Infinite-dimensional generalized continued fractions, distribution of quadratic residues and non-residues and ergodic theory

L.D. Pustyl'nikov*

Abstract

A new theory of generalized continued fractions for infinite-dimensional vectors with integer components is constructed. The results of this theory are applied to the classical problem on the distribution of quadratic residues and non-residues modulo a prime number and based on the study of ergodic properties of some infinite-dimensional transformations.

Introduction

While usual continuous fractions enable to approximate real numbers by rational ones, it is the mission of generalized continuous fractions to approximate more complicated objects through the elements of a countable dense subset in the relevant space.

Generalized continued fractions are a classic object of mathematics. The generalizations of continued fractions for number vectors were studied by Euler, Dirichlet, Jacobi, Perron, Poincaré, Hermite, Hurwitz, Klein, Minkowski, Solotaryov, Voronoi and many other mathematicians [1]–[17].

In papers [18]–[22] the notion of an (A, ω)-continued fraction was introduced for an arbitrary real n-dimensional vector depending on a map A of n-dimension space and on a vector ω that belongs to the unit n-dimensional torus. This notion was applied in [18]–[21] to solutions of problems of analysis and number theory, which were set up by Weyl [24] on finding strong estimates of Weyl sums and the remainder

*BiBoS, Universität Bielefeld, D-33615 Bielefeld, Germany

term in the law of distribution of fractional parts of the values of polynomial. In the special case where x is number ($n = 1$) and A has the form $A : x \to \frac{1}{x}$, the (A, ω)-continued fraction is an ordinary continued fraction [25].

In this paper we use the construction of an (A, ω)-continued fraction for the definition of an (A, p, ω)-continued fraction of an arbitrary infinite-dimensional vector $x = (x_1, x_2, \ldots)$ with integer components $x_n (n = 1, 2, \ldots)$. (A, p, ω)-continued fraction depends on infinite-dimensional map A, on infinite sequence $p = (p_1, p_2, \ldots)$ of pairwise relative prime natural numbers $p_1, p_2, \ldots$ and on a vector $\omega = (\omega_1, \omega_2, \ldots)$ that belongs to some infinite-dimensional torus T, which is defined with the help of the sequence p. We establish the basic properties of (A, p, ω)-continued fractions which are connected with ergodic properties of the map $\overline{A}$ obtained from map A by the projection onto the torus T.

This theory is applied to classical problem of the distribution of quadratic residues and non-residues modulo a prime number going from Euler and Gauss. It is well-know that if $\tilde{p}$ is prime more than 2 then there are the same number of quadratic residues and quadratic non-residues in the row $1, \ldots, \tilde{p} - 1$ [26]. The main problem is to prove the analogue of this assertion for arbitrarily slowly increasing lengths of intervals $X_{\tilde{p}} = (x_{\tilde{p}} + 1, \ldots, x_{\tilde{p}} + r_{\tilde{p}})$ of sequentically placed integer numbers as $\tilde{p} \to \infty$. It means that for $\tilde{p} \to \infty$ the number $Q'_{\tilde{p}}$ of quadratic residues and the number $Q''_{\tilde{p}}$ of quadratic non-residues modulo $\tilde{p}$ which are contained in $X_{\tilde{p}}$ have the forms

$$Q'_{\tilde{p}} = \frac{r_{\tilde{p}}}{2} + o(\tau_{\tilde{p}}) \ , \qquad Q''_{\tilde{p}} = \frac{r_{\tilde{p}}}{2} + o(r_{\tilde{p}}) \ , \tag{1}$$

where $\lim_{\tilde{p} \to \infty} \frac{o(r_{\tilde{p}})}{r_{\tilde{p}}} = 0$. Thre are no many results obtained in the direction of solution of this problem.

First of all we mention that if the sequence of numbers $r_{\tilde{p}}$ is bounded as $\tilde{p} \to \infty$ then for many number of intervals $X_{\tilde{p}}$ the equalities (1) are not valid as $\tilde{p} \to \infty$ ([28], [29]). If $r_{\tilde{p}} = \tilde{p}^{\epsilon}$, and $\epsilon > \frac{1}{4}$ then the equalities (1) were proved in [30] for intervals $X_{\tilde{p}}$ with arbitrary value $x_{\tilde{p}}$ and originally this assertion was proved in [27] for $\epsilon > \frac{1}{2}$. The main result of this paper is theorem 8 (section 5) in which the equalities (1) are proved for arbitrarily slowly increasing function $\psi(\tilde{p}) = r_{\tilde{p}}$, for any subsequence of prime numbers p_n (instead of all prime $\tilde{p}$) such that its speed of growth more then some function depended on the function $\psi(\tilde{p})$, and for almost all collections of interval X_{p_n} in sense of natural measure on torus T.

Moreover, the assertion of theorem 8 is valid for any collection of intervals X_{p_n} but concrete lengths r_{p_n} of these intervals are defined by means of expansion of vector $x = (x_{p_1}, x_{p_2}, \ldots) \in T$ (the n-th component of x is x_{p_n}) in (A, p, ω)-continued fraction in which $A = A_*$ is the some concrete map, $p = (p_1, p_2, \ldots)$ is the sequence consisting of prime numbers p_n, and the vector $\omega \in \Omega$, where Ω is some subset of torus T which is explicitly constructed in section 5.

If for $\omega \in \Omega$ the (A_*, p, ω)-continued fraction of vector x is finite (it is true for almost all $x \in T$) then equalities (1) are valid for all elements p_n of the sequence p, but if for $\omega \in \Omega$ the (A_*, p, ω)-continued fraction of vector x is infinite then equalities (1) are valid for arbitrarily long initial interval of numbers of the sequence p, which is defined by means of (A_*, p, ω)-convergents of the continued fractions of the vector x. Theorem 8 is the corollary of theorem 7 having independent significance on the estimates of shot sums of Legendre symbols by means of (A, p, ω)-continued fractions.

Ergodic theory plays important role in theory of (A, p, ω)-continued fraction. It is applied to the map $\overline{A}$ on torus T. In section 2 it is proved that the map $\overline{A}$ is ergodic on T with respect natural measure and the Birkhoff ergodic theorem is valid everywhere for the characteristic function of a cylinder (the definition of cylinder is given at the beginning of section 2). The map $\overline{A}$ is the group shift on the compact group with respect to topology for which all cylinders are closed and open sets simultaneously, and therefore according to general ergodic theory ([31]) the unique ergodicity of the map $\overline{A}$ is the corollary of its ergodicity. It means that the map $\overline{A}$ has the unique Borel normalized invariant measure. In theorem 1 we use the following main conditions which are necessary for ergodicity of map $\overline{A}$: any two numbers in the sequence $p = (p_1, p_2, \ldots)$ are pairwise relatively prime and for any n the number p_n is relatively prime to the number γ_n which is used in the definition of the map A (§ 1). In the opposite case the map $\overline{A}$ can be not ergodic.

The paper contains five sections . In sections 3 and 4 the theory of (A, p, ω)-continued fraction is constructed, in section 2 the ergodic properties of the map $\overline{A}$ are studied and section 5 is devoted to quadratic residues and non-residues.

We present the list of notation and definitions used in this paper in section 1. In addition, other notation and definitions having the meaning for some sections are introduced there.

The numbering of formulas, theorems, lemmas, corollaries, remarks and definitions is throughout the paper.

