On the spectrum of Schrödinger operator with periodic surface potential

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January 6, 2000

Abstract

We consider a discrete Schrödinger operator $H = -\Delta + V$ acting in $l^2(\mathbb{Z}^d)$, with periodic potential V supported by the subspace "surface" $\{0\} \times \mathbb{Z}^{d_2}$. We prove that the spectrum of H is purely absolutely continuous, and that surface waves (see [8] for definition) oscillate in the longitudinal directions to the "surface". We find also an explicit formula for the generalized spectral shift function introduced in [4].

1 Introduction

In this paper we will primarily discuss the discrete Schrödinger operator H with a surface potential V acting on the Hilbert space $l^2(\mathbb{Z}^d)$

$$H = H_0 + V, \tag{1.1}$$

$$V(X) = \delta(x)v(\xi), \tag{1.2}$$

where

$$\mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} = \{X = (x, \xi) \mid x \in \mathbb{Z}^{d_1},\ \xi \in \mathbb{Z}^{d_2}\}, \tag{1.3}$$

In other words

$$H\psi(X) = \sum_{Y \in \mathbb{Z}^d, |Y - X| = 1} \psi(Y) + \delta(x)v(\xi)\psi(X), \tag{1.4}$$

for all $\psi \in l^2(\mathbb{Z}^d)$, where $\delta(x)$ is Kronecker symbol.

It is well known that H_0 is a bounded self-adjoint operator on $l^2(\mathbb{Z}^d)$, and

$$\begin{aligned}\sigma_{ac}(H_0) &= \sigma(H_0) = [-2d, 2d], \\ \sigma_{pp}(H_0) &= \sigma_{sc}(H_0) = \varnothing.\end{aligned}$$

For every real-valued potential v, it is clear that the operator H is self-adjoint in $l^2(\mathbb{Z}^d)$. Using Weyl's criterion one can see that $\sigma(H_0)$ is contained in the spectrum of H. Moreover, in [8] the authors prove that for bounded potential v, $\sigma(H_0)$ is always contained in the absolutely continuous component of the spectrum of H. This result was generalized for an arbitrary unbounded potential v in [5, 12].

In fact, in this model we have two special parts of the spectrum $\sigma(H)$ of the operator H

- Bulk branches of the spectrum whose generalized eigenfunctions (polynomially bounded solutions of the equation $H\psi = \lambda\psi, \quad \lambda \in \sigma(H)$) are plane waves, i.e. they oscillate in all directions.
- Surface branches of the spectrum (or more simple "surface spectrum") whose generalized eigenfunction decay in the transversal directions x and either oscillate or decay in the longitudinal directions ξ. These solutions are called surface waves (see [6, 8, 11] for results and references).

There is a large literature on the spectral properties of H and the geometry of surface branches of the spectrum of H (see [2, 8, 11, 9, 10, 14]). For example in [2] the authors study the case where v belongs to a special class of unbounded quasiperiodic potentials. In that case they prove that away from $\sigma(H_0)$ the surface spectrum of H is pure point dense and the corresponding generalized spectral functions are exponentially localized. A typical example of this class is

$$v(\xi) = \lambda \tan(\pi\alpha \cdot \xi + \theta)$$

with $\alpha = (\alpha_1, \cdots, \alpha_d) \in [0,1]^d$ and $\theta \in \mathbb{R}$. In that case H is the Maryland surface model. This model was also studied in [14] and [11]. In [14] the authors prove that if α has typical Diophantine properties, i.e. if there exist constants $C, k > 0$ such that

$$|\xi \cdot \alpha - n| > C|\xi|^{-k}, \quad \forall \xi \in \mathbb{Z}^{d_2}, \quad \forall n \in \mathbb{Z},$$

then the surface spectrum of H is dense and pure point outside $\sigma(H_0)$ for any $\lambda \neq 0$ and $\theta \in \mathbb{R}$ and the corresponding surface waves are localized. In [11] the authors proved that if $\alpha_1, \cdots, \alpha_d$ are $\mathbb{Q}$-linearly independent, then the spectrum of H is purely absolutely continuous on $\sigma(H_0)$.

