CLASSICAL LIMIT FOR EUCLIDEAN GIBBS STATES OF QUANTUM CONTINUOUS SYSTEMS

YU. G. KONDRATIEV, T. KUNA, AND G. V. SHCHEPAN'UK

ABSTRACT. We consider finite volume Euclidean Gibbs states corresponding to quantum continuous systems of particles obeying Boltzmann, Bose and Fermi statistics and show that in the classical limit they converge to the corresponding finite volume Gibbs states of classical continuous systems.

KEY WORDS: Classical limit, quantum continuous systems, Euclidean Gibbs states.

1. INTRODUCTION

Consider an infinite system of quantum particles of mass m in $\mathbb{R}^d$, dynamics of which can be described by the following heuristic Hamiltonian

$$\widehat{H} = -\frac{\hbar^2}{2m}\sum_i \Delta_i + \sum_{i,j} v(x_i, x_j),$$

where Δ_i is the Laplace operator with respect to the variable $x_i \in \mathbb{R}^d$, $\hbar$ is the Planck constant and $v : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a pair potential, which is supposed to be a symmetric function.

According to the Bohr correspondence principle one can expect that in a classical limit, *ie* when $\hbar \to 0$ or, what is the same, $m \to \infty$, the equilibrium behaviour of this system at inverse temperature β and fugasity z, should be similar to the equilibrium behaviour of the corresponding classical system with inverse temperature β and activity $\tilde{z}$.

The connection between quantum fugasity z and classical activity is usually given by the following formula

$$z = \left(\frac{2\pi\beta\hbar^2}{m}\right)^{d/2}\tilde{z}, \tag{1}$$

see *eg* [11, 12], and this means that undertaking the classical limit we should regard z to be a function of $\hbar$, $\tilde{z}$, β and m as given by (1)

In the present paper we show that this is really the case by proving that the finite volume Euclidean Gibbs states (with empty boundary condition) corresponding to quantum continuous systems of particles with Boltzmann, Bose and Fermi statistics converge, in the classical limit, to the corresponding finite volume Gibbs states of classical continuous systems.

Date: 22 September 1999.

Bonn University, D–53013 Bonn, Germany, e-mail: kondratiev@uni-bonn.de; BiBoS research center, Bielefeld University, D–33615, Bielefeld, Germany; Institute for Mathematics of the Ukrainian National Science Academy, 3 Tereshchenkivs'ka St, MSP, Kyiv-4, 252601, Ukraine.

Bonn University, D–53013 Bonn, Germany; e-mail: tkuna@wiener.iam.uni-bonn.de; BiBoS research center, Bielefeld University, D–33615, Bielefeld, Germany;

BiBoS research center, Bielefeld University, D–33615, Bielefeld, Germany, e-mail: shchepan@physik.uni-bielefeld.de; Institute for Mathematics of the Ukrainian National Science Academy, 3 Tereshchenkivs'ka St, MSP, Kyiv-4, 252601, Ukraine, e-mail: gena@imath.kiev.ua.

It is also shown that the limiting state does not depend on the statistics of the corresponding quantum system, which explains again that in the case of classical systems we have only one statistics.

The concept of Euclidean Gibbs states naturally follows from the classical works by J. Ginibre [7] and C. Gruber [9] on reduced density matrices and Euclidean Green functions for quantum gases, see [10]. Let us also mention that in the case of quantum lattice systems study of the classical limit for corresponding Euclidean Gibbs states was initiated by S. Albeverio and R. Høegh-Krohn in [1]. They considered the case of so-called quantum lattice systems with gentle anharmonic interaction. Recently the classical limit for more general quantum lattice systems was analyzed in [2].

The contents of the present work is as follows. In Section 2 we introduce the notions of Euclidean Gibbs states corresponding to the quantum continuous systems of particles obeying Boltzmann statistics. In Section 3 we formulate and prove our result about convergence (in the classical limit) of the Euclidean Gibbs states corresponding to quantum continuous systems with Boltzmann statistics to the corresponding Gibbs states of classical continuous systems. Section 4 deals with the corresponding fact about quantum continuous systems with Bose statistics. Finally, Section 5 deals with the case of Fermi statistics.