The main results of this paper were announced in [23].

The author thanks Forschungszentrum Bielefeld-Bonn-Stochastik for the support.

1 General definitions and notation

We introduce the following objects:

1) a sequence $p = (p_1, p_2, \ldots)$ of infinitely many pairwise distinct, pairwise relatively prime natural numbers $p_n (n = 1, 2, \ldots)$;

2) a vector $\gamma = (\gamma_1, \gamma_2, \ldots)$ with integer components such that the number γ_n is relatively prime to p_n for every $n = 1, 2, \ldots$;

3) the discrete circle S_n of the residues $0, 1, \ldots, p_{n-1}$ modulo p_n and the measure μ_n on S_n such that the measure of every element of S_n inequal to p_n^{-1};

4) the torus T that is the Cartesian product of the circles S_n for all $n = 1, 2, \ldots$, a point of which is a vector $\omega = (\omega_1, \omega_2, \ldots)$ such that $\omega_n \in S_n$ $(n = 1, 2, \ldots)$, and the measure mes on T that is the product of measures μ_n for all n;

5) the map A that sends a vector $x = (x_1, x_2, \ldots)$ to the vector $x' = Ax = (x'_1, x'_2, \ldots)$ with components $x'_n = x_n + \gamma_n$ $(n = 1, 2, \ldots)$;

6) the map $\overline{A}$ of T that sends $\omega = (\omega_1, \omega_2, \ldots) \in T$ to $\overline{\omega}_n = \omega_n + \gamma_n \text{mod } p_n$ $(n = 1, 2, \ldots)$;

7) the map $\hat{A}$ of T which is the invers of $\overline{A}$ and sends $\omega = (\omega_1, \omega_2, \ldots) \in T$ to $\hat{\omega} = \hat{A}\omega = (\hat{\omega}_1, \hat{\omega}_2, \ldots)$ with components $\hat{\omega}_n = \omega_n - \gamma_n \text{mod } p_n (n = 1, 2, \ldots)$;

8) for a real number z, the symbol [z] means the integer part of z and the symbol $\{z\}$ is its fractional part;

9) for any vector $x = (x_1, x_2, \ldots)$ with integer components $x_n (n = 1, 2, \ldots)$ we introduce the vectors $[[x]] = ([[x]]_1, [[x]]_2, \ldots)$ $\{\{x\}\} = (\{\{x\}\}_1, \{\{x\}\}_2, \ldots)$, where for $n = 1, 2, \ldots$ the components $[[x]]_n = \left[\frac{x_n}{p_n}\right]$ and $\{\{x\}\}_n = x_n - p_n [[x]]_n$;

10) the assertion which is valid everywhere on T means that it holds for all vectors $x \in T$ and the assertion which is valid for almost all $x \in T$ means that it holds for all vectors $x \in T$, except for a set of measure mes zero.

2 Ergodic properties of the map $\overline{A}$

Let s be a natural number and $k_1, \ldots, k_s$ be pairwise distinct natural numbers.

Definition 1. We introduce s-dimensional torus $T_{k_1,\ldots,k_s} = S_{k_1} \times \ldots \times S_{k_s}$ that is the direct product of the circles $S_{k_1}, \ldots, S_{k_s}$, and the measure $\mu_{k_1,\ldots,k_s}$ on $T_{k_1,\ldots,k_s}$ that is the direct product of measures $\mu_{k_1}, \ldots, \mu_{k_s}$.

Definition 2. Let $c_{k_1,\ldots,k_s}$ be a subset of torus $T_{k_1,\ldots,k_s}$. We introduce the cylinder $C_{k_1,\ldots,k_s} \subset T$ that is the subset of T consisting of all such vectors $x = (x_1, x_2, \ldots)$ for which s-dimensional vector $x_{k_1}, \ldots, x_{k_s}$ for with components $x_{k_1}, \ldots, x_{k_s}$ belongs to subset $c_{k_1,\ldots,k_s}$.

Remark 1. From the definitions of torus T and measure mes on T it follows that σ-algebra of mes-measurable sets is generated by all cylinders $C_{k_1,\ldots,k_s}$, and for any cylinder $C_{k_1,\ldots,k_s}$ the equality mes $C_{k_1,\ldots,k_s} = \mu_{k_1,\ldots,k_s}(C_{k_1,\ldots,k_s})$ holds.

Definition 3. The characteristic function of a set is the function that takes the value 1 on this set and the value 0 in other points.

Lemma 1. Let $\Omega \subset T$, Ω be a mes-measurable set. Then the set $\overline{A}(\Omega)$ is mes-measurable on T and mes $\Omega =$ mes $\overline{A}(\Omega)$.

Proof. If Ω - a cylinder, then the statement of lemma 1 follows from the definition of map $\overline{A}$ in section 1 and from the Definition 2. For arbitrary mes-measurable set Ω this statement is corollary of the remark 1. Lemma 1 is proved.

Definition 4. The map acting in a space with a finite measure and conserving this measure is called ergodic if there are no subsets having the measure differing from 0 and the measure of all space, which are invariant with respect to this map.

We formulate the Birkhoff ergodic theorem relating to the transformation $\overline{A}$ acting on the torus T with measure mes, which is used in what follows.

Birkhoff ergodic theorem ([31]). Let $f(x)$ be a function on T such that $|f(x)|$ is integrable function on T with respect to measure mes. Then for almost all $x \in T$ there exists $\lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{A}^k x) = \overline{f}(x)$, and if the transformation $\overline{A}$ is ergodic then the function $\overline{f}(x)$ is a constant which is equal to the integral of the function $f(x)$ over the torus T with respect to measure mes.

Now we prove Theorem 1 from which it follows that the map $\overline{A}$ is ergodic and consequently the second part of Birkhoff ergodic theorem is valid.

Theorem 1. The transformation $\overline{A}$ on the torus T is ergodic with respect to measure mes, and if $f(x)$ is the characteristic function of a cylinder then for it the Birkhoff ergodic theorem is valid everywhere on T.

Proof.

Lemma 2. Let $k_1, \ldots, k_s$ be pairwise distinct natural numbers, numbers $p_{k_1}, \ldots, p_{k_s}$ are differed from 1, $t_{k_\nu}(\nu = 1, \ldots, s)$ are integer numbers such that $1 \leq t_{k_\nu} \leq p_{k_\nu} - 1$. Then the sum $R_{k_1,\ldots,k_s} = \sum_{\nu=1}^{s} \frac{\gamma_{k_\nu} t_{k_\nu}}{p_{k_\nu}}$ cannot be integer.

Proof of lemma 2. Let $\frac{a_\nu}{b_\nu} = \frac{t_{k_\nu}}{p_{k_\nu}}$ be the fraction in which the numerator a_ν and the denominator b_ν are pairwise relatively simple numbers. Then by virtue of the numbers γ_n and p_n are relatively simple and the numbers $p_{k_1}, \ldots, p_{k_s}$ are pairwise relatively simple, we obtain that the quantity $R_{k_1,\ldots,k_s} = \sum_{\nu=1}^{s} \frac{\gamma_{k_\nu} a_\nu}{b_\nu} = \frac{\gamma_{k_1} a_1 \prod_{\nu=2}^{s} b_\nu + L}{b_1 \ldots b_s}$ is the fraction in which the denominator divides by number $b_1 \neq 1$ and the numerator does not divide by b_1, because its first term does not divide by b_1 and second ones L divides by b_1. Therefore, the number $R_{k_1,\ldots,k_s}$ cannot be integer.