Our goal in this paper is to study the geometry of the spectrum of H in the case of a surface periodic potential, i.e. we assume that there exist $N_1, \cdots, N_{d_2} \in \mathbb{N}^*$ such that

$$v(\xi + N_j \mathbf{e_j}) = v(\xi) \qquad \forall j = 1, \cdots, d_2 \tag{1.5}$$

where $\{\mathbf{e_j}\}$ are the canonical basis of $\mathbb{R}^{d_2}$. We prove in the first section, that the spectrum of H is purely absolutely continuous, and that surface waves oscillate in the longitudinal directions ξ and are localized in the transversal directions x. A similar problem in the continuous case was studied in [7].

In the second section we find an explicit formula for the generalized spectral shift function which was introduced in [4] for a homogeneous surface potential (periodic, quasi-periodic, random ergodic).

Acknowledgments

The author is grateful to A. Boutet de Monvel and L. Pastur for useful discussions.

2 Study of the spectrum:

In this section we will show that the spectrum of H is purely absolutely continuous if $v(\xi)$ has property (1.5). Let Ω^{d_2} be the periodic cell i.e.

$$\Omega^{d_2} = \{\xi = (\xi_1, \cdots, \xi_{d_2}) \in \mathbb{Z}^{d_2}, \quad 0 \leq \xi_j \leq N_j - 1, \quad j = 1, \cdots, d_2\}.$$

Let $\mathbb{T}^{d_2} = [0, 2\pi]^{d_2}$ be the torus in $\mathbb{R}^{d_2}$ and let us consider following spaces

$$\mathcal{H}'_1 = l^2(\Omega^{d_2}), \qquad \mathcal{H}_1 = \int_{\mathbb{T}^{d_2}}^{\oplus} \mathcal{H}'_1 \frac{d\theta}{(2\pi)^{d_2}}.$$

$$\mathcal{H}'_2 = l^2(\mathbb{Z}^{d_1}) \times l^2(\Omega^{d_2}), \qquad \mathcal{H}_2 = \int_{\mathbb{T}^{d_2}}^{\oplus} \mathcal{H}'_2 \frac{d\theta}{(2\pi)^{d_2}}.$$

And let U_1 be the following operator

$$\begin{aligned} U_1 \quad &: \quad l^2(\mathbb{Z}^{d_2}) \longrightarrow \mathcal{H}_1 \\ (U_1 f)_\theta(\xi) \quad &= \quad \sum_{n \in \mathbb{Z}^{d_2}} e^{-i\theta \cdot n} f(\xi + n|N), \quad \theta \in \mathbb{T}^{d_2}, \end{aligned}$$

where $n|N = (n_1 N_1, \cdots, n_{d_2} N_{d_2}) \in \mathbb{Z}^{d_2}$. Let U_2 be the following operator

$$\begin{aligned} U_2 &: l^2(\mathbb{Z}^d) \longrightarrow \mathcal{H}_2 \\ U_2 &= 1 \otimes U_1 \end{aligned}$$

i.e.

$$(U_2 f)_\theta(x, \xi) = \sum_{n \in \mathbb{Z}^{d_2}} e^{-i\theta \cdot n} f(x, \xi + n|N).$$

Let us denote by h_0 the Laplacian on $l^2(\mathbb{Z}^{d_2})$. Then we have the following

Lemma 2.1

$$U_1 h_0 U_1^{-1} = \int_{\mathbb{T}^{d_2}}^{\oplus} h_0(\theta) \frac{d\theta}{(2\pi)^{d_2}},$$

where $h_0(\theta)$ is the Laplacian on $\mathcal{H}_1'$ with the Bloch-Floquet conditions:

$$\begin{aligned} \psi_\theta(\xi_1, \cdots, N_j, \cdots, \xi_{d_2}) &= e^{i\theta_j} \psi(\xi_1, \cdots, 0, \cdots, \xi_{d_2}), \\ \psi_\theta(\xi_1, \cdots, N_j - 1, \cdots, \xi_{d_2}) &= e^{i\theta_j} \psi(\xi_1, \cdots, -1, \cdots, \xi_{d_2}). \end{aligned}$$

