UNITARIZING MEASURES FOR REPRESENTATIONS OF THE VIRASORO ALGEBRA, ACCORDING TO KIRILLOV AND MALLIAVIN : STATE OF THE PROBLEM

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This note essentially reproduces the contents of a talk at Bielefeld University on July 27th, 2004. Most of the material comes from [1], [2] and [5] ; I have tried to uniformize notation, sign conventions, etc. , and I have slightly amended the definition of ρ given in [1].

1.Preliminaries

Let $\mathcal{D}iff(S^1)$ denote the group of $\mathcal{C}^\infty$, orientation–preserving diffeomorphisms of the circle S^1.Its Lie algebra $diff(S^1)$ can be naturally identified with the set of $\mathcal{C}^\infty$ vector fields on S^1, *i.e.* :

$$diff(S^1) = \{\phi(\theta)\frac{d}{d\theta}|\phi : \mathbf{R} \to \mathbf{R}\ ,\mathcal{C}^\infty, 2\pi - \text{periodic}\}\ .$$

We shall often identify, without further warning, the function ϕ and the vector field $\phi(\theta)\frac{d}{d\theta}$. A topological basis (for the obvious Fréchet space topology) of $diff(S^1)$ is given by the $(f_k)_{k\geq 0}$ and the $(g_k)_{k\geq 1}$, where :

$$f_k =_{def} \cos(k\theta)\frac{d}{d\theta}$$

and

$$g_k =_{def} \sin(k\theta)\frac{d}{d\theta}\ .$$

Let $diff_{\mathbf{C}}(S^1)\ =_{def}\ diff(S^1) \otimes_{\mathbf{R}} \mathbf{C}$ denote the complexified Lie algebra of $diff(S^1)$; it is now clear that a topological basis of $diff_{\mathbf{C}}(S^1)$ is given by the $(e_k)_{k\in\mathbf{Z}}$, where :

$$e_k =_{def} e^{ik\theta}\frac{d}{d\theta}\ .$$

One has the commutation relations :

$$[e_k, e_{k'}] = i(k^{'} - k)e_{k+k'} \ .$$

The Lie algebra $diff(S^1)$ contains :

$$\mathcal{A} =_{def} Vect_{\mathbf{C}}(e_k)_{k \in \mathbf{Z}}$$

as a Lie subalgebra, dense for the natural Fréchet space topology.

Setting $L_k = -ie_k$, one finds that :

$$[L_k, L_{k'}] = (k^{'} - k)L_{k+k'} \ ,$$

whence

$$\mathcal{A} \simeq Der_{\mathbf{C}}(\mathbf{C}[t, t^{-1}])$$

(L_k corresponding, through this isomorphism, to $t^{k+1}\dfrac{d}{dt}$, which is equivalent to setting $t = e^{i\theta}$).

The algebra $\mathcal{V}ir_{c,h}$ is defined by :

$$\mathcal{V}ir_{c,h} =_{def} diff_{\mathbf{C}}(S^1) \oplus \mathbf{C}\kappa$$

as a vector space, with the following bracket :

$$\forall (f, g) \in diff_{\mathbf{C}}(S^1)^2$$

$$[\kappa, f] = 0,$$

and :

$$[f, g]_{\mathcal{V}ir_{c,h}} = [f, g] + \omega_{c,h}(f, g)\kappa \ ,$$

where

$$\omega_{c,h}(f, g) =_{def} \int_0^{2\pi} ((2h - \frac{c}{12})f^{'}(\theta) - \frac{c}{12}f^{'''}(\theta))g(\theta)\frac{d\theta}{2\pi} \ .$$

The so–called *Gelfand–Fuks cocycle* is $\omega_{0,1}$.