Lemma 2 is proved.

According to Remark 1 the orthogonal basis of the space L_2 on T with respect to measure mes consists of the functions

$$f_{k_1,\ldots,k_s}(x) = \exp\left(2\pi i \left(\frac{x_{k_1} t_{k_1}}{p_{k_1}} + \ldots + \frac{x_{k_s} t_{k_s}}{p_{k_s}}\right)\right) , \tag{2}$$

where $x = (x_1, x_2, \ldots) \in T$; $k_1, \ldots, k_s$ are pairwise distinct natural numbers; $0 \leq t_{k_\nu} \leq p_{k_\nu} - 1$; t_{k_ν} is integer; $\nu = 1, \ldots, s$. We consider the operator $\mathcal{U}$ which is conjugated to $\overline{A}$ and acts in L_2 on T so that if $f(x) \in L_2$ then $(\mathcal{U}f)(x) = f(\overline{A}x)$. From the definition of $\overline{A}$ in the section 1 it follows that the functions $f_{k_1,\ldots,k_s}(x)$ from (2) are the eigenfunctions of the operator $\mathcal{U}$ and their eigenvalues $\lambda_{k_1,\ldots,k_s} = \exp(2\pi i R_{k_1,\ldots,k_s})$, where $R_{k_1,\ldots,k_s}$ are numbers introduced in the formulation of Lemma 2. By virtue of Lemma 2, if the numbers $p_{k_1}, \ldots, p_{k_s}$ are differed from 1 and numbers $t_{k_1}, \ldots, t_{k_s}$ are differed from 0 then $\lambda_{k_1,\ldots,k_s} \neq 1$. Therefore, there is no function differing from a constant which is invariant with respect to the operator $\mathcal{U}$, and the map $\overline{A}$ is ergodic ([31]).

Now we prove that for characteristic function $\chi_{k_1,\dots,k_s}(x)$ of a cylinder $C_{k_1,\dots,k_s}$ the Birckhoff ergodic theorem is valid everywhere. According to definitions 1, 2 and to Remark 1 it means that for any vector $x \in T$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{k_1,\dots,k_s}(\overline{A}^k x) = \text{mes } C_{k_1,\dots,k_s} = \mu_{k_1,\dots,k_s}(c_{k_1,\dots,k_s}) \ , \tag{3}$$

where $c_{k_1,\dots,k_s}$ is the subset of torus $T_{k_1,\dots,k_s}$ corresponding to the cylinder $C_{k_1,\dots,k_s}$ by virtue of Definition 2. As any function of a finite number variables $x_{k_1}, \dots, x_{k_s}$ taking the finite number of values $x_{k_\nu} \in S_{k_\nu} (\nu = 1, \dots, s)$ can be represented by the finite linear combination of the functions $f_{k_1,\dots,k_s}(x)$ introduced in (2), then for the proof of the equality (3) it is sufficiently to prove the following statement: if the numbers $p_{k_1}, \dots, p_{k_s}$ are differed from 1 and $t_{k_1}, \dots, t_{k_s}$ are integer numbers satisfying the inequalities

$$1 \le t_{k_\nu} \le p_{k_\nu} - 1 \qquad (\nu = 1, \dots, s) \tag{4}$$

then the function $f_{k_1,\dots,k_s}(x)$ satisfies equalities

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_{k_1,\dots,k_s}(\overline{A}^k x) = 0 \ , \tag{5}$$

$$\int_{T_{k_1,\dots,k_s}} f_{k_1,\dots,k_s}(x_{k_1}, \dots, x_{k_s}) d\mu_{k_1,\dots,k_s} = 0 \ , \tag{6}$$

where $f_{k_1,\dots,k_s}(x_{k_1}, \dots, x_{k_s})$ is the function introduced in (2) and considered as the function of arguments $x_{k_1}, \dots, x_{k_s}$ only. To prove the equality (5) we use the Lemma 2 according to which the number $R_{k_1,\dots,k_s} = \frac{a}{b}$ (a and b are integer numbers) is irreducible fraction differing from integer ($b \ne 1$), and we use the equality

$$f_{k_1,\dots,k_s}(\overline{A}^k x) = \exp(2\pi i k R_{k_1,\dots,k_s}) f_{k_1,\dots,k_s}(x) \ .$$

By virtue of this equality for any natural m the equality $\sum_{k=m}^{m+b-1} f_{k_1,\dots,k_s}(\overline{A}^k x) = 0$ holds. Therefore there exists a constant M depending on the function $f_{k_1,\dots,k_s}(x)$ only such that for all natural numbers n the inequality $|\sum_{k=1}^{n} f_{k_1,\dots,k_s}(\overline{A}^k x)| < M$ holds, and the equality (5) is proved. Now we prove (6). As the numbers $p_{k_1}, \dots, p_{k_s}$ are differed from 1 and the numbers $t_{k_1}, \dots, t_{k_s}$ satisfy the inequality (4) then from the Definition 1 of measure $\mu_{k_1,\dots,k_s}$ we obtain:

$$\int_{T_{k_1,\dots,k_s}} f_{k_1,\dots,k_s}(x_{k_1} \dots, x_{k_s}) d\mu_{k_1,\dots,k_s} = \prod_{\nu=1}^{s} \left(p_{k_\nu}^{-1} \sum_{x_{k_\nu}=0}^{p_{k_\nu}-1} \exp\left(2\pi i \frac{x_{k_\nu} t_{k_\nu}}{p_{k_\nu}}\right)\right) = 0 \ .$$

Theorem 1 is proved.

Corollary 1. Any trajectory of the map $\overline{A}$ is everywhere dense on the torus T in the following sense: the trajectory intersects any cylinder.

3 Generalized continued fractions of infinite-dimensional vectors with integer components

Let $x = (x_1, x_2, \dots)$ be a vector with integer components $x_n (n = 1, 2, \dots)$. We represent x in the form of (A, p, ω)-continued fraction which can be finite or infinite. We denote it by $x = \left[q^{(0)}, \dots, q^{(\nu)}\right]_{A,p,\omega}$ if (A, p, ω)-continued fraction of x is finite and by $x = \left[q^{(0)}, q^{(1)}, \dots\right]_{A,p,\omega}$ if it is infinite. Let $x^{(0)} = x$, $q^{(0)} = \left[\left[x^{(0)}\right]\right]$, $\delta^{(0)} = \{\{x^{(0)}\}\}$. If $\delta^{(0)} = \omega$, then the process of constructing of the (A, p, ω)-continued fraction is completed, $x = \left[q^{(0)}\right]_{A,p,\omega}$ and the (A, p, ω)-continued fraction is finite. Otherwise, if $\delta^{(0)} \neq \omega$, then we suppose $x^{(1)} = Ax^{(0)}$.

We suppose now that for some integer $k \geq 0$ the infinite-dimensional vectors $q^{(0)}, q^{(1)}, \dots, q^{(k)}$ with integer components and infinite-dimensional vectors $x^{(0)}, \dots, x^{(1)}$ are constructed so that $q^{(s)} = \left[\left[x^{(s)}\right]\right]$ and $\delta^{(s)} = \{\{x^{(s)}\}\} \neq \omega$ for $s = 0, \dots, k$.

