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This note essentially reproduces the contents of a talk at Bielefeld University on July 27th, 2004. Most of the material comes from [1], [2] and [5] ; I have tried to uniformize notation, sign conventions, etc. , and I have slightly amended the definition of ρ given in [1].

1. Preliminaries

Let $Diff(S^1)$ denote the group of $C^\infty$, orientation–preserving diffeomorphisms of the circle $S^1$. Its Lie algebra $diff(S^1)$ can be naturally identified with the set of $C^\infty$ vector fields on $S^1$, i.e. :

$$diff(S^1) = \{ \phi(\theta) \frac{d}{d\theta} | \phi : \mathbb{R} \to \mathbb{R} , C^\infty, 2\pi - \text{periodic} \} .$$

We shall often identify, without further warning, the function $\phi$ and the vector field $\phi(\theta) \frac{d}{d\theta}$. A topological basis (for the obvious Fréchet space topology) of $diff(S^1)$ is given by the $(f_k)_{k \geq 0}$ and the $(g_k)_{k \geq 1}$, where :

$$f_k =_{def} \cos(k\theta) \frac{d}{d\theta}$$

and

$$g_k =_{def} \sin(k\theta) \frac{d}{d\theta} .$$

Let $diff_\mathbb{C}(S^1) =_{def} diff(S^1) \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexified Lie algebra of $diff(S^1)$; it is now clear that a topological basis of $diff_\mathbb{C}(S^1)$ is given by the $(e_k)_{k \in \mathbb{Z}}$, where :

$$e_k =_{def} e^{ik\theta} \frac{d}{d\theta} .$$
One has the commutation relations:

\[ [e_k, e_{k'}] = i(k' - k)e_{k+k'} \]

The Lie algebra \( \text{diff}(S^1) \) contains:

\[ A = \text{def} \, \text{Vect}_\mathbb{C}(e_k)_{k \in \mathbb{Z}} \]

as a Lie subalgebra, dense for the natural Fréchet space topology.

Setting \( L_k = -ie_k \), one finds that:

\[ [L_k, L_{k'}] = (k' - k)L_{k+k'} \]

whence

\[ A \simeq \text{Der}_\mathbb{C}(\mathbb{C}[t, t^{-1}]) \]

\((L_k \text{ corresponding, through this isomorphism, to } t^{k+1} \frac{d}{dt}, \text{ which is equivalent to setting } t = e^{i\theta})\).

The algebra \( \text{Vir}_{c,h} \) is defined by:

\[ \text{Vir}_{c,h} = \text{def} \, \text{diff}_\mathbb{C}(S^1) \oplus \mathbb{C}\kappa \]

as a vector space, with the following bracket:

\[ \forall (f, g) \in \text{diff}_\mathbb{C}(S^1)^2 \]

\[ [\kappa, f] = 0, \]

and:

\[ [f, g]_{\text{Vir}_{c,h}} = [f, g] + \omega_{c,h}(f, g)\kappa, \]

where

\[ \omega_{c,h}(f, g) = \text{def} \int_0^{2\pi} \left( (2h - \frac{c}{12})f'(\theta) - \frac{c}{12} f'''(\theta) \right) g(\theta) \frac{d\theta}{2\pi}. \]

The so-called \textit{Gelfand–Fuks cocycle} is \( \omega_{0,1} \).

An easy computation yields:

\textbf{Proposition 1.2.}

\[ \forall (m, n) \in \mathbb{Z}^2 \omega_{c,h}(e_m, e_n) = i[2hm + \frac{m^3 - m}{12}]\delta_{m,-n}. \]

It is easy to deduce from [4] chapter 7, exercises 7.1 and 7.13, that the \( \omega_{c,h} \) are exactly the continuous cocycles \( \alpha \) on \( \text{diff}_\mathbb{C}(S^1) \) such that

\[ \forall f \in \text{diff}(S^1) \alpha(e_0, f) = 0. \]

