neq \emptyset \qquad \Leftrightarrow \qquad \mathbf{M}_{J,h} \neq \emptyset \quad \wedge \quad \text{the limit (5.2) exists.}$$

The next proposition improves condition (5.2) in case of finite range. Later, we are only interested in a coupling function that exhibits finite range.

Proposition 5.19. ([Georgii, 88], Theorem 13.26) *Let $h \in \Omega$ and $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be a positive definite symmetric function such that $\{j \in \mathbf{S} : J(i,j) \neq 0\} \in \mathcal{S}$ for all $i \in \mathbf{S}$. Then $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$ if and only if $\mathbf{M}_{J,h} \neq \emptyset$ and*

$$\sup_{\Lambda \in \mathcal{S}} \mathscr{J}_\Lambda^{-1}(i,i) < \infty \quad \forall i \in \mathbf{S}. \tag{5.4}$$

Proof. [Georgii, 88], pp. 269,270 □

The following example is a generalization of the Ising ferromagnet in Section 2.7. This is explicitly shown in Example 5.33.

Example 5.20 (The harmonic oscillator)**.** ([Georgii, 88], Example 13.29) *Let $d \geq 1$, $\mathbf{S} = \mathbb{Z}^d$, $\beta > 0$ and define the coupling function*

$$J(i,j) = \begin{cases} -\ \frac{\beta}{2d} & \text{if } |i-j| = 1, \\ \quad\ \beta & \text{if } i = j, \\ \quad\ 0 & \text{else.} \end{cases}$$

For all $h \in \Omega$ we obtain that $\mathcal{G}(\gamma^{J,h}) = \emptyset$ if $d \leq 2$ and $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$ if $d \geq 3$. Thus, whenever $d \geq 3$, we may apply the characterization of Gibbs states in Proposition 5.15. For $t \in \mathbb{R}$ define the constant configuration $m^t \in \Omega$ by $m_i^t = t$ for all $i \in \mathbf{S}$. $\mathbf{M}_{J,0}$ contains all constant configurations and thus $\gamma^{J,h}$ exhibits the (spin translation) symmetry $\tau^{m^t} : \omega \mapsto (\omega_i + t)_{i \in \mathbf{S}}$, $t \in \mathbb{R}$. Moreover, we have $\mathbf{M}_{J,h} \neq \emptyset$ for all h and hence, whenever $d \geq 3$, $\gamma^{J,h}$ exhibits breakdown of the symmetry τ^{m^t}, $t \in \mathbb{R} \setminus \{0\}$.

Proof. J is symmetric, i.e. $J(i,j) = J(j,i)$ for all $i, j \in \mathbf{S}$, exhibits finite range and satisfies

1. $J(i,j) \leq 0 \quad \forall i \neq j$ and
2. $\sum_{j \in \mathbf{S}} J(i,j) \geq 0 \quad \forall i$

Thus, we may follow that J is positive definite: Let $(u_i)_{i \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $|\{i \in \mathbf{S} : u_i \neq 0\}| < \infty$. Then

$$\sum_{i,j \in \mathbf{S}} u_i J(i,j) \bar{u}_j \overset{2}{\geq} -\frac{1}{2} \sum_{i \neq j} J(i,j) |u_i - u_j|^2 \overset{1}{\geq} 0.$$

The term on the right hand side equals 0 if and only if $u_i = u_j$ whenever $|i-j| = 1$, i.e. $J(i,j) < 0$. By definition of J, this can only be the case if $u_i = u_j$ for all $i \neq j \in \mathbf{S}$. Since we assumed $\{i \in \mathbf{S} : u_i \neq 0\}$ to be finite, this can only be the case if $u_i = 0$ for all $i \in \mathbf{S}$.

We now define a stochastic matrix $Q := (Q(i,j))_{i,j \in \bar{\mathbf{S}}}$, where $\bar{\mathbf{S}} := \mathbf{S} \cup \{\infty\}$, by

$$Q(i,j) = \begin{cases} -\quad \frac{J(i,j)}{J(i,i)} & \text{if } i \neq j \in \mathbf{S}, \\ \quad \sum_{j \in \mathbf{S}} \frac{J(i,j)}{J(i,i)} & \text{if } i \in \mathbf{S},\ j = \infty, \\ \quad 1 & \text{if } i = j = \infty, \\ \quad 0 & \textit{else}. \end{cases}$$

Now let $(X_n^i)_{n \in \mathbb{N}}$ be the symmetric random walk on $\bar{\mathbf{S}}$ with transition matrix Q and starting point i. Let P_i denote the underlying probability measure, i.e. $P_i \circ (X_0, \ldots X_n)^{-1} := \delta_i \otimes Q \otimes \ldots \otimes Q$ for all $n \in \mathbb{N}$. For $\Lambda \in \mathcal{S}$ define τ_Λ, the time of first exit from Λ, by

$$\tau_\Lambda := \min\{n \in \mathbb{N} : X_n \in \bar{\mathbf{S}} \setminus \Lambda\}.$$

The following equivalence holds:

$$\sup_{\Lambda \in \mathcal{S}} \mathscr{J}_\Lambda^{-1}(i,i) < \infty \quad \forall i \in \mathbf{S} \qquad \Leftrightarrow \qquad \forall i \in \mathbf{S} \quad i \quad \text{is transient for} \quad Q, \tag{5.5}$$

i.e.

$$\sum_{n \geq 0} Q^n(i,i) = P_i \left(\sum_{n \geq 0} 1_{\{X_n = i\}} \right) < \infty \quad \forall i \in \mathbf{S} :$$

For all $i, j \in \Lambda \in \mathcal{S}$, we have

$$\mathscr{J}_\Lambda^{-1}(i,j) = \frac{P_i(\sum_{n=0}^{\tau_\Lambda - 1} 1_{\{X_n = j\}})}{J(j,j)}. \tag{5.6}$$

Equation (5.6) will not be shown here. An outline of the proof can be found in [Georgii, 88], p. 271. Applying this representation for $\mathscr{J}_\Lambda^{-1}$ to condition (5.4), we obtain the equivalent condition

$$\sup_{\Lambda \in \mathcal{S}} P_i \left(\sum_{n=0}^{\tau_\Lambda - 1} 1_{\{X_n = i\}} \right) < \infty \quad \forall i \in \mathbf{S}. \tag{5.7}$$

By definition of τ_Λ, equivalence (5.5) follows since

$$\sup_{\Lambda \in \mathcal{S}} P_i \left(\sum_{n=0}^{\tau_\Lambda - 1} 1_{\{X_n = i\}} \right) = P_i \left(\sum_{n \geq 0} 1_{\{X_n = i\}} \right).$$

By Theorem 5.19 this yields:

$$\mathcal{G}(\gamma^{J,h}) \neq \emptyset \quad \Leftrightarrow \quad \mathbf{M}_{J,h} \neq \emptyset \quad \wedge \quad \forall i \in \mathbf{S},\ i \text{ is transient for } Q.$$

As it is well known (e.g. in [Röckner, 05], Example 7.7.4), every $i \in \mathbb{Z}^d$ is recurrent for Q if and only if $d \leq 2$ and transient if and only if $d \geq 3$. Thus it follows $\mathcal{G}(\gamma^{J,h}) = \emptyset$ if and only if $d \leq 2$. $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$ whenever $\mathbf{M}_{J,h} \neq \emptyset$ and $d \geq 3$. Proposition 5.15 yields that $\mathbf{M}_{J,h}$ and $ex\mathcal{G}(\gamma^{J,h})$ are isomorphic in this case.

Since J exhibits finite range we have $\Omega_J = \Omega$. If $m \in \Omega$ is constant, i.e. $m_i = m_j$ for all $i, j \in \mathbf{S}$, we have $\sum_{j \in \mathbf{S}} J(i,j) m_j = m_i \sum_{j \in \mathbf{S}} J(i,j) = 0$ for all $i \in \mathbf{S}$. Consequently, we obtain that $\mathbf{M}_{J,0}$ contains all constant configurations.

By Remark 5.11(c), τ^m, $m \in \mathbf{M}_{J,0}$, is a symmetry for $\gamma^{J,h}$. Thus, τ^{m^t} defined by $\omega \mapsto \tau^{m^t}(\omega) := (\omega_i + t)_{i \in \mathbf{S}}$, $t \in \mathbb{R}$, is a symmetry for $\gamma^{J,h}$.

Obviously, $\mathbf{M}_{J,h} \neq \emptyset$ for all $h \in \Omega$. Moreover, since $\mathbf{M}_{J,h} = m + \mathbf{M}_{J,0}$ for $m \in \mathbf{M}_{J,h}$, we see that $\mathbf{M}_{J,h}$ is uncountable. Thus, by Proposition 5.15 a Gaussian field μ with covariance C and mean $m \in \mathbf{M}_{J,h}$ belongs to $ex\mathcal{G}(\gamma^{J,h}) \subset \mathcal{G}(\gamma^{J,h})$ if $d \geq 3$. We have seen that τ^{m^t} is a symmetry for $\gamma^{J,h}$ since $m^t \in \mathbf{M}_{J,0}$. Let $\mu \in ex\mathcal{G}(\gamma^{J,h})$, then μ is Gaussian with mean $m \in \mathbf{M}_{J,h}$ and covariance C. By Remark 5.10, $\tau^{m^t}(\mu)$ is Gaussian with mean $(m_i + t)_{i \in \mathbf{S}} \neq m$ for $t \neq 0$. Thus, $\tau^{m^t}(\mu) \neq \mu$ for $t \neq 0$ by Remark 5.2 and hence, τ^{m^t} is not a symmetry for $\mu \in ex\mathcal{G}(\gamma^{J,h})$: $\gamma^{J,h}$ exhibits uncountably many symmetry breakdowns. □

Remark 5.21. *In the setting of Example 5.20, equations (5.6) and (5.2) show that the covariance function C is given by*

$$C(i,j) = \frac{\sum_{n \geq 0} Q^n(i,j)}{J(j,j)} \quad \forall i, j \in \mathbf{S}.$$

The coupling functions discussed in the next section are designed for tackling the existence problem when J may exhibit infinite range. We have seen that the harmonic oscillator, that constitutes the corner stone of our economic model, exhibits finite range. Nevertheless, coupling in the harmonic oscillator is homogeneous and thus, the next section may generate some more insights in the structure of Gibbs states for the harmonic oscillator. In particular, we characterize homogeneous extreme Gibbs states.

5.2.3 Homogeneous Coupling Functions

For the rest of this section assume $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$, and J and h being homogeneous in the sense given below. Recall from Example 2.28 that θ_j, $j \in \mathbf{S}$, denotes the *(lattice) shift* or *(lattice) translation.* Moreover, homogeneity of a specification was introduced in Example 2.31.

Definition 5.22. ([Georgii, 88], p. 273) *Coupling function $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and external field $h \in \Omega$ are called* homogeneous *if there exists an even function $\tilde{J} : \mathbf{S} \to \mathbb{R}$, i.e. $\tilde{J}(s) = \tilde{J}(-s)$ for all $s \in \mathbf{S}$, and $\tilde{h} \in \mathbb{R}$ such that*

$$J(i,j) = \tilde{J}(i-j) \quad \textit{and} \quad h_j = \tilde{h} \quad \forall i,j \in \mathbf{S}.$$

Note that the coupling function J as defined in 5.20 is homogeneous.

Remark 5.23. ([Georgii, 88], p. 273) *Let $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ be symmetric and positive definite, $h \in \Omega$. A Gaussian specification $\gamma^{J,h}$ is homogeneous if and only if J and h are homogeneous.*

Proof. Since a Gaussian field is uniquely determined by mean and covariance, we have to assure that the means and covariances of $\gamma_{\Lambda+j}(\theta_j A|\theta_j\omega)$ and $\gamma_\Lambda(A|\omega)$ coincide. Applying the explicit characterization of mean and covariance in Proposition 5.6 to Example 2.31 yields the assertion. □

Given an even function $\tilde{J} : \mathbf{S} \to \mathbb{R}$ and $\tilde{h} \in \mathbb{R}$ we can determine the corresponding homogeneous coupling function J and external field h. Thus, we call $\tilde{J}$ positive definite, whenever the corresponding J is positive definite. By definition, a homogeneous function is symmetric. The notions $\Omega_{\tilde{J}}$, $M_{\tilde{J},\tilde{h}}$ and $\gamma^{\tilde{J},\tilde{h}}$ are well defined for $\tilde{J}$ and $\tilde{h}$ as they are for J and h.

We have assumed J and h to be homogeneous. Thus, we may identify J with its corresponding $\tilde{J}$ and h with its corresponding $\tilde{h}$. By this token and following [Georgii, 88], we henceforth denote $\tilde{J}$ by J and $\tilde{h}$ by h.

We want to achieve a result similar to Proposition 5.15 when $\gamma^{J,h}$ is a homogeneous Gaussian specification. In particular, we want to characterize the elements in $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h})$. We would like to obtain a characterization of $ex\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h})$, the set of all pure states. However, we obtain all homogeneous extreme Gibbs states. Since homogeneous extreme distributions are also extreme homogeneous, we obtain the set of all pure states (ergodic Gibbs states) that are phases (extreme Gibbs states). Thus, we obtain

$$ex\mathcal{G}(\gamma^{J,h}) \cap \mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h}) \subset ex\mathcal{G}_{\mathbf{\Theta}}\gamma^{J,h}.$$

However, extreme homogeneous Gibbs states (pure states) do not necessarily have to be homogeneous extreme Gibbs states (homogeneous phases). Thus, we do not necessarily obtain the set of all pure states.