Let $q^{(k+1)} = \left[\left[x^{(k+1)}\right]\right], \delta^{(k+1)} = \{\{x^{(k+1)}\}\}$. If $\delta^{(k+1)} = \omega$, then the process of constructing of (A, p, ω)-continued fraction is completed, $x = \left[q^{(0)}, \dots, q^{(k+1)}\right]_{A,p,\omega}$, and the $[A, p, \omega)$-continued fraction of vector x is finite. Otherwise, if $\delta \neq \omega$, we suppose that $x^{(k+2)} = A\delta^{(k+1)}$. If $\delta^{(k)} \neq \omega$ for all integers $k \geq 0$, then $x = \left[q^{(0)}, q^{(1)}, \dots\right]_{A,p,\omega}$, and the (A, p, ω)-continued fraction of vector x is infinite. The representation of vector x in the form of an (A, p, ω)-continued fraction if fully described.

It follows from the constructing of (A, p, ω)-continued fraction that it is sufficient to have a map A of the set $T \backslash \omega$ only. In the special case where x is a real number, $[[x]] = [x]$, $\{\{x\}\} = \{x\}$, A is the map of interval $0 \leq y < 1$, which has the form $A : y \to \frac{1}{y}$, and $\omega = 0$, the (A, p, ω)-continued fraction is the ordinary continued fraction [25].

Theorem 2. The (A, p, ω)-continued fraction of x is finite if and only if $\{\{x\}\}$ belongs to the trajectory $\hat{A}^k \omega$ $(k = 0, 1, \dots)$, where $\hat{A}$ is the map introduced in Definition 7) of section 1 ($\hat{A}^0$ is the identitiy map of T).

Proof. Suppose that the (A,p,ω)-continued fraction $x = \left[q^{(0)},\dots,q^{(m)}\right]_{A,p,\omega}$ of vector x is finite. By virtue of Definition 6) of map $\overline{A}$ (section 1) for $k = 0,\dots,m$ we have: $\delta^{(k)} = \overline{A}^k\{\{x\}\}$ and $\delta^{(m)} = \omega$, where $\delta^{(k)}$ are the quantities introduced in the process of constructing of (A,p,ω)-continued fraction. Therefore, according to the definition of the map $\hat{A}$ we have $\{\{x\}\} = \hat{A}^m\omega$. Conversely, suppose that for some integer $m \geq 0$ the equality $\{\{x\}\} = \hat{A}^m\omega$ holds, and for all nonnegative $s < m$ $\hat{A}^s\omega \neq \{\{x\}\}$. Then we have $\delta^{(m)} = \omega$ and $\delta^{(s)} \neq \omega$ for $s < m$. This proves the finiteness of (A,p,ω)-continued fraction of the vector x.

Theorem 2 is proved.

Theorem 3. For any distinct vectors, their (A,p,ω)-continued fraction are distinct.

Proof. Let $\overline{x} = (\overline{x}_1,\overline{x}_2,\dots)$ and $\overline{\overline{x}} = (\overline{\overline{x}}_1,\overline{\overline{x}}_2,\dots)$ are two distinct vectors with integer components $\overline{x}_n$ and $\overline{\overline{x}}_n$ $(n = 1,2,\dots)$, $\overline{q} = [[\overline{x}]]$, $\overline{\delta} = \{\{\overline{x}\}\}$, $\overline{\overline{q}} = [[\overline{\overline{x}}]]$, $\overline{\overline{\delta}} = \{\{\overline{\overline{x}}\}\}$, where $\overline{q} = (\overline{q}_1,\overline{q}_2,\dots)$, $\overline{\delta} = (\overline{\delta}_1,\overline{\delta}_2,\dots)$, $\overline{\overline{q}} = (\overline{\overline{q}}_1,\overline{\overline{q}}_2,\dots)$, $\overline{\overline{\delta}} = (\overline{\overline{\delta}}_1,\overline{\overline{\delta}}_2,\dots)$. Suppose that the (A,p,ω)-continued fraction is finite for at least one of the vectors $\overline{x},\overline{\overline{x}}$. If $\overline{q} \neq \overline{\overline{q}}$, then the assertion of theorem 3 follows from constructing of the (A,p,ω)-continued fraction. If $\overline{q} = \overline{\overline{q}}$, then we have $\overline{\delta} \neq \overline{\overline{\delta}}$. Therefore, assuming that the (A,p,ω)-continued fraction of vectors $\overline{x}$ and $\overline{\overline{x}}$ coincide, by virtue of theorem 2 we have $\overline{\delta} = \overline{\overline{\delta}} = \hat{A}^k\omega$, where k is a natural number. This contradiction proves Theorem 3 in the case where the (A,p,ω)-continued fraction of a least one of the vectors $\overline{x},\overline{\overline{x}}$ is finite.

Let us suppose that the (A,p,ω)-continued fraction of both vectors $\overline{x},\overline{\overline{x}}$ are infinite, $\overline{x} = \left[\overline{q}^{(0)},\overline{q}^{(1)},\dots\right]_{A,p,\omega}$, $\overline{\overline{x}} = \left[\overline{\overline{q}}^{(0)},\overline{\overline{q}}^{(1)},\dots\right]_{A,p,\omega}$. Assuming that the (A,p,ω)-continued fraction of the vectors $\overline{x}$ and $\overline{\overline{x}}$ coincide, we find that $\overline{q}^{(k)} = \overline{\overline{q}}^{(k)}$ for all integers $k \geq 0$ and, hence, $\overline{d} \neq \overline{\overline{\delta}}$.

Let s be the smallest natural number for which $\overline{\delta}_s \neq \overline{\overline{\delta}}_s$. For $k = 0,1,\dots$ we introduce the vectors $\overline{\delta}^{(k)} = \overline{A}^k\overline{\delta} = (\overline{\delta}_1^{(k)},\overline{\delta}_2^{(k)},\dots)$, $\overline{\overline{\delta}}^{(k)} = \overline{A}^k\overline{\overline{\delta}} = (\overline{\overline{\delta}}_1^{(k)},\overline{\overline{\delta}}_2^{(k)},\dots)$ and the vectors $\overline{x}^{(k)} = (\overline{x}_1^{(k)},\overline{x}_2^{(k)},\dots)$, $\overline{\overline{x}}^{(k)} = (\overline{\overline{x}}_1^{(k)} = (\overline{\overline{x}}_1^{(k)},\overline{\overline{x}}_2^{(k)},\dots)$ with components $\overline{x}_n^{(k)} = p_n\overline{q}_n^{(k)} + \overline{\delta}_n^{(k)}$, $\overline{\overline{x}}_n^{(k)} = p_n\overline{\overline{q}}_n^{(k)} + \overline{\overline{\delta}}_n^{(k)}$ $(n = 1,2,\dots)$. From the construction of (A,p,ω)-continued fraction and the definition of map A (Section 1) for any $k = 1,2,\dots$ we have

$$\overline{x}^{(k)} = A\overline{\delta}^{(k-1)} = \overline{\delta}^{(k-1)} + \gamma, \qquad \overline{\overline{x}}^{(k)} = A\overline{\overline{\delta}}^{(k-1)} + \gamma,$$

$$p_s\overline{q}_s^{(k)} \leq \overline{x}_s^{(k)} < p_s\overline{q}_s^{(k)} + p, \qquad p_s\overline{\overline{q}}_s^{(k)} \leq \overline{\overline{x}}_s^{(k)} < p_s\overline{\overline{q}}_s^{(k)} + p_s,$$

$$\overline{x}_s^{(k)} - \overline{\overline{x}}_s^{(k)} = \overline{\delta}_s - \overline{\overline{\delta}}_s \neq 0 \; . \tag{7}$$