2. Euclidean Gibbs states for quantum continuous systems with Boltzmann statistics

Let $\mathfrak{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$ and let $\Omega_\beta := C^{\mathrm{per}}\big([0,\beta] \mapsto \mathbb{R}^d\big)$ be the set of all continuous periodic functions from $[0,\beta]$ into $\mathbb{R}^d$ equipped with the uniform norm

$$\|\omega\|_{\mathrm{u},\beta} := \max_{t\in[0,\beta]} \big|\omega(t)\big|.$$

We introduce a measure $W^\hbar_{z,\beta}$ on the σ-algebra $\mathcal{C}(\Omega_\beta)$ generated by all cylinder sets of the form $\{\omega \in \Omega_\beta \mid \omega(t_1) \in B_1, \dots, \omega(t_n) \in B_n\}$, $B_1, \dots, B_n \in \mathfrak{B}(\mathbb{R}^d)$ and $0 \leqslant t_1 < t_2 < \cdots < t_n < \beta$, by the following formula

$$W^\hbar_{z,\beta}\big(\{\omega \in \Omega_\beta \mid \omega(t_1) \in B_1, \dots, \omega(t_n) \in B_n\}\big) = \\ = z \int\limits_{B_1} dx_1 \cdots \int\limits_{B_n} dx_n\, p^\hbar_{t_2-t_1}(x_2|x_1) \dots p^\hbar_{t_n-t_{n-1}}(x_n|x_{n-1}) p^\hbar_{t_1+\beta-t_n}(x_1|x_n),$$

where

$$p^\hbar_t(x|y) = \exp\left[\frac{\hbar^2 t}{2m}\Delta\right](x,y) = \Big(\frac{m}{2\pi\hbar^2 t}\Big)^{d/2} \exp\Big[-\frac{m}{2\hbar^2 t}(x-y)^2\Big]$$

is the kernel of the heat semigroup

$$\exp\left[\frac{\hbar^2 t}{2m}\Delta\right],$$

see *eg* [8], and z is given by (1).

Let for any $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$

$$\Omega_{\beta,\Lambda} := \big\{\omega \in \Omega_\beta \mid \omega(0) \in \Lambda\big\}.$$

Note that $\Omega_{\beta,\mathbb{R}^d} = \Omega_\beta$ and

$$W^\hbar_{z,\beta}(\Omega_{\beta,\Lambda}) = |\Lambda| \tilde{z},$$

where $|\Lambda|$ is the Lebesgue measure of the set Λ.

For the description of Euclidean Gibbs states corresponding to quantum continuous systems with Boltzmann statistics we introduce for each, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, the Euclidean configuration space

$$\Gamma_{\Omega_{\beta,\Lambda}} := \big\{\gamma \subset \Omega_{\beta,\Lambda} \mid \#(\gamma \cap \Omega_{\beta,\Lambda'}) < \infty \text{ for any } \Lambda' \in \mathfrak{B}_b(\Lambda)\big\},$$

where $\mathfrak{B}_b(\Lambda)$ stands for the set of all bounded Borel $\Lambda' \subset \Lambda$.

Let $\mathcal{G}_{\Lambda'}(\Gamma_{\Omega_{\beta,\Lambda}})$, $\Lambda, \Lambda' \in \mathfrak{B}(\mathbb{R}^d)$, be the σ-algebra generated by all mappings of the form

$$\Gamma_{\Omega_{\beta,\Lambda}} \ni \gamma \mapsto \#(\gamma \cap A) \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

where $A \in \mathcal{C}(\Omega_{\beta,\Lambda'})$ is such that $A \subset \Omega_{\beta,\Lambda''}$ for some $\Lambda'' \in \mathfrak{B}_b(\Lambda')$.

To simplify the notations we systematically omit the lower index Λ at $\mathcal{G}_\Lambda$ in case if $\Lambda = \mathbb{R}^d$.

Let us denote by $\bar{C}_{\mathrm{bs}}(\Omega_\beta)$ the set of all continuous bounded functions on Ω_β with bounded support and introduce for any $\varphi \in \bar{C}_{\mathrm{bs}}(\Omega_\beta)$, $\varphi > -1$, the following functions on Γ_{Ω_β}:

$$e^{\hbar}_{z,\beta}(\varphi, \gamma) := \exp\Big[\sum_{\omega\in\gamma} \ln\big(1 + \varphi(\omega)\big) - \int_{\Omega_\beta} \varphi(\omega) W^{\hbar}_{z,\beta}(d\omega)\Big],$$

which are usually called coherent states. They form a separating set of functions in the sense of [5, Subsection 3.4], and therefore two probability measures P and Q on $\mathcal{G}(\Gamma_{\Omega_\beta})$ coincide if

$$\int_{\Omega_\beta} e^{\hbar}_{z,\beta}(\varphi,\gamma) P(d\gamma) = \int_{\Omega_\beta} e^{\hbar}_{z,\beta}(\varphi,\gamma) Q(d\gamma)$$

for all $\varphi \in \bar{C}_{\mathrm{bs}}(\Omega_\beta)$, $\varphi > -1$.