An easy computation yields :

Proposition 1.2.

$$\forall (m, n) \in \mathbf{Z}^2 \ \omega_{c,h}(e_m, e_n) = i[2hm + c\frac{m^3 - m}{12}]\delta_{m,-n} \ .$$

It is easy to deduce from [4],chapter 7,exercises 7.1 and 7.13, that the $\omega_{c,h}$ are exactly the continuous cocycles α on $diff_{\mathbf{C}}(S^1)$ such that

$$\forall f \in diff(S^1) \ \alpha(e_0, f) = 0 \ .$$

We shall denote by $\mathcal{V}ir^{\mathbf{R}}_{c,h}$ the obvious “real”Lie subalgebra of $\mathcal{V}ir_{c,h}$, *i.e.* :

$$\mathcal{V}ir^{\mathbf{R}}_{c,h} =_{def} diff(S^1) \oplus \mathbf{R}\kappa.$$

¿From now on, we shall assume $c > 0$ and $h \geq 0$. Let

$$diff_0(S^1) =_{def} \{\phi \in diff(S^1) | \int_0^{2\pi} \phi(\theta) d\theta = 0\} .$$

On $diff_0(S^1)$, one defines a complex structure in the usual way : for

$$\phi(\theta) = \sum_{k=1}^{+\infty} (a_k \cos(k\theta) + b_k \sin(k\theta))$$

(a_k, b_k rapidly decreasing), we set :

$$J\phi(\theta) =_{def} \sum_{k=1}^{+\infty} (-a_k \sin(k\theta) + b_k \cos(k\theta)) .$$

Lemma 1.3([2],p.630).

$$\forall f \in diff_0(S^1) \ \omega_{c,h}(f, Jf) = \frac{1}{2} \sum_{k=1}^{+\infty} [2hk + \frac{c}{12}(k^3 - k)](a_k^2 + b_k^2) \geq 0 .$$

Proof. Let us set, as usual, $c_k(f) = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{d\theta}{2\pi}$; then $f(\theta) = \sum_{k \in \mathbf{Z}} c_k e^{ik\theta}$ and

$$Jf(\theta) = \sum_{k \geq 1} (ic_k e^{ik\theta} - ic_{-k} e^{-ik\theta})$$

whence (by Proposition 1.2):

$$\begin{aligned} \omega_{c,h}(f, Jf) &= \sum_{k \geq 1} (c_k(-ic_{-k})(i(2h - \frac{c}{12})k + \frac{ic}{12}k^3) + \sum_{k \geq 1} (c_{-k}(-ic_k)(i(2h - \frac{c}{12})(-k) - \frac{ic}{12}k^3) \\ &= 2 \sum_{k \geq 1} (c_k c_{-k})(2hk + \frac{c}{12}(k^3 - k)) . \end{aligned}$$

Taking into account the obvious relations: $\forall k \geq 1$, $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$, the result follows. □

2.Kirillov's construction

(Kirillov , [5] ; Airault and Malliavin , [2])

Let $\mathcal{M}$ denote the set of $\mathcal{C}^\infty$ functions $f : \bar{D} \to \mathbf{C}$, injesctive, holomorphic on D, with $f(0) = 0$, $f'(0) = 1$, and $\forall z \in \bar{D}\ f'(z) \neq 0$. Each $f \in \mathcal{M}$ can be written as :

$$\forall z \in D \ f(z) = z(1 + \sum_{n=1}^{+\infty} c_n z^n) ,$$

whence an imbedding :

$$\mathcal{M} \hookrightarrow \mathbf{C}^{N^*}$$
$$f \mapsto (c_1, c_2, ...)$$

(in fact, by De Branges'solution of Bieberbach's conjecture, one has $|c_n| \leq n$, thus $\mathcal{M}$ is identified with an open subset of $\Pi_{n\geq 1}\mathcal{B}_{\mathbf{C}}(0, n+1)$; one therefore obtains a structure of (contractible) manifold on $\mathcal{M}$).

Let $D = \mathcal{D}(0,1)$ denote the unit disk. For $f \in \mathcal{M}$, $\Gamma = f(S^1) = f(\partial D)$ is a Jordan curve, therefore one has a decomposition into connected components :

$$\mathbf{C} \cup \{\infty\} = \Gamma^+ \cup \Gamma^-$$

with $0 \in \Gamma^+$ and $\infty \in \Gamma^-$. By a combination of Riemann's representation Theorem and Caratheodory's Theorem, there exists an holomorphic mapping

$$\phi_f : (\mathbf{C} \cup \{\infty\} \setminus D) \to \overline{\Gamma^-} = \Gamma^- \cup \Gamma$$

such that $\phi_f(\infty) = \infty$. Let us then define g_f by :

$$g_f : S^1 \to S^1$$
$$e^{i\theta} \mapsto f^{-1}(\phi_f(e^{i\theta})) \ .$$

Then $g \in Diff(S^1)$, and g_f is well–defined up to multiplication on the right by an holomorphic automorphism of $\mathbf{C} \setminus \bar{D}$ stabilizing ∞, *i.e.* a rotation, whence a mapping :

$$\mathcal{K} : \mathcal{M} \to Diff(S^1)/S^1$$

.

