Mild Solutions of SPDE's Driven by Poisson Noise in Infinite Dimensions and their Dependence on Initial Conditions

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Chapter 0

Introduction

Stochastic partial differential equations (abbreviated SPDE's) driven by Gaussian noise are well studied (see [Wa 86], [Pe 95], [DaPrZa 92], [DaPrZa 96] and the references therein) whereas SPDE's driven by a noise of jump type are less well understood. But within the last years SPDE's driven for example by a compensated Poisson random measure or a Lévy noise draw more attention, one reason for which may be the prospect of numerous applications: "White noise perturbations, however, are not always appropriate to interpret real data in a reasonable way. This is the case for example if the nature of the underlying perturbation process has to model abrupt pulses or extreme events."(see [ImPa 04, p.2, l.9-11])

Already in the 80's even infinite dimensional SPDE's perturbed with a stochastic integral with respect to a compensated Poisson random measure were used to model the membrane potential of a neuron. In the earliest models a neuron was represented by a single point. Walsh was one of the first who considered spatially extended neurons. As proposed by Rall in [Ra 59], he treated the dendritic tree as an infinitely thin cylinder of length L (see [Wa 81]). In [KaWo 84] Kallianpur and Wolpert proposed, for the purpose of more realistic models, other choices of the surface membrane of a neuron, for example it can be any smooth, compact, d-dimensional manifold. But already in the simplest spatially extended case the solution of the corresponding SPDE at time t, which describes the membrane potential at time t, takes values in an infinite dimensional space.

A further class of models, where SPDE's with noise of jump type are needed, are the stochastic climate models, for example to explain the so-called Dansgaard-Oeschger events during a glacial period. "In fact, paleoclimatic records from the Greenland ice-core show that the climate of the last glacial period experienced rapid transitions between cold basic glacial periods and several warmer interstadials (the so-called Dansgaard -Oeschger events)"([ImPa 04, p.2, l.16-18]). So far, this phenomenon is not completely understood. There are several suggestions for an explanation, e.g. the concept of stochastic resonance. This concept consists in modelling the paleoclimatic temperature process as the solution of an SPDE of the following type

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s))\, ds + \varepsilon\eta_t \text{ (for details see [ImPa 04])},$$

where the question arises which noise term is to choose. First in [Di 99a], [Di 99b] and some years later in [ImPa 04] the authors model the noise by a Lévy process L.

Finally, we have to mention the class of financial market models. Indeed, in the area of the stochastic financial markets the Brownian motion, traditionally, plays a dominant role, but "although very elegant the Black-Scholes-Merton model has limitations and possible defects that have led many probabilists to query it. Indeed, empirical studies of stock prices have found evidence of heavy tails which is incompatible with a Gaussian model." ([Ap 04, p.1341, l.50-55]) This is carried out in more detail in [Ap 04]. See also for example [EbRa 99],[Ra 00].

In this paper we study mild solutions of SPDE's in infinite dimensions driven by a compensated Poisson random measure and their dependence on the initial value. Apart from applications, SPDE's with Poisson noise are of independent interest and basic investigations and a better understanding of stochastic integrals w.r.t. a compensated Poisson random measure and of SPDE's with Poisson noise is an important step for the study of SPDE's with Lévy noise. There is quite a substantial amount of work that has been done in this field (see e.g. [IkWa 81], [AlWuZh 97], [Mu 98], [ApWu 00], [ApTa 01], [MaRu 03] and references therein and the discussion below for their relation with our results).

Let us first introduce our setting, then we will summarize our main results.
Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$. Moreover, let $(U, \mathcal{B}, \nu)$ be a σ-finite measure space and p an $(\mathcal{F}_t)$-Poisson point process on $((0,\infty) \times U, \mathcal{B}((0,\infty)) \otimes \mathcal{B})$ with intensity measure $\nu \otimes \lambda$ where λ denotes the Lebesgue measure. Denote by N_p the to p associated Poisson random measure.
Let $T > 0$ and consider the following SPDE in a separable Hilbert space $(H, \langle\, ,\, \rangle)$

$$\begin{cases} dX(t) & = [AX(t) + F(X(t))]\, dt + B(X(t), y)\, q(dt, dy) \\ X(0) & = \xi \end{cases} \tag{1}$$

where
1.) $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of linear, bounded operators on H,
2.) $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable,
3.) $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable,
4.) $q(t, B) := N_p(t, B) - t\nu(B) := N_p(]0, t] \times B) - t\nu(B)$, $t \geq 0$, $B \in \mathcal{B}$, $\nu(B) < \infty$, and
5.) ξ is an H-valued, $\mathcal{F}_0$-measurable random variable.

We are interested in the existence and uniqueness of a mild solution of (1) in

$$\begin{aligned}\mathcal{H}^2(T, H) := \{Y(t), t \in [0, T] \mid\ & Y \text{ has an } H\text{-predictable version,} \\ & Y(t) \in L^p(\Omega, \mathcal{F}_t, P; H) \text{ and} \\ & \sup_{t \in [0,T]} E[\|Y(t)\|^2] < \infty\}.\end{aligned}$$

Our main interest is directed towards the analysis of its dependence on the initial value ξ. Since a mild solution $X(\xi)$ is given implicitly by

$$\begin{aligned}X(\xi) = \mathcal{F}(\xi, X(\xi)) := \big(S(t)\xi + & \int_0^{\cdot} S(\cdot - s)F(X(\xi)(s))\, ds \\ & + \int_0^{\cdot+} \int_U S(\cdot - s)B(X(\xi)(s), y)\, q(ds, dy)\big)_{t \in [0,T]}\end{aligned}$$

these questions can be treated on the very abstract level of a general contracting mapping $G : \Lambda \times E \to E$ on arbitrary Banach spaces Λ and E. Existence of an implicit function and its differentiability properties can then be deduced from properties of the mapping G. For this purpose we consider the Banach space $(H^2(T, H), \| \|_{\mathcal{H}^2})$ of equivalence classes of elements in $\mathcal{H}^2(T, H)$ with respect to the seminorm

$$\|Y\|_{\mathcal{H}^2} := \sup_{t \in [0,T]} \big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}$$

and for $\bar{\xi} \in L_0^2 := L^2(\Omega, \mathcal{F}_0, P; H)$ and $\bar{Y} \in H^2(T, H)$ we define $\bar{\mathcal{F}}(\bar{\xi}, \bar{Y})$ to be the equivalence class of $\mathcal{F}(\xi, Y)$ w.r.t. $\| \|_{\mathcal{H}^2}$ for arbitrary $\xi \in \bar{\xi}$ and arbitrary predictable $Y \in \bar{Y}$.

Now we summarize our main results.

0.1 Existence and uniqueness of the mild solution in $\mathbf{H^2(T, H)}$

Under Lipschitz assumptions on the coefficients F and $S(t)B : H \to L^2(U, \mathcal{B}, \nu; H)$, $t \in]0, T]$, we show the contraction property of $\bar{\mathcal{F}}$ by the help of the isometric property of the stochastic integral and we prove the existence and uniqueness of the mild solution X as a mapping from L^2 to $H^2(T,H)$ (see theorem 4.4).

Though the above existence and uniqueness in $H^2(T,H)$, is of their own interest, our main interest is the analysis of the dependence on the initial condition $\xi \in L_0^2$. This constitutes the second set of our main results which we shall desribe now.

0.2 Dependence on the initial condition and analytic consequences

Our first result is the Gâteaux differentiability of the mild solution as a mapping $X : L_0^2 \to H^2(T,H)$ (see theorem 5.1). As a consequence we obtain a gradient estimate for the Gâteaux derivative ∂X of X and for the resolvent (R_α) associated to the mild solution. Under the additional assumptions that $S(t)$, $t \geq 0$, is quasicontractive, $\nu(U) < \infty$, B is constant and F is dissipative we get that

$$\|\partial X(x)h(t)\| \leq e^{\omega_0 t}$$

for all $x, h \in H$ and $t \geq 0$. Moreover, for all $f \in C_b^1(H,\mathbb{R})$, $R_\alpha f : H \to \mathbb{R}$ is Gâteaux differentiable for all $\alpha \geq 0$ and

$$\|\partial R_\alpha f(x)\|_{L(H,\mathbb{R})} \leq \frac{1}{\alpha - \omega_0} \sup_{x \in H} \|Df(x)\|_{L(H)} \text{ for all } \alpha > \omega_0,\ x \in H$$

(see chapter 7).

Before we describe our results more precisely we go into the details of some results that have been achieved in this field.

In [AlWuZh 97] the authors analyze SPDE's in $\mathbb{R}$ driven by a Poisson noise. Under Lipschitz assumptions, existence and uniqueness of a mild solution in L^2 is proved. This is done by using the method of Banach's fixed point theorem, i.e. the mild solution is obtained as L^2-limit of an iterating sequence.
Applebaum and Wu study in [ApWu 00] the following parabolic SPDE in $\mathbb{R}$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = a(t,x,u(t,x)) + B(t,x,u(t,x)) F_{t,x} \tag{2}$$

where $F_{t,x}$ is a so-called Lévy space-time white noise. The authors give a meaning to (2) as a stochastic integral equation of jump type, where the jump part is described by a stochastic integral with respect to a compensated Poisson random measure. As in [AlWuZh 97], again under Lipschitz assumptions on the coefficients, the unique mild solution is constructed by iteration. In this way the authors get the unique mild solution of their problem in L^2.
In [ApTa 01] the authors study stochastic differential equations driven by

infinite dimensional semimartingales with jumps on a finite dimensional smooth manifold. Existence of a unique maximal solution which has a modification which is a stochastic flow of local C^m-diffeomorphisms is proved. In [MaRu 03] the authors investigate Banachspace valued stochastic integral equations of the following type

$$\begin{aligned} X(t,\omega) =& \phi(t,\omega) + \int_0^t F(s, X(s,\omega),\omega)\,ds \\ &+ \int_0^t \int_A B(s,y,X(s,\omega))(N(ds,dy)(\omega) - \mu(ds,dy)) \end{aligned} \tag{3}$$

where $N(ds,dy) - \mu(ds,dy)$ is a compensated Poisson random measure. Under the assumption that the Banach space is separable and of type 2 and under Lipschitz assumptions on the coefficients, it is proved by Banach's fixed point theorem, that there exists an up to stochastic equivalence unique solution of (3) in L^2.

Now we go into the particulars of the structure of this work summarizing the contents and results chapterwise.

In *chapter 1* we recall some basic terminology and standard notations on stochastic processes. Our main references are the books [DaPrZa 92], [DeMe 82], [EtKu 86], [IkWa 81] and [Pr 90]. Moreover, we give a brief insight without proofs into the construction of the stochastic integral w.r.t. a real-valued local martingale as presented in [Pr 90].

In *chapter 2* we give an introduction to the theory of Poisson random measures and Poisson point processes where we shall follow largely the organization of [IkWa 81]. In the third section we present the construction of the stochastic integral of Hilbert space valued integrands w.r.t. a compensated Poisson random measure. In the style of the definition of the integral w.r.t. a Wiener process (cf. [DaPrZa 92]) or w.r.t. a square-integrable martingale (cf. [Me 82]) we define the integral by an L^2-isometry, which, in the case of the Wiener process, is just the classical Itô isometry. Independently, this was done in [Ru 04].

In *chapter 3* we present some useful properties of the stochastic integral, with detailed proofs.

In *chapter 4* we are now able to treat the question of existence and uniqueness of a mild solution in $H^2(T,H)$. In the first section we prove that under the assumption that F and $S(t)B : H \to L^2(U,\mathcal{B},\nu;H)$, $t \in]0,T]$, are Lipschitz continuous $\bar{\mathcal{F}} : L_0^2 \times H^2(T,H) \to H^2(T,H)$ is well defined, which implies the existence of a predictable version of the stochastic integral and that $\bar{\mathcal{F}}$ is a contraction in the second variable. Hence, there exists a unique mild solution $X : L_0^2 \to H^2(T,H)$, which is Lipschitz (see theorem 4.4).
This existence result as well as the definition of the stochastic integral are

subject of the preprint [Kn 03].

In *chapter 5* we analyze the first order differentiability of the mapping $\xi \mapsto X(\xi)$. Under the assumption that F and $B(\cdot, y)$ are Gâteaux differentiable such that $\partial F : H \times H \to H$, $S(t)\partial_1 B(\cdot, y)z : H \to H$ and $S(t)\partial_1 B(\cdot, \cdot)z : H \to L^2(U, \mathcal{B}, \nu, H)$, $t \in]0, T]$, are continuous, in the first section we prove the Gâteaux differentiability of $X : L_0^2 \mapsto H^2(T, H)$ (see theorem 5.1).

Chapter 6 is devoted to an analytic consequence. We show that under the additional conditions that $(A, D(A))$ is the generator of a quasi-contractive semigroup, $\nu(U) < \infty$, B is constant and F is dissipative, the Gâteaux derivative of $X : H \to H^2(T, H)$ can be estimated ω-wise in the following way

$$\|\partial X(x)h(t)\| \leq e^{\omega_0 t} \quad P\text{-a.s.} \tag{4}$$

From (4) we deduce for the resolvent $(R_\alpha)_{\alpha > \omega_0}$ associated to the mild solution that

$$\|\partial R_\alpha f(x)\|_{L(H,\mathbb{R})} \leq \frac{1}{\alpha - \omega_0} \sup_{x \in H} \|Df(x)\|_{L(H,\mathbb{R})}$$

for all $\alpha > \omega_0$, $x \in H$ and $f \in C_b^1(H)$.

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Chapter 1

Fundamentals on Stochastic Processes

In this chapter we recall some fundamental definitions and results on stochastic processes. Moreover, this chapter includes the definition of the stochastic integral w.r.t. a real-valued local martingale as presented in [Pr 90] and the well-known Itô-formula in $\mathbb{R}$. For more details we refer to the books [DaPrZa 92], [DeMe 82], [EtKu 86], [IkWa 81] and [Pr 90].

1.1 Stochastic processes

Let $(E, \| \ \|)$ be a separable Banach space and $(\Omega, \mathcal{F}, P)$ a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$.

Definition 1.1. Let $X(t)$, $t \in I$, and $Y(t)$, $t \in I$, be two E-valued stochastic processes with index set $I \subset \mathbb{R}$. X is called a *modification* or *version* of Y if $P(X(t) = Y(t)) = 1$ for all $t \in I$.
X and Y are said to be *indistinguishable* or *P-equal* if there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ $X(t,\omega) = Y(t,\omega)$ for all $t \in I$.
We say that a process X is defined *P-uniquely* by certain properties if every further process fulfilling these properties and the process X are P-equal.

Definition 1.2.

(i) An E-valued process $X(t)$, $t \geq 0$, is said to have *left* (*right*) *limits* if for P-a.e. $\omega \in \Omega$ the mapping $[0, \infty[\to E$, $t \mapsto X(t,\omega)$ has left (right) limits, i.e. the paths of X have P-a.s. left (right) limits.

(ii) An E-valued process $X(t)$, $t \geq 0$, is called *continuous*, *right-continuous* or *left-continuous* if for P-a.e. $\omega \in \Omega$ the mapping $[0,\infty[\to E$, $t \mapsto X(t,\omega)$ is continuous, right-continuous or left-continuous, respectively.

(iii) An E-valued right-continuous process $X(t)$, $t \geq 0$, with paths having left limits is called *cádlág*.

(iv) An E-valued left-continuous process $X(t)$, $t \geq 0$, with paths having right limits is called *cáglád*.

Definition 1.3. Let $X(t)$, $t \geq 0$, be an E-valued process having left limits. For $t > 0$ we define $X(t-) := \lim_{\substack{s\uparrow t \\ s<t}} X(s)$ and $\Delta X(t) := X(t) - X(t-)$.
For $t = 0$ we make the convention $X(0-) := 0$ and $\Delta X(0) := X(0)$.

Definition 1.4 (Increasing process). An $\mathbb{R}$-valued process $A(t)$, $t \geq 0$, is called *increasing process* if it is $(\mathcal{F}_t)$-adapted and has P-a.s. positive, increasing, finite and cádlág paths.

Theorem 1.5. *Let A be an increasing process. Then there exists a continuous increasing process A^c, a sequence T_n, $n \in \mathbb{N}$, of $(\mathcal{F}_t)$-stopping times and a sequence λ_n, $n \in \mathbb{N}$, of strictly positive constants such that*

$$A(t) = A^c(t) + \sum_{n=1}^{\infty} \lambda_n 1_{\{T_n \leq t\}}.$$

The process A^c is P-unique and is called the path by path continuous part of A. *The process $A - A^c$ is denoted by A^d and is called the* purely discontinuous part *or* jump part of A. *If $A^c \equiv 0$ then A is called* purely discontinuous.

Proof. [DeMe 82, VI.52, p.115] □

Remark 1.6. In the proof of the above theorem the authors define A^c and A^d in the following way. For allmost every $\omega \in \Omega$ the increasing function $A(\cdot,\omega)$ has a unique decomposition into a continuous increasing function $A^c(\cdot,\omega)$ and a purely discontinuous increasing function $A^d(\cdot,\omega)$ and moreover

$$A^d(t,\omega) = \sum_{0 \leq s \leq t} \Delta A(s,\omega).$$

This derivation of A^c and A^d has the consequence that if A and A' are two increasing processes which are P-equal then A^c and $(A')^c$ (A^d and $(A')^d$ respectively) are P-equal.

1.2 Martingales

In this section we give the basic notions of Banachspace-valued martingales and real-valued submartingales and some of their basic properties.

As in the previous section let $(E, \| \, \|)$ be a separable Banach space and $(\Omega, \mathcal{F}, P)$ a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$.

Definition 1.7 (Martingale). An E-valued stochastic process M with index set $I \subset \mathbb{R}_+$ is called $(\mathcal{F}_t)$-martingale if it is an integrable $(\mathcal{F}_t)$-adapted process such that for all $s, t \in I$ with $0 \leq s \leq t < \infty$

$$E[M(t)|\mathcal{F}_s] = M(s) \quad P\text{-a.s.}$$

Remark 1.8. For the existence and uniqueness of the conditional expectation we refer to [St 93, 5.1.22 Theorem, p.262].

Definition 1.9 (Submartingale). An $\mathbb{R}$-valued stochastic process $M(t)$, $t \in I$, with index set $I \subset \mathbb{R}_+$ is called $(\mathcal{F}_t)$-submartingale if it is an integrable $(\mathcal{F}_t)$-adapted process such that for all $s, t \in I$ with $0 \leq s \leq t < \infty$

$$E[M(t)|\mathcal{F}_s] \geq M(s) \quad P\text{-a.s.}$$

Proposition 1.10. *Let $M(t)$, $t \in I$, be an E-valued $(\mathcal{F}_t)$-martingale. Then $\|M(t)\|$, $t \in I$, is a real-valued $(\mathcal{F}_t)$-submartingale.*

Proof. [DaPrZa 92, Proposition 3.7 (i), p.78] □

Proposition 1.11 (Doob-inequality). *Let $p \in]1, \infty[$ and $M(t)$, $t \geq 0$, a right-continuous $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-submartingale. Then for $T > 0$*

$$E[\sup_{0 \leq t \leq T} M(t)^p] \leq \Big(\frac{p}{p-1}\Big)^p E[M(T)^p].$$

Proof. [EtKu 86, 2.16 Proposition (b), p.63] □

Definition 1.12. An E-valued $(\mathcal{F}_t)$-martingale $M(t)$, $t \geq 0$, is called L^2-*martingale* if $\|M(t)\|_{L^2} < \infty$ for all $t \geq 0$. We denote by $\mathcal{M}^2(E)$ the space of all E-valued cádlág L^2-martingales (with respect to the filtration $\mathcal{F}_t$, $t \geq 0$).

An E-valued $(\mathcal{F}_t)$-martingale $M(t)$, $t \geq 0$, is called *square integrable* if $\sup_{t \geq 0} \|M(t)\|_{L^2} < \infty$. We denote by $\mathcal{M}^2_\infty(E)$ the space of all E-valued cádlág, square integrable $(\mathcal{F}_t)$-martingales.

Let $T > 0$. We denote by $\mathcal{M}^2_T(E)$ the space of all E-valued cádlág $(\mathcal{F}_t)$-martingales $M(t)$, $t \in [0, T]$, such that $\sup_{t \in [0,T]} \|M(t)\|_{L^2} = \|M(T)\|_{L^2} < \infty$.

Proposition 1.13. *The space $\mathcal{M}_T^2(E)$ equipped with the norm*

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} E[\|M(t)\|^2]^{\frac{1}{2}}$$

is a Banachspace.

Proof. Clearly, $\| \ \|_{\mathcal{M}_T^2}$ defines a semi-norm on $\mathcal{M}_T^2(E)$. By considering equivalence classes with respect to $\| \ \|_{\mathcal{M}_T^2}$ $\mathcal{M}_T^2(E)$ becomes a normed space. To prove completeness assume that $(M_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_T^2(E)$, i.e.

$$\sup_{t\in[0,T]} E[\|M_n(t) - M_m(t)\|^2]^{\frac{1}{2}} \longrightarrow 0 \text{ as } n, m \to \infty.$$

Hence, for each $t \in [0, T]$ there exists $M(t) \in L^2(\Omega, \mathcal{F}_t, P; E)$ such that $\|M_n(t) - M(t)\|_{L^2} \longrightarrow 0$ as $n \to \infty$.
Obviously, the process $M(t)$, $t \in [0, T]$, has the martingale property. By the Doob-inequality 1.11 and proposition 1.10 we even know that

$$E[\sup_{t\in[0,T]} \|M_n(t) - M_m(t)\|^2]^{\frac{1}{2}} \longrightarrow 0 \text{ as } n, m \to \infty.$$

Hence, we can find a subsequence n_k, $k \in \mathbb{N}$, such that

$$P(\sup_{t\in[0,T]} \|M_{n_{k+1}}(t) - M_{n_k}(t)\| \geq 2^{-k}) \leq 2^{-k}$$

and by the lemma of Borel-Cantelli we can conclude that M_{n_k} converges P-a.s. uniformly on $[0, T]$ which implies the existence of an $(\mathcal{F}_t)$-adapted cádlág version of M which we denote again by M.
It remains to check the convergence of M_n to M in $\| \ \|_{\mathcal{M}_T^2}$:

$$\begin{aligned}
\sup_{t\in[0,T]} E[\|M(t) - M_n(t)\|^2] &\leq E[\|M(T) - M_n(T)\|^2] \\
&= \lim_{m\to\infty} E[\|M_m(T) - M_n(T)\|^2] \\
&\longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

□

Proposition 1.14. *(i) Let $M \in \mathcal{M}^2(\mathbb{R})$. Then there exists an integrable, increasing, predictable process $A(t)$, $t \geq 0$, (i.e. $A : [0, \infty[\times\Omega \to \mathbb{R}$ is measurable w.r.t. the predictable σ-field*

$$\mathcal{P}_T := \sigma(g : [0, T] \times \Omega \to \mathbb{R}, \mid g \text{ is } (\mathcal{F}_t)\text{-adapted and left-continuous}))$$

such that $M(t)^2 - A(t)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale. A is uniquely determined.

(ii) Let $M, N \in \mathcal{M}^2(\mathbb{R})$. Then there exists a process $A(t)$, $t \geq 0$, which is expressible as the difference of two predictable, integrable, increasing processes such that $M(t)N(t) - A(t)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale. A is uniquely determined.

A in (i) is denoted by $< M >$ and A in (ii) by $< M, N >$. Then $< M > = < M, M >$. $< M, N >$ is called the quadratic variation of M and N *and $< M >$ the* quadratic variation of M.

Proof. [IkWa 81, II. Proposition 2.1., p.53] □

Definition 1.15 (Local martingale). An E-valued $(\mathcal{F}_t)$-adapted process $M(t)$, $t \geq 0$, is called a local $(\mathcal{F}_t)$-martingale if there exists an increasing sequence of $(\mathcal{F}_t)$-stopping times T_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} T_n = +\infty$ P-a.s. and for $n \in \mathbb{N}$ the process $M(t \wedge T_n)1_{\{T_n>0\}}$, $t \geq 0$, is a uniformly integrable $(\mathcal{F}_t)$-martingale for each $n \in \mathbb{N}$.

Proposition 1.16. *Every E-valued $(\mathcal{F}_t)$-martingale $M(t)$, $t \geq 0$, is a local $(\mathcal{F}_t)$-martingale with localizing sequence $T_n := n$, $n \in \mathbb{N}$.*

Proof. Since $\|M(t)\|$, $t \geq 0$, is a submartingale the assertion is obvious. □

Definition 1.17. Let X be a stochastic process. *A property $\mathcal{P}$ is said to hold locally* if there exists a sequence of stopping times T_n, $n \in \mathbb{N}$, with $T_n \uparrow \infty$ P-a.s. as $n \to \infty$ such that $X(t \wedge T_n)1_{\{T_n>0\}}$, $t \geq 0$, has property $\mathcal{P}$ for each $n \in \mathbb{N}$.

In the two following sections we introduce the definition of the stochastic integral with respect to an $\mathbb{R}$-valued, cádlág local martingale and the notion of the bracket process of $\mathbb{R}$-valued, cádlág local martingales. The approach here presented and detailed proofs can be found in [Pr 90, Chapter II, Section 4-6] where the author defines the stochastic integral and the bracket process for a more general class of processes, namely semimartingales. Since by [Pr 90, III.5 Corollary, p.105] every local martingale is a semimartingale we may reduce the definitions to the class of local martingales.

1.3 The stochastic integral w.r.t. an L^2-martingale: The real-valued case

Let $M(t)$, $t \geq 0$, be a cádlág local real $(\mathcal{F}_t)$-martingale.
We define the space $\mathcal{S}$ of simple predictable processes in the following way.

Definition 1.18. A real-valued process Φ is said to be *simple predictable* if it has a representation of the following form:

$$\Phi = 1_{\{0\}}\Phi_0 + \sum_{i=1}^{n-1} 1_{]T_i,T_{i+1}]}\Phi_i$$

where $0 \leq T_1 \leq \cdots \leq T_n$ are $(\mathcal{F}_t)$-stopping times and for each $0 \leq i \leq n$ Φ_i is an $\mathcal{F}_{T_i}$-measurable real-valued random variable, where for an arbitrary $(\mathcal{F}_t)$-stopping time T, $\mathcal{F}_T$ is defined as $\{A \in \mathcal{F} \mid A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$.
Then the space $\mathcal{S}$ of simple predictable processes is a linear space.

For a simple predictable process $\Phi \in \mathcal{S}$ we define the stochastic integral process w.r.t. M by

$$\mathrm{Int}_M(\Phi)(t) := \Phi_0 M(0) + \sum_{i=1}^{n-1} \Phi_i\big(M(T_{i+1} \wedge t) - M(T_i \wedge t)\big),\ t \geq 0.$$

$\mathrm{Int}_M(\Phi)$ does not depend on the representation of Φ and

$$\mathrm{Int}_M : \mathcal{S} \to \mathcal{R} := \{X(t), t \geq 0 \mid X \text{ is a } (\mathcal{F}_t)\text{-adapted, cádlág process}\}$$

is a linear mapping.
For the extension of Int_M to a more general class of integrands

$$\mathcal{L} := \{X(t), t \geq 0 \mid X \text{ is an } (\mathcal{F}_t)\text{-adapted, cáglád process}\}$$

we need the notion of uniform convergence on compacts in probability.

Definition 1.19. A sequence of $(\mathcal{F}_t)$-adapted processes X_n, $n \in \mathbb{N}$, converges to an $(\mathcal{F}_t)$-adapted process X *uniformly on compacts in probability* (abbreviated *ucp*) if for all $t > 0$ $\sup_{0 \le s \le t} |X_n(s) - X(s)| \underset{n\to\infty}{\longrightarrow} 0$ in probability.
To emphazise that the spaces $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{L}$ are endowed with the ucp-topology we denote this spaces by $\mathcal{S}_{ucp}$,$\mathcal{R}_{ucp}$ and $\mathcal{L}_{ucp}$

Remark 1.20. The space $\mathcal{R}_{ucp}$ endowed with the topology induced by the uniform convergence on compacts in probability is a metrizable space. A compatible metric is given by

$$d_{ucp}(X,Y) := \sum_{n=1}^{\infty} \frac{1}{2^n} E[\sup_{0 \le s \le n} |X(s) - Y(s)| \wedge 1], \; X, Y \in \mathcal{R}_{ucp}.$$

The metric space $(\mathcal{R}_{ucp}, d_{ucp})$ is complete.

To extend the mapping Int_M uniquely to $\mathcal{L}$ one has to show that the linear mapping $\mathrm{Int}_M : \mathcal{S}_{ucp} \to \mathcal{R}_{ucp}$ is continuous and $\mathcal{S}_{ucp}$ is dense in $\mathcal{L}_{ucp}$. This is done in [Pr 90, II.4 Theorem 10, p.49; II.4 Theorem 11, p.50].

Definition 1.21. The continuous linear mapping $\mathrm{Int}_M : \mathcal{L}_{ucp} \to \mathcal{R}_{ucp}$ obtained as the unique extension of $\mathrm{Int}_M : \mathcal{S}_{ucp} \to \mathcal{R}_{ucp}$ is called *the stochastic integral with respect to* M.
The image of $X \in \mathcal{L}$ under the mapping Int_M will be denoted by $\int X \, dM$ and the random variable of the process $\int X \, dM$ at time $t \ge 0$ by $\int_0^t X(s) \, dM(s) = \int_{[0,t]} X(s) \, dM(s)$.
To exclude 0 in the integral we write

$$\int_{0+}^t X(s) \, dM(s) := \int_{]0,t]} X(s) \, dM(s) := \int_0^t 1_{]0,t]}(s) X(s) \, dM(s).$$

Notice that

$$\int_{]0,t]} X(s) \, dM(s) = \int_{[0,t]} X(s) \, dM(s) - X(0)M(0).$$

Proposition 1.22. *Let $M(t)$, $t \ge 0$, be a cádlág local martingale with $M(0) = 0$ P-a.s.*
Then $Int_M(X)(0) = 0$ P-a.s. for all $X \in \mathcal{L}$.

Proof. If X is a simple predictable process the assertion is obvious. If X is an arbitrary element of $\mathcal{L}$ then there exists a sequence Φ_k, $k \in \mathbb{N}$, of simple predictable processes such that $\Phi_k \longrightarrow X$ uniformly on compacts in probability as $k \to \infty$ which implies by the definition of the mapping Int_M that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} E[\sup_{0 \le s \le n} |\mathrm{Int}_M(X)(s) - \mathrm{Int}_M(\Phi_k)(s)| \wedge 1] \longrightarrow 0 \text{ as } k \to \infty.$$

Hence, there exists a subsequence k_l, $l \in \mathbb{N}$, such that

$$|\mathrm{Int}_M(X)(0) - \mathrm{Int}_M(\Phi_{k_l})(0)| \longrightarrow 0 \text{ as } l \to \infty$$

which implies that $\mathrm{Int}_M(X)(0) = 0$ P-a.s $\square$

Theorem 1.23. *Let $M \in \mathcal{M}^2_\infty(\mathbb{R})$ and $X \in \mathcal{L}$, P-a.s. bounded, then $Int_M(X) \in \mathcal{M}^2_\infty(\mathbb{R})$.*

Proof. [Pr 90, II.5 Theorem 20, p.56] $\square$

Theorem 1.24. *Let $X \in \mathcal{R}$ or $X \in \mathcal{L}$ and let Π_n, $n \in \mathbb{N}$, a sequence of partitions of $[0, \infty[$ given by $0 = t_0^n \le t_1^n \le \dots \le t_{k_n}^n < \infty$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} t_{k_n}^n = \infty$ and $\sup_{0 \le i \le k_n - 1} |t_{i+1}^n - t_i^n|$ converges to 0 as $n \to \infty$. Then*

$$\sum_{i=1}^{k_n - 1} X(t_i^n)\big(M(t_{i+1}^n \wedge \cdot) - M(t_i^n \wedge \cdot)\big) \to \int_{0+}^{\cdot} X(s-)\, dM(s)$$

as $n \to \infty$ uniformly on compacts in probability.

Proof. [Pr 90, II.5. Theorem 21, p.57] $\square$

1.4 Square bracket

As in 1.3 in this section all processes are real-valued.

Definition 1.25. Let M, N be cádlág local $(\mathcal{F}_t)$-martingales. The *bracket process* of M, N, also called simply the *bracket* of M, N, is defined by

$$[M, N]_t := M(t)N(t) - \int_0^t M(s-)\, dN(s) - \int_0^t N(s-)\, dM(s).$$

$[M, M]$ will be denoted by $[M]$ and called the *square bracket* of M.

Obviously, the mapping $(M, N) \mapsto [M, N]$ is bilinear and symmetric.

Theorem 1.26. *Let M be a cádlág local $(\mathcal{F}_t)$-martingale. The square bracket $[M]$ of M is a cádlág, $(\mathcal{F}_t)$-adapted process with P-a.s. increasing paths such that* *(i)* $[M]_0 = M(0)^2$ *and* $\Delta[M] = (\Delta M)^2$ *P-a.s.,*

(ii) *if* Π_n, $n \in \mathbb{N}$, *is a sequence of random partitions* $0 \leq T_0^n \leq T_1^n \leq \cdots \leq T_{k_n}^n$, $n \in \mathbb{N}$, *where* T_i^n, $1 \leq i \leq k_n$, *are* $(\mathcal{F}_t)$*-stopping times, such that* $\lim_{n\to\infty} T_{k_n}^n = +\infty$ *and* $\sup_{1\leq i\leq k_n-1}|T_{i+1}^n - T_i^n| \underset{n\to\infty}{\longrightarrow} 0$ *P-a.s., then*

$$M(0)^2 + \sum_{i=1}^{k_n-1} \big(M(T_{i+1}^n \wedge \cdot) - M(T_i^n \wedge \cdot)\big)^2 \longrightarrow [M]. \text{ as } n \to \infty$$

uniformly on compacts in probability.

In particular, $[M]$ *is an increasing process in the sense of Definition 1.4.*

Proof. [Pr 90, Theorem 22, p.59] □

Theorem 1.27. *Let* M, N *be cádlág, locally square integrable local* $(\mathcal{F}_t)$*-martingales. The bracket* $[M, N]$ *of* M *is the* P*-unique,* $(\mathcal{F}_t)$*-adapted, cádlág process* $A(t)$, $t \geq 0$, *with paths of finite variation on compacts such that*

(i) $MN - A$ *is a local* $(\mathcal{F}_t)$*-martingale,*

(ii) $\Delta A(t) = \Delta M(t)\Delta N(t)$ *for all* $t \geq 0$ *P-a.s.*

Proof. [Pr 90, II.6 Corollary 2, p.65] □

Remark 1.28. Let M be a cádlág local martingale and T a $(\mathcal{F}_t)$-stopping time. Then $[M]_{\cdot\wedge T} = [M(\cdot \wedge T)]$.

Proof. $[M]_{\cdot\wedge T} = [M(\cdot\wedge T)]$ is an obvious consequence of theorem 1.26 which approximates $[M]$ by sums. □

At this point, we may introduce the notion of a purely discontinuous local martingale and of the continuous part of a local martingale.

Definition 1.29. Let M be a cádlág local martingale. If $[M]$ is purely discontinuous then M is called *quadratic pure jump.*

Theorem 1.30. *Let* M *be a cádlág local martingale. Then* M *has a* P*-unique decomposition as a sum of a continuous local martingale, called the* continuous part of M *and denoted by* M^c *, and a quadratic pure jump local martingale, called the* jump part of M *and denoted by* M^d.

Proof. [DeMe 82, VIII.43 Theorem (a), p.353] □

To close this section about the bracket process we want to consider the square bracket of the stochastic integral process $\int X\,dM$.

Proposition 1.31. *Let M be a real-valued, locally square integrable, cádlág local $(\mathcal{F}_t)$-martingale and $X \in \mathcal{L}$, real-valued. Then $\int X\,dM$ is a locally square integrable, cádlág local $(\mathcal{F}_t)$-martingale and*

$$[\int_0^{\cdot} X(s)\,dM(s)]_t = \int_{[0,t]} X(s)^2\,d[M]_s,\ t \geq 0,$$

where the integral on the right-hand side is a Stieltjes-integral taken for every $\omega \in \Omega$.

Proof. [Pr 90, II.5 Theorem 20, p.56; II.6 Theorem 29, p.68] $\square$

Chapter 2

The Stochastic Integral w.r.t. Poisson Point Processes

In the first two sections of this chapter we present the notions of random measures and point processes. Our main reference is [IkWa 81, I.8 and I.9] and we shall follow the set-up presented therein. In the third section we define the stochastic integral with respect to a compensated Poisson random measure.

2.1 Poisson random measures

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(E, \mathcal{S})$ a measurable space.

Let $\mathbb{M}$ be the space of $\mathbb{Z}_+ \cup \{+\infty\}$-valued measures on $(E, \mathcal{S})$ and

$$\mathcal{B}_{\mathbb{M}} := \sigma(\mathbb{M} \ni \mu \mapsto \mu(B) \,|\, B \in \mathcal{S}).$$

Definition 2.1 (Poisson random measure). A random variable $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ is called *Poisson random measure* on $(E, \mathcal{S})$ (and $(\Omega, \mathcal{F}, P)$) if the following conditions hold.

(i) For all $B \in \mathcal{S}$: $\Pi(B) : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is Poisson distributed with parameter $E[\Pi(B)]$, i.e.:

$$P(\Pi(B) = n) = exp\big(-E[\Pi(B)]\big)E[\Pi(B)]^n/n!, \; n \in \mathbb{N} \cup \{0\}.$$

If $E[\Pi(B)] = +\infty$ then $\Pi(B) = +\infty$ P-a.s.

(ii) If $B_1, \ldots, B_m \in \mathcal{S}$ are pairwise disjoint then $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.