Proof. The proof of this lemma is the same proof for the continuous Laplacian developed in [18]. ■

It is clear that the spectrum of $h_0(\theta)$ is pure point, moreover

$$(h_0(\theta)\phi_n^\theta)(\xi) = \alpha_n(\theta)\phi_n^\theta(\xi),$$

where

$$\phi_n^\theta(\xi) = \prod_{j=1}^{d_2} \exp(i\frac{\theta_j}{N_j} + \frac{2\pi i n_j}{N_j})\xi_j, \tag{2.1}$$

$$\alpha_n(\theta) = -2\sum_{j=1}^{d_2} \cos(\frac{\theta_j}{N_j} + \frac{2\pi n_j}{N_j}). \tag{2.2}$$

Corollary 2.1 *We have*

$$U_2 H_0 U_2^{-1} = \int_{\mathbb{T}^{d_2}}^{\oplus} H_0(\theta) \frac{d\theta}{(2\pi)^{d_2}},$$

where $H_0(\theta)$ is the Laplacian on $l^2(\mathbb{Z}^{d_1}) \times l^2(\Omega^{d_2})$ with Bloch-Floquet conditions for longitudinal directions ξ:

$$\begin{aligned}\psi_\theta(x,\xi_1,\cdots,N_j,\cdots,\xi_{d_2}) &= e^{i\theta_j}\psi(x,\xi_1,\cdots,0,\cdots,\xi_{d_2}),\\ \psi_\theta(x,\xi_1,\cdots,N_j-1,\cdots,\xi_{d_2}) &= e^{i\theta_j}\psi(x,\xi_1,\cdots,-1,\cdots,\xi_{d_2}),\end{aligned}$$

for all $j=1,\cdots,d_2$.

It is clear that $H_0(\theta) = (-\Delta^{d_1})\otimes 1 + 1\otimes h_0(\theta)$ where $-\Delta_{d_1}$ is the discrete Laplacian on $l^2(\mathbb{Z}^{d_1})$. Therefore the spectrum of $H_0(\theta)$ is purely absolutely continuous and

$$\sigma_{ac}(H_0(\theta)) = [-2d_1 + \min_{n\in\Omega^{d_2}} \alpha_n(\theta), 2d_1 + \max_{n\in\Omega^{d_2}} \alpha_n(\theta)] \subset [-2d, 2d]. \quad (2.3)$$

Lemma 2.2 *The operator H defined in (1.1)-(1.4) is decomposable in direct integral.*

Proof. Let A be a multiplication operator on $\mathcal{H}_2$ by a measurable function f, and let F be an operator of $l^2(\mathbb{Z}^d)$ into itself defined by $F = U_2^{-1}AU_2$, then one has

$$(F\varphi)(x,\xi+n|N) = \sum_{n'\in\mathbb{Z}^{d_2}} \varphi(x,\xi+n'|N)\widetilde{f}(n-n'),$$

where

$$\tilde{f}(n) = \int_{\mathbb{T}^{d_2}} e^{-i\theta\cdot n} f(\theta)\frac{d\theta}{(2\pi)^{d_2}}.$$

By a direct calculation one finds that the commutator $[H,F]=0$, this shows that the operator H is decomposable according to the Theorem XIII.84 of [18]. ■

Lemma 2.3 *We have*

$$U_2 H U_2^{-1} = \int_{\mathbb{T}^{d_2}}^{\oplus} H(\theta)\frac{d\theta}{(2\pi)^{d_2}},$$

where $H(\theta) = H_0(\theta) + V_\theta(X)$, and $V_\theta(X)$ is the potential $\delta(x)v(\xi)$ on $\mathcal{H}_2'$.

