Continuous dependence of the eigenvalues of generalized Schrödinger operators

A. Ben Amor

January 6, 2000

Abstract

A modified definition of a μ-eigenvalue is introduced where μ is a nonnegative measure in the local Kato class. Dependence of the μ-eigenvalues of $-\triangle + \nu$, where ν is a positive measure on the domain Ω and on the measure μ are investigated. It is also proved that the smallest μ-eigenvalue is non degenerate with positive associated μ-eigenfunction.

1 Introduction

Answering a question asked by R.E.Davis [4](p.129), recently B.Fuglede in [5] could give necessary and sufficient conditions for the eigenvalues of the Dirichlet Laplacian on Ω_n with Ω_n a decreasing bounded subsets of $\mathbb{R}^d$ to converge to the eigenvalues of the Dirichlet Laplacian on Ω where Ω is the interior of $\cap_{n\in\mathbb{N}}\Omega_n$. For the smallest ν-eigenvalue $\lambda_1(\Omega)$ of $-\triangle+\mu$ on Ω where ν is a positive measure in the local Kato class whose fine support is $\mathbb{R}^d$ and μ is in the local Kato class, W.Hansen [8] proved that if Ω is an open subset of $\mathbb{R}^d$ and if Ω_n is decreasing such that the fine interior of $\cap_{n\in\mathbb{N}}\Omega_n$ is Ω then $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$, in [7] the same author proved the same result under the condition stated by B.Fuglede in [5], namely Ω is stable. Let μ be a positive measure in the local Kato class whose fine support is $\mathbb{R}^d$. In this paper a modified definition of the μ-eigenvalue is given. We first prove that the smallest μ-eigenvalue is simple. The monotonicity of the μ-eigenvalues is studied and aspect of their continuity are investigated with respect to the measure μ and to the domain Ω. It is proved that the results stated in [8] (proposition3.2, 3.4) hold true for all the μ-eigenvalues extending thereby the recently proved results [5] about the eigenvalues of the Dirichlet Laplacian on a bounded open subset of $\mathbb{R}^d$. In the third section it is proved that the properties of the μ-eigenvalues of $-\triangle$ hold true for the μ-eigenvalues of $-\triangle+\nu$ where ν is a positive measure in the local Kato class. At the end of the paper it is proved that the same thing holds true even if the fine support of μ is not the whole $\mathbb{R}^d$ provided that μ is non zero.
We now give some basic notations and definition. Denote G the Green function of the Laplacian on $\mathbb{R}^d$ i.e: $-\triangle G(.,y)=\delta_y$

Definition 1.1 *: A positive measure μ is in the local Kato class if the function $\int_\Omega G(x,y)d\mu(y)$ is real continuous on Ω, for every open bounded subset Ω of $\mathbb{R}^d$.*

In all the paper a domain Ω is just an open subset of $\mathbb{R}^d$, and μ a non negative measure in the local Kato class. The notion of quasi-continuity is related to the classical Dirichlet capacity, and $H_0^1(\Omega)$ is the classical Sobolev space, the Lebesgue measure on $\mathbb{R}^d$ is denoted λ^d and dx under the integral sign. We denote $C_b(\Omega)$ the space of bounded continuous functions on Ω, and for every linear operator T, $\sigma(T)$ its spectrum. For a compact self-adjoint operator with eigenvalues $(\lambda_k)_{1\le k\le r(T)}$ where $r(T)$ is the dimension of the range of T

we enumerate the eigenvalues in a decreasing way. In this enumeration every eigenvalue is repeated as many times as its algebraic multiplicity. If $r(T)$ is finite we put $\lambda_k = 0$ for $k \geq r(T)$. For every function f we denote supp(f) its support.