Remark 2.2. Notice that if Π is a Poisson random measure then the mapping $\Omega \to \mathbb{Z}_+ \cup \{+\infty\}$, $\omega \mapsto \Pi(\omega)(B)$, $B \in \mathcal{B}$, is $\mathcal{F}$-measurable by the measurability of $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ and the definition of $\mathcal{B}_{\mathbb{M}}$.

After giving the definition of a Poisson random measure we have to check the existence of such an object. For this purpose we need the following two lemmas.

Lemma 2.3. *Let $m \in \mathbb{N}$ and μ and ν be two probability measures on $[0,\infty[^m$. If for all $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$*

$$\int_{[0,\infty[^m} e^{-\langle \alpha, x\rangle}\, \mu(dx) = \int_{[0,\infty[^m} e^{-\sum_{j=1}^m \alpha_j x_j}\, \mu(d(x_1, \dots, x_m))$$
$$= \int_{[0,\infty[^m} e^{-\sum_{j=1}^m \alpha_j x_j}\, \nu(d(x_1, \dots, x_m)) = \int_{[0,\infty[^m} e^{-\langle \alpha, x\rangle}\, \nu(dx)$$

then $\mu = \nu$.

Proof. Denote by $\mathcal{H}$ the space of all $\mathcal{B}(\mathbb{R}_+^m)$-measurable, bounded functions $f : \mathbb{R}_+^m \to \mathbb{R}$ such that $\int_{\mathbb{R}_+^m} f\, d\mu = \int_{\mathbb{R}_+^m} f\, d\nu$. Then $\mathcal{H}$ is a monotone vector space. Moreover, define

$$\mathcal{A} := \{\mathbb{R}_+^m \to \mathbb{R}, x \mapsto \exp(-\sum_{j=1}^m \alpha_j x_j) \,|\, \alpha_j \in \mathbb{Q}_+, 1 \le j \le m\}.$$

Then $\mathcal{A}$ is a class of bounded, measurable functions, which is closed under multiplication and which is a subset of $\mathcal{H}$ by assumption. By the monotone class theorem it follows that $\sigma(\mathcal{A})_b \subset \mathcal{H}$.
Moreover, $\mathcal{A}$ as a subset of $\{f : \mathbb{R}_+^m \to \mathbb{R} \,|\, f$ is bounded,$\mathcal{B}(\mathbb{R}_+^m)$-measurable$\}$ is countable and separates the points of $\mathbb{R}_+^m$. Thus, we obtain that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}_+^m)$ and $\mathcal{B}(\mathbb{R}_+^m)_b \subset \mathcal{H}$. In particular, we get for $A \in \mathcal{B}(\mathbb{R}_+^m)$ that $\mu(A) = \nu(A)$. □

Lemma 2.4. *Let X be a Poissonian random variable on $(\Omega, \mathcal{F}, P)$ with parameter $c > 0$, i.e. $X : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ such that for all $n \in \mathbb{N} \cup \{0\}$: $P(X = n) = exp(-c)\dfrac{c^n}{n!}$. Then*

$$E\left[e^{\alpha X}\right] = \int_0^\infty e^{\alpha x}\, P \circ X^{-1}(dx) = \sum_{n=0}^\infty e^{\alpha n} e^{-c} \frac{c^n}{n!} = exp(c\,(e^\alpha - 1))$$

for all $\alpha \in \mathbb{R}$.

Theorem 2.5. *Given a σ-finite measure m on $(E, \mathcal{S})$ there exists a complete probability space $(\Omega, \mathcal{F}, P)$ such that there exists a Poisson random measure Π on $(E, \mathcal{S})$ and $(\Omega, \mathcal{F}, P)$ with $E[\Pi(B)] = m(B)$ for all $B \in \mathcal{S}$. m is then called the* mean measure *or* intensity measure *of the Poisson random measure Π.*

Proof. [IkWa 81, I. Theorem 8.1, p.42]

Step 1. $m(E) < \infty$.
There exists a complete probability space $(\Omega, \mathcal{F}, P)$ such that there exist the following family of independent random variables: a Poissonian random variable N with parameter $c := m(E)$ and a sequence of independent E-valued random variables $\xi_1, \xi_2, \ldots$ with distribution $\frac{1}{c}\, m$, also independent of N.
Define $\Pi := \sum_{k=1}^{N} \delta_{\xi_k}$. If $N = 0$ then $\sum_{k=1}^{N} \delta_{\xi_k}(B) := 0$.

Claim 1. Let $B \in \mathcal{S}$. Then $\Pi(B)$ is Poisson distributed with parameter $m(B)$.

Let $\alpha \in \mathbb{R}_+$, then

$$\int_{[0,\infty[} e^{-\alpha x}\, P \circ \Pi(B)^{-1}(dx) = E[e^{-\alpha \Pi(B)}]$$

$$\begin{aligned}
&= E\big[\exp(-\alpha \sum_{k=1}^{N} \delta_{\xi_k}(B))\big] = E\big[\sum_{n=0}^{\infty} \exp(-\alpha \sum_{k=1}^{n} 1_B(\xi_k)) 1_{\{N=n\}}\big] \\
&= \sum_{n=0}^{\infty} E\big[\prod_{k=1}^{n} \exp(-\alpha 1_B(\xi_k)) 1_{\{N=n\}}\big] \\
&= \sum_{n=0}^{\infty} \prod_{k=1}^{n} E\big[\exp(-\alpha 1_B(\xi_k))\big] P(N=n), \text{ since } N, \xi_k, k \in \mathbb{N}, \text{ are independent,} \\
&= \sum_{n=0}^{\infty} E\big[\exp(-\alpha 1_B(\xi_1))\big]^n e^{-c} \frac{c^n}{n!}, \text{ since } \xi_k, k \in \mathbb{N}, \text{ are i.i.d.,} \\
&= \exp\big(c\,(E[\exp(-\alpha 1_B(\xi_1))] - 1)\big) \\
&= \exp\big(c\, P(\xi_1 \in B) e^{-\alpha} + c\, P(\xi_1 \in B^c) - c)\big) \\
&= \exp\big(c\, \frac{m(B)}{c} e^{-\alpha} + c\,(1 - \frac{m(B)}{c}) - c\big) \\
&= \exp\big(m(B)(e^{-\alpha} - 1)\big).
\end{aligned}$$

By lemma 2.4 and lemma 2.3 the assertion follows.

Claim 2. Let $B_1, \ldots, B_m \in \mathcal{S}$ pairwise disjoint. Then $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.

Let $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_+$, then:

$$\int_{[0,\infty[^m} \exp(-\sum_{j=1}^{m} \alpha_j x_j)\, P \circ (\Pi(B_1), \ldots, \Pi(B_m))^{-1} d(x_1, \ldots, x_m)$$

$$= E\big[\exp(-\sum_{j=1}^{m} \alpha_j \Pi(B_j))\big]$$

$$
\begin{aligned}
&= E\big[\sum_{n=0}^{\infty} \exp\big(-\sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} 1_{B_j}(\xi_k)\big) 1_{\{N=n\}}\big] \\
&= \sum_{n=0}^{\infty} \prod_{k=1}^{n} E\big[\exp\big(-\sum_{j=1}^{m} \alpha_j 1_{B_j}(\xi_k)\big)\big] e^{-c}\, \frac{c^n}{n!} \\
&= \sum_{n=0}^{\infty} E\big[\exp(-\sum_{j=1}^{m} \alpha_j 1_{B_j}(\xi_1))\big]^n e^{-c}\, \frac{c^n}{n!} \\
&= \exp\big\{c\,\big(E\big[\exp(-\sum_{j=1}^{m} \alpha_j 1_{B_j}(\xi_1))\big] - 1\big)\big\} \\
&= \exp\Big\{c\,\big(E\big[\; 1_{\{\xi_1 \in \bigcup_{j=1}^{m} B_j\}} \exp(-\sum_{j=1}^{m} \alpha_j 1_{B_j}(\xi_1)) \\
&\qquad\qquad + 1_{\{\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c\}} \exp(-\sum_{j=1}^{m} \alpha_j 1_{B_j}(\xi_1))\big] - 1\big)\Big\} \\
&= \exp\Big\{c\big(E\big[\sum_{j=1}^{m} 1_{\{\xi_1 \in B_j\}} e^{-\alpha_j} + 1_{\{\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c\}}\big] - 1\big)\Big\} \\
&= \exp\Big\{c\big(\sum_{j=1}^{m} P(\xi_1 \in B_j) e^{-\alpha_j} + P(\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c) - 1\big)\Big\} \\
&= \exp\Big\{c\big(\sum_{j=1}^{m} \frac{m(B_j)}{c}\, e^{-\alpha_j} + 1 - \sum_{j=1}^{m} \frac{m(B_j)}{c} - 1\big)\Big\} \\
&= \exp\Big(\sum_{j=1}^{m} m(B_j)(e^{-\alpha_j} - 1)\Big) = \prod_{j=1}^{m} \exp\big(m(B_j)(e^{-\alpha_j} - 1)\big) \\
&= \prod_{j=1}^{m} \int_0^{\infty} \exp(-\alpha_j x_j)\, P \circ \Pi(B_j)^{-1}(dx_j), \text{ by lemma 2.4 and claim 1,} \\
&= \int_{[0,\infty[^m} \exp(-\sum_{j=1}^{m} \alpha_j x_j)\; P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1} \\
&\qquad\qquad\qquad\qquad\qquad d(x_1, \ldots, x_m)
\end{aligned}
$$

Hence, by lemma 2.3, we can conclude that

$$P \circ (\Pi(B_1), \ldots, \Pi(B_m))^{-1} = P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1}$$

which implies the required independence.

Step 2. m is σ-finite.
There exist $E_i \in \mathcal{S}$, $i \in \mathbb{N}$, pairwise disjoint such that $m(E_i) < \infty$ for all $i \in \mathbb{N}$ and $E = \bigcup_{i=1}^{\infty} E_i$. Set $m_i := m(\cdot \cap E_i)$, $i \in \mathbb{N}$.
As in step 1 there exists a complete probability space $(\Omega, \mathcal{F}, P)$ such that there exist the following families of random variables.

For each $i \in \mathbb{N}$ there exists a Poissonian random variable N_i with parameter $c_i := m(E_i)$ and a family of independent E_i-valued random variables $\xi_1^i, \xi_2^i, \dots$ with distribution $\frac{1}{c_i} m_i$, also independent of N_i. Moreover, the families of random variables $\{N_i, \xi_1^i, \xi_2^i, \dots\}$, $i \in \mathbb{N}$, shall be independent.
Let Π_i be the Poisson random measure on E_i associated with N_i and $\xi_1^i, \xi_2^i, \dots$ with intensity measure m_i as defined in step 1.
Define $\Pi := \sum_{i=1}^{\infty} \Pi_i := \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{\xi_k^i}$. Then one has for $B \in \mathcal{S}$ that

$$\begin{aligned}\Pi(B) &= \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{\xi_k^i}(B) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_B(\xi_k^i) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_{B \cap E_i}(\xi_k^i) \\ &= \sum_{i=1}^{\infty} \Pi_i(B \cap E_i)\end{aligned}$$

and

$$\begin{aligned}m(B) &= \sum_{i=1}^{\infty} m(B \cap E_i) = \sum_{i=1}^{\infty} E[\Pi_i(B \cap E_i)], \text{ by step 1, claim 1} \\ &= E[\Pi(B)].\end{aligned}$$

Claim 1. Let $B \in \mathcal{S}$ with $E[\Pi(B)] < \infty$ then $\Pi(B)$ is Poisson distributed with parameter $m(B)$.

Let $\alpha \in \mathbb{R}_+$, then:

$$\begin{aligned}&E[e^{-\alpha \Pi(B)}] \\ &= \lim_{m \to \infty} E\big[\exp(-\alpha \sum_{i=1}^{m} \Pi_i(B \cap E_i))\big] \\ &= \lim_{m \to \infty} \prod_{i=1}^{m} E\big[\exp(-\alpha\, \Pi_i(B \cap E_i))\big]\text{, since } \{N_i, \xi_1^i, \xi_2^i, \dots\},\ i \in \mathbb{N}, \text{ are independent,} \\ &= \lim_{m \to \infty} \prod_{i=1}^{m} \exp(m(B \cap E_i)(e^{-\alpha} - 1)), \text{ by step 1, claim 1} \\ &= \exp(m(B)(e^{-\alpha} - 1)).\end{aligned}$$

By lemma 2.4 and lemma 2.3 the assertion follows.

Claim 2. Let $B \in \mathcal{S}$ with $m(B) = E[\Pi(B)] = +\infty$. Then $\Pi(B) = +\infty$ P-a.s.

$$P(\Pi(B) = +\infty) = P(\bigcap_{m \in \mathbb{N}} \bigcup_{i \geq m} \{\Pi_i(B \cap E_i) > 0\})$$

since

$$\begin{aligned}
&P(\bigcap_{i\geq m}\{\Pi_i(B\cap E_i)>0\}^c) = P(\bigcap_{i\geq m}\{\Pi_i(B\cap E_i)=0\}) \\
&= \lim_{n\to\infty} P(\bigcap_{i=m}^{m+n}\{\Pi_i(B\cap E_i)=0\}) = \lim_{n\to\infty}\prod_{i=m}^{m+n} e^{-m(B\cap E_i)} \\
&= \lim_{n\to\infty}\exp(-\sum_{i=m}^{m+n} m(B\cap E_i)) = 0
\end{aligned}$$

it follows that $P(\bigcup_{i\geq m}\{\Pi_i(B\cap E_i)>0\}) = 1$ for all $m\in\mathbb{N}$ and therefore $P(\Pi(B)=+\infty)=1$.

Claim 3. Let $B_1,\ldots,B_m\in\mathcal{S}$ pairwise disjoint. Then $\Pi(B_1),\ldots,\Pi(B_m)$ are independent.

Since $\Pi(B)=+\infty$ P-a.s. if $m(B)=+\infty$, without loss of generalization we can assume that $E[\Pi(B_j)]=m(B_j)<\infty$ for all $j\in\{1,\ldots,m\}$ then one gets for all $\alpha_1,\ldots,\alpha_m\in\mathbb{R}_+$ that

$$\begin{aligned}
&E\big[\exp(-\sum_{j=1}^{m}\alpha_j\Pi(B_j))\big] \\
&= E\big[\exp(-\sum_{i=1}^{\infty}\sum_{j=1}^{m}\alpha_j\Pi_i(B_j\cap E_i))\big] \\
&= \lim_{n\to\infty} E\big[\exp(-\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_j\Pi_i(B_j\cap E_i))\big] \\
&= \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{j=1}^{m} E\big[\exp(-\alpha_j\Pi_i(B_j\cap E_i))\big], \text{ by step 1, claim 1,} \\
&= \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{j=1}^{m}\exp\big(m(B_j\cap E_i)(e^{-\alpha_j}-1)\big), \text{ by step 1, claim 2,} \\
&= \prod_{j=1}^{m}\exp\big(m(B_j)(e^{-\alpha_j}-1)\big).
\end{aligned}$$

As in step 1, claim 2, this implies the stated independence.

$\square$

2.2 Point processes and Poisson point processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(U, \mathcal{B})$ a measurable space.

Definition 2.6 (Point function on U). A *point function p on U* is a mapping $p : D_p \subset (0, \infty) \to U$ where the domain D_p is a countable subset of $(0, \infty)$.
p defines a measure $N_p(dt, dy)$ on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{B})$ in the following way.
Define $\bar{p} : D_p \to (0, \infty) \times U$, $t \mapsto (t, p(t))$ and denote by c the counting measure on $(D_p, \mathcal{P}(D_p))$, i.e. $c(A) := \#A$ for all $A \in \mathcal{P}(D_p)$.
For $\bar{B} \in \mathcal{B}([0, \infty)) \otimes \mathcal{B}$ define

$$N_p(\bar{B}) := c(\bar{p}^{-1}(\bar{B})).$$

Then, in particular, we have for all $A \in \mathcal{B}([0, \infty))$ and $B \in \mathcal{B}$

$$N_p(A \times B) = \#\{t \in D_p | t \in A, p(t) \in B\}.$$

Notation: $N_p(t, B) := N_p(]0, t] \times B)$, $t \geq 0$, $B \in \mathcal{S}$.

Let $\mathcal{P}_U$ be the space of all point functions on U and

$$\mathcal{B}_{\mathcal{P}_U} := \sigma(\mathcal{P}_U \ni p \mapsto N_p(t, B) \,|\, t > 0, B \in \mathcal{B})$$

Definition 2.7 (Point process). A *point process on U* (and $(\Omega, \mathcal{F}, P)$) is a random variable $p : (\Omega, \mathcal{F}) \to (\mathcal{P}_U, \mathcal{B}_{\mathcal{P}_U})$.

Remark 2.8. Notice that if p is a point process the mapping $\Omega \to \mathbb{Z}_+ \cup \{+\infty\}$, $\omega \mapsto N_{p(\omega)}(t, B)$ is $\mathcal{F}$-measurable for all $t > 0$ and $B \in \mathcal{B}$ by the $\mathcal{F}/\mathcal{B}_{\mathcal{P}_U}$-measurability of p and the definition of $\mathcal{B}_{\mathcal{P}_U}$.

Definition 2.9. Let p be a point process on U and $(\Omega, \mathcal{F}, P)$.

(i) p is called *stationary* if for every $t > 0$, p and $\theta_t p$ have the same probability law, where θ_t is given by $\theta_t : (0, \infty) \to (0, \infty)$, $s \mapsto s + t$ and $\theta_t p$ is defined by $D_{\theta_t p} := \{s \in (0, \infty) \,|\, \theta_t(s) = s + t \in D_p\}$ and $(\theta_t p)(s) := p(\theta_t(s)) = p(s + t)$.

(ii) p is called *σ-finite* if there exist $U_i \in \mathcal{B}$, $i \in \mathbb{N}$, such that $U_i \uparrow U$ as $i \to \infty$ and $E[N_p(t, U_i)] < \infty$ for all $t > 0$ and $i \in \mathbb{N}$.

(iii) p is called *Poisson point process* if there exists a Poisson random measure Π on $((0, \infty) \times U, \mathcal{B}(0, \infty) \otimes \mathcal{B})$ such that there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and for all $\bar{B} \in \mathcal{B}(0, \infty) \otimes \mathcal{B}$: $N_{p(w)}(\bar{B}) = \Pi(\omega)(\bar{B})$.

The next proposition characterizes the stationary Poisson point processes.

Proposition 2.10. *Let p be a σ-finite Poisson point process on U and $(\Omega, \mathcal{F}, P)$. Then p is stationary if and only if there exists a σ-finite measure ν on $(U, \mathcal{B})$ such that*

$$E[N_p(dt, dy)] = \lambda(dt) \otimes \nu(dy)$$

where λ denotes the Lebesgue-measure on $(0, \infty)$. In this case ν is unique and called characteristic measure *of p.*

Proof. "$\Leftarrow$" Suppose that there exists a σ-finite measure ν on $(U, \mathcal{B})$ such that

$$E[N_p(dt, dy)] = \lambda(dt) \otimes \nu(dy).$$

We have to show that p is stationary.
Let $t > 0$.

$$\begin{aligned}\mathcal{B}_{\mathcal{P}_U} &:= \sigma(\,\mathcal{P}_U \to \mathbb{Z}_+ \cup \{\infty\}, p \mapsto N_p(t, B) \,|\, t > 0, B \in \mathcal{B}) \\ &= \sigma\big(\underbrace{\bigcap_{i=1}^{n} \{p \in \mathcal{P}_U \,|\, N_p(t_i, B_i) = m_i\} \mid t_i > 0, B_i \in \mathcal{B}, m_i \in \mathbb{Z}_+, 1 \le i \le n, \; n \in \mathbb{N}}_{=:\mathcal{E}}\big)\end{aligned}$$

Since $\mathcal{E}$ is stable under intersections it is enough to check that for all $A \in \mathcal{E}$

$$P(p \in A) = P(\theta_t p \in A)$$

If $A \in \mathcal{E}$ then there exists $m \in \mathbb{N}$ such that for all $1 \le l \le m$ there exist $0 \le s_j^l < t_j^l < \infty$, $k_j^l \in \mathbb{N}$ and $C_j^l \in \mathcal{B}$, $1 \le j \le n^l$, such that $]s_j^l, t_j^l] \times C_j^l$, $1 \le j \le n^l$ are pairwise disjoin and such that

$$A = \bigcup_{1 \le l \le m} \underbrace{\bigcap_{1 \le j \le n^l} \{N_\cdot(]s_j^l, t_j^l] \times C_j^l) = k_j^l\}}_{=:A_l}$$

where A_l, $1 \le l \le m$, are pairwise disjoint. To prove that $P(p \in A) = P(\theta_t p \in A)$ for all $A \in \mathcal{E}$ it suffices to consider the case $A = \bigcap_{i=1}^n \{N_\cdot(]s_i, t_i] \times B_i) = m_i\}$, $0 \le s_i < t_i < \infty$, $B_i \in \mathcal{B}$, such that $]s_i, t_i] \times B_i$, $1 \le i \le n$, are pairwise disjoint, $m_i \in \mathbb{Z}_+, 1 \le i \le n$ Then

$$\begin{aligned}&P(p \in A) \\ =\,& P(\bigcap_{i=1}^{n} \{N_p(]s_i, t_i] \times B_i) = m_i\}\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} P(N_p(]s_i,t_i] \times B_i) = m_i), \quad \text{by Definition 2.1(ii)} \\
&= \prod_{i=1}^{n} E\big[N_p(]s_i,t_i] \times B_i)\big]^{m_i} \frac{\exp\big(-E[N_p(]s_i,t_i] \times B_i)]\big)}{m_i!} \\
&= \prod_{i=1}^{n} ((t_i - s_i)\nu(B_i))^{m_i} \frac{\exp(-(t_i - s_i)\nu(B_i))}{m_i!} \\
&= \prod_{i=1}^{n} E\big[N_p(]s_i+s,t_i+s] \times B_i)\big]^{m_i} \frac{\exp\big(-E[N_p(]s_i+s,t_i+s] \times B_i)]\big)}{m_i!} \\
&= \prod_{i=1}^{n} P(N_p(]s_i+s,t_i+s] \times B_i) = m_i) \\
&= P(\bigcap_{i=1}^{n} \{N_{\theta_s p}(]s_i,t_i] \times B_i) = m_i\} \\
&= P(\theta_s p \in A)
\end{aligned}
$$

"$\Rightarrow$" Suppose that p is stationary.
Define for fixed $B \in \mathcal{B}$ a measure on $\big([0,\infty), \mathcal{B}([0,\infty))\big)$ by

$$\mu_B(A) := E[N_p(A \times B)].$$

Then, for all $t > 0$ and $A \in \mathcal{B}([0,\infty))$

$$
\begin{aligned}
\mu_B(A) &= E[N_p(A \times B)] = E[N_{\theta_t p}(A \times B)] \\
&= E[N_p(\theta_t(A) \times B)] = \mu_B(\theta_t(A)),
\end{aligned}
$$

i.e. μ_B is translation invariant and hence there exists a unique constant $\nu(B) \geq 0$ such that $\mu_B = \nu(B)\lambda$. ν defines a measure on $(U, \mathcal{B})$ (the σ-additivity is a consequence of the uniqueness of $\nu(B)$).
Moreover, from the σ-finiteness of p follows the σ-finiteness of ν by the fact that for all $B \in \mathcal{B}$, $\nu(B) = E[N_p(1, B)]$. □

Theorem 2.11. *Given a σ-finite measure ν on $(U, \mathcal{B})$ there exists a complete probability space $(\Omega, \mathcal{F}, P)$ such that there exists a stationary, σ-finite Poisson point process on U and $(\Omega, \mathcal{F}, P)$ with characteristic measure ν.*

Proof. By theorem 2.5 there exists a complete probability space $(\Omega, \mathcal{F}, P)$ such there exists a Poisson random measure Π on $((0,\infty) \times U, \mathcal{B}(0,\infty) \otimes \mathcal{B})$ (and $(\Omega, \mathcal{F}, P)$) with intensity measure $\lambda \otimes \nu$. Remember the construction of Π in the proof of theorem 2.5.
There exist U_j, $j \in \mathbb{N}$, pairwise disjoint such that $U = \bigcup_{j \in \mathbb{N}} U_j$ and $c_j := \nu(U_j) < \infty$.

For $i, j \in \mathbb{N}$ let

- $N_{i,j}$ be a Poissonian random variable with parameter c_j,
- $\xi_k^{i,j} = (t_k^{i,j}, x_k^{i,j})$, $k \in \mathbb{N}$, i.i.d. $]i-1, i] \times U_j$-valued random variables with distribution $\lambda(\cdot \cap]i-1, i]) \otimes (\frac{1}{c_j}\nu(\cdot \cap U_j))$, also independent of $N_{i,j}$.

Moreover, the families of random variables $\{N_{i,j}, \xi_1^{i,j}, \xi_2^{i,j}, \dots\}_{i,j \in \mathbb{N}}$, are independent.
Then

$$\Pi := \sum_{i,j=1}^{\infty} \Pi_{i,j} := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{N_{i,j}} \delta_{(t_k^{i,j}, x_k^{i,j})}$$

is a Poisson random measure on $((0,\infty) \times U, \mathcal{B}(0,\infty) \otimes \mathcal{B})$ with intensity measure $\lambda \otimes \nu$ and for $\bar{B} \in \mathcal{B}(0,\infty) \otimes \mathcal{B}$

$$\Pi(\bar{B}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Pi_{i,j}(\bar{B} \cap (]i-1, i] \times U_j)). \tag{2.1}$$

.
Then there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$

$$\Pi(\omega)(\{t\} \times U) = 1 \text{ or } 0 \text{ for all } t > 0, \text{ since}$$

$$\begin{aligned}
& P(\{\omega \in \Omega \mid \exists t > 0 \text{ s.t. } \Pi(\{t\} \times U) > 1\}) \\
&= P(\bigcup_{i=1}^{\infty} \{\omega \in \Omega \mid \exists t \in]i-1, i] \text{ s.t. } \Pi(\{t\} \times U) > 1\}) \\
&\leq \sum_{i=1}^{\infty} P(\,\{\omega \in \Omega \mid \exists t \in]i-1, i] \text{ with } \sum_{j=1}^{\infty} \Pi_{i,j}(\{t\} \times U_j) > 1\}) \\
&\leq \sum_{i=1}^{\infty} P(\bigcup_{j,k=1}^{\infty} \{\omega \in \Omega \mid \exists t \in]i-1, i] \text{ with } \Pi_{i,j}(\{t\} \times U_j) \geq 1, \\
&\qquad\qquad \Pi_{i,k}(\{t\} \times U_k) \geq 1\}) \\
&\leq \sum_{i=1}^{\infty} \sum_{j,k=1}^{\infty} P(\bigcup_{n,m} \{\omega \in \Omega \mid \exists t \in]i-1, i] \text{ with } \delta_{\xi_n^{i,j}(\omega)}(\{t\} \times U_j) = 1 \text{ and} \\
&\qquad\qquad \delta_{\xi_m^{i,k}(\omega)}(\{t\} \times U_k) = 1\}) \\
&\leq \sum_{i=1}^{\infty} \sum_{j,k=1}^{\infty} \sum_{n,m=1}^{\infty} P(\,\{\omega \in \Omega \mid \exists t \in]i-1, i] \text{ with } t_n^{i,j}(\omega) = t_m^{i,k}(\omega) = t\}) \\
&= \sum_{i=1}^{\infty} \sum_{j,k=1}^{\infty} \sum_{n,m=1}^{\infty} P \circ (t_n^{i,j}, t_m^{i,k})^{-1}(\{(t,t) \mid t \in]i-1, i]\})
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \sum_{j,k=1}^{\infty} \sum_{n,m=1}^{\infty} \lambda \otimes \lambda(\{(t,t) \,|\, t \in]i-1,i]\})$$
$$= 0.$$

If $\omega \in N^c$ and $t > 0$, then there exists $i \in \mathbb{N}$ such that $t \in]i-1,i]$. Then $\Pi(\omega)(\{t\} \times U) = 1$ if and only if

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{k=1}^{N_{i,j}(\omega)} \delta_{(t_k^{i,j}(\omega), x_k^{i,j}(\omega))}(\{t\} \times U_j) &= \sum_{j=1}^{\infty} \Pi_{i,j}(\omega)(\{t\} \times U_j) \\
&= \Pi(\omega)(\{t\} \times U) \text{ , by equation (2.1),} \\
&= 1,
\end{aligned}$$

i.e. $\Pi(\omega)(\{t\} \times U) = 1$ if and only if $\exists!\, j \in \mathbb{N}$, $\exists!\, k \in \{1, \dots, N_{i,j}(\omega)\}$ such that $t = t_k^{i,j}(\omega)$.
Now we can define

$$\begin{aligned}
D_{p(\omega)} &:= \{t \in (0,\infty) \,|\, \Pi(\omega)(\{t\} \times U) \neq 0\} \\
&= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{t_k^{i,j}(\omega) \,|\, k \in \{1, \dots, N_{i,j}(\omega)\}\}
\end{aligned}$$

and

$$p(\omega)(t_k^{i,j}) := x_k^{i,j}(\omega),\ k \in \{1, \dots, N_{i,j}(\omega)\},\ i,j \in \mathbb{N}.$$

By the above considerations $p(\omega)$ is well defined.
If $\omega \in N$ then define $p_0 \in \mathcal{P}_U$ by $D_p := \{t_0\} \subset (0,\infty)$ and $p_0(t_0) = x_0 \in U$ and set $p(\omega) = p_0$.

Claim 1. $N_p = \Pi$ P-a.s.

Since Π is a Poisson random measure on $(0,\infty) \times U$ with intensity measure $\lambda \otimes \nu$ we know that $E[\Pi(]0,i] \times U_j)] < \infty$ for all $i,j \in \mathbb{N}$. Hence there exists a P-nullset $\tilde{N} \in \mathcal{F}$ such that for all $\omega \in \tilde{N}^c$ $\Pi(\omega)(]0,i] \times U_j) < \infty$ for all $i,j \in \mathbb{N}$.
Let $\omega \in (N \cup \tilde{N})^c$, $A \in \mathcal{B}(0,\infty)$ and $B \in \mathcal{B}$ then:

$$\begin{aligned}
&\Pi(\omega)(A \times B) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{N_{i,j}(\omega)} \delta_{(t_k^{i,j}, x_k^{i,j})(\omega)}\big((A \cap]i-1,i]) \times (B \cap U_j)\big) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \#\{s \in]i-1,i] \,|\, s \in A, \exists k \in \{1, \dots, N_{i,j}(\omega)\} \text{ such that } s = t_k^{i,j}(\omega) \\
&\qquad\qquad \text{and } x_k^{i,j}(\omega) \in B \cap U_j\}
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \#\{s \in]i-1, i] \cap D_{p(\omega)} \,|\, s \in A, p(\omega)(s) \in B \cap U_j\},$$
$$= \sum_{i=1}^{\infty} \#\{s \in]i-1, i] \cap D_{p(\omega)} \,|\, s \in A, p(\omega)(s) \in B\},$$
$$= \#\{s \in D_{p(\omega)} \,|\, s \in A, p(\omega)(s) \in B\}$$
$$= N_{p(\omega)}(A \times B).$$

Since $\{A \times B | A \in \mathcal{B}(0,\infty), B \in \mathcal{B}\}$ is a $\cap$-stable generator of $\mathcal{B}(0,\infty) \otimes \mathcal{B}$ and $N_{p(\omega)}(]0,i] \times \bigcup_{j=1}^{i} U_j) = \Pi(\omega)(]0,i] \times \bigcup_{j=1}^{i} U_j) < \infty$ where $]0,i] \times \bigcup_{j=1}^{i} U_j \uparrow (0,\infty) \times U$ we get that $N_{p(\omega)} = \Pi(\omega)$.

Claim 2. For all $\bar{B} \in \mathcal{B}(0,\infty) \otimes \mathcal{B}$ the mapping $N_p(\bar{B})$ is $\mathcal{F}$-measurable and $E[N_p(dt, dx)] = \lambda(dt) \otimes \nu(dx)$.

Since $N_p(\bar{B}) = \Pi(\bar{B})$ P-a.s. the measurability is obvious by remark 2.2 and the completness of $(\Omega, \mathcal{F}, P)$. Now $E[N_p(\bar{B})]$ is well defined and we obtain that $E[N_p(\bar{B})] = E[\Pi(\bar{B})] = \lambda \otimes \nu(\bar{B})$, since Π is a Poisson random measure with intensity measure $\lambda \otimes \nu$.

Claim 3. $p : \Omega \to \mathcal{P}_U$ is $\mathcal{F}/\mathcal{B}_{\mathcal{P}_U}$-measurable.

$$\begin{aligned} \mathcal{B}_{\mathcal{P}_U} &= \sigma(\mathcal{P}_U \to \mathbb{Z}_+ \cup \{+\infty\}, p \mapsto N_p(t, B) \,|\, t > 0, B \in \mathcal{B}) \\ &= \sigma(\{p \in \mathcal{P}_U \,|\, N_p(t, B) = m\} \,|\, t > 0, B \in \mathcal{B}, m \in \mathbb{Z}_+) \end{aligned}$$

and for $t > 0$, $B \in \mathcal{B}$ and $m \in \mathbb{Z}_+$ one gets by claim 2 that

$$\{p \in \{N_{\cdot}(t, B) = m\}\} = \{N_p(t, B) = m\} \in \mathcal{F}.$$

By claim 1 - 3 it follows that p is a Poisson point process with characteristic measure ν. By proposition 2.10 p is stationary. □

Definition 2.12. Let $\mathcal{F}_t$, $t \geq 0$, be a filtration on $(\Omega, \mathcal{F})$ and p a point process on U and $(\Omega, \mathcal{F}, P)$. p is called $(\mathcal{F}_t)$-adapted if for every $t \geq 0$ and $B \in \mathcal{B}$ $N_p(t, B)$ is $\mathcal{F}_t$-measurable.

Definition 2.13 ($(\mathcal{F}_t)$-Poisson point process). Let $\mathcal{F}_t$, $t \geq 0$, be a filtration on $(\Omega, \mathcal{F})$ and p a point process on U and $(\Omega, \mathcal{F}, P)$. p is called an $(\mathcal{F}_t)$-*Poisson point process* if it is an $(\mathcal{F}_t)$-adapted, σ-finite Poisson point process such that $\{N_p(]t, t+h] \times B) \,|\, h > 0, B \in \mathcal{B}\}$ is independent of $\mathcal{F}_t$ for all $t \geq 0$.

Remark 2.14. Let p be a σ-finite Poisson point process on U and $(\Omega, \mathcal{F}, P)$. Then there exists a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$ and p is an $(\mathcal{F}_t)$-Poisson point process.

Proof. Define $\mathcal{N} := \{N \in \mathcal{F} \mid P(N) = 0\}$ and for $t \geq 0$

$$\mathcal{A}_t := \sigma(N_p(s,B) \mid 0 < s \leq t, B \in \mathcal{B}) \vee \mathcal{N} \text{ and } \mathcal{F}_t := \bigcap_{\varepsilon > 0} \mathcal{A}_{t+\varepsilon}.$$

Then $(\Omega, \mathcal{F}, P)$ is a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$. Moreover, p is $(\mathcal{F}_t)$-adapted.
It remains to show that for all $t > 0$ $N_p(]t, t+h] \times B)$ is independent of $\mathcal{F}_t$ for all $h > 0$ and $B \in \mathcal{B}$.
Let $B \in \mathcal{B}$ and $h > 0$. For $n \in \mathbb{N}$ $N_p(]t + \frac{h}{n}, t+h] \times B)$ is independent of $\mathcal{A}_{t+\frac{h}{m}}$ for all $m \geq n$ and therefore also of $\mathcal{F}_t$. Since $N_p(]t, t+h] \times B) = \sup_{n \in \mathbb{N}} N_p(]t + \frac{h}{n}, t+h] \times B)$ it is easy to see that $N_p(]t, t+h] \times B)$ is independent of $\mathcal{F}_t$. □

For an arbitrary point process p define the following set

$$\Gamma_p := \{B \in \mathcal{B} \mid E[N_p(t,B)] < \infty \text{ for all } t > 0\}.$$

To motivate the next definition of *point processes of class (QL)* we want to recall the Doob-Meyer-decomposition theorem and give an application of it to the process $N_p(t,B)$, $t \geq 0$, if $B \in \Gamma_p$.
Let $\mathcal{F}_t$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F})$. If p is a σ-finite $(\mathcal{F}_t)$-adapted point process on U then for $B \in \Gamma_p$ $N_p(t,B)$, $t \geq 0$, is a right-continuous $(\mathcal{F}_t)$-submartingale with the property that for all $a > 0$ the family of random variables

$$\{N_p(\sigma, B) \mid \sigma \text{ is a } (\mathcal{F}_t)\text{-stopping time, s.t. } \sigma \leq a\}$$

is uniformly integrable. Then by the Doob-Meyer-decomposition theorem (vgl. [IkWa 81, I. Theorem 6.12, p.36]) there exists an $(\mathcal{F}_t)$-martingale $M(t)$, $t \geq 0$, and a process $A(t)$, $t \geq 0$, with the following properties

(i) A is $(\mathcal{F}_t)$-adapted,

(ii) $A(0) = 0$ and $t \mapsto A(t)$ is right continuous and increasing P-a.s.,

(iii) $E[A(t)] < \infty$ for all $t \geq 0$,

such that $N_p(t,B) = M(t) + A(t)$ for all $t \geq 0$ P-a.s.
Furthermore, A can be chosen natural, i.e. for every bounded, cádlág $(\mathcal{F}_t)$-martingale $N(t)$, $t \geq 0$,

$$E[\int_0^t N(s)\, dA(s)] = E[\int_0^t N(s-)\, dA(s)],\ t \geq 0,$$

and in this case the decomposition of $N_p(\cdot, B)$ is unique in the following sense.
If $\tilde{M}$ is a further $(\mathcal{F}_t)$-martingale and $\tilde{A}$ a further natural process which fulfills the conditions (i)-(iii) such that $N_p(t, B) = M(t) + A(t)$ for all $t \geq 0$, then $M(t) = \tilde{M}(t)$ and $A(t) = \tilde{A}(t)$ for all $t \geq 0$ P-a.s.
A continuous process A which fulfills the conditions (i)-(iii) is natural. (vgl. [IkWa 81, p.35])
Now we give the definition of a point process of class (QL).