Proof. In view of Corollary 2.1 It suffices to verify that

$$U_2 V U_2 = \int_{\mathbb{T}^{d_2}}^{\oplus} V_\theta \frac{d\theta}{(2\pi)^{d_2}}.$$

This follows from

$$(U_2 V f)_\theta(X) = V_\theta(X)(U_2 f)_\theta(X)$$

which is obvious from a direct calculation. ■

Theorem 2.2 *We have*

$$\begin{aligned}\sigma_{ac}(H(\theta)) &= \sigma_{ac}(H_0(\theta)) = [-2d_1 + \min_{n\in\Omega^{d_2}} \alpha_n(\theta), 2d_1 + \max_{n\in\Omega^{d_2}} \alpha_n(\theta)],\\ \sigma_{sc}(H(\theta)) &= \varnothing,\end{aligned}$$

and $H(\theta)$ has at most a finite number of eigenvalues situated outside of $[-2d_1 - \alpha(\theta), 2d_1 + \alpha(\theta)]$.

Proof. We have $H(\theta) = H_0(\theta) + V_\theta(X)$ where $V_\theta(X)$ is the multiplication operator by a finite-rank matrix whose rank

$$r = \text{rank} V_\theta(X) = |\Omega^{d_2}|$$

is the volume of Ω^{d_2}. Then by the Theorem XI.10 of [18] we have

$$\begin{aligned}\sigma_{ac}(H(\theta)) &= \sigma_{ac}(H_0(\theta)) = [-2d_1 + \min_{n\in\Omega^{d_2}} \alpha_n(\theta), 2d_1 + \max_{n\in\Omega^{d_2}} \alpha_n(\theta)],\\ \sigma_{sc}(H(\theta)) &= \varnothing.\end{aligned}$$

Moreover, $H(\theta)$ has at most r eigenvalues. Let us show that $\sigma_{pp}(H(\theta)) \cap \sigma_{ac}(H(\theta)) = \varnothing$. Fix $E \in \sigma_{pp}(H(\theta)) \cap \sigma_{ac}(H_0(\theta))$. By the Green's formula for the pair $H(\theta)$ and $H_0(\theta)$ we obtain

$$u_E(x,\xi) = \sum_{\eta\in\Omega^{d_2}} G_E(x, \xi - \eta, H_0(\theta)) v(\eta) u_E(0,\eta),$$

where G_E is the Green function of $H_0(\theta)$ and u_E is the eigenfunction of $H(\theta)$ corresponding to E. We notice that $u_E \in l^2(\mathbb{Z}^{d_1}) \otimes l^2(\Omega^{d_2})$ if and only if $G_E(x, \xi, H_0(\theta))$ decay sufficiently fast in x which is possible only if $E \notin \sigma(H_0(\theta)) = [-2d_1 + \min_{n\in\Omega^{d_2}} \alpha_n(\theta), 2d_1 + \max_{n\in\Omega^{d_2}} \alpha_n(\theta)]$ ■

In fact a part of the eigenvalues of $H(\theta)$ can be plunged in the spectrum of H_0, i.e. in $[-2d, 2d]$. A priori the Theorem XIII.85-(f) of [18] does not assure us that the spectrum of H is purely absolutely continuous on $[-2d, 2d]$, therefore first of all we will study these eigenvalues and show that they generate an absolutely continuous spectrum for H.

Let $E \in \sigma_{pp}(H(\theta))$. For all $n \in \Omega^{d_2}$ we define

$$k_E^\theta(n) = \left(\int_{\mathbb{T}^{d_1}} \frac{dp}{\Phi_{d_1}(p) + \alpha_n(\theta) - E}\right)^{-1}, \tag{2.4}$$

where $\alpha_n(\theta)$ are the eigenvalues of $h_0(\theta)$ defined in (2.2), and

$$\Phi_{d_1}(p) = -2\sum_{j=0}^{d_1} \cos p_j.$$

$k_E^\theta(n)$ is well defined because according to Theorem 2.2 one has $E \notin \sigma(H_0(\theta))$ this means that $\forall n \in \Omega^{d_2}$, $\Phi_{d_1}(p) + \alpha_n(\theta) - E \neq 0$. Let us now define the following operator

$$\begin{aligned} K_{E,v}^\theta \quad &: \quad \mathcal{H}_1^{'} \longrightarrow \mathcal{H}_1^{'} \\ (K_{E,v}^\theta \psi)(n) \quad &= \quad k_E^\theta(n)\psi(n) + \sum_{n' \in \Omega^{d_2}} \tilde{v}(n - n')\psi(n'), \end{aligned} \tag{2.5}$$

where

$$\tilde{v}(n) = \sum_{\xi \in \Omega^{d_2}} \overline{\phi_n^\theta(\xi)} v(\xi).$$

and $\phi_n^\theta(\xi)$ are the eigenfunctions of $h_0(\theta)$ defined in the equation (2.1). The spectrum of this operator is clearly pure point. Moreover we have

