NON-UNIQUENESS FOR NONLINEAR FOKKER–PLANCK EQUATIONS AND THEIR ASSOCIATED DISTRIBUTION-DEPENDENT SDES

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Abstract. In this paper, we study distribution-dependent stochastic differential equations on the domain $\mathcal{O} = \mathbb{T}^d$ or $\mathbb{R}^d$, $d \geqslant 2$, of the form

$$\mathrm{d}X_t = v(t, X_t, \rho_t)\,\mathrm{d}t + \sqrt{2}\,\sigma(t, X_t, \rho_t)\,\mathrm{d}W_t, \qquad \rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}x},$$

where $\mu_t = \mathrm{Law}(X_t)$. Our main construction is carried out at the level of the associated nonlinear Fokker–Planck equations. We first build non-unique probability solutions to these PDEs and then use the superposition principle to obtain non-unique martingale solutions to the corresponding DDSDEs.

We establish two main non-uniqueness results concerning stationary states, both on the torus and in the whole space, under the corresponding structural assumptions. First, we construct a divergence-free drift $v \in C_t L^{d-}$ such that the DDSDE admits *infinitely many* distinct solutions starting from the stationary initial density. This result lies at the natural critical regularity threshold: in several models, well-posedness is expected for drifts in $C_t L^{d+}$. Second, for $d \geqslant 3$ and every prescribed $N \in \mathbb{N}$, we construct a divergence-free drift for which the DDSDE admits at least N distinct stationary martingale solutions. The resulting multiplicity of equilibrium states is reminiscent of multistability and phase-transition phenomena in physical systems.

Contents

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1. Introduction

Interacting particle systems provide effective models for a wide range of collective phenomena arising in physics, biology, kinetic theory, and other scientific fields. As a prototypical example, consider a system of N weakly interacting particles on $\mathbb{R}^d$ governed by

$$\mathrm{d}X_t^{i,N} = (V * \mu_t^N)(X_t^{i,N})\,\mathrm{d}t + \sqrt{2}\,(B * \mu_t^N)(X_t^{i,N})\,\mathrm{d}W_t^i, \qquad i = 1, \dots, N,$$

where

$$\mu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}$$

is the empirical measure of the particle system, $V : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $B : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable interaction kernels, and $\{W^i\}_{i\in\mathbb{N}}$ is a family of independent standard Brownian motions. Under suitable assumptions on the coefficients and the initial particle configuration, the propagation of chaos principle asserts that, as $N \to \infty$, the empirical measure μ_t^N converges to a deterministic probability measure μ_t, which is the law of a solution to a McKean–Vlasov stochastic differential equation, also known as a distribution-dependent stochastic differential equation (DDSDE):

$$\mathrm{d}X_t = b\big(t, X_t, \mu_t\big)\,\mathrm{d}t + \sqrt{2}\,\sigma\big(t, X_t, \mu_t\big)\,\mathrm{d}W_t, \qquad \mu_t = \mathrm{Law}(X_t). \tag{1.1}$$

Here $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ are measurable coefficients. In the convolution-type example above, they are given by $b(t,x,\mu) = \int_{\mathbb{R}^d} V(t, x-y)\,\mu(\mathrm{d}y), \sigma(t,x,\mu) = \int_{\mathbb{R}^d} B(t,x-y)\,\mu(\mathrm{d}y)$.

The (linear) Kolmogorov operator associated with (1.1) is

$$L_\mu = \sigma(t,x,\mu)\sigma(t,x,\mu)^T : \nabla^2 + b(t,x,\mu)\cdot\nabla.$$

Formally, and under suitable integrability assumptions, Itô's formula shows that the marginal laws $\mu_t = \mathrm{Law}(X_t)$ satisfy the nonlinear Fokker–Planck equation (nFPE)

$$\partial_t \mu_t = L_{\mu_t}^* \mu_t.$$

Besides arising as mean-field limits of interacting particle systems, nonlinear Fokker–Planck equations also appear in many fundamental models from physics, biology, and kinetic theory. At the macroscopic level, these equations describe the time evolution of particle densities under the combined effects of diffusion, transport, and self-consistent interaction forces. Typical examples include nonlinear porous-medium type equation,

$$\partial_t \rho = \Delta\beta(\rho) + \mathrm{div}(E\rho);$$

the Keller–Segel equation

$$\partial_t \rho = \Delta\rho - \mathrm{div}(\rho\nabla c), \qquad -\Delta c = \rho;$$

and the 2D vorticity Navier–Stokes equation

$$\partial_t \rho = \Delta\rho - \mathrm{div}(\rho\nabla^\perp c), \qquad -\Delta c = \rho.$$

Such equations possess rich analytic structures and natural probabilistic interpretations: the associated DDSDE describes the stochastic evolution of a representative particle, cell, or agent, while the nonlinear Fokker–Planck equation governs the corresponding density.