We shall denote by \( \text{Vir}_{c,h}^\mathbb{R} \) the obvious “real” Lie subalgebra of \( \text{Vir}_{c,h} \), \textit{i.e.}:

\[ \text{Vir}_{c,h}^\mathbb{R} = \text{def} \, \text{diff}(S^1) \oplus \mathbb{R}\kappa. \]
From now on, we shall assume $c > 0$ and $h \geq 0$. Let

$$\text{diff}_0(S^1) = \{ \phi \in \text{diff}(S^1) | \int_0^{2\pi} \phi(0)d\theta = 0 \}. $$

On $\text{diff}_0(S^1)$, one defines a complex structure in the usual way: for

$$\phi(\theta) = \sum_{k=1}^{+\infty} (a_k \cos(k\theta) + b_k \sin(k\theta))$$

($a_k, b_k$ rapidly decreasing), we set:

$$J\phi(\theta) = \sum_{k=1}^{+\infty} (-a_k \sin(k\theta) + b_k \cos(k\theta)).$$

Lemma 1.3([2], p.630).

$$\forall f \in \text{diff}_0(S^1), \omega_{c,h}(f,Jf) = \frac{1}{2} \sum_{k=1}^{+\infty} [2hk + \frac{c}{12}(k^3 - k)](a_k^2 + b_k^2) \geq 0.$$ 

**Proof.** Let us set, as usual, $c_k(f) = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{d\theta}{2\pi}$; then $f(\theta) = \sum_{k \in \mathbb{Z}} c_ke^{ik\theta}$ and

$$Jf(\theta) = \sum_{k \geq 1} (ic_k e^{ik\theta} - ic_{-k} e^{-ik\theta})$$

whence (by Proposition 1.2):

$$\omega_{c,h}(f,Jf) = \sum_{k \geq 1} (c_k(-ic_{-k})(i(2h - \frac{c}{12})k + \frac{ic}{12} k^3) + \sum_{k \geq 1} (c_{-k}(-ic_k)(i(2h - \frac{c}{12})(-k) - \frac{ic}{12} k^3)$$

$$= 2 \sum_{k \geq 1} (c_k c_{-k})(2hk + \frac{c}{12}(k^3 - k)).$$

Taking into account the obvious relations: $\forall k \geq 1, a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$, the result follows.

2. **Kirillov’s construction**

(Kirillov, [5]; Airault and Malliavin, [2])

Let $\mathcal{M}$ denote the set of $C^\infty$ functions $f : \tilde{D} \to \mathbb{C}$, injective, holomorphic on $D$, with $f(0) = 0$, $f'(0) = 1$, and $\forall z \in \tilde{D} f'(z) \neq 0$. Each $f \in \mathcal{M}$ can be written as:

$$\forall z \in D, f(z) = z(1 + \sum_{n=1}^{+\infty} c_n z^n),$$

whence an imbedding:
\[ \mathcal{M} \hookrightarrow \mathbb{C}^{N^*} \]
\[ f \mapsto (c_1, c_2, \ldots) \]

(in fact, by De Branges’ solution of Bieberbach’s conjecture, one has \(|c_n| \leq n\), thus \(\mathcal{M}\) is identified with an open subset of \(\Pi_{n \geq 1} \mathbb{B}_\mathbb{C}(0, n + 1)\); one therefore obtains a structure of (contractible) manifold on \(\mathcal{M}\)).

Let \(D = D(0, 1)\) denote the unit disk. For \(f \in \mathcal{M}\), \(\Gamma = f(S^1) = f(\partial D)\) is a Jordan curve, therefore one has a decomposition into connected components:

\[ \mathbb{C} \cup \{\infty\} = \Gamma^+ \cup \Gamma^- \]

with \(0 \in \Gamma^+\) and \(\infty \in \Gamma^-\). By a combination of Riemann’s representation Theorem and Caratheodory’s Theorem, there exists an holomorphic mapping

\[ \phi_f : \mathbb{C} \cup \{\infty\} \setminus D \to \overline{\Gamma^-} = \Gamma^- \cup \Gamma \]

such that \(\phi_f(\infty) = \infty\). Let us then define \(g_f\) by :

\[ g_f : S^1 \to S^1 \]
\[ e^{i\theta} \mapsto f^{-1}(\phi_f(e^{i\theta})) \]