Definition 1.2 *: ([1]) λ is said to be a μ-eigenvalue of $-\triangle$ on Ω if there is $f \in H_0^1(\Omega)$, $\mu(\mathrm{supp(f)}) > 0$ such that $-\triangle f = \lambda f \mu$ in the sense of distribution on Ω . Such a function f will be called a μ-eigenfunction.*

It is known that every element of $H_0^1(\Omega)$ has a quasi-continuous modification we can admit that every $f \in H_0^1(\Omega)$ is quasi-continuous (This lieds to pick each time the quasi-continuous representative). Moreover, it was proved [1] that if μ is in the local Kato class then every quasi-continuous function $f \in H_0^1(\Omega)$ is in $L^2(\Omega,\mu)$. Now the fact that the fine support of μ is $I\!R^d$ implies that $H_0^1(\Omega) \subset L^2(\Omega,\mu)$ (Cf [2]). Let us mention that for every $f \in L^2(\mu)$ we have $K_\Omega^\mu f \in H_0^1(\Omega)$. All this implies that the definition of the μ-eigenvalues is well posed. A direct computation yields that λ is a μ-eigenvalue if and only if λ^{-1} is an eigenvalue of K_Ω^μ

$$K_\Omega^\mu :\ L^2(\Omega,\mu) \longrightarrow\ L^2(\Omega,\mu),\ K_\Omega^\mu f(x) = \int_\Omega G(x,y)f(y)d\mu(y)$$

Remark 1.1 *If λ is a μ-eigenvalue of $-\triangle$ on Ω then there is $f \in H_0^1(\Omega)$ such that $\int_\Omega \nabla f \bar{\nabla} g dx = \lambda \int_\Omega f \bar{g} d\mu$ for every $g \in H_0^1(\Omega)$ thus λ is an eigenvalue of the form:*

$$\mathcal{E},\ D(\mathcal{E}) = H_0^1(\Omega),\ \mathcal{E}(f,g) = \int_\Omega \nabla f \bar{\nabla} g dx \tag{1}$$

where $\mathcal{E}$ is defined in $L^2(\Omega,\mu)$. In [2] it is proved that $\mathcal{E}$ is closable provided that the fine support of μ is $I\!R^d$. Now denote $\bar{\mathcal{E}}$ the closure of $\mathcal{E}$, then every μ-eigenvalue is an eigenvalue of $\bar{\mathcal{E}}$ but the converse is not true. Thus the characterization of the μ-eigenvalues as being the inverses of the eigenvalues of K_Ω^μ is more natural. Moreover if we denote $D(\bar{\mathcal{E}})$ the domain of $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}_1$ the norm $\bar{\mathcal{E}}[.] + \|.\|^2_{L^2(\Omega,\mu)}$ then for every closed subspace F of $(D(\bar{\mathcal{E}}), \bar{\mathcal{E}}_1^{\frac{1}{2}})$ the injection $(F, \bar{\mathcal{E}}_1^{\frac{1}{2}}) \longrightarrow\ L^2(\Omega,\mu)$ is compact. For instance it is proved in [2] that the operator $-\triangle$ associated to the form $\bar{\mathcal{E}}$ has compact resolvent. So in the case where the fine support of μ is $I\!R^d$ we are in the setting described by B.Fuglede [5]. But the method we propose is different and simpler. However *if the fine support of μ is not required to be $I\!\!R^d$ then it is no more possible to define $-\triangle$ in $L^2(\Omega,\mu)$ by means of a closed form since one can not identify $H^1_0(\Omega)$ to a subspace of $L^2(\Omega,\mu)$. In this situation the method given in [5] is no longer valid, but ours still works.*

Now the operator K^μ_Ω is compact [1] therefore the μ-eigenvalues of $-\triangle$ on a bounded subset forms at most a discrete set accumulating at infinity. We enumerate them $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega)... \leq \lambda_k(\Omega) \leq \lambda_{k+1}(\Omega)...$ repeated according to their multiplicities. Now one can assert as in the case where $\mu = \lambda^d$ (where a λ^d-eigenvalue is simply an eigenvalue) the Lebesgue measure on $I\!\!R^d$.