The results in this section rely on Fourier analysis. Thus we need some further notation. However, we do not go much into detail. Let now $\mathbf{K} = \{z \in \mathbb{C} : |z| = 1\}$. Then, set $\mathbf{G} = \mathbf{K}^d$.

Definition 5.24. ([Georgii, 88], p. 274) *Let $J : \mathbf{S} \to \mathbb{R}$ be such that*

$$\sum_{i \in \mathbf{S}} |J(i)| < \infty. \tag{5.8}$$

The Fourier transform *$\hat{J} : \mathbf{G} \to \mathbb{C}$ of J is given by*

$$\hat{J}(z) := \sum_{i \in \mathbf{S}} z^i J(i)$$

for all $z \in \mathbf{G}$, where $z^i := (z_1^{i_1}, \dots, z_d^{i_d})$ for $i = (i_1, \dots, i_d) \in \mathbf{S}$ and $z = (z_1, \dots, z_d) \in \mathbf{G}$.

Remark 5.25. ([Georgii, 88], p. 274) *Let J be even. Then J is positive definite if and only if $\hat{J} \geq 0$ and there exists $z \in \mathbf{G}$ such that $\hat{J}(z) \neq 0$.*

We can now turn to the main result in this section. It provides a necessary and sufficient condition for existence of Gibbs measures. However, our main interest is in the explicit characterization of homogeneous Gibbs states.

Proposition 5.26. ([Georgii, 88], Theorem 13.36) *Let $\mathbf{S} = \mathbb{Z}^d$, $h \in \mathbb{R}$ and $J : \mathbf{S} \to \mathbb{R}$ be a positive definite even function satisfying $\sum_{i \in \mathbf{S}} |J(i)| < \infty$. Then $\mathcal{G}(\gamma^{J,h}) \neq 0$ if and only if $\mathbf{M}_{J,h} \neq 0$ and*

$$\int_{\mathbf{G}} \hat{J}(z)^{-1} dz < \infty,$$

where dz denotes the image measure on $\mathbf{G}$ of the normalized Lebesgue measure $\tilde{\lambda}$ on $]-1,+1]^d$ with respect to the mapping

$$]-1,+1] \ni p = (p_1, \dots, p_d) \mapsto z_p := (e^{\iota\pi p_1}, \dots, e^{\iota\pi p_d}) \in \mathbf{G},$$

i.e. $dz = z_p(\tilde{\lambda})(dp)$. In this case we have

$$\mathcal{G}(\gamma^{J,h}) = \{\mu_C * \nu : \nu \in \mathcal{P}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1\}$$

as in Proposition 5.15 and

$$\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h}) = \{\mu_C * \nu : \nu \in \mathcal{P}_{\mathbf{\Theta}}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1\},$$

where μ_C denotes the unique centered Gauss field with covariance function

$$C(i,j) = \int_{\mathbf{G}} z^{j-i} \hat{J}(z)^{-1} dz \quad \forall i,j \in \mathbf{S}.$$

Proof. [Georgii, 88], pp. 275,276, shows that $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$. The characterization of Gibbs states then follows as in Proposition 5.15.

The characterization of $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h})$ is shown immediately: $\mu_C * \nu \in \mathcal{G}(\gamma^{J,h})$ is homogeneous if and only if $\nu \in \mathcal{P}_{\mathbf{\Theta}}(\Omega, \mathcal{F})$. This equivalence directly follows from Corollary 2.41 and Note 5.17. □

Remark 5.27. ([Georgii, 88], Remark 13.39) *Let $h \in \mathbb{R}$ and $J : \mathbf{S} \to \mathbb{R}$ be a positive definite even function exhibiting finite range. Then $\mathbf{M}_{J,h} \neq \emptyset$. Assuming there exists $i \neq 0$ such that $J(i) \neq 0$, then $\mathbf{M}_{J,h}$ is uncountable.*

Proof. [Georgii, 88], pp. 276,277 □

We now proof a remark given in [Georgii, 88].

Remark 5.28. ([Georgii, 88], p. 277) *Let $J : \mathbf{S} \to \mathbb{R}$ be even, positive definite and exhibit finite range. Then we have three distinct cases:*

1. $\mathcal{G}(\gamma^{J,h}) = \emptyset$,
2. $|\mathcal{G}(\gamma^{J,h})| = 1$,
3. $ex\mathcal{G}(\gamma^{J,h})$ *is uncountable.*

Proof. Remark 5.27 shows $\mathbf{M}_{J,h} \neq \emptyset$. By Theorem 5.26, the first case occurs whenever $\hat{J}^{-1}$ is not integrable, i.e. $\int_{\mathbf{G}} \hat{J}(z)^{-1} dz = \infty$.

Second, assume $\mathbf{M}_{J,h}$ to be countable. Then, by Remark 5.27, $J(i) = 0$ for all $i \neq 0$, i.e. there is no interaction. Since $\mathbf{M}_{J,h} = m + \mathbf{M}_{J,0}$ for each $m \in \mathbf{M}_{J,h}$, $\mathbf{M}_{J,0}$ is countable, and thus, as a linear subspace of Ω, we obtain $\mathbf{M}_{J,0} = \{0\}$. From the proof of Proposition 5.15, we infer that $|ex\mathcal{G}(\gamma^{J,0})| = 1$ and hence $|\mathcal{G}(\gamma^{J,0})| = 1$. Remark 5.11(b) yields $|\mathcal{G}(\gamma^{J,h})| = 1$.

The last case occurs whenever J is non-trivial, i.e. there actually exists interaction among agents, and $\hat{J}^{-1}$ is integrable. Remark 5.27 shows that $\mathbf{M}_{J,h}$ is uncountable in this case. But then, since $\mathcal{G}(\gamma^{J,h}) \neq \emptyset$, we obtain $ex\mathcal{G}(\gamma^{J,h})$ to be uncountable by Remark 5.11(d). □

As in the Ising economy in Section 3.5, we set $h = 0$. The following three remarks that can be found within the lines of [Georgii, 88], p 278, motivate this assumption of a vanishing external field.

Remark 5.29. *Let $h \neq 0$ and $\hat{J}(1) = 0$. Then $\mathbf{M}_{J,h}$ does not contain any constant element.*

Proof. Assume there exists $m \in \mathbf{M}_{J,h}$ and $c \in \mathbb{R}$ such that $m_i = c$ for all $i \in \mathbf{S}$. Then for all $i \in \mathbf{S}$

$$\begin{aligned} 0 &= h_i + \sum_{j \in \mathbf{S}} J(i-j) m_j \\ &= h_i + c \underbrace{\sum_{k \in \mathbf{S}} J(k)}_{=\hat{J}(1)=0} . \end{aligned}$$

Thus, $h_i = 0$ for all $i \in \mathbf{S}$. This contradicts $h \neq 0$. □

Remark 5.30. *Let $h \neq 0$ and $\hat{J}(1) = 0$. In this case, there does not exist any $\mu \in \mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h})$ such that $\mu(|\sigma_0|) < \infty$.*

Proof. Taking into account the result of Remark 5.29, Georgii refers to the proof of Proposition 5.8, [Georgii, 88], pp. 266,267. □

Remark 5.31. *Coupling J of the harmonic oscillator satisfies $\hat{J}(1) = 0$:*

$$\hat{J}(z) = \sum_{i \in \mathbf{S}} z^i J(i) \bigg|_{z=1} = \sum_{i \in \mathbf{S}} J(i) = \beta - 2d \frac{\beta}{2d} = 0.$$

Thus, when considering the harmonic oscillator, in light of Remarks 5.29 and 5.30, it makes sense to confine ourselves to $h = 0$.

Roots of $\hat{J}$ on $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,h})$ exhibit even stronger consequences when $h = 0$:

Corollary 5.32 (of Proposition 5.26)**.** ([Georgii, 88], Corollary 13.40) *Let $J : \mathbf{S} \to \mathbb{R}$ be even, positive definite and absolutely summable in the sense of equation (5.8). If $\hat{J}^{-1}$ is integrable and there exists some $z \in \mathbf{G}$ such that $\hat{J}(z) = 0$, then $\mathrm{ex}\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0})$ contains uncountably many Gibbs measures with finite second moment, i.e. $\mu(\sigma_i^2) < \infty$ for all $i \in \mathbf{S}$. Conversely, if $\hat{J}$ has no root in $\mathbf{G}$ then μ_C is the only element of $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0})$ with finite second moment.*

Proof. [Georgii, 88], pp. 278,279 □

We again consider the harmonic oscillator but restate the homogeneous coupling function J in terms of the corresponding even function $\tilde{J}$, denoted by J, too. We have already motivated to assume $h = 0$.

Example 5.33. ([Georgii, 88], pp. 281,282) *Let $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$, $\beta > 0$ and*

$$J(i) = \begin{cases} -\ \frac{\beta}{2d} & \text{if } |i| = 1, \\ \quad \beta & \text{if } i = 0, \\ \quad 0 & \text{else.} \end{cases}$$

We have already mentioned that the harmonic oscillator can be seen as a generalization of the Ising ferromagnet: By Proposition 5.6 we can show that $\gamma_\Lambda^{J,0} = \rho_\Lambda \lambda_\Lambda$*, where*

$$\rho_\Lambda(\omega) = Z_\Lambda^{J,0}(\omega)^{-1} \exp\left[-\frac{\beta}{4d} \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ |i-j|=1}} (\omega_i - \omega_j)^2 \right]$$

for all $\Lambda \in \mathcal{S}$ *and* $\omega \in \Omega = \Omega_J$*: Using Proposition 5.6 we obtain for* $A \in \mathcal{F}$

$$\begin{aligned}
\int_A \rho_\Lambda(x)\lambda_\Lambda(dx|\omega) &\overset{(2.2)}{=} \rho_\Lambda\lambda_\Lambda(A|\omega) \\
&= \gamma_\Lambda^{J,0}(A|\omega) \\
&\overset{5.6}{=} Z_\Lambda^{J,0}(\omega)^{-1}\lambda_\Lambda(1_A e^{-H_\Lambda^{J,0}}|\omega) \\
&= Z_\Lambda^{J,0}(\omega)^{-1}\int_{\Omega_\Lambda} 1_A(x) e^{-H_\Lambda^{J,0}(x)}\lambda_\Lambda(dx|\omega) \\
&= Z_\Lambda^{J,0}(\omega)^{-1}\int_A e^{-H_\Lambda^{J,0}(x)}\lambda_\Lambda(dx|\omega).
\end{aligned}$$

Thus,

$$\rho_\Lambda(x) = Z_\Lambda^{J,0}(\omega)^{-1} e^{-H_\Lambda^{J,0}(x)}.$$

Moreover, we have

$$\Phi_A^{J,0}(\omega) = \begin{cases} \frac{1}{2}\beta\omega_i^2 & \text{if } A = \{i\}, \\ -\frac{1}{2d}\beta\omega_i\omega_j & \text{if } A = \{i,j\},\ |i-j| = 1, \\ 0 & \text{else.} \end{cases}$$

by equation (5.1) and the assumption of a vanishing external field, i.e. $h = 0$*. For* $\mathbf{E} = \{-1, +1\}$*, this potential is equivalent to the Ising potential given for the Ising ferromagnet in Section 2.7: On first view, the main difference is that the potential for the Ising ferromagnet was given just by a two-body-potential and here, we also have a self-potential. Moreover, we have multiplied the potential by* $\frac{\beta}{2d}$*. However, for* $\mathbf{E} = \{-1, +1\}$, $\frac{1}{2}\beta\omega_i^2 = \frac{\beta}{2}$ *and thus constant. This shows the equivalence of these potentials and hence, the coincidence of corresponding sets of Gibbs states.*

However, here $\mathbf{E} = \mathbb{R}$*. The Hamiltonian is obtained as*

$$\begin{aligned}
H_\Lambda^{J,0}(\omega) &= \sum_{\substack{A\in\mathcal{S}\\ A\cap\Lambda\neq\emptyset}} \Phi_A^{J,0}(\omega) \\
&= \frac{\beta}{2}\sum_{i\in\Lambda}\omega_i^2 - \frac{\beta}{2d}\sum_{\substack{\{i,j\}\subset\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} \omega_i\omega_j \\
&= \frac{\beta}{2}\sum_{i\in\Lambda}\omega_i^2 - \frac{\beta}{4d}\sum_{\substack{i,j\in\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} \omega_i\omega_j \\
&= \frac{\beta}{4}\sum_{i\in\Lambda}\omega_i^2 + \frac{\beta}{4}\sum_{j\in\Lambda}\omega_j^2 - \frac{\beta}{8d}\sum_{\substack{i,j\in\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} 2\omega_i\omega_j \\
&= \frac{\beta}{8d}\left(\sum_{i\in\Lambda}2d\omega_i^2 + \sum_{j\in\Lambda}2d\omega_j^2 - \sum_{\substack{i,j\in\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} 2\omega_i\omega_j\right) \\
&= \frac{\beta}{8d}\sum_{\substack{i,j\in\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} (\omega_i-\omega_j)^2 \\
&= \frac{\beta}{4d}\sum_{\substack{\{i,j\}\subset\mathbf{S}:\\ \{i,j\}\cap\Lambda\neq\emptyset,\\ |i-j|=1}} (\omega_i-\omega_j)^2.
\end{aligned}$$