Definition 2.15. Let $\mathcal{F}_t$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ and p a point process on U and $(\Omega, \mathcal{F}, P)$. p is said to be of *class (QL)* (quasi-left-continuous) with respect to $\mathcal{F}_t$, $t \geq 0$, if it is $(\mathcal{F}_t)$-adapted, σ-finite and there exists for all $B \in \mathcal{B}$ a process $\hat{N}_p(t, B)$, $t \geq 0$, such that

(i) for $B \in \Gamma_p$, $\hat{N}_p(t, B)$, $t \geq 0$, is a continuous $(\mathcal{F}_t)$-adapted increasing process with $\hat{N}_p(0, B) = 0$ P-a.s.,

(ii) for all $t \geq 0$ and P-a.e. $\omega \in \Omega$, $\hat{N}_p(\omega)(t, \cdot)$ is a σ-finite measure on $(U, \mathcal{B})$,

(iii) for $B \in \Gamma_p$, $q(t, B) := N_p(t, B) - \hat{N}_p(t, B)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale.

$\hat{N}_p$ is called the *compensator* of the point process p and q the *compensated Poisson random measure* of p.

Proposition 2.16. *The compensator of a point process p on U of class (QL) is unique in the following sense.*
If there exists a further process $X(t, B)$, $t \geq 0$, $B \in \mathcal{B}$, which fulfills the conditions (i)-(iii) of Definition 2.15 then, for all $B \in \mathcal{B}$,

$$\hat{N}_p(t, B) = X(t, B) \text{ for all } t \geq 0 \; P\text{-a.s.}$$

Proof. Let $B \in \Gamma_p$ then, by the Doob-Meyer-decomposition theorem, $\hat{N}_p(t, B) = X(t, B)$ for all $t \geq 0$ P-a.s.
Let now be B an arbitrary element of $\mathcal{B}$. Since p is σ-finite there exist $U_n \in \Gamma_p$, $n \in \mathbb{N}$, such that $U_n \uparrow U$. Therefore, we get

$$\begin{aligned}
&\hat{N}_p(t, B) \\
&= \lim_{n \to \infty} \hat{N}_p(t, B \cap U_n), \text{ as } \hat{N}_p(t, \cdot) \text{ is a measure on } \mathcal{B} \text{ for all } t \geq 0 \; P\text{-a.s.}, \\
&= \lim_{n \to \infty} X(t, B \cap U_n) \text{ for all } t \geq 0 \; P\text{-a.s. as } B \cap U_n \in \Gamma_p, \\
&= X(t, B) \text{ for all } t \geq 0 \; P\text{-a.s.}
\end{aligned}$$

$\square$

The next proposition gives us a criterium to decide if an $(\mathcal{F}_t)$-Poisson point process w.r.t. a right-continuous filtration is of class (QL): the continuity of $[0,T] \to \mathbb{R}, t \mapsto E[N_p(t,B)]$, $B \in \Gamma_p$. In this case $\hat{N}_p(t,B) = E[N_p(t,B)]$, $t \geq 0$, $B \in \mathcal{B}$.
In fact, as a subset of the set of point processes of class (QL) the $(\mathcal{F}_t)$-Poisson point processes are characterized by the property that their compensator is a *non random* σ-finite measure on $[0,\infty) \times U$. (see [IkWa 81, II. Theorem 6.2, p.75]).

Proposition 2.17. *Let $\mathcal{F}_t$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F})$ and p an $(\mathcal{F}_t)$-Poisson point process. p is of class (QL) if and only if the mapping $[0,T] \to \mathbb{R}, t \mapsto E[N_p(t,B)]$ is continuous for all $B \in \Gamma_p$. And in this case $\hat{N}_p(t,B) = E[N_p(t,B)]$ for all $t \geq 0$ P-a.s. for all $B \in \mathcal{B}$.*

Proof. "$\Leftarrow$" Suppose that $[0,T] \to \mathbb{R}, t \mapsto E[N_p(t,B)]$ is continuous for all $B \in \Gamma_p$.
Define $\hat{N}_p(t,B) := E[N_p(t,B)]$ for all $t \geq 0$ and $B \in \mathcal{B}$. Then the conditions (i) and (ii) of Definition 2.15 are fulfilled. Moreover, for $B \in \Gamma_p$ $q(t,B) := N_p(t,B) - \hat{N}_p(t,B)$, $t \geq 0$, is $(\mathcal{F}_t)$-adapted. It remains to check that for $B \in \Gamma_p$ $q(t,B)$, $t \geq 0$, has the martingale property.
For this end let $0 \leq s < t < \infty$ and $F_s \in \mathcal{F}_s$, then

$$\begin{aligned}
&E[q(t,B)1_{F_s}] = E[(N_p(t,B) - \hat{N}_p(t,B))1_{F_s}] \\
=\,& E[N_p(t,B)1_{F_s}] - E[N_p(t,B)]P(F_s) \\
=\,& E[N_p(]s,t] \times B)1_{F_s}] + E[N_p(s,B)1_{F_s}] - E[N_p(t,B)]P(F_s) \\
=\,& E[N_p(t,B)]P(F_s) - E[N_p(s,B)]P(F_s) + E[N_p(s,B)1_{F_s}] \\
&- E[N_p(t,B)]P(F_s), \text{ since } N_p(]s,t] \times B) \text{ is independent of } \mathcal{F}_s, \\
=\,& E[N_p(s,B)1_{F_s}] - E[N_p(s,B)]P(F_s) \\
=\,& E[(N_p(s,B) - \hat{N}_p(s,B))1_{F_s}] \\
=\,& E[q(s,B)1_{F_s}].
\end{aligned}$$

"$\Rightarrow$" Suppose now that p is of class (QL). Then $E[N_p(t,B)] = E[\hat{N}_p(t,B)]$ for all $t \geq 0$ and $B \in \Gamma_p$ since $N_p(t,B) - \hat{N}_p(t,B)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale which starts in 0.
Since $\hat{N}_p(t,B)$ is continuous in t for all $B \in \Gamma_p$ and $E[\hat{N}_p(t,B)] = E[N_p(t,B)]$ $< \infty$ for all $t \geq 0$ we get the desired continuity of $E[N_p(\cdot,B)]$ by Lebesgues dominated convergence theorem. $\square$

As an easy consequence of the previous proposition we obtain the following corollary which gives us the existence of a point process of class (QL).

Corollary 2.18. *Let $\mathcal{F}_t$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F})$. Moreover, let ν be a σ-finite measure on $(U, \mathcal{B})$ and p a stationary $(\mathcal{F}_t)$-Poisson point process on U with characteristic measure ν. Then p is of* *class (QL) w.r.t.* $\mathcal{F}_t$*,* $t \geq 0$*, with compensator* $\hat{N}_p(t, B) = t\nu(B)$*,* $t \geq 0$*,* $B \in \mathcal{B}$*.*

Proposition 2.19. *Let* p *be a point process on* U *of class (QL) w.r.t. a right-continuous filtration* $\mathcal{F}_t$*,* $t \geq 0$*, on* $(\Omega, \mathcal{F})$*.*
For $B \in \Gamma_p$ $q(t, B)$*,* $t \geq 0$*, is an element of* $\mathcal{M}^2(\mathbb{R})$ *and we have for* $B_1, B_2 \in \Gamma_p$ *that*

$$\langle q(\cdot, B_1), q(\cdot, B_2)\rangle(t) = \hat{N}_p(t, B_1 \cap B_2), \ t \geq 0.$$

In particular, this means that for all $B \in \Gamma_p$
$M(t) := q(t, B)^2 - \hat{N}_p(t, B)$*,* $t \geq 0$*, is an* $(\mathcal{F}_t)$*-martingale which starts in* 0 *since* $q(0, B) = 0 = \hat{N}_p(0, B)$ P*-a.s.*

Proof. [IkWa 81, II. Theorem 3.1, p.60] □

2.3 Stochastic integrals with respect to Poisson point processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$ and $(U, \mathcal{B})$ a measurable space. Moreover, let p be an $(\mathcal{F}_t)$-Poisson point process on $(U, \mathcal{B})$ and $(\Omega, \mathcal{F}, P)$ of class (QL) with compensator $\hat{N}_p(t, B) = E[N_p(t, B)]$, $t \geq 0$, and $B \in \mathcal{B}$.

Notation: In the following we will use the following notations.
If $\bar{B} \in \mathcal{B}([0, \infty)) \otimes \mathcal{B}$ we define $\hat{N}_p(\bar{B}) := E[N_p(\bar{B})]$. Then $\hat{N}_p$ is a σ-finite measure on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{B})$.
Moreover, we set $q(]s, t] \times B) := N_p(]s, t] \times B) - \hat{N}_p(]s, t] \times B)$, $0 \leq s \leq t < \infty$, $B \in \Gamma_p$.

Remark 2.20. If

$$\begin{aligned} B \in \Gamma_p =& \{B \in \mathcal{B} \,|\, E[N_p(t, B)] < \infty \text{ for all } t > 0\} \\ =& \{B \in \mathcal{B} \,|\, \hat{N}_p(t, B) < \infty \text{ for all } t > 0\} \end{aligned}$$

then $q(s, B) \in \mathbb{R}$ for all $s \geq 0$ P-a.s. since $q(s, B) = N_p(s, B) - \hat{N}_p(s, B)$ where $N_p(s, B) < \infty$ for all $s \geq 0$ P-a.s. as $E[N_p(n, B)] < \infty$ for all $n \in \mathbb{N}$.

If $0 \leq s \leq t < \infty$ and $B \in \Gamma_p$ then

$$\begin{aligned} q(t,B) - q(s,B) &= N_p(t,B) - N_p(s,B) - (\hat{N}_p(t,B) - \hat{N}_p(s,B)) \\ &= N_p(]s,t] \times B) - E[N_p(]s,t] \times B)] \quad P\text{-a.s.} \\ &= N_p(]s,t] \times B) - \hat{N}_p(]s,t] \times B) \\ &= q(]s,t] \times B) \end{aligned}$$

Step 1. Definition of the stochastic integral for elementary processes

Let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space with $\|\ \| = \langle\ ,\ \rangle^{\frac{1}{2}}$ and fix $T > 0$.
The class $\mathcal{E}$ of all elementary processes is determined by the following definition.

Definition 2.21. An H-valued process $\Phi(t) : \Omega \times U \to H$, $t \in [0,T]$, on $(\Omega \times U, \mathcal{F} \otimes \mathcal{B})$ is said to be *elementary* if there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0,T]$ and for $m \in \{0, \ldots, k-1\}$ there exist $B_1^m, \ldots, B_{I(m)}^m \in \Gamma_p$, pairwise disjoint, such that

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m,t_{m+1}] \times B_i^m}$$

where $\Phi_i^m \in L^2(\Omega, \mathcal{F}_{t_m}, P; H)$, $1 \leq i \leq I(m)$, $0 \leq m \leq k-1$.
$\mathcal{E}$ is a linear space.

For $\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m,t_{m+1}] \times B_i^m} \in \mathcal{E}$ define the stochastic integral process by

$$\begin{aligned} &\operatorname{Int}(\Phi)(t) \\ &:= \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) := \int_{]0,t]} \int_U \Phi(s,y)\, q(ds,dy) \\ &:= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1} \wedge t, B_i^m) - q(t_m \wedge t, B_i^m)), \end{aligned}$$

$t \in [0,T]$.
Then $\operatorname{Int}(\Phi)$ is P-a.s. well-defined and Int is linear in $\Phi \in \mathcal{E}$.

Proposition 2.22.
If $\Phi \in \mathcal{E}$ then $X(t) := \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)$, $t \in [0,T]$, is an element of $\mathcal{M}_T^2(H)$ with $X(0) = 0$ P-a.s. and

$$\|\operatorname{Int}(\Phi)\|_{\mathcal{M}_T^2}^2 := \sup_{t \in [0,T]} E[\| \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) \|^2] \tag{2.2}$$

$$= E[\int_0^T \int_U \|\Phi(s,y)\|^2\, \hat{N}_p(ds,dy)] =: \|\Phi\|_T^2$$

Proof. Obviously, $\mathrm{Int}(\Phi)$ is a cádlág process.

Claim 1. $\mathrm{Int}(\Phi)$ is $(\mathcal{F}_t)$-adapted.

Let $t \in [0,T]$ then

$$\begin{aligned}&\mathrm{Int}(\Phi)(t)\\ &= \sum_{\substack{m=0\\ t_m \le t}}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m \big(N_p(t_{m+1} \wedge t, B_i^m) - \hat{N}_p(t_{m+1} \wedge t, B_i^m)\\ &\qquad\qquad - N_p(t_m, B_i^m) + \hat{N}_p(t_m, B_i^m)\big)\end{aligned}$$

which is $\mathcal{F}_t$-measurable since p is $(\mathcal{F}_t)$-adapted and Φ_i^m is $\mathcal{F}_{t_m}/\mathcal{B}(H)$-measurable for all $1 \le i \le I(m)$ and $0 \le m \le k-1$ such that $t_m \le t$.

Claim 2. For all $t \in [0,T]$

$$E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big] = E\big[\int_0^t \int_U \|\Phi(s,y)\|^2\, \hat{N}_p(ds,dy)\big] < \infty.$$

$$\begin{aligned}&E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big]\\ &= E\big[\|\sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m q(]t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2\big]\\ &= E\Big[\sum_{\substack{m=0\\ t_m \le t}}^{k-1} \Big(\sum_{i=1}^{I(m)} \|\Phi_i^m q(]t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2\\ &\qquad + 2 \sum_{1 \le i < j \le I(m)} \langle \Phi_i^m \Delta_i^m, \Phi_j^m \Delta_j^m \rangle \Big)\\ &\quad + 2 \sum_{\substack{0 \le m < n \le k-1\\ t_n \le t}} \sum_{\substack{(i,j) \in \{1,\dots,I(m)\}\\ \times \{1,\dots,I(n)\}}} \langle \Phi_i^m \Delta_i^m, \Phi_j^n \Delta_j^n \rangle \Big]\end{aligned}$$

where $\Delta_h^l := q(]t_l \wedge t, t_{l+1} \wedge t] \times B_h^l)$, $1 \le h \le I(l)$, $0 \le l \le k-1$.

1.: For $m \in \{0,\dots,k-1\}$ such that $t_m \le t$ and $i \in \{1,\dots,I(m)\}$

$$E\big[\|\Phi_i^m q(]t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2\big] = E\big[\|\Phi_i^m\|^2 |\Delta_i^m|^2\big] < \infty :$$

Since $\|\Phi_i^m\|^2$ is $\mathcal{F}_{t_m}$-measurable and $|\Delta_i^m|^2$ is independent of $\mathcal{F}_{t_m}$ we get that

$$E\big[\|\Phi_i^m\|^2 |\Delta_i^m|^2\big] = E\big[\|\Phi_i^m\|^2\big] E\big[|\Delta_i^m|^2\big]$$

where $E\big[\|\Phi_i^m\|^2\big] < \infty$. It remains to show that $E\big[|\Delta_i^m|^2\big] < \infty$.
For this purpose let $0 \le s \le t \le T$ and $B \in \Gamma_p$, then:

$$\begin{aligned}E\big[q(]s,t] \times B)^2\big] &= E\big[(q(t,B) - q(s,B))^2\big]\\ &= E\big[\underbrace{q(t,B)^2}_{(a)} - 2\underbrace{q(t,B)q(s,B)}_{(b)} + q(s,B)^2\big]\end{aligned}$$

(a) By proposition 2.19 it follows for $u \in [0,T]$ and $B \in \Gamma_p$ that

$$E[q(u,B)^2] = E[\hat{N}_p(u,B)] = E[N_p(u,B)] < \infty.$$

(b) Since $|q(]s,t] \times B)|$ and $|q(s,B)|$ are independent we get that

$$\begin{aligned} E\big[|q(t,B)q(s,B)|\big] &\le E\big[q(s,B)^2\big] + E\big[|q(]s,t] \times B)q(s,B)|\big] \\ &= E\big[|q(]s,t] \times B)|\big] E\big[|q(s,B)|\big] + E\big[q(s,B)^2\big] \\ &< \infty. \end{aligned}$$

From (a) and (b) it follows that $E\big[q(]s,t] \times B)^2\big] < \infty$. Moreover, we obtain that

$$\begin{aligned} &E\big[q(]s,t] \times B)^2\big] \\ =&E\big[q(t,B)^2\big] - 2E\big[q(t,B)q(s,B)\big] + E\big[q(s,B)^2\big] \\ =&E\big[q(t,B)^2\big] - 2E\big[q(]s,t] \times B)q(s,B)\big] - E\big[q(s,B)^2\big] \\ =&E[\hat{N}_p(t,B)] - 2E\big[q(]s,t] \times B)\big]E\big[q(s,B)\big] - E[\hat{N}_p(s,B)] \\ =&\hat{N}_p(]s,t] \times B), \text{ as } E\big[q(s,B)\big] = E[N_p(s,B)] - \hat{N}_p(s,B) = 0. \end{aligned} \tag{2.3}$$

This will be useful later on.

2.: For $m \in \{0, \dots, k-1\}$ such that $t_m \le t$ and $i,j \in \{1, \dots, I(m)\}$, $i < j$,

$$\begin{aligned} &E[|\langle \Phi_i^m \Delta_i^m, \Phi_j^m \Delta_j^m \rangle|] \\ \le&\big(E[\|\Phi_i^m \Delta_i^m\|^2]\big)^{\frac{1}{2}} \big(E[\|\Phi_j^m \Delta_j^m\|^2]\big)^{\frac{1}{2}} < \infty, \end{aligned}$$

by **1.**.

3.: For $m,n \in \{0, \dots, k-1\}$, $m < n$, such that $t_n \le t$ and $i \in \{1, \dots, I(m)\}$, $j \in \{1, \dots, I(n)\}$,

$$E\big[|\langle \Phi_i^m \Delta_i^m, \Phi_j^n \Delta_j^n \rangle|\big] = E\big[|\langle \Phi_i^m \Delta_i^m, \Phi_j^n \rangle||\Delta_j^n|\big] < \infty :$$

Since $m < n$ and $t_m < t_{m+1} \le t_n \le t$ $\langle \Phi_i^m \Delta_i^m, \Phi_j^n \rangle$ is $\mathcal{F}_{t_n}/\mathcal{B}(H)$-measurable. In addition, $|\Delta_j^n|$ is independent of $\mathcal{F}_{t_n}$. Therefore, we get that

$$\begin{aligned} E\big[|\langle \Phi_i^m \Delta_i^m, \Phi_j^n \rangle||\Delta_j^n|\big] &= E\big[|\langle \Phi_i^m \Delta_i^m, \Phi_j^n \rangle|\big] E\big[|\Delta_j^n|\big] \\ &\le E\big[\|\Phi_i^m \Delta_i^m\|^2\big]^{\frac{1}{2}} E\big[\|\Phi_j^n\|^2\big]^{\frac{1}{2}} E\big[|\Delta_j^n|\big] \\ &< \infty, \text{ by } \mathbf{1.}. \end{aligned}$$

4.: For $m \in \{0,\dots,k-1\}$ such that $t_m \le t$ and $i,j \in \{1,\dots,I(m)\}$, $i<j$,

$$E[\langle \Phi_i^m, \Phi_j^m\rangle \Delta_i^m \Delta_j^m] = 0 :$$

Since $\langle \Phi_i^m, \Phi_j^m\rangle \in L^1(\Omega, \mathcal{F}_{t_m}, P)$ and $\Delta_i^m\Delta_j^m \in L^1(\Omega,\mathcal{F},P)$ is independent of $\mathcal{F}_{t_m}$ we get that

$$E[\langle \Phi_i^m, \Phi_j^m\rangle \Delta_i^m \Delta_j^m] = E[\langle \Phi_i^m, \Phi_j^m\rangle] E\big[\Delta_i^m \Delta_j^m\big].$$

Moreover, as B_i^m and B_j^m are disjoint if $i \ne j$, we know that Δ_i^m and Δ_j^m are independent. Therefore

$$E[\Delta_i^m \Delta_j^m] = E[\Delta_i^m] E[\Delta_j^m] = 0$$

and we obtain that

$$E[\langle \Phi_i^m, \Phi_j^m\rangle \Delta_i^m \Delta_j^m] = 0$$

5.: For $m,n \in \{0,\dots,k-1\}$, $m<n$, such that $t_n \le t$ and $i \in \{1,\dots,I(m)\}$, $j \in \{1,\dots,I(n)\}$,

$$\begin{aligned} &E\big[\langle \Phi_i^m \Delta_i^m, \Phi_j^n \Delta_j^n\rangle\big] \\ = &E\big[\langle \Phi_i^m \Delta_i^m, \Phi_j^n\rangle \Delta_j^n\big] = 0 : \end{aligned}$$

Since $\langle \Phi_i^m \Delta_i^m, \Phi_j^n\rangle \in L^1(\Omega, \mathcal{F}_{t_n}, P)$ and $\Delta_j^n \in L^1(\Omega,\mathcal{F},P)$ is independent of $\mathcal{F}_{t_n}$ we get that

$$\begin{aligned} &E\big[\langle \Phi_i^m \Delta_i^m, \Phi_j^n\rangle \Delta_j^n\big] \\ = &E\big[\langle \Phi_i^m \Delta_i^m, \Phi_j^n\rangle\big] E[\Delta_j^n] \\ = &0. \end{aligned}$$

By **1.-5.** one gets for all $t \in [0,T]$ that

$$\begin{aligned} &E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big] \\ = &E\Big[\|\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} \Phi_i^m q(]t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2\Big] \\ = &E\Big[\sum_{\substack{m=0\\ t_m \le t}}^{k-1} \Big(\sum_{i=1}^{I(m)} \|\Phi_i^m q(]t_m \wedge t, t_{m+1}\wedge t] \times B_i^m)\|^2 \\ &\qquad + 2 \sum_{1 \le i<j \le I(m)} \langle \Phi_i^m \Delta_i^m, \Phi_j^m \Delta_j^m\rangle\Big) \\ &\quad + 2 \sum_{\substack{0 \le m<n \le k-1\\ t_n \le t}} \sum_{\substack{(i,j)\in\{1,\dots,I(m)\}\\ \times\{1,\dots,I(n)\}}} \langle \Phi_i^m \Delta_i^m, \Phi_j^n \Delta_j^n\rangle\Big] \end{aligned}$$

$$= \sum_{\substack{m=0\\t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} E\big[\|\Phi_i^m q(]t_m\wedge t, t_{m+1}\wedge t]\times B_i^m)\|^2\big]$$

$$= \sum_{\substack{m=0\\t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} E\big[\|\Phi_i^m\|^2\big]E\big[q(]t_m, t_{m+1}\wedge t]\times B_i^m)^2\big],$$

since $\|\Phi_i^m\|^2\in L^1(\Omega,\mathcal{F}_{t_m},P)$ and $q(]t_m,t_{m+1}\wedge t]\times B_i^m)^2\in L^1(\Omega,\mathcal{F},P)$ is independent of $\mathcal{F}_{t_m}$,

$$= \sum_{\substack{m=0\\t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} E\big[\|\Phi_i^m\|^2\big]\hat{N}_p(]t_m, t_{m+1}\wedge t]\times B_i^m),$$

by equation (2.3),

$$= \int_0^t\int_U \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} E\big[\|\Phi_i^m\|^2\big]1_{]t_m,t_{m+1}]\times B_i^m}(s,y)\,\hat{N}_p(ds,dy)$$

$$= \int_0^t\int_U E\big[\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)}\|\Phi_i^m\|^2 1_{]t_m,t_{m+1}]\times B_i^m}(s,y)\big]\,\hat{N}_p(ds,dy)$$

$$= \int_0^t\int_U E\big[\|\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)}\Phi_i^m 1_{]t_m,t_{m+1}]\times B_i^m}(s,y)\|^2\big]\,\hat{N}_p(ds,dy)$$

$$= \int_0^t\int_U E\big[\|\Phi(s,y)\|^2\big]\,\hat{N}_p(ds,dy)$$

$$= E\big[\int_0^t\int_U \|\Phi(s,y)\|^2\,\hat{N}_p(ds,dy)\big]$$

Claim 3. $\mathrm{Int}(\Phi)(t)$, $t\in[0,T]$, is an $(\mathcal{F}_t)$-martingale.

Let $0\le s<t\le T$ and $F_s\in\mathcal{F}_s$ then:

$$E\big[1_{F_s}\int_0^{t+}\int_U \Phi(r,y)\,q(dr,dy)\big]$$

$$= \int_{F_s}\sum_{\substack{m=0\\t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \Phi_i^m(q(t_{m+1}\wedge t, B_i^m)-q(t_m\wedge t, B_i^m))\,dP$$

$$= \sum_{\substack{m=0\\t_m\le s}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} \Phi_i^m(q(t_{m+1}\wedge t, B_i^m)-q(t_m\wedge s, B_i^m))\,dP$$

$$+ \sum_{\substack{m=0\\s<t_m\le t}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} \Phi_i^m(q(t_{m+1}\wedge t, B_i^m)-q(t_m, B_i^m))\,dP$$

$$
\begin{aligned}
&= \sum_{\substack{m=0\\ t_m\le s}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s}\Phi_i^m\big(E\big[q(t_{m+1}\wedge t,B_i^m)|\mathcal{F}_s\big]-q(t_m\wedge s,B_i^m)\big)\,dP\\
&+\sum_{\substack{m=0\\ s<t_m\le t}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s}\Phi_i^m\,\underbrace{\big(E\big[q(t_{m+1}\wedge t,B_i^m)|\mathcal{F}_{t_m}\big]-q(t_m,B_i^m)\big)}_{\substack{=0,\text{ since } q(\cdot,B_i^m)\text{ is an }(\mathcal{F}_t)\text{-martingale}\\ \text{and } 1_{F_s}\Phi_i^m\in L^1(\Omega,\mathcal{F}_{t_m},P;H)}}\,dP\\
&= \sum_{\substack{m=0\\ t_m\le s}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s}\Phi_i^m(q(t_{m+1}\wedge s,B_i^m)-q(t_m\wedge s,B_i^m))\,dP,\\
&\qquad\text{since } q(t_{m+1}\wedge\cdot,B_i^m)\text{ is an }(\mathcal{F}_t)\text{-martingale and } 1_{F_s}\Phi_i^m\in L^1(\Omega,\mathcal{F}_s,P;H),\\
&= E\Big[1_{F_s}\int_0^{s+}\int_U\Phi(r,y)\,q(dr,dy)\Big].
\end{aligned}
$$

□

In this way one has found a seminorm $\|\ \|_T$ on $\mathcal{E}$ such that Int : $(\mathcal{E},\|\ \|_T)\to(\mathcal{M}_T^2(H),\|\ \|_{\mathcal{M}_T^2})$ is an isometric transformation. To get a norm on $\mathcal{E}$ one has to consider equivalence classes of elementary processes with respect to $\|\ \|_T$. For simplicity, the space of equivalence classes will be denoted by $\mathcal{E}$, too.
Since $\mathcal{E}$ is dense in the abstract completion $\bar{\mathcal{E}}^{\|\ \|_T}$ of $\mathcal{E}$ w.r.t. $\|\ \|_T$ it is clear that there is a unique isometric extension of Int to $\bar{\mathcal{E}}^{\|\ \|_T}$.

Step 2. Characterization of $\bar{\mathcal{E}}^{\|\ \|_T}$
Define the predictable σ-field on $[0,T]\times\Omega\times U$ by

$$
\begin{aligned}
&\mathcal{P}_T(U)\\
&:=\sigma(g:[0,T]\times\Omega\times U\to\mathbb{R}\,|\,g\text{ is }(\mathcal{F}_t\otimes\mathcal{B})-\text{adapted and left-continuous})\\
&=\sigma(\{]s,t]\times\tilde F_s\,|\,0\le s\le t\le T,\tilde F_s\in\mathcal{F}_s\otimes\mathcal{B}\}\cup\{\{0\}\times\tilde F_0\,|\,\tilde F_0\in\mathcal{F}_0\otimes\mathcal{B}\})\\
&=\sigma(\ \{]s,t]\times F_s\times B\,|\,0\le s\le t\le T,F_s\in\mathcal{F}_s,B\in\mathcal{B}\}\\
&\qquad\cup\{\{0\}\times F_0\times B\,|\,F_0\in\mathcal{F}_0,B\in\mathcal{B}\})
\end{aligned}
$$

At this point, we also define the predictable σ-field $\mathcal{P}_T$ on $[0,T]\times\Omega$ by

$$
\begin{aligned}
\mathcal{P}_T &:= \sigma(g:[0,T]\times\Omega\to\mathbb{R},|\,g\text{ is }(\mathcal{F}_t)\text{-adapted and left-continuous})\\
&= \sigma\big(\underbrace{\{]s,t]\times F_s\,|\,0\le s\le t\le T,F_s\in\mathcal{F}_s\}\cup\{\{0\}\times F_0\,|\,F_0\in\mathcal{F}_0\}}_{=:\mathcal{A}}\big)
\end{aligned}
$$

Let $\tilde H$ be an arbitrary Hilbert space. If $Y:[0,T]\times\Omega\to\tilde H$ is $\mathcal{P}_T/\mathcal{B}(\tilde H)$-measurable it is called ($\tilde H$-)predictable.

Remark 2.23. (i) If $B \in \mathcal{B}([0,T])$ then $B \times \Omega \times U \in \mathcal{P}_T(U)$.

(ii) If $A \in \mathcal{P}_T$ and $B \in \mathcal{B}$ then $A \times B \in \mathcal{P}_T(U)$.

Proposition 2.24. *If Φ is a $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable process and*

$$E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \hat{N}_p(ds,dy)] < \infty$$

then there exists a sequence of elementary processes Φ_n, $n \in \mathbb{N}$, such that $\|\Phi - \Phi_n\|_T \longrightarrow 0$ as $n \to \infty$.

Proof. There exist $U_n \in \mathcal{B}$, $n \in \mathbb{N}$, with $\hat{N}_p(t,U_n) = E[N_p(t,U_n)] < \infty$ for all $t \geq 0$ and $n \in \mathbb{N}$ such that $U_n \uparrow U$ as $n \to \infty$. Then $1_{U_n}\Phi : [0,T] \times \Omega \times U_n \to H$ is $\mathcal{P}_T(U) \cap ([0,T] \times \Omega \times U_n)/\mathcal{B}(H)$-measurable.
Moreover,

$$\begin{aligned} &\mathcal{P}_T(U) \cap ([0,T] \times \Omega \times U_n) \\ =&\,\sigma(\,\{]s,t] \times F_s \times B \,|\, 0 \leq s \leq t \leq T, F_s \in \mathcal{F}_s, B \in \mathcal{B} \cap U_n\} \\ &\quad \cup \{\{0\} \times F_0 \times B \,|\, F_0 \in \mathcal{F}_0, B \in \mathcal{B} \cap U_n\}) \\ =&\,\mathcal{P}_T(U_n). \end{aligned} \tag{2.4}$$

Therefore, one gets that $1_{U_n}\Phi : [0,T] \times \Omega \times U_n \to H$ is $\mathcal{P}_T(U_n)/\mathcal{B}(H)$-measurable. Then there exists a sequence Φ_k^n, $k \in \mathbb{N}$, of simple random variables of the following form

$$\sum_{m=1}^{M} x_m 1_{A_m},\; x_m \in H,\; A_m \in \mathcal{P}_T(U_n),\; 1 \leq m \leq M,$$

such that $\|1_{U_n}\Phi - \Phi_k^n\| \downarrow 0$ as $k \to \infty$ by lemma B.5. Since

$$\begin{aligned} \|1_{U_n}\Phi - \Phi_k^n\| &\leq \|1_{U_n}\Phi - \Phi_1^n\| \leq \|1_{U_n}\Phi\| + \|\Phi_1^n\| \\ &\qquad \in L^2([0,T] \times \Omega \times U_n, \mathcal{P}_T(U_n), P \otimes \hat{N}_p(ds,d\omega,dy)), \end{aligned}$$

where for $A \in \mathcal{P}_T(U)$ we define $P \otimes \hat{N}_p(A) := E[\int_0^T \int_U 1_A(s,y)\, \hat{N}_p(ds,dy)]$, one gets by Lebesgue's dominated convergence theorem that

$$\begin{aligned} \|1_{U_n}(\Phi - \Phi_k^n)\|_T^2 &= E[\int_0^T \int_U \|1_{U_n}(\Phi(s,y) - \Phi_k^n(s,y))\|^2 \, \hat{N}_p(ds,dy)] \\ &= E[\int_0^T \int_{U_n} \|1_{U_n}\Phi(s,y) - \Phi_k^n(s,y)\|^2 \, \hat{N}_p(ds,dy)] \underset{k \to \infty}{\longrightarrow} 0. \end{aligned}$$

Choose for $n \in \mathbb{N}$ $k(n) \in \mathbb{N}$ such that $\|1_{U_n}(\Phi - \Phi_{k(n)}^n)\|_T < \dfrac{1}{n}$, then

$$\|\Phi - 1_{U_n}\Phi_{k(n)}^n\|_T \leq \|\Phi - 1_{U_n}\Phi\|_T + \|1_{U_n}(\Phi - \Phi_{k(n)}^n)\|_T$$

where the first summand converges to 0 by Lebesgue's dominated convergence theorem and the second summand is smaller than $\frac{1}{n}$.
Thus, the assertion of the proposition is reduced to the case $\Phi = x1_A$ where $x \in H$ and $A \in \mathcal{P}_T(U_n)$ for some $n \in \mathbb{N}$. We have to show that there is a sequence of elementary processes Φ_k, $k \in \mathbb{N}$, such that $\|\Phi - \Phi_k\|_T \longrightarrow 0$ as $k \to \infty$.
To get this result it is sufficient to prove that for any $\varepsilon > 0$ there is a finite sum $\Lambda = \bigcup_{i=1}^N A_i$ of predictable rectangles

$$\begin{aligned} A_i \in \mathcal{A}_n := &\{]s,t] \times F_s \times B \,|\, F_s \in \mathcal{F}_s, 0 \le s \le t \le T, B \in \mathcal{B} \cap U_n\} \\ &\cup \{\{0\} \times F_0 \times B \,|\, F_0 \in \mathcal{F}_0, B \in \mathcal{B} \cap U_n\},\ 1 \le i \le N, \end{aligned}$$

such that $P \otimes \hat{N}_p(A \triangle \Lambda) \le \varepsilon$, since then one obtains that $\sum_{i=1}^N x1_{A_i}$ is an elementary process, as $x1_{A_i}$, $1 \le i \le N$, are elementary processes and $\mathcal{E}$ is a linear space, and

$$\begin{aligned} \|x1_A - \sum_{i=1}^N x1_{A_i}\|_T &= E[\int_0^T \int_U \|x(1_A - \sum_{k=1}^N 1_{A_i})\|^2 \, d\hat{N}_p]^{\frac{1}{2}} \\ &\le \|x\| P \otimes \hat{N}_p(A \triangle \Lambda) \le \|x\|\varepsilon \end{aligned}$$

Hence define $\mathcal{K} := \{\bigcup_{i \in I} A_i \,|\, |I| < \infty, A_i \in \mathcal{A}_n, i \in I\}$ then $\mathcal{K}$ is stable under finite intersections. Now let $\mathcal{G}$ be the family of all $A \in \mathcal{P}_T(U_n)$ which can be approximated by elements of $\mathcal{K}$ in the above sense. Then $\mathcal{G}$ is a Dynkin system and therefore $\mathcal{P}_T(U_n) = \sigma(\mathcal{K}) = \mathcal{D}(\mathcal{K}) \subset \mathcal{G}$ as $\mathcal{K} \subset \mathcal{G}$. □

Define

$$\begin{aligned} \mathcal{N}_q^2(T,U,H) := \{\Phi : [0,T] \times \Omega \times U \to H \,|\, &\Phi \text{ is } \mathcal{P}_T(U)/\mathcal{B}(H)\text{-measurable} \\ &\text{and } \|\Phi\|_T = E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \hat{N}_p(ds,dy)]^{\frac{1}{2}} < \infty\} \end{aligned}$$

Then $\mathcal{E} \subset \mathcal{N}_q^2(T,U,H)$ and

$$\mathcal{N}_q^2(T,U,H) = L^2([0,T] \times \Omega \times U, P_T(U), P \otimes \hat{N}_p; H)$$

is complete w.r.t. $\|\,\|_T$ since $(H, \|\,\|)$ is complete. Therefore, $\bar{\mathcal{E}}^{\|\,\|_T} \subset \mathcal{N}_q^2(T,U,H)$ and by the previous proposition it follows that $\bar{\mathcal{E}}^{\|\,\|_T} \supset \mathcal{N}_q^2(T,U,H)$. So finally, one gets that $\bar{\mathcal{E}}^{\|\,\|_T} = \mathcal{N}_q^2(T,U,H)$

Example 2.25. If ν is a σ-finite measure on $(U,\mathcal{B})$ and p a stationary $(\mathcal{F}_t)$-Poisson point process with characteristic measure ν. Then by corollary 2.18 p is of class (QL) with compensator $\hat{N}_p(t,B) = t\nu(B)$, $t \ge 0$, $B \in \mathcal{B}$. Then the class of processes which are integrable with respect to $q(ds,dy)$ is

$$\mathcal{N}_q^2(T,U,H) = \{\Phi : [0,T] \times \Omega \times U \to H \mid \Phi \text{ is } \mathcal{P}_T(U)/\mathcal{B}(H)\text{-measurable}$$
$$\text{and } \|\Phi\|_T = E[\int_0^T \int_U \|\Phi(s,y)\|^2 \,\nu(dy)\,\lambda(ds)]^{\frac{1}{2}} < \infty\}$$

and we have by theorem 2.22 the following isometric formula for $\Phi \in \mathcal{N}_q^2(T,U,H)$

$$\begin{aligned} \|\mathrm{Int}(\Phi)\|^2_{\mathcal{M}_T^2} &= \sup_{t\in[0,T]} E[\|\int_0^{t+} \int_U \Phi(s,y)\,q(ds,dy)\|^2] \\ = E[\int_0^T \int_U \|\Phi(s,y)\|^2 \,\nu(dy)\,ds] &= \|\Phi\|_T. \end{aligned} \tag{2.5}$$

Chapter 3

Properties of the Stochastic Integral and of the Integral w.r.t. N_p

Let $(U, \mathcal{B})$ be a measurable space and $(\Omega, \mathcal{F}, P)$ a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$. Moreover, let p be an $(\mathcal{F}_t)$-Poisson point process of class (QL) on $(U, \mathcal{B})$ and $(\Omega, \mathcal{F}, P)$.