Proposition 1.1 *For every bounded domain Ω of $I\!\!R^d$, $\lambda_1(\Omega)$ is simple with positive associated μ-eigenfunction.*

Proof: It suffices to prove that K^μ_Ω is positivity preserving then conclude by theorem(XIII.43) in [10]. Obviously if $f \geq 0,\ \mu - a.e$ then $K^\mu_\Omega \geq 0,\ \mu - a.e.$ Now if $K^\mu_\Omega f = 0$ then $f = 0,\ \mu - a.e$ thus K^μ_Ω is positivity preserving. □
One can ask what is about the dependence of the μ-eigenvalues on the measure μ and on the domain Ω? For the dependence on the measure a sort of continuity can be observed:

Proposition 1.2 *Let (μ_n) be a sequence of measures in the local Kato class increasing to μ then $\lambda_{k,n}(\Omega) \rightarrow \lambda_k(\Omega)$, where $\lambda_{k,n}(\Omega)$ is the k'th μ_n-eigenvalue of $-\triangle$ on Ω.*

First we give a lemma that will be used along the paper. We say that an operator T defined on a Hilbert space $\mathcal{H}$ is positive if $(Tf,f) > 0$ for $f \neq 0$. A sequence of operators (T_k) is said to be decreasing if $(T_k f - T_{k+1}f, f) \geq 0$ for every $f \in \mathcal{H}$.

Lemma 1.1 *Let (T_k) be a decreasing sequence of compact self-adjoint operators on a Hilbert space $\mathcal{H}$ converging to T where T is positive . Let $(\beta_{k,n})$ be such that $\beta_{k,n}$ is the n-th eigenvalue of T_k for every k. Then $(\beta_{k,n})$ converges to β_n the n-th eigenvalue of T.*

Although the proof of this lemma can be derived from the proof of theorem(4.2) (p.18) in [6] we will give here an independent proof.
Proof: Remark first that the convergence of $(\beta_{k,n})$ is a consequence of the

monotonicity of (T_k). Now fix $n > 1$ suppose that for every j, β_j has algebraic multiplicity $r(j)$. Then we have

$$\beta_1 = ... = \beta_{r(1)} > \beta_{r(1)+1} = ... = \beta_{r(1)+r(2)} > ... > \beta_n = \beta_{r(s-1)+1} = ...\beta_{r(s-1)+r(n)}$$

For some integer $s > 1$. It is known that for large k there is at $r(1)$ eigenvalues of T_k near β_1, say $\beta_{k,1}, ..., \beta_{r(1)}$. Since the sequence (T_k) is decreasing then for large k we have $\beta_{k,j} \geq \beta_1$ for $1 \leq j \leq r()1$. Now using theorem(4.10) (p.291) in [9] we get thet $\beta_{k,j}$ tends to an element of the spectrum of T for every $1 \leq j \leq r(1)$ thereby it tends to β_1. Now we can retrieve the same arguments for $\lambda_{r(2)}$ using the fact that there is $\delta < \beta_1$ with $\beta_{r(2)} \leq \beta_{k,j} < \delta$ for $r(1) + 1 \leq j \leq r(2)$. Now by recursion we get the result for β_n. □

Remark 1.2 *: Clearly lemma(1.1) holds true if (T_k) is monotone. We also mention that the hypothesis that T_k, T are monotone self-adjoint can be dropped provided that the eigenvalues of the operators under consideration are all real monotone.*

Proof: (of theorem(1.2)) Since μ_n increases to μ then $L^2(\Omega, \mu) \subset L^2(\Omega, \mu_n)$ and the ordered eigenvalues are real monotone . Now using the spectral invariance [3] one can prove that the eigenvalues of $K_\Omega^{\mu_n}$ are the eigenvalues of the operator $K_n := : L^2(\Omega, \mu) \longrightarrow L^2(\Omega, \mu)$, $K_n f = \int_\Omega G(., y) f(y) d\mu_n(y)$, hence according to lemma(1.1) and the remark that follows it, it suffices to prove that $\|K_n - K_\Omega^\mu\| \to 0$. A direct computation shows that if we put $\nu_n = \mu - \mu_n$ then $\|K_n - K\| \leq \|G_\Omega^{\nu_n}\|_\infty^{\frac{1}{2}} \|G_\Omega^\mu\|_\infty^{\frac{1}{2}}$. Clearly $\|G_\Omega^{\nu_n}\|_\infty^{\frac{1}{2}} \to 0$ and the proof is complete. □

2 Dependence on the domain

The main tool to use to prove monotonicity and continuity of the μ-eigenvalues relatively to the domain is the mini-max principle.