Thus, the present model coincides with "Shlosman's stair case model" as defined in [Georgii, 88], chapter 6.3, if we would set $\mathbf{E} = \mathbb{Z}$ *and* λ *the counting measure. Considering the above, the harmonic oscillator may be seen as a generalization of the Ising ferromagnet in Section 2.7.*

Having a look at the Hamiltonian, we see that energy in the system is minimal when spins align.[6] *Thus, in economic terms, a model based on the harmonic oscillator exhibits preference for conformity among agents. Consequently, an economy based on the harmonic oscillator would generalize Föllmer's Ising economy.*

We now could use some results from Fourier analysis and in particular Proposition 5.26 to show that $\mathcal{G}(\gamma^{J,0}) \neq \emptyset$ *if and only if* $d \geq 3$*: Obviously,* J

[6] In section 2.7, we have motivated the connection between Gibbs states and states of minimal energy.

is absolutely summable, i.e. $\sum_{i \in \mathbf{S}} J(i) < \infty$. *We have shown that* $\mathbf{M}_{J,0} \neq \emptyset$. *We can furthermore show that* $\hat{J}^{-1}$ *is integrable for* $d \geq 3$. *Then Theorem 5.26 implies that* $\mathcal{G}(\gamma^{J,0}) \neq \emptyset$. *Although the proof is quite elegant, we have already shown the assertion in Example 5.20.*

We obtain that $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0}) \cap ex\mathcal{G}(\gamma^{J,0})$ *contains uncountably many Gibbs states breaking the symmetry* τ^{m^t}, $t \in \mathbb{R}$, *of* $\gamma^{J,0}$: *As in Example 5.20 we have that* $\mathbf{M}_{J,0}$ *contains all constant configurations. From Proposition 5.26 we obtain a characterization of* $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0})$ *and from the proof of Proposition 5.15 one of* $ex\mathcal{G}(\gamma^{J,0})$. *We see that uncountably many Gaussian fields belong to* $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0}) \cap ex\mathcal{G}(\gamma^{J,0})$, *namely all* $\mu_C * \delta_m$, *with* m *a constant configuration; the Dirac measure with mass on a constant configuration is homogeneous. However, the characterization of* $\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0}) \cap ex\mathcal{G}(\gamma^{J,0})$ *will explicitly be shown in the next chapter. As shown in Example 5.20, the symmetries* τ^{m^t}, $t \in \mathbb{R}$, *for* $\gamma^{J,0}$ *are broken for Gaussian fields in* $\mathcal{G}(\gamma^{J,0})$ *if and only if* $t \neq 0$.

We have obtained the tools to introduce an economic model taking into account locally interacting agents with an unbounded and uncountable state space.

5.3 Modeling Local Interactions

In this this section we introduce a local random economy $\mathscr{E}^{lr}$ generalizing the model in Section 3.5 with respect to the space of individual agents' states. Within this new model, we endogenize the distribution of α-transforms that is given exogenously in the distribution economy introduced in [Grandmont, 92]. Grandmont suggests:

> An important issue to investigate would then be how such macroeconomic distributions might arise endogenously from specific socioeconomic interactive processes at the micro-level. One could for instance envision a more 'adaptive' viewpoint [...] in which the decision rule [...] of an individual [...] (is) influenced in a stochastic (Markovian) fashion by those (decision rules) of his immediate neighbor(s), and generate endogenously a macroeconomic distribution by looking for invariant distributions. ([Grandmont, 92], p.40)

In formalizing these lines, we use the concepts and insights from Gibbsian theory by appropriately generalizing Föllmer's model.

The Baseline Model

To carry out an analysis of market demand when the global distribution of individual demand functions is endogenized, we adopt the notion of a Gibbsian random (pure exchange) economy from [Hohnisch, 03]. As a global random economy, this concept turns out to exhibit a macroeconomic character.

To explicitly describe how the global distribution of demand arises by virtue of a local interaction structure, we introduce the notion of a *local unbounded spin Ising economy* as a generalization of Föllmer's Ising economy when the state space is given by $\mathbb{R}^l$. This type of economy is an application of the harmonic oscillator in Example 5.20. As in [Föllmer, 74] the model is purely microeconomic in the sense that only local data is needed to completely determine the economy.

The procedure is the following: First, we specify a local random economy, i.e. basically a configuration space with specification. Our special type of economy will be called the *local unbounded spin Ising economy* and is given by the state space $\mathbf{E} = \mathbb{R}^l$ and a family of specifications each with spin space $\mathbb{R}$ and determined as for the harmonic oscillator. In this sense the local unbounded spin Ising economy turns out to be a so called *l-fold local random economy.* We then specify a resulting global random economy, i.e. a probability measure on $\mathbf{E}^{\mathbf{S}}$. Here, we consider Gibbsian random economies: the probability distribution is given by a Gibbs state with respect to the specification of the corresponding local random economy. In this sense, the Gibbsian random economy corresponding to the local unbounded spin Ising economy is an *l-fold Gibbsian random economy* as the emerging distribution is Gibbsian with respect to a product specification, where the factors are the specifications in the local unbounded spin Ising economy.

In order to apply Grandmont's analysis, we have to find a way to link global random economies with distribution economies. In Chapter 6 we encounter a solution to this problem.

Local Random Economies

Again, we assume an arbitrary underlying probability space $(\Omega, \mathcal{F}, P)$.

We first specify the primitives needed for a Gibbsian approach to locally interacting agents when individual states or characteristics are given by demand-income pairs: Let the parameter set $\mathbf{S} := \mathbb{Z}^d$ be the set of economic agents.

In Grandmont's setup each agent is determined by some individual demand function $\xi : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ and income $w \in \mathbb{R}_{++}$. Thus,

$$\mathbf{E} = \{(\xi, w) | \xi : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+, \quad (p, w) \mapsto \xi(p, w), \quad w \in \mathbb{R}_{++}\}$$

would make sense as state space but exhibits insufficient structure for our attempt. As in [Grandmont, 92], we consider the equivalence class of α-transforms for a given demand function ξ, a distribution on which is equivalently given by a distribution on $\mathbb{R}^l$. As already mentioned in the introduction to this chapter, we assume an underlying demand function for the economy. This reflects some kind of consensus on a specific consumption pattern within society. We furthermore assume a fixed underlying type, i.e. not only a demand function but also a fixed income $w \in \mathbb{R}_{++}$.

Thus, since income is fixed and the set of all possible demand functions may now be identified with the set of α-transforms of the underlying demand, we set

$$\mathbf{E} = \mathbb{R}^l, \quad \mathcal{E} = \mathcal{B}(\mathbb{R}^l). \tag{5.9}$$

Definition 5.34. *A* random state *σ_s of agent $s \in \mathbf{S}$ is a random variable σ_s on $(\Omega, \mathcal{F}, P)$ with values in $(\mathbf{E}, \mathcal{E})$. We assume the random variable σ_s to be integrable with respect to P. Since an agent is entirely characterized by her state, we also call σ_s a* random agent*.*

Here, we always consider $\mathbf{E}$ as defined above. Thus, $\sigma_s(\omega)$ corresponds to some α-transform of the underlying demand function. However, the (economic) concepts introduced in this chapter are applicable to all kinds of spin spaces, as for example spaces of preferences, utility functions or sites of residence, among many others.

Remark 5.35. *As in Chapter 2, we use the following canonical representation:*

$$\Omega = \mathbf{E}^{\mathbf{S}} = \{\omega = (\omega_s)_{s \in \mathbf{S}} : \omega_s \in \mathbf{E}\},$$
$$\mathcal{F} = \mathcal{E}^{\mathbf{S}},$$
$$\sigma_s : \Omega \to \mathbf{E}, \quad \omega \mapsto \sigma_s(\omega) = \omega_s.$$

Ω is called the configuration space, $\mathbf{S}$ the parameter set and $\mathbf{E}$ the (individual) state or spin space. In a deterministic context an economy is given by a configuration $\omega \in \Omega$, i.e. a map $\omega : \mathbf{S} \to \mathbf{E}$.

Following the approach laid out in Chapter 2, the local interaction structure in an economy or social system is given by a specification γ, i.e. by a family $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ of probability kernels satisfying the consistency conditions in Definition 2.6. This methodology endogenizes the global distribution of agents' states; in our context, the distribution of consumption behavior.

The following definition is a general version of Definition 3.22.

Definition 5.36. (a) *A* local random pure exchange economy *$\mathscr{E}^{lr}$ is given by a tuple $\mathscr{E}^{lr} = (\mathbf{E}^{\mathbf{S}}, \gamma)$, where $\mathbf{E}^{\mathbf{S}}$ is the configuration space and γ a specification with state space $\mathbf{E}$ as introduced in Definition 2.6.*
(b) *The set of Gibbs measures $\mathcal{G}(\gamma)$ of a random pure exchange economy $\mathscr{E}^{lr}$ with specification γ is denoted by $\mathcal{G}(\mathscr{E}^{lr})$. An element $\mu \in \mathcal{G}(\mathscr{E}^{lr})$ is called Gibbs state of $\mathscr{E}^{lr}$.*
(c) *We call a local random pure exchange economy $\mathscr{E}^{lr}$* homogeneous *whenever γ is homogeneous. We call $\mathscr{E}^{lr}$* Gaussian *whenever γ is Gaussian.*
(d) *When there is not necessarily an economic meaning of the elements in $\mathbf{E}$, we refer to $\mathscr{E}^{lr}$ as a* local (random) system.

We are now faced with the problem of determining a specification for state space $\mathbf{E} = \mathbb{R}^l$ that is justifiable in economic terms. We want to use this specification to generate distributions of α-transforms, i.e. distributions on $\mathbb{R}^l$. We have obtained explicit characterizations of Gibbs states in case of Gaussian specifications. However, these are only admissible for spin space $\mathbb{R}$. We have already indicated that we obtain the distribution of $\alpha \in \mathbb{R}^l$ as follows: For every $i \leq l$, we consider the distribution μ_i of $\alpha_i \in \mathbb{R}$ given by a Gibbs measure generated by the harmonic oscillator as in Example 5.20, i.e. $\mu_i \in \mathcal{G}(\gamma^{J_i,h_i})$, where the homogeneous Gaussian specification γ^{J_i,h_i} can be inferred from Example 5.20 together with Proposition 5.6. Then, the distribution μ of $\alpha \in \mathbb{R}^l$ is given by the product distribution $\mu = \mu_1 \otimes \ldots \otimes \mu_l$. This implicitly assumes the distributions of α_i's to be mutually independent. Note, that the independence of these marginal distributions does not imply demand to be independent across commodities in any reasonable economic meaning, as e.g. vanishing price effects or the like. It just means that the mechanisms used to obtain the distributions of demand for a commodity are independent.

In the next chapter, we show that this procedure leads to a distribution economy as considered in [Grandmont, 92], where the distribution of demand is obtained endogenously.

To recall: Let $\gamma^{J,h}$ be a specification as given in Proposition 5.6. There, we have shown that $\gamma^{J,h}$ is Gaussian if J is positive definite and symmetric. In this case, the potential $\Phi^{J,h}$, as defined in (5.1) and generating $\gamma^{J,h}$, is called Gaussian, too. Moreover, the Gaussian specification $\gamma^{J,h}$ was shown to be homogeneous if and only if J and h are homogeneous. Henceforth, we assume $\gamma^{J,h}$ to be a homogeneous Gaussian specification as obtained in Proposition 5.6; Thus, assume the coupling function J to be symmetric and positive definite and, as the external field h, homogeneous

Definition 5.37. *An* l-fold local random pure exchange economy *is a tuple* $\mathscr{E}^{llr} = (\mathbf{E}^{\mathbf{S}}, \gamma_1, \ldots, \gamma_l)$*, where* $\mathbf{E} = \tilde{\mathbf{E}}_1 \times \ldots \times \tilde{\mathbf{E}}_l$ *and* $((\tilde{\mathbf{E}}_k)^{\mathbf{S}}, \gamma_k)$*,* $k = 1, \ldots, l$*, is a local system, i.e.* γ_k *is a specification with state space* $\tilde{\mathbf{E}}_k$*.* $\mathscr{E}^{llr}$ *is called (homogeneous) Gaussian whenever the family of specifications* $(\gamma_k)_{k=1,\ldots,l}$ *is a family of (homogeneous) Gaussian specifications. When* $\gamma_1 = \ldots = \gamma_l =: \gamma$ *and thus* $\tilde{\mathbf{E}}_1 = \ldots = \tilde{\mathbf{E}}_l$*, we write* $\mathscr{E}^{llr} = (\mathbf{E}^{\mathbf{S}}, \gamma)$*.*

As already mentioned, our framework leaves the opportunity to consider distinct mechanisms to obtain a distribution for the transform α_k of each commodity k: One may use the harmonic oscillator for the distribution of demand for the first good to indicate a preference for conformity in this commodity. The transform for second good may be represented by a preference for antagonism and for a further good independence among agents. However, in our approach we assume preference for conformity among agents in all goods.

The following definition is an application of the harmonic oscillator.