Then \(g \in Diff(S^1)\), and \(g_f\) is well-defined up to multiplication on the right by an holomorphic automorphism of \(\mathbb{C} \setminus \bar{D}\) stabilizing \(\infty\), i.e. a rotation, whence a mapping :

\[ \mathcal{K} : \mathcal{M} \to Diff(S^1)/S^1 \]

\[ \mathbf{Theorem\ 2.1(Kirillov,[5],p.736).} \ \mathcal{K} \ \text{is a bijection.} \]

Therefore, by transport of structure, \(Diff(S^1)/S^1\) acquires a structure of contractible complex manifold. Using \(J\) and \(\omega_{c,h}\), this manifold can be equipped with a Kählerian structure (see [2]).

\[ \mathbf{Definition\ 2.2(Kirillov\ action).} \ \text{For} \ v = \phi(\theta) \frac{d}{d\theta} \in \text{diff}(S^1) \ \text{and} \ f \in \mathcal{M}, \ \text{let us write} \ w(e^{i\theta}) = \phi(\theta), \ \text{and define} \ K_v(f) \ \text{by :} \]

\[ K_v(f)(z) = \frac{f(z)^2}{2\pi} \int_{S^1} \frac{tf'(t)}{f(t)} \cdot \frac{w(t)}{f(t) - f(z)} dt \]

\[ \mathbf{Definition\ 2.3.} \ \text{For} \ n \in \mathbb{Z}, \ \text{let} \]

\[ L_n =_{def} -iK_{e_n} \]

For nonnegative \(n\), it is very easy to compute \(L_n\) :
Proposition 2.4.

(1) For \( n \geq 1 \),
\[
L_n = \frac{\partial}{\partial c_n} + \sum_{k=1}^{\infty} (k+1) c_k \frac{\partial}{\partial c_{n+k}} ;
\]

(2)
\[
L_0 = \sum_{n \geq 1} n c_n \frac{\partial}{\partial c_n} .
\]

Proof:

(1) In this case, the expression for \( K_v \) becomes
\[
K_{c_n}(f)(z) = \frac{f(z)^2}{2\pi} \int_{S^1} \left( \frac{tf'(t)}{f(t)} \right)^2 \frac{t^n}{f(t) - f(z)} dt
\]
\[
= \frac{f(z)^2}{2\pi} \int_{S^1} \left( \frac{tf'(t)}{f(t)} \right)^2 \frac{t^n}{f(t) - f(z)} dt
\]
\[
= \frac{f(z)^2}{2\pi} 2i\pi \text{Res}_z \left[ \left( \frac{tf'(t)}{f(t)} \right)^2 \frac{t^n}{f(t) - f(z)} \right] \text{(by Cauchy's formula)}
\]
\[
= \frac{f(z)^2}{2\pi} 2i\pi \left( \frac{zf'(z)}{f(z)} \right)^2 \frac{z^{n-1}}{f'(z)}
\]
\[
= iz^{n+1}f'(z)
\]
\[
= iz^{n+1} + i \sum_{k=1}^{\infty} (k+1)c_k z^{k+n+1}
\]

therefore
\[
L_n(f)(z) = z^{n+1} + \sum_{k=1}^{\infty} (k+1)c_k z^{k+n+1},
\]

whence the result.

(2) The computation is similar, taking into account the pole at 0, and yields
\[
L_0(f)(z) = zf'(z) - f(z) ,
\]

whence the result.

\[ \square \]

Lemma 2.5. One has the commutation relations :
\[
\forall (m, n) \in \mathbb{Z}^2 \ [L_m, L_n] = (m - n)L_{m+n} .
\]

(*)

Proof. [2],p. 655. \[ \square \]
3. The Neretin polynomials and the representation $\rho$

Let $\gamma_k = \frac{c}{12}(k^3 - k)$, and $P_k = 0$ for $k < 0$.