Suppose, for definiteness, that

$$\overline{\delta}_s > \overline{\overline{\delta}}_s \; . \tag{8}$$

By virtue of Corollary 1 of Theorem 1 (section 2) and the definition of the vectors $\overline{\delta}^{(k)}$, there exists a natural number k_0 such that

$$\overline{\delta}_s^{(k_0)} < \overline{\delta}_s - \overline{\overline{\delta}}_s \; . \tag{9}$$

Since

$$\overline{x}_s^{(k)} = p_s \overline{q}_s^{(k)} + \overline{\delta}_s^{(k)}, \overline{\overline{x}}_s^{(k)} = p_s \overline{\overline{q}}_s^{(k)} + \overline{\overline{\delta}}_s^{(k)} = p_s \overline{q}_s^{(k)} + \overline{\overline{\delta}}_s^{(k)}$$

for all $k \geq 1$, in accordance with (7) and (8) we have: $\overline{\delta}_s^{(k_0)} - \overline{\overline{\delta}}_s^{(k_0)} = \overline{\delta}_s - \overline{\overline{\delta}}_s > 0$, and this contradicts inequality (9). Theorem 3 is proved.

Theorem 4. The infinite (A, p, ω)-continued fraction $x = \left[q^{(0)}, q^{(1)}, \ldots\right]_{A,p,\omega}$ of the vector x cannot be periodic, i.e. there are no natural numbers k_0 and h such that $q^{k+h} = q^{(k)}$ for any natural $k \geq k_0$.

Proof. Let us suppose the contrary, i.e. that the infinite (A, p, ω)-continued fraction $x = \left[q^{(0)}, q^{(1)}, \ldots\right]_{A,p,\omega}$ is such that $q^{(k+h)} = q^{(k)}$ for $k \geq k_0$. From the definition of the sequence p (section 1) it follows that there exists a natural number s such that $p_s > 1$ and if $n \geq s$ then the number p_n is relatively prime to h. For any vectors $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots) \in T$, $\overline{\overline{x}} = (\overline{\overline{x}}_1, \overline{\overline{x}}_2, \ldots) \in T$ such that their components $\overline{x}_s$ and $\overline{\overline{x}}_s$ satisfy the equalities $\overline{x}_s = 0$, $\overline{\overline{x}}_s = p_s - 1$ their images $\overline{y} = (\overline{y}_1, \overline{y}_2, \ldots) = A\overline{x}, \overline{\overline{y}} = (\overline{\overline{y}}_1, \overline{\overline{y}}_2, \ldots) = A\overline{\overline{x}}$ with respect to the map A satisfy the inequality

$$\left[\frac{\overline{y}_s}{p_s}\right] \neq \left[\frac{\overline{\overline{y}}_s}{p_s}\right] \; . \tag{10}$$

We introduce the vectors $\delta = \{\{x\}\}, \delta^{(k)} = \overline{A}^k \delta$, two sets $\overline{M} \subset T$ $\overline{\overline{M}} \subset T$ such that

$$\overline{M} = \{y = (y_1, y_2, \ldots) : y_s = 0\}, \overline{\overline{M}} = \{y = (y_1, y_2, \ldots) : y_s = p_s - 1\} \tag{11}$$

the torus $T^{(s)}$ that is the Cartesian product of the circles S_n for all $n \geq s$ and the map $\tilde{A}$ of $T^{(s)}$ that sends a vector $y = (y_s, y_{s+1}, \ldots) \in T^{(s)}$ to the vector $\tilde{y} = \tilde{A}y =$

$(\tilde{y}_s, \tilde{y}_{s+1}, \ldots)$ with components $\tilde{y}_n = y_n + h\gamma_n \bmod p_n$ $(n \geq s)$. As according to definitions of numbers γ_n (section 1) and the number s the number $h\gamma_n$ is relatively prime to p_n for every $n \geq s$ then the assertion of Corollary 1 (section 2) is applied to map $\tilde{A}$ on the torus $T^{(s)}$. Therefore, by virtue of Corollary 1 there exist the natural numbers $\overline{n}$ and $\overline{\overline{n}}$ such that

$$\overline{A}^{h\overline{n}}\delta^{(k)} \in \overline{M} \ , \qquad \overline{A}^{h\overline{\overline{n}}}\delta^{(k)} \in \overline{\overline{M}} \ . \tag{12}$$

From the construction of (A, p, ω)-continued fractions, it follows that the vectors $q^{(h\overline{n}+k+1)} = (q_1^{(h\overline{n}+k+1)}, q_2^{(h\overline{n}+k+1)}, \ldots)$, $q^{(h\overline{\overline{y}}+k+1)} = (q_1^{(h\overline{\overline{y}}+k+1)}, q_2^{(h\overline{\overline{y}}+k+1)}, \ldots)$, arising from the (A, p, ω)-continued fraction $x = \left[q^{(0)}, q^{(1)}, \ldots\right]_{A,p,\omega}$ are connected with the vectors $x^{(h\overline{n}+k+1)} = (x_1^{(h\overline{n}+k+1)}, x_2^{(h\overline{n}+k+1)}, \ldots) = A\overline{A}^{h\overline{n}}\delta^{(k)}$, $x^{(h\overline{\overline{y}}+k+1)} = (x_1^{(h\overline{\overline{y}}+k+1)}, x_2^{(h\overline{\overline{y}}+k+1)}, \ldots) = A\overline{A}^{h\overline{\overline{y}}}\delta^{(k)}$, by the following relations:

$$q_\nu^{(h\overline{n}+k+1)} = \left[\frac{x_\nu^{(h\overline{n}+k+1)}}{p_\nu}\right] \ , \qquad q_\nu^{h\overline{\overline{y}}+k+1)} = \left[\frac{x_\nu^{(h\overline{\overline{y}}+k+1)}}{p_\nu}\right] \ ,$$

where $\nu = 1, 2, \ldots$. Therefore, by virtue of relations (10), (11), and (12), we have the inequalities $q_s^{(h\overline{n}+k+1)} \neq q_s^{(h\overline{\overline{y}}+k+1)}, q^{(h\overline{n}+k+1)} \neq q^{(h\overline{\overline{y}}+k+1)}$, which contradict the assumption concerning the periodicity of the infinite (A, p, ω)-continued fraction of the vector x. Theorem 4 is proved.

Theorem 5. Let Ω be a measurable set on T, mes $\Omega > 0$. Then:

1) for almost all $x \in T$ there exists $\omega = \omega(x) \in \Omega$ such that the (A, p, ω)-continued fraction of x is finite;

2) if the set Ω is a cylinder then assertion 1) holds for all $x \in T$.

Proof. Let

$$\chi(y) = \begin{cases} 1 & \text{if} \quad y \in \Omega \\ 0 & \text{if} \quad y \notin \Omega \end{cases}$$

Applying Theorem 1 (section 2) to the map $\overline{A}$ and the function $\chi(y)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N} \chi(\overline{A}^k x) = \text{mes } \Omega > 0 \ . \tag{13}$$

Therefore, the assertion 1) follows from Theorem 2.