Definition 5.38. *Let* $\mathbf{E} = \mathbb{R}^l$, $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$. *A homogeneous Gaussian* l*-fold local random pure exchange economy* $\mathscr{E}^{llr} = (\mathbf{E}^{\mathbb{Z}^d}, \gamma^{J_{\beta_1}, h_1}, \ldots, \gamma^{J_{\beta_l}, h_l})$ *is called a* local unbounded spin Ising economy *if the homogeneous Gaussian specification* $\gamma^{J_{\beta_k}, h_k}$ *is generated by* $\Phi^{J_{\beta_k}, h_k}$*, given in equation (5.1), as in Proposition 5.6, where* $h_k \in \mathbb{R}^{\mathbb{Z}^d}$*, constant, and*

$$J_{\beta_k}(i,j) = \begin{cases} -\ \frac{\beta_k}{2d} & \text{if } |i-j| = 1, \\ \beta_k & \text{if } i = j, \\ 0 & \text{else} \end{cases}$$

for some constants $\beta_k > 0$, $k \leq l$. *If* $\beta_k = \beta$ *for some* $\beta > 0$ *and* $h_k = h$ *for some* $h \in \mathbb{R}^{\mathbb{Z}^d}$*, constant, for all* $k \leq l$*, we write* $\mathscr{E}^{\beta,h} := (\mathbf{E}^{\mathbb{Z}^d}, \gamma^{J_\beta, h})$.

In other words, the local unbounded spin Ising economy is an "l-fold harmonic oscillator" where each spin in $\mathbb{R}^l$ represents an α-transform of the underlying demand function and thus represents some demand function. For the analysis of this type of random economy we may apply the results obtained in Section 5.2. Henceforth, as in Section 3.5, we assume $(h_k)_i = 0$ for all $i \in \mathbf{S}$ and $k \leq l$, i.e. the "grand external field" vanishes.

Global Random Economies

So far, we have established a microeconomic framework for the analysis of macroeconomic variables as market demand. Using this microeconomic concept, we can determine the set of Gibbs states that may emerge. In view of l-fold local random economies, we may consider the product specification as introduced in Proposition 2.7 and then consider the corresponding Gibbs states as obtained in Proposition 2.8. Thus, assuming the emergence of a specific Gibbs state, we may evaluate macroscopic observables. The following definition is the economic analogon to Definition 2.1.

Definition 5.39. *Given a probability space* $(\Omega, \mathcal{F}, P)$*, a* global random (pure exchange) economy $\mathscr{E}^{gr}$ *is given by a family of random variables* $\sigma := (\sigma_s)_{s\in\mathbf{S}}$ *on* $(\Omega, \mathcal{F}, P)$ *with values in* $(\mathbf{E}, \mathcal{E})$. *The distribution of* σ *is denoted by* $\mu \in \mathcal{P}(\mathbf{E}^{\mathbf{S}}, \mathcal{E}^{\mathbf{S}})$.[7]

Applying the canonical representation in Remark 5.35, a global random economy $\mathscr{E}^{gr}$ is equivalently defined by a probability measure μ on $(\mathbf{E}^{\mathbf{S}}, \mathcal{E}^{\mathbf{S}})$.

Recall, we denote probability measure μ ergodic if it is ergodic with respect to the lattice shift group $\mathbf{\Theta}$ on $\mathbf{S} = \mathbb{Z}^d$.

Definition 5.40. (a) ([Hohnisch, 03], Definition 1) *A* Gibbsian random (pure exchange) economy *with specification* γ *is a global random (pure exchange) economy* $\sigma := (\sigma_s)_{s\in\mathbf{S}}$*, the distribution* μ *of which is a Gibbs measure with respect to* γ. *Thus, a Gibbsian random economy is a family* $(\sigma_s)_{s\in\mathbf{S}}$

[7] Again, we assume each σ_s to be "sufficiently integrable".

with distribution $\mu \in \mathcal{G}(\mathscr{E}^{lr})$, where $\mathscr{E}^{lr} = (\mathbf{E}^{\mathbf{S}}, \gamma)$ is called the corresponding *local random economy.*
(b) *A Gibbsian random economy is called* homogeneous *if the ruling Gibbs measure μ is homogeneous. The economy is called* ergodic *if μ is ergodic.*

A Gibbsian random economy with specification γ is equivalently defined by a probability measure μ on $(\mathbf{E}^{\mathbf{S}}, \mathcal{E}^{\mathbf{S}})$, where $\mu \in \mathcal{G}(\gamma)$.

If γ is such that $|\mathcal{G}(\gamma)| = 1$, Definition 5.40(a) is again of microeconomic character since no knowledge of aggregates is necessary to determine the economy. On the other hand, if we do not have uniqueness of Gibbs measures, i.e. $\mathcal{G}(\gamma)$ is not singleton, γ does not uniquely determine the Gibbs measure that emerges as the ruling distribution for $(\sigma_s)_{s\in\mathbf{S}}$. Stated another way, a local random economy does not uniquely specify a global random economy. Then we need some knowledge of the aggregate in order to determine the emerging Gibbs state and thus the Gibbsian random economy. In this sense, the concept is not purely microeconomic and thus, we call it macroeconomic.

Definition 5.41. (a) *An l*-fold Gibbsian random (pure exchange) economy *with a family of specifications $(\gamma_k)_{k=1,\dots,l}$ is a global random pure exchange economy $\sigma = (\sigma_i)_{i\in\mathbf{S}}$ on $(\Omega, \mathcal{F}, P)$ with values in $\mathbf{E} = \tilde{\mathbf{E}}_1 \otimes \ldots \otimes \tilde{\mathbf{E}}_l$, the distribution μ of which is of product form $\mu := \mu_1 \otimes \ldots \otimes \mu_l$, where $\mu_k \in \mathcal{G}(\gamma^k)$; γ_k is a specification with state space $\tilde{\mathbf{E}}_k$.*
(b) *A* global unbounded spin Ising economy *is an l-fold Gibbsian random (pure exchange) economy where $\mathbf{E} = \mathbb{R}^l$ and $\gamma^k = \gamma^{J_{\beta_k}, h_k}$ is given as in Definition 5.38.*

Remark 5.42. *By virtue of Propositions 2.7 and 2.8 an l-fold Gibbsian random economy is a Gibbsian random economy with product specification $\gamma = \gamma_1 \otimes \ldots \otimes \gamma_l$, where the distribution of $(\sigma_s)_{s\in\mathbf{S}}$ is of the form $\mu = \mu_1 \otimes \ldots \otimes \mu_l \in \mathcal{G}(\gamma)$.*

Given the l-fold local random economy $\mathscr{E}^{llr} = (\mathbf{E}^{\mathbf{S}}, \gamma_1, \ldots, \gamma_l)$, $\mathbf{E} = \tilde{\mathbf{E}}_1 \otimes \ldots \otimes \tilde{\mathbf{E}}_l$, $\mu \in \mathcal{G}(\gamma_1 \otimes \ldots \otimes \gamma_l)$ defines an (l-fold) Gibbsian random economy. Then we call $\mathscr{E}^{llr}$ the corresponding l-fold local random economy for a Gibbsian random economy ruled by μ, vice versa.

In this token, a global unbounded spin Ising economy is just a global random economy that corresponds to the local unbounded spin Ising economy.

Summary

Based on the notion of a random economy, we have introduced several new economic concepts in this section. To keep account of these concepts, let us summarize the notions used in the next chapter: Let $\mathbf{S}$ be a parameter set, $l \in \mathbb{N}$, $(\tilde{\mathbf{E}}_k)_{k\le l}$ a family of state spaces, $(\gamma_k)_{k\le l}$ a family of specifications, each γ_k with state space $\tilde{\mathbf{E}}_k$. Furthermore let γ be a specification with state space $\mathbf{E}$.

Then we have defined the *local random economy* $\mathscr{E}^{lr}$ by $(\mathbf{E}^{\mathbf{S}}, \gamma)$ and a corresponding *Gibbsian random economy* by $\mu \in \mathcal{G}(\gamma)$. A Gibbsian random economy is a specific *global random economy.*

Furthermore, we have defined the *l-fold local random economy* $\mathscr{E}^{llr}$ by $(\mathbf{E}^{\mathbf{S}}, \gamma_1, \ldots \gamma_l)$, $\mathbf{E} = \tilde{\mathbf{E}}_1 \times \ldots \times \tilde{\mathbf{E}}_l$, and a corresponding *(l-fold) Gibbsian random economy* by $\mu = \mu_1 \otimes \ldots \otimes \mu_l$, where $\mu_k \in \mathcal{G}(\gamma_k)$, $k \leq l$. Thus, $\mu \in \mathcal{G}(\gamma_1 \otimes \ldots \otimes \gamma_l)$ and this in turn yields a Gibbsian random economy.

In our analysis in the next chapter we consider a special class of local and global economies: the local unbounded spin Ising economy and the global unbounded spin Ising economy. Here, the parameter set $\mathbf{S}$ was assumed to be $\mathbb{Z}^d$, $d \geq 1$, and the state space $\mathbf{E} = \mathbb{R}^l$. Interaction is given by an l-fold harmonic oscillator to take into account preference for conformity. In view of [Grandmont, 92], a probability distribution on $\mathbf{E}$ is a distribution on the equivalence class of α-transforms for an underlying type; in this sense, a distribution on the space of admissible demand functions. Thus, the global unbounded spin Ising economy is a global random economy specified by an overall distribution of demand functions for all agents in $\mathbf{S}$, that is consistent with the specifications in the corresponding local unbounded spin Ising economy when there is assumed an underlying type of agent. In this sense, the global unbounded spin Ising economy is the “random analogon” of the distribution economy in [Grandmont, 92], when only one equivalence class of α-transforms is admissible.

For a further analysis of market demand in spirit of Grandmont, we need a result combining Gibbsian random economies and distribution economies. Such a result is given in [Hohnisch, 03] in terms of a convergence theorem for empirical distributions of ergodic Gibbsian random economies.

Chapter 6

Analysis of Market Demand

6.1 Introduction

In the last chapter, we have defined the concept under consideration: The global unbounded spin Ising economy. Given an underlying type $a = (b, w) \in \mathbf{A}$ as in Section 4.2, $\mathbf{E} = \mathbb{R}^l$ represents the space of all individual demand functions $\xi_b^\alpha : \mathbb{R}_{++}^l \times \mathbb{R}_{++} \to \mathbb{R}^l$ obtained from ξ_b by an α-transform with $\alpha \in \mathbf{E}$, and these are the admissible ones. By this token, the unbounded spin Ising economy allows to endogenize the distribution of α-transforms that is given exogenously in [Grandmont, 92]. To conclude, the global unbounded spin Ising economy is an overall distribution on demand, whenever the admissible demand functions are those obtained from an underlying demand by α-transforms.

However, we have to be very diligent with the different economic concepts introduced so far: In Section 5.3 we consider (Gibbsian) global random economies, i.e. families $(\sigma_s)_{s\in\mathbf{S}}$ of random variables with state space $(\mathbf{E}, \mathcal{E})$ distributed by $\mu \in \mathcal{P}(\mathbf{E}^\mathbf{S}, \mathcal{E}^\mathbf{S})$. In [Grandmont, 92], we are faced with a distribution economy, i.e. a probability distribution $\nu \in \mathcal{P}(\mathbf{E}, \mathcal{E})$.

For a distribution economy, an observable is introduced as a bounded continuous function on $\mathbf{E}$. As argued in Section 2.5, in context of a global random economy, a (macroeconomic) observable is given by a tail-measurable function on Ω. Throughout, we assume the canonical representation in Remark 5.35.

When trying to apply Grandmont's analysis to our set-up, we have to explore the linkage between random and distribution economies. In case of independent and identically distributed random agents σ_s, a convergence result based on the law of large numbers is given in Proposition 3.19. In a more general case, an elegant way is presented in [Hohnisch, 03]. The proof is based on a multidimensional ergodic theorem, an elaborate discussion of which can be found in [Georgii, 88], pp. 302-307. An ergodic theorem usually comes along with the catchphrase "time average equals space average". However, in our context it should be rephrased as "average over agents (parameter average) equals space average".

The chapter is structured as follows: First, we encounter the relation between random and distribution economies. Then, we consider a specific class of global unbounded spin Ising economies and generate the corresponding distribution economies for which we mimic Grandmont's analysis. The question at hand is: Do we obtain enough behavioral heterogeneity when assuming particular Gaussian specifications to endogenize consumption behavior that imply preference for conformity? At first glance, this seems to be quite paradox. However, in Section 5.2 we have seen an explicit characterization of (ergodic extremal) Gibbs states induced by Gaussian specifications.

Moreover, we will have a look at another approach to demand analysis related to the "low temperature limit" as already discussed in Section 2.7. For this thermodynamic limit, we see that the strong result in [Grandmont, 92], where the weak axiom is actually "created", cannot be obtained, but the weak axiom is "inherited" by market demand.

6.2 Ergodic Global Random Economies

This section is actually tailored for a more general state space than $\mathbf{E} = \mathbb{R}^l$. Here, we assume $\mathbf{E}$ to be a polish space and $\mathbf{S} = \mathbb{Z}^d$.

A distribution economy $\nu \in \mathcal{P}(\mathbf{E}, \mathcal{E})$ generates a global random economy, when considering random agents σ_s, $s \in \mathbf{S}$, that are independent and identically distributed with law ν. The question at hand is whether we can construct a corresponding distribution economy for a given global random economy. The result in [Hohnisch, 03] is motivated as follows:

> As far as aggregate variables of the economy [...] are concerned, any infinite random exchange economy with converging limiting empirical distribution ν is equivalent to a Hildenbrand distribution economy characterized by the same distribution ν. [...] The linkage between a distribution economy and an ergodic random exchange economy [...] is provided by a convergence result for the empirical distribution [...]. ([Hohnisch, 03], pp. 2,3)

Convergence of the empirical distribution for ergodic random economies is obtained by means of a multidimensional ergodic theorem.