**Theorem 3.1 (Kirillov–Neretin).** There exists a unique sequence $(P_n)_{n \geq 0}$ of polynomials in the $(c_i)_{i \geq 1}$ such that:

1. $P_k$ depends only upon $c_1, \ldots, c_k$;
2. $P_0 = h$;
3. $\forall k \geq 1 \forall n \geq 1, L_k(P_n) = (n + k)P_{n-k} + \gamma_k \delta_{k,n}$;
4. $\forall n \geq 1, P_n(0) = 0$.

**Proof.** Given $P_0, \ldots, P_n (n \geq 0)$, the relation (3) (with $n+1$ in place of $n$) is trivially satisfied for any polynomial $P_{n+1}$ in $c_1, \ldots, c_{n+1}$ and any $k > n+1$; for $1 \leq k \leq n+1$, the relations determine, by descending induction on $k$, the $\frac{\partial P_{n+1}}{\partial c_k}$ in a unique way, therefore they determine $P_{n+1}$ up to a constant; (4) for $n+1$ now determines a unique $P_{n+1}$. □

The first few terms of the sequence are easily computed:

$$P_0 = h,$$

$$P_1 = 2hc_1,$$

$$P_2 = (4h + \frac{c}{2})c_2 - (h + \frac{c}{2})c_1^2.$$

If each $c_k$ is given the weight $k$, it is easily seen that $P_k$ is homogeneous of weight $k$.

Let us remind the reader of the definition of the *Schwarzian derivative* of an holomorphic function $f$:

$$S(f)(z) = \frac{f'''(z)}{f''(z)} - 3\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The following result could have been used as definition of the polynomials $P_k$:

**Proposition 3.2 ([5], p.742, Theorem).**

$$\forall f \in \mathcal{M}, \sum_{n=0}^{+\infty} P_n(c_1, \ldots, c_n)z^n = h\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{c^2}{12}S(f)(z).$$

**Proposition 3.3.**

$$\forall k \geq 0 \forall p \geq 0, L_{-k}(P_p) - L_{-p}(P_k) = (p - k)P_{p+k};$$

in particular, the formula of Theorem 3.1(3) remains valid for $k = 0$.

**Proof.** [2], p.663. □

Let

$$Q_k = \begin{cases} P_k & \text{for } k \neq 0 \\ 0 & \text{for } k = 0. \end{cases}$$
**Theorem 3.4.** Let us set, for each \( k \in \mathbb{Z} \):

\[
\rho(e_k) = -i(L_k + Q_{-k})
\]

and

\[
\rho(\kappa) = i \text{Id}.
\]

Then \( \rho \) defines a representation of the Lie algebra \( \text{Vir}_{c,h} \) into the Lie algebra of differential operators on \( \mathcal{M} \).

**Proof.** As, obviously, \( [\rho(e_k), \rho(\kappa)] = 0 \) is enough to prove that

\[
[\rho(e_m), \rho(e_n)] = \rho([e_m, e_n]).
\]

Taking Proposition 1.2 into account, this is easily reduced to checking the relation:

\[
L_n(Q_{-m}) - L_m(Q_{-n}) = (n - m)Q_{-m-n} - [2hm + \gamma_m]\delta_{m,-n}.
\]

But, for \( m \geq 0 \) and \( n \geq 0 \), that relation is trivially satisfied; for \( m = 0 \) and \( n < 0 \), as well as for \( n = 0 \) and \( m < 0 \), it follows from the relation

\[
\forall n \geq 1 \; L_0(P_n) = nP_n;
\]

in the case \( m < 0 \) and \( n < 0 \), setting \( p = -m \) and \( k = -n \), we have to prove that:

\[
\forall p \geq 1 \forall k \geq 1 \; L_{-k}(P_p) - L_{-p}(P_k) = (p - k)P_{p+k},
\]

but both these facts follow from Proposition 3.3.