The Poisson measure $\pi^{\hbar}_{z,\beta}$ on $\mathcal{G}(\Gamma_{\Omega_\beta})$ with intensity $W^{\hbar}_{z,\beta}$ can be defined through its characteristic functional

$$\int_{\Gamma_{\Omega_\beta}} e^{i\langle f,\gamma\rangle} \pi^{\hbar}_{z,\beta}(d\gamma) = \exp\Big[\int_{\Omega_\beta} (e^{if(\omega)} - 1) W^{\hbar}_{z,\beta}(d\omega)\Big], \tag{2}$$

where f is any bounded, $\mathcal{C}(\Omega_\beta)$-measurable function with bounded support and

$$\langle f, \gamma\rangle := \sum_{\omega\in\gamma} f(\omega).$$

The measure $\pi^{\hbar}_{z,\beta}$ corresponds to the quantum continuous system of non-interacting particles with Boltzmann statistics at fugasity z.

Lemma 1. *For any $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$ and $F \in L^1\big(\Gamma_{\Omega_\beta}, \mathcal{G}_\Lambda(\Gamma_{\Omega_\beta}), \pi^{\hbar}_{z,\beta}\big)$*

$$\int_{\Gamma_{\Omega_\beta}} F(\gamma \cap \Omega_{\beta,\Lambda}) \pi^{\hbar}_{z,\beta}(d\gamma) = \int_{\Gamma_{\Omega_{\beta,\Lambda}}} F(\gamma) \pi^{\hbar,\Lambda}_{z,\beta}(d\gamma),$$

where $\pi^{\hbar,\Lambda}_{z,\beta}$ is the projection of $\pi^{\hbar}_{z,\beta}$ onto $\Omega_{\beta,\Lambda}$, ie,

$$\int_{\Gamma_{\Omega_{\beta,\Lambda}}} F(\gamma) \pi^{\hbar,\Lambda}_{z,\beta}(d\gamma) = e^{-W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Omega_{\beta,\Lambda})^n} F\big(\{\omega_1, \dots, \omega_n\}\big) W^{\hbar}_{z,\beta}(d\omega_1) \dots W^{\hbar}_{z,\beta}(d\omega_n).$$

Consider now a continuous system of point particles interacting via a continuous potential $v : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfying the following stability condition:

$$\exists B > 0: \quad \forall n \geqslant 2, \quad \forall x_1, \dots, x_n \in \mathbb{R}^d \quad \sum_{1\leqslant i<j\leqslant n} v(x_i, x_j) \geqslant -Bn. \tag{3}$$

and define for any $\omega, \omega' \in \Omega_\beta$

$$V(\omega, \omega') := \int_0^\beta v\big(\omega(t), \omega'(t)\big) dt.$$

A finite volume Euclidean Gibbs state $G^{\hbar}_{z,\beta,\Lambda}(\cdot)$, $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$ corresponding to the quantum continuous system of particles (with Boltzmann statistics) at the inverse temperature β and fugasity z is a measure on $\mathcal{G}(\Gamma_{\Omega_\beta})$ such that for any $B \in \mathcal{G}(\Gamma_{\Omega_\beta})$

$$G^{\hbar}_{z,\beta,\Lambda}(B) = \frac{1}{\Xi^{\hbar}_{z,\beta,\Lambda}} \int\limits_B \exp\Big[-\sum_{\{\omega,\omega'\}\subset\gamma} V(\omega,\omega')\Big]\pi^{\hbar,\Lambda}_{z,\beta}(d\gamma), \tag{4}$$

where

$$\Xi^{\hbar}_{z,\beta,\Lambda} = \int\limits_{\Gamma_{\Omega_{\beta,\Lambda}}} \exp\Big[-\sum_{\{\omega,\omega'\}\subset\gamma} V(\omega,\omega')\Big]\pi^{\hbar,\Lambda}_{z,\beta}(d\gamma). \tag{5}$$

Remark 1. Note that under our conditions on the potential v, $\Xi^{\hbar}_{z,\beta,\Lambda} < \infty$, and so $G^{\hbar}_{z,\beta,\Lambda}$ is defined correctly.

Remark 2. In our work we regard finite volume Euclidean Gibbs states only with empty boundary condition. For more general settings as well as definition of infinite volume Euclidean Gibbs states see [10].

3. Classical limit for quantum continuous systems with Boltzmann statistics

All constructions of the configuration spaces, Poisson measures and Gibbs states corresponding to classical continuous systems can be obtained from that of the previous section if instead of the measurable space $\big(\Omega_\beta, \mathcal{C}(\Omega_\beta), W_\beta\big)$ one takes $\big(\Omega^0_\beta, \mathcal{C}(\Omega^0_\beta), W^0_{\tilde{z},\beta}\big)$, where

$$\Omega^0_\beta := \big\{\omega \in \Omega_\beta \mid \exists x \in \mathbb{R}^d : \omega(t) = x \text{ for all } t \in [0,\beta]\big\}$$

is the space of all constant functions from $[0,\beta]$ to $\mathbb{R}^d$ (which is naturally isomorphic to $\mathbb{R}^d$) and $W^0_{\tilde{z},\beta}(d\omega) := \tilde{z}dx$.