Recall that a homogeneous distribution μ on $(\mathbf{E}^{\mathbb{Z}^d}, \mathcal{E}^{\mathbb{Z}^d})$ is called ergodic if it is trivial on the σ-algebra of lattice-shift-invariant events.

We say a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of finite volumes in $\mathcal{S}$ *exhausts* $\mathbf{S}$, if $\Lambda_n \subset \Lambda_{n+1}$, and $\bigcup_{n\in\mathbb{N}} \Lambda_n = \mathbf{S}$.[1] The following ergodic theorem is a version of Theorem 14.A8 in [Georgii, 88].

[1] Proposition 6.1 is stated for sequences of cubes in [Georgii, 88]. However, it is stated that the result still holds for sequences of arbitrary finite volumes.

Proposition 6.1. *Let μ be an ergodic probability measure on $(\Omega, \mathcal{F})$ and $f \in \mathscr{L}^1(\Omega, \mu)$. Then for any sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of finite volumes that exhausts* $\mathbf{S}$*, we have*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{s\in\Lambda_n} f \circ \theta_s = \mu(f|\mathscr{I}) = \mu(f) \quad \mu\text{-a.s.},$$

where $\mathscr{I} := \{A \in \mathcal{F} : \theta_s(A) = A \quad \forall s \in \mathbf{S}\}$ is the σ-algebra of lattice-shift-invariant or homogeneous sets. $\mu(\cdot|\mathscr{I})$ denotes the expectation with respect to μ conditional on $\mathscr{I}$ and $\mu(\cdot)$ the expectation with respect to μ.

Proof. The first equality is shown in [Georgii, 88], pp. 306,307. The second follows immediately since $\mu(f|\mathscr{I})$ is $\mathscr{I}$-measurable and thus μ-a.s. constant. Hence, we obtain for some $c \in \mathbb{R}$

$$\mu(f|\mathscr{I}) = c = \mu(\mu(f|\mathscr{I})) = \mu(f) \quad \mu\text{-a.s.}$$

□

The following definitions are generalized versions of Definitions 2 and 3 in [Hohnisch, 03]. Recall, $\mathbf{S} = \mathbb{Z}^d$, $d \geq 1$.

Definition 6.2. (a) *Let $\Lambda \in \mathcal{S}$ be a finite volume and $\sigma := (\sigma_s)_{s\in\Lambda}$ be a family of random variables on some probability space $(\Omega, \mathcal{F}, P)$ with values in $(\mathbf{E}, \mathcal{E})$. The* empirical distribution *of σ, denoted by $Y_\Lambda(\cdot,\cdot)$, is a random distribution on $(\mathbf{E}, \mathcal{E})$, i.e. a map $Y_\Lambda : \mathcal{E} \times \Omega \to [0, 1]$ which is specified by*

$$Y_\Lambda(B, \cdot) = \frac{1}{|\Lambda|} \sum_{s\in\Lambda} 1_B \circ \sigma_s(\cdot) \quad \forall B \in \mathcal{E}.$$

Note that $Y_\Lambda(\cdot, \omega)$ is a probability distribution on $(\mathbf{E}, \mathcal{E})$ for all $\omega \in \Omega$, i.e. Y_Λ is a distribution-valued random variable.
(b) *Let $\sigma := (\sigma_s)_{s\in\mathbf{S}}$ be a random field on some probability space $(\Omega, \mathcal{F}, P)$ with values in $(\mathbf{E}, \mathcal{E})$. If there exists a probability measure Y on $(\mathbf{E}, \mathcal{E})$ such that for any sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of finite volumes in $\mathcal{S}$ that exhausts $\mathbf{S}$, the sequence of empirical distributions $(Y_{\Lambda_n})_{n\in\mathbb{N}}$, where Y_{Λ_n} is the empirical distribution of the family $(\sigma_s)_{s\in\Lambda_n}$, converges to Y with respect to the weak topology for P-a.e. $\omega \in \Omega$, i.e.*

$$P\{\omega \in \Omega | Y_{\Lambda_n}(\cdot, \omega) \overset{weakly}{\rightarrow} Y(\cdot)\} = 1,$$

then Y is called limiting empirical distribution *of σ (or simply* empirical distribution *of σ).*

The following result is a generalization of Proposition 3.19.

Proposition 6.3. ([Hohnisch, 03], Proposition 2) *Let* $(\sigma_s)_{s\in\mathbf{S}}$ *denote an ergodic Gibbsian random economy with distribution* μ *on* $(\mathbf{E}^{\mathbf{S}}, \mathcal{E}^{\mathbf{S}})$. *Let* ν *denote the marginal distribution* $\sigma_s(\mu) = \mu \circ \sigma_s^{-1}$ *of each* σ_s *on* $(\mathbf{E}, \mathcal{E})$. *Then for any sequence of finite volumes* $(\Lambda_n)_{n\in\mathbb{N}}$ *that exhausts* $\mathbf{S}$ *the limiting empirical distribution* Y *exists and is equal to* ν μ*-a.s. on* Ω.

Proof. The complete proof can be read after in [Hohnisch, 03], pp. 14,15. However, it seems worthwhile to elaborately reconsider the first part of the proof: Note that the marginal distributions $\nu := \mu \circ \sigma_s^{-1}$ of σ_s, $s \in \mathbf{S}$, are identical: Let μ be ergodic, $s \in \mathbb{Z}^d$ and μ_k denote the marginal distribution of σ_k, $\mu_k = \mu \circ \sigma_k^{-1}$, $k \in \mathbb{Z}^d$. By homogeneity of μ we obtain

$$\begin{aligned} & \mu \circ \theta_{-s} = \mu \circ \theta_s^{-1} = \mu \\ \Rightarrow \quad & \mu_{k+s} = \mu \circ (\sigma_k \circ \theta_s)^{-1} = (\mu \circ \theta_{-s}) \circ \sigma_k^{-1} = \mu_k. \end{aligned}$$

Let $g : \mathbf{E} \to \mathbb{R}$ bounded and continuous. Then $g \circ \sigma_s : \Omega \to \mathbb{R}$ is integrable as σ_s is assumed to be. Then, we have

$$\begin{aligned} Y_{\Lambda_n}(g) &= \int_{\mathbf{E}} g dY_{\Lambda_n} \\ &= \frac{1}{|\Lambda_n|} \sum_{s\in\Lambda_n} g \circ \sigma_s = \frac{1}{|\Lambda_n|} \sum_{s\in\Lambda_n} (g \circ \sigma_0) \circ \theta_s \\ &\xrightarrow{6.1} \mu(g \circ \sigma_0) = \int (g \circ \sigma_0) d\mu \\ &= \int g d(\mu \circ \sigma_0^{-1}) = \int g d\nu = \nu(g). \end{aligned}$$

For the rest of the proof, we refer to [Hohnisch, 03]. □

This result only takes care of ergodic Gibbsian random economies. However, as already mentioned in section 3.5:

> Whereas for our present argument we take ergodicity as an assumption, it is argued elsewhere using a dynamical framework (Hohnisch and Kondratiev (2003)) that ergodic Gibbs measures are the only appropriate measures within the class of shift-invariant Gibbs measures to represent equilibrium states of real economic systems. ([Hohnisch, 03], p. 14)

Proposition 6.3 has a deep impact on the analysis of market demand in Gibbsian economies:

> The convergence of the empirical distribution of an ergodic Gibbsian random exchange economy allows to associate to any such economy a Hildenbrand distribution economy [...] in the

> sense that properties of the latter, e.g. equilibrium prices, per-capita aggregate demand [...], could be as well obtained from a random economy with the same distribution ν as limiting empirical distribution. ([Hohnisch, 03], p. 15)

Corollary 6.4. *Given an ergodic Gibbsian random economy $\sigma = (\sigma_s)_{s \in \mathbf{S}}$ with state space $\mathbf{E}$, distribution μ on $(\mathbf{E}^{\mathbf{S}}, \mathcal{E}^{\mathbf{S}})$ and limiting empirical distribution ν. Then the economy is equivalent to a distribution economy ν on $(\mathbf{E}, \mathcal{E})$ in the sense that (per capita) macroeconomic variables of the distribution economy can be approximated by variables of sufficiently large sample populations in the random economy.*

Let us give an example for the statement in Corollary 6.4: Let $\mathbf{E}$ be the space of demand functions ξ. Then (per capita) market demand for the distribution economy $\nu \in \mathcal{P}(\mathbf{E}, \mathcal{E})$ is given by $\nu(\xi) = \int_{\mathbf{E}} \xi d\nu$. Consider now an ergodic Gibbsian random economy with marginal distributions ν. Then by Proposition 6.3, we have $Y_{\Lambda_n}(\xi) \overset{n\to\infty}{\longrightarrow} \nu(\xi)$, where $Y_{\Lambda_n}(\xi) = \int_{\mathbf{E}} \xi dY_{\Lambda_n}$ is the (per capita) market demand of the sample population Λ_n.

The crucial part in the proof of Proposition 6.3 is ergodicity not the Gibbsian property. Thus, we may apply this theorem to all global random economies with ergodic distribution:

Corollary 6.5. *Given an ergodic global random economy σ with distribution μ and limiting empirical distribution ν. Then the economy is equivalent to a distribution economy ν on $(\mathbf{E}, \mathcal{E})$ in the sense of Corollary 6.4.*

Corollary 6.4 allows for the following procedure: First specify a local random economy, as for example the local unbounded spin Ising economy. Have a look at the set of Gibbs states that may emerge. Consider the pure states, i.e. ergodic Gibbs states, and construct a corresponding ergodic Gibbsian random economy, as for example an ergodic global unbounded spin Ising economy. Specify the limiting empirical distribution by means of Proposition 6.3. Apply Grandmont's analysis to the distribution economy determined by this limiting empirical distribution. The local unbounded spin Ising economy allows for the theory of Gaussian fields and we thus obtain explicit characterizations of the resulting distribution economies in terms of mean and covariance.

6.3 The Weak Axiom of Revealed Preference

On first glance, our model does not seem as general as Grandmont's since we assume an underlying type $(b, w) \in \mathbf{A}$ for the economy. Nevertheless, this restriction is not restrictive at all:[2] The distribution of α-transforms within

[2] If b is such that ξ_b is of Cobb-Douglas type, then market demand satisfies all properties we are seeking for, irrespective of the distribution of α-transforms.

the equivalence class of α-transforms for the underlying type is generated endogenously. If we would assume the distribution of types being exogenous as done in [Grandmont, 92], it would not make any difference to the analysis conducted there, as it is stated in [Hildenbrand, 94], p. 46. In other words, when doing an analysis as in [Grandmont, 92], one may without loss of generality consider the one-type-case. This is already motivated by (A6). For our approach we only consider local interactions in demand behavior among agents. Assuming distribution of types being exogenous, if we would allow for more than one type, we would not consider interaction in types and in particular in income among agents.

If we would like to endogenize the distribution of types via local interactions, we would have to come up with an appropriate local interaction structure for types among agents and moreover with an interaction structure between a specific type and corresponding α-transforms for each agent and among the agents. This calls for a more complex interaction structure than we consider here and is beyond the scope of this diploma thesis: In particular, one would first have to answer the question how interaction in income across agents should be modeled and how an individual agent's income interacts with her other economic variables $\alpha_1, \ldots, \alpha_l$.

To summarize: We consider the case $|\mathbf{A}| = 1$ and thus, total market demand equals conditional market demand as introduced in Definition 4.17.

The Unbounded Spin Ising Distribution Economy

We consider the global unbounded spin Ising economy and make use of the notions and results in Section 5.2. We would like to apply Proposition 6.3. Thus, the question at hand is: Can we characterize ergodic Gibbs states for homogeneous Gaussian specifications? Recall, that we assume $\mathbf{E} = \mathbb{R}^l$ and $\mathbf{S} = \mathbb{Z}^d$. Furthermore, assume $d \geq 3$ for the rest of the chapter, to have existence of Gibbs measures for the harmonic oscillator.

Assumption 6.6. *Henceforth, let the specification $\gamma^{J,h}$ be given as in Proposition 5.6, i.e. generated by the "potential" in equation (5.1), where $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ is symmetric and positive definite and $h \in \mathbb{R}^{\mathbf{S}}$. Thus, $\gamma^{J,h}$ is a Gaussian specification. Moreover, we assume J and h to be homogeneous. In this case, we have shown that $\gamma^{J,h}$ is a homogeneous Gaussian specification. Again, we identify $J : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ with the even function $\tilde{J} : \mathbf{S} \to \mathbb{R}$ and $h \in \mathbb{R}^{\mathbf{S}}$ with $\tilde{h} \in \mathbb{R}$ and write J for $\tilde{J}$ and h for $\tilde{h}$. Here, we consider an "l-fold harmonic oscillator"; thus, we consider homogeneous positive definite coupling functions $J_{\beta_k} =: J_k$ as given in Definition 5.38 and assume $h_k = 0$ for each harmonic oscillator $k \leq l$.*

In Theorem 5.15 it is shown that a random field μ belongs to $ex\mathcal{G}(\gamma^{J,h})$ if and only if μ is Gaussian with covariance function C as in equation (5.2) and mean $m \in \mathbf{M}_{J,h}$. Moreover, in Theorem 5.26 we have obtained that

$\mathcal{G}_{\boldsymbol{\Theta}}(\gamma^{J,h}) = \{\mu_C * \nu : \nu \in \mathcal{P}_{\boldsymbol{\Theta}}(\Omega, \mathcal{F}), \nu(\mathbf{M}_{J,h}) = 1\}$ in the homogeneous case. We then see that μ is extremal in $\mathcal{G}(\gamma^{J,h})$ and ergodic, i.e. extremal in $\mathcal{G}_{\boldsymbol{\Theta}}(\gamma^{J,h})$, if and only if μ is Gaussian with covariance C as in equation (5.2) and homogeneous mean $m \in \mathbf{M}_{J,h}$.