Lemma 2.4 *We have*

$$0 \in \sigma_{pp}(K_{E,v}^\theta) \Leftrightarrow E \in \sigma_{pp}(H(\theta)).$$

Proof. Let ψ_E^θ be the eigenfunction of $K_{E,v}^\theta$ corresponding to the eigenvalue 0. By a simple calculation and by using (2.4) and (2.5) we find that the function

$$u_E^\theta(x, \xi) = \int_{\mathbb{T}^{d_1}} dp e^{ix.p} \sum_{n \in \Omega^{d_2}} \phi_n^\theta(\xi) \frac{k_E^\theta(n)}{\Phi_{d_1}(p) + \alpha_n(\theta) - E} \psi_E^\theta(n)$$

is an eigenfunction of $H(\theta)$ corresponding to the eigenvalue E. ■

Let us suppose that $\theta = \theta(t) = a + tb$ where a and b are two fixed vectors in $\mathbb{R}^{d_2}$, and $t \in \mathbb{R}$.

Lemma 2.5 *Let $E \in \sigma_{pp}(H(\theta))$ and let $A_E(t) = K_E^{\theta(t)}$. Then for any $t \in \mathbb{R}$ there exists a neighborhood of the real axis where the eigenvalues $\{\lambda_E^n(\cdot)\}$ are analytic not identically constant in t.*

Proof. Let $K_{E,0}^{\theta(t)}$ be the operator $K_{E,v=0}^{\theta(t)}$. Obviously, the eigenvalues of this operator are

$$k_E^{\theta(t)}(n) = (\int_{\mathbb{T}^{d_1}} \frac{dp}{\Phi_{d_1}(p) + \alpha_n(a+tb) - E})^{-1}. \tag{2.6}$$

To show that $\lambda_E^n(t)$ is analytic on a neighborhood of $\mathbb{R}$ it is enough to show that they are bounded for a finite $t \in \mathbb{R}$. Let $\psi_E^n(t)$ be the normalized eigenfunctions of $A_E(t)$, i.e.

$$\begin{aligned} A_E(t)\psi_E^n(t) &= \lambda_E^n(t)\psi_E^n(t), \\ \|\psi_E^n(t)\| &= 1. \end{aligned}$$

Then we have

$$(A_E(t)\psi_E^n(t), \psi_E^n(t)) = \lambda_E^n(t).$$

And (see [13])

$$\begin{aligned} \frac{d\lambda_E^n(t)}{dt} &= (\frac{dA_E(t)}{dt}\psi_E^n(t), \psi_E^n(t)) \\ &= (\frac{dK_{E,0}^{\theta(t)}}{dt}\psi_E^n(t), \psi_E^n(t)) \\ &= \frac{dk_E^{\theta(t)}(n)}{dt}. \end{aligned}$$

And this last quantity $\frac{dk_E^{\theta(t)}(n)}{dt}$ is explicitly calculable. By deriving the equation (2.6) in t one finds

$$\frac{dk_E^{\theta(t)}(n)}{dt} = \frac{d\alpha_n(a+tb)}{dt} \int_{\mathbb{T}^{d_1}} \frac{dp}{(\Phi_{d_1}(p) + \alpha_n(a+tb) - E)^2} (k_E^{\theta(t)}(n))^{-2}.$$

This derivative is obviously bounded. Thus there exists $C > 0$ such that

$$|\frac{d\lambda_E^n(t)}{dt}| = |\frac{dk_E^{\theta(t)}(n)}{dt}| \leq C.$$

Then $\lambda_E^n(t)$ can not grow up to infinity for a finite $t \in \mathbb{R}$.