Proposition 2.1 . *Let Ω_1, Ω_2 be two bounded domains in $\mathbb{R}^d$ with $\Omega_1 \subset \Omega_2$. Then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$.*

Proof: It is equivalent to the facts that the $\beta_k(\Omega)$'s are increasing with respect to Ω where $\beta_k(\Omega)$ are the eigenvalues of K^μ_Ω ordered in a decreasing way. By the mini-max principle we have

$$\beta_k(\Omega_1) = \min_{\mathrm{dimL}\leq \mathrm{k}-1} \max\{(K^\mu_{\Omega_1} f, f)_{L^2(\Omega_1,\mu)},\ f \in L^\perp,\ \|f\| = 1,\ L \subset L^2(\Omega_1, \mu)\}$$

Choose $f \in L^2(\Omega_1, \mu)$ since $\Omega_1 \subset \Omega_2$ the extension $\tilde{f}$ of f to Ω_1 by 0 in $\Omega_2 \setminus \Omega_1$ is in $L^2(\Omega_2, \mu)$. Furthermore

$$K^\mu_{\Omega_1} f(x) = \int_{\Omega_1} G(x, y) f(y) d\mu(y) = \int_{\Omega_2} G(x, y)\tilde{f}(y) d\mu(y) = K^\mu_{\Omega_2} f(x) \qquad (2)$$

Now $(K^\mu_{\Omega_1} f, f)_{L^2(\Omega_1,\mu)} = (K^\mu_{\Omega_2} \tilde{f}, \tilde{f})_{L^2(\Omega_2,\mu)}$, it follows that

$$\max\{(K^\mu_{\Omega_1} f, f)_{L^2(\Omega_1,\mu)},\ f \in L^\perp,\ \|f\| = 1,\ L \subset L^2(\Omega_1, \mu)\} \qquad (3)$$

$$\leq\ \max\{(K^\mu_{\Omega_2} f, f)_{L^2(\Omega_2,\mu)},\ f \in L^\perp,\ \|f\| = 1,\ L \subset L^2(\Omega_2, \mu)\} \qquad (4)$$

and thereby $\beta_k(\Omega_1) \leq \beta_k(\Omega_2)$. □

In the sequel we denote $K^\mu_{\Omega_n}$ by K_n and K^μ_Ω by K, also $\lambda_k(\Omega_n)$ is denoted $\lambda_{k,n}$ and $\lambda_k(\Omega)$ is denoted λ_k.

Theorem 2.1 *Let* (Ω_n) *be an increasing sequence of domains of* $\mathbb{R}^d$ *with* $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$, *then* $\lim_{k\to\infty} \lambda_{k,n} = \lambda_k$.

Proof: We will prove the convergence for the $\beta_k(\Omega_n)$'s. Denote $\beta_{k,n} = \beta_k(\Omega_n)$, given $\epsilon > 0$ there is $L \subset L^2(\Omega_n, \mu)$, $\mathrm{dimL} \leq \mathrm{k} - 1$, such that for every $f \in L^\perp$, $\beta_{k,n} \leq (K^\mu_{\Omega_n} f, f)_{L^2(\Omega_n,\mu)} + \epsilon \leq (K^\mu_{\Omega_n} \tilde{f}, \tilde{f})_{L^2(\Omega,\mu)} \leq \beta_k + \epsilon$, where $\tilde{f}$ is the extension of f by 0 in $\Omega \setminus \Omega_n$. The reversed inequality is due to proposition(2.1). □

For a sequence of decreasing domains the convergence of the μ-eigenvalues to the previous limit is less obvious.