In light of Remark 5.29 and the significance of the harmonic oscillator for our approach, we assume $h = 0$ for the rest of this chapter.

The next proposition formalizes these ideas in completely characterizing ergodic extreme Gibbs states for a homogeneous Gaussian specification $\gamma^{J,0}$. Again, Gibbs states are ergodic extreme if they are homogeneous extreme, since homogeneous extreme distributions are also extreme homogeneous. The following result originates in the explicit characterization of Gibbs states given in Proposition 5.26. Recall the proof of Proposition 5.15 for a better understanding.

Proposition 6.7. *Suppose $\mathcal{G}(\gamma^{J,0}) \neq \emptyset$. Then an element $\mu \in \mathcal{G}(\gamma^{J,0})$ is extreme in $\mathcal{G}(\gamma^{J,0})$ and ergodic if and only if μ belongs to the set*

$$\{\underbrace{\mu_C * \delta_m}_{=:\mu_{C,m}} : m \in \mathbf{M}_{J,0}, m \textit{ constant}\},$$

i.e. μ is Gaussian with constant mean $m \in \mathbf{M}_{J,0}$ and covariance C as given in equation (5.2). A mean $m = (m_s)_{s\in\mathbf{S}} \in \Omega$ is said to be constant *if $m_i = m_j$ for all $i, j \in \mathbf{S}$.*

Proof. From the proof of Proposition 5.15, we know that

$$ex\mathcal{G}(\gamma^{J,0}) = \{\mu_C * \delta_m : m \in \mathbf{M}_{J,0}\}.$$

Since $\mu_C * \delta_m \in ex\mathcal{G}(\gamma^{J,0})$ and $\mathcal{G}_{\boldsymbol{\Theta}}(\gamma^{J,0}) \subset \mathcal{G}(\gamma^{J,0})$ we have that $\mu_C * \delta_m$ is ergodic if it is homogeneous.

$\mu_c * \nu \in \mathcal{G}(\gamma^{J,0})$, $\nu \in \mathcal{P}(\Omega, \mathcal{F})$, is homogeneous if and only if $\nu \in \mathcal{P}_{\boldsymbol{\Theta}}(\Omega, \mathcal{F})$:[3] By Note 5.17, we may identify ν with the probability measure ψ on $ex\mathcal{G}(\gamma^{J,0})$ representing $\mu_C * \nu$ in Proposition 2.40. By Corollary 2.41, we have that a Gibbs measure $\mu \in \mathcal{G}(\gamma)$ is homogeneous if and only if the representing measure ν is homogeneous.

Thus, it leaves to show that δ_m is homogeneous if and only if m is constant: Let $s \in \mathbb{Z}^d$, $m \in \mathbf{M}_{J,0}$, $A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$. Then

$$\begin{aligned} \theta_s(\delta_m)(A) &= (\delta_m \circ \theta_s^{-1})(A) = \delta_{(m_{i+s})_{i\in\mathbf{S}}}(A) \\ &= \begin{cases} 1 & \text{if } (m_{i+s})_{i\in\mathbf{S}} \in A, \\ 0 & \text{if } (m_{i+s})_{i\in\mathbf{S}} \notin A. \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} & \delta_m = \theta_s(\delta_m) \quad \forall s \in \mathbf{S} \\ \Leftrightarrow \quad & [(m_i)_{i\in\mathbf{S}} \in A \Leftrightarrow (m_{i+s})_{i\in\mathbf{S}} \in A] \quad \forall s \in \mathbf{S}, \quad \forall A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}) \\ \Leftrightarrow \quad & m \text{ is constant.} \end{aligned}$$

[3] This was already shown in the proof of Proposition 5.26.

□

Actually, we have characterized the homogeneous extreme Gibbs states. Again, homogeneous extreme Gibbs states are extreme homogeneous and thus ergodic, but in general not vice versa.

Remark 6.8. *In chapter 2.6, we have introduced the notion* phase *for elements in* $ex\mathcal{G}(\gamma)$ *and* pure state *for elements in* $ex\mathcal{G}_\Theta(\gamma)$*. Thus, let us now call elements in* $ex\mathcal{G}(\gamma) \cap ex\mathcal{G}_\Theta(\gamma)$*, i.e. ergodic extremal Gibbs states,* pure phases*.*

In case of the harmonic oscillator, we have seen that $\mathbf{M}_{J,0}$ contains all constant configurations and $\mathcal{G}(\gamma^{J,0}) \neq \emptyset$ whenever $d \geq 3$.

In Chapter 2.6 we have discussed why we may confine the analysis to phases, i.e. extreme Gibbs states. Moreover, in [Hohnisch, 03] it is argued that only pure states, i.e. ergodic Gibbs states, shall be considered. As shown in the foregoing section, ergodicity generates some kind of stability of the empirical distribution of a random economy. Such an empirical distribution as a state of a finite subpopulation is the entity that can be observed (within empirical studies) in our “finite real world”.

Remember that $\gamma^{J,0}$ is assumed to be homogeneous. Thus, it makes sense to consider the set of all homogeneous extreme[4] and thus ergodic extreme Gibbs states. For our analysis of market demand, we assume that such a *pure phase* emerges within a local random economy and gives rise to an ergodic Gibbsian random economy.

So far the framework is elaborated for local random economies and resulting Gibbsian random economies with distribution μ that is extreme and ergodic. However, the economy under consideration, the local unbounded spin Ising economy, is an l-fold local random economy and the resulting global unbounded spin Ising economy is l-fold, too. We have shown in Proposition 2.8, that the product of Gibbs states is again Gibbsian with respect to the product specification. For our analysis, we need to show that the product of ergodic distributions is again ergodic and analogous for phases. This is done within the following two lemmata. Given two sets A_1 and A_2, we define $A_1 \times A_2 := \{(a_1, a_2) : a_i \in A_i\}$ and analogously for the product of σ-algebras.

Lemma 6.9. *Let* μ_i*,* $i = 1, 2$*, be ergodic measures on* $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}))$*. Then, the product measure* $\mu := \mu_1 \otimes \mu_2$ *is ergodic on* $((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d}, \mathcal{B}((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d}))$*.*

Proof. Within this proof, we use the same term “θ_s” for the lattice-shift on $\mathbb{R}^{\mathbb{Z}^d}$ and on $(\mathbb{R}\times\mathbb{R})^{\mathbb{Z}^d}$. By definition, μ_i is ergodic if and only if $\mu_i(A) \in \{0, 1\}$ for every event A in

$$\mathscr{I} := \{A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}) : \theta_s(A) = A \quad \forall s \in \mathbb{Z}^d\},$$

[4]We have to be very exact in distinguishing the terms “extreme homogeneous” and “homogeneous extreme”.

the σ-algebra of homogeneous events. Now, consider $\mu := \mu_1 \otimes \mu_2$ on $((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d}, \mathcal{B}((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d}))$ given by $\mu(E_1 \times E_2) := \mu_1(E_1)\mu_2(E_2)$ for all $E_1 \times E_2 \in \mathcal{B}((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d})$. Then define

$$\mathscr{I}^* := \{A \in \mathcal{B}((\mathbb{R} \times \mathbb{R})^{\mathbb{Z}^d}) | \theta_s(A) = A \quad \forall s \in \mathbb{Z}^d\}.$$

Since $\theta_s(A_1 \times A_2) = A_1 \times A_2$ if and only if $\theta_s(A_i) = A_i$, $i = 1, 2$, $s \in \mathbb{Z}^d$, we obtain $\mathscr{I}^* = \mathscr{I} \times \mathscr{I}$.

Hence, $\mu(A) = \mu_1(A_1)\mu_2(A_2) \in \{0, 1\}$ for all $A = A_1 \times A_2 \in \mathscr{I}^*$. $\square$

Lemma 6.10. *Given specifications γ^1 and γ^2, let $\mu_1 \in ex\mathcal{G}(\gamma^1)$ and $\mu_2 \in ex\mathcal{G}(\gamma^2)$. Then $\mu_1 \otimes \mu_2 \in ex\mathcal{G}(\gamma^1 \otimes \gamma^2)$.*

Proof. The proof is conducted as the foregoing when applying Proposition 2.39. $\square$

For the local unbounded spin Ising economy we consider a family $(\gamma^{J_k, h_k})_{k \leq l}$ of specifications each with state space $\mathbb{R}$, generated as in Proposition 5.6. Here, $h_k \in \mathbb{R}^{\mathbb{Z}^d}$, constant, and J_k given by

$$J_k(i) = \begin{cases} -\ \frac{\beta_k}{2d} & \text{if } |i| = 1, \\ \ \ \beta_k & \text{if } i = 0, \\ \ \ 0 & else \end{cases}$$

for $\beta_k > 0$. However, we confine ourselves to the case $h_k = 0$ for $k \leq l$.

We now consider $\mu_{C,m} := \mu_{C_1,m_1} \otimes \ldots \otimes \mu_{C_l,m_l}$, $m_k \in M_{J_k,0}$, constant, and C_k as given in Proposition 5.15 for $\gamma^{J_k,0}$. Here, $\mu_{C_k,m_k} := \mu_{C_k} * \delta_{m_k}$ is the homogeneous extreme, and thus ergodic, Gaussian measure on $(\mathbb{R}^{\mathbf{S}}, \mathcal{B}(\mathbb{R}^{\mathbf{S}}))$ obtained in Proposition 6.7. By Lemma 6.9, $\mu_{C,m}$ defines an ergodic Gibbsian random economy with product specification $\gamma^{J,0} := \gamma^{J_1,0} \otimes \ldots \otimes \gamma^{J_k,0}$ with state space $\mathbf{E} = \mathbb{R}^l$, i.e. $\mu_{C,m} \in ex\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J,0})$. We may now apply Proposition 6.3 to obtain the corresponding distribution economy $\nu_{C,m} := \nu_{C_1,m_1} \otimes \ldots \otimes \nu_{C_l,m_l}$ as the one-dimensional marginal distribution $\nu_{C,m} := \mu_{C,m} \circ \sigma_0^{-1}$ of $\mu_{C,m}$ on $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$, when $(\sigma_s)_{s \in \mathbf{S}} := ((\sigma_{k_s})_{k \leq l})_{s \in \mathbf{S}}$ with state space $\mathbb{R}^l$ is the corresponding global random economy ruled by $\mu_{C,m}$. Equivalently, we may say that the corresponding distribution economy $\nu_{C,m}$ is given by the random variable σ_0 and its marginal distribution $\nu_{C,m} = \nu_{C_1,m_1} \otimes \ldots \otimes \nu_{C_l,m_l}$ under $\mu_{C,m} = \mu_{C_1,m_1} \otimes \ldots \otimes \mu_{C_l,m_l}$. Furthermore, each ν_{C_k,m_k} is the one-dimensional marginal distribution of the ergodic Gaussian field μ_{C_k,m_k}, i.e. $\nu_{C_k,m_k} = \mu_{C_k,m_k} \circ \sigma_{k_0}^{-1}$.

We now characterize these one-dimensional marginal distributions $\nu_{C_k,m_k} \in \mathcal{P}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Recall the assumption $\mathbf{S} = \mathbb{Z}^d$.

Proposition 6.11. *Let $(\sigma_{k_s})_{s \in \mathbf{S}}$ be a Gaussian field in $\mathbb{R}^{\mathbf{S}}$ with mean $m_k : \mathbf{S} \to \mathbb{R}$, $i \mapsto (m_k)_i$, and covariance $C_k : \mathbf{S} \otimes \mathbf{S} \to \mathbb{R}$, $(i, j) \mapsto (C_k)_{i,j}$. Then,* *for each* $i \in \mathbf{S}$, σ_{k_i} *is Gaussian with mean* $(m_k)_i$ *and variance* $(C_k)_{i,i}$. *In particular,* σ_{k_i} *is distributed with density*

$$f_k(x) = \frac{1}{\sqrt{2\pi(C_k)_{i,i}}} e^{-\frac{(x-(m_k)_i)^2}{2(C_k)_{i,i}}}$$

Proof. Let $u \in \mathbb{R}$ and $y \in \mathbf{S}$ with $y_j = 1$ for $i = j$ and $y_j = 0$ else. Then

$$\begin{aligned} \varphi_{\sigma_{k_i}}(u) &= \int e^{\iota u \sigma_{k_i}} dP \\ &= \int e^{\iota u \sum_{j \in \mathbf{S}} y_j \sigma_{k_j}} dP \\ &= e^{\iota \sum_{j \in \mathbf{S}} u y_j (m_k)_j - \frac{1}{2} \sum_{j,l \in \mathbf{S}} u y_j (C_k)_{i,l} u y_l} \\ &= e^{\iota u (m_k)_i - \frac{1}{2} u^2 (C_k)_{i,i}}, \end{aligned}$$

where again ι denotes the imaginary unit. □

The proposition shows that each ν_{C_k,m_k} is a Gaussian measure on $\mathbb{R}$ with mean $(m_k)_i$ and variance $(C_k)_{i,i}$, where $i \in \mathbf{S}$ may be chosen arbitrarily due to homogeneity of μ_{C_k,m_k}.