Proposition 3.1. *Let $\Phi : [0,T] \times \Omega \times U \to H$ $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. Then, for all $t \in [0,T]$*

$$E[\int_0^t \int_U \|\Phi(s,y)\| \, \hat{N}_p(ds,dy)] = E[\int_{]0,t]} \int_U \|\Phi(s,y)\| \, N_p(ds,dy)], \qquad (3.1)$$

where $\int_{]0,t]} \int_U \|\Phi(s,y)\| \, N_p(ds,dy)$ is defined ω-wise as $\mathbb{R}$-valued Lebesgues integral.

Proof. Define

$$\begin{aligned}\mathcal{H} := \{\Phi : [0,T] \times \Omega \times U \to \mathbb{R}_+ \mid \Phi \text{ is } \mathcal{P}_T(U)\text{-measurable, bounded and} \\ E[\int_0^t \int_U \Phi(s,y) \, \hat{N}_p(ds,dy)] = E[\int_{]0,t]} \int_U \Phi(s,y) \, N_p(ds,dy)] \\ \text{for all } t \in [0,T]\}.\end{aligned}$$

Then $\mathcal{H}$ is a monotone vector space.

Besides, define

$$\mathcal{A} := \{ \sum_{k=0}^{K-1} x_k 1_{A_k} 1_{]t_k,t_{k+1}]\times B_k} + x 1_A 1_{\{0\}\times B} \,|\, 0 \le t_0 < \cdots < t_K \le T,$$
$$x_k, x \in \mathbb{R}_+, B_k, B \in \mathcal{B}, A_k \in \mathcal{F}_{t_k}, 1 \le k \le K, A \in \mathcal{F}_0, K \in \mathbb{N}\}.$$

Then $\mathcal{A}$ is closed under multiplication and $\mathcal{A} \subset \mathcal{H}$, since for $t \in [0,T]$

$$\begin{aligned}
&E[\int_{]0,t]} \int_U \sum_{k=0}^{K} x_k 1_{A_k} 1_{]t_k,t_{k+1}]\times B_k}(s,y) + x 1_A 1_{\{0\}\times B}(s,y)\, N_p(ds,dy)] \\
&= \sum_{\substack{k=0 \\ t_k \le t}}^{K} x_k E[1_{A_k} \underbrace{N_p(]t_k, t_{k+1} \wedge t] \times B_k)}_{\text{independent of } \mathcal{F}_{t_k}}] \\
&= \sum_{\substack{k=0 \\ t_k \le t}}^{K} x_k P(A_k) E[N_p(]t_k, t_{k+1} \wedge t] \times B_k)] \\
&= \sum_{\substack{k=0 \\ t_k \le t}}^{K} x_k P(A_k) \hat{N}_p(]t_k, t_{k+1} \wedge t] \times B_k) \\
&= E[\int_0^t \int_U \sum_{k=0}^{K} x_k 1_{A_k} 1_{]t_k,t_{k+1}]\times B_k}(s,y) + x 1_A 1_{\{0\}\times B}(s,y)\, \hat{N}_p(ds,dy)].
\end{aligned}$$

Then by a monotone class argument we get that $\sigma(\mathcal{A})_b \subset \mathcal{H}$. Moreover,

$$\begin{aligned}
\mathcal{P}_T(U) = &\sigma(\{]s,t] \times F_s \times B \,|\, 0 \le s \le t \le T, B \in \mathcal{B}, F_s \in \mathcal{F}_s\} \\
&\cup \{\{0\} \times F_0 \times B \,|\, B \in \mathcal{B}, F_0 \in \mathcal{F}_0) \\
&\subset \sigma(\mathcal{A}) \subset \mathcal{P}_T(U).
\end{aligned}$$

Hence we get that all $\Phi : [0,T] \times \Omega \times U \to \mathbb{R}_+$ which are $\mathcal{P}_T(U)$-measurable and bounded are elements of $\mathcal{H}$.
Finally, by the monotone convergence theorem (B.Levi), we obtain that equation (3.1) holds for all $\Phi : [0,T] \times \Omega \times U \to \mathbb{R}_+$ which are $\mathcal{P}_T(U)$-measurable. □

Proposition 3.2. *Let $\Phi : [0,T] \times \Omega \times U \to \mathbb{R}$ $\mathcal{P}_T(U)$-measurable such that $E[\int_{]0,T]} \int_U |\Phi(s,y)|\, N_p(ds,dy)] < \infty$, then*

$$\int_{]0,t]} \int_U \Phi(s,y)\, N_p(ds,dy) = \sum_{\substack{s \in D_p \\ s \le t}} \Phi(s,p(s)) \text{ for all } t \in [0,T] \tag{3.2}$$

P-a.s. where $\int_{]0,t]} \int_U \Phi(s,y)\, N_p(ds,dy)$ is defined ω-wise as $\mathbb{R}$-valued Lebesgues integral.

Proof. Since $E[\int_{]0,T]}\int_U |\Phi(s,y)|\, N_p(ds,dy)] < \infty$ there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$

$$\int_{]0,t]}\int_U |\Phi(s,\omega,y)|\, N_{p(\omega)}(ds,dy) < \infty \quad \forall t \in [0,T].$$

We fix $\omega \in N^c$. The mapping $\Phi(\cdot,\omega,\cdot) : [0,T] \times U \to \mathbb{R}$ is $\mathcal{B}([0,T]) \otimes \mathcal{B}$-measurable.
Suppose that Φ is non-negativ, then there exists a sequence of simple processes Φ_n, $n \in \mathbb{N}$, of the following form

$$\Phi_n = \sum_{k=1}^{K(n)} x_k^n 1_{A_k^n},\ x_k^n \geq 0,\ A_k^n \in \mathcal{B}([0,T]) \otimes \mathcal{B},\ 1 \leq k \leq K(n),\ n \in \mathbb{N},$$

such that $\Phi_n \uparrow \Phi(\cdot,\omega,\cdot)$. Then

$$\begin{aligned}
&\int_{]0,t]}\int_U \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy) = \lim_{n\to\infty}\int_{]0,t]}\int_U \Phi_n(s,y)\, N_{p(\omega)}(ds,dy)\\
&= \lim_{n\to\infty}\sum_{k=1}^{K(n)} x_k^n N_{p(\omega)}(A_k^n \cap (]0,t] \times U))\\
&= \lim_{n\to\infty}\sum_{k=1}^{K(n)} x_k^n \#\{s \in D_{p(\omega)} | s \leq t, (s,p(\omega)(s)) \in A_k^n\}\\
&= \lim_{n\to\infty}\sum_{k=1}^{K(n)} x_k^n \sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}} 1_{A_k^n}(s,p(\omega)(s)) = \lim_{n\to\infty}\sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}}\sum_{k=1}^{K(n)} x_k^n 1_{A_k^n}(s,p(\omega)(s))\\
&= \lim_{n\to\infty}\sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}} \Phi_n(s,p(\omega)(s)) = \sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}} \lim_{n\to\infty}\Phi_n(s,p(\omega)(s))\\
&= \sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}} \Phi(s,\omega,p(\omega)(s)).
\end{aligned}$$

If Φ is not necessarily non-negativ then equality (3.2) can be shown by splitting Φ up into its positiv and its negativ part. □

Proposition 3.3. *Let $\Phi : [0,T] \times \Omega \times U \to \mathbb{R}$ $\mathcal{P}_T(U)$-measurable such that $E[\int_{]0,T]}\int_U |\Phi(s,y)|\, N_p(ds,dy)] < \infty$, then*

$$\Delta \int_{]0,t]}\int_U \Phi(s,y)\, N_p(ds,dy) = \begin{cases} \Phi(t,p(t)) & , \text{ if } t \in D_p, \\ 0 & , \text{ otherwise.}\end{cases}$$

for all $t \in [0,T]$ P-a.s.

Proof. Since $E[\int_{]0,T]}\int_U |\Phi(s,y)|\, N_p(ds,dy)] < \infty$ there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and all $t \in [0,T]$

$$\int_{]0,T]}\int_U 1_{\{t\}}(s)|\Phi(s,\omega,y)|\, N_{p(\omega)}(ds,dy) \leq \int_{]0,t]}\int_U |\Phi(s,\omega,y)|\, N_{p(\omega)}(ds,dy)$$
$$\leq \int_{]0,T]}\int_U |\Phi(s,\omega,y)|\, N_{p(\omega)}(ds,dy) < \infty.$$

We fix $\omega \in N^c$. Then, for all $t \in [0,T]$

$$\begin{aligned}
&\Delta \int_{]0,t]}\int_U \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy) \\
&= \lim_{r\uparrow t}\Big(\int_{]0,t]}\int_U \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy) - \int_{]0,r]}\int_U \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy)\Big) \\
&= \lim_{r\uparrow t}\int_{]0,T]}\int_U 1_{]r,t]}(s)\, \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy) \\
&= \int_{]0,T]}\int_U 1_{\{t\}}(s)\, \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy),
\end{aligned}$$

by Lebesgue's dominated convergence theorem since

$$\int_{]0,t]}\int_U |\Phi(s,\omega,y)|\, N_{p(\omega)}(ds,dy) < \infty.$$

By proposition 3.2 and the definition of N we know that for $\omega \in N^c$

$$\int_{]0,T]}\int_U 1_{\{t\}}(s)\, \Phi(s,\omega,y)\, N_{p(\omega)}(ds,dy) = \sum_{\substack{s\in D_{p(\omega)}\\ s\leq t}} 1_{\{t\}}(s)\, \Phi(s,\omega,p(\omega)(s))$$
$$= \begin{cases} \Phi(t,\omega,p(\omega)(t)) & \text{, if } t \in D_{p(\omega)}, \\ 0 & \text{, otherwise.} \end{cases}$$

□

As an easy consequence of the previous two propositions we obtain the following corollary.

Corollary 3.4. *Let $\Phi : [0,T] \times \Omega \times U \to \mathbb{R}$ $\mathcal{P}_T(U)$-measurable such that $E[\int_{]0,T]}\int_U |\Phi(s,y)|\, N_p(ds,dy)] < \infty$, then*

$$\int_{]0,t]}\int_U \Phi(s,y)\, N_p(ds,dy) = \sum_{\substack{s\in D_p\\ s\leq t}} \Delta \int_{]0,s]}\int_U \Phi(s,y)\, N_p(ds,dy)$$

for all $t \in [0,T]$ P-a.s.

In particular, if Φ is non-negativ then

$$A(t) := \int_{]0,t]} \int_U \Phi(s,y)\, N_p(ds,dy),\ t \in [0,T],$$

is an increasing process in the sense of definition 1.4 with $A^c \equiv 0$.

Proposition 3.5. *Assume that $\Phi \in \mathcal{N}_q^2(T,U,H)$ and that τ is an $(\mathcal{F}_t)$-stopping time such that $P(\tau \leq T) = 1$. Then $1_{]0,\tau]}\Phi \in \mathcal{N}_q^2(T,U,H)$ and*

$$\int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y)\, q(ds,dy) = \int_0^{(t\wedge\tau)+} \int_U \Phi(s,y)\, q(ds,dy)$$

for all $t \in [0,T]$ P-a.s.

Proof.

Step 1. Let Φ be an elementary process, i.e.

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m,t_{m+1}]\times B_i^m} \in \mathcal{E},$$

and τ a simple stopping time, i.e.

$$\tau(\Omega) = \{a_0, \ldots, a_n\} \text{ and } \tau = \sum_{j=0}^{n} a_j 1_{A_j}$$

where $0 \leq a_j < a_{j+1} \leq T$ and $A_j = \{\tau = a_j\} \in \mathcal{F}_{a_j}$. Then

$$1_{]\tau,T]}\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \sum_{j=0}^{n} \Phi_i^m 1_{A_j} 1_{]t_m\vee a_j,t_{m+1}\vee a_j]\times B_i^m}$$

is an elementary process since $\Phi_i^m 1_{A_j}$ is $\mathcal{F}_{t_m\vee a_j}/\mathcal{B}(H)$-measurable. Concerning the integral of $1_{]0,\tau]}\Phi$ one then obtains for $t \in [0,T]$ that

$$\begin{aligned}
&\int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y)\, q(ds,dy) \\
&= \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{t+} \int_U 1_{]\tau,T]}(s)\Phi(s,y)\, q(ds,dy) \\
&= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1}\wedge t, B_i^m) - q(t_m \wedge t, B_i^m)) \\
&\quad - \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \sum_{j=0}^{n} \Phi_i^m 1_{A_j} \big(q((t_{m+1}\vee a_j)\wedge t, B_i^m) - q((t_m\vee a_j)\wedge t, B_i^m)\big)
\end{aligned}$$

$$= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1} \wedge t, B_i^m) - q(t_m \wedge t, B_i^m))$$
$$- \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \sum_{j=0}^{n} \Phi_i^m 1_{A_j} \big(q((t_{m+1} \vee \tau) \wedge t, B_i^m) - q((t_m \vee \tau) \wedge t, B_i^m) \big)$$
$$= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1} \wedge t, B_i^m) - q(t_m \wedge t, B_i^m))$$
$$- \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m \big(q((t_{m+1} \vee \tau) \wedge t, B_i^m) - q((t_m \vee \tau) \wedge t, B_i^m) \big)$$
$$= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m \big(q(t_{m+1} \wedge t, B_i^m) - q(t_m \wedge t, B_i^m) - q((t_{m+1} \vee \tau) \wedge t, B_i^m) + q((t_m \vee \tau) \wedge t, B_i^m) \big)$$
$$= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1} \wedge \tau \wedge t, B_i^m) - q(t_m \wedge \tau \wedge t, B_i^m))$$
$$= \int_0^{(t \wedge \tau)+} \int_U \Phi(s, y)\, q(ds, dy)$$

Step 2. Now we consider the case that Φ is still an elementary process while τ is an arbitrary stopping time with $P(\tau \leq T) = 1$. Then there exists a sequence $\tau_n = \sum_{k=0}^{2^n - 1} T(k+1)2^{-n} 1_{]Tk2^{-n}, T(k+1)2^{-n}]} \circ \tau$, $n \in \mathbb{N}$, of simple stopping times such that $\tau_n \downarrow \tau$ as $n \to \infty$.

By the right-continuity of the stochastic integral we get that

$$\int_0^{(t \wedge \tau_n)+} \int_U \Phi(s, y)\, q(ds, dy) \underset{n \to \infty}{\longrightarrow} \int_0^{(t \wedge \tau)+} \int_U \Phi(s, y)\, q(ds, dy)$$

for all $t \in [0, T]$ P-a.s.

Besides we obtain (even for non-elementary processes Φ) that

$$\|1_{]0, \tau_n]} \Phi - 1_{]0, \tau]} \Phi\|_T^2 = E\big[\int_0^T \int_U 1_{]\tau, \tau_n]}(s) \|\Phi(s, y)\|^2 \, \hat{N}_p(ds, dy) \big] \underset{n \to \infty}{\longrightarrow} 0$$

which, by the definition of the integral and proposition 1.11, implies that

$$E\big[\sup_{t \in [0, T]} \| \int_0^{t+} \int_U 1_{]0, \tau_n]}(s) \Phi(s, y)\, q(ds, dy) - \int_0^{t+} \int_U 1_{]0, \tau]}(s) \Phi(s, y)\, q(ds, dy) \|^2 \big] \underset{n \to \infty}{\longrightarrow} 0.$$

As by step 1

$$\int_0^{t+} \int_U 1_{]0, \tau_n]}(s) \Phi(s, y)\, q(ds, dy) = \int_0^{(t \wedge \tau_n)+} \int_U \Phi(s, y)\, q(ds, dy)$$

for all $n \in \mathbb{N}$ the assertion follows.

Step 3. Let now $\Phi \in \mathcal{N}_q^2(T,U,H)$, then $1_{]0,\tau]}\Phi \in \mathcal{N}_q^2(T,U,H)$.
There exists a sequence of elementary processes Φ_n, $n \in \mathbb{N}$, such that $\|\Phi_n - \Phi\|_T \longrightarrow 0$ as $n \to \infty$. Then it is clear that $\|1_{]0,\tau]}\Phi_n - 1_{]0,\tau]}\Phi\|_T \longrightarrow 0$ as $n \to \infty$. By the definition of the stochastic integral and proposition 1.11 it follows that

$$
\begin{aligned}
&E\big[\sup_{t\in[0,T]} \| \int_0^{t+} \int_U \Phi_n(s,y)\, q(ds,dy) - \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)\|^2\big] \\
&+ E\big[\sup_{t\in[0,T]} \| \int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi_n(s,y)\, q(ds,dy) \\
&\qquad\qquad - \int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y)\, q(ds,dy)\|^2\big] \\
&\underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

This implies the existence of a subsequence n_k, $k \in \mathbb{N}$, such that P-a.s.

$$
\begin{aligned}
\int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy) &\underset{k\to\infty}{\longrightarrow} \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) \\
\int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi_{n_k}(s,y)\, q(ds,dy) &\underset{k\to\infty}{\longrightarrow} \int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y)\, q(ds,dy)
\end{aligned}
$$

for all $t \in [0,T]$. In particular,

$$
\int_0^{(t\wedge\tau)+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy) \underset{k\to\infty}{\longrightarrow} \int_0^{(t\wedge\tau)+} \int_U \Phi(s,y)\, q(ds,dy)
$$

for all $t \in [0,T]$ P-a.s.
Then by step 2 we get that

$$
\int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y)\, q(ds,dy) = \int_0^{(t\wedge\tau)+} \int_U \Phi(s,y)\, q(ds,dy)
$$

for all $t \in [0,T]$ P-a.s.

□

Proposition 3.6. *Let $\Phi \in \mathcal{N}_q^2(T,U,H)$ and define $X(t) := \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)$, $t \in [0,T]$. Then X is cádlág and $X(t) = X(t-)$ P-a.s. for all $t \in [0,T]$.*

Proof. Let $t \in [0,T]$ and t_n, $n \in \mathbb{N}$, a sequence in $[0,t[$ such that $t_n \uparrow t$. Define

$$
\begin{aligned}
Y_n &:= \int_0^{T+} \int_U 1_{]t_n,t]}(s)\Phi(s,y)\, q(ds,dy) \\
&= \int_0^{t+} \int_U \Phi(s,y) - \int_0^{t_n+} \int_U \Phi(s,y) \ P\text{-a.s.},\ n \in \mathbb{N},\text{, by proposition 3.5,}
\end{aligned}
$$

$Y := X(t) - X(t-)$.

Then $Y_n \underset{n\to\infty}{\longrightarrow} Y$ P-a.s. and the sequence Y_n, $n \in \mathbb{N}$, is uniformly integrable since

$$\sup_{n\in\mathbb{N}} E[\|Y_n\|^2] \le \|\Phi\|_T^2 < \infty.$$

Therefore $Y_n \underset{n\to\infty}{\longrightarrow} Y$ in $L^1(\Omega, \mathcal{F}, P)$ and

$$\begin{aligned}
E[\|Y\|] &= \lim_{n\to\infty} E[\|Y_n\|] \le \limsup_{n\to\infty} E[\|Y_n\|^2]^{\frac{1}{2}} \\
&= \limsup_{n\to\infty} E[\|\int_0^{T+}\int_U 1_{]t_n,t]}(s)\Phi(s,y)\, q(ds,dy)\|^2]^{\frac{1}{2}} \\
&= \limsup_{n\to\infty} E\Big[\int_0^{T}\int_U 1_{]t_n,t]}(s)\|\Phi(s,y)\|^2\, \nu(dy)\, ds\Big]^{\frac{1}{2}} = 0,
\end{aligned}$$

by Lebesgue's dominated convergence theorem since $\|\Phi\|_T < \infty$. Hence, $Y = 0$ P-a.s., i.e. $X(t) = X(t-)$ P-a.s. $\square$

Proposition 3.7. *Let $\Phi \in \mathcal{N}_q^2(T, U, H)$, $(\tilde{H}, \langle\,,\,\rangle_{\tilde{H}})$ a further Hilbert space and $L \in L(H, \tilde{H})$. Then $L(\Phi) \in \mathcal{N}_q^2(T, U, \tilde{H})$ and*

$$L\Big(\int_0^{t+}\int_U \Phi(s,y)\, q(ds,dy)\Big) = \int_0^{t+}\int_U L(\Phi(s,y))\, q(ds,dy)$$

for all $t \in [0,T]$ P-a.s.

Proof. Since $\Phi \in \mathcal{N}_q^2(T, U, H)$ and $\|L(\Phi(s,\omega,y))\|_{\tilde{H}} \le \|L\|_{L(H,\tilde{H})}\|\Phi(s,\omega,y)\|$ for all $(s,\omega,y) \in [0,T] \times \Omega \times U$ it is obvious that $L(\Phi) \in \mathcal{N}_q^2(T, U, \tilde{H})$.

Step 1. Let Φ be an elementary process, i.e.

$$\Phi = \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m,t_{m+1}]\times B_i^m} \in \mathcal{E}.$$

Then

$$L(\Phi) = \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} L(\Phi_i^m) 1_{]t_m,t_{m+1}]\times B_i^m} \in \mathcal{E}$$

and

$$
\begin{aligned}
&L\Big(\int_0^{t+}\int_U \Phi(s,y)\,q(ds,dy)\Big)\\
&= L\Big(\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} \Phi_i^m\big(q(t_{m+1}\wedge t, B_i^m) - q(t_m\wedge t, B_i^m)\big)\Big)\\
&= \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} L(\Phi_i^m)(q(t_{m+1}\wedge t, B_i^m) - q(t_m\wedge t, B_i^m))\\
&= \int_0^{t+}\int_U L(\Phi(s,y))\,q(ds,dy) \;\text{ for all } t\in[0,T].
\end{aligned}
$$

Step 2. Let $\Phi \in \mathcal{N}_q^2(T,U,H)$. Then there exists a sequence of elementary processes Φ_n, $n\in\mathbb{N}$, such that $\|\Phi_n - \Phi\|_T \longrightarrow 0$ as $n\to\infty$. Then $L(\Phi_n)$, $n\in\mathbb{N}$, is a sequence of elementary processes with values in $\tilde{H}$ and

$$
\|L(\Phi_n) - L(\Phi)\|_T \le \|L\|_{L(H,\tilde{H})}\|\Phi_n - \Phi\|_T \longrightarrow 0 \;\text{ as } n\to\infty.
$$

By the definition of the stochastic integral and proposition 1.11 we get the existence of a subsequence n_k, $k\in\mathbb{N}$, such that P-a.s. for all $t\in[0,T]$

$$
\begin{aligned}
&\int_0^{t+}\int_U L(\Phi(s,y))\,q(ds,dy)\\
&= \lim_{k\to\infty}\int_0^{t+}\int_U L(\Phi_{n_k}(s,y))\,q(ds,dy)\\
&= \lim_{k\to\infty} L\Big(\int_0^{t+}\int_U \Phi_{n_k}(s,y)\,q(ds,dy)\Big), \;\text{ by step 1,}\\
&= L\Big(\lim_{k\to\infty}\int_0^{t+}\int_U \Phi_{n_k}(s,y)\,q(ds,dy)\Big), \;\text{ by the continuity of } L,\\
&= L\Big(\int_0^{t+}\int_U \Phi(s,y)\,q(ds,dy)\Big).
\end{aligned}
$$

□

Proposition 3.8. *Let $B\in\Gamma_p$ then $([q(\cdot,B)]_t)_{t\ge 0} = (N_p(t,B))_{t\ge 0}$.*

Proof. By theorem 1.27 $([q(\cdot,B)]_t)_{t\ge 0}$ is the P-unique $(\mathcal{F}_t)$-adapted, cádlág process of finite variation on compacts with the following properties:

(i) $q(t,B)^2 - [q(\cdot,B)]_t$, $t\ge 0$, is a local $(\mathcal{F}_t)$-martingale,

(ii) $\Delta[q(\cdot,B)]_t = \big(\Delta q(t,B)\big)^2$ for all $t\ge 0$ P-a.s.

Since $N_p(\cdot \times B)$ is a measure on $([0,\infty), \mathcal{B}([0,\infty))$ such that $N_p(]0,t] \times B) < \infty$ for all $t \geq 0$ P-a.s. the process $N_p(t,B) = N_p(]0,t] \times B)$, $t \geq 0$, is cádlág and increasing thus, in particular, of finite variation on compacts.
Moreover,

$$\begin{aligned}\Delta N_p(t,B) &= N_p(t,B) - \lim_{s \uparrow t} N_p(s,B) \\ &= N_p(t,B) - \hat{N}_p(t,B) - \lim_{s \uparrow t} \big(N_p(s,B) - \hat{N}_p(t,B)\big) = \Delta q(t,B)\end{aligned}$$

for all $t \geq 0$ P-a.s. Since

$$\Delta N_p(t,B) = \begin{cases} 0 & , \text{ if } p(t) \notin B, \\ 1 & , \text{ if } p(t) \in B, \end{cases}$$

we get that $\Delta N_p(t,B) = \Delta N_p(t,B)^2 = \Delta q(t,B)^2$ for all $t > 0$ P-a.s. and $N_p(0,B) = 0 = q(0,B)^2$.
It remains to check that $q(t,B)^2 - N_p(t,B)$, $t \geq 0$, is a local $(\mathcal{F}_t)$-martingale. Since $B \in \Gamma_p$ $q(t,B)^2 - N_p(t,B)$ is integrable for all $t \geq 0$:

$$\begin{aligned}&E\big[|(q(t,B)^2 - N_p(t,B)|\big] \leq E\big[q(t,B)^2\big] + E\big[N_p(t,B)\big] \\ =&E\big[\hat{N}_p(t,B)\big] + E\big[N_p(t,B)\big], \text{ by proposition 2.19,} \\ =&2E\big[N_p(t,B)\big] < \infty.\end{aligned}$$

To show the martingale property let $0 \leq s < t \leq T$ and $A \in \mathcal{F}_s$ then, again by proposition 2.19, we get that

$$\begin{aligned}&E\big[1_A\big(q(t,B)^2 - N_p(t,B)\big)\big] \\ =&E\big[1_A\big(q(t,B)^2 - \hat{N}_p(t,B)\big)\big] + P(A)\hat{N}_p(t,B) - E\big[1_A N_p(s,B)\big] \\ &- E\big[1_A\big(N_p(t,B) - N_p(s,B)\big)\big] \\ =&E\big[1_A\big(q(s,B)^2 - \hat{N}_p(s,B)\big)\big] + P(A)\hat{N}_p(t,B) - E\big[1_A N_p(s,B)\big] \\ &- P(A)\big(\hat{N}_p(t,B) - \hat{N}_p(s,B)\big), \text{ since } N_p(]s,t] \times B) \text{ is independent of } \mathcal{F}_s, \\ =&E\big[1_A\big(q(s,B)^2 - N_p(s,B)\big)\big].\end{aligned}$$

By proposition 1.16 $q(t,B)^2 - [q(\cdot,B)]_t$, $t \geq 0$, is a local $(\mathcal{F}_t)$-martingale. $\square$

Proposition 3.9. *Let* $\Phi \in \mathcal{N}_q^2(T,U,\mathbb{R})$. *Then*
$\big(X(t)\big)_{t \geq 0} := \big(\int_0^{(t \wedge T)+} \int_U \Phi(s,y)\, q(ds,dy)\big)_{t \geq 0} \in \mathcal{M}^2(\mathbb{R})$ *and*

$$\Big[\int_0^{(\cdot \wedge T)+} \int_U \Phi(s,y)\, q(ds,dy)\Big] = \int_{]0,\cdot \wedge T]} \int_U |\Phi(s,y)|^2\, N_p(ds,dy).$$

Proof.

Step 1. Let $\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m,t_{m+1}] \times B_i^m} \in \mathcal{E}$.

Then

$$
\begin{aligned}
&[\int_{]0,\cdot\wedge T]} \int_U \Phi(s,y)\, q(ds,dy)] \\
=&[\sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big)] \\
=&\sum_{m=0}^{k-1} \Big(\sum_{i=1}^{I(m)} [\Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big)] \\
&\qquad + 2 \sum_{1\le i<j\le I(m)} [\, \Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big), \\
&\qquad\qquad \Phi_j^m \big(q(t_{m+1}\wedge\cdot, B_j^m) - q(t_m \wedge \cdot, B_j^m)\big)]\Big) \\
&+ 2 \sum_{0\le m<n\le k-1} \sum_{\substack{(i,j)\in\{1,\dots,I(m)\} \\ \times\{1,\dots,I(n)\}}} [\, \Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big), \\
&\qquad \Phi_j^n \big(q(t_{m+1}\wedge\cdot, B_j^n) - q(t_m \wedge \cdot, B_j^n)\big)].
\end{aligned}
$$

Claim 1. Let $0 \le m \le k-1$ and $1 \le i \le I(m)$ then

$$
\begin{aligned}
&[\Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big)] \\
=&|\Phi_i^m|^2 \big(N_p(t_{m+1}\wedge\cdot, B_i^m) - N_p(t_m \wedge \cdot, B_i^m)\big).
\end{aligned}
$$

By theorem 1.27 the square bracket of the process $Y(t) := \Phi_i^m \big(q(t_{m+1}\wedge\cdot, B_i^m) - q(t_m \wedge \cdot, B_i^m)\big)$, $t \ge 0$, is defined as the P-unique $(\mathcal{F}_t)$-adapted, cádlág process A of finite variation on compacts with the following properties:

(i) $Y(t)^2 - A(t)$, $t \ge 0$, is a right-continuous, local $(\mathcal{F}_t)$-martingale,

(ii) $\Delta A(t) = \big(\Delta Y(t)\big)^2$ for all $t \ge 0$.

$A(t) := |\Phi_i^m|^2 \big(N_p(t_{m+1}\wedge t, B_i^m) - N_p(t_m \wedge t, B_i^m)$, $t \ge 0$, is a cádlág $(\mathcal{F}_t)$-adapted process. Moreover, it is increasing in t what can be shown by considering A on the intervalls $[0,t_m]$, $]t_m,t_{m+1}]$ and $]t_{m+1},\infty[$. As increasing process it is of finite variation on compacts.
As next step we check property (2), i.e. we show that $\Delta A(t) = \big(\Delta Y(t)\big)^2$ for all $t \ge 0$ P-a.s.
If $t = 0$ then $Y(0)^2 = 0 = A(0)$.
If $0 < t \le t_m$ then $Y(t) = 0 = A(t)$ and thus $\big(\Delta Y(t)\big)^2 = 0 = A(t)$.
If $t_m < t \le t_{m+1}$ then $Y(t) = \Phi_i^m \big(q(t, B_i^m) - q(t_m, B_i^m)\big)$ and $A(t) = |\Phi_i^m|^2 \big(N_p(t, B_i^m) - N_p(t_m, B_i^m)\big)$. Hence, by proposition 3.8,

$$
\begin{aligned}
&\big(\Delta Y(t)\big)^2 \\
&= |\Phi_i^m|^2\big(\Delta q(t,B_i^m)\big)^2 = |\Phi_i^m|^2 \Delta N_p(t,B_i^m)\,,\ \text{for all } t\in]t_m,t_{m+1}]\ P\text{-a.s.,}\\
&= \Delta A(t).
\end{aligned}
$$

If $t_{m+1} < t < \infty$ then $Y(t) = \Phi_i^m\big(q(t_{m+1},B_i^m) - q(t_m,B_i^m)\big)$ and $A(t) = |\Phi_i^m|^2\big(N_p(t_{m+1},B_i^m) - N_p(t_m,B_i^m)\big)$. Thus $(\Delta Y(t))^2 = 0 = \Delta A(t)$.
It remains to check that

$$
\begin{aligned}
&\big(\Phi_i^m\big(q(t_{m+1}\wedge t,B_i^m) - q(t_m\wedge t,B_i^m)\big)\big)^2\\
&- |\Phi_i^m|^2\big(N_p(t_{m+1}\wedge t,B_i^m) - N_p(t_m\wedge t,B_i^m),\ t\geq 0,
\end{aligned}
$$

is a local $(\mathcal{F}_t)$-martingale. For this purpose let $0\leq s<t<\infty$ and $B\in\mathcal{F}_s$. We show the martingale property by differentiating between four cases.

Case 1. Let $0\leq s<t\leq t_m$ then $Y(t)^2 - A(t) = 0 = Y(s)^2 - A(s)$ and therefore

$$
E\big[1_B\big(Y(t)^2 - A(t)\big)\big] = E\big[1_B\big(Y(s)^2 - A(s)\big)\big].
$$

Case 2. Let $0\leq s\leq t_m < t$.

$$
\begin{aligned}
&E\big[1_B|\Phi_i^m|^2\big(\,q(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)^2\\
&\qquad\qquad - N_p(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)\big)\big]\\
&= E\big[1_B|\Phi_i^m|^2\big]\Big(E\big[q(]t_m,t_{m+1}\wedge t]\times B_i^m)^2\big] - E\big[N_p(]t_m,t_{m+1}\wedge t]\times B_i^m)\big]\Big),\\
&\quad \text{since } q(]t_m,t_{m+1}\wedge t]\times B_i^m)^2, N_p(]t_m,t_{m+1}\wedge t]\times B_i^m) \text{ are in-}\\
&\quad \text{dependent of } \mathcal{F}_{t_m} \text{ and } 1_B|\Phi_i^m|^2\in L^1(\Omega,\mathcal{F}_{t_m},P),\\
&= 0,\ \text{by equation 2.3},\\
&= E\big[1_B|\Phi_i^m|^2\big(\,q(]t_m\wedge s,t_{m+1}\wedge s]\times B_i^m)^2\\
&\qquad\qquad - N_p(]t_m\wedge s,t_{m+1}\wedge s]\times B_i^m)\big)\big]
\end{aligned}
$$

Case 3. Let $0\leq t_m < s < t$ and $s\leq t_{m+1}$.

$$
\begin{aligned}
&E\big[1_B|\Phi_i^m|^2\big(\,q(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)^2\\
&\qquad\qquad - N_p(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)\big)\big]\\
&= E\big[1_B|\Phi_i^m|^2\big(q(]t_m,s]\times B_i^m)^2 - N_p(]t_m,s]\times B_i^m)\big)\big]\\
&+ E\big[1_B|\Phi_i^m|^2\big(q(]s,t_{m+1}\wedge t]\times B_i^m)^2 - N_p(]s,t_{m+1}\wedge t]\times B_i^m)\big)\big]\\
&+ E\big[1_B|\Phi_i^m|\,2\,q(]t_m,s]\times B_i^m)q(]s,t_{m+1}\wedge t]\times B_i^m)\big]
\end{aligned}
$$

where

$$\begin{aligned}
&E\big[1_B|\Phi_i^m|^2\big(q(]s,t_{m+1}\wedge t]\times B_i^m)^2-N_p(s,t_{m+1}\wedge t]\times B_i^m)\big)\big]\\
=&E\big[1_B|\Phi_i^m|^2\big]\Big(E\big[q(]s,t_{m+1}\wedge t]\times B_i^m)^2\big]-E\big[N_p(s,t_{m+1}\wedge t]\times B_i^m)\big]\Big)\\
=&0
\end{aligned}$$

since $q(]s,t_{m+1}\wedge t]\times B_i^m)^2-N_p(]s,t_{m+1}\wedge t]\times B_i^m)$ is independent of $\mathcal{F}_s$ and $1_B|\Phi_i^m|^2\in L^1(\Omega,\mathcal{F}_s,P)$ and

$$\begin{aligned}
&E\big[1_B|\Phi_i^m|^2\,2\,q(]t_m,s]\times B_i^m)q(]s,t_{m+1}\wedge t]\times B_i^m)\big]\\
=&E\big[1_B|\Phi_i^m|^2\,2\,q(]t_m,s]\times B_i^m)\big]E\big[q(]s,t_{m+1}\wedge t]\times B_i^m)\big]\\
=&0
\end{aligned}$$

since $q(]s,t_{m+1}\wedge t]\times B_i^m)$ is independent of $\mathcal{F}_s$ and $1_B|\Phi_i^m|^2\,2\,q(]t_m,s]\times B_i^m)\in L^1(\Omega,\mathcal{F}_s,P)$.

Case 4. Let $0\le t_m<t_{m+1}<s<t$.

$$\begin{aligned}
&E\big[1_B|\Phi_i^m|^2\big(q(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)^2\\
&\qquad\qquad-N_p(]t_m\wedge t,t_{m+1}\wedge t]\times B_i^m)\big)\big]\\
=&E\big[1_B|\Phi_i^m|^2\big(q(]t_m,t_{m+1}]\times B_i^m)^2\\
&\qquad\qquad-N_p(]t_m,t_{m+1}]\times B_i^m)\big)\big]\\
=&E\big[1_B|\Phi_i^m|^2\big(q(]t_m\wedge s,t_{m+1}\wedge s]\times B_i^m)^2\\
&\qquad\qquad-N_p(]t_m\wedge s,t_{m+1}\wedge s]\times B_i^m)\big)\big]
\end{aligned}$$

Hence

$$\begin{aligned}
&\big(\Phi_i^m\big(q(t_{m+1}\wedge t,B_i^m)-q(t_m\wedge t,B_i^m)\big)\big)^2\\
&-|\Phi_i^m|^2\big(N_p(t_{m+1}\wedge t,B_i^m)-N_p(t_m\wedge t,B_i^m)\big),\ t\ge 0,
\end{aligned}$$

is an $(\mathcal{F}_t)$-martingale and therefore, by proposition 1.16 a local $(\mathcal{F}_t)$-martingale.