Theorem 2.2 *Let* (Ω_n) *be a decreasing sequence of bounded subsets of* $\mathbb{R}^d$ *such that* $\cap_{n\in\mathbb{N}}\Omega_n = \Omega_\infty$. *Put* Ω *the interior with respect to the usual topology on* $\mathbb{R}^d$ *of* Ω_∞ *which we suppose to be non void. Put* $\mathrm{int_f}(\Omega_\infty)$ *the interior with respect to the fine topology of* Ω_∞. *Suppose that* $\mathrm{int_f}(\Omega_\infty) \setminus \Omega$ *is polar then* $\lambda_{k,n} \to \lambda_k(\Omega)$.

Remark 2.1 *It was first proved in [7] that if* $\mathrm{int_f}(\Omega_\infty)$ *and* Ω *differ by a polar set (this is* Ω *is stable) then the smallest* μ*-eigenvalue tend to the previous one. Later on B.Fuglede proved [5] that for the* λ^d*-eigenvalues,* $\lambda_{k,n} \to \lambda_k$ *if and only if* $\cap_{n\in\mathbb{N}} H_0^1(\Omega_n) = H_0^1(\Omega)$ *which in turns equivalent to the fact that* Ω *is stable.*

Before giving the proof we give some observations. Denote $L^2(\mu)$ the space $L^2(\mathbb{R}^d, \mu)$. For every open subset U of $\mathbb{R}^d$ we identify $L^2(U, \mu)$ to the subspace of $L^2(\mu)$ defined by $\{f \in L^2(\mu),\ f = 0,\ on\ U^c\}$, where $U^c = \mathbb{R}^d \backslash U$. The space $H_0^1(U)$ with $\{f \in H^1(\mathbb{R}^d),\ f = 0,\ on\ U^c\}$. The extension of K_U^μ on $L^2(\mu)$ will be defined as follows for every $f \in L^2(\mu)$:

$$\widetilde{K_U^\mu} f(x) = \begin{cases} K_U^\mu f(x) & ; x \in U,\ \mu - a.e \\ 0 & ; x \in U^c \end{cases}$$

Clearly for every $f \in L^2(U, \mu)$, $\widetilde{K_U^\mu} f = K_U^\mu f$, moreover $\widetilde{K_U^\mu} : L^2(\mu) \longrightarrow L^2(\mu)$ continuously. Indeed $\int_{\mathbb{R}^d} |\widetilde{K_U^\mu} f(x)|^2 d\mu(x) = \int_U |K_U^\mu f(x)|^2 d\mu(x) \leq \|G_U^\mu\|_\infty \int_{\mathbb{R}^d} |f|^2 d\mu(x)$. Remark that $\widetilde{K_U^\mu}$ maps also $L^2(\mu)$ into $L^2(\lambda^d)$ continuously. Now by [1] we have for every $f \in L^2(\mu)$, $\widetilde{K_U^\mu} f_{|U} \in H_0^1(U)$ and $\widetilde{K_U^\mu} f \in H^1(\mathbb{R}^d)$.

Lemma 2.1 *Under the hypothesis of theorem(2.2) we have*

$$\|\widetilde{K_n} - \widetilde{K}\| \longrightarrow 0$$

Proof: Let $f \in L^2(\mu)$ then:

$$\widetilde{K_n} f(x) - \widetilde{K} f(x) = \begin{cases} K_n f(x) - K f(x) & ; x \in \Omega,\ \mu - a.e \\ K_n f(x) & ; x \in \Omega_n \setminus \Omega,\ \mu - a.e \\ 0 & ; x \in \Omega_n^c,\ \mu - a.e \end{cases}$$

We thereby get:
$\|\widetilde{K_n} f - \widetilde{K} f\|^2 = \int_\Omega |K_n f(x) - K f(x)|^2 d\mu(x) + \int_{\Omega_n \setminus \Omega} |K_n f(x)|^2 d\mu(x) \leq \int_\Omega |\int_{\Omega_n \setminus \Omega} G(x,y) f(y) d\mu(y)|^2 d\mu(x) + \int_{\Omega_n \setminus \Omega} |\int_{\Omega_n} G(x,y) f(y) d\mu(y)|^2 d\mu(x).$