Theorem 2.1(Kirillov,[5]**,p.736).** *$\mathcal{K}$ is a bijection.*

Therefore, by transport of structure, $Diff(S^1)/S^1$ acquires a structure of contractible complex manifold. Using J and $\omega_{c,h}$, this manifold can be equipped with a Kählerian structure(see [2]).

Definition 2.2(Kirillov action). *For $v = \phi(\theta)\dfrac{d}{d\theta} \in diff(S^1)$ and $f \in \mathcal{M}$, let us write $w(e^{i\theta}) = \phi(\theta)$, and define $K_v(f)$ by :*

$$K_v(f)(z) = \frac{f(z)^2}{2\pi}\int_{S^1}(\frac{tf^{'}(t)}{f(t)})^2\frac{w(t)}{f(t)-f(z)}\frac{dt}{t} \ .$$

Definition 2.3. *For $n \in \mathbf{Z}$, let*

$$L_n =_{def} -iK_{e_n} \ .$$

For nonnegative n, it is very easy to compute L_n :

Proposition 2.4.

(1) *For* $n \geq 1$,

$$L_n = \frac{\partial}{\partial c_n} + \sum_{k=1}^{+\infty}(k+1)c_k \frac{\partial}{\partial c_{n+k}} \ ;$$

(2)

$$L_0 = \sum_{n \geq 1} n c_n \frac{\partial}{\partial c_n} \ .$$

Proof.

(1) In this case, the expression for K_v becomes

$$\begin{aligned} K_{e_n}(f)(z) &= \frac{f(z)^2}{2\pi}\int_{S^1}(\frac{tf'(t)}{f(t)})^2\frac{t^n}{f(t)-f(z)}\frac{dt}{t} \\ &= \frac{f(z)^2}{2\pi}\int_{S^1}(\frac{tf'(t)}{f(t)})^2\frac{t^{n-1}}{f(t)-f(z)}dt \\ &= \frac{f(z)^2}{2\pi}2i\pi Res_z[(\frac{tf'(t)}{f(t)})^2\frac{t^{n-1}}{f(t)-f(z)}](\text{by Cauchy's formula}) \\ &= \frac{f(z)^2}{2\pi}2i\pi(\frac{zf'(z)}{f(z)})^2\frac{z^{n-1}}{f'(z)} \\ &= iz^{n+1}f'(z) \\ &= iz^{n+1} + i\sum_{k=1}^{+\infty}(k+1)c_k z^{k+n+1} \end{aligned}$$

therefore

$$L_n(f)(z) = z^{n+1} + \sum_{k=1}^{+\infty}(k+1)c_k z^{k+n+1} \ ,$$

whence the result.

(2) The computation is similar, taking into account the pole at 0, and yields

$$L_0(f)(z) = zf'(z) - f(z) \ ,$$

whence the result.

□

Lemma 2.5. *One has the commutation relations :*

$$\forall (m,n) \in \mathbf{Z}^2 \ [L_m, L_n] = (m-n)L_{m+n} \ . \qquad (*)$$

Proof. [2],p. 655. □

3.The Neretin polynomials and the representation ρ

Let $\gamma_k =_{def} \frac{c}{12}(k^3-k)$, and $P_k = 0$ for $k<0$.

Theorem 3.1(Kirillov–Neretin). *There exists a unique sequence $(P_n)_{n\geq 0}$ of polynomials in the $(c_i)_{i\geq 1}$ such that :*

(1) *P_k depends only upon c_1,...,c_k;*
(2) *$P_0 = h$;*
(3)
$$\forall k\geq 1\ \forall n\geq 1\ L_k(P_n) = (n+k)P_{n-k} + \gamma_k\delta_{k,n}\ ;$$
(4) $\forall n\geq 1\ P_n(0) = 0$.