Thus, we can write $\lambda_E^n(\tau)$ where $\tau \in \mathbb{C}$ belongs to a certain neighborhood of $t \in \mathbb{R}$. We have to show that $\lambda_E^n(\tau)$ is not identically constant. Let us suppose that $\lambda_E^n(\tau)$ is constant

$$\lambda_E^n(\tau) = \lambda. \tag{2.7}$$

We have according to the relation (2.2)

$$\alpha_n(a + \tau b) = -2 \sum_{j=1}^{d_2} \cos(\frac{a_j + \tau b_j}{N_j} + \frac{2\pi n_j}{N_j})$$

Let us suppose that $\tau = \mu + iy \in \mathbb{C}$. So there exists C_1, m two positive constants such that

$$|\alpha_n(a + \tau b)| \geq C_1(e^{m|y|} + 1).$$

Then if y_0 is big enough, there is $C(y_0) > 0$ such that

$$|\int_{\mathbb{T}^{d_1}} \frac{dp}{\Phi_{d_1}(p) + \alpha_n(a + \tau b) - E}| \leq \int_{\mathbb{T}^{d_1}} \frac{dp}{|\Phi_{d_1}(p) + \alpha_n(a + \tau b) - E|} \leq \frac{1}{C(y_0)(e^{m|y|} + 1)}.$$

Thus

$$|k_E^{\theta(\tau)}(n)| \geq C_2(e^{m|y|} + 1). \tag{2.8}$$

And for any $\zeta \in \mathbb{C} \backslash \mathbb{R}$ there exist a positive constant C such that we have the bound

$$\| (K_{E,0}^{\theta(\tau)} - \zeta)^{-1} \| \leq \frac{C}{e^{m|y|} + 1}.$$

By taking y to infinity we obtain

$$\lim_{y \to \infty} \| (K_{E,0}^{\theta(\tau)} - \zeta)^{-1} \| = 0. \tag{2.9}$$

Let $\tilde{v}$ be the following operator

$$\begin{aligned} \tilde{v} \quad &: \quad \mathcal{H}_1' \to \mathcal{H}_1' \\ (\tilde{v}\psi)(n) \quad &= \quad \sum_{n' \in \Omega^{d_2}} \tilde{v}(n - n')\psi(n'). \end{aligned}$$

This operator is a finite-rank matrix. Thus we have also

$$\lim_{y\to\infty} \| \tilde{v}(K_{E,0}^{\theta(\tau)} - \zeta)^{-1} \| = 0. \tag{2.10}$$

By (2.9), (2.10), and the resolvent identity one finds

$$\lim_{y\to\infty} \| (K_{E,v}^{\theta(\tau)} - \zeta)^{-1} \| = 0. \tag{2.11}$$

Since $K_{E,v}^{\theta(\tau)}$ is a finite dimensional operator we have

$$\| (K_{E,v}^{\theta(\tau)} - \zeta)^{-1} \| \geq \frac{1}{|\lambda - \zeta|},$$

where λ is defined in (2.7). This relation contradicts (2.11). Therefore $\lambda_E^n(t)$ cannot be constant function. ■

Lemma 2.6 *For any $t \in \mathbb{R}$ the eigenvalues $\lambda_E^n(t)$ are strictly monotonous in E.*

Proof. With the same notations of the proof of Lemma 2.5 one has (e.g. [13])

$$\begin{aligned} \frac{d\lambda_E^n(t)}{dE} &= (\frac{dA_E(t)}{dE}\psi_E^n(t), \psi_E^n(t)) \\ &= (\frac{dK_{E,0}^{\theta(t)}}{dE}\psi_E^n(t), \psi_E^n(t)) \\ &= \frac{dk_E^{\theta(t)}(n)}{dE}. \end{aligned}$$

By the direct calculation of the derivative of $k_E^{\theta(t)}(n)$ in E from the relation (2.6) we obtain

$$\frac{dk_E^{\theta(t)}(n)}{dE} = -\int_{\mathbb{T}^{d_1}} \frac{dp}{(\Phi_{d_1}(p) + \alpha_n(a+tb) - E)^2} (\int_{\mathbb{T}^{d_1}} \frac{dp}{\Phi_{d_1}(p) + \alpha_n(a+tb) - E})^{-2}.$$