Claim 2. Let $0\le m\le k-1$ and $1\le i<j\le I(m)$, then

$$\begin{aligned}
&[\Phi_i^m q(]t_m\wedge\cdot,t_{m+1}\wedge\cdot]\times B_i^m),\\
&\ \Phi_j^m q(]t_m\wedge\cdot,t_{m+1}\wedge\cdot]\times B_j^m)]\equiv 0.
\end{aligned}$$

Claim 3. Let $0\le m<n\le k-1$, $1\le i\le I(m)$ and $1\le j\le I(n)$ then

$$\begin{aligned}
&[\Phi_i^m q(]t_m\wedge\cdot,t_{m+1}\wedge\cdot]\times B_i^m),\\
&\ \Phi_j^n q(]t_n\wedge\cdot,t_{n+1}\wedge\cdot]\times B_j^n)]\equiv 0.
\end{aligned}$$

Claim 2 and claim 3 can be shown analoguously to claim 1.

Step 2. Let $\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$.

Define $A(t) := \int_{]0,T\wedge t]} \int_U |\Phi(s,y)|^2 \, N_p(ds,dy)$, $t \geq 0$.
Then A is an increasing process. Moreover it is cádlág, which can be shown by Lebesgue's dominated convergence theorem since, by proposition 3.1, $E[\int_{]0,T]} \int_U \Phi^2(s,y) \, N_p(ds,dy)] = E[\int_0^T \int_U \Phi^2(s,y) \, \hat{N}_p(ds,dy)] < \infty$ and therefore $\int_{]0,T]} \int_U \Phi^2(s,y) \, N_p(ds,dy) < \infty$ P-a.s.

Since $\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$ there exists a sequence Φ_n, $n \in \mathbb{N}$, in $\mathcal{E}$, such that

$$
\begin{aligned}
&E[\int_{]0,T]} \int_U \|\Phi(s,y) - \Phi_n(s,y)\|^2 \, N_p(ds,dy)] \\
&= E[\int_{]0,T]} \int_U \|\Phi(s,y) - \Phi_n(s,y)\|^2 \, \hat{N}_p(ds,dy)], \text{ by proposition 3.1,} \\
&= \|\Phi - \Phi_n\|_T \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

By the definition of the intergal with respect to q we obtain that

$$
\|\int_0^{\cdot+} \int_U \Phi(s,y) \, q(ds,dy) - \int_0^{\cdot+} \int_U \Phi_n(s,y) \, q(ds,dy)\|_{\mathcal{M}_T^2} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.3}
$$

Hence, we get the existence of a subsequence n_k, $k \in \mathbb{N}$, such that

$$
\begin{aligned}
&\sup_{0\leq t\leq T} |\int_0^{t+} \int_U \Phi_{n_k}(s,y) \, q(ds,dy) - \int_0^{t+} \int_U \Phi(s,y) \, q(ds,dy)| \\
&\underset{k\to\infty}{\longrightarrow} 0 \quad P\text{-a.s.}
\end{aligned} \tag{3.4}
$$

and

$$
\int_{]0,t]} \int_U (\Phi_{n_k}(s,y) - \Phi(s,y))^2 \, N_p(ds,dy) \underset{k\to\infty}{\longrightarrow} 0 \text{ for all } t \in [0,T] \; P\text{-a.s.}
$$

Then

$$
\begin{aligned}
&|\big(\int_{]0,t]} \int_U 1_{\{t\}} \Phi_{n_k}^2(s,y) \, N_p(ds,dy)\big)^{\frac{1}{2}} \\
&\qquad - \big(\int_{]0,t]} \int_U 1_{\{t\}} \Phi^2(s,y) \, N_p(ds,dy)\big)^{\frac{1}{2}}| \\
&\underset{k\to\infty}{\longrightarrow} 0 \text{ for all } t \in [0,T] \; P\text{-a.s.}
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
&|\big(\int_{]0,t]} \int_U \Phi_{n_k}^2(s,y) \, N_p(ds,dy)\big)^{\frac{1}{2}} - \big(\int_{]0,t]} \int_U \Phi^2(s,y) \, N_p(ds,dy)\big)^{\frac{1}{2}}| \\
&\underset{k\to\infty}{\longrightarrow} 0 \text{ for all } t \in [0,T]
\end{aligned}
$$

which, in particular, implies for all $t \in [0,T]$ the $\mathcal{F}_t$-measurability of $\int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy)$.
Moreover, for all $t \in [0,T]$

$$
\begin{aligned}
&|E[\int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy)]^{\frac{1}{2}} - E[\int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy)]^{\frac{1}{2}}| \\
=&\,|E[\int_0^t \int_U \Phi^2_{n_k}(s,y)\, \hat{N}_p(ds,dy)]^{\frac{1}{2}} - E[\int_0^t \int_U \Phi^2(s,y)\, \hat{N}_p(ds,dy)]^{\frac{1}{2}}| \\
\leq&\, E[\int_0^t \int_U (\Phi_{n_k}(s,y) - \Phi(s,y))^2\, \hat{N}_p(ds,dy)]^{\frac{1}{2}},
\end{aligned}
$$

i.e. for all $t \in [0,T]$

$$
\underbrace{\int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy)}_{\geq 0} \underset{k\to\infty}{\longrightarrow} \int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy) \ P\text{-a.s. and}
$$

$$
E[\int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy)] \underset{k\to\infty}{\longrightarrow} E[\int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy)].
$$

Thus, we can conclude that for all $t \in [0,T]$

$$
\int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy) \underset{k\to\infty}{\longrightarrow} \int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy) \tag{3.6}
$$

in $L^1(\Omega, \mathcal{F}, P)$. Now we show that $\big(\int_0^{\cdot\wedge T} \int_U \Phi(s,y)\, q(ds,dy)\big)^2 - A$ has the martingale property. For this purpose let $0 \leq r < t \leq T$ and $B \in \mathcal{F}_r$.
By (3.3) and (3.6) and step 1 we get that

$$
\begin{aligned}
&E\big[1_B\big((\int_0^t \int_U \Phi(s,y)\, q(ds,dy))^2 - A(t)\big)\big] \\
=&\lim_{k\to\infty} E\big[1_B\big((\int_0^t \int_U \Phi_{n_k}(s,y)\, q(ds,dy))^2 - \int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy)\big)\big] \\
=&\lim_{k\to\infty} E\big[1_B\big((\int_0^r \int_U \Phi_{n_k}(s,y)\, q(ds,dy))^2 - \int_{]0,r]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy)\big)\big] \\
=&\,E\big[1_B\big((\int_0^r \int_U \Phi(s,y)\, q(ds,dy))^2 - A(r)\big)\big].
\end{aligned}
$$

It remains to check that

$$
\Delta A(t) = \Big(\Delta \int_0^t \int_U \Phi(s,y)\, q(ds,dy)\Big)^2 \text{ for all } 0 \leq t \leq T \ P\text{-a.s.}
$$

If $t = 0$ then

$$
\Delta A(0) = A(0) = 0 = \Big(\Delta \int_0^0 \int_U \Phi(s,y)\, q(ds,dy)\Big)^2.
$$

We already showed in the proof of proposition 3.3 that

$$\begin{aligned}
&\Delta \int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy) \\
&= \int_{]0,T]} \int_U 1_{\{t\}}(s)\, \Phi^2(s,y)\, N_p(ds,dy) \text{ for all } t \in]0,T] \ P\text{-a.s.}
\end{aligned}$$

and

$$\begin{aligned}
&\Delta \int_{]0,t]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy) \\
&= \int_{]0,T]} \int_U 1_{\{t\}}(s)\, \Phi^2_{n_k}(s,y)\, N_p(ds,dy) \text{ for all } t \in]0,T], k \in \mathbb{N} \ P\text{-a.s.}
\end{aligned}$$

Hence, by (3.5) and step 1 we obtain that

$$\begin{aligned}
&\Delta \int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy) = \int_{]0,T]} \int_U 1_{\{t\}}(s)\, \Phi^2(s,y)\, N_p(ds,dy) \\
&= \lim_{k\to\infty} \int_{]0,T]} \int_U 1_{\{t\}}(s)\, \Phi^2_{n_k}(s,y)\, N_p(ds,dy) = \lim_{k\to\infty} \Delta \int_{]0,T]} \int_U \Phi^2_{n_k}(s,y)\, N_p(ds,dy) \\
&= \lim_{k\to\infty} \Big(\Delta \int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)\Big)^2 \text{ for all } t \in]0,T] \ P\text{-a.s.}
\end{aligned}$$

Since by (3.4) $\int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)$ converges to $\int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)$ P-a.s. uniformly in $t \in [0,T]$ we get that

$$\begin{aligned}
&|\Delta \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) - \Delta \int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)| \\
&= |\lim_{r\uparrow t} \big(\int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{r+} \int_U \Phi(s,y)\, q(ds,dy)\big) \\
&\quad - \lim_{r\uparrow t} \big(\int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy) - \int_0^{r+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)\big)| \\
&= \lim_{r\uparrow t} | \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy) \\
&\quad - \big(\int_0^{r+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{r+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)\big)| \\
&\leq 2 \sup_{0\leq t\leq T} | \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)| \\
&\quad \longrightarrow 0 \text{ for all } t \in]0,T] \ P\text{-a.s. as } k \to \infty.
\end{aligned}$$

Finally, we obtain that

$$\begin{aligned}
&\Delta \int_{]0,t]} \int_U \Phi^2(s,y)\, N_p(ds,dy) \\
&= \lim_{k\to\infty} \Big(\Delta \int_0^{t+} \int_U \Phi_{n_k}(s,y)\, q(ds,dy)\Big)^2 \\
&= \Big(\Delta \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)\Big)^2 \text{ for all } t \in]0,T] \; P\text{-a.s.}
\end{aligned}$$

□

Proposition 3.10. *Let $\Phi \in \mathcal{N}_q^2(T,U,\mathbb{R})$. Denote by X the integral process $(X(t))_{t\geq 0} := (\int_{]0,t\wedge T]} \int_U \Phi(s,y)\, q(ds,dy))_{t\geq 0} \in \mathcal{M}^2(\mathbb{R})$.*
Moreover, let Y be an $(\mathcal{F}_t)$-adapted, left continuous, bounded process ($|Y(t,\omega)| \leq K < \infty$ for all $t \geq 0$ and $\omega \in \Omega$).
Then

(i) $Y \in \mathcal{L}_{ucp}$ *and* $Y\Phi \in \mathcal{N}_q^2(T,U,\mathbb{R})$,

(ii)

$$\int_{]0,t]} Y(s)\, dX(s) = \int_0^{t+} \int_U Y(s)\Phi(s,y)\, q(ds,dy) \text{ for all } t \in [0,T] \; P\text{-a.s.}$$

Proof. Let Π_n, $n \in \mathbb{N}$, a sequence of partitions of $[0,\infty[$ given by $0 = t_0^n \leq t_1^n \leq \cdots \leq t_{k_n}^n < \infty$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} t_{k_n}^n = \infty$ and $\sup_{0\leq i\leq k_n-1} |t_{i+1}^n - t_i^n|$ converges to 0 as $n \to \infty$. Then we obtain by Lebesgue's dominated convergence theorem that

$$E\Big[\int_0^T \int_U \Big| \sum_{j=0}^{k(n)-1} 1_{]t_j^n\wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n)\Phi(s,y) - Y(s)\Phi(s,y)\Big|^2 \nu(dy)\, ds\Big] \longrightarrow 0$$

as $n \to \infty$ since $\Phi \in \mathcal{N}_q^2(T,U,\mathbb{R})$ and Y is left continuous and bounded. By the definition of the stochastic integral we get that

$$\begin{aligned}
\sup_{0\leq t\leq T} E\Big[\Big| &\int_0^{t+} \int_U \sum_{j=0}^{k(n)-1} 1_{]t_j^n\wedge T, t_{j+1}^n\wedge T]}(s) Y(t_j^n)\Phi(s,y)\, q(ds,dy) \\
&- \int_0^{t+} \int_U Y(s)\Phi(s,y)\, q(ds,dy)\Big|^2\Big] \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

In particular, we obtain for $t \in [0, T]$ that

$$\int_0^{t+} \int_U \sum_{j=0}^{k(n)-1} 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n) \Phi(s,y)\, q(ds,dy) \longrightarrow \int_0^{t+} \int_U Y(s) \Phi(s,y)\, q(ds,dy)$$

P-stochastically as $n \to \infty$.
Moreover, for $t \in [0, T]$,

$$\int_0^{t+} \int_U \sum_{j=0}^{k(n)-1} 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n) \Phi(s,y)\, q(ds,dy) = \sum_{j=0}^{k(n)-1} Y(t_j^n)\big(X(t_{j+1}^n \wedge t) - X(t_j \wedge t)\big)$$

since

$$\begin{aligned}
&\sum_{j=0}^{k(n)-1} Y(t_j^n)\big(X(t_{j+1}^n \wedge t) - X(t_j \wedge t)\big)\\
&= \sum_{j=0}^{k(n)-1} Y(t_j^n)\Big(\int_0^{(t_{j+1}^n \wedge t)+} \int_U \Phi(s,y)\, q(ds,dy) - \int_0^{(t_j^n \wedge t)+} \int_U \Phi(s,y)\, q(ds,dy)\Big)\\
&= \sum_{j=0}^{k(n)-1} Y(t_j^n) \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) \Phi(s,y)\, q(ds,dy) \ P\text{-a.s.},\\
&\quad \text{by proposition 3.5,}\\
&= \sum_{j=0}^{k(n)-1} \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n)\, \Phi(s,y)\, q(ds,dy).
\end{aligned}$$

To show the last equality assume first that $\Phi \in \mathcal{E}$. Then $1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]} Y(t_j^n)\, \Phi \in \mathcal{E}$ and the stated equality holds obviously. If $\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$ then there exists a sequence $\Phi_m \in \mathcal{E}$, $m \in \mathbb{N}$, such that $\|\Phi - \Phi_m\|_T \to 0$ as $m \to \infty$. Then

$$\|1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]} \Phi - 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]} \Phi_m\|_T \to 0 \text{ and}$$
$$\|1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]} Y(t_j^n)\, \Phi - 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]} Y(t_j^n)\, \Phi\|_T \to 0.$$

Hence there exist a subsequence m_k, $k \in \mathbb{N}$, such that

$$\begin{aligned}
&Y(t_j^n) \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s)\, \Phi(s,y)\, q(ds,dy) \\
&= \lim_{k\to\infty} Y(t_j^n) \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s)\, \Phi_{m_k}(s,y)\, q(ds,dy) \\
&= \lim_{k\to\infty} \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n)\, \Phi_{m_k}(s,y)\, q(ds,dy) \\
&= \int_0^{t+} \int_U 1_{]t_j^n \wedge T, t_{j+1}^n \wedge T]}(s) Y(t_j^n)\, \Phi(s,y)\, q(ds,dy) \quad P\text{-a.s.}
\end{aligned}$$

By theorem 1.24 $\int_{0+}^t Y(s)\, dX(s)$ can be approximated by the sums

$$\sum_{j=0}^{k(n)-1} Y(t_j^n)\big(X(t_{j+1}^n \wedge t) - X(t_j \wedge t)\big)$$

P-stochastically. Hence, since limits in probability are P-a.s. unique we obtain for all $t \in [0,T]$ that

$$\int_{]0,t]} Y(s)\, dX(s) = \int_0^{t+} \int_U Y(s)\Phi(s,y)\, q(ds,dy)\ P\text{-a.s.}$$

By the right-continuity of both sides of the above equation the assertion follows. □

Chapter 4

Existence of the Mild Solution

As in the previous chapter let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space, $(U, \mathcal{B}, \nu)$ a σ-finite measure space and $(\Omega, \mathcal{F}, P)$ a complete probability space with a right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$.
Moreover, let p be a stationary $(\mathcal{F}_t)$-Poisson point process on U and $(\Omega, \mathcal{F}, P)$ with characteristic measure ν. Let $T > 0$ and consider the following type of stochastic differential equation in H

$$\begin{cases} dX(t) & = [AX(t) + F(X(t))]\, dt + B(X(t), y)\, q(dt, dy) \\ X(0) & = \xi \end{cases} \tag{4.1}$$

where we always assume that

- $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of linear, bounded operators on H.
- $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable.
- $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable.
- $q(t, B) = N_p(t, B) - t\nu(B)$, $t \geq 0$, $B \in \Gamma_p$.
- ξ is an H-valued, $\mathcal{F}_0$-measurable random variable.

Remark 4.1. If we set $M_T := \sup_{t\in[0,T]} \|S(t)\|_{L(H)}$ then $M_T < \infty$.

Proof. By [Pa 83, Theorem 2.2, p.4] there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\|_{L(H)} \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

□

We interpret (4.1) as an integral equation and search for a mild solution.

Definition 4.2 (Mild solution). An H-valued predictable process $X(t)$, $t \in [0,T]$, is called a *mild solution of equation (4.1)* if

$$\begin{aligned} X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\, ds \\ + \int_0^{t+} \int_U S(t-s)B(X(s),y)\, q(ds,dy) \quad P\text{-a.s.} \end{aligned} \tag{4.2}$$

for all $t \in [0,T]$. In particular, the appearing integrals have to be well defined.

The idea to interpret (4.1) by (4.2) can be justified by the following proposition.

Proposition 4.3. *Let $X(t)$, $t \in [0,T]$, be a mild solution of (4.1).*
Assume that
$\int_0^t S(t-s)F(X(s))\, ds$ *and* $\int_0^{T+} \int_U 1_{]0,t]}(s)S(t-s)B(X(s),y)\, q(ds,dy)$, $t \in [0,T]$, *have predictable versions and that for all* $\zeta \in D(A^*)$

$$\begin{aligned} &\int_0^T \|F(X(s))\|\, ds < \infty \text{ and} \\ &\int_0^T E\big[\int_0^t \int_U |\langle S(t-s)B(X(s),y), A^*\zeta\rangle|^2\, \nu(dy)\, ds\big]\, dt < \infty \end{aligned}$$

then X is a weak solution, i.e.

$$\begin{aligned} \langle X(t),\zeta\rangle = \langle \xi,\zeta\rangle + \int_0^t \langle X(s), A^*\zeta\rangle + \langle F(X(s)),\zeta\rangle\, ds \\ + \int_0^t \langle B(X(s),y),\zeta\rangle\, q(ds,dy) \quad P\text{-a.s.} \end{aligned}$$

for all $t \in [0,T]$ and $\zeta \in D(A^*)$.

Proof. Since $\int_0^{T+} \int_U 1_{]0,t]}(s)S(t-s)B(X(s),y)\;\; q(ds,dy)$, $t \in [0,T]$, has a predictable version we know by proposition 3.7 that for all $\zeta \in D(A^*)$

$$\int_0^{T+} \int_U \langle 1_{]0,t]}(s)S(t-s)B(X(s),y), A^*\zeta\rangle\; q(ds,dy),\; t \in [0,T],$$

has a predictable version. By the notations

$$\begin{aligned} &\int_0^t S(t-s)F(X(s))\, ds,\; t \in [0,T], \\ &\int_0^{T+} \int_U \langle 1_{]0,t]}(s)S(t-s)B(X(s),y), A^*\zeta\rangle\; q(ds,dy),\; t \in [0,T],\; \zeta \in D(A^*), \end{aligned}$$

we understand the predictable versions of the respective processes.
For all $\zeta \in D(A^*)$

$$\begin{aligned}
&\int_0^T |\int_0^t \langle S(t-s)F(X(s)), A^*\zeta\rangle \, ds| \, dt \\
= &\int_0^T |\langle \int_0^t S(t-s)F(X(s)) \, ds, A^*\zeta\rangle| \, dt \\
\le &\|A^*\zeta\| M_T \int_0^T \int_0^t \|F(X(s))\| \, ds \, dt \\
\le &\|A^*\zeta\| M_T T \int_0^T \|F(X(s))\| \, ds < \infty \quad P\text{-a.s.}.
\end{aligned}$$

By the isometry for stochastic integrals we have that

$$\begin{aligned}
&E\big[\int_0^T |\int_0^{T+} \int_U \langle 1_{]0,t]}(s) S(t-s)B(X(s),y), A^*\zeta\rangle \, q(ds,dy)| \, dt\big] \\
\le &T^{\frac{1}{2}} \Big(\int_0^T E\big[|\int_0^T \int_U \langle 1_{]0,t]}(s) S(t-s)B(X(s),y), A^*\zeta\rangle \, q(ds,dy)|^2\big] \, dt\Big)^{\frac{1}{2}} \\
= &T^{\frac{1}{2}} \Big(\int_0^T E\big[\int_0^t \int_U |\langle S(t-s)B(X(s),y), A^*\zeta\rangle|^2 \, \nu(dy) \, ds\big] \, dt\Big)^{\frac{1}{2}} < \infty
\end{aligned}$$

for all $\zeta \in D(A^*)$. Therefore the processes $\int_0^t \langle S(t-s)F(X(s)), A^*\zeta\rangle \, ds$ and $\int_0^{T+} \int_U \langle 1_{]0,t]}(s) S(t-s)B(X(s),y), A^*\zeta\rangle \, q(ds,dy)$, $t \in [0,T]$, are P-a.s. Bochner integrable and we obtain that

$$\begin{aligned}
&E\big[\,|\int_0^t \langle X(s), A^*\zeta\rangle \, ds - \int_0^t \langle S(s)\xi, A^*\zeta\rangle \, ds \\
&- \int_0^t \int_0^s \langle S(s-u)F(X(u)), A^*\zeta\rangle \, du \, ds \\
&- \int_0^t \int_0^{T+} \int_U \langle 1_{]0,s]}(u) S(s-u)B(X(u),y), A^*\zeta\rangle \, q(du,dy) \, ds|\,\big] \\
\le &\int_0^t E\big[\,|\langle X(s), A^*\zeta\rangle - \langle S(s)\xi, A^*\zeta\rangle \\
&- \langle \int_0^s S(s-u)F(X(u)) \, du, A^*\zeta\rangle \\
&- \int_0^{T+} \int_U \langle 1_{]0,s]}(u) S(s-u)B(X(u),y), A^*\zeta\rangle \, q(du,dy)|\,\big] \, ds
\end{aligned}$$

where for each $s \in [0,T]$ by proposition 3.7 and proposition 3.5

$$
\begin{aligned}
&E\big[\,|\langle X(s), A^*\zeta\rangle - \langle S(s)\xi, A^*\zeta\rangle \\
&\quad - \langle \int_0^s S(s-u)F(X(u))\,du, A^*\zeta\rangle \\
&\quad - \int_0^{T+}\int_U \langle 1_{]0,s]}(u)S(s-u)B(X(u),y), A^*\zeta\rangle\, q(du,dy)|\,\big] \\
=\,&E\big[\,|\langle X(s) - S(s)\xi - \int_0^s S(s-u)F(X(u))\,du \\
&\quad - \int_0^{s+}\int_U S(s-u)B(X(u),y)\,q(du,dy), A^*\zeta\rangle|\,\big] \\
=\,&0
\end{aligned}
$$

since $X(t)$, $t \in [0,T]$, is a mild solution. Thus we get for all $\zeta \in D(A^*)$ and $t \in [0,T]$

$$
\begin{aligned}
&\int_0^t \langle X(s), A^*\zeta\rangle\; ds \\
=&\int_0^t \langle S(s)\xi, A^*\zeta\rangle\; ds + \int_0^t\int_0^s \langle S(s-u)F(X(u)), A^*\zeta\rangle\; du\; ds \\
&+\int_0^t\int_0^{T+}\int_U \langle 1_{]0,s]}(u)S(s-u)B(X(u),y), A^*\zeta\rangle\; q(du,dy)\; ds\; P\text{-a.s.}
\end{aligned}
$$

By [Pa 83, Corollary 10.6, p.41] $S^*(t)$, $t \in [0,T]$, is a C_0-semigroup with infinitesimal generator A^*. Then by proposition C.1 we get that $S^*(t)\zeta \in D(A^*)$ for all $t \in [0,T]$ and $\frac{d}{dt}S^*(t)\zeta = A^*S^*(t)\zeta = S^*(t)A^*\zeta$ for all $\zeta \in D(A^*)$.
Thus we can conclude by the fundamental theorem for Bochner integrals B.8 that

$$
\begin{aligned}
&\int_0^t \langle S(s)\xi, A^*\zeta\rangle\; ds = \int_0^t \langle \xi, S^*(s)A^*\zeta\rangle\; ds \\
=\,&\langle \xi, S^*(t)\zeta - \zeta\rangle = \langle S(t)\xi, \zeta\rangle - \langle \xi, \zeta\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\int_0^s \langle S(s-u)F(X(u)), A^*\zeta\rangle\; du\; ds \\
=&\int_0^t\int_0^t 1_{[0,s]}(u)\langle F(X(u)), S^*(s-u)A^*\zeta\rangle\; du\; ds \\
=&\int_0^t\int_u^t \langle F(X(u)), \frac{d}{ds}S^*(s-u)\zeta\rangle\; ds\; du \\
=\,&\langle \int_0^t S(t-s)F(X(s))\; ds, \zeta\rangle - \int_0^t \langle F(X(s)), \zeta\rangle\; ds.
\end{aligned}
$$

To calculate

$$\int_0^t \int_0^{T+} \int_U \langle 1_{]0,s]}(u) S(s-u) B(X(u),y), A^*\zeta\rangle \; q(du,dy)\; ds$$

we need a stochastic Fubini theorem. For an adequate version we refer to [Ap 05, Theorem 5]. Then we get that

$$\begin{aligned}
&\int_0^t \int_0^{T+} \int_U \langle 1_{]0,s]}(u) S(s-u) B(X(u),y), A^*\zeta\rangle \; q(du,dy)\; ds \\
=& \int_0^{T+} \int_U \int_0^t 1_{]0,s]}(u) \langle B(X(u),y), S^*(s-u)A^*\zeta\rangle \; ds \; q(du,dy) \\
=& \int_0^{T+} \int_U 1_{]0,t]}(u) \int_u^t \langle B(X(u),y), S^*(s-u)A^*\zeta\rangle \; ds \; q(du,dy) \\
=& \int_0^{T+} \int_U 1_{]0,t]}(u) \langle B(X(u),y), S^*(t-u)\zeta - \zeta\rangle \; q(du,dy) \\
=& \langle \int_0^{T+} \int_U 1_{]0,t]}(s) S(t-s) B(X(s),y)\; q(ds,dy), \zeta\rangle \\
&- \int_0^{T+} \int_U 1_{]0,t]}(s) \langle B(X(s),y), \zeta\rangle \; q(ds,dy) \; P\text{-a.s.}
\end{aligned}$$

where in the last step we used proposition 3.7.
Hence the mild solution $X(t)$, $t \in [0,T]$, fulfills the following equation P-a.s.:

$$\begin{aligned}
&\int_0^t \langle X(s), A^*\zeta\rangle \; ds \\
=& \langle S(t)\xi, \zeta\rangle + \langle \int_0^t S(t-s)F(X(s))\; ds, \zeta\rangle \\
&+ \langle \int_0^{T+} \int_U 1_{]0,t]}(s) S(t-s) B(X(s),y)\; q(ds,dy), \zeta\rangle \\
&- \langle \xi, \zeta\rangle - \int_0^t \langle F(X(s)), \zeta\rangle \; ds - \int_0^{T+} \int_U 1_{]0,t]}(s) \langle B(X(s),y), \zeta\rangle \; q(ds,dy) \\
=& \langle X(t), \zeta\rangle - \langle \xi, \zeta\rangle - \int_0^t \langle F(X(s)), \zeta\rangle \; ds - \int_0^{t+} \int_U \langle B(X(s),y), \zeta\rangle \; q(ds,dy)
\end{aligned}$$

P-a.s., where in the last step we used proposition 3.7 and 3.5 and the fact that X is a mild solution. Finally, we get that for all $t \in [0,T]$ and $\zeta \in D(A^*)$

$$\begin{aligned}
\langle X(t), \zeta\rangle =& \langle \xi, \zeta\rangle + \int_0^t \langle X(s), A^*\zeta\rangle + \langle F(X(s)), \zeta\rangle \; ds \\
&+ \int_0^{t+} \int_U \langle B(X(s),y), \zeta\rangle \; q(ds,dy) \quad P\text{-a.s.}
\end{aligned}$$

$\square$

Before stating the theorems about existence and uniqueness of a mild solution we give some notations and present the idea of the proof, where for the details we refer to the proofs of the theorems 4.4.
First, we introduce the spaces where we want to find the mild solution of the above problem. For $p \geq 2$ we define

$$\begin{aligned}\mathcal{H}^2(T,H) := \{Y(t), t \in [0,T] \mid Y &\text{ has an } H\text{-predictable version,}\\ &Y(t) \in L^2(\Omega, \mathcal{F}_t, P; H) \text{ and}\\ &\sup_{t\in[0,T]} E[\|Y(t)\|^2] < \infty\}\end{aligned}$$

and for $Y \in \mathcal{H}^2(T,H)$ define a seminorm on $\mathcal{H}^2(T,H)$ by

$$\|Y\|_{\mathcal{H}^2} := \sup_{t\in[0,T]} \big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}.$$

For technical reasons we also consider the seminorms $\| \ \|_{2,\lambda,T}$, $\lambda \geq 0$, on $\mathcal{H}^2(T,H)$ given by

$$\|Y\|_{2,\lambda,T} := \sup_{t\in[0,T]} e^{-\lambda t}\big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}.$$

Then $\| \ \|_{\mathcal{H}^2} = \| \ \|_{2,0,T}$ and all seminorms $\| \ \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent.
Let $\zeta \in \mathcal{L}_0^2 := \mathcal{L}^2(\Omega, \mathcal{F}_0, P; H)$ and $Z \in \mathcal{H}^2(T,H)$. Then Z has at least one predictable version which we denote again by Z. Define

$$\begin{aligned}\mathcal{F}(\zeta, Z) := \Big(S(t)\zeta + &\int_0^t S(t-s)F(Z(s))\,ds\\ &+ \int_0^{t+}\int_U S(t-s)B(Z(s),y)\,q(ds,dy)\Big)_{t\in[0,T]}.\end{aligned} \tag{4.3}$$

Later we will prove that under certain conditions on F and B the appearing integrals are well-defined and the processes on the right hand side of (4.3) are elements of $\mathcal{H}^2(T,H)$. Moreover, under the assumption that all integrals are well-defined, $\mathcal{F}$ is well-defined in the sense of version, i.e. taking another $\tilde{\zeta}$ such that $\tilde{\zeta} = \zeta$ P-a.s. and another predictable version $\tilde{Z}$ of Z, then $\mathcal{F}(\zeta, Z)$ is a version of $\mathcal{F}(\tilde{\zeta}, \tilde{Z})$ since we have that

$$\begin{aligned}\Big(E\Big[\ \|S(t)\zeta + &\int_0^t S(t-s)F(Z(s))\,ds\\ &\qquad + \int_0^{t+}\int_U S(t-s)B(Z(s),y)\,q(ds,dy)\\ -S(t)\tilde{\zeta} + &\int_0^t S(t-s)F(\tilde{Z}(s))\,ds\\ &\qquad + \int_0^{t+}\int_U S(t-s)B(\tilde{Z}(s),y)\,q(ds,dy)\|^2\Big]\Big)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
\leq& \Big(E[\|S(t)(\zeta-\tilde{\zeta})\|^2]\Big)^{\frac{1}{2}}\\
&+ M_T T^{\frac{1}{2}}\Big(E[\int_0^T \|F(Z(s))-F(\tilde{Z}(s))\|^2\,ds\Big)^{\frac{1}{2}}\\
&+ \Big(\int_0^t\int_U E[\|S(t-s)B(Z(s),y)-S(t-s)B(\tilde{Z}(s),y)\|^2]\,\nu(dy)\,ds\Big)^{\frac{1}{2}}\\
\leq& M_T\Big(E[\|\zeta-\tilde{\zeta}\|^2]\Big)^{\frac{1}{2}}\\
&+ M_T T^{\frac{1}{2}}\Big(\int_0^T E[\|F(Z(s))-F(\tilde{Z}(s))\|^2]\,ds\Big)^{\frac{1}{2}}\\
&+ M_T\Big(\int_0^T\int_U E[\|B(Z(s),y)-B(\tilde{Z}(s),y)\|^2]\,\nu(dy)\,ds\Big)^{\frac{1}{2}}\\
=& 0
\end{aligned}
$$

A mild solution of problem (4.1) with initial condition $\xi \in \mathcal{L}_0^2$ is by definition 4.2 an H-predictable process $X(\xi)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$ in the sense of versions.
Thus, we have to search for an implicit function $X : \mathcal{L}_0^2 \to \mathcal{H}^2(T,H)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$ in $\mathcal{H}^2(T,H)$ for all $\xi \in \mathcal{L}_0^2$.
The idea to prove this is to use Banach's fixed point theorem. This approach requires that $\mathcal{H}^2(T,H)$ is a Banach space. For this purpose we consider equivalence classes in $\mathcal{H}^2(T,H)$ w.r.t. $\| \|_{2,\lambda,T}$, $\lambda \geq 0$. We denote the space of equivalence classes by $H^2(T,H)$. $(H^2(T,H), \| \|_{2,\lambda,T})$, $\lambda \geq 0$, are Banach spaces.
For simplicity we use the following notations

$$
H^2(T,H) := (H^2(T,H), \| \|_{\mathcal{H}^2})
$$

and

$$
H^{2,\lambda}(T,H) := (H^2(T,H), \| \|_{2,\lambda,T}),\ \lambda > 0.
$$

Now we define for $\xi \in L_0^2 := L^2(\Omega, \mathcal{F}_0, P; H)$ and $Y \in H^2(T,H)$, $\bar{\mathcal{F}}(\xi, Y)$ as the equivalence class of $\mathcal{F}(\zeta, Z)$ w.r.t. $\| \|_{\mathcal{H}^2}$ for an arbitrary $\zeta \in \xi$ and an arbitrary predictable representative $Z \in Y$. By the above considerations, in $\mathcal{H}^2(T,H)$, $\mathcal{F}(\zeta, Z)$ is independent of the representatives ζ and Z.
Now, we search for an implicit function $X : L_0^2 \to H^2(T,H)$ such that $\bar{\mathcal{F}}(\xi, X(\xi)) = X(\xi)$ in $H^2(T,H)$ for all $\xi \in L_0^2$.
For this purpose we prove that $\bar{\mathcal{F}}$ as a mapping from $L_0^2 \times H^2(T,H)$ to $H^2(T,H)$ is well-defined and we show that there exists $\lambda_{T,2} =: \lambda \geq 0$ such that

$$
\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T,H) \to H^{2,\lambda}(T,H)
$$

is a contraction in the second variable, i.e. that there exists $L_{T,\lambda} < 1$ such that for all $\xi \in L_0^2$ and $Y, \tilde{Y} \in H^{2,\lambda}(T,H)$

$$\|\bar{\mathcal{F}}(\xi, Y) - \bar{\mathcal{F}}(\xi, \tilde{Y})\|_{2,\lambda,T} \leq L_{T,\lambda}\|Y - \tilde{Y}\|_{2,\lambda,T}.$$

Then the existence and uniqueness of the mild solution $X(\xi) \in H^{2,\lambda}(T,H)$ of (4.1) with initial condition $\xi \in L_0^2$ follows by Banach's fixed point theorem.
Since the norms $\| \ \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent we may consider $X(\xi)$ then as an element of $H^2(T,H)$ and get the existence of the implicit function $X : L_0^2 \to H^2(T,H)$ such that $\bar{\mathcal{F}}(\xi, X(\xi)) = X(\xi)$.

To get the existence of a mild solution on $[0,T]$ in $H^2(T,H)$ we make the following assumptions.

Hypothesis H.0

- $F : H \to H$ is Lipschitz-continuous, i.e. there exists a constant $C > 0$ such that

$$\begin{aligned} \|F(x) - F(y)\| &\leq C\|x - y\| \\ \|F(x)\| &\leq C(1 + \|x\|) \quad \text{for all } x, y \in H. \end{aligned}$$

- There exists an integrable mapping $K : [0,T] \to [0,\infty[$ such that for all $t \in]0,T]$ and for all $x, z \in H$

$$\begin{aligned} \int_U \|S(t)(B(x,y) - B(z,y))\|^2 \, \nu(dy) &\leq K(t)\|x - z\|^2 \\ \int_U \|S(t)B(x,y)\|^2 \, \nu(dy) &\leq K(t)(1 + \|x\|)^2. \end{aligned}$$

Theorem 4.4. *Assume that the coefficients A, F and B fulfill the conditions of hypothesis H.0 then for every initial condition $\xi \in L_0^2$ there exists a unique mild solution $X(\xi)(t)$, $t \in [0,T]$, of equation (4.1) in $H^2(T,H)$.*
In addition, we even obtain that the mapping

$$X : L_0^2 \to H^2(T,H)$$

is Lipschitz continuous.

For the proof of the theorem we need the following lemmas.

Lemma 4.5. *If $Y : [0,T] \times \Omega \times U \to H$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable then the mapping*

$$[0,T] \times \Omega \times U \to H, \ (s, \omega, y) \mapsto 1_{]0,t]}(s)S(t-s)Y(s,\omega,y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$.