Our main result concerning quantum continuous systems of particles obeying Boltzmann statistics can now be formulated in the following

Theorem 1. *Finite volume Euclidean Gibbs states $G^{\hbar}_{z,\beta,\Lambda}(\cdot)$, $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$, corresponding to the equilibrium states of the quantum continuous systems of particles (obeying Boltzmann statistics and interacting via a stable continuous potential v) at the inverse temperature β and fugasity*

$$z = z(\hbar) = \Big(\frac{2\pi\beta\hbar^2}{m}\Big)^{d/2}\tilde{z} \tag{6}$$

converge, as $\hbar \to 0$ to the Gibbs states $G^0_{\tilde{z},\beta,\Lambda}(\cdot)$ corresponding to the classical continuous systems of particles at inverse temperature β and activity $\tilde{z}$ on all coherent states $e^{\hbar}_{z,\beta}(\varphi,\cdot)$ with $\varphi \in \bar{C}_{\mathrm{bs}}(\Omega_\beta)$, $\varphi > -1$.

Proof. Let us first prove a weak convergence of the corresponding intensity measures $W^{\hbar}_{z,\beta}$ to $W^0_{\tilde{z},\beta}$, as $\hbar \to 0$. To do this we just need to show [5, Section 3, Subsection 4, Proposition 4.4] that

$$\lim_{\hbar\to 0}\int\limits_{\Omega_\beta} f(\omega)W^{\hbar}_{z,\beta}(d\omega) = \int\limits_{\Omega_\beta} f(\omega)W^0_{\tilde{z},\beta}(d\omega) \tag{7}$$

for all uniformly continuous bounded functions $f : \Omega_\beta \mapsto \mathbb{R}^d$ with bounded support, or equivalently, that for any uniformly continuous bounded function $f : \Omega_{\beta,\Lambda} \mapsto \mathbb{R}^d$ with bounded support and $\varepsilon > 0$ there exists such positive constant $\hbar_0$ that for all $\hbar \in [0,\hbar_0[$

$$I := \Big|\int\limits_\Lambda dx \int\limits_{\Omega_{\beta,0|0}} W^{\hbar}_{\beta,0|0}(d\omega)f(x+\omega) - \int\limits_\Lambda dx f(x)\Big| < \varepsilon, \tag{8}$$

where $W^{\hbar}_{\beta,0|0}$ is the Wiener bridge measure over the space $\Omega_{\beta,0|0} := \Omega_{\beta,\{0\}}$ of all continuous loops of 'length' β starting at zero.

Using a standard probabilistic technique, see *eg* [8] or [3], one can show that for any $r > 0$

$$W^{\hbar}_{\beta,0|0}\big(\{\omega \in \Omega_{\beta,0|0} \mid \|\omega\|_{\mathrm{u},\beta} \geqslant r\}\big) \leqslant 2\Omega(d)\Big(\frac{2}{\pi}\Big)^{d/2}\Gamma\Big(\frac{d}{2}, \frac{mr^2}{4\beta\hbar^2}\Big), \tag{9}$$

where $\Omega(d)$ denotes the volume of d-dimensional unit sphere, and

$$\Gamma(d,\alpha) := \int\limits_{\alpha}^{\infty} x^{d-1}e^{-x}dx, \qquad \alpha > 0$$

is the incomplete gamma-function having the following asymptotics:

$$\Gamma(d+1,\alpha) = \frac{\alpha^{d+1}e^{-\alpha}}{\alpha-d}\left[1 - \frac{d}{(\alpha-d)^2} + \frac{2d}{(\alpha-d)^3} + O\Big(\frac{d^2}{(\alpha-d)^4}\Big)\right], \qquad \alpha \to +\infty, \tag{10}$$

see *eg* [4, Chapter IX]. Note that as a function on $\mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ :=]0,+\infty[$, the gamma-function is decreasing with respect to the second variable.