Hence $|\int_{\Omega_n\setminus\Omega} G(x,y)f(y)d\mu(y)|^2 \leq \|G_{\Omega_n\setminus\Omega}\|_\infty \int_{\Omega_n\setminus\Omega} G(x,y)|f|^2 d\mu(y)$ making the same obervation for the other term we get

$$\|K_n - K\|^2 \leq \|G_{\Omega_n\setminus\Omega}\|_\infty (\|G_{\Omega_n\setminus\Omega}\|_\infty + \|G^\mu_{\Omega_1}\|). \tag{5}$$

Now

$$G_{\Omega_n\setminus\Omega} = \int_{\Omega_n\setminus\Omega} G(.,y)d\mu(y) \leq \int_{\Omega_n} G(.,y)1_{\Omega_n\setminus\Omega} d\mu(y) = K_n 1_{\Omega_n\setminus\Omega} \tag{6}$$

Since $\mathrm{int_f}\Omega_\infty \setminus \Omega$ is polar we get $K_n 1_{\Omega_n\setminus\Omega} = K_n 1_{\Omega_n\setminus \mathrm{int_f}\Omega_\infty}$. Using lemma(3.3) in [8] we get $\|G^\mu_{\Omega_n\setminus\Omega}\|_\infty \to 0$. And the proof is complete. □

Lemma 2.2 *For every bounded open subset U of $\mathbb{R}^d$, $\widetilde{K^\mu_U}$ is compact.*

Proof: Let (f_n) be a sequence in $L^2(\mu)$ with $\|f_n\| \leq 1$. Put $g_n = f_{n|_U}$, clearly (g_n) is bounded in $L^2(U,\mu)$. Now the compactness of K^μ_U on $L^2(U,\mu)$ implies the existence a a subsequence g_{n_k} such that $K^\mu_U g_{n_k} \to \varphi$ in $L^2(U,\mu)$ but $\widetilde{K^\mu_U}(f_{n_k|_U}) = \widetilde{K^\mu_U} f_{n_k}$ which yields $\widetilde{K^\mu_U} f_{n_k} \to \varphi$ in $L^2(\mu)$. □

As a consequence we get that $\widetilde{K}$ is compact. This observation yields the following

$$\sigma(\widetilde{K}) = \sigma(K)$$

One can consider only non zero elements of the both spectra, those are eigenvalues. By the identification of $L^2(U,\mu)$ one easily get $\sigma(K) \subset \sigma(\widetilde{K})$. Now given $\lambda \in \sigma(\widetilde{K})$ there is $f \in L^2(\mu)$ such that $Kf = \lambda f = \widetilde{K}(f_{|_U})$. Hence $f \in L^2(U,\mu)$ and $Kf = \lambda f$ thereby $\lambda \in \sigma(K)$.

Proof(of 2.2): By lemma (1.1) and (2.1) the (enumerated) eigenvalues of $\widetilde{K_n}$ converge to those of $\widetilde{K}$ and the last remark gives the result. □

Using theorem(2.2) in [5] one get the following interesting result

Corollary 2.1 *Under the assumptions of theorem(2.2), denote $\Lambda_{k,n}$ the ordered eigenvalues of $-\triangle$ on Ω_n with Dirichlet boundary condition and Λ_k the ordered eigenvalues of the corresponding operator on Ω. Then if $\Lambda_{k,n} \to \Lambda$ we get $\lambda_{k,n} \to \lambda_k$ for every positive measure in the local Kato class whose fine support is $\mathbb{R}^d$.*