Proof. Given $P_0, ..., P_n (n\geq 0)$, the relation (3) (with $n+1$ in place of n) is trivially satisfied for any polynomial P_{n+1} in $c_1, ..., c_{n+1}$ and any $k > n+1$; for $1\leq k\leq n+1$, the relations determine, by descending induction on k, the $\frac{\partial P_{n+1}}{\partial c_k}$ in a unique way, therefore they determine P_{n+1} up to a constant ; (4) for $n+1$ now determines a unique P_{n+1}. □

The first few terms of the sequence are easily computed :

$$P_0 = h\ ,$$

$$P_1 = 2hc_1\ ,$$

$$P_2 = (4h+\frac{c}{2})c_2 - (h+\frac{c}{2})c_1^2\ .$$

If each c_k is given the weight k, it is easily seen that P_k is homogeneous of weight k.

Let us remind the reader of the definition of the *Schwarzian derivative* of an holomorphic function f :

$$S(f)(z) =_{def} \frac{f^{'''}(z)}{f'(z)} - 3(\frac{f^{''}(z)}{f'(z)})^2\ .$$

The following result could have been used as definition of the polynomials P_k :

Proposition 3.2([5],p.742,Theorem).

$$\forall f\in\mathcal{M}\ \sum_{n=0}^{+\infty} P_n(c_1, ..., c_n)z^n = h(\frac{zf'(z)}{f(z)})^2 + \frac{cz^2}{12}S(f)(z)\ .$$

Proposition 3.3.

$$\forall k\geq 0\ \forall p\geq 0\ \ L_{-k}(P_p) - L_{-p}(P_k) = (p-k)P_{p+k}\ ;$$

in particular, the formula of Theorem 3.1(3) remains valid for $k = 0$.

Proof. [2], p.663. □

Let

$$Q_k =_{def} \begin{cases} P_k \text{ for } k\neq 0 \\ 0 \text{ for } k = 0\ . \end{cases}$$

Theorem 3.4. *Let us set , for each $k \in \mathbf{Z}$:*

$$\rho(e_k) = -i(L_k + Q_{-k}.)$$

and

$$\rho(\kappa) = i \; Id \; .$$

Then ρ defines a representation of the Lie algebra $\mathcal{V}ir_{c,h}$ into the Lie algebra of differential operators on $\mathcal{M}$.

Proof. As, obviously, $[\rho(e_k), \rho(\kappa)] = 0$ is enough to prove that

$$[\rho(e_m), \rho(e_n)] = \rho([e_m, e_n]) \; .$$

Taking Proposition 1.2 into account, this is easily reduced to checking the relation :

$$L_n(Q_{-m}) - L_m(Q_{-n}) = (n-m)Q_{-m-n} - [2hm + \gamma_m]\delta_{m,-n} \; .$$

But , for $m \geq 0$ and $n \geq 0$, that relation is trivially satisfied ; for $m = 0$ and $n < 0$, as well as for $n = 0$ and $m < 0$, it follows from the relation

$$\forall n \geq 1 \; L_0(P_n) = nP_n \; ;$$

in the case $m < 0$ and $n < 0$, setting $p = -m$ and $k = -n$, we have to prove that :

$$\forall p \geq 1 \; \forall k \geq 1 \;\; L_{-k}(P_p) - L_{-p}(P_k) = (p-k)P_{p+k} \; ,$$

but both these facts follow from Proposition 3.3 .

There remains the case $m \leq -1$ and $n \geq 1$ (or the other way round) ; in this case, we need to prove, setting $k = -m \geq 1$, that :

$$\forall k \geq 1 \; \forall n \geq 1 \; L_n(P_k) = (n+k)Q_{k-n} + (2hk + \gamma_k)\delta_{k,n}$$

i.e.

$$\forall k \geq 1 \; \forall n \geq 1 \; L_n(P_k) \begin{cases} = (n+k)P_{k-n} \text{ if } n \neq k \\ = 2hk + \gamma_k \text{ if } n = k \; . \end{cases}$$

As $P_0 = h$, this follows from Theorem 3.1(3). ☐

4. Unitarizing measure(s)?

Definition 4.1. *A Borel probability measure μ on $\mathcal{M}$ is said to be unitarizing for ρ if and only if*

$$\forall v \in \mathcal{V}^{\mathbf{R}}_{c,h} \; \rho(v)^* = -\rho(v)$$

on $\mathcal{H}L^2_\mu(\mathcal{M})$.