As it follows from (9) and (10)

$$W^{\hbar}_{\beta,0|0}\big(\{\omega \in \Omega_{\beta,0|0} \mid \|\omega\|_{\mathrm{u},\beta} \geqslant r\}\big) \to 0, \text{ as } \hbar \to 0.$$

Let us choose $r \in \mathbb{R}_+$ in such a way that

$$\big|f(x+\omega) - f(x)\big| < \frac{\varepsilon}{2|\Lambda|}$$

for all $x \in \Lambda$ and $\omega \in \Omega_{\beta,0|0}$ such that $\|\omega\|_{\mathrm{u},\beta} < r$ (this is possible because f is uniformly continuous), and afterwards, $\hbar_0 > 0$ such that

$$W^{\hbar_0}_{\beta,0|0}\big(\{\omega \in \Omega_{\beta,0|0} \mid \|\omega\|_{\mathrm{u},\beta} \geqslant r\}\big) < \frac{\varepsilon}{4|\Lambda|\,\|f\|_\infty}.$$

Then, for all positive $\hbar < \hbar_0$

$$I \leqslant \int\limits_{\Lambda} dx \int\limits_{\{\omega\in\Omega_{\beta,0|0} \mid \|\omega\|_{\mathrm{u},\beta}\geqslant r\}} W^{\hbar}_{\beta,0|0}(d\omega)\big|f(x+\omega) - f(x)\big| + \\ + \int\limits_{\Lambda} dx \int\limits_{\{\omega\in\Omega_{\beta,0|0} \mid \|\omega\|_{\mathrm{u},\beta}< r\}} W^{\hbar}_{\beta,0|0}(d\omega)\big|f(x+\omega) - f(x)\big| < \varepsilon,$$

which proves (7).

Next, for any $\varphi \in \bar{C}_{\mathrm{bs}}(\Omega_\beta)$, $\varphi > -1$,

$$\int\limits_{\Gamma_{\Omega_\beta}} e^{\hbar}_{z,\beta}(\varphi,\gamma)G^{\hbar}_{z,\beta,\Lambda}(d\gamma) = \frac{1}{\Xi^{\hbar}_{z,\beta,\Lambda}} \exp\Big[- W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda}) - \int\limits_{\Omega_\beta} \varphi(\omega)W^{\hbar}_{z,\beta}(d\omega)\Big]\Big\{1 + \\ + \sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{(\Omega_{\beta,\Lambda})^n} \exp\Big[\sum_{i=1}^{n} \ln\big(1+\varphi(\omega_i)\big) - \sum_{1\leqslant i<j\leqslant n} V(\omega_i,\omega_j)\Big] W^{\hbar}_{z,\beta}(d\omega_1)\ldots W^{\hbar}_{z,\beta}(d\omega_n)\Big\}. \tag{11}$$

As it follows from Lemma 2 below the function

$$\exp\Big[- \sum_{1\leqslant i<j\leqslant n} V(\omega_i,\omega_j)\Big]$$

is continuous on $(\Omega_\beta)^n$. The same is true for the function

$$\exp\Big[\sum_{i=1}^{n}\ln\big(1+\varphi(\omega_i)\big)\Big].$$

These functions are not bounded, but using the stability condition we can choose such $c>0$ that

$$\exp\Big[-\sum_{1\leqslant i<j\leqslant \#\gamma} V(\omega_i,\omega_j)+\sum_{i=1}^{\#\gamma}\ln\big(1+\varphi(\omega_i)\big)\Big]e^{-c\cdot(\#\gamma)}$$

is a bounded function on

$$\widetilde{\Gamma}_{\Omega_{\beta,\Lambda}}:=\bigsqcup_{n=0}^{\infty}(\Omega_{\beta,\Lambda})^n.$$

Next, on the natural σ-algebra over $\widetilde{\Gamma}_{\Omega_{\beta,\Lambda}}$ we introduce the following measure

$$\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}:=\exp\big[-e^c W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda})\big]\sum_{n=0}^{\infty}\frac{e^{cn}}{n!}(W^{\hbar}_{z,\beta})^{\otimes n}.$$

Then, one can interpret (11) as the integral with respect to $\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}$.

So, to finish the proof of the theorem we just have to show the weak convergence, as $\hbar\to 0$, of the family of measures $\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}$, $\Lambda\in\mathfrak{B}_b$.

For every $M\in\mathbb{N}$ there exists a compact set $K_M\subset\Omega_{\beta,\Lambda}$ such that

$$\frac{W^{\hbar}_{z,\beta}(K_M)}{W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda})}\geqslant 1-\frac{1}{M}.$$

Hence, for the set

$$\widetilde{K}_M:=\bigsqcup_{n=0}^{M}(K_M)^n,$$

which is obviously compact in $\widetilde{\Gamma}_{\Omega_{\beta,\Lambda}}$,

$$\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}(\widetilde{K}_M)\geqslant \exp\big[-e^c W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda})\big]\sum_{n=0}^{M}\frac{e^{cn}}{n!}\Big(1-\frac{1}{M}\Big)^n\big(W^{\hbar}_{z,\beta}(\Omega_{\beta,\Lambda})\big)^n\to 1,\qquad \text{as } M\to\infty,$$

uniformly in $\hbar$. This yields the tightness of $\{\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}\}_{\hbar}$.