A central problem in the theory of DDSDEs and the associated nFPEs is their well-posedness. Classical results on McKean–Vlasov equations in the whole space can be found in [Fun84, Szn84, Sch87, Chi94]. Since then, a substantial theory has been developed for equations with singular coefficients; see [RZ21, CRF22, HW23, Wan23, Zha25]. These works extend, in various directions, the classical regularization theory for SDEs with singular drifts developed in[KR05]. On the PDE side, Fokker–Planck–Kolmogorov equations with measure-valued or probability-valued solutions have been studied extensively; see [BR21, BR22, BR23a, BSS23] and the references therein. For results on stationary distributions of DDSDEs, we refer to [Wan18, LM22, HWY24].

A related but conceptually different problem concerns the multiplicity of stationary distributions. The coexistence of several stationary states is a mathematical signature of multistability and is commonly associated with phase transitions in mean-field systems. Moreover, at criticality the fluctuations of mean-field models need not obey the usual Gaussian central-limit scaling; see, for instance, [EN78a, EN78b, Daw83]. A classical example is Dawson's model with double-well confinement and Curie–Weiss interaction on the line. It can be written in the form

$$\mathrm{d}X_t = -\left[\nabla U(X_t) + \theta\big(X_t - m_t\big)\right]\mathrm{d}t + \sigma_0\,\mathrm{d}W_t, \qquad m_t := \int_{\mathbb{R}} y\,\mu_t(\mathrm{d}y),$$

where

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2, \qquad \theta > 0.$$

Dawson [Daw83] proved that there exists a critical noise strength $\sigma_c > 0$ such that the system has a unique stationary distribution when $\sigma_0 > \sigma_c$, whereas three stationary distributions coexist when $0 < \sigma_0 < \sigma_c$. This multiplicity of equilibrium states is referred to as a phase transition. Further quantitative studies of phase transitions and long-time behavior can be found in [HT10, Tug10, Tug13, DT16]. We also refer to [CP10, CGPS20] for related results in the periodic setting. Most of these phase-transition results concern constant scalar diffusion. The stationary problem for genuinely distribution-dependent diffusion coefficients is considerably less understood; see, in particular, [Zha23].

Another natural phenomenon is the possible non-uniqueness of the time evolution starting from a fixed initial law. For rough drift fields, even if the system is initialized at a stationary distribution, it is natural to ask whether every weak solution must remain stationary. An even more striking question is whether infinitely many distinct evolutions can start from a common stationary state. In recent work, the authors [GG25, LRZ25, RZZ25, LR25] established weak non-uniqueness for SDEs with constant diffusion $\sigma = \mathrm{Id}$ and supercritical drifts. In that setting, the associated Fokker–Planck equation is linear. Consequently, the existence of two solutions automatically yields infinitely many solutions. For general distribution-dependent coefficients, however, the associated Fokker–Planck equation is genuinely nonlinear in the density, and convex combinations of solutions are no longer solutions in general. The construction of infinitely many solutions requires additional ideas and cannot be obtained by a direct application of the methods developed for a single linear Fokker–Planck equation.

The main purpose of the present paper is to establish non-uniqueness in law for a broad class of DDSDEs with nonlinear or nonlocal diffusion and interaction mechanisms. We show that an arbitrarily small but sufficiently rough external drift can destroy both uniqueness of the evolution starting from a stationary initial law and uniqueness of stationary states.

We treat both the periodic and the whole-space settings. The periodic setting is technically simpler and allows us to cover a broader class of nonlinear and singular interaction mechanisms, since the volume measure provides a natural stationary reference state for many equations. Then in the latter whole-space case, we work within the confining framework studied by Barbu and Röckner [BR23a], in which the equation enjoys well-posedness under suitable assumptions. In contrast, our results show that these conclusions may fail after adding an arbitrarily small but sufficiently rough, compactly supported divergence-free perturbation to the drift.