Step 1. Consider the case that Y is a simple process given by

$$Y = \sum_{k=1}^{n} x_k 1_{A_k}$$

where $x_k \in H$, $1 \le k \le n$, and $A_k \in \mathcal{P}_T(U)$, $1 \le k \le n$, is a disjoint covering of $[0,T] \times \Omega \times U$. Then we obtain that

$$\begin{aligned} \tilde{Y} : [0,T] \times \Omega \times U &\to H \\ (s,\omega,y) &\mapsto 1_{]0,t]}(s) S(t-s) Y(s,\omega,y) \\ &= 1_{]0,t]}(s) \sum_{k=1}^{n} S(t-s) x_k 1_{A_k}(s,\omega,y) \end{aligned}$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable since for $B \in \mathcal{B}(H)$ we get that

$$\tilde{Y}^{-1}(B) = \bigcup_{k=1}^{n} \big(\{s \in [0,T] \,|\, 1_{]0,t]}(s) S(t-s) x_k \in B\} \times \Omega \times U\big) \cap A_k$$

where $\{s \in [0,T] \,|\, 1_{]0,t]}(s) S(t-s) x_k \in B\} \in \mathcal{B}([0,T])$ by the strong continuity of the semigroup $S(t)$, $t \in [0,T]$. By remark 2.23 (i) we can conclude that $\tilde{Y}^{-1}(B) \in \mathcal{P}_T(U)$.

Step 2. Let Y be an arbitrary $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable process.
Then there exists a sequence Y_n, $n \in \mathbb{N}$, of simple $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable random variables such that $Y_n \to Y$ pointwise as $n \to \infty$ by lemma B.5. Since $S(t) \in L(H)$ for all $t \in [0,T]$ the assertion follows.

□

Lemma 4.6. *Let $Y(t)$, $t \ge 0$, be a process on $(\Omega, \mathcal{F}, P)$ with values in a separable Banach space E. If Y is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and stochastically continuous then there exists a predictable version of Y.*
In particular, if Y is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and continuous in the square mean then there exists a predictable version of Y.

Proof. [DaPrZa 92, Proposition 3.6 (ii), p.76] □

Proof of theorem 4.4:
To prove the first statement of theorem 4.4 we show that there exists $\lambda_{T,2} =: \lambda \ge 0$ such that

$$\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T,H) \to H^{2,\lambda}(T,H)$$

is well-defined and a contraction in the second variable.

Step 1. We show that the mapping $\mathcal{F} : \mathcal{L}_0^2 \times \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is well-defined.

Let $\xi \in \mathcal{L}_0^2$ and $Y \in \mathcal{H}^2(T,H)$, predictable, then, by theorem D.3 (i), $(S(t)\xi)_{t\in[0,T]} \in \mathcal{H}^2(T,H)$, $1_{[0,t]}(\cdot)S(t-\cdot)F(Y(\cdot))$ is P-a.s. Bochner integrable on $[0,T]$ and the process

$$\Big(\int_0^t S(t-s)F(Y(s))\,ds\Big)_{t\in[0,T]}$$

has a version which is an element of $\mathcal{H}^2(T,H)$.
Therefore it remains to prove that
$(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$ for all $t \in [0,T]$ and that

$$\Big(\int_0^{t+}\int_U S(t-s)B(Y(s),y)\;q(ds,dy)\Big)_{t\in[0,T]}$$

is an element of $\mathcal{H}^2(T,H)$.

Claim 1. If $Y \in \mathcal{H}^2(T,H)$, predictable, then
$\Phi := \big(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot)\big)_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$ for all $t \in [0,T]$.

Let $t \in [0,T]$. First, we prove that the mapping

$$[0,T]\times\Omega\times U \to H,\; (s,\omega,y) \mapsto 1_{]0,t]}(s)S(t-s)B(Y(s,\omega),y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. By lemma 4.5 we have to check if the mapping $(s,\omega,y) \mapsto B(Y(s,\omega),y)$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable.
The mapping $G : [0,T]\times\Omega\times U \to H \times U$, $(s,\omega,y) \mapsto (Y(s,\omega),y)$ is $\mathcal{P}_T(U)/\mathcal{B}(H)\otimes\mathcal{B}$-measurable since for $A \in \mathcal{B}(H)$ and $C \in \mathcal{B}$ we have that

$$G^{-1}(A\times C) = \underbrace{Y^{-1}(A)}_{\in\mathcal{P}_T} \times C \in \mathcal{P}_T(U) \;\text{ by lemma 2.23 (ii).}$$

Moreover, B is $\mathcal{B}(H)\otimes\mathcal{B}/\mathcal{B}(H)$-measurable by assumption.

With respect to the norm $\| \|_T$ of Φ we obtain

$$\begin{aligned}\|\Phi\|_T^2 &= E\big[\int_0^T \int_U \|1_{]0,t]}(s)S(t-s)B(Y(s),y)\|^2\,\nu(dy)\,ds\big]\\ &\le E\big[\int_0^t K(t-s)(1+\|Y(s)\|)^2\,ds\big]\\ &\le \int_0^t K(t-s)2(1+E[\|Y(s)\|^2])\,ds\\ &\le 2(1+\|Y\|_{\mathcal{H}^2}^2)\int_0^T K(s)\,ds\\ &< \infty.\end{aligned}$$

Claim 2. If $Y \in \mathcal{H}^2(T,H)$, predictable, then there is a predictable version of

$$(Z(t))_{t\in[0,T]} := \big(\int_0^{t+}\int_U S(t-s)B(Y(s),y)\;q(ds,dy)\big)_{t\in[0,T]}$$

which is an element of $\mathcal{H}^2(T,H)$.

To prove the existence of a predictable version of Z we want to apply lemma 4.6. For this reason we will show that the process Z is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and continuous as a mapping from $[0,T]$ to $L^2(\Omega,\mathcal{F},P;H)$. Let $1<\alpha<2$ and define for $t\in[0,T]$

$$\begin{aligned}Z^\alpha(t) &:= \int_0^{(\frac{t}{\alpha})+}\int_U S(t-s)B(Y(s),y)\,q(ds,dy)\\ &= \int_0^{(\frac{t}{\alpha})+}\int_U S(t-\alpha s)S((\alpha-1)s)B(Y(s),y)\,q(ds,dy),\end{aligned}$$

where we used the semigroup property of $S(t)$, $t\ge 0$.
Set $\Phi^\alpha(s,\omega,y) := S((\alpha-1)s)B(Y(s,\omega),y)$ then one can show analogously to the proof of the $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurability of the mapping $(s,\omega,y)\mapsto 1_{]0,t]}(s)S(t-s)B(Y(s,\omega),y)$ that Φ^α is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. Moreover,

$$\begin{aligned}&E\big[\int_0^T\int_U \|\Phi^\alpha(s,y)\|^2\,\nu(dy)\,ds\big]\\ =\,&E\big[\int_0^T\int_U \|S((\alpha-1)s)B(Y(s),y)\|^2\,\nu(dy)\,ds\big]\\ \le\,&2(1+\|Y\|_{\mathcal{H}^2}^2)\int_0^T K((\alpha-1)s)\,ds\\ =\,&2(1+\|Y\|_{\mathcal{H}^2}^2)\frac{1}{\alpha-1}\int_0^{(\alpha-1)T} K(s)\,ds\\ <\,&\infty.\end{aligned}$$

Now we show that the mapping $Z^\alpha : [0,T] \to L^2(\Omega, \mathcal{F}, P; H)$ is continuous for all $\alpha > 1$. For this reason let $0 \le u \le t \le T$.

$$\begin{aligned}
&\Big(E\big[\| \int_0^{(\frac{t}{\alpha})+} \int_U S(t-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy) \\
&\qquad - \int_0^{(\frac{u}{\alpha})+} \int_U S(u-\alpha s)\Phi^\alpha(s,y)\ q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}}, \\
&= \Big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y) - 1_{]0,\frac{u}{\alpha}]}(s) S(u-\alpha s)\Phi^\alpha(s,y) \\
&\qquad\qquad q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&= \Big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{u}{\alpha}]}(s) (S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y) \\
&\qquad\qquad + 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&\le \Big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{u}{\alpha}]}(s) (S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\, q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&\quad + \Big(E\big[\| \int_0^{T+} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&= \Big(E\big[\int_0^{T} \int_U 1_{]0,\frac{u}{\alpha}]}(s) \|(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big]\Big)^{\frac{1}{2}} \\
&\quad + \Big(E\big[\int_0^{T} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) \|S(t-\alpha s)\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big]\Big)^{\frac{1}{2}}\ , \text{ by (2.5).}
\end{aligned}$$

The first summand converges to 0 as $u \uparrow t$ or $t \downarrow u$ by Lebesgue's dominated convergence theorem since the integrand converges pointwisely to 0 as $u \uparrow t$ or $t \downarrow u$ by the strong continuity of the semigroup and can be estimated independently of u and t by $4M_T^2\|\Phi^\alpha(s,\omega,y)\|^2$, $(s,\omega,y) \in [0,T] \times \Omega \times U$, where $E\big[\int_0^T \int_U \|\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big] < \infty$.
The second summand can be estimated by

$$\Big(E\big[\int_0^{T} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) M_T^2 \|\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big]\Big)^{\frac{1}{2}}$$

and therefore converges to 0 by Lebesgue's dominated convergence theorem as $u \uparrow t$ or $t \downarrow u$.
To obtain the continuity of $Z : [0,T] \to L^2(\Omega, \mathcal{F}, P; H)$ we prove the uniform convergence of Z^{α_n}, $n \in \mathbb{N}$, to Z in $L^2(\Omega, \mathcal{F}, P; H)$ for an arbitrary sequence α_n, $n \in \mathbb{N}$, with $\alpha_n \downarrow 1$ as $n \to \infty$.

$$
\begin{aligned}
&\|Z(t) - Z^{\alpha_n}(t)\|^2_{L^2(\Omega,\mathcal{F},P;H)}\\
=&E\big[\| \int_0^{t+}\int_U S(t-s)B(Y(s),y)\, q(ds,dy)\\
&\quad - \int_0^{(\frac{t}{\alpha_n})+}\int_U S(t-\alpha_n s)\Phi^{\alpha_n}(s,y)\, q(ds,dy)\|^2\big]\\
=&E\big[\| \int_0^{T+}\int_U 1_{]0,t]}(s)S(t-s)B(Y(s),y) - 1_{]0,\frac{t}{\alpha_n}]}(s)S(t-s)B(Y(s),y)\\
&\qquad q(ds,dy)\|^2\big]\\
=&E\big[\| \int_0^{T+}\int_U 1_{]\frac{t}{\alpha_n},t]}(s)S(t-s)B(Y(s),y)\, q(ds,dy)\|^2\big]\\
=&E\big[\int_{\frac{t}{\alpha_n}}^{t}\int_U \|S(t-s)B(Y(s),y)\|^2\,\nu(dy)\,ds\big]\\
\leq&2\big(1+\|Y\|^2_{\mathcal{H}^2}\big)\int_{\frac{t}{\alpha_n}}^{t} K(t-s)\,ds\\
\leq&2\big(1+\|Y\|^2_{\mathcal{H}^2}\big)\int_0^{\frac{\alpha_n-1}{\alpha_n}T} K(s)\,ds
\end{aligned}
$$

where $\int_0^{\frac{\alpha_n-1}{\alpha_n}T} K(s)\,ds \to 0$ as $n\to\infty$.
Moreover, we know for all $t\in[0,T]$ that

$$\Big(\int_0^{u+}\int_U 1_{]0,t]}(s)S(t-s)B(Y(s),y)\,q(ds,dy)\Big)_{u\in[0,T]} \in \mathcal{M}^2_T(H)$$

since $(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}^2_q(T,U,H)$. In particular, this means that the process

$$Z(t) = \int_0^{t+}\int_U S(t-s)B(Y(s),y)\,q(ds,dy),\ t\in[0,T],\ \text{is } (\mathcal{F}_t)\text{-adapted.}$$

Together with the continuity of $Z:[0,T]\to L^2(\Omega,\mathcal{F},P;H)$, by lemma 4.6, this implies the existence of a predictable version of $Z(t)$, $t\in[0,T]$, which we denote by

$$\Big(\int_0^{t-}\int_U 1_{]0,t]}(s)S(t-s)B(Y(s),y)\ q(ds,dy)\Big)_{t\in[0,T]}.$$

Altogether, we proved that

$$\bar{\mathcal{F}}: L^2_0\times H^2(T,H)\to H^2(T,H)$$

is well defined.

Step 2. We show that there exists $\lambda_{T,2} =: \lambda \geq 0$ such that for all $\xi \in L_0^2$

$$\bar{\mathcal{F}}(\xi, \cdot) : H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$$

is a contraction where the contraction constant does not depend on ξ.

Let $Y, \tilde{Y} \in \mathcal{H}^2(T, H)$, predictable, and $\xi \in \mathcal{L}_0^2$. Then we get for $\lambda \geq 0$ that

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\lambda t} \|\big(\mathcal{F}(\xi, Y) - \mathcal{F}(\xi, \tilde{Y})\big)(t)\|_{L^2} \\
\leq &\sup_{t\in[0,T]} e^{-\lambda t} \|\int_0^t S(t-s)[F(Y(s)) - F(\tilde{Y}(s))]\, ds\|_{L^2} \\
&+ \sup_{t\in[0,T]} e^{-\lambda t} \|\int_0^{t+} \int_U S(t-s)[B(Y(s), y) - B(\tilde{Y}(s), y)]\, q(ds, dy)\|_{L^2}.
\end{aligned}$$

By theorem D.3 (ii) the first summand can be estimated by

$$\underbrace{M_T C T^{\frac{1}{2}} \big(\frac{1}{2\lambda}\big)^{\frac{1}{2}}}_{\to 0 \text{ as } \lambda\to\infty} \|Y - \tilde{Y}\|_{2,\lambda,T}.$$

By the isometric formula (2.5) we get the following estimation for the second summand:

$$\begin{aligned}
&E\big[\| \int_0^{t+} \int_U S(t-s)\big(B(Y(s), y) - B(\tilde{Y}(s), y)\big)\, q(ds, dy)\|^2\big] \\
=&E\big[\int_0^t \int_U \|S(t-s)\big(B(Y(s), y) - B(\tilde{Y}(s), y)\big)\|^2\, \nu(dy)\, ds\big] \\
\leq&E\big[\int_0^t K(t-s)\|Y(s) - \tilde{Y}(s)\|^2\, ds\big] \\
=&E\big[\int_0^t e^{2\lambda s} K(t-s) e^{-2\lambda s}\|Y(s) - \tilde{Y}(s)\|^2\, ds\big] \\
\leq&\int_0^t e^{2\lambda s} K(t-s)\, ds \|Y - \tilde{Y}\|_{2,\lambda,T}^2 \\
\leq&e^{2\lambda t} \int_0^T e^{-2\lambda s} K(s)\, ds \|Y - \tilde{Y}\|_{2,\lambda,T}^2.
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\lambda t} \|\int_0^{t+} \int_U S(t-s)\big(B(Y(s), y) - B(\tilde{Y}(s), y)\big)\, q(ds, dy)\|_{L^2} \\
\leq&\underbrace{\Big(\int_0^T e^{-2\lambda s} K(s)\, ds\Big)^{\frac{1}{2}}}_{\to 0 \text{ as } \lambda\to\infty} \|Y - \tilde{Y}\|_{2,\lambda,T}.
\end{aligned}$$

Thus, we have finally proved that there exists $\lambda_{T,2} =: \lambda \geq 0$ such that there exists $L_{T,\lambda} < 1$ with

$$\|\bar{\mathcal{F}}(\xi, Y) - \bar{\mathcal{F}}(\xi, \tilde{Y})\|_{2,\lambda,T} \leq L_{T,\lambda} \|Y - \tilde{Y}\|_{2,\lambda,T}$$

for all $Y, \tilde{Y} \in H^{2,\lambda}(T, H)$ and $\xi \in L_0^2$. Hence the existence of a unique implicit function

$$\begin{aligned} X : L_0^2 &\to H^2(T, H) \\ \xi &\mapsto X(\xi) = \bar{\mathcal{F}}(\xi, X(\xi)) \end{aligned}$$

is verified.

Step 3. We show that the mapping $X : L_0^2 \to H^2(T, H)$ is Lipschitz continuous.

By theorem A.1 (ii) and the equivalence of the norms $\| \ \|_{2,\lambda,T}$, $\lambda \geq 0$, we only have to check that for all $Y \in H^2(T, H)$ the mapping

$$\bar{\mathcal{F}}(\cdot, Y) : L_0^2 \to H^2(T, H)$$

is Lipschitz continuous where the Lipschitz constant does not depend on Y. But this assertion is true as for all $\xi, \zeta \in \mathcal{L}_0^2$ and $Y \in \mathcal{H}^2(T, H)$, predictable,

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\zeta, Y)\|_{\mathcal{H}^2} = \|S(\cdot)(\xi - \zeta)\|_{\mathcal{H}^2} \leq M_T \|\xi - \zeta\|_{L^2}.$$

$\square$

Chapter 5

First Order Differentiability of the Mild Solution

The principal object of this chapter is the analysis of the first order differentiability of the mild solution with respect to the initial condition. We consider the mild solution as a mapping from L_0^2 to $H^2(T, H)$ and prove Gâteaux differentiability (see theorem 5.1). To this end we make the following assumptions.

Hypothesis H.1

- F is Gâteaux differentiable and
$$\partial F : H \times H \to H$$
is continuous.
- For all $y \in U$ $B(\cdot, y) : H \to H$ is Gâteaux differentiable and for all $y \in U$, $z \in H$ and $t \in]0, T]$
$$S(t)\partial_1 B(\cdot, y)z : H \to H$$
is continuous.
- For all $t \in]0, T]$ and $z \in H$ the mapping
$$\begin{aligned} S(t)\partial_1 B(\cdot, \cdot)z : H &\to L^2(U, \mathcal{B}, \nu; H) \\ x &\mapsto S(t)\partial_1 B(x, \cdot)z \end{aligned}$$
is continuous.

Theorem 5.1. *Assume that the coefficients A, F and B fulfill the conditions of hypothesis H.0 and H.1. Then the following statements hold.*

(i) *The mild solution of (4.1)*

$$X : L_0^2 \to H^2(T, H)$$
$$\xi \mapsto X(\xi)$$

is Gâteaux differentiable and the mapping

$$\partial X : L_0^2 \times L_0^2 \to H^2(T, H)$$

is continuous.

(ii) *For all $\bar{\xi}, \bar{\zeta} \in L_0^2$ the Gâteaux derivative of X fulfills the following equation*

$$\begin{aligned}\partial X(\bar{\xi})\bar{\zeta} = &\Big(S(t)\bar{\zeta} + \int_0^t S(t-s)\partial F(X(\bar{\xi})(s))\partial X(\bar{\xi})\bar{\zeta}(s)\, ds \\ &+ \int_0^{t+} \int_U S(t-s)\partial B(X(\bar{\xi})(s), y)\partial X(\bar{\xi})\bar{\zeta}(s)\, q(ds, dy)\Big)_{t\in[0,T]}\end{aligned}$$

in $\mathcal{H}^2(T, H)$ where the right-hand side is defined as the equivalence class of

$$\begin{aligned}&\Big(S(t)\zeta + \int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds \\ &+ \int_0^{t+} \int_U S(t-s)\partial B(Y(s), y)Z(s)\, q(ds, dy)\Big)_{t\in[0,T]}\end{aligned}$$

w.r.t. $\| \|_{\mathcal{H}^2}$ for arbitrary $\zeta \in \bar{\zeta}$ and arbitrary predictable $Y \in X(\bar{\xi})$, $Z \in \partial X(\bar{\xi})\bar{\zeta}$.

(iii) *In addition, the following estimate is true*

$$\|\partial X(\xi)\zeta\|_{\mathcal{H}^2} \leq K_{T,2}\|\zeta\|_{L^2}$$

for all $\xi, \zeta \in L_0^2$ where $K_{T,2}$ denotes the Lipschitz constant of the mapping $X : L_0^2 \to H^2(T, H)$.

For the proof of the above theorem we need the following lemmas.

Lemma 5.2. (i) *If F satisfies H.0 and H.1 we obtain that $\|\partial F(x)\|_{L(H)} \leq C$ for all $x \in H$.*

(ii) *If we assume that $B : H \times U \to H$ satisfies hypothesis H.0 and is Gâteaux differentiable in the first variable then we get for all $t \in]0, T]$ and $x \in H$ that $H \ni z \mapsto S(t)\partial_1 B(x, \cdot)z \in L(H, L^2(U, \mathcal{B}, \nu; H))$ with*

$$\|S(t)\partial_1 B(x, \cdot)\|_{L(H,L^2(U,\mathcal{B},\nu;H))} \leq \sqrt{K(t)}.$$

In particular, we obtain for all $t \in [0,T]$ and for all predictable $Y, Z \in \mathcal{H}^2(T,H)$ that the mapping

$$\begin{aligned} G_t : [0,T] \times \Omega \times U &\to H \\ (s,\omega,y) &\mapsto 1_{]0,t]}(s) S(t-s)\partial_1 B(Y(s,\omega),y) Z(s,\omega) \end{aligned}$$

is an element of $\mathcal{N}_q^2(T,U,H)$.

Proof. (ii) Let $x, z \in H$ and $t \in]0,T]$ then

$$\begin{aligned} &\int_U \|S(t)\partial_1 B(x,y)z\|^2 \, \nu(dy) \\ &= \int_U \liminf_{h\to 0} \frac{1}{h^2} \|S(t)B(x+hz,y) - S(t)B(x,y)\|^2 \, \nu(dy) \\ &\le \liminf_{h\to 0} \frac{1}{h^2} \int_U \|S(t)B(x+hz,y) - S(t)B(x,y)\|^2 \, \nu(dy) \\ &\le K(t)\|z\|^2. \end{aligned}$$

Since, by remark 2.23(ii), Y and Z as mappings from $[0,T] \times \Omega \times U$ to H are $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable and $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable, we get that $\partial_1 B(Y,\cdot)Z : [0,T] \times \Omega \times U \to H$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. Then, by lemma 4.5, the mapping G_t is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. Moreover,

$$\begin{aligned} E\Big[\int_0^T \int_U \|G_t(s,y)\|^2 \, \nu(dy)\, ds\Big] &\le E\Big[\int_0^t K(t-s)\|Z(s)\|^2 \, ds\Big] \\ &\le \int_0^T K(s)\, ds \|Z\|^2_{\mathcal{H}^2} < \infty. \end{aligned}$$

□

Lemma 5.3. *Assume that the mapping B satisfies the conditions of H.0 and H.1. Then for all $t \in]0,T]$ and $x, z \in H$*

$$\begin{aligned} &\Big\|\frac{1}{h}\big(S(t)B(x+hz,\cdot) - S(t)B(x,\cdot)\big) - S(t)\partial_1 B(x,\cdot)z\Big\|^2_{L^2(U,\mathcal{B},\nu;H)} \\ &\le \frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,\cdot)z - S(t)\partial_1 B(x,\cdot)z\|^2_{L^2(U,\mathcal{B},\nu;H)} \, ds \end{aligned}$$

and therefore, in particular, one has that for all $t \in]0,T]$

$$\frac{S(t)B(x+hz,\cdot) - S(t)B(x,\cdot)}{h} \underset{h\to 0}{\longrightarrow} S(t)\partial_1 B(x,\cdot)z$$

in $L^2(U,\mathcal{B},\nu;H)$.

Proof. Let $t \in]0,T]$. Since $S(t)\partial_1 B(\cdot,y)z : H \to H$ is continuous we obtain by the fundamental theorem for Bochner integrals B.8 that

$$\begin{aligned}
&\int_U \|\frac{1}{h}\big(S(t)B(x+hz,y) - S(t)B(x,y)\big) - S(t)\partial_1 B(x,y)z\|^2 \,\nu(dy)\\
&= \int_U \|\frac{1}{h}\int_0^h S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z \,ds\|^2 \,\nu(dy)\\
&\leq \int_U \frac{1}{h^2}\Big(\int_0^h \|S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z\| \,ds\Big)^2 \,\nu(dy)\\
&\leq \int_U \frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z\|^2 \,ds\,\nu(dy)\\
&= \frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,\cdot)z - S(t)\partial_1 B(x,\cdot)z\|^2_{L^2(U,\mathcal{B},\nu;H)} \,ds.
\end{aligned}$$

Since

$$\begin{aligned}
S(t)\partial_1 B(x+\cdot z,\cdot)z : [0,1] &\to L^2(U,\mathcal{B},\nu;H)\\
s &\mapsto S(t)\partial_1 B(x+sz,\cdot)z
\end{aligned}$$

is uniformly continuous by hypothesis H.1 the second part of the assertion follows. □

Lemma 5.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let (E, d) be a polish space.*
Moreover, let $Y, Y_n, n \in \mathbb{N}$, be E-valued random variables on $(\Omega, \mathcal{F}, \mu)$ such that

$$Y_n \longrightarrow Y \quad \text{in measure as } n \to \infty.$$

Let $(\tilde{E}, \tilde{d})$ be an arbitrary metric space and $f : (E, d) \to (\tilde{E}, \tilde{d})$ a continuous mapping. Then

$$f \circ Y_n \longrightarrow f \circ Y \quad \text{in measure as } n \to \infty.$$

Proof. [FrKn 02, Lemma 4.6, p.95] □

Proof of theorem 5.1:
In order to prove the stated differentiability of the mild solution X we apply theorem A.6 (i) to the spaces $\Lambda = L_0^2$ and $E = H^{2,\lambda}(T, H)$ and to the mapping $G = \bar{\mathcal{F}}$, where $\lambda \geq 0$ is such that $\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$ is a contraction in the second variable. In this way we obtain that $X : L_0^2 \to H^{2,\lambda}(T, H)$ is Gâteaux differentiable. By the equivalence of the norms $\| \, \|_{2,\lambda,T}$, $\lambda \geq 0$, we then also get the Gâteaux differentiability of X as a mapping from L_0^2 to $H^2(T, H)$.
For simplicity, we check that $\bar{\mathcal{F}} : L_0^2 \times H^2(T, H) \to H^2(T, H)$ fulfills the conditions of theorem A.6 which implies, again by the equivalence of the norms $\| \, \|_{2,\lambda,T}$, $\lambda \geq 0$, that $\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$ satisfies them, too.

Proof of (i):

Step 1. We show the existence of the directional derivatives of $\bar{\mathcal{F}}$. For this purpose let $\bar{\xi}, \bar{\zeta} \in L_0^2$ and $\bar{Y}, \bar{Z} \in H^2(T, H)$. We show that there exist the directional derivatives $\partial_1 \mathcal{F}(\xi, Y; \zeta)$ and $\partial_2 \mathcal{F}(\xi, Y; Z)$ in $\mathcal{H}^2(T, H)$ for $\xi \in \bar{\xi}$, $\zeta \in \bar{\zeta}$, $Y \in \bar{Y}$ and $Z \in \bar{Z}$, where Y and Z are predictable. Then there exist the directional derivatives of $\bar{\mathcal{F}}$ as the respective equivalence classes w.r.t. $\| \, \|_{\mathcal{H}^2}$.

(a) It is obvious that $\partial_1 \mathcal{F}(\xi, Y; \zeta) = S(\cdot)\zeta \in \mathcal{H}^2(T, H)$.
(b) The integrals

$$\int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds, \ t \in [0, T], \text{ and}$$

$$\int_0^{t+} \int_U 1_{]0,t]}(s) S(t-s)\partial_1 B(Y(s), y) Z(s)\, q(ds, dy), \ t \in [0, T],$$

are well defined by H.0, H.1 theorem D.4 (i) and lemma 5.2 (ii). In the following we show that

$$\partial_2\mathcal{F}(\xi,Y;Z) = \Big(\int_0^t S(t-s)\partial F(Y(s))Z(s)\,ds + \int_0^{t+}\int_U S(t-s)\partial_1 B(Y(s),y)Z(s)\,q(ds,dy)\Big)_{t\in[0,T]} \in \mathcal{H}^2(T,H)$$

Let $t\in[0,T]$ and $h\neq 0$. Then we get that

$$\begin{aligned}
&\Big\|\frac{\mathcal{F}(\xi,Y+hZ)(t)-\mathcal{F}(\xi,Y)(t)}{h} - \int_0^t S(t-s)\partial F(Y(s))Z(s)\,ds \\
&- \int_0^{t+}\int_U S(t-s)\partial_1 B(Y(s),y)Z(s)\,q(ds,dy)\Big\|_{L^2(\Omega,\mathcal{F},P;H)} \\
\le\ & \Big\|\int_0^t S(t-s)\big(\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\big)\,ds\Big\|_{L^2} \\
&+ \Big\|\int_0^{t+}\int_U S(t-s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad - \partial_1 B(Y(s),y)Z(s)\big)\,q(ds,dy)\Big\|_{L^2}
\end{aligned}$$

The first summand can be estimated independently of $t\in[0,T]$ by

$$M_T T^{\frac{1}{2}} E\Big[\int_0^T \Big\|\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\Big\|^2\,ds\Big]^{\frac{1}{2}}$$

and converges to 0 as $h\to 0$ by Lebesgue's dominated convergence theorem (see theorem D.4 (ii)).
To get the convergence to 0 of the second summand as $h\to 0$ we first fix $\alpha>1$ and get by the isometric formula (2.5)

$$\begin{aligned}
&\Big(E\Big[\Big\|\int_0^{t+}\int_U S(t-s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad - \partial_1 B(Y(s),y)Z(s)\big)\,q(ds,dy)\Big\|^2\Big]\Big)^{\frac{1}{2}} \\
=\ & \Big(E\Big[\int_0^{\frac{t}{\alpha}}\int_U \Big\|S(t-\alpha s)S((\alpha-1)s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad - \partial_1 B(Y(s),y)Z(s)\big)\Big\|^2\,\nu(dy)\,ds\Big] \\
&+E\Big[\int_{\frac{t}{\alpha}}^t\int_U \Big\|S(t-s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad - \partial_1 B(Y(s),y)Z(s)\big)\Big\|^2\,\nu(dy)\,ds\Big]\Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the semigroup property of $S(t)$, $t \geq 0$.
The first integral can be estimated by

$$M_T^2 E\Big[\int_0^T \int_U \|S((\alpha-1)s)\Big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} - \partial_1 B(Y(s),y)Z(s)\Big)\|^2 \,\nu(dy)\, ds\Big].$$

If we fix $s \in]0,T]$ we know by lemma 5.3 that

$$\begin{aligned}&\|\frac{1}{h}\big(S((\alpha-1)s)B(Y(s)+hZ(s),\cdot) - S((\alpha-1)s)B(Y(s),\cdot)\big)\\ &- S((\alpha-1)s)\partial_1 B(Y(s),\cdot)Z(s)\|^2_{L^2(U,\mathcal{B},\nu;H)}\\ &\to 0 \quad \text{as } h \to 0.\end{aligned}$$

Since, by lemma 5.2 (ii), the above sequence can be estimated by the mapping

$$[0,T]\times\Omega \to \mathbb{R}, (s,\omega) \mapsto 4K((\alpha-1)s)\|Z(s,\omega)\|^2,$$

which is an element of $L^1([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}, \lambda\otimes P)$, we get by Lebesgue's dominated convergence theorem that

$$\begin{aligned}&M_T^2 E\Big[\int_0^T \int_U \|S((\alpha-1)s)\Big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} - \partial_1 B(Y(s),y)Z(s)\Big)\|^2 \,\nu(dy)\, ds\Big].\\ &\to 0 \quad \text{as } h \to 0.\end{aligned}$$

Again by lemma 5.2 (ii), the second integral can be estimated independently of $h \neq 0$ and $t \in [0,T]$ in the following way

$$\begin{aligned}&E\Big[\int_{\frac{t}{\alpha}}^t \int_U \|S(t-s)\Big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} - \partial_1 B(Y(s),y)Z(s)\Big)\|^2 \,\nu(dy)\, ds\Big]\\ &\leq \int_{\frac{t}{\alpha}}^t 4K(t-s)E[\|Z(s)\|^2]\, ds\\ &\leq 4\int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\, ds \|Z\|^2_{\mathcal{H}^2}\end{aligned}$$

where $\|Z\|_{\mathcal{H}^2} < \infty$ and $\int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\, ds \to 0$ as $\alpha \downarrow 1$ since $K \in L^1([0,T])$.
Altogether, we have an estimation of the second summand which is independent of $t \in [0,T]$ and we get the desired convergence in $\mathcal{H}^2(T,H)$:

$$\begin{aligned}
&\sup_{t\in[0,T]} \| \int_0^{t+} \int_U S(t-s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\ q(ds,dy)\|_{L^2(\Omega,\mathcal{F},P;H)} \\
&\le \Big(M_T^2 E\Big[\int_0^T \int_U \|S((\alpha-1)s)\big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\
&\qquad\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\|^2\ \nu(dy)\ ds\Big] \\
&\quad + 4 \int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\ ds \|Z\|_{\mathcal{H}^2}^2 \Big)^{\frac{1}{2}}
\end{aligned}$$

where the right hand side tends to zero if $\alpha \downarrow 1$ and $h \to 0$.

Step 2. We show that the directional derivatives

$$\begin{aligned}
\partial_1 \bar{\mathcal{F}} : L_0^2 \times H^2(T,H) \times L_0^2 \to H^2(T,H) \\
\partial_2 \bar{\mathcal{F}} : L_0^2 \times H^2(T,H) \times H^2(T,H) \to H^2(T,H)
\end{aligned}$$

are continuous.

(a) The continuity of $\partial_1 \bar{\mathcal{F}}$ is obvious.
(b) To analyze the continuity of $\partial_2 \bar{\mathcal{F}}$ let $Y, Y_n, Z, Z_n \in \mathcal{H}^2(T,H)$, $n \in \mathbb{N}$, and $\xi, \xi_n \in \mathcal{L}_0^2$, $n \in \mathbb{N}$, such that $Y_n \to Y$ and $Z_n \to Z$ in $\mathcal{H}^2(T,H)$ and $\xi_n \to \xi$ in L_0^2 as $n \to \infty$. Then we have for all $t \in [0,T]$ that

$$\begin{aligned}
&\|\partial_2 \mathcal{F}(\xi_n, Y_n; Z_n) - \partial_2 \mathcal{F}(\xi, Y; Z)\|_{\mathcal{H}^2} \\
&\le \sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\ ds\|_{L^2} \\
&\quad + \sup_{t\in[0,T]} \| \int_0^{t+} \int_U S(t-s)\big(\partial_1 B(Y_n(s),y)Z_n(s) \\
&\qquad\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\ q(ds,dy)\|_{L^2}.
\end{aligned}$$

The first summand converges to 0 as $n \to \infty$ (see theorem D.4 (iii)).
In order to estimate the second summand we fix $\alpha > 1$ and use the isometric formula (2.5) to get that

$$\begin{aligned}
&\| \int_0^{t+} \int_U S(t-s)\big(\partial_1 B(Y_n(s),y)Z_n(s) - \partial_1 B(Y(s),y)Z(s)\big)\ q(ds,dy)\|_{L^2} \\
&= \big(E\big[\int_0^t \int_U \|S(t-s)(\partial_1 B(Y_n(s),y)Z_n(s) - \partial_1 B(Y(s),y)Z(s))\|^2 \\
&\qquad\qquad \nu(dy)\ ds\big]\big)^{\frac{1}{2}} \\
&\le \big(E\big[\int_0^t \int_U \|S(t-s)\partial_1 B(Y_n(s),y)(Z_n(s)-Z(s))\|^2\ \nu(dy)\ ds\big]\big)^{\frac{1}{2}} \\
&\quad + \big(E\big[\int_0^t \int_U \|S(t-s)(\partial_1 B(Y_n(s),y) - \partial_1 B(Y(s),y))Z(s)\|^2\ \nu(dy)\ ds\big]\big)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(E\Big[\int_0^t K(t-s)\|Z_n(s)-Z(s)\|^2\,ds\Big]\right)^{\frac{1}{2}}\ \text{, by lemma 5.2(ii),}\\
&\quad+\Bigg(E\Big[\int_0^{\frac{t}{\alpha}}\int_U \|S(t-s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\,\nu(dy)\,ds\Big]\\
&\quad+E\Big[\int_{\frac{t}{\alpha}}^t\int_U \|S(t-s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\,\nu(dy)\,ds\Big]\Bigg)^{\frac{1}{2}}\\
&\leq \Big(\int_0^t K(s)\,ds\Big)^{\frac{1}{2}}\|Z_n-Z\|_{\mathcal{H}^2}\\
&\quad+\Bigg(M_T^2 E\Big[\int_0^{\frac{t}{\alpha}}\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\\
&\qquad\qquad\nu(dy)\,ds\Big]\\
&\quad+E\Big[\int_{\frac{t}{\alpha}}^t 4K(t-s)\|Z(s)\|^2\,ds\Big]\Bigg)^{\frac{1}{2}}\\
&\leq \Big(\int_0^T K(s)\,ds\Big)^{\frac{1}{2}}\|Z_n-Z\|_{\mathcal{H}^2}\\
&\quad+\Bigg(M_T^2 E\Big[\int_0^{T}\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\\
&\qquad\qquad\nu(dy)\,ds\Big]\\
&\quad+4\int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\,ds\|Z\|_{\mathcal{H}^2}^2\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

$\|Z_n-Z\|_{\mathcal{H}^2}\to 0$ as $n\to\infty$ by assumption and $\int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\,ds\to 0$ as $\alpha\downarrow 1$ by Lebesgue's theorem since $K\in L^1([0,T])$.
To show the convergence of the third term to 0 as $n\to\infty$ we use lemma 5.4.
For fixed $s\in]0,T]$ the sequence of random variables $(Y_n(s),Z(s))$, $n\in\mathbb{N}$, converges in probability to $(Y(s),Z(s))$. Moreover, the mapping

$$
\begin{aligned}
f: H\times H &\to L^2(U,\mathcal{B},\nu;H)\\
(x,z) &\mapsto S((\alpha-1)s)\partial_1 B(x,\cdot)z
\end{aligned}
$$

is continuous. Hence, by lemma 5.4 it follows that

$$
\|S((\alpha-1)s)(\partial_1 B(Y_n(s),\cdot)-\partial_1 B(Y(s),\cdot))Z(s)\|^2_{L^2(U,\mathcal{B},\nu;H)}\underset{n\to\infty}{\longrightarrow}0
$$

in probability. In addition, this sequence is bounded by $4K((\alpha-1)s)\|Z(s)\|^2\in L^1(\Omega,\mathcal{F},P)$ which implies the uniform integrability. Therefore we get that

$$
E\big[\|S((\alpha-1)s)(\partial_1 B(Y_n(s),\cdot)-\partial_1 B(Y(s),\cdot))Z(s)\|^2_{L^2(U,\mathcal{B},\nu;H)}\big]\underset{n\to\infty}{\longrightarrow}0.
$$

Since the above expectation is bounded by $4K((\alpha-1)s)\|Z\|^2_{\mathcal{H}^2}$ where $4K((\alpha-1)\cdot)\|Z\|^2_{\mathcal{H}^2} \in L^1([0,T])$ we finally obtain that

$$\int_0^T E\big[\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y) - \partial_1 B(Y(s),y))Z(s)\|^2\,\nu(dy)\big]\,ds$$
$$\underset{n\to\infty}{\longrightarrow} 0.$$

Proof of (ii): Let $\bar{\xi}, \bar{\zeta} \in L^2_0$. Then by theorem A.6 (i) we have the following representation of the Gâteaux derivative of X:

$$\partial X(\bar{\xi})\bar{\zeta} = \big[I - \partial_2 \bar{\mathcal{F}}(\bar{\xi}, X(\bar{\xi}))\big]^{-1} \partial_1 \bar{\mathcal{F}}(\bar{\xi}, X(\bar{\xi}))\bar{\zeta}$$

and therefore we have that

$$\partial X(\bar{\xi})\bar{\zeta} = \partial_1 \bar{\mathcal{F}}(\bar{\xi}, X(\bar{\xi}))\bar{\zeta} + \partial_2 \mathcal{F}(\bar{\xi}, X(\bar{\xi}))\partial X(\bar{\xi})\bar{\zeta}.$$

By (i) the assertion follows.
Proof of (iii): By theorem 4.4 the mild solution $X : L^2_0 \to H^2(T,H)$ is Lipschitz continuous. We denote the Lipschitz constant of X by $K_{T,2}$. Hence, we get that

$$\|\partial X(\xi)\zeta\|_{\mathcal{H}^2} \le K_{T,2}\|\zeta\|_{L^2} \quad \text{for all } \xi, \zeta \in L^2_0$$

□

Chapter 6

Gradient Estimates for the Resolvent Corresponding with the Mild Solution

As in the previous chapters let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space, $(U, \mathcal{B}, \nu)$ a σ-finite measure space and $(\Omega, \mathcal{F}, P)$ a complete probability space with right-continuous filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$. Moreover, let p be a stationary $(\mathcal{F}_t)$-Poisson point process on U with characteristic measure ν. We denote as in the previous chapters with q the compensated Poisson random measure of p.