Therefore, as the weak convergence of $W^{\hbar}_{z,\beta}$ implies the weak convergence of $(W^{\hbar}_{z,\beta})^n$ for all $n\in\mathbb{N}$, and the continuous functions with support in $(\Omega_{\beta,\Lambda})^n$ for some $n\in\mathbb{N}$ separate points, *cf* [5, Section 3, Subsection 4, Lemma 4.3], $\tilde{\pi}^{\hbar,\Lambda}_{z,\beta}$ weakly converge to $\tilde{\pi}^{0,\Lambda}_{z,\beta}$. □

Lemma 2. *For any continuous potential* $v:\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}$ *the function*

$$V(\omega,\omega')=\int_0^\beta v\big(\omega(t),\omega'(t)\big)dt$$

is continuous on $(\Omega_\beta)^2$.

Proof. Let $\{(\omega_n,\omega'_n)\}_{n\in\mathbb{N}}$ be a sequence from $(\Omega_\beta)^2$ converging to $(\omega,\omega')\in(\Omega_\beta)^2$ in $\|\cdot\|_{\mathrm{u},\beta}$. Then, there exists a compact set $K\subset\mathbb{R}^d$ such that $\omega_1(t),\omega'_1(t),\dots\omega_n(t),\omega'_n(t),\omega(t),\omega'(t)\in K$ for all $t\in[0,\beta]$. By continuity of v we have that

$$\sup_{x,y\in K}\big|v(x,y)\big|<\infty.$$

So, we always can interchange the limit with respect to n and integration, which proves the statement of the lemma. $\square$

4. Classical limit for quantum continuous systems with Bose statistics

Let for any $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$

$$\Omega_\Lambda := \bigsqcup_{p=1}^{\infty} \Omega_{p\beta,\Lambda},$$

and

$$\mathcal{C}(\Omega_\Lambda) := \sigma\Big(\bigsqcup_{p=1}^{\infty} \mathcal{C}(\Omega_{p\beta,\Lambda})\Big).$$

The Euclidean configuration space Γ_{Ω_Λ}, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, corresponding to the quantum continuous system of point particles with Bose statistics and the σ-algebra $\mathcal{G}_{\Lambda'}(\Gamma_{\Omega_\Lambda})$, $\Lambda, \Lambda' \in \mathfrak{B}(\mathbb{R}^d)$, is introduced in the same way as the corresponding objects for quantum continuous system with Boltzmann statistics, one just need to drop β from all the corresponding definitions of Section 2.

Remark 3. Just as in Section 3 we systematically omit the lower index Λ in $\mathcal{G}_\Lambda$ and Ω_Λ if it is equal to $\mathbb{R}^d$.

Definition 1. We say that a set $A \subset \Omega$ is bounded if there exist $p \in \mathbb{N}$ and $C \in \mathbb{R}_+$ such that

$$\max_{\omega\in A} p(\omega) \leqslant p \qquad \text{and} \qquad \sup_{\omega\in A} \|\omega\|_{\mathrm{u},p(\omega)\beta} \leqslant C,$$

where the function $p : \Omega_\Lambda \mapsto \mathbb{N}$ is defined in such a way that $\omega \in \Omega_{p(\omega)\beta,\Lambda}$.

The Poisson measure $\pi_z^\hbar$ on $\mathcal{G}(\Gamma_\Omega)$ corresponding to the quantum continuous system of non-interacting particles is defined via the following characteristic functional

$$\int_{\Gamma_\Omega} e^{i\langle f,\gamma\rangle} \pi_z^\hbar(d\gamma) = \exp\left[\sum_{p=1}^{\infty} \frac{z^{p-1}}{p} \int_{\Omega_{p\beta}} (e^{if(\omega)} - 1) W_{z,p\beta}^\hbar(d\omega)\right],$$

where f is any bounded $\mathcal{C}(\Omega)$-measurable function with bounded support. In other words, the intensity measure here is as follows

$$W_z^\hbar = \sum_{p=1}^{\infty} \frac{z^{p-1}}{p} W_{z,p\beta}^\hbar.$$

Remark 4. By $\pi_z^{\hbar,\Lambda}$ we denote the projection of $\pi_z^\hbar$ on Γ_{Ω_Λ}, *cf* Lemma 1.