1.1. **Main results on the torus.** From now on, we consider the following distribution-dependent SDE on $\mathbb{T}^d$:

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + (V * \rho_t)(X_t)\,\mathrm{d}t + \sqrt{2}\,\sigma(t, X_t, \rho_t)\,\mathrm{d}W_t, \tag{1.2}$$

where $\mu_t = \mathrm{Law}(X_t)$ and we denote by ρ_t its density with respect to the Lebesgue measure on $\mathbb{T}^d$, namely $\mu_t(\mathrm{d}x) = \rho_t(x)\,\mathrm{d}x$. Here $v : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$, $V : \mathbb{T}^d \to \mathbb{R}^d$, and $\sigma : [0, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^{d\times d}$ are measurable functions. The process W is a standard $\mathbb{T}^d$-valued Brownian motion. The displayed SDE should be understood through the corresponding martingale problem on the manifold $\mathbb{T}^d$. We adopt the latter interpretation throughout this paper. We refer to [Hsu02] for background on Brownian motion and stochastic analysis on manifolds. With a slight abuse of notation, we shall still refer to (1.2) as a DDSDE.

Definition 1.1 (Martingale solution). *Let $C([0, T]; \mathbb{T}^d)$ be the space of continuous paths on the torus, equipped with its Borel σ-algebra. Denote by*

$$\Pi_t(\omega) := \omega(t), \qquad \omega \in C([0, T]; \mathbb{T}^d),$$

the canonical process, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the natural filtration generated by $(\Pi_t)_{t\in[0,T]}$.

The corresponding (linear) diffusion operator is defined by

$$L_{\rho_t} := \sigma\sigma^T(t, x, \rho_t) : \nabla^2 + (v(t, x) + V * \rho_t) \cdot \nabla.$$

We say that a probability measure $\mathbf{Q}$ on $C([0, T]; \mathbb{T}^d)$ is a martingale solution associated with L_{ρ_t} if

$$\mathrm{d}\mathbf{Q} \circ \Pi_t^{-1} = \rho_t\,\mathrm{d}x, \qquad t \in [0, T],$$

and, for every $f \in C^2(\mathbb{T}^d)$, the process

$$f(\Pi_t) - f(\Pi_0) - \int_0^t L_{\rho_s} f(\Pi_s)\mathrm{d}s$$

is a $\mathbf{Q}$-martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

In this paper, we mainly focus on the associated nonlinear Fokker–Planck equation. In distributional form, it reads

$$\partial_t\rho - \mathrm{divdiv}((\sigma\sigma^T)(\rho)\rho) + \mathrm{div}(v\rho) + \mathrm{div}((V * \rho)\rho) = 0 \qquad \text{on } (0, T) \times \mathbb{T}^d. \tag{1.3}$$

For a matrix-valued function $A = (A_{ij})_{1\leqslant i,j\leqslant d}$, we define $\mathrm{divdiv}A = \sum_{i,j} \partial_i\partial_j A_{ij}$. By the superposition principle, any probability solution ρ to the nFPE satisfying the standard integrability conditions on the coefficients gives rise to a martingale solution of the DDSDE (1.2).

We impose the following assumptions on the coefficients.

Assumption 1.2. *Assume that the interaction kernel satisfies $V \in L^{d,\infty}(\mathbb{T}^d; \mathbb{R}^d)$.*

We assume that the diffusion coefficient $\sigma : [0, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^{d\times d}$ satisfies one of the following conditions.

(1) (Nemytskii-type) $\sigma(t, x, \rho) = \sigma_1(t, x, \rho_t(x))$, where $\sigma_1 : [0, T]\times\mathbb{T}^d\times\mathbb{R} \to \mathbb{R}^{d\times d}$ is a measurable function satisfying $\sigma_1 \in C_b^2([0, T] \times \mathbb{T}^d \times \mathbb{R})$, and $\|\partial_\rho[\sigma_1(t, x, \rho)\sigma_1^T(t, x, \rho)\rho]\|_{C^1_{t,x}C^0_\rho} < \infty$.

(2) *(Convolution-type)* $\sigma(t,x,\rho) = \int_{\mathbb{T}^d} \sigma_2(t,x-y)\rho_t(y)\,\mathrm{d}y$, *or* $\sigma(t,x,\rho) = \sqrt{\int_{\mathbb{T}^d} \sigma_2(t,x-y)\rho_t(y)\,\mathrm{d}y}$, *where* $\sigma_2 : [0,T]\times\mathbb{T}^d \to \mathbb{R}^{d\times d}$ *is a measurable function satisfying* $\sigma_2 \in C^2([0,T];C(\mathbb{T}^d))$. *In the second case, we assume that the matrix inside the square root is symmetric and non-negative. The notation* $\sqrt{B}$ *denotes a matrix square root of* B*, namely a matrix* A *satisfying* $AA^T = B$.