In order to prove assertion 2) it suffices to prove that if the set Ω is a cylinder then there exists the limit in (13) for all $x \in T$. This fact follows from the Theorem 1 as well. Theorem 5 is proved.

4 Convergents of Generalized infinite-dimensional continued fractions and their properties

Definition 5. Let $x = \left[q^{(0)}, q^{(1)}, \ldots\right]_{A,p,\omega}$. For all $\nu = 0, 1, \ldots$ we introduce the (A, p, ω)-convergents of the continued fraction $S^{(\nu)} = \left[q^{(0)}, \ldots, q^{(v)}\right]_{A,p,\omega}$ of the vector x as follows: $S^{(\nu)} = (S_1^{(\nu)}, S_2^{(\nu)}, \ldots)$ is an infinite-dimensional vector whose n-th $(n = 0, 1, \ldots)$ component is

$$S_n^{(\nu)} = q_n^{(0)} p_n + (\ldots(-\gamma_n + q_n^{(\nu-1)} p_n + (-\gamma_n + q_n^{(\nu)} p_n + \omega_n))\ldots) \ , \tag{14}$$

where $q_n^{(k)}$ $(k = 0, \ldots, \nu)$ and ω_n are components of the vectors

$$q^{(k)} = (q_1^{(k)}, q_2^{(k)}, \ldots) \ , \qquad \omega = (\omega_1, \omega_2, \ldots) \ .$$

Theorem 6. Let the (A, p, ω)-continued fraction of the vector x be infinite, $S^{(\nu)}$ $(\nu = 0, 1, \ldots)$ be (A, p, ω)-convergents of the continued fraction of x, N and m be natural numbers, $U_N(x) = \{y = (y_1, y_2, \ldots) \in T : y_n = x_n, 1 \leq n \leq N\}$ be the cylinder, $D_{N,m}(x)$ be the number of $\nu \in \{1, \ldots, m\}$ such that $S^{(\nu)} \in U_N(x)$. Then the equality $\lim_{n\to\infty} \frac{D_{N,m}(x)}{m} = (p_1, \ldots, p_N)^{-1}$ holds.

Proof. From the construction of (A, p, ω)-continued fraction and from the definition of map A, it follows that the components x_n $(n \geq 1)$ of vector x can be represented by the form

$$x = q_n^{(0)} p_n + (\ldots(-\gamma_n + q_n^{(\nu-1)} p_n + (-\gamma_n + q_n^{(\nu)} p_n + \delta_n^{(\nu)}))\ldots) \ , \tag{15}$$

where $q_n^{(k)}$ and $\delta_n^{(\nu)}$ are components of vectors $q^{(k)} = (q_1^{(k)}, q_2^{(k)}, \ldots)$ and $\delta^{(\nu)} = (\delta_1^{(\nu)}, \delta_2^{(\nu)}, \ldots)$. Therefore according to the equalities (14) and (15) we have: $x - S^{(\nu)} = \delta^{(\nu)} - \omega$. By virtue of this equality the quantity $D_{N,m}(x)$ introduced in the formulation of Theorem 6 is equal to the quantity $Q_{N,m}(x)$ that is the number of $\nu \in \{1, \ldots, m\}$ for which $\delta^{(\nu)} \in U_N(\omega)$, where the cylinder $U_N(\omega)$ is also defined in the formulation of Theorem 6 for $\omega = x$. Applying Theorem 1 and Birkhoff ergodic theorem to the map $\overline{A}$ and the characteristic function $\chi(y)$ of the cylinder $U_N(\omega)$ by virtue of the equality $\delta^{(\nu)} = \overline{A}^{\nu}\{\{x\}\}$ we have the equality

$$\lim_{m\to\infty} \frac{Q_{N,m}(x)}{m} = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \chi(\overline{A}^k \delta^{(0)}) = \int_T \chi(y) dy = (p_1, \ldots, p_N)^{-1} \ ,$$

from which the assertion of theorem 6 follows. Theorem 6 is proved.

Corollary 2 Let the (A, p, ω)-continued fraction of a vector x be infinite. Then x is the limit point of the sequence $S^{(\nu)} = (S_1^{(\nu)}, S_2^{(\nu)}, \ldots)$ $(\nu = 0, 1, \ldots)$ its (A, p, ω)-convergents of continued fraction in the following sense: for any natural nubmer N there exists the natural number ν such that the component $S^{(\nu)} = x_n$ if $1 \leq n \leq N$.

5 Estimates of sums of Legendre symbols and the distribution of quadratic residues and non-residues modulo a prime number

Definition 6. Let $\tilde{p}$ be a prime. Integer number z not dividing by $\tilde{p}$ is called the quadratic residue modulo $\tilde{p}$, if it is congruent with square of integer modulo $\tilde{p}$, and the number z is the quadratic non-residues modulo $\tilde{p}$, if it is not congruent with any square of integer modulo $\tilde{p}$.

Definition 7. Let $\tilde{p}$ be prime, d be integer. We define the Legendre symbol $\left(\frac{d}{\tilde{p}}\right)$ as follows: this symbol is equal to 1, if d is a quadratic residue modulo $\tilde{p}$, it is equal to -1, if d is a non-residue modulo $\tilde{p}$, and it is equal to 0, if d divides by $\tilde{p}$.

Definition 8. We introduce the map A_* sending an infinite-dimensional vector $x = (x_1, x_2, \ldots)$ to the vector $x^* = A_* x = (x_1^*, x_2^*, \ldots)$ with components $x_n^* = x + 1 (n = 1, 2, \ldots)$.

Remark 2. The map A_* coincide with map A (section 1) if in its definition we put $\gamma_n = 1 (n = 1, 2, \ldots)$.

We introduce the following objects:

r is a natural number,

ϵ is a real number satisfying the inequality $0 < \epsilon < \frac{1}{2}$;

$\psi(n)$ is an arbitrary real function such that $\psi(n) \geq 1$, $\lim_{n\to\infty} \psi(n) = \infty$ and $[\psi(\tau)]^r \leq c\sqrt{n}$, where $n \geq 1$, and c is a constant not depending on n;

$p = (p_1, p_2, \ldots)$ is a sequence consisting of pairwise distinct prime number p_n such that $\sum_{n=1}^{\infty} [\psi(p_n)]^{-r(1-2\epsilon)} < \infty$;

n_0 is a natural number such that for all $n \geq 0$ the inequalities $\psi(p_n) < p_n$ and $\sum_{n=n_0}^{\infty} [\psi(p_n)]^{-r(1-2\epsilon)} < ((2r)^r + 4rc)^{-1}$ hold;

Γ_n is the subset of the discrete circle S_n (section 1) consisting of numbers $k \in S_n$ for which the inequality $|\sum_{m=1}^{[\psi(p_n)]} \left(\frac{k+m}{p_n}\right)| \geq [\psi(p_n)]^{1-\epsilon}$ holds;

Π_n is the cylinder on the torus T such that if $x = (x_1, x_2, \ldots) \in \Pi_n$, then $x_n \in \Gamma_n$.