Remark 2.2 *If Ω is as in theorem(2.2) then $H_0^1(\Omega) = \cap_{n\in\mathbb{N}} H_0^1(\Omega_n)$. Indeed if $\widetilde{P}_{k,n}$ are the eigenprojections associated to $\beta_{k,n}$, then clearly $\widetilde{P}_{k,n} \to \widetilde{P_k}$ where $\widetilde{P_k}$ are the eigenprojections associated to β_k (this is simply by lemma(2.1)). Clearly the eigenprojections $\widetilde{P}_{k,n}$ are the natural extension of $P_{k,n}$. Moreover since $\widetilde{K_n}$, $\widetilde{K}$ are compact self-adjoint then they have the following development*

$$\widetilde{K_n} = \sum_{k\geq 1} \beta_{k,n} \widetilde{P}_{k,n} \tag{7}$$

and

$$\widetilde{K} = \sum_{k\geq 1} \beta_k \widetilde{P}_k \tag{8}$$

Thus if f is in the range of $\widetilde{P}_{k,n}$ then for $j \neq k$ we have $\|\widetilde{P}_{j,n} f - \widetilde{P}_j f\| = \|\widetilde{P}_j f\| \to 0$ hence the corresponding eigenspace of $\beta_{k,n}$ is included in that associated to β_k thereby $H_0^1(\Omega_n) \subset H_0^1(\Omega)$ for every n and then $\cap_{n\in\mathbb{N}} H_0^1(\Omega) \subset H_0^1(\Omega)$. The reversed inclusion is obvious.

3 Perturbations

The above proved properties of the μ-eigenvalues of $-\triangle$ on a bounded open subset of $\mathbb{R}^d$ still hold true after perturbation of $-\triangle$ by a positive measure in the local Kato class. Indeed let ν be a positive measure in the local Kato class. We say that λ is a μ-eigenvalue of $-\triangle + \nu$ if there is $f \in H_0^1(\Omega)$, $\mu(\text{supp(f)}) > 0$ such that $-\triangle f + f\nu = \lambda f \mu$ in the sense of distribution on Ω. Or equivalently λ^{-1} is an eigenvalue of

$$K = (I + K_\Omega^\nu)^{-1} K_\Omega^\mu$$

Since for every $f \in L^2(\Omega, \mu)$, $K_\Omega^\mu f \in H_0^1(\Omega)$ then the operator K is well defined on $L^2(\Omega, \mu)$. It is known [8] that K is a kernel operator on $L^2(\Omega, \mu)$, more precisely there is a real symmetric kernel G_Ω^ν such that for every $f \in L^2(\Omega, \mu)$

$$Kf = \int_\Omega G_\Omega^\nu(., y) f(y) d\mu(y) \tag{9}$$

Moreover there is a constant $c > 0$ such that $G^\nu_\Omega(x, y) \leq cG(x, y)$ for $x \neq y$. This yields that μ is in the local Kato class with respect to G^ν_Ω and thereby [2] K is compact on $L^2(\Omega, \mu)$ with a nonnegative smallest μ-eigenvalue.
Now fix an open bounded subset U of $I\!R^d$ denote $\mathcal{U}$ the set of all open subsets included in U. Clearly for every $\Omega \in \mathcal{U}, G^\nu_\Omega = G^\nu_U$. Changing G by G^ν_U and taking $\Omega_n \in \mathcal{U},\ \Omega \in \mathcal{U}$, one get

Theorem 3.1 *The μ-eigenvalues of $-\triangle + \nu$ on subsets of $\mathcal{U}$ are increasing and satisfy proposition(1.1), (1.2) and theorem(2.1), (2.2).*

Proof: For the dependence on the measure μ one has just to observe that $(I + K^\nu_\Omega)^{-1} K^{\mu_n}_\Omega f = \int_\Omega G^\nu_\Omega(., y) f(y) d\mu_n(y)$. For the other assertions the proof is substantially like those of proposition(1.1), (1.2) and theorem(2.1), (2.2). □

4 For measures μ whose fine support is not $I\!R^d$

In this section we no longer suppose that the fine support of the measure μ is $I\!R^d$ but we suppose that the support of μ is $I\!R^d$. It then follows that it is no more possible to define $-\triangle$ as an operator acting in $L^2(\Omega, \mu)$. The assumption, $\mathrm{supp}(\mu) = \mathrm{R^d}$ is crucial for the Hilbert space $L^2(\Omega, \mu)$ to be a non trivial Hilbert space for every domain Ω. We adopt the same definition of a μ-eigenvalue of $-\triangle$ on Ω, we denote ν the measure $\mu + \lambda^d$. This yields the following characterization of the μ-eigenvalues (Cf. [1]): λ is a μ-eigenvalue of $-\triangle$ if and only if λ^{-1} is an eigenvalue of the operator K_Ω:

$$K_\Omega :\ L^2(\Omega, \nu) \longrightarrow\ L^2(\Omega, \nu),\ K_\Omega f = \int_\Omega G(., y) f(y) d\mu(y)$$