Thus

$$\frac{d\lambda_E^n(t)}{dE} < 0.$$

This yields the result. ■

Theorem 2.3 *Fix $\theta(t) = a + tb$ where a, b are two vectors in $\mathbb{R}^{d_2}$, and let $B(t) = H(\theta(t))$. Then for any $t_0 \in \mathbb{R}$ there exist a neighborhood of the real axis in t such that the eigenvalues $\{E_n(t)\}_n$ of $B(t)$ are analytic and not identically constant in this neighborhood.*

Proof. By Lemma 2.4 one has

$$E \in \sigma_{pp}(B(t)) \Longleftrightarrow 0 \in \sigma_{pp}(A_E(t)),$$

where $A_E(t) = K_E^{\theta(t)}$. Let $\{\lambda_E^n(t)\}$ the set of the eigenvalues of $A(t)$. According to Lemmas 2.5 and 2.6 $\lambda_E^n(t)$ is an analytic function not identically constant on a neighborhood of the real axis in t, and strictly monotonous in E. By the theorem of implicit functions there exists $E_n(t)$ an analytic function not identically constant in t such that $E = E_n(t)$. ■

Now we can follow the schema the demonstration of the Theorem XIII.100 of [18]:

Theorem 2.4 *The spectrum of H is purely absolutely continuous.*

Proof. Let $b, K_2, \cdots, K_{d_2}$ be a basis of $\mathbb{R}^{d_2}$, thus $\mathbb{T}^{d_2} = \{\theta = s_1 b + s_2 K_2 + \cdots + s_{d_2} K_{d_2} | s_1 \in M(s_\perp), s_\perp = (s_2, \ldots, s_{d_2}) \in N\}$, then we have

$$H = \int_{s_\perp \in N} \int_{s_1 \in M(s_\perp)} H(s_1 b + \cdots + s_{d_2} K_{d_2}) \frac{ds_1 ds_\perp}{(2\pi)^{d_2}},$$

According to Theorem 2.2 and Theorem 2.3 the spectrum of $B(s_1) = H(s_1 b + \cdots + s_{d_2} K_{d_2})$ is the union of a purely absolutely continuous spectrum and a set of analytic eigenvalues not identically constant in s_1. According to the two Theorems XIII.86 and XIII.85-(f) of [18] the spectrum of

$$\int_{s_1 \in M(s_\perp)} H(s_1 b + \cdots + s_{d_2} K_{d_2}) \frac{ds_1}{2\pi}$$

is purely absolutely continuous. By applying XIII.85-(f) of [18] once again to the direct integral on $s_\perp \in N$ one finds the result. ■

Remark. In fact the part of $\sigma(H)$ coming from the direct integral of the eigenvalues of $H(\theta)$ is the surface spectrum of H because the corresponding generalized eigenfunctions decay in transversal directions x . This follows from the fact that the direct integration of the eigenfunctions of $H(\theta)$ does not act on x. The other part of the spectrum of $H(\theta)$ which comes from the direct integration of the absolutely continuous spectrums of $H(\theta)$ is the bulk spectrum and is equal to $[-2d, 2d]$. The intersection of these two parts is not necessary empty because a part of $H(\theta)$'s eigenvalues can be plunged in $[-2d, 2d]$.

3 Generalized spectral shift function:

The spectral shift function ξ was introduced by I.Lifchitz [16] and M.Krein [15] for the trace class perturbations i.e. for a couple of operators (A, B) such that $\mathrm{Tr}\{B-A\} < \infty$. This function verifies the trace formula (see [3, 19] for more results and references), i.e. for any function f in certain class of real functions ($C^\infty(\mathbb{R})$ with compact support for example), one has

$$\int_R f'(\lambda)\xi(\lambda)d\lambda = \mathrm{Tr}\{f(B) - f(A)\}. \tag{3.1}$$

We showed in [4] that when one perturbs the discrete Schrödinger operator by a surface homogeneous (ergodic or periodic for example) potential a quantity $\bar{\xi}$ exists in the distribution's sense. This quantity is the analogue of the spectral shift function, and we called it the generalized spectral shift function. In the particular case of a periodic surface potential a formula similar to the trace formula (3.1) exists and has the form

$$\int f'(\lambda)\bar{\xi}(\lambda)d\lambda = \frac{1}{|\Omega^{d_2}|}\,\mathrm{Tr}\,P_\Omega\{f(H) - f(H_0)\}, \tag{3.2}$$

where P_Ω is the orthogonal projection on the slab $\Omega = \mathbb{Z}^{d_1} \times \Omega^{d_2}$.