Lemma 4.2([1],Theorem 1,p.433). *If μ exists, then, setting $Z_k = L_k - \overline{L_{-k}}(k \geq 0)$, one has :*

$$\forall F \in \mathcal{C}^\infty(\mathcal{M}) \int_{\mathcal{M}} Z_k(F) d\mu = -\int_{\mathcal{M}} F \beta_k d\mu \ , \tag{4.2.1}$$

where

$$\beta_k = \begin{cases} -\bar{P}_k \text{ if } k \geq 1 \ , \\ 0 \text{ if } k = 0 \ . \end{cases}$$

Proof. From the definition follows that :

$$\forall v \in \mathcal{V}ir_{c,h} \ \rho(v)^* = -\rho(\bar{v}) \ .$$

By a density argument, one may assume that $F = \varphi\bar{\psi}$, with φ and ψ holomorphic; then one has :

$$\begin{aligned}
\int_{\mathcal{M}} Z_k(F) d\mu &= \int_{\mathcal{M}} L_k(\varphi\bar{\psi}) d\mu - \int_{\mathcal{M}} \overline{L_{-k}}(\varphi\bar{\psi}) d\mu \\
&= \int_{\mathcal{M}} L_k(\varphi\bar{\psi}) d\mu - \overline{\int_{\mathcal{M}} L_{-k}(\bar{\varphi}\psi) d\mu} \\
&= \int_{\mathcal{M}} (L_k(\varphi)\bar{\psi} + \varphi L_k(\bar{\psi})) d\mu - \overline{\int_{\mathcal{M}} (L_{-k}(\bar{\varphi})\psi + \bar{\varphi} L_{-k}(\psi)) d\mu} \\
&= \int_{\mathcal{M}} (L_k(\varphi)\bar{\psi} + \varphi L_k(\bar{\psi})) d\mu - \overline{\int_{\mathcal{M}} (L_{-k}(\bar{\varphi})\psi + \bar{\varphi} L_{-k}(\psi)) d\mu} \\
&= \int_{\mathcal{M}} L_k(\varphi)\bar{\psi} d\mu - \overline{\int_{\mathcal{M}} \bar{\varphi} L_{-k}(\psi) d\mu}
\end{aligned}$$

(because φ is holomorphic and ψ anti–holomorphic)

$$\begin{aligned}
&= i\int_{\mathcal{M}} (\rho(e_k)(\varphi) - Q_{-k}\varphi)\bar{\psi} d\mu - \overline{\int_{\mathcal{M}} \bar{\varphi}(i\rho(e_{-k})(\psi) - Q_k\psi) d\mu} \\
&= i(\rho(e_k)(\varphi), \psi) + \int_{\mathcal{M}} \varphi(\bar{Q_k} - Q_{-k})\bar{\psi} d\mu + i\overline{(\rho(\overline{e_k})(\psi), \varphi)} \\
&= i(\rho(e_k)(\varphi), \psi) + \int_{\mathcal{M}} \varphi(\bar{Q_k} - Q_{-k})\bar{\psi} d\mu + i(\varphi, \rho(\bar{e_k})(\psi)) \\
&= \int_{\mathcal{M}} \varphi(\bar{Q_k} - Q_{-k}) F d\mu
\end{aligned}$$

by the hypothesis on μ .

Whence the result with :

$$\beta_k = Q_{-k} - \overline{Q_k} = \begin{cases} -\overline{P_k} \text{ for } k \geq 1 \\ 0 \text{ for } k = 0 \ . \end{cases}$$

□

Theorem 4.3([1],Theorem 3 and Corollary 4, p.234).