Let for any $\omega, \omega' \in \Omega$

$$V_q(\omega) = \sum_{0\leqslant i<j\leqslant p(\omega)-1} \int_0^\beta v\big(\omega(i\beta+t), \omega(j\beta+t)\big)dt$$

and

$$V_q(\omega,\omega') = \sum_{i=0}^{p(\omega)-1} \sum_{j=0}^{p(\omega')-1} \int_0^\beta v\big(\omega(i\beta+t), \omega'(j\beta+t)\big)dt.$$

A finite volume Euclidean Gibbs state $G^{\hbar,s}_{z,\beta,\Lambda}(\cdot)$, $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$, corresponding to the quantum continuous system of point particles (with Bose statistics) the inverse temperature β and fugasity $z < e^{-\beta B}$ is a measure on $\mathcal{G}(\Gamma_\Omega)$ such that for any $B \in \mathcal{G}(\Gamma_\Omega)$

$$G^{\hbar,s}_{z,\beta,\Lambda}(B) = \frac{1}{\Xi^{\hbar,s}_{z,\beta,\Lambda}} \int_B \exp\Big[-\sum_{\omega\in\gamma} V_q(\omega) - \sum_{\{\omega,\omega'\}\subset\gamma} V_q(\omega,\omega')\Big]\pi_z^{\hbar,\Lambda}(d\gamma), \tag{12}$$

where

$$\Xi^{\hbar,s}_{z,\beta,\Lambda} = \int_{\Gamma_{\Omega_\Lambda}} \exp\Big[-\sum_{\omega\in\gamma} V_q(\omega) - \sum_{\{\omega,\omega'\}\subset\gamma} V_q(\omega,\omega')\Big]\pi_z^{\hbar,\Lambda}(d\gamma). \tag{13}$$

Remark 5. Our restriction on parameter z to be less than $e^{-\beta B}$ appears as a sufficient condition ensuring that $\Xi^{\hbar,s}_{z,\beta,\Lambda} < \infty$ for all $\Lambda \in \mathfrak{B}_b$, which is necessary for $G^{\hbar,s}_{z,\beta,\Lambda}$ to be correctly defined, *cf* Remark 1. For the free Bose gas this restriction on the fugasity: $z < 1$, is well-known in physics, see *eg* [12, Chapter V, § 54] or [11, Chapter III, § 1].

Our result about a convergence of the finite volume Euclidean Gibbs states corresponding to quantum continuous systems of particles obeying Bose statistics is essentially the same as in the case of Boltzmann statistics:

Theorem 2. *Finite volume Euclidean Gibbs states* $G^{\hbar,s}_{z,\beta,\Lambda}(\cdot)$, $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$, *corresponding to the equilibrium states of the quantum continuous systems of particles (obeying Bose statistics and interacting via a stable continuous potential* v*) at the inverse temperature* β *and fugasity*

$$z = z(\hbar) = \Big(\frac{2\pi\beta\hbar^2}{m}\Big)^{d/2}\tilde{z}$$

converge, as $\hbar \to 0$, *to the Gibbs states* $G^0_{\tilde{z},\beta,\Lambda}(\cdot)$ *corresponding to the classical continuous systems of particles at inverse temperature* β *and activity* $\tilde{z}$ *on all coherent states*

$$e_z^{\hbar} := \exp\Big[\sum_{\omega\in\gamma} \ln\big(1+\varphi(\omega)\big) - \int_\Omega \varphi(\omega)W_z^{\hbar}(d\omega)\Big]$$

with $\varphi \in \bar{C}_{bs}(\Omega)$, $\varphi > -1$.

Proof. The proof of Theorem 2 is essentially the same as the proof of Theorem 1, with the only difference that now we have to show that for any uniformly continuous function $f : \Omega_\Lambda \mapsto \mathbb{R}^d$ with bounded support

$$\lim_{\hbar\to 0}\sum_{p=1}^{\infty}\frac{z^{p-1}}{p}\int_{\Omega_{p\beta,\Lambda}} f(\omega)W^{\hbar}_{z,p\beta}(d\omega) = \int_{\Omega_{\beta,\Lambda}} f(\omega)W^0_{\tilde{z},\beta}(d\omega),$$

or, taking into account (7), that

$$\lim_{\hbar\to 0}\sum_{p=2}^{\infty}\frac{z^{p-1}}{p}\int_{\Omega_{p\beta,\Lambda}} f(\omega)W^{\hbar}_{z,p\beta}(d\omega) = 0,$$

where z is a function of β, m, $\hbar$ and $\tilde{z}$ as follows from (1). But the latter is a trivial consequence of (1) and the fact that the function $f \in \bar{C}_{\mathrm{bs}}(\Omega_\Lambda)$. □