Due to Gaussian property, ν_{C_k,m_k} is uniquely characterized by $(C_k)_0$ and $(m_k)_0$, as μ_{C_k,m_k} by C_k and m_k. For our purposes we do not need to characterize the entire covariance function C_k but just the diagonal elements, i.e. $(C_k)_{i,i}$ for all $i \in \mathbf{S}$, and, due to homogeneity, it suffices to specify $(C_k)_{0,0}$.

Applying equation (5.2) we obtain $(C_k)_{0,0} = \frac{1}{J_k(0,0)}$. In case of the unbounded spin Ising economy as defined in 5.38, this yields $(C_k)_{0,0} = \frac{1}{\beta_k}$, $\beta_k > 0$, $k \leq l$.

The above discussion shows the following definition to be consistent with the concepts introduced in the last chapter.

Definition 6.12. (a) *The* ergodic (extreme) global unbounded spin Ising economy *is given by a family* $(\sigma_s)_{s \in \mathbb{Z}^d}$ *of random variables each with state space* $\mathbb{R}^l$, *the distribution of which is of the form* $\mu_{C,m} = \mu_{C_1,m_1} \otimes \ldots \otimes \mu_{C_l,m_l}$ *with ergodic extremal Gaussian distributions* $\mu_{C_k,m_k} \in \mathcal{G}(\gamma^{J_k,0})$, $\gamma^{J_k,0}$, $k = 1, \ldots, l$, *as given in Definition 5.38, where mean* $m_k \in M_{J_k,0}$ *is a constant configuration and covariance function* C_k *is as given in equation (5.2) for* J_k.
(b) *A distribution economy* $\nu_{C,m} \in \mathcal{P}(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ *corresponding to the ergodic global unbounded spin Ising economy is given by a probability distribution* $\nu_{C,m} := \nu_{C_1,m_1} \otimes \ldots \otimes \nu_{C_l,m_l}$ *on* $\mathbb{R}^l$, ν_{C_k,m_k} *mutually independent and Gaussian on* $\mathbb{R}$, *each with variance* $(C_k)_{(0,0)} = \frac{1}{\beta_k} \in \mathbb{R}_{++}$ *and mean* $(m_k)_0 \in \mathbb{R}$, $k \leq l$. *Thus,* $\nu_{C,m}$ *is Gaussian with mean* $m = ((m_1)_0, \ldots, (m_k)_0)$ *and covariance*

$$C_{j,k} = \begin{cases} \frac{1}{\beta_k} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

$j, k \leq l$. $\nu_{C,m}$ is called the unbounded spin Ising distribution economy *with parameters $\beta_1, \ldots, \beta_l, m_1, \ldots, m_l$.*

Proof of consistency. The proof of part (a) just summarizes the discussion above: The relevant Gibbs states for the Gaussian specification $\gamma^{J_k,0}$ as defined by 5.38 are given by the ergodic extremal elements in $\mathcal{G}(\gamma^{J_k,0})$ that are characterized in Proposition 6.7. $m_k \in \mathbf{M}_{J_k,0}$ is well defined by Example 5.20. Note, $\mu_{C,m} \in \mathcal{G}(\otimes_{k \leq l}\gamma^{J_k,0})$ by Proposition 2.8 and ergodic since each μ_{C_k,m_k} is ergodic. Part (b) then follows directly by Proposition 6.3 and Proposition 6.11. □

Intuitively, an unbounded spin Ising distribution economy specifies the distribution $\nu_{C,m}$ of demand functions, more precisely of α-transforms on the underlying demand function.

For convenience let us consider $\beta_1 = \ldots = \beta_l =: \beta$. In terms of Section 2.7, β is the "inverse temperature" of the system. As done in [Grandmont, 92] we want to achieve more heterogeneity among agents, i.e. we basically want to increase variance $(C_k)_{(0,0)} = \frac{1}{\beta}$ and thus decrease β. In other words, we want to raise temperature. This is exactly the opposite of what was done in Section 2.7, where we had a look at the low temperature or thermodynamic limit. Intuitively, it is clear that more heterogeneity among agents corresponds to a decreasing β in our approach: The lower β, the weaker coupling among agents and thus the less possibilities for "preference for conformity" to be transmitted. When β approaches zero, there is no coupling and random agents are independently distributed with a priory measure; as assumed the Lebesgue measure on $\mathbb{R}$. The unique Gibbs state is the given by the product measure. Thus, heuristically we obtain the weak axiom of revealed preference in the aggregate for the unbounded spin Ising distribution economy the "more independent" agents and thus the weaker local interactions.

At first view, one may remark that this would contradict our assumption of preference for conformity. However, we do not think that this objection is admissible: It is not the assumption of preference for conformity that is weakened when β decreases. It is more that "social links" among agents get weaker as β decreases.

Since we only consider unbounded spin Ising economies, we henceforth assume $\gamma^{J_k,0}$ to be the specification introduced in Definition 5.38 and $d \geq 3$ to assure existence of Gibbs states. We have already seen that the existence of Gibbs states in $\mathcal{G}(\gamma^{J_k,0})$ does not depend on the value of $\beta > 0$. We could even set $\beta = 0$.[5] In this case, random agents are independent an identically distributed with reference measure. As already mentioned in Section 3.5, the unique Gibbs state is then given by the product distribution of reference measures.

[5]Of course, this contradicts Definition 5.38.

Cooling down Society

In light of Grandmont's analysis, we have argued heuristically that market demand gets closer to satisfying the weak axiom of revealed preference the higher temperature. The common question in statistical mechanics is the opposite one: What happens when a system reaches the thermodynamic or low temperature limit? Stated in other words: What can be said about satisfaction of the weak axiom for market demand when β converges to infinity?

We may infer from the specific form of the Hamiltonian $H^{J,0}$ in Example 5.33 that constant configurations are *ground states*, i.e. configurations of minimal energy. In chapter 2.7 we have seen the connection between ground states and Gibbs measures in the low temperature limit for the 2-dimensional Ising ferromagnet: When β approaches infinity, the set of Gibbs measures gets arbitrarily close to the Dirac measures with mass on ground states; this is shown in Proposition 2.53. In this sense, the system gets more and more static as β approaches ∞ since coupling gets arbitrarily strong. In other words, the system is attracted by states of minimal energy.

For our purposes, the analysis of thermodynamic limit is only admissible if it admits an economic interpretation: In case of the local unbounded spin Ising economy, an increase in β means that coupling among agents gets stronger and agents prefer more and more to be similar. Thus, we obtain the opposite situation as needed for Grandmont's analysis. The interpretation of this limit might be that society settles down in the sense that agents are more and more consensus orientated. Thus, $\beta \to \infty$ literally means "cooling down society". In terms of [Kindermann & Snell, 80b], p.5: Minimum tension (caused by agents' interaction) means maximum boredom and occurs when agents agree, i.e. when the configuration is constant.

Let ω^{c_k}, $c_k \in \mathbb{R}$, denote the constant configuration with $\omega_s^{c_k} = c_k$ for all $s \in \mathbf{S}$. The potential $\Phi^{J_k,0}$ in Example 5.33[6] exhibits a ground state degeneracy in that all ω^{c_k} are ground states for $\Phi^{J_k,0}$. This would imply phase transition if $\mathcal{G}(\gamma^{J_k,0})$ is attracted by each $\delta_{\omega^{c_k}}$ when $\beta \to \infty$; in other words ω^{c_k}'s are stable. However, we have already shown the existence of phase transition in Example 5.20.

We now obtain a result analogous to Proposition 2.53: Given the ergodic extremal Gaussian Gibbs measure $\mu_{C_k,\omega^{c_k}} = \mu_{C_k} * \delta_{\omega^{c_k}} \in \mathcal{G}(\gamma^{J_k,0})$, $\gamma^{J_k,0}$ as given in Definition 5.38, we have

$$\mu_{C_k,\omega^{c_k}} \to \delta_{\omega^{c_k}}$$

weakly as $\beta \to \infty$. This weak convergence is immediate since $\mu_{C_k,\omega^{c_k}}$ is Gaussian on $\mathbb{R}^l$ with mean ω^{c_k} and covariance C_k as given above. Thus, $\nu_{C_k,\omega^{c_k}} \to \delta_{c_k}$ weakly as $\beta \to \infty$, In other words, the distribution of $\alpha_k \in \mathbb{R}$

[6] Here, $\Phi^{J_k,0}$ is a potential since $J_{k,0}$ exhibits finite range.

converges to a Dirac measure with mass in $c_k \in \mathbb{R}$.[7] Let the one-dimensional marginal distribution $\nu_{C_k,\omega^{c_k}}$ of $\mu_{C_k,\omega^{c_k}}$ be denoted by ν_{C_k,c_k}, i.e. $\nu_{C_k,c_k} := \mu_{C_k,\omega^{c_k}} \circ (\sigma_k)_0^{-1}$.

We have shown that the unbounded spin Ising distribution economy $\nu_{C,c} := \nu_{C_1,c_1} \otimes \ldots \otimes \nu_{C_l,c_l}$ weakly converges to $\delta_c := \delta_{c_1} \otimes \ldots \otimes \delta_{c_l}$ as $\beta \to \infty$ and for conditional (in our case total) market demand the following convergence result holds: Let $c = (c_1, \ldots, c_l) \in \mathbb{R}^l$

$$\begin{aligned} X(p, w) &= \int_{\mathbb{R}^l} \xi^\alpha(p, w) \nu_{C,c}(d\alpha) \\ &\overset{\beta\to\infty}{\longrightarrow} \int_{\mathbb{R}^l} \xi^\alpha(p, w) \delta_c(d\alpha) = \xi^c(p, w). \end{aligned} \tag{6.1}$$

Proposition 6.13. *Given an unbounded spin Ising distribution economy. Assume the underlying individual demand ξ to satisfy the weak axiom of revealed preference, then market demand X approaches a demand function satisfying the weak axiom as the inverse temperature β approaches infinity, i.e. X satisfies the weak axiom in the low temperature limit.*

Proof. Using the discussion above and in particular equation (6.1), it leaves to show that any α-transform of ξ satisfies the weak axiom of revealed preference: Let ξ satisfy the weak axiom and recall $\xi^\alpha(p, w) := e^\alpha \otimes \xi(e^\alpha \otimes p, w)$, $\alpha \in \mathbb{R}^l$. Let $(p, w), (\tilde{p}, \tilde{w}) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ be given. Assuming

$$[e^\alpha \otimes p] \cdot \xi(e^\alpha \otimes \tilde{p}, \tilde{w}) = p \cdot [e^\alpha \otimes \xi(e^\alpha \otimes \tilde{p}, \tilde{w})] = p \cdot \xi^\alpha(\tilde{p}, \tilde{w}) \le w$$

and

$$\xi^\alpha(\tilde{p}, \tilde{w}) = e^\alpha \otimes \xi(e^\alpha \otimes \tilde{p}, \tilde{w}) \neq e^\alpha \otimes \xi(e^\alpha \otimes p, w) = \xi^\alpha(p, w),$$

the weak axiom for ξ implies

$$\tilde{p}\xi^\alpha(p, w) = \tilde{p} \cdot [e^\alpha \otimes \xi(e^\alpha \otimes p, w)] = [e^\alpha \otimes \tilde{p}] \cdot \xi(e^\alpha \otimes p, w) > \tilde{w}.$$

Thus, ξ^α satisfies the weak axiom of revealed preference. □

This result is weaker than that in [Grandmont, 92]: Whereas Grandmont shows that the weak axiom for market demand is "created", we have shown that it is "inherited" by market demand in the low temperature limit. Even though we have not assumed identical demand functions for agents, we have posed assumptions that imply identical α-transforms in the low temperature limit. In other words: Every agent exhibits the same demand function represented by $\alpha = (c_1, \ldots, c_l)$, although, due to phase transition, we do not

[7] Here, we have obtained the result by first applying the low temperature limit to the harmonic oscillator and then consider the one-dimensional marginal distribution. However, we could turn this approach upside down and first consider the one-dimensional marginal distribution $\nu_{C,\omega^{c_k}}$ and then apply the low temperature limit.

know a priori which one. Of course, when every agent has the same demand function that satisfies the weak axiom of revealed preference, per capita aggregate demand will satisfy the weak axiom, too. However, the strength of this result is that we do not know a priori which pure phase emerges, but in the low temperature limit it forces market demand to satisfy the weak axiom.