Now the operator K_Ω is no longer self-adjoint and the arguments used before are no more valid. So one need an other characterization of the μ-eigenvalues. Put $K^\mu_\Omega :\ L^2(\Omega, \mu) \longrightarrow\ L^2(\Omega, \mu),\ K^\mu_\Omega f = \int_\Omega G(., y) f(y) d\mu(y)$ then we have:

Lemma 4.1 *λ is an eigenvalue of K_Ω if and only if it is an eigenvalue of K^μ_Ω having the same multiplicity.*

Proof: Since $\mu \leq \nu$ then $L^2(\Omega,\nu) \subset L^2(\Omega,\mu)$ and $K^{\mu}_{\Omega|_{L^2(\Omega,\nu)}} = K_\Omega$ which implies that if f is an eigenfunction associated to λ then $f \in L^2(\Omega,\mu)$ and $K^{\mu}_{\Omega}f = \lambda f$ to get that λ is an eigenvalue of K^{μ}_{Ω} it suffices to prove that the measure with respect to μ of the support of f is non zero. Suppose that $\mu(\text{suppf}) = 0$ then $K_\Omega f = 0$ every where hence $\lambda f = 0$ every where since $\lambda \neq 0$ this implies that $f = 0$ which contradicts the fact f is an eigenfunction. On the other hand by the spectral invariance we get $\sigma(K^{\mu}_{\Omega}) = \sigma(K^{\mu})$ where

$$K^{\mu} := C_b(\Omega) \longrightarrow C_b(\Omega), \ K^{\mu}f = \int_\Omega G(.,y)f(y)d\mu(y) \tag{10}$$

with the same multiplicity for the eigenvalues. The inclusion $C_b(\Omega) \subset L^2(\Omega,\nu)$ gives the reversed part. □

Now since the operator K^{μ}_{Ω} is compact self-adjoint we can retrieve our arguments. We immediately conclude

Theorem 4.1 *Let μ be a positive measure in the local Kato class whose support is $\mathbb{R}^d$. Then the μ-eigenvalues of $-\triangle$ satisfy proposition(1.1), (1.2) and theorem(2.1), (2.2).*

References

[1] F.Ammar Sassi, A. Ben Amor: Sur l'estimation du nombre des μ-valeurs propres de $-\triangle + V\mu$. Preprint.

[2] A.Ben Amor: On the μ-eigenvalues of $-\triangle + V\mu$. Preprint.

[3] A.Ben Amor, W.Hansen: On compactness of operators related to Kato measures. Preprint.

[4] R.E.Davies: Spectral Theory and Differential Operators. Cambridge University Press. 1995.

[5] B.Fuglede: Continuous domain dependence of the eigenvalues of the Dirichlet Laplacian and related operators in Hilbert spaces. J.Functional.Anal. 167,183-200. (1999).

[6] I.C.Gohberg, M.G.Krein: Introduction to the theory of linear nonselfadjoint operators. Translations of Mathematical Monographs. Vol.18.1969.

[7] W.Hansen: Prived note. 1995.

[8] W.Hansen: Valeurs propres pour les opérateurs de Schrödinger. Lectures Notes in math.1393, Springer-Verlag, 117-134, 1989.

[9] T.Kato: Perturbation theory for linear operators. Springer, 1982.

[10] M.Reed, B.Simon: Methods of Modern Mathematical Physics. Vol.IV. Academic Press. 1978.

Universität Bielefeld. Fakultät für Physik.BiBoS.
Universitätsstrasse 25. D-33615. Germany.
E-mail: amor@physik.uni-bielefeld.de