In the first part of this chapter we make the following assumptions on the coefficients A, F and B.

Hypothesis H.2

- $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, on H, i.e. there exists $\omega_0 \geq 0$ such that $\|S(t)\|_{L(H)} \leq e^{\omega_0 t}$ for all $t \geq 0$.
- F is Lipschitz continuous and Gâteaux differentiable such that
$$\partial F : H \times H \to H$$
is continuous.
- F is dissipativ, i.e. $\langle \partial F(x)y, y\rangle \leq 0$ for all $x, y \in H$.
- $B : H \times U \to H$ such that
  - for all $y \in U$ $B(\cdot, y) : H \to H$ is constant,

- there exists an integrable mapping $K : [0,T] \to [0,\infty[$ such that for all $t \in]0,T]$ and $x \in H$ holds

$$\int_U \|S(t)B(x,y)\|^2 \,\nu(dy) \leq K(t)\big(1+\|x\|\big)^2.$$

It is easy to check that, on condition that the assumptions of hypothesis H.2 are fulfilled,the coefficients A, F and B satisfy H.0 and H.1.
Under the assumptions of hypothesis H.2 we already proved in theorem 4.4 the existence of a mild solution of the following stochastic differential equation

$$\begin{cases} dX(t) &= [AX(t)+F(X(t))]\,dt + B(X(t),y)\,q(dt,dy) \\ X(0) &= x \in H. \end{cases} \tag{6.1}$$

Moreover, the mild solution $X : H \to H^2(T,H)$ is Gâteaux differentiable by theorem 5.1(i).

Notation: In the following we denote by $X(x)$ and $\partial X(x)h$ predictable representatives in $\mathcal{H}^2(T,H)$ of the respective equivalence classes in $H^2(T,H)$.

The Gâteaux derivative $\partial X(x)h$ of X in $x \in H$ in direction $h \in H$ fulfills the following equation:

$$\partial X(x)h(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)\partial X(x)h(s)\,ds \quad P\text{-a.s.}$$

for all $t \in [0,T]$ (see theorem 5.1(ii)).

Proposition 6.1. *There exists a continuous version* $Y \in \mathcal{H}^2(T,H)$ *of* $\partial X(x)h$, $x,h \in H$, *such that*

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \textit{ for all } t \in [0,T]$$

P-a.s.

Proof. Let $h \in H$ and $Y \in \mathcal{H}^2(T,H)$. Then Y has at least one predictable version which we denote again by Y. Define

$$\mathcal{G}(h,Y) := \Big(S(t)h + \int_0^t S(t-s)\partial F(X(x)(s))Y(s)\,ds\Big)_{t\in[0,T]}. \tag{6.2}$$

Then the appearing integral is well defined and an element of $\mathcal{H}^2(T,H)$. Moreover, $\mathcal{G}$ is well defined in the sense of version, i.e. taking another predictable version $\tilde{Y}$ of Y, then $\mathcal{G}(h,Y)$ is a version of $\mathcal{G}(h,\tilde{Y})$.

Define for $h \in H$ and $Y \in H^2(T,H)$, $\bar{\mathcal{G}}(h,Y)$ as the equivalence class of $\mathcal{G}(h,Z)$ w.r.t. $\| \, \|_{\mathcal{H}^2}$ for an arbitrary predictable representative $Z \in Y$. By the above considerations, in $\mathcal{H}^2(T,H)$, $\mathcal{G}(h,Z)$ is independent of the representative Z, i.e. $\bar{\mathcal{G}}$ is well defined. Moreover, there exists $\lambda_T > 0$ such that $\bar{\mathcal{G}} : H \times H^2_{\lambda_T}(T,H) \to H^2_{\lambda_T}(T,H)$ is a contraction in the second variable. By Banach's fixed point theorem we get the existence and uniqueness of an equivalence class $\bar{Z} \in \mathcal{H}^2_{\lambda_T}(T,H)$ such that for all $Y \in \bar{Z}$

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \quad P\text{-a.s.}$$

for all $t \in [0,T]$. In particular, $\partial X(x)h \in \bar{Z}$.
Define now

$$Y(t) := S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)\partial X(x)h(s)\,ds,\ t \in [0,T].$$

Obviously, Y is a version of $\partial X(x)h$ and by the previous considerations we know that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds\ P\text{-a.s.}$$

for all $t \in [0,T]$.
Moreover, both Y and the process $\Big(S(t)h+\int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds\Big)_{t\in[0,T]}$ are continuous. To show this let $Z \in \mathcal{H}^2(T,H)$.
Since

$$E[\int_0^T \|Z(s)\|\,ds] \le T\|Z\|_{\mathcal{H}^2} < \infty$$

we get that

$$\int_0^t \|Z(s)\|\,ds < \infty \text{ for all } t \in [0,T]\ P\text{-a.s.}$$

Let now $u,t \in [0,T]$ with $u \le t$ then

$$\begin{aligned}
&\|S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Z(s)\,ds - S(u)h \\
&\qquad - \int_0^u S(u-s)\partial F\big(X(x)(s)\big)Z(s)\,ds\| \\
&\le \|S(t)h - S(u)h\| \\
&\quad + \|\int_0^u \big(S(t-s) - S(s-u)\big)\partial F(X(x)(s))Z(s)\,ds\| \\
&\quad + \|\int_u^t S(t-s)\partial F(X(x)(s))Z(s)\,ds\|.
\end{aligned}$$

The first summand converges to 0 as $u \uparrow t$ or $t \downarrow u$ by the strong continuity of the semigroup.
As $\|Z(\cdot)\| \in L^1([0,T])$ P-a.s. the second and third summand converge to 0 as $u \uparrow t$ or $t \downarrow u$ by Lebesgue's dominated convergence theorem where the P-nullset does not depend on t and u.
Thus, we proved the existence of a continuous version Y of $\partial X(x)h$ such that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \text{ } P\text{-a.s.}$$

for all $t \in [0,T]$ where by the above considerations also the right-hand side is continuous. By the continuity of both sides we get that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds$$

for all $t \in [0,T]$ P-a.s. □

In the following we have to distinguish between the case $A \in L(H)$ and the case of an arbitrary, possibly unbounded generator $(A, D(A))$.

6.1 First Case: $A \in L(H)$

Proposition 6.2. *Let $Y \in \mathcal{H}^2(T,H)$ be a continuous version of $\partial X(x)h$ such that*

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \textit{ for all } t \in [0,T]$$

P-a.s. Then

$$Y(t) = h + \int_0^t AY(s)\,ds + \int_0^t \partial F(X(x)(s))Y(s)\,ds \textit{ for all } t \in [0,T]$$

P-a.s.

Proof. Since

$$E\Big[\int_0^T \|Y(s)\|\,ds\Big] \leq T\|Y\|_{\mathcal{H}^2} < \infty$$

we get that

$$\int_0^t \|Y(s)\|\,ds < \infty \text{ for all } t \in [0,T] \text{ } P\text{-a.s.}$$

and therefore we have that P-a.s.

$$S(t-\cdot)\partial F\big(X(x)(\cdot)\big)Y(\cdot) \in L^1([0,t]) \text{ for all } t \in [0,T]. \tag{6.3}$$

Then we obtain that P-a.s. for all $t \in [0,T]$ that

$$\begin{aligned}
&\int_0^t AY(s)\,ds \\
&= \int_0^t AS(s)h\,ds + \int_0^t A\Big(\int_0^s S(s-u)\partial F\big(X(x)(u)\big)Y(u)\,du\Big)\,ds \\
&= \int_0^t AS(s)h\,ds + \int_0^t \int_0^s AS(s-u)\partial F\big(X(x)(u)\big)Y(u)\,du\,ds, \\
&\quad \text{by proposition B.7, the fact that } A \in L(H) \text{ and (6.3),} \\
&= \int_0^t \frac{d}{ds}S(s)h\,ds + \int_0^t \int_u^t \frac{d}{ds}S(s-u)\partial F\big(X(x)(u)\big)Y(u)\,ds\,du, \\
&\quad \text{by proposition C.1,} \\
&= S(t)h - h + \int_0^t S(t-u)\partial F\big(X(x)(u)\big)Y(u)\,du \\
&\quad - \int_0^t \partial F\big(X(x)(u)\big)Y(u)\,du, \text{ by proposition B.10,} \\
&= Y(t) - h - \int_0^t \partial F\big(X(x)(u)\big)Y(u)\,du.
\end{aligned}$$

Finally, we get that

$$Y(t) = h + \int_0^t AY(s)\,ds + \int_0^t \partial F(X(x)(s))Y(s)\,ds \text{ for all } t \in [0,T]$$

P-a.s. □

Let now $Y \in \mathcal{H}^2(T,H)$ be a version of $\partial X(x)h$ such that there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and $t \in [0,T]$

(i) $Y(\cdot,\omega)$ is continuous and $Y(0,\omega) = h$

(ii) $\int_0^t \|Y(s,\omega)\|\,ds < \infty$ and

(iii) $Y(t,\omega) = h + \int_0^t AY(s,\omega)\,ds + \int_0^t \partial F(X(x)(s,\omega))Y(s,\omega)\,ds \quad (6.4)$

Then, using proposition B.10 and differentiating both sides of (6.4) we obtain that for all $\omega \in N^c$:

$$Y'(t,\omega) = AY(t,\omega) + \partial F(X(x)(t,\omega))Y(t,\omega) \text{ for } \lambda\text{-a.e. } t \in [0,T]$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|Y(t,\omega)\|^2 = \langle Y'(t,\omega), Y(t,\omega)\rangle \tag{6.5}$$
$$= \langle AY(t,\omega) + \partial F(X(x)(t,\omega))Y(t,\omega), Y(t,\omega)\rangle$$
$$\text{for } \lambda\text{-a.e. } t \in [0,T]. \tag{6.6}$$

Proposition 6.3. *For all $\omega \in N^c$ and $t \in [0,T]$*

$$\|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 = \int_0^t \frac{d}{ds}\|Y(s,\omega)\|^2\, ds.$$

Proof. Let $\omega \in N^c$ and $t \in [0,T]$. By proposition B.12 we have the show that the mapping $f : [0,t] \to \mathbb{R}, s \mapsto \|Y(s,\omega)\|^2$ is absolutely continuous.
As first step we prove that $g : [0,t] \to \mathbb{R}, s \mapsto \|Y(s,\omega)\|$ is absolutely continuous, i.e. we show that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n |g(t_i) - g(s_i)| < \varepsilon$ whenever $\sum_{i=1}^n |t_i - s_i| < \delta$ for any finite set of disjoint intervals such that $]s_i, t_i[\subset [0,t]$ for each i.
Let $\varepsilon > 0$. For any set of disjoint intervals such that $]s_i, t_i[\subset [0,t]$ for each i we have

$$\begin{aligned}
\sum_{i=1}^n |g(t_i) - g(s_i)| &= \sum_{i=1}^n |\|Y(t_i,\omega)\| - \|Y(s_i,\omega)\||\\
&\le \sum_{i=1}^n \|Y(t_i,\omega) - Y(s_i,\omega)\|\\
&\le \sum_{i=1}^n \int_{s_i}^{t_i} \|AY(s,\omega) + \partial F(X(x)(s,\omega))Y(s,\omega)\|\, ds\\
&= \int_{\bigcup_{i=1}^n]s_i,t_i[} \|AY(s,\omega) + \partial F(X(x)(s,\omega))Y(s,\omega)\|\, ds.
\end{aligned}$$

Since $\|AY(\cdot,\omega) + \partial F(X(x)(\cdot,\omega))Y(\cdot,\omega)\| \in L^1([0,T], d\lambda)$ there exists $\delta > 0$ such that

$$\int_{\bigcup_{i=1}^n]s_i,t_i[} \|AY(\omega,s) + \partial F(X(x)(\omega,s))Y(\omega,s)\|\, ds < \varepsilon$$

provided $\sum_{i=1}^n |t_i - s_i| = \lambda(\bigcup_{i=1}^n]s_i, t_i[) < \delta$.
Now we use the fact that the product of two functions which are absolutely continuous on a finite interval $[a,b]$ is again absolutely continuous (see [deBa 81, 9.3 Example 7, p.161]) and obtain that
$\|Y(\cdot,\omega)\|^2 = \|Y(\cdot,\omega)\|\|Y(\cdot,\omega)\|$ is absolutely continuous on $[0,t]$. Now, the assertion follows by proposition B.12. □

Integrating both sides of equation (6.5), using the previous proposition and taking into account the dissipativity of F we obtain for all $\omega \in N^c$ and

$t \in [0,T]$ that

$$\|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 = \int_0^t \frac{d}{ds}\|Y(s,\omega)\|^2\, ds$$
$$= 2\int_0^t \langle AY(s,\omega) + \partial F(X(x)(s,\omega))Y(s,\omega), Y(s,\omega)\rangle\, ds$$
$$\le 2\int_0^t \langle AY(s,\omega), Y(s,\omega)\rangle\, ds.$$

Since A is the generator of the quasi-contractive C_0-semigroup $S(t)$, $t \ge 0$, we get by the following calculation that $\langle Ax, x\rangle \le \omega_0\|x\|^2$ for all $x \in H$:

$$\langle Ax, x\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle S(t)x - x, x\rangle \le \lim_{t\downarrow 0}\frac{1}{t}(\|S(t)x\|\|x\| - \|x\|^2)$$
$$\le \lim_{t\downarrow 0}\frac{1}{t}(e^{\omega_0 t} - 1)\|x\|^2 = (\frac{d}{dt}e^{\omega_0 t})_{|\,t=0}\,\|x\|^2 = \omega_0\|x\|^2.$$

Consequently,

$$\|Y(t,\omega)\|^2 - \|h\|^2 = \|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 \le 2\int_0^t \omega_0\|Y(s,\omega)\|^2\, ds.$$

Using Gronwall's lemma (see [HaTh 94, Lemma 6.12]) we can conclude that $\|Y(t)\|^2 \le e^{2\omega_0 t}\|h\|^2$ for all $t \in [0,T]$ P-a.s. Since Y is a version of $\partial X(x)h$, finally, we have an exponentially estimation for $\|\partial X(x)h(t)\|$, $t \in [0,T]$:

$$\|\partial X(x)h(t)\| \le e^{\omega_0 t}\|h\| \quad P\text{-a.s. for all } t \in [0,T].$$

6.2 Second case: $(A, D(A))$ is a (possibly) unbounded operator

In this section we need stronger assumptions on the measure ν and the coefficient B.
For the second part of this chapter we make the following assumptions on the coefficients A, F and B and the measure ν.

Hypothesis H.2'

- $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \ge 0$, on H, i.e. there exists $\omega_0 \ge 0$ such that $\|S(t)\|_{L(H)} \le e^{\omega_0 t}$ for all $t \ge 0$.
- F is Lipschitz continuous and Gâteaux differentiable such that

  $$\partial F : H \times H \to H$$

  is continuous.

- F is dissipativ, i.e. $\langle \partial F(x)y, y\rangle \leq 0$ for all $x, y \in H$.
- $\nu(U) < \infty$.
- $B : H \times U \to H$, $(x, y) \mapsto z$ is constant.

If ν and B satisfy hypothesis H.2' then we obtain for every C_0-semigroup $T(t)$, $t \geq 0$, on H that

$$\int_U \|T(t)B(x,y)\|^2 \, \nu(dy) \leq \sup_{t\in[0,T]} \|T(t)\|^2_{L(H)} \|z\|^2 \nu(U)(1 + \|x\|)^2$$

for all $t \in [0, T]$ and $x \in H$, i.e $T(t)B$, $t \in [0, T]$, satisfies hypothesis H.2.

Since $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, there is a constant $\omega_0 \geq 0$ such that $\|S(t)\|_{L(H)} \leq e^{\omega_0 t}$ for all $t \geq 0$. By C.3 A can be approximated by the Yosida-approximation A_n, $n \in \mathbb{N}$, $n > \omega_0$. Each A_n, $n > \omega_0$, is an element of $L(H)$ and, by proposition C.4, again the infinitesimal generator of a quasi-contractive C_0-semigroup $S_n(t)$, $t \geq 0$, $n \in \mathbb{N}$, $n > \omega_0$, such that

$$\|S_n(t)\|_{L(H)} \leq exp(\frac{\omega_0 n t}{n - \omega_0}) \text{ for all } t \geq 0,\ n > \omega_0.$$

Thus, we get that the coefficients A_n, F and B, $n \in \mathbb{N}$, $n > \omega_0$, fulfill the assumptions of H.2. and so those of H.0 and H.1.
Now, we can derive for $n > \omega_0$ the existence of a unique mild solution $X_n(x)$ of the following stochastic differential equation

$$\begin{cases} dX(t) &= [A_n X(t) + F(X(t))]\, dt + z\, q(dt, dy) \\ X(0) &= x \in H \end{cases} \tag{6.7}$$

which is Gâteaux differentiable as a mapping from H to $H^2(T, H)$.

We define $\mathcal{F}_n$ and $\bar{\mathcal{F}}_n : H \times H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$, $n > \omega_0$, as in chapter 5, section 1 for the coefficients A_n, $n > \omega_0$, F and B. Since A_n, $n > \omega_0$, F and B fulfill H.0 and H.1 we get by theorem 4.4 the existence of a unique mild solution $X_n : H \to H^2(T, H)$ of (6.7) as the implicit function of $\bar{\mathcal{F}}_n$, i.e. $\bar{\mathcal{F}}_n(x, X_n(X)) = X_n(x)$ in $H^2(T, H)$. By theorem 5.1 $X_n : H \to H^2(T, H)$, $n > \omega_0$, is Gâteaux differentiable.

Notation: In the following we denote by $X_n(x)$ and $\partial X_n(x)H$, $n > \omega_0$, $x, h \in H$, predictable representatives in $\mathcal{H}^2(T, H)$ of the respective equivalence classes in $H^2(T, H)$.

Since $A_n \in L(H)$ for all $n \in \mathbb{N}$, $n > \omega_0$, we already know by section 6.1 that for all $x, h \in H$, $t \in [0,T]$ and $n > \omega_0$ holds

$$\|\partial X_n(x)h(t)\| \le e^{\omega_n t}\|h\| \quad P\text{-a.s.} \tag{6.8}$$

where $\omega_n := \dfrac{\omega_0 n}{n - \omega_0}$.
Our next aim is to show that $X(x)$ and $\partial X(x)h$ are the limits in $\mathcal{H}^2(T,H)$ of $(X_n(x))_{n\in\mathbb{N},n>\omega_0}$ and $(\partial X_n(x)h)_{n\in\mathbb{N},n>\omega_0}$, respectively. For this purpose we use theorem A.8.
We have to check that the mappings $\mathcal{F}, \mathcal{F}_n$, $n \in \mathbb{N}$, fulfill the conditions of theorem A.8 if we set $\Lambda := H$ and $E := H^2_{\lambda_0}(T,H)$ for an appropriate $\lambda_0 \ge 0$.

Proposition 6.4. *There exists $\lambda_0 \ge 0$ and $\alpha \in [0,1[$ such that for all $n > \omega_0$ and predictable $Y, Z \in \mathcal{H}^2(T,H)$*

$$\begin{aligned}
&\|\mathcal{F}_n(x,Y) - \mathcal{F}_n(x,Z)\|_{2,\lambda_0,T} \le \alpha\|Y - Z\|_{2,\lambda_0,T} \quad \textit{and}\\
&\|\mathcal{F}(x,Y) - \mathcal{F}(x,Z)\|_{2,\lambda_0,T} \le \alpha\|Y - Z\|_{2,\lambda_0,T}.
\end{aligned}$$

Proof. By the proof of theorem 4.4 we know that for all $x \in H$ and predictable $Y, Z \in \mathcal{H}^2(T,H)$,

$$\begin{aligned}
&\|\mathcal{F}(x,Y) - \mathcal{F}(x,Z)\|_{2,\lambda,T} \le M_T C\big(\frac{T}{2\lambda}\big)^{\frac{1}{2}}\|Y - Z\|_{2,\lambda,T} \quad \text{and}\\
&\|\mathcal{F}_n(x,Y) - \mathcal{F}_n(x,Z)\|_{2,\lambda,T} \le M_{T,n} C\big(\frac{T}{2\lambda}\big)^{\frac{1}{2}}\|Y - Z\|_{2,\lambda,T},\ n \in \mathbb{N},
\end{aligned}$$

where

$$\begin{aligned}
M_T &:= \sup_{t\in[0,T]} \|S(t)\|_{L(H)} \le e^{\omega_0 T} \quad \text{and}\\
M_{T,n} &:= \sup_{t\in[0,T]} \|S_n(t)\|_{L(H)} \le exp\big(\frac{\omega_0 n T}{n - \omega_0}\big),\ n \in \mathbb{N},\ n > \omega_0.
\end{aligned}$$

As the sequence $exp(\frac{\omega_0 n T}{n-\omega_0})$, $n \in \mathbb{N}$, $n > \omega_0$, is convergent with limit $e^{\omega_0 T}$ it is bounded from above by a constant $K > 0$. If we choose $\lambda_0 \ge 0$ such that

$$\alpha := (K \vee M_T) C\big(\frac{T}{2\lambda_0}\big)^{\frac{1}{2}} \in [0,1[$$

then the assertion follows. □

Proposition 6.5. *For all $x, y \in H$, $Z \in \mathcal{H}^2(T,H)$, predictable, and $\lambda \ge 0$ the mappings*

$$\begin{aligned}
\partial_1 \mathcal{F}_n(x,\cdot)y &: \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)\\
\partial_2 \mathcal{F}_n(x,\cdot)Z &: \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)
\end{aligned}$$

are continuous uniformly in $n \in \mathbb{N}$, $n > \omega_0$.

Proof. Since for $x, y \in H$ and $Z \in \mathcal{H}^2(T,H)$, predictable, $\partial_1 \mathcal{F}_n(x,Z)y = (S_n(t)y)_{t\in[0,T]}$ the continuity of $\partial_1\mathcal{F}_n(x,\cdot)y$ uniformly in $n \in \mathbb{N}$, $n > \omega_0$, is obvious.
We have to show the continuity of

$$\partial_2\mathcal{F}_n(x,\cdot)Z\mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$$
$$Y \mapsto \Big(\int_0^t S_n(t-s)\partial F(Y(s))Z(s)\,ds\Big)_{t\in[0,T]}.$$

Let $x \in H$ and $Y, Y_k, Z \in \mathcal{H}^2(T,H)$, predictable, $k \in \mathbb{N}$, such that $Y_k \underset{k\to\infty}{\longrightarrow} Y$ in $\mathcal{H}^2(T,H)$. Then we get for all $n > \omega_0$ that

$$\begin{aligned}
&\|\partial_2\mathcal{F}_n(x,Y)Z - \partial_2\mathcal{F}_n(x,Y_k)Z\|_{\mathcal{H}^2}\\
\le\,& M_{T,n}T^{\frac{1}{2}}E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\,ds]^{\frac{1}{2}}\\
\le\,& KT^{\frac{1}{2}}E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\,ds]^{\frac{1}{2}}.
\end{aligned}$$

(For the definition of $M_{T,n}$ and K see the proof of proposition 6.4.)
Since $\partial F : H \times H \to H$ is continuous we obtain by lemma 5.4 that $\|\partial F(Y)Z - \partial F(Y_k)Z\| \underset{k\to\infty}{\longrightarrow} 0$ in $\lambda_{|[0,T]} \otimes P$-measure.
Moreover,

$$\|\partial F(Y)Z - \partial F(Y_k)Z\|^2 \le 4C^2\|Z\|^2 \in L^1([0,T]\times\Omega, \lambda_{|[0,T]}\otimes P).$$

Hence we obtain that

$$E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\,ds] \to 0 \text{ as } k \to \infty.$$

□

Proposition 6.6. *For all $x, y \in H$ and predictbale $Y, Z \in \mathcal{H}^2(T,H)$*

(i) $\mathcal{F}_n(x,Y) \to \mathcal{F}(x,Y)$ *as* $n \to \infty$, $n > \omega_0$,

(ii) $\partial_1\mathcal{F}_n(x,Y)y \to \partial_1\mathcal{F}(x,Y)y$ *as* $n \to \infty$, $n > \omega_0$,

(iii) $\partial_2\mathcal{F}_n(x,Y)Z \to \partial_2\mathcal{F}(x,Y)Z$ *as* $n \to \infty$, $n > \omega_0$,

in $\mathcal{H}^2(T,H)$.

Proof.
(i) Let $x \in H$ and $Y \in \mathcal{H}^2(T,H)$, predictable, then

$$
\begin{aligned}
&\big(E[\|\mathcal{F}_n(x,Y)(t) - \mathcal{F}(x,Y)(t)\|^2]\big)^{\frac{1}{2}} \\
\le\ &\big(E[\|S_n(t)x - S(t)x\|^2]\big)^{\frac{1}{2}} \\
&+ \big(E[\|\int_0^t S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\,ds\|^2]\big)^{\frac{1}{2}} \\
&+ \big(E[\|\int_0^{t+}\int_U S_n(t-s)z - S(t-s)z\,q(ds,dy)\|^2]\big)^{\frac{1}{2}} \\
\le\ &\sup_{t\in[0,T]}\|S_n(t)x - S(t)x\| \\
&+ \big(E[T\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\,ds]\big)^{\frac{1}{2}} \\
&+ \big(E[\int_0^t\int_U \|S_n(t-s)z - S(t-s)z\|^2\,\nu(dy)\,ds]\big)^{\frac{1}{2}} \\
\le\ &\sup_{t\in[0,T]}\|S_n(t)x - S(t)x\| \\
&+ T^{\frac{1}{2}}\big(E[\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\,ds]\big)^{\frac{1}{2}} \\
&+ \nu(U)^{\frac{1}{2}}\Big(\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\|^2\,ds\Big)^{\frac{1}{2}}.
\end{aligned}
$$

$\sup_{t\in[0,T]}\|S_n(t)x - S(t)x\| \to 0$ as $n \to \infty$ for all $x \in H$ by proposition C.4. Again by proposition C.4, for fixed $s \in [0,T]$

$$
\begin{aligned}
&\sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\| \underset{n\to\infty}{\longrightarrow} 0 \quad (1) \\
\text{and } &\sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\| \underset{n\to\infty}{\longrightarrow} 0 \qquad (2).
\end{aligned}
$$

Moreover, the first sequence (1) of mappings from $[0,T]\times\Omega$ to $\mathbb{R}$ is bounded by $(K+M_T)C(1+\|Y\|) \in L^2([0,T]\times\Omega, \lambda_{|[0,T]}\otimes P)$.
Hence, by Lebesgue's dominated convergence theorem we get that

$$
E[\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\,ds] \to 0
$$

as $n \to \infty$, $n > \omega_0$.
The second sequence (2): $\sup_{t\in[0,T]} 1_{[0,t]}(\cdot)\|S_n(t-\cdot)z - S(t-\cdot)z\|$, $n \in \mathbb{N}$, $n > \omega_0$, is bounded by $(K+M_T)\|z\| \in L^2([0,T])$, thus, we obtain again by Lebesgue's theorem that $\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\|^2\,ds \to 0$ as $n \to \infty$, $n > \omega_0$.

The proof of (ii) and (iii) can be done analoguously. $\square$

By proposition 6.5 and proposition 6.6 we justified that the mappings

$$\bar{\mathcal{F}}_n : H \times H^2_{\lambda_0}(T,H) \to H^2_{\lambda_0}(T,H),\ n \in \mathbb{N},\ n > \omega_0, \text{ and}$$
$$\bar{\mathcal{F}} : H \times H^2_{\lambda_0}(T,H) \to H^2_{\lambda_0}(T,H)$$

fulfill the conditions of theorem A.8 and, finally, we obtain that for all $x, h \in H$

$$X_n(x) \to X(x) \text{ and } \partial X_n(x)h \to \partial X(x)h \text{ in } \mathcal{H}^2_{\lambda_0}(T,H) \text{ as } n \to \infty.$$

In particular, we get for each $t \in [0,T]$ the existence of a subsequence $(n_k(t))_{k\in\mathbb{N}}$ such that

$$\partial X_{n_k(t)}(x)h(t) \underset{\substack{k\to\infty \\ n_k(t)>\omega_0}}{\longrightarrow} \partial X(x)h(t) \ P\text{-a.s.}$$

Thus, by (6.8), it follows that for all $t \in [0,T]$

$$\begin{aligned} \|\partial X(x)h(t)\| &= \lim_{\substack{k\to\infty \\ n_k(t)>\omega_0}} \|\partial X_{n_k(t)}(x)h(t)\| \leq \lim_{\substack{k\to\infty \\ n_k(t)>\omega_0}} exp(\frac{\omega_0 n_k(t)}{n_k(t)-\omega_0}t)\|h\| \\ &= e^{\omega_0 t}\|h\| \quad P\text{-a.s.} \end{aligned} \tag{6.9}$$

6.3 Gradient estimates for the resolvent

We define the transition kernels and the "resolvent" corresponding with the mild solution $X(x)$, $x \in H$, in the following way.
Let $f : (H, \mathcal{B}(H)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, bounded. Define

$$p_t f(x) := E[f(X(x)(t)],\ t \in [0,T],\ x \in H, \text{ and}$$
$$R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x)\, dt,\ \alpha \geq 0.$$

Proposition 6.7. *If $f \in C_b^1(H,\mathbb{R})$ where*

$$C_b^1 := \{g : H \to \mathbb{R} \mid g \textit{ is continuously Fréchet differentiable such that } \sup_{x\in H} \|Dg(x)\|_{L(H,\mathbb{R})} < \infty\}$$

then $R_\alpha f : H \to \mathbb{R}$ is Gâteaux differentiable for all $\alpha \geq 0$ and for all $x, h \in H$ and $\alpha \geq 0$

$$\partial R_\alpha f(x)h = \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt.$$

Proof. Let $\alpha \geq 0$, $x, h \in H$ and $\varepsilon > 0$ then we get that

$$
\begin{aligned}
&\Big|\frac{R_\alpha f(x+\varepsilon h) - R_\alpha f(x)}{\varepsilon} - \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\,dt\Big| \\
\leq &\int_0^\infty e^{-\alpha t} E\big[\Big|\frac{f(X(x+\varepsilon h)(t)) - f(X(x)(t))}{\varepsilon} - Df(X(x)(t))\partial X(x)h(t)\Big|\big]\,dt,
\end{aligned}
$$

where by proposition B.8

$$
\begin{aligned}
&E\big[\Big|\frac{f(X(x+\varepsilon h)(t)) - f(X(x)(t))}{\varepsilon} - Df(X(x)(t))\partial X(x)h(t)\Big|\big] \\
=&E\big[\Big|\int_0^1 Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad \big(\frac{X(x+\varepsilon h)(t) - X(x)(t)}{\varepsilon}\big) - Df(X(x)(t))\partial X(x)h(t)\,d\sigma\Big|\big] \\
\leq &E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big)\|_{L(H,\mathbb{R})} \\
&\qquad \|\frac{X(x+\varepsilon h)(t) - X(x)(t)}{\varepsilon} - \partial X(x)h(t)\|\,d\sigma\big] \\
&+E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|_{L(H,\mathbb{R})}\|\partial X(x)h(t)\|\,d\sigma\big] \\
\leq &\sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|\frac{X(x+\varepsilon h) - X(x)}{\varepsilon} - \partial X(x)h\|_{\mathcal{H}^2} \\
&+ \big(E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\,d\sigma\big]\big)^{\frac{1}{2}}\|\partial X(x)h\|_{\mathcal{H}^2}.
\end{aligned}
$$

Thus, we get that

$$
\begin{aligned}
&\Big|\frac{R_\alpha f(x+\varepsilon h) - R_\alpha f(x)}{\varepsilon} - \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\,dt\Big| \\
\leq &\int_0^\infty e^{-\alpha t}\,dt \sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|\frac{X(x+\varepsilon h) - X(x)}{\varepsilon} - \partial X(x)h\|_{\mathcal{H}^2} \\
&+ \int_0^\infty e^{-\alpha t}\big(E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\,d\sigma\big]\big)^{\frac{1}{2}}\,dt\,\|\partial X(x)h\|_{\mathcal{H}^2}.
\end{aligned}
$$

The first summand converges to 0 as $\varepsilon \to 0$ as $X : H \to \mathcal{H}^2(T, H)$ is Gâteaux-differentiable.
To prove the convergence to 0 of the second summand we use lemma 5.4.
Since $X(t) : H \to L^2(\Omega, \mathcal{F}_t, P; H)$ is continuous we can conclude that for fixed $\sigma \in [0, 1]$

$$
X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t)) \underset{\varepsilon\to 0}{\to} X(x)(t) \text{ in } P\text{-measure.}
$$

Moreover, $Df : H \to L(H, \mathbb{R})$ is continuous and we obtain by lemma 5.4 that

$$\|Df(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in P-measure. As this sequence is bounded by $4\sup_{x\in H}\|Df(x)\|^2_{L(H,\mathbb{R})} < \infty$ it follows that

$$E\big[\|Df(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\big] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Since this expectation is bounded by $4\sup_{x\in H}\|Df(x)\|^2_{L(H,\mathbb{R})} < \infty$ we get by Lebesgue's dominated convergence theorem that

$$\int_0^1 E\big[\|Df(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\big]\, d\sigma \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Finally, again by Lebesgue's theorem, we obtain that

$$\begin{aligned}\int_0^\infty e^{-\alpha t} E\Big[\int_0^1 &\|Df(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t)))\\ &- Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\, d\sigma\Big]^{\frac{1}{2}}\, dt\, \|\partial X(x)h\|_{\mathcal{H}^2}\\ \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

We proved the existence of the directional derivatives $\partial R_\alpha f(x,;h)$, $x, h \in H$. Obviously, $\partial R_\alpha f(x,;h) \in L(H,\mathbb{R})$ and therefore the assertion of the proposition follows. □

Using the gradient estimate (6.9) for the mild solution and the representation of $\partial R_\alpha f(x)h$ we get, if $f \in C_b^1(H,\mathbb{R})$ and $\alpha > \omega_0$, that

$$\begin{aligned}\|\partial R_\alpha f(x)h\| &= \|\int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt\|\\ &\le \int_0^\infty e^{-\alpha t} E[\sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|\partial X(x)h(t)\|]\, dt\\ &\le \int_0^\infty e^{-\alpha t} \sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})} e^{\omega_0 t}\|h\|\, dt\\ &= \frac{1}{\alpha-\omega_0}\sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|h\|\end{aligned}$$

Finally, we have

$$\|\partial R_\alpha f(x)\|_{L(H,\mathbb{R})} \le \frac{1}{\alpha-\omega_0}\sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})} \text{ for all } \alpha > \omega_0 \text{ and } f \in C_b^1(H,\mathbb{R}).$$

Appendix A

Existence, Continuity and Differentiability of Implicit Functions

Let $(E, \| \ \|)$ and $(\Lambda, \| \ \|_\Lambda)$ be two Banach spaces. In the whole chapter we consider a mapping $G : \Lambda \times E \to E$ which is a contraction in the second variable, i.e. there exists an $\alpha \in [0, 1[$ such that

$$\|G(\lambda, x) - G(\lambda, y)\| \leq \alpha \|x - y\| \text{ for all } \lambda \in \Lambda,\ x, y \in E. \tag{A.1}$$

Then, by Banach's fixed point theorem, we get the existence of a unique implicit function $\varphi : \Lambda \to E$, i.e.

$$\varphi(\lambda) = G(\lambda, \varphi(\lambda)) \text{ for all } \lambda \in \Lambda.$$

A.1 Continuity of the implicit function

Theorem A.1 (Continuity of the implicit function). *(i) If for all $x \in E$ the mapping $G(\cdot, x) : \Lambda \to E$ is continuous then $\varphi : \Lambda \to E$ is continuous.*

(ii) If there exists a constant $L \geq 0$ such that

$$\|G(\lambda, x) - G(\tilde{\lambda}, x)\|_E \leq L \|\lambda - \tilde{\lambda}\|_\Lambda \text{ for all } x \in E$$

then $\varphi : \Lambda \to E$ is Lipschitz continuous.