There remains the case \( m \leq -1 \) and \( n \geq 1 \) (or the other way round); in this case, we need to prove, setting \( k = -m \geq 1 \), that:

\[
\forall k \geq 1 \forall n \geq 1 \; L_n(P_k) = (n + k)Q_{k-n} + (2hk + \gamma_k)\delta_{k,n}
\]

i.e.

\[
\forall k \geq 1 \forall n \geq 1 \; L_n(P_k) \begin{cases} (n + k)P_{k-n} & \text{if } n \neq k \\ 2hk + \gamma_k & \text{if } n = k. \end{cases}
\]

As \( P_0 = h \), this follows from Theorem 3.1(3).

4. **Unitarizing measure(s)?**

**Definition 4.1.** A Borel probability measure \( \mu \) on \( \mathcal{M} \) is said to be unitarizing for \( \rho \) if and only if

\[
\forall v \in \mathcal{V}^R_{c,h} \; \rho(v)^* = -\rho(v)
\]

on \( \mathcal{H}L^2_{\mu}(\mathcal{M}) \).
Lemma 4.2([1],Theorem 1,p.433). If $\mu$ exists, then, setting $Z_k = L_k - \overline{L}_{-k}$ ($k \geq 0$), one has:

$$
\forall F \in C^\infty(M) \int_M Z_k(F)d\mu = -\int_M F\beta_k d\mu ,
$$

(4.2.1)

where

$$
\beta_k = \begin{cases} 
- \tilde{P}_k & \text{if } k \geq 1 , \\
0 & \text{if } k = 0 .
\end{cases}
$$

Proof. From the definition follows that:

$$
\forall v \in \mathfrak{vir}_{r,h} \rho(v)^* = -\rho(\bar{v}) .
$$

By a density argument, one may assume that $F = \varphi \bar{\psi}$, with $\varphi$ and $\psi$ holomorphic; then one has:

$$
\int_M Z_k(F)d\mu = \int_M L_k(\varphi \bar{\psi})d\mu - \int_M \overline{L}_{-k}(\varphi \bar{\psi})d\mu \\
= \int_M L_k(\varphi \bar{\psi})d\mu - \int_M \overline{L}_{-k}(\varphi \bar{\psi})d\mu \\
= \int_M (L_k(\varphi \bar{\psi}) + \varphi L_k(\bar{\psi}))d\mu - \int_M (L_{-k}(\varphi \bar{\psi}) + \varphi L_{-k}(\bar{\psi}))d\mu \\
= \int_M L_k(\varphi \bar{\psi})d\mu - \int_M \overline{\varphi L_{-k}(\bar{\psi})}d\mu \\
= \int_M L_k(\varphi \bar{\psi})d\mu - \int_M \overline{\varphi L_{-k}(\bar{\psi})}d\mu \\
= i \int_M (e_k(\varphi)(\varphi) - Q_{-k} \varphi \bar{\psi})d\mu - \int_M \overline{\varphi (e_{-k})(\psi) - Q_{k} \varphi \bar{\psi})d\mu \\
= i(e_k(\varphi)(\varphi) + \int_M \varphi (Q_k - Q_{-k}) \bar{\psi}d\mu + i(\overline{\varphi (e_{-k})(\bar{\psi})}) \\
= i(e_k(\varphi)(\varphi) + \int_M \varphi (Q_k - Q_{-k}) \bar{\psi}d\mu + i(\overline{\varphi (e_{-k})(\bar{\psi})}) \\
= \int_M \varphi (Q_k - Q_{-k}) Fd\mu
$$

by the hypothesis on $\mu$ .

Whence the result with:

$$
\beta_k = Q_{-k} - \overline{Q}_k = \begin{cases} 
- \overline{P}_k & \text{for } k \geq 1 \\
0 & \text{for } k = 0 .
\end{cases}
$$

$\square$
Theorem 4.3([1], Theorem 3 and Corollary 4, p.234).

(1) If \( \mu \) exists then the sequence \( 1, P_1, P_2, \ldots \) is a sequence of orthogonal polynomials in \( L^2(\mu) \); more precisely:

\[
(P_m, P_k)_{L^2(\mu)} = \begin{cases} 
0 & \text{if } m \neq k \\
\gamma_k + 2hk & \text{if } m = k \geq 1 \\
h^2 & \text{if } m = k = 0
\end{cases}
\]

(2) If \( h = 0 \) then there is no unitarizing measure on \( \mathcal{M} \) for \( \rho \).