(3) σ *admits the decomposition* $\sigma\sigma^T = \tilde{\sigma}_1(\tilde{\sigma}_1)^T + \tilde{\sigma}_2(\tilde{\sigma}_2)^T$, *where* $\tilde{\sigma}_1$ *satisfies condition* (1) *above and* $\tilde{\sigma}_2$ *satisfies condition* (2) *above.*

In particular, it is straightforward to verify that, for every diffusion coefficient σ satisfying Assumption 1.2(2), or Assumption 1.2(1) in the space-independent case $\sigma(t,x,\mu) = \sigma_1(t,\rho(x))$, the following property holds:

$$\operatorname{div}\operatorname{div}\big(\sigma\sigma^T(t,x,1)\big) \equiv 0. \tag{1.4}$$

Let us also point out that condition (1.4) covers many distribution-independent cases. For example,

$$\sigma\sigma^T(t,x) = \operatorname{diag}\big(1+\varepsilon h(t,x),1,\ldots,1\big),$$

where $h(t,x)$ is any smooth periodic function independent of x_1, and $|\varepsilon|$ is sufficiently small. Or

$$\sigma\sigma^T(t,x) = \mathrm{Id} + \varepsilon\,\operatorname{diag}(B,0,..,0),\ B = \begin{pmatrix} \partial_{22}\psi(x_1,x_2), & -\partial_{12}\psi(x_1,x_2) \\ -\partial_{12}\psi(x_1,x_2), & \partial_{11}\psi(x_1,x_2) \end{pmatrix}.$$

where $\psi \in C^\infty(\mathbb{T}^2)$ and $|\varepsilon|$ is sufficiently small.

Consequently, under (1.4), if v is divergence-free, then the constant density $\rho \equiv 1$ is a stationary probability solution of the nonlinear Fokker–Planck equation (1.3).

1.1.1. *Non-uniqueness of weak solutions.* Our first main result shows that the DDSDE may admit *infinitely many* distinct weak solutions, even when starting from this stationary state.

Theorem 1.3. *Let* $T > 0$, $\epsilon_0 \in (0,1)$. *Let* $d \geqslant 2$, $1 < s < d$, *and let* $p,r \in [1,\infty]$ *satisfy* $\frac{d}{p}+\frac{1}{r} > 1$. *Assume that* V *and* σ *satisfy Assumption 1.2, and also* (1.4). *Let* $\overline{v} \in C^\infty([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$ *be divergence-free. Then there exists a distribution-independent, divergence-free vector field* $v \in L^r(0,T;L^p(\mathbb{T}^d;\mathbb{R}^d)) \cap C([0,T];L^s(\mathbb{T}^d;\mathbb{R}^d))$ *satisfying*

$$\|v-\overline{v}\|_{L^r_t L^p(\mathbb{T}^d)} + \|v-\overline{v}\|_{C_t L^s(\mathbb{T}^d)} \leqslant \epsilon_0, \tag{1.5}$$

such that the following assertions hold.

(1) *The nonlinear Fokker–Planck equation* (1.3) *admits infinitely many distinct probability density solutions* ρ^i, $i\in\mathbb{N}$, *with initial condition* $\rho_0 \equiv 1$. *More precisely, there exists* $\epsilon > 0$ *such that, for every* $i \in \mathbb{N}$, $\rho^i \in C\big([0,T];L^{1+\epsilon}(\mathbb{T}^d)\big) \cap L^1\big((0,T);W^{1,1+\epsilon}(\mathbb{T}^d)\big)$, *and* $|v|^{1+\epsilon}\rho^i \in L^1([0,T];L^1(\mathbb{T}^d))$.

(2) *The DDSDE* (1.2) *admits infinitely many distinct martingale solutions* $\mathbf{Q}^i$, $i\in\mathbb{N}$, *starting from the volume measure on* $\mathbb{T}^d$. *Moreover, these solutions satisfy* $\mathbf{E}^{\mathbf{Q}^i}\Big[\int_0^T |v(s,\Pi_s)|^{1+\epsilon}\,\mathrm{d}s\Big] < \infty, i\in\mathbb{N}$.

For sufficiently regular diffusion coefficients satisfying suitable uniform ellipticity assumptions, the corresponding initial value problem is known to be well posed for drifts $v \in L^r_t L^p$ with $\frac{d}{p}+\frac{2}{r} < 1$; see [KR05, RZ21, Zha25] and the references therein. In contrast, our result shows that, for any divergence-free drift field $\overline{v} \in C_t L^{d+}$, every sufficiently small neighborhood of $\overline{v}$ in the low-regularity topology $C_t L^{d-}$ contains “bad” drift fields for which the corresponding DDSDE admits non-unique weak solutions. Our result is essentially sharp in view of the expected well-posedness for drifts. Several examples covered by our results are presented in Section 2.