Definition 9. We introduce the sets Π and Ω on the torus T such that

$$\Pi = \bigcup_{n=n_0}^{\infty} \Pi_n \ , \qquad \Omega = T \backslash \Pi \ .$$

Theorem 7. For any vector $x = (x_1, x_2, \ldots) \in T$ the following assertions hold:

1) if there exists a vector $\omega \in \Omega$ such that (A_*, p, ω)-continued fraction of vector x is finite and has the form

$$x = \left[q^{(0)}, \ldots, q^{(\nu)}\right]_{A_*, p, \omega} \tag{16}$$

then for all $n \geq n_0$ the inequality

$$|\sum_{k=\nu+1}^{\nu+[\psi(p_n)]} \left(\frac{x_n + k}{p}\right)| < [\psi(p_n)]^{1-\epsilon} \tag{17}$$

holds, and the set of vectors $x \in T$ for which the relations (16) and (17) are valid, is a complement of a set of mes-measure zero on T;

2) if for $\omega \in \Omega$ the vector x is expanded into the infinite (A_*, p, ω)-continued fraction, and $S^{(\nu)} = (S_1^{(\nu)}, S_2^{(\nu)}, \ldots)$ is its (A_*, p, ω)-convergent of a continued fraction such that the equalities $S_n^{(\nu)} = x_n$ are satisfied for $n_0 \leq n \leq N$, then the inequality (17) holds for $n_0 \leq n \leq N$.

Proof.

Lemma 3. ([30]). Let $\tilde{p}$ be a prime, k be an integer, ℓ be an integer from the interval $(0, \tilde{p})$, r be a natural, $D_\ell(k) = \sum_{m=1}^{\ell} \left(\frac{k+m}{\tilde{p}}\right)$. Then $\sum_{k=0}^{\tilde{p}-1} D_\ell^{2r}(k) < (2r)^r \tilde{p} \ell^r + 4r\sqrt{\tilde{p}}\ell^{2r}$.

Lemma 4. Let $\tilde{p}$ be a prime, ℓ be an integer from the interval $(0, \tilde{p})$, ϵ be a real number satisfying the inequality $0 < \epsilon < \frac{1}{2}$; $N_{\ell,\tilde{p}}^{(\epsilon)}$ be the number of integers k from interval $0 \le k \le \tilde{p}-1$ for which the inequality $|\sum_{m=1}^{\ell}\binom{k+m}{\tilde{p}}| \ge \ell^{1-\epsilon}$ holds. Then for any natural r the inequality

$$N_{\ell,\tilde{p}}^{(\epsilon)} < (2r)^r \frac{\tilde{p}}{\ell^{r-2\epsilon r}} + 4r\sqrt{\tilde{p}}\ell^{2\epsilon\tilde{p}}$$

holds.

Proof of Lemma 4. According to lemma 3 we have the inequality

$$N_{\ell,\tilde{p}}^{(\epsilon)}\ell^{2r-2\epsilon r} < (2r)^r\tilde{p}\ell^r + 4r\sqrt{\tilde{p}}\ell^{2r} \ ,$$

from which the assertion of lemma 4 follows.

Applying the lemma 4 in the case, where $\tilde{p} = p_n$, $\ell = [\psi(p_n)]$, and using the definition of the measure μ_n on the discrete circle S_n, we obtain that the measure μ_n of the set Γ_n introduced at the beginning of this section, satisfies the inequality

$$\mu_n(\Gamma_n) < \frac{(2r)^r}{[\psi(p_n)]^{r(1-2\epsilon)}} + \frac{4r\,[\psi(p_n)]^{2\epsilon r}}{\sqrt{p_n}} \ .$$

Therefore, according to the definitions of the function $\psi(n)$, the number n_0, the cylinders Π_n at the beginning of this section and according to the Definition 9 and the definitions of the measure mes on the torus T, we have the following inequalities:

$$\text{mes}\ \Pi < \sum_{n=n_0}^{\infty}\mu_n(\Gamma_n) < \sum_{n=n_0}^{\infty}\frac{(2r)^r + 4rc}{[\psi(p_n)]^{r(1-2\epsilon)}} < 1 \ ,$$

$$\text{mes}\ \Omega > 0 \ . \tag{18}$$

It follows from the inequality (18) and the Theorem 5 (section 2), that for almost all $x \in T$ there exists a vector $x = (x_1, x_2, \ldots) \in \Omega$ such that (A_*, p, ω)-continued fraction of x is finite and has the form (16), and according to definitions of maps $\overline{A}$ and $\hat{A}$ (section 1) and Theorem 2 (section 2) the equality $\omega = \overline{A}_*^{\nu}x$ holds for this case, and this equality is equivalent to the relations

$$\omega_n \equiv x_\nu + \nu \mod p_n \ ; \qquad n = 1, 2, \ldots \ . \tag{19}$$

From the Definition 9 of the set Ω it follows that for any component $\omega_n (n = 1, 2, \ldots)$ of the vector $\omega \in \Omega$ the inequality

$$|\sum_{m=1}^{[\psi(p_n)]}\left(\frac{\omega_n + m}{p_n}\right)| < [\psi(p_n)]^{1-\epsilon} \tag{20}$$

holds. Now we deduce the assertion 1) of Theorem 7 from the relations (19) and (20).

We prove the assertion 2). From the Definition 5 of the (A_*, p, ω)-convergents of the continued fraction of the vector $x = (x_1, x_2, \ldots)$ and from Theorem 2 it follows that

$$\overline{A}_*^{\nu}\{\{S^{(\nu)}\}\} = \omega \ . \tag{21}$$

Therefore, if this (A_*, p, ω)-convergent of the continued fraction $S^{\nu} = (S_1^{(\nu)}, S_2^{(\nu)}, \ldots)$ is such that the equality $S^{(\nu)} = x_n$ holds for $n_0 \leq n \leq N$, then from (21) and the Definition 8 of map A_* it follows that the congruences

$$\omega_n = x_n + \nu \ \mathrm{mod} p_n \ ; n_0 \leq n \leq N \ , \tag{22}$$

hold. Now the assertion 2) follows from the inequality (20) and the relations (22). Theorem 7 is proved.

Theorem 8. For any vector $x = (x_1, x_2, \ldots) \in T$ the following assertions hold:

1) if there exists a vector $\omega \in \Omega$ such that (A_*, p, ω)-continued fraction of x is finite and has the form (16), then for all $n \geq n_0$ among the integers z situated in the region
$$x_n < z < x_n + [\psi(p_n)] + \nu + 1 \ , \tag{23}$$
there are $\frac{\nu+[\psi(p_n)]}{2} + \theta_n'(\nu + [\psi(p_n)]^{1-\epsilon})$ of quadratic residues mod p_n and $\frac{\nu+[\psi(p_n)]}{2} + \theta_n''(\nu + [\psi(p_n)]^{1-\epsilon})$ of quadratic non-residues mod p_n, where θ_n' and θ_n'' are constants satisfying the inequalities $|\theta_n'| \leq 1$, $|\theta_n''| \leq 1$;

2) the set of $x \in T$ for which there exists $\omega \in \Omega$ such that the inequality (16) and the assertion 1) are valid is a complement of a set of measure mes zero;

3) if for $\omega \in \Omega$ the vector x can be expanded into an infinite (A_*, p, ω)-continued fraction and if $S^{(\nu)} = (S_1^{(\nu)}, S_2^{(\nu)}, \ldots)$ is its (A_*, p, ω)-convergent of continued fraction such that the equalities $S_n^{(\nu)} = x_n$ are satisfied for $n_0 \leq n \leq N$ then, the assertions 1) holds for $n_0 \leq n \leq N$.