Grandmont's Analysis for Unbounded Spin Ising Economies, or: Heating up Society

In view of the discussion so far, we apply the following procedure: Consider the set of α-transforms $\{\alpha|\alpha \in \mathbb{R}^l\}$ of the underlying demand function. The crucial assumption is that we obtain the distribution of every $\alpha_k \in \mathbb{R}$ independently within a social system that is given by an harmonic oscillator with specification $\gamma^{J_k,0}$ as in Example 5.20, representing preference for conformity. This is just the local unbounded spin Ising economy and we consider a corresponding ergodic (extreme) global unbounded spin Ising economy, i.e. an ergodic (extreme) Gibbs state $\mu_{C_1,m_1} \otimes \ldots \otimes \mu_{C_k,m_k} \in ex\mathcal{G}_{\mathbf{\Theta}}(\gamma^{J_1,0} \otimes \ldots \otimes \gamma^{J_k,0})$, where C_k is again as given in Proposition 5.15 for J_k as in Definition 5.38 and $m_k \in \mathbf{M}_{J_k,0}$ constant.

To recall, we then apply Proposition 6.3 to obtain a distribution economy $\nu_{C,m} := \nu_{C_1,m_1} \otimes \ldots \otimes \nu_{C_l,m_l}$ on the space of individual characteristics $\alpha \in \mathbb{R}^l$, where $\nu_{C_k,m_k} := \mu_{C_k,m_k} \circ \sigma_0^{-1}$ is the distribution of $\alpha_k \in \mathbb{R}$, $k \leq l$. We have called $\nu_{C,m}$ the unbounded spin Ising distribution economy and may now apply Grandmont's analysis to this economy.

Recall $\beta := \beta_1 = \ldots = \beta_l$. Intuitively this assumption means that the willingness to conform with neighboring agents is equal not only across agents but also across distinct commodities. Then, the analysis of market demand is done for an unbounded spin Ising distribution economy $\nu_{C,m}$, i.e. $\nu_{C,m}$ is Gaussian with mean $m = ((m_1)_0, \ldots, (m_l)_0)$ and covariance C given by

$$C_{j,k} = \begin{cases} \frac{1}{\beta} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

$j, k \leq l$.

Let $\varphi^2 := \frac{1}{\beta}$. By Proposition 6.11, each $\nu_{C_k,m_k} = N((m_k)_0, \varphi^2)$, the Normal distribution with mean $(m_k)_0$ and variance φ^2. Thus, the distribution of α_k is given by the density

$$f_k(\alpha_k) = \frac{1}{\sqrt{2\pi}\varphi} e^{-\frac{(\alpha_k - (m_k)_0)^2}{2\varphi^2}}.$$

Due to the independence assumption, the distribution of α as a random variable with values in $\mathbb{R}^l$ is ruled by the product density

$$f(\alpha) = \frac{1}{(2\pi\varphi^2)^{\frac{l}{2}}} e^{-\frac{\sum_{k=1}^{l}(\alpha_k-(m_k)_0)^2}{2\varphi^2}}. \tag{6.2}$$

We now have to check whether the density f satisfies the assumptions in [Grandmont, 92]: We have assumed that there exists only one type $a \in \mathbf{A}$ for the unbounded spin Ising distribution economy, i.e. one underlying demand-income-pair (ξ, w) or simply $|\mathbf{A}| = 1$. Thus, we may set $f(\alpha) = f(\alpha|a)$, the density conditional on type $a \in \mathbf{A}$. Assuming $w \in \mathbb{R}_{++}$ to be the fixed income, Definition 4.17 implies conditional market demand to equal total market demand, i.e.

$$\begin{aligned}
X(a,p,w) &= \int_{\mathbb{R}^l} \xi_a^\alpha(p,w) f(\alpha) d\alpha \\
&= \int_{\mathbb{R}^l} \xi_a^\alpha(p,w) f(\alpha|a) d\alpha \\
&= \int_{\mathbf{A}} \int_{\mathbb{R}^l} \xi_b^\alpha(p,w) f(\alpha|b) d\alpha \delta_a(db) \\
&= X(p).
\end{aligned}$$

Obviously, f is continuous in α. Moreover, its partial derivatives $\frac{\partial f}{\partial \alpha_j}$, $j = 1, \dots, l$, are given by

$$\frac{\partial f}{\partial \alpha_j}(\alpha) = -\frac{\alpha_j - (m_j)_0}{(2\pi)^{\frac{l}{2}} \varphi^{l+2}} e^{-\frac{\sum_{k=1}^{l}(\alpha_k-(m_k)_0)^2}{2\varphi^2}}.$$

These are again continuous. Thus, (*A2*) holds.

Satisfaction of (*A3*) and (*A4*) can be seen as follows:

$$\begin{aligned}
\mathfrak{v}_j(a) &= \mathfrak{v}_j = \int_{\mathbb{R}^l} \left| \frac{\partial f}{\partial \alpha_j}(\alpha) \right| d\alpha \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left| \frac{(m_j)_0 - \alpha_j}{(2\pi)^{\frac{l}{2}} \varphi^{l+2}} \right| e^{\frac{\sum_{k=1}^{l}(\alpha_k-(m_k)_0)^2}{2\varphi^2}} d\alpha_1 \dots d\alpha_l \\
&= \frac{1}{\sqrt{2\pi}\varphi^3} \int_{\mathbb{R}} |(m_j)_0 - \alpha_j| e^{-\frac{(\alpha_j-(m_j)_0)^2}{2\varphi^2}} d\alpha_j \\
&= \frac{1}{\varphi}\sqrt{\frac{2}{\pi}} = \sqrt{\frac{2\beta}{\pi}}.
\end{aligned}$$

It is worth noting that $\mathfrak{v}_j$ does not depend on the particular commodity j when assuming $\beta_k = \beta > 0$ for all $k = 1, \dots, l$. Also note that $\mathfrak{v}_j$ is independent of m. Again, $\mathfrak{v}_j$ is a measure for the concentration of the distribution of α around the mean $((m_1)_0, \dots, (m_l)_0)$: the higher the value of each $\mathfrak{v}_j$, the more concentrated the distribution.

Due to the one-type-assumption, (*A6*) and (*A5*) hold trivially. (*A1*) and (*A7*) are not posed in terms of f; these will just be assumed as done in [Grandmont, 92]. Due to the one-type-assumption, (*A7*) is rephrased as: For every $k = 1, ..., l$, there exists $\varpi_k > 0$, with $\sum_{k=1}^{l} \varpi_k \leq 1$, such that for all price systems $p \in \mathbb{R}^l_{++}$ we have

$$p_k \xi_k(p, w) \geq \varpi_k w > 0.$$

The following Corollary of Theorem 4.34 can equivalently be stated for corresponding local or ergodic global unbounded spin Ising economies.

Corollary 6.14 (of Theorem 4.34)**.** *Given the unbounded spin Ising distribution economy with parameters* $\frac{1}{\beta}, m_1, \ldots, m_l$*. Let* $\varpi_k > 0$ *be given as in (A7). If* $0 < \beta < \varpi_k^2 \frac{\pi}{2}$*, market demand is strictly decreasing in its own price, i.e.* $\frac{\partial X_k}{\partial p_k}(p) < 0$*.*

If the coupling constant $0 < \beta < \frac{\varpi_k^2 \pi}{2l^2}$ *for all* $k = 1, \ldots, l$*, then market demand satisfies the weak axiom of revealed preference.*

Proof. Apply Theorem 4.34 when $\mathfrak{v}_k = \sqrt{\frac{2\beta}{\pi}}$, $\beta > 0$ by Definition 5.38. □

Irrespective of the emerging phase that rules the economy, aggregate demand satisfies the weak axiom for specific values of the parameter β. This is just because $\mathfrak{v}_j$ is independent of m. For market demand to satisfy the weak axiom of revealed preference coupling has to be smaller the more goods available in the economy. We could also have stated the remaining assertions in Theorem 4.34 for our model

A further analysis of existence and uniqueness of market exchange equilibrium as conducted in Section 4.2 for our set-up does not seem appropriate within this diploma thesis: Varying income w, and moreover allowing for several types, would either call for a more complex interaction structure as already laid out or one would just repeat the analysis in [Grandmont, 92] for an exogenously given distribution of types and obtain the same results with just applying the normal distribution for α-transforms that we have obtained endogenously. Nevertheless, this would be an approach to show existence, uniqueness and stability of equilibrium in a private ownership competitive exchange economy with not necessarily rational agents when (at least) the distribution of demand is specified endogenously by virtue of Gibbsian local interaction.

Thus, if we would go further in our analysis and would show uniqueness of equilibrium for an unbounded spin Ising distribution economy, the problem of non-uniqueness of equilibrium would only arise because of non-uniqueness of Gibbs states. In other words, we can show uniqueness of equilibrium for a given pure phase. However, if the set of pure phases is not singleton, we have a priori multiplicity of equilibrium due to phase transition.

Applying a mechanism to endogenize the distribution of demand behavior leading to a set of Gibbs states that is singleton would imply a priori uniqueness of equilibrium in a local random pure exchange economy.

Let us just for a moment consider how one may generalize the local interaction structure as given in the local unbounded spin Ising economy to an interaction structure that takes account of income or more general, of types. Two main problems have to be solved: First, one has to specify an appropriate local interactions structure of income among agents and then an interaction structure between demand functions and income for each agent on her own. However, the question arises, whether the functional form of demand depends on the level of income. Considering our approach, we might say that the functional form of demand is independent of income. Thus, an appropriate interaction structure between demand and income for our model might be independence. Nevertheless, we still have to find an interaction structure for income among agents. Since income is not (always) chosen by an agent herself it does not make sense to endogenize the distribution of income using preference for conformity.

However, it would make sense to come up with a model where agents are given a fixed income and then choose where to be sited on $\mathbb{Z}^d$. This model would be similar to Shelling's neighborhood segregation model. As argued in [Georgii, 88], under specific assumptions, such a model is equivalent to a model where agents choose their states on fixed sites. But still, this model of endogenous income formation would then have to be combined with our model of endogenous demand formation.

Chapter 7

Conclusions

In this thesis, we followed a fundamental idea by Grandmont:

> An alternative research strategy might be indeed to rely more on particular features of the distribution of behavioral characteristics [...]. An important issue to investigate would then be how such macroeconomic distributions might arise endogenously [...]. One could for instance envision a more 'adaptive' viewpoint [...] in which the decision rule [...] of an individual (is) influenced [...] by those of his immediate neighbor(s), and generate endogenously a macroeconomic distribution by looking for invariant distributions. The properties of these invariant distributions might in turn generate enough strong macroeconomic structure [...]. ([Grandmont, 92], p. 40)

We have modeled a framework to endogenize the distribution of demand behavior: In light of Föllmer's Ising economy, we came up with a model taking account of a more general state space that, in a sense, represents the space of all demand functions. This model, that we refer to as the global unbounded spin Ising economy, is based on the harmonic oscillator and thus formalizes "preference for conformity" among agents. We then have obtained the unbounded spin Ising distribution economy that enabled us to apply Grandmont's analysis to our framework of endogenous demand formation. We have seen that, for market demand to satisfy the weak axiom of revealed preference, coupling among agents has to be smaller than some constant, without assuming the weak axiom for individual demand. This result makes intuitively sense when having in mind that Grandmont's analysis calls for heterogeneous agents whereas our interaction structure assumes agents to exhibit a desire for conforming with peers. In particular, we obtain a unique market exchange equilibrium for every pure phase. Non-uniqueness of market exchange equilibrium then emerges due to non-uniqueness of Gibbs states. However, we have not explicitly conducted the uniqueness analysis for our set-up.

We should be conscious that this result has an impact on social sciences: Individual rationality is not at all fundamental to achieve a rationally behaving society. Further investigation should be carried out on the empirical content of this model: Do we actually obtain distinct structural properties of market demand for societies that are more consensus-based than for those with less social ties?

Furthermore, we have conducted an analysis of market demand that relies on the thermodynamic limit: Since coupling constant coincides with inverse temperature, this is just the opposite of Grandmont's analysis. Letting the inverse temperature approach infinity, we have shown that market demand satisfies the weak axiom of revealed preference if individual demand is assumed to do so. This result was immediate since in thermodynamic limit all agents exhibits the same demand function.

Our model relies on very restrictive assumptions:First, we have assumed preference for conformity for all commodities. Then, the product specifications used in the model imply independence of the distribution of demand for different commodities. and we have assumed this distribution to be independent of the exogenously given distribution of income. However, the model can be generalized in several directions: We do not have to assume preference for conformity for each commodity. Within the Gibbsian approach we can develop models with different local interaction structures for distinct commodities. Among many other concepts, we may model "preference for antagonism" or just "independence" in demand behavior for several goods. Our concept of product specifications implies that the distribution of demand for specific commodities is obtained independently. Thus, to overcome this assumption, we have to design an entirely new interaction structure. One way would be to assume $\mathbb{Z}^d$ not to be just the graph of agents but to consist of disjoint boxes with l sites. Each box represents an agent; each site within such a box represents the demand of that agent for a specific commodity. Then, the difficulty is to find an appropriate interaction structure among l-boxes, i.e. among agents, and within each box, i.e. among demand for distinct goods. A simpler way may be conducted for the unbounded spin Ising distribution economy: If we would like to obtain correlation among goods, we could just assume the distribution to be multivariate normal with an appropriate correlation. Within the text, we have already argued which difficulties may arise when endogenizing the exogenously given distribution of income and particularly of types: We have to come up with a local interaction structure not only specifying interactions in income among agents, but also interactions between income and demand for each agent on her own. Then we have to combine this set-up with our model of endogenous demand formation.

However, we hope that our model reveals a fruitful analysis of market demand with endogenous demand formation.

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