Proof. (1) Let us set, for each \( k \geq 0 \), and \( H_k = Z^2_k + \beta_k Z_k \); it follows from Lemma 4.2 applied to \( Z_k(F) \) that, for each \( k \geq 0 \), one has:

\[
\forall F \in C^\infty(\mathcal{M}) \ \forall k \geq 0 \int_{\mathcal{M}} H_k(F) d\mu = 0 . \tag{4.3.1}
\]

But it follows from the definition of the Neretin polynomials (Theorem 3.1(3)) and from the last remark in Proposition 3.3 that:

\[
\forall k \geq 0 \ \forall n \geq 1
\]

\[
H_k(P_n) = L_k((n+k)P_{n-k} + \gamma_k \delta_{k,n}) + \beta_k((n+k)P_{n-k} + \gamma_k \delta_{k,n}) \\
= (n+k)nP_{n-2k} + (n+k)\gamma_k \delta_{k,n-k} + (n+k)\beta_k P_{n-k} + \beta_k \gamma_k \delta_{k,n} . \tag{4.3.2}
\]

By (4.3.1) one has

\[
\forall k \geq 0 \ \forall n \geq 1 \int_{\mathcal{M}} H_k(P_n) d\mu = 0 . \tag{4.3.3}
\]

Applying (4.3.2) for \( k = 0 \) and \( n \geq 1 \), one finds that:

\[
\forall n \geq 1 \ H_0(P_n) = n^2 P_n ,
\]

whence (4.3.3) yields that:

\[
\forall n \geq 1 \int_{\mathcal{M}} P_n d\mu = 0 . \tag{4.3.4}
\]

From Lemma 4.2 applied to \( F = 1 \) follows:

\[
\forall k \geq 0 \int_{\mathcal{M}} \beta_k d\mu = 0 . \tag{4.3.5}
\]

Taking now \( k \geq 1, m \geq 1 \) and \( n = m + k \), (4.3.2) and (4.3.3) together yield:

\[
\int_{\mathcal{M}} [(2k+m)(k+m) P_{m-k} + (m+2k)\gamma_k \delta_{m,k} + (m+2k)\beta_k P_m + \beta_k \gamma_k \delta_{m,0}] d\mu = 0 ;
\]
from the fact that
\[ \int_{\mathcal{M}} P_n d\mu = \begin{cases} 0 & \text{for } n \geq 1 \quad (4.3.4) \\ h & \text{for } n = 0 \\ 0 & \text{for } n < 0 \text{ (by definition)} \end{cases} \]
and from (4.3.5), we get:
\[ \int_{\mathcal{M}} \beta_k P_m d\mu = \begin{cases} 0 & \text{if } m \neq k \\ -\gamma_k - 2kh & \text{if } m = k \neq 0 \end{cases} . \]

Remembering that \( \beta_k = -\tilde{P}_k \) for \( k \geq 1 \), the result follows.

(2) Let us remind the reader that \( P_1 = 2hc_1 \). Clearly,
\[ Z_1(c_1) = (L_1 - \overline{L_{-1}})(c_1) = L_1(c_1) = 1 , \]
whence:
\[
1 = \int_{\mathcal{M}} d\mu \\
= \int_{\mathcal{M}} Z_1(c_1) d\mu \\
= -\int_{\mathcal{M}} c_1 \beta_1 d\mu \quad \text{(by Lemma 4.2)} \\
= \int_{\mathcal{M}} c_1 \tilde{P}_1 d\mu \\
= 2h \int_{\mathcal{M}} c_1 \tilde{c}_1 d\mu ,
\]
which is impossible for \( h = 0 \). A more geometrical proof of this nonexistence result had previously been given in [3], Theorem 2.2, p.625. □

References
4. V.G.Kac, Infinite dimensional Lie algebras.

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