Proof. To prove the assertions 1) and 2) of Theorem 8 we use the assertion 1) of Theorem 7, and to prove the assertion 3) of Theorem 8 we use the assertion 2) of Theorem 7. For both cases the estimate

$$| \sum_{k=1}^{\nu+[\psi(p_n)]} \left(\frac{x_n + k}{p_n} \right) < \nu + [\psi(p_n)]^{1-\epsilon} \ , \tag{24}$$

follows from the inequality (17).

We denote the number of quadratic residues mod p_n situated in the region (23) by Q'_n, and the number of quadratic non-residues mod p_n situated in the region (23) by Q''_n.

By the virtue of Definition 7 of Legendre symbol we have:

$$Q'_n - Q''_n = \sum_{k=1}^{\nu+[\psi(p_n)]} \left(\frac{x_n+k}{p_n}\right) \ , \tag{25}$$

and from the definitions of numbers Q'_n and Q''_n for $n \geq n_0$ it follows the equality

$$Q'_n + Q''_n = Q \ , \tag{26}$$

where Q_n takes one of two values:

$$Q_n = \begin{cases} \nu+[\psi(p_n)] \ , & \text{if there are no numbers } z \text{ in the} \\ & \text{region (23) dividing by } p_n \ , \\ \nu+[\psi(p_n)]-1 \ , & \text{if there is the number in the region (23)} \\ & \text{dividing by } p_n \end{cases}$$

Now, adding and subtracting the relations (25) and (26) between themselves and using (24), we obtain all assertions of Theorem 8. Theorem 8. is proved.

Remark 3. By virtue of Corollary 2 of Theorem 6 (section 6), the assertions 2 of Theorem 7 and 3 of Theorem 8 are valid for any arbitrarily large N.

References

[1] A.D. Brjuno, V.I. Parusnikov. Polyhedrons of Klein for two cubic Davenport forms, Matem. Zametki, 1994, Vol. 56, issue 4, p. 9–27 (in Russian).

[2] A.D. Brjuno,V.I. Parusnikov. Comparison of Various Generalizations of Continued Fraction, Mathematical Notes, Vol. 61, N. 3, (1997), 278–286.

[3] L. Charve. De la reduciton des formes quadratiques ternaires positives et de leur application aux irrationelles du troisieme degree, Annales Scientifiques de l'Ecole Normale Superieure, 1880, Ser. 2, Vol. 9, Supplement, p. 1–156.

[4] T.W. Cusik. Diophantine approximations of linear forms over algebraic number field, Mathematika, 1973, Vol. 20, N. 1, p. 16–23.

[5] L. Dirichlet. Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einigen Anwendungen auf die Theorie der Zahlen, Dirchlet's Werke, Bd. I, Berlin, 1889, S. 633.

[6] H. Davenport. On the product of three homogeneous linear forms, I, Proc. London Math. Soc., 1938, Vol. 44, p. 412–431.

[7] L. Euler. De relatione inter ternas pluresve quantitates instituenda. Petersburger Academie Notiz: Exhib. August 14, 1775 = Opuscula Analytica von Euler, tom 2, p. 91, 1785 = Commentationes arithmeticae collectae, t. II, p. 99–104, St. Petersburg, 1849 = Commentationes arithmeticae, t. 3, p. 136–145, 1941.

[8] C. Hermite. Sur l'introduction des variables continues dans la théorie des nombres, Journal f. d. Mathematik, Bd. 41, p. 191–216.

[9] A. Hurwitz. Über die angenäherte Darstellung der Zahlen durch rationale Brüche, Mathematische Annalen, Bd. 44, p. 417–436.

[10] C.G.J. Jacobi. Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird, Journ. f. d. Mathematik, Bd. 69, S. 29–64.

[11]F. Klein. Über eine geometrische Auffassung der gewöhnlichen Kettenbruchentwicklung, Nachr. Ges. Wiss. Göttingen, Mathem.-Phys., K1, 1895, N. 3, S. 357–359.

[12] F. Klein. Sur une representation geometrique du developpement en fraction continue ordinaire, Nouv. Ann. Math., 1896, V. 15, N. 3, p. 321–331.

[13] H. Minkovski. Generalisation de la theorie des fractions continues, Annales Sci. de l'Ecole Normale Superieure, 1986, Ser. 3, v. 13, N. 2, p. 41–60.

[14] L.J. Mordell. The product of three gomogeneous linear ternary forms, J. London Math. Soc., 1942, V. 17, N. 2, p. 107–115.

[15] H. Poincare. Sur une géneralisation des fractions continues., Comptes rendus des séances de l'Académie de Paris, t. 99, p. 1014.

[16] B.F. Skubenko. Minimums of decomposable forms of power n of n variables for $n \geq 3$, Zapiski nauch. semin. LOMI, 1990, T. 183, p. 142–154.

[17] G.F. Voronoi. On a generalization of an algorithm of continued fractions, 1896 Publishing House of Warsaw University; Collected Works, Kiev, AN USSR, 1952, T. 1, p. 197–391 (in Russian).

[18] L.D. Pustyl'nikov. Generalized continued fractions and the estimates of Weyl sums and remainder term in the law of distribution of fractional parts of the values of a polynomial, Matemat. Sametki, 1995, vol. 56–6, p. 144–148 (in Russian); english translation in Mathematical Notes, Vol. 56, No. 5–6, 1994, p. 1315–1317.

[19] L.D. Pustyl'nikov. Ergodic Theory and Some Number Theory Problems. Preprint, Forschungsinstitut für Mathematik, ETH Zürich, July 1994.

[20] L.D. Pustyl'nikov. Generalized continued fraction and differential equation, Preprint N. 96 of the Keldysh Institute of Applied Mathematics of RAS, Moscow, 1997.

[21] L.D. Pustyl'nikov. Generalized continued fraction and ergodic theory, Journal of Mathematical Science, to appear.

[22] L.D. Pustyl'nikov. Generalized continued fractions connected with the Gauss transformation, Preprint N. 35 of the Keldysh Institute of Applied Mathematics of RAS, Moscow, 1998.

[23] L.D. Pustyl'nikov. Infinite-dimensional generalized continued fractions, quadratic residues and non-residues, and ergodic theory, Russian Math. Surveys 52:2, 420–421.

[24] H. Weyl. Über die Gleichverteilung der Zahlen mod. Eins, Math. Ann. 77 (1916), 313–252.

[25] A.Ya. Khinchin. Continued fractions, State Publishing House of Physical-Mathematical Literature, Moscow 1961.

[26] I.M. Vinogradov. Elements of Number Theory. Moscow, 1965.

[27] I.M. Vinogradov. On the Distribution of Power Residues and Non-Residues, Journal of the Physical-Mathematical Society, Perm University, 1 (1918), 94–98.

[28] H. Davenport. On the distribution of quadratic residues (mod. p), Journ. of London Math. Soc., 8 (1933), 46–52.

[29] A.O. Gel'fond and Yu.V. Linnik. Elementary methods in analytic number theory, Pergamon Press, Oxford-New York-Toronto, 1966.

[30] D. Burgess. The distribution of quadratic residues and non-residues, Mathematica, 4, p. 2, N. 8 (1957), 106–112.

[31] I.P. Kornfel'd, S.V. Fomin, Ya.G. Sinai. Ergodic Theory, New York, Springer 1982.