Proof. [FrKn 2002, Theorem D.1, p.164] □

A.2 Different concepts of differentiability in general Banach spaces

Let $(E_1, \| \|_{E_1})$ and $(E_2, \| \|_{E_2})$ be two real Banach spaces and let $H : E_1 \to E_2$.

Definition A.2. $L(E_1, E_2)$ is defined as the space of all bounded, linear operators from E_1 to E_2. If $E_2 = E_1$ we write $L(E_1) := L(E_1, E_1)$.

Definition A.3 (Directional derivatives). H is said to be *differentiable in $x_0 \in E_1$ and in the direction $y \in E_1$* if there exists

$$\lim_{h \to \infty} \frac{H(x_0 + hy) - H(x_0)}{h} =: \partial H(x_0; y) \in E_2.$$

$\partial H(x_0; y)$ is called the *directional derivative of H (in x_0 and direction y).*

Definition A.4 (Gâteaux differentiability). H is said to be *Gâteaux differentiable in $x_0 \in E_1$* if there exist all directional derivatives $\partial H(x_0; y)$, $y \in E_1$, and if $\partial H(x_0; \cdot) \in L(E_1, E_2)$. Then we write $\partial H(x_0)y$ instead of $\partial H(x_0; y)$, $y \in E_1$, and $\partial H(x_0)$ is called *Gâteaux derivative* of H in x_0.
If $H : E_1 \to E_2$ is Gâteaux differentiable in all $x \in E_1$ we call H *Gâteaux differentiable.*

Lemma A.5. *(i) If $H : E_1 \to E_2$ is differentiable in $x_0 \in E_1$ and in direction $y \in E_1$ then there exist all directional derivatives $\partial H(x_0; \lambda y)$, $\lambda \in \mathbb{R}$, and*

$$\partial H(x_0; \lambda y) = \lambda \partial H(x_0; y)$$

(ii) If there exist all directional derivatives $\partial H(x; y)$, $x, y \in E_1$, such that the mapping $x \mapsto \partial H(x; y)$ is continuous from E_1 to E_2 for each $y \in E_1$ then $\partial H(x; \cdot)$ is additive for all $x \in E_1$, i.e.

$$\partial H(x; y_1 + y_2) = \partial H(x; y_1) + \partial H(x; y_2) \quad \text{for all } x, y_1, y_2 \in E_1$$

Proof. [FrKn 2002,Lemma D.4, p.165] □

Theorem A.6 (First order differentiability). *We assume that the mapping $G : \Lambda \times E \to E$ fulfills the following conditions.*

1. *$G(\cdot, x) : \Lambda \to E$ is continuous for all $x \in E$,*
2. *for all $\lambda, \mu \in \Lambda$ and all $x, y \in E$ there exist the directional derivatives*

$$\partial_1 G(\lambda, x; \mu) = E - \lim_{h \to \infty} \frac{G(\lambda + h\mu, x) - G(\lambda, x)}{h}$$

$$\partial_2 G(\lambda, x; y) = E - \lim_{h\to\infty} \frac{G(\lambda, x+hy) - G(\lambda, x)}{h}$$

and $\partial_1 G : \Lambda \times E \times \Lambda \to E$ *and* $\partial_2 G : \Lambda \times E \times E \to E$ *are continuous.*

Then the implicit function $\varphi : \Lambda \to E$ *is Gâteaux differentiable such that the mapping* $\Lambda \times \Lambda \to E$, $(\lambda, \mu) \mapsto \partial\varphi(\lambda)\mu$ *is continuous and*

$$\partial\varphi(\lambda)\mu = [I - \partial_2 G(\lambda, \varphi(\lambda))]^{-1} \partial_1 G(\lambda, \varphi(\lambda))\mu \tag{A.2}$$

for all $\lambda, \mu \in \Lambda$.

Proof. [FrKn 2002, Theorem D.8, p.168] □

Corollary A.7. *If the assumptions of theorem A.6 are fulfilled and if there exists* $C \geq 0$ *such that* $\|\partial_1 G(\lambda, x)\|_{L(\Lambda,E)} \leq C$ *for all* $\lambda \in \Lambda$ *and* $x \in E$ *then* $\partial\varphi : \Lambda \to L(\Lambda, E)$ *is also bounded.*

Proof. [FrKn 2002, Corollary D.11, p.173] □

Theorem A.8. *Let* $G_n : \Lambda \times E \to E$, $n \in \mathbb{N}$, *such that*

$$\|G_n(\lambda, x) - G_n(\lambda, y)\| \leq \alpha\|x - y\| \qquad \text{for all } \lambda \in \Lambda \text{ and all } x, y \in E \text{ and } n \in \mathbb{N}.$$

Moreover, assume that the mappings G *and* G_n, $n \in \mathbb{N}$, *fulfill the following conditions.*

1. $G(\cdot, x)$ *and* $G_n(\cdot, x)$, $n \in \mathbb{N}$, *are continuous for all* $x \in E$,
2. G, G_n, $n \in \mathbb{N}$, *are Gâteaux differentiable such that*

$$\partial_1 G : \Lambda \times E \times \Lambda \to E \text{ and } \partial_2 G : \Lambda \times E \times E \to E$$
$$\partial_1 G_n : \Lambda \times E \times \Lambda \to E \text{ and } \partial_2 G_n : \Lambda \times E \times E \to E,\ n \in \mathbb{N},$$

   are continuous,

3. $\partial_1 G_n(\lambda, \cdot)\mu$ *and* $\partial_2 G_n(\lambda, \cdot)x$, $\lambda, \mu \in \Lambda$, $x \in E$, *are continuous uniformly in* $n \in \mathbb{N}$,
4. $G_n \to G$, $\partial_1 G_n \to \partial_1 G$ *and* $\partial_2 G_n \to \partial_1 G$ *pointwisely as* $n \to \infty$.

Then there exist unique implicit functions $\varphi, \varphi_n : \Lambda \to E$, $n \in \mathbb{N}$, *such that* $G(\lambda, \varphi(\lambda)) = \varphi(\lambda)$ *and* $G_n(\lambda, \varphi_n(\lambda)) = \varphi_n(\lambda)$, $n \in \mathbb{N}$, *for all* $\lambda \in \Lambda$.
φ *and* φ_n, $n \in \mathbb{N}$, *are Gâteaux differentiable.*

Moreover, $\varphi_n(\lambda) \to \varphi(\lambda)$ *and* $\partial\varphi_n(\lambda)\mu \to \partial\varphi(\lambda)\mu$ *as* $n \to \infty$ *for all* $\lambda, \mu \in \Lambda$.

Proof. For all $\lambda \in \Lambda$ we have that

$$\begin{aligned}
&\|\varphi_n(\lambda) - \varphi(\lambda)\| = \|G_n(\lambda, \varphi_n(\lambda)) - G(\lambda, \varphi(\lambda))\| \\
\leq &\|G_n(\lambda, \varphi_n(\lambda)) - G_n(\lambda, \varphi(\lambda))\| + \|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\| \\
\leq &\alpha\|\varphi_n(\lambda) - \varphi(\lambda)\| + \|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\|.
\end{aligned}$$

Subtracting on both sides of the above equation $\alpha\|\varphi_n(\lambda) - \varphi(\lambda)\|$ and dividing by $(1 - \alpha)$ we get that

$$\|\varphi_n(\lambda) - \varphi(\lambda)\| \leq \frac{1}{1 - \alpha}\|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\| \underset{n\to\infty}{\longrightarrow} 0$$

by assumption.
By theorem A.6 (i) φ and φ_n, $n \in \mathbb{N}$, are Gâteaux differentiable. Using the representation (A.2) of the Gâteaux derivatives of φ_n, $n \in \mathbb{N}$, and φ we can estimate $\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\|$, $\lambda, \mu \in \Lambda$, in the following way:

$$\begin{aligned}
&\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\| \\
\leq &\|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \|\partial_1 G_n(\lambda, \varphi_n(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\| \\
\leq &\|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \sup_{m\in\mathbb{N}}\|\partial_2 G_m(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G_m(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \|\partial_2 G_n(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \sup_{m\in\mathbb{N}}\|\partial_1 G_m(\lambda, \varphi_n(\lambda))\mu - \partial_1 G_m(\lambda, \varphi(\lambda))\mu\| \\
&+ \|\partial_1 G_n(\lambda, \varphi(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\|
\end{aligned}$$

Since
$\|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu\| \leq \alpha\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\|$
we obtain that

$$\begin{aligned}
&\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\| \\
\leq &\frac{1}{1 - \alpha}\Big(\sup_{m\in\mathbb{N}}\|\partial_2 G_m(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G_m(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&\qquad + \|\partial_2 G_n(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&\qquad + \sup_{m\in\mathbb{N}}\|\partial_1 G_m(\lambda, \varphi_n(\lambda))\mu - \partial_1 G_m(\lambda, \varphi(\lambda))\mu\| \\
&\qquad + \|\partial_1 G_n(\lambda, \varphi(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\|\Big) \\
&\underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

since $\varphi_n(\lambda) \to \varphi(\lambda)$ as $n \to \infty$ and by the assumptions on the mappings G_n, $n \in \mathbb{N}$, and G. □

Appendix B

The Bochner Integral

Let $(X, \| \ \|)$ be a Banach space, $\mathcal{B}(X)$ the Borel σ-field of X and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ.

B.1 Definition of the Bochner integral

Step 1:
As first step we want to define the integral for simple functions which are defined as follows. Set

$$\mathcal{E} := \{f : \Omega \to X | f = \sum_{k=1}^{n} x_k 1_{A_k}, x_k \in X, A_k \in \mathcal{F}, 1 \le k \le n, n \in \mathbb{N}\}$$

and define a semi-norm $\| \ \|_{\mathcal{E}}$ on the vector space $\mathcal{E}$ by

$$\|f\|_{\mathcal{E}} := \int \|f\| \, d\mu, \ f \in \mathcal{E}.$$

To get that $(\mathcal{E}, \| \ \|_{\mathcal{E}})$ is a normed vector space we consider equivalence classes with respect to $\| \ \|_{\mathcal{E}}$. For simplicity we will not change the notations.
For $f \in \mathcal{E}$ we define now the Bochner integral to be

$$\int f d\mu := \sum_{k=1}^{n} x_k \mu(A_k).$$

In this way we get a mapping

$$\begin{aligned} \text{int} : (\mathcal{E}, \| \ \|_{\mathcal{E}}) &\to (X, \| \ \|) \\ f &\mapsto \int f \, d\mu \end{aligned}$$

which is linear and uniformly continuous since $\|\int f\, d\mu\| \leq \int \|f\|\, d\mu$ for all $f \in \mathcal{E}$.
Therefore we can extend the mapping int to the abstract completion of $\mathcal{E}$ with respect to $\|\ \|_{\mathcal{E}}$ which we denote by $\overline{\mathcal{E}}$.

Step 2: We give an explicit representation of $\overline{\mathcal{E}}$.

Definition B.1. A function $f : \Omega \to X$ is called strongly measurable if it is Borel measurable and $f(\Omega) \subset X$ is separable.

Definition B.2. Let $1 \leq p < \infty$. Then we define

$$\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X) := \{f : \Omega \to X \mid f \text{ is strongly measurable with respect to } \mathcal{F} \text{ and } \int \|f\|^p\, d\mu < \infty\}$$

and the semi-norm $\|f\|_{L^p} := \left(\int \|f\|^p\, d\mu\right)^{\frac{1}{p}}$, $f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu; X)$. The space of all equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X)$ with respect to $\|\ \|_{L^p}$ is denoted by $L^p(\Omega, \mathcal{F}, \mu; X)$. The elements of $L^p(\Omega, \mathcal{F}, \mu; X)$ are called p-integrable or just integrable if $p = 1$.

Notation B.3. Let $1 \leq p < \infty$. We use the following notations:
$L^p(\Omega, \mathcal{F}, \mu) := L^p(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and if confusion is impossible $L^p(\Omega) := L^p(\Omega, \mu) := L^p(\Omega, \mathcal{F}, \mu)$.

Claim: $L^1(\Omega, \mathcal{F}, \mu; X) = \overline{\mathcal{E}}$.
Step 1: $(L^1(\Omega, \mathcal{F}, \mu; X), \|\ \|_{L^1})$ is complete.
The proof is just a modification of the proof of the Fischer-Riesz theorem by the help of the following proposition.

Proposition B.4. *Let $(\Omega, \mathcal{F})$ be a measurable space and let X be a Banach space. Then*

(i) the set of Borel measurable functions from Ω to X is closed under the formation of pointwise limits, and

(ii) the set of strongly measurable functions from Ω to X is closed under the formation of pointwise limits.

Proof. [Co 80, Proposition E.1., p.350] □

Step 2: $\mathcal{E}$ is a dense subset of $L^1(\Omega, \mathcal{F}, \mu; X)$ with respect to $\|\ \|_{L^1}$.
This can be shown by the help of the following lemma.

Lemma B.5. *Let E be a metric space with metric d and let $f : \Omega \to E$ be strongly measurable. Then there exists a sequence f_n, $n \in \mathbb{N}$, of simple E-valued functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(E)$-measurable and takes only a finite number of values) such that for arbitrary $\omega \in \Omega$ the sequence $d(f_n(\omega), f(\omega))$, $n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. [DaPrZa 92, Lemma 1.1, p.16] □

Let now $f \in L^1(\Omega, \mathcal{F}, \mu; X)$. By the above lemma B.5 we get the existence of a sequence of simple functions f_n, $n \in \mathbb{N}$, such that

$$\|f_n(\omega) - f(\omega)\| \downarrow 0 \text{ for all } \omega \in \Omega \text{ as } n \to \infty$$

Hence $f_n \underset{n\to\infty}{\longrightarrow} f$ in $\| \ \|_{L^1}$ by Lebesgue's dominated convergence theorem.

B.2 Properties of the Bochner integral

Proposition B.6. *Let $f \in L^1(\Omega, \mathcal{F}, \mu; X)$. Then*

$$\int \varphi \circ f \, d\mu = \varphi\Big(\int f \, d\mu\Big)$$

holds for all $\varphi \in X^* = L(X, \mathbb{R})$.

Proof. [Co 80, Proposition E.11, p.356] □

Proposition B.7. *Let Y be a further Banach space, $\varphi \in L(X, Y)$ and $f \in L^1(\Omega, \mathcal{F}, \mu; X)$ such that $\varphi \circ f$ is strongly measurable. Then*

$$\int \varphi \circ f \, d\mu = \varphi\Big(\int f \, d\mu\Big).$$

Proof. [DaPrZa 92, Proposition 1.6, p.21] □

Proposition B.8 (Fundamental theorem). *Let $-\infty < a < b < \infty$ and $f \in C^1([a, b]; X)$. Then*

$$f(t) - f(s) = \int_s^t f'(u) \, du := \begin{cases} \int 1_{[s,t]}(u) f'(u) \, du & \text{if } s \le t \\ -\int 1_{[t,s]}(u) f'(u) \, du & \text{otherwise} \end{cases}$$

for all $s, t \in [a, b]$ where du denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

Proof. [FrKn 02, Proposition A.7, p.152] □

Proposition B.9. *Let $[a,b]$ be a finite interval and $f \in L^1([a,b],\mathcal{B}([a,b]),\lambda;\mathbb{R})$, where λ denotes the Lebesgue measure. Then the mapping $F : [a,b] \to \mathbb{R}$, $s \mapsto \int_a^s f(t)\,dt$, is differentiable λ-a.e. on $[a,b[$ and $F'(s) = f(s)$ for λ-a.e. $s \in [a,b[$.*

Proof. [deBa 81, Chapter 4, Theorem 12, p.89] □

Proposition B.10. *Let $[a,b]$ be a finite interval and let $f \in L^1([a,b],\mathcal{B}([a,b]),\lambda;X)$, where λ denotes the Lebesgue measure. Then the mapping $F : [a,b] \to X$, $s \mapsto \int_a^s f(t)\,dt$, is differentiable λ-a.e. on $[a,b[$ and $F'(s) = f(s)$ for λ-a.e. $s \in [a,b[$.*

Proof. Since $f([a,b])$ is separable there exist x_n, $n \in \mathbb{N}$, such that $\{x_n \,|\, n \in \mathbb{N}\}$ is a dense subset of $f([a,b])$. Then $\|f - x_n\| \in L^1([a,b],\lambda)$ for all $n \in \mathbb{N}$. Consequently, by proposition B.9 the mappings $F_n : [a,b] \to \mathbb{R}$, $s \mapsto \int_a^s \|f(t) - x_n\|\,dt$, $n \in \mathbb{N}$, are differentiable λ-a.e. on $[a,b[$ and $F_n(s) = \|f(s) - x_n\|$ for all $n \in \mathbb{N}$ and for λ-a.e. $s \in [a,b[$.
Then we get for λ-a.e. $s \in [a,b[$ that

$$\begin{aligned}
&\limsup_{h\to 0} \Big\|\frac{1}{h}\Big(\int_a^{s+h} f(t)\,dt - \int_a^s f(t)\,dt\Big) - f(s)\Big\| \\
&= \limsup_{h\to 0} \Big\|\frac{1}{h}\int_s^{s+h} (f(t) - f(s)\,dt\Big\| \\
&\leq \limsup_{h\to 0} \frac{1}{h}\int_s^{s+h} \|f(t) - f(s)\|\,dt \\
&\leq \limsup_{h\to 0} \frac{1}{h}\int_s^{s+h} \|f(t) - x_n\|\,dt - \|f(s) - x_n\| \\
&= 2\|f(s) - x_n\|.
\end{aligned}$$

Choosing a subsequence x_{n_k}, $k \in \mathbb{N}$, such that $\|f(s) - x_{n_k}\| \to 0$ as $k \to \infty$ we obtain that for λ-a.e. $s \in [a,b[$ holds

$$\Big\|\frac{1}{h}\Big(\int_a^{s+h} f(t)\,dt - \int_a^s f(t)\,dt\Big) - f(s)\Big\| \to 0 \text{ as } h \to 0.$$

□

Definition B.11 (Absolut continuity). Let $-\infty \leq a < b \leq \infty$. A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous (on $[a,b]$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n |f(x_i) - f(y_i)| < \varepsilon$ whenever $\sum_{i=1}^n |x_i - y_i| < \delta$ for any set of disjoint intervals such that $(x_i, y_i) \subset [a,b]$ for each $i \in \{1,\dots,n\}$.

Proposition B.12. *Let* $[a, b]$ *be a finite interval and* $f : [a, b] \to \mathbb{R}$ *absolutely continuous, then if* $x \in [a, b]$

$$f(x) - f(a) = \int_a^x f'(t)\, dt$$

Proof. [deBa 81, Chapter 9, Corollary 3, p.162] $\square$

Appendix C

The Theorem of Hille-Yosida

Let $(E, \| \ \|)$ be a separable Banach space.

Proposition C.1. *Let $S(t)$, $t \geq 0$ be a C_0-semigroup on E and let $(A, D(A))$ be its infinitesimal generator. If $x \in D(A)$ then $S(t)x \in D(A)$ and*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax \text{ for all } t \geq 0.$$

Proof. [Pa 83, I. Theorem 2.4, p.4/5] □

Proposition C.2 (Hille-Yosida). *Let $(A, D(A))$ be a linear operator on E. Then the following statements are equivalent.*
(i) A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$.
(ii) A is closed and $D(A)$ is dense in E, the resolvent set $\rho(A)$ contains the interval $]\omega, \infty[$ and the following estimates for the resolvent $G_\alpha := (\alpha - A)^{-1}$, $\alpha \in \rho(A)$, associated to A hold

$$\|G_\alpha^k\|_{L(H)} \leq \frac{M}{(\alpha - \omega)^k}\ ,\ k \in \mathbb{N},\ \alpha > \omega.$$

Proof. [Pa 83, I. Theorem 5.3, p.20] □

Let $(A, D(A))$ be the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. We define now the Yosida-approximation of A. For $n \in \mathbb{N}$, $n > \omega$, define

$$A_n := nAG_n = nG_nA.$$

Proposition C.3. *Let $(A, D(A))$ be the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. Then*

$$\lim_{n\to\infty} A_n x = Ax \text{ for all } x \in D(A).$$

Proof. Let $x \in D(A)$ and $n > \omega$, then

$$\begin{aligned} \|nG_n x - x\|_E &= \|G_n(nx - Ax) + G_n Ax - x\|_E \\ = \|G_n Ax\|_E &\leq \frac{M}{n-\omega}\|Ax\|_E \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

But, by proposition C.2, $D(A)$ is dense in E and $\|nG_n x\|_{L(E)} \leq \frac{Mn}{n-\omega}$, where the sequence $\frac{Mn}{n-\omega}$, $n > \omega$, is convergent and therefore bounded. Hence we get for arbitrary $x \in E$ that $\|nG_n x - x\|_E \to 0$.
In particular, we obtain for all $x \in D(A)$ that

$$A_n x = nG_n Ax \underset{n\to\infty}{\longrightarrow} Ax.$$

□

Proposition C.4. *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. Moreover, let A_n, $n \in \mathbb{N}$, $n > \omega$, be the Yosida-approximation of A. Then*

$$S(t)x = \lim_{n\to\infty} S_n(t)x \text{ locally uniformly in } t \geq 0 \text{ for all } x \in E$$

where $S_n(t) := e^{tA_n}$, $t \geq 0$, and the following estimate holds

$$\|S_n(t)\|_{L(E)} \leq M exp(\frac{\omega n t}{n-\omega}) \text{ for all } t \geq 0,\, n > \omega.$$

Proof. [Pa 83, I. Theorem 5.5, p.21] □

Appendix D

Complements

In this chapter we present some results, needed in the theorems 4.4, 4.4 and 5.1, for the drift part $\int_0^t S(t-s)F(X(s))\,ds$, $t \in [0,T]$, of equation (4.1). They can also be found in [FrKn 2002].

Lemma D.1. *If a mapping $g : [0,T] \times \Omega \to \mathbb{R}$ is $\mathcal{P}_T/\mathcal{B}(\mathbb{R})$-measurable then the mapping*

$$\begin{aligned} \tilde{Y} : \Omega_T &\to \mathbb{R} \\ (s,\omega) &\mapsto 1_{]0,t]}(s) g(s,\omega) \end{aligned}$$

is $\mathcal{B}([0,T]) \otimes \mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable for each $t \in [0,T]$.

Proof. We have to show that $(]0,t] \times \Omega) \cap \mathcal{P}_T \subset \mathcal{B}([0,T]) \otimes \mathcal{F}_t$.
Let $t \in [0,T]$. If we set

$$\mathcal{A} := \{A \in \mathcal{P}_T \mid A \cap (]0,t] \times \Omega) \in \mathcal{B}([0,T]) \otimes \mathcal{F}_t\}$$

it is clear that $\mathcal{A}$ is a σ-field which contains the predictable rectangles $]s,u] \times F_s$, $F_s \in \mathcal{F}_s$, $0 \le s \le u \le T$ and $\{0\} \times F_0$, $F_0 \in \mathcal{F}_0$. Therefore $\mathcal{A} = \mathcal{P}_T$. $\square$

Lemma D.2. *Let Φ be a predictable H-valued process which is P-a.s. Bochner integrable. Then the process given by*

$$\int_0^t S(t-s)\Phi(s)\,ds, \quad t \in [0,T],$$

is P-a.s. continuous and adapted to $\mathcal{F}_t$, $t \in [0,T]$. This especially implies that it is predictable.

Proof. [FrKn 02, Lemma 3.9, p.70] $\square$

Theorem D.3. *Assume that F fulfills hypothesis H.0. Moreover, let $\xi \in L_0^2$ and $Y, \tilde{Y} \in \mathcal{H}^2(T,H)$, predictable, then*

(i) $(S(t)\xi)_{t\in[0,T]} \in \mathcal{H}^2(T,H)$, $1_{[0,t]}(\cdot)S(t-\cdot)F(Y(\cdot))$ *is P-a.s. Bochner integrable on $[0,T]$ and the process*

$$\Big(\int_0^t S(t-s)F(Y(s))\,ds\Big)_{t\in[0,T]}$$

is an element of $\mathcal{H}^2(T,H)$,

(ii) for $\lambda \geq 0$

$$\|\int_0^{\cdot} S(\cdot - s)\big(F(Y(s)) - F(\tilde{Y}(s))\big)\,ds\|_{2,\lambda,T} \leq M_T C T^{\frac{1}{2}} (\frac{1}{\lambda 2})^{\frac{1}{2}} \|Y - \tilde{Y}\|_{2,\lambda,T}.$$

Proof. (i)

Claim 1. $S(t)\xi$, $t \in [0,T]$, is an element of $\mathcal{H}^2(T,H)$.

The mapping

$$(s,\omega) \mapsto S(t)\xi(\omega)$$

is predictable since for fixed $\omega \in \Omega$

$$t \mapsto S(t)\xi(\omega)$$

is a continuous mapping from $[0,T]$ to H and for fixed $t \in [0,T]$

$$\omega \mapsto S(t)\xi(\omega)$$

is not only $\mathcal{F}_t$- but even $\mathcal{F}_0$-measurable.
With respect to the norm we obtain that

$$\|S(\cdot)\xi\|_{\mathcal{H}^2} = \sup_{t\in[0,T]} E[\|S(t)\xi\|^2]^{\frac{1}{2}} \leq M_T \|\xi\|_{L^2} < \infty$$

Claim 2. The Bochner integral $\int_0^t S(t-s)F(Y(s))\,ds$, $t \in [0,T]$, is well defined and has a version which is an element of $\mathcal{H}^2(T,H)$.

Because of the measurability of $F : H \to H$ it is clear that $F(Y(t))$, $t \in [0,T]$, is predictable and the process $F(Y(t))$, $t \in [0,T]$, is P-a.s. Bochner integrable since

$$E[\int_0^t \|F(Y(s))\|\,ds] \leq \int_0^t E[C(1 + \|Y(s)\|)]\,ds \leq CT(1 + \|Y\|_{\mathcal{H}^2}) < \infty.$$

Hence, by lemma D.2 the Bochner-integral is well-defined and has a predictabel version.
Concerning the norm we obtain that

$$\begin{aligned}
&E[\|\int_0^t S(t-s)F(Y(s))\,ds\|^2]^{\frac{1}{2}}\\
&\leq E[C^2T^1M_T^2\int_0^t (1+\|Y(s)\|)^2\,ds]^{\frac{1}{2}}\\
&\leq CT^{\frac{1}{2}}M_T\big(E[\int_0^T 1\,ds]^{\frac{1}{2}}+\big(\int_0^T E[\|Y(s)\|^2]\,ds\big)^{\frac{1}{2}}\big)\\
&\leq CTM_T(1+\|Y\|_{\mathcal{H}^2})<\infty.
\end{aligned}$$

Thus, $\|\int_0^{\cdot} S(\cdot-s)F(Y(s))\,ds\|_{\mathcal{H}^2}\leq CTM_T\big(1+\|Y\|_{\mathcal{H}^2}\big)<\infty$.

(ii) For $t\in[0,T]$

$$\|\int_0^t S(t-s)[F(Y(s))-F(\tilde{Y}(s))]\,ds\|^2\leq M_T^2C^2T\int_0^t\|Y(s)-\tilde{Y}(s)\|^2\,ds$$

This implies that

$$\begin{aligned}
&E[\|\int_0^t S(t-s)[F(Y(s))-F(\tilde{Y}(s))]\,ds\|^2]^{\frac{1}{2}}\\
&\leq M_TCT^{\frac{1}{2}}(\int_0^t E[\|Y(s)-\tilde{Y}(s)\|^2]\,ds)^{\frac{1}{2}}\\
&= M_TCT^{\frac{1}{2}}(\int_0^t e^{\lambda 2s}\underbrace{e^{-\lambda 2s}\|Y(s)-\tilde{Y}(s)\|_{L^2}^2}_{\leq\|Y-\tilde{Y}\|_{2,\lambda,T}^2}\,ds)^{\frac{1}{2}}\\
&\leq M_TCT^{\frac{1}{2}}(\int_0^t e^{\lambda 2s}\,ds)^{\frac{1}{2}}\|Y-\tilde{Y}\|_{2,\lambda,T}\\
&= M_TCT^{\frac{1}{2}}e^{\lambda t}(\frac{1}{\lambda 2})^{\frac{1}{2}}\|Y-\tilde{Y}\|_{2,\lambda,T}
\end{aligned}$$

Dividing by $e^{\lambda t}$ provides that

$$\|\int_0^{\cdot} S(\cdot-s)[F(Y(s))-F(\tilde{Y}(s))]\,ds\|_{2,\lambda,T}\leq\underbrace{M_TCT^{\frac{1}{2}}(\frac{1}{\lambda 2})^{\frac{1}{2}}}_{\to 0\quad\text{as }\lambda\to\infty}\|Y-\tilde{Y}\|_{2,\lambda,T}$$

□

Theorem D.4. *Assume that F fulfills hypotheses H.0 and H.1.*

(i) Let $Y,Z\in\mathcal{H}^2(T,H)$, predictable. Then $1_{[0,t]}(\cdot)S(t-\cdot)\partial F(Y(\cdot))Z(\cdot)$ is P-a.s. Bochner integrable on $[0,T]$.

(ii) Let $Y, Z \in \mathcal{H}^2(T,H)$, predictable. Then

$$\begin{aligned}&\sup_{t\in[0,T]} \| \int_0^t S(t-s)\big(\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\big)\, ds\|_{L^2}\\ &\leq M_T T^{\frac{1}{2}} E\big[\int_0^T \|\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\|^2\, ds\big]^{\frac{1}{2}}\\ &\xrightarrow[h\to 0]{} 0.\end{aligned}$$

(iii) Let $Y, Y_n, Z, Z_n \in \mathcal{H}^2(T,H)$, predictable, $n \in \mathbb{N}$, such that $Y_n \to Y$ and $Z_n \to Z$ in $\mathcal{H}^2(T,H)$. Then

$$\begin{aligned}&\sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\, ds\|_{L^2}\\ &\xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Proof. (i) Since Y is predictable and F is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable the process $\partial F(Y(\cdot))Z(\cdot)$ is predictable. Moreover, $\|\partial F(Y)Z\| \leq C\|Z\| \in L^1(\Omega \times [0,T], P\otimes\lambda)$. Hence, $\partial F(Y(\cdot))Z(\cdot)$ is P-a.s. Bochner integrable.

(ii) The estimate is an easy calculation. Then by Lebesgue's dominated convergence theorem the convergence to 0 follows (see also [FrKn 02, Proof of Theorem 4.3.(i), Step 1, (b), (1.), p.97]).

(iii)

$$\sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\, ds\|_{L^p}$$

can be estimated by

$$\begin{aligned}M_T T^{\frac{p-1}{p}} \Big[&C T^{\frac{1}{p}} \|Z_n - Z\|_{\mathcal{H}^p}\\ &+ \big(E\big[\int_0^T \|\partial F(Y_n(s))Z(s) - \partial F(Y(s))Z(s)\|^p\, ds\big]\big)^{\frac{1}{p}}\Big].\end{aligned}$$

$\|Z_n - Z\|_{\mathcal{H}^p} \to 0$ as $n \to \infty$ by assumption. The second summand converges to 0 as $n \to \infty$, by the continuity of ∂F, lemma 5.4 and the fact that

$$\|\partial F(Y_n(s))Z(s) - \partial F(Y(s))Z(s)\|^p \leq 2^p C^p \|Z\|^p \in L^1(\Omega \times [0,T], \mathcal{P}_T, P \times \lambda)$$

(see also [FrKn 02, Proof of Theorem 4.3.(i), Step 2, (b), (1.), p.100/101]).

$\square$

Bibliography

[AlWuZh 97] Albeverio, S., Wu, J.-L., Zhang, T.-S.: Parabolic SPDEs driven by Poisson white noise. Stochastic Processes and their Applications 74 (1998), p.21-36

[ApWu 00] Applebaum, D., Wu, J.-L.: Stochastic partial differential equations driven by Lévy space-time white noise. Random Operators and Stochastic Equations 8 (2000), p.245-261

[ApTa 01] Applebaum, D., Tang, F.: Stochastic flows of diffeomorphisms on manifolds driven by infinite-dimensional semimartingales with jumps. Stochastic Processes and their Applications 92 (2001), p.219-236

[Ap 04] Applebaum, D.: Lévy processes - from probability to finance and quantum groups. Notices of the AMS 51 (2004) number 11, p.1336-1342

[Ap 05] Applebaum, D.: Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space, to appear in Seminaire de Probabilites

[ChTe 78] Chow, Y., Teicher, H.: Probability Theory. New York-Heidelberg-Berlin: Springer Verlag 1978

[Co 80] Cohn, D.L.: Measure Theory. Boston: Birkhäuser 1980

[DaPrZa 92] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge: Cambridge University Press 1992

[DaPrZa 96] Da Prato, G., Zabczyk, J.: Ergodicity for Infinite Dimensional Systems. Cambridge: Cambridge University Press 1996 [deBa 81] de Barra,G.: Measure Theory and Integration. Chichester: Ellis Horwood Limited . Distributed by: New York-Chichester-Brisbane-Toronto: John Wiley & Sons 1977

[DeMe 82] Dellacherie, C., Meyer, P.-A.: Probability and Potential. Theory of Martingales. Amsterdam-New York-Oxford: North-Holland Publishing Company 1982

[Di 99a] Ditlevsen, P.D.: Anomalous jumping in a double-well potential. Physical Review E, 60 (1) (1999), p.172-179

[Di 99b] Ditlevsen, P.D.: Observation of α-stable noise induced millenial climate changes from an ice record. Geophysical Research Letters, 26 (10) (1999), p.1441-1444

[EbRa 99] Eberlein, E., Raible, S.: Term structure models driven by general Lévy processes. Math. Finance, 9 (1) (1999), p.31-53

[EtKu 86] Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. New York-Chichester-Brisbane-Toronto-Singapore: John Wiley & Sons 1986

[FrKn 02] Frieler, K., Knoche, C.: Solutions of stochastic differential equations in infinite dimensional Hilbert spaces and their dependence on initial data. Diplomathesis, Bielefeld 2001, Preprint in der Ergänzungsreihe des Forschungszentrums BiBoS (Bielefeld-Bonn-Stochastics), No.03-06-119

[Ha 05] Hausenblas, E.: Existence, uniqueness and regularity of parabolic spdes driven by Poisson random measure. To appear in Electronic Journal of Probability. Accepted on September, 16, 2005

[HaTh 94] Hackenbroch, W., Thalmaier, A.: Stochastische Analysis: eine Einführung in die Theorie der stetigen Semimartingale. Stuttgart : Teubner 1994

[IkWa 81] Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Amsterdam-Oxford-New York: North-Holland Publishing Company 1981

[ImPa 04] Imkeller, P., Pavlyukevich, I.: First exit times of solutions of non-linear stochastic differential equations driven by symmetric Lévy processes with α-stable components. arXiv: math.PR/0409246 v1, 15. September 2004

[KaWo 84] Kallianpur, G., Wolpert, R.: Infinite dimensional stochastic differential equation models for spatially distributed neurons. Applied Mathematics and Optimization 12 (1984), p.125-172

[Kn 03] Knoche, C.: Stochastic integrals and stochastic differential equations with respect to compensated Poisson random measures in infinite dimensional Hilbert spaces. Preprint des Forschungszentrums BiBoS (Bielefeld-Bonn-Stochastics), no. 03-06-119

[Kn 04] Knoche, C.: SPDE's in infinite dimensions with Poisson noise. Comptes Rendus Mathématique. Académie des Sciences. Paris, Serie I 339 (2004), p.647-652

[MaRu 03] Mandrekar, V., Rüdiger, B.: Existence and uniqueness of path wise solutions for stochastic integral equations driven by non Gaussian noise on separable Banach spaces. Preprint No. 61 des SFB 611, Fakultät für Mathematik, Universität Bonn. (2003)

[Me 82] Metivier, M.: Semimartingales. A course on Stochastic Processes.Berlin-New York: W. de Gruyter 1982

[Mu 98] Müller, C.: The heat equation with Lévy noise. Stochastic Processes and their Applications 74 (1998), p.67-82

[Pa 83] Pazy, A.: Semigroups of Linear Operators. Berlin-Heidelberg-New York-Tokyo: Springer Verlag 1983

[Pe 95] Peszat, S.: Existence and uniqueness of the solution for stochastic equations on Banach spaces. Stochastics and Stochastics Reports 55 (1995), p.167-193

[Pr 90] Protter, P.: Stochastic Integration and Differential Equations : a New Approach. Berlin [u.a.] : Springer Verlag 1990

[Ra 00] Raible, S.: Lévy processes in finance: Theory, numerics and empirical facts. PhD Thesis. Fakultät für Mathematik, Universität Freiburg 2000

[Ra 59] Rall, W.: Branching dendritic trees and motoneuron resistivity. Experimental Neurology 1 (1959) , p.491-527

[Ru 04] Rüdiger, B.: Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. Stochastics and Stochastics Reports 76 (2004), no. 3, p.213-242

[St 93] Strook, D.W.: Probability Theory, an Analytic Approach. Cambridge: Cambridge University Press 1993

[Wa 81] Walsh, J.B.: A stochastic model of neural response. Adv. Appl. Prob. 13 (1981), p.231-281

[Wa 86] Walsh, J.B.: An introduction to stochastic partial differential equations. In: Ecole dÉté de Probabilité de St. Flour XIV, Lecture Notes in Math. 1180, p.266-439. Berlin: Springer Verlag 1986

Symbols