Let $H_0(\theta), H(\theta)$ be the two operators defined in the preceding section. In fact the perturbation $(H(\theta), H_0(\theta))$ is of a finite-rank, and thus according to [3] the spectral shift function $\xi(\lambda, \theta)$ of this couple exists.

In [4] we showed, in particular, that for the simplest case ($v(\xi) = Const.$ the generalized spectral shift function $\bar{\xi}$ is a usual function (not distribution) and is given by the relation

$$\bar{\xi}(\lambda) = \int_{\mathbb{R}} \xi_{d_1}(\lambda - \mu)N_{d_2}(d\mu) \tag{3.3}$$

where ξ_{d_1} is the spectral shift function of the couple $(-\Delta_{d_1} + a\delta(x), -\Delta_{d_1})$ and N_{d_2} is the integrated density of states of $h_0 = -\Delta_{d_2}$. We will prove the next Theorem which is a generalization of the relation (3.3) for a periodic potential. We can rewrite (3.3) as following

$$\bar{\xi}(\lambda) = \int_{\mathbb{T}^{d_2}} \xi_{d_1}(\lambda - \Phi(\theta))\frac{d\theta}{(2\pi)^{d_2}},$$

where $\Phi(\theta) = -2\sum_{j=1}^{d_2} \cos\theta_j$.

Theorem 3.1 *Let $\bar{\xi}(\lambda)$ be the generalized spectral shift function of (H, H_0). Then*

$$\bar{\xi}(\lambda) = \frac{1}{|\Omega^{d_2}|} \int_{\mathbb{T}^{d_2}} \xi(\lambda, \theta) \frac{d\theta}{(2\pi)^{d_2}}.$$

Proof. As we mentioned before the theorem the spectral shift function $\xi(\lambda, \theta)$ of the pair $(H(\theta), H_0(\theta))$ exists and verifies the trace formula (3.1), thus $\forall f \in C^\infty(\mathbb{R})$ with compact support

$$\int f'(\lambda)\xi(\lambda, \theta) d\lambda = \mathrm{Tr}\{f(H(\theta) - f(H_0(\theta))\}.$$

In the other hand

$$\begin{aligned}
\int f'(\lambda)\bar{\xi}(\lambda) d\lambda &= \frac{1}{|\Omega^{d_2}|} \mathrm{Tr}\, P_\Omega \{f(H) - f(H_0)\} \\
&= \frac{1}{|\Omega^{d_2}|} \int_{\mathbb{T}^{d_2}} \frac{d\theta}{(2\pi)^{d_2}} \mathrm{Tr}\{f(H(\theta) - f(H_0(\theta))\} \\
&= \frac{1}{|\Omega^{d_2}|} \int_{\mathbb{T}^{d_2}} \frac{d\theta}{(2\pi)^{d_2}} \int f'(\lambda)\xi(\lambda, \theta) d\lambda.
\end{aligned}$$

By applying Fubini's Theorem one finds that for any function $f \in C^\infty(\mathbb{R})$ with compact support

$$\int f'(\lambda)(\bar{\xi}(\lambda) - \frac{1}{|\Omega^{d_2}|} \int_{\mathbb{T}^{d_2}} \xi(\lambda, \theta) \frac{d\theta}{(2\pi)^{d_2}}) d\lambda = 0.$$

This relation is equivalent to the assertion of the theorem. ■

This theorem shows that studying the smoothness and asymptotic properties of $\xi(\lambda, \theta)$ allows us to study the smoothness and the asymptotic properties of $\bar{\xi}(\lambda)$. This will be discussed in a later work.

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