(1) *If μ exists then the sequence $1, P_1, P_2, \dots$ is a sequence of orthogonal polynomials in $L^2(\mu)$; more precisely :*

$$(P_m, P_k)_{L^2(\mu)} = \begin{cases} 0 \text{ if } m \neq k \\ \gamma_k + 2hk \text{ if } m = k \geq 1 \\ h^2 \text{ if } m = k = 0 \end{cases}$$

(2) *If $h = 0$ then there is no unitarizing measure on $\mathcal{M}$ for ρ.*

Proof. (1) Let us set, for each $k \geq 0$, and $H_k = Z_k^2 + \beta_k Z_k$; it follows from Lemma 4.2 applied to $Z_k(F)$ that, for each $k \geq 0$, one has :

$$\forall F \in \mathcal{C}^\infty(\mathcal{M}) \ \forall k \geq 0 \int_{\mathcal{M}} H_k(F) d\mu = 0 \ . \tag{4.3.1}$$

But it follows from the definition of the Neretin polynomials (Theorem 3.1(3)) and from the last remark in Proposition 3.3 that:

$$\forall k \geq 0 \ \forall n \geq 1$$

$$\begin{aligned} H_k(P_n) &= L_k((n+k)P_{n-k} + \gamma_k \delta_{k,n}) + \beta_k((n+k)P_{n-k} + \gamma_k \delta_{k,n}) \\ &= (n+k)nP_{n-2k} + (n+k)\gamma_k \delta_{k,n-k} + (n+k)\beta_k P_{n-k} + \beta_k \gamma_k \delta_{k,n} \ . \end{aligned} \tag{4.3.2}$$

By (4.3.1) one has

$$\forall k \geq 0 \ \forall n \geq 1 \int_{\mathcal{M}} H_k(P_n) d\mu = 0 \ . \tag{4.3.3}$$

Applying (4.3.2) for $k = 0$ and $n \geq 1$, one finds that :

$$\forall n \geq 1 \ H_0(P_n) = n^2 P_n \ ,$$

whence (4.3.3) yields that :

$$\forall n \geq 1 \ \int_{\mathcal{M}} P_n d\mu = 0 \ . \tag{4.3.4}$$

From Lemma 4.2 applied to $F = 1$ follows :

$$\forall k \geq 0 \ \int_{\mathcal{M}} \beta_k d\mu = 0 \ . \tag{4.3.5}$$

Taking now $k \geq 1$, $m \geq 1$ and $n = m + k$, (4.3.2) and (4.3.3) together yield :

$$\int_{\mathcal{M}} [(2k+m)(k+m)P_{m-k} + (m+2k)\gamma_k \delta_{m,k} + (m+2k)\beta_k P_m + \beta_k \gamma_k \delta_{m,0}] d\mu = 0 \ ;$$

from the fact that

$$\int_{\mathcal{M}} P_n d\mu = \begin{cases} 0 \text{ for } n \geq 1 \text{ (4.3.4)} \\ h \text{ for } n = 0 \\ \ 0 \text{ for } n < 0 \text{ (by definition)} \end{cases}$$

and from (4.3.5), we get :

$$\int_{\mathcal{M}} \beta_k P_m d\mu \begin{cases} = 0 \text{ if } m \neq k \\ = -\gamma_k - 2kh \text{ if } m = k \neq 0 \ . \end{cases}$$

Remembering that $\beta_k = -\bar{P}_k$ for $k \geq 1$, the result follows.
(2) Let us remind the reader that $P_1 = 2hc_1$. Clearly,

$$Z_1(c_1) = (L_1 - \overline{L_{-1}})(c_1) = L_1(c_1) = 1 \ ,$$

whence :

$$\begin{aligned} 1 &= \int_{\mathcal{M}} d\mu \\ &= \int_{\mathcal{M}} Z_1(c_1) d\mu \\ &= -\int_{\mathcal{M}} c_1 \beta_1 d\mu \ \text{ (by Lemma 4.2)} \\ &= \int_{\mathcal{M}} c_1 \bar{P}_1 d\mu \\ &= 2h \int_{\mathcal{M}} c_1 \bar{c_1} d\mu \ , \end{aligned}$$

which is impossible for $h = 0$. A more geometrical proof of this nonexistence result had previously been given in [3], Theorem 2.2, p.625. □

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