5. Classical limit for quantum continuous systems with Fermi statistics

The results of the present paper stated in Theorems 1 and 2 can be easily extended to include the case of quantum continuous system of point particles obeying the Fermi statistics. The only difference here consists in the fact that finite volume Euclidean Gibbs states corresponding to quantum continuous systems with Fermi statistics are no more measures but charges (signed measures) over configuration space Γ_Ω, defined by the following formulae, *cf* (12) and (13),

$$G^{\hbar,a}_{z,\beta,\Lambda}(B) = \frac{1}{\Xi^{\hbar,a}_{z,\beta,\Lambda}} \int_B (-1)^{\langle p-1,\gamma\rangle} \exp\Big[-\sum_{\omega\in\gamma} V_q(\omega) - \sum_{\{\omega,\omega'\}\subset\gamma} V_q(\omega,\omega')\Big]\pi^{\hbar}_z(d\gamma),$$

$$\Xi^{\hbar,a}_{z,\beta,\Lambda} = \int_{\Gamma_{\Omega_\Lambda}} (-1)^{\langle p-1,\gamma\rangle} \exp\Big[-\sum_{\omega\in\gamma} V_q(\omega) - \sum_{\{\omega,\omega'\}\subset\gamma} V_q(\omega,\omega')\Big]\pi^{\hbar}_z(d\gamma),$$

where $B \in \mathcal{G}(\Gamma_\Omega)$.

It should be mentioned here that because of the fact that for all $\Lambda \in \mathfrak{B}_b(\mathbb{R}^d)$,

$$\Xi^{\hbar,a}_{z,\beta,\Lambda} \to \Xi^0_{\tilde{z},\beta,\Lambda}, \quad \text{as} \quad \hbar \to 0,$$

and $\Xi^0_{\tilde{z},\beta,\Lambda} > 0$ is well-defined, the signed measure $G^{\hbar,a}_{z,\beta,\Lambda}(\cdot)$ is correctly defined at least for sufficiently small $\hbar$.

Acknowledgments.

The first author, Yu.K., was supported by DFG (Deutsche Forschungsgemeinschaft) through the project no. AL 214/9–2 and SFB 256, Bonn University. The third author, G.S., was partially supported by a DAAD research fellowship and SFB 343, Bielefeld University. He also gratefully acknowledges the kind hospitality of the Research Center BiBoS, Bielefeld University.

References

[1] S. Albeverio and R. Høegh-Krohn “Homogeneous random fields and quantum statistical mechanics” *Journal of Functional Analysis* **19** (1975), P. 242–272.

[2] S. Albeverio, Yu. G. Kondratiev and Yu. Kozitsky “Classical limits for Gibbs states of quantum lattice systems.” (To appear in Letters in Mathematical Physics.)

[3] A. N. Borodin *Handbook of Brownian motion: facts and formulae.* Birkhauser, 1996, 462 p.

[4] A. Erdélyi *et al. Higher transcendental functions.* Vol. 2. McGraw-Hill Book Company, Inc.: New York–St. Louis–San Francisco–Düsseldorf–Johannesburg–Kuala Lumpur–London–Mexico–Monreal–New Delhi–Panama–Rio de Janeiro–Singapure–Sidney–Toronto, 1953, 396 p.

[5] S. N. Ethier and T. G. Kurtz *Markov processes. Characterization and convergence.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons: New York–Chichester–Brisbane–Toronto–Singapore, 1986, 534 p.

[6] H.-O. Georgii *Canonical Gibbs measures*, Lecture Notes in Mathematics **760**, Springer: Berlin–Heidelberg–New York, 1979.

[7] J. Ginibre “Reduced density matrices of quantum gases I. Limit of infinite volume,” *Journal of Mathematical Physics* **6**, 238–251 (1965), “II. Cluster property,” *ibid.* **6**, 252–262 (1965), “III. Hard core potentials,” *ibid.* **6**, 1432–1446 (1965).

[8] J. Ginibre “Some applications of functional integration in statistical mechanics.” In *Statistical Mechanics and Quantum Field Theory*, Lectures delivered at Summer School in Les Houches, France, 1970, edited by C. De Witt and R. Stora. Gordon and Breach: New York, 1971, pp. 327–427.

[9] C. Gruber *Green functions in quantum statistical mechanics.* PhD theses. Princeton, 1968.

[10] Yu. G. Kondratiev, E. W. Lytvynov, A. L. Rebenko, M. Röckner and G. V. Shchepan'uk “Euclidean Gibbs states for quantum systems with Boltzmann statistics via cluster expansions” *Methods of Functional Analysis and Topology* **3**, no. 1 (1997), P. 62–81.

[11] I. A. Kvasnikov *Thermodynamics and statistical physics. Equilibrium systems.* Moscow University: Moscow, 1991, 793 p.

[12] E. M. Lifshits and L. P. Pitaevskii *Statistical physics*, Part 1. Series: Course of Theoretical Physics, L. D. Landau and E. M. Lifshits Eds. Pergamon Press: Oxford–New York–Toronto–Sydney–Paris–Frankfurt, 1980, 544 p.

[13] C. Preston *Random fields*, Lecture Notes in Mathematics **534**. Springer-Verlag: Berlin–Heidelberg–New York, 1976.

[14] C. Preston *Specifications and their Gibbs states*, manuscript (unpublished).