We emphasize that our construction yields *infinitely many* weak solutions. Since the equation is nonlinear, this is fundamentally different from merely producing finitely many distinct solutions: convex combinations of solutions are no longer solutions in general. As a consequence, the construction of infinitely many solutions requires additional ideas and constitutes one of the main technical difficulties of the present work. The proof of Theorem 1.3 is given in Section 4- Section 5.

The previous results apply to a large class of interaction kernels. However, in many physical models, the interaction kernel may be more singular. For instance, if G denotes the Green function, then

$$\nabla G(x) \lesssim |x|^{1-d} \in L^{\frac{d}{d-1},\infty}, \qquad d \geqslant 2.$$

The Green function on the torus exhibits the same type of singularity near the origin; see Section 2.4 for a detailed discussion. Therefore, when $d \geqslant 3$, such kernels are not covered by Theorem 1.3. The following result shows that our non-uniqueness mechanism remains valid even for more singular interactions.

Theorem 1.4. *Let $T > 0$, $\epsilon_0 \in (0,1)$, and $N \in \mathbb{N}$. Let $d \geqslant 3$ and $1 < d_0 < d$. Assume that the coefficients σ and V are given by $\sigma(\rho) = \sqrt{a * \rho}, V = \operatorname{div} b$, where $a, b \in L^{\frac{d}{d-\gamma},\infty}(\mathbb{T}^d; \mathbb{R}^{d\times d})$ and $1 < \frac{d}{\gamma} < d_0' := \frac{d_0}{d_0-1}$. Assume moreover that $a * \rho$ admits a measurable matrix square root for all densities ρ considered below.*

Then, for any smooth divergence-free vector field $\overline{v}(t,x)$, there exists a divergence-free vector field $v \in C([0,T]; L^{d_0}(\mathbb{T}^d; \mathbb{R}^d))$ satisfying $\|v - \overline{v}\|_{C_t L^1(\mathbb{T}^d)} \leqslant \epsilon_0$, such that the following assertions hold.

1. *The nonlinear Fokker–Planck equation (1.3) admits at least N distinct nontrivial probability density solutions ρ^i, $1 \leqslant i \leqslant N$, with initial condition $\rho_0^i \equiv 1$. More precisely, for some small constant $\epsilon > 0$, one has $\rho^i \in C\big([0,T]; L^{d_0'} \cap W^{1,1+\epsilon}(\mathbb{T}^d)\big), 1 \leqslant i \leqslant N$.*
2. *The DDSDE (1.2) admits at least N distinct martingale solutions starting from the volume measure on $\mathbb{T}^d$.*

Compared with Theorem 1.3, the time regularity estimates here are upgraded to C_t bounds. Consequently, we obtain stronger regularity estimates for the densities ρ^i, which are sufficient to make sense of the nonlinear interaction terms involving the singular kernels above. At the same time, we still keep the drift in the regime $C_t L^{d-}$. The proof of Theorem 1.4 is deferred to Section 6- Section 7.

1.1.2. *Non-uniqueness of stationary solutions.* Our second main result concerns the existence of non-unique stationary solutions to the DDSDE (1.2). More precisely, by a stationary solution we mean a stationary martingale solution $\mathbf{Q}$ whose time marginals are time-independent, namely

$$\mathrm{d}\mathbf{Q} \circ \Pi_t^{-1} = \rho \, \mathrm{d}x, \qquad t \geqslant 0,$$

for some probability density ρ on $\mathbb{T}^d$. The non-uniqueness of stationary solutions means that the system admits several different equilibrium states. From the physical viewpoint, this is reminiscent of multistability and phase-transition phenomena. In this setting, we study the following nonlinear stationary Fokker–Planck equation:

$$-\operatorname{div}\operatorname{div}\big((\sigma\sigma^T)(\rho)\rho\big) + \operatorname{div}(v\rho) + \operatorname{div}\big((V * \rho)\rho\big) = 0. \tag{1.6}$$

Since the problem is posed on the compact torus, strongly coercive diffusion operators often impose strong rigidity on stationary probability solutions. For example, for the aforementioned case

$$-\Delta\beta(\rho) = 0,\ \beta' > 0, \quad \text{or} \quad -\Delta((a*\rho)\rho) = 0,\ a(x) > 0, a \in L^1,$$

one immediately obtains that the unique probability solution is $\rho \equiv 1$. In contrast, in this part, we show that, for many systems, small but rough perturbations of the drift can produce multiple stationary states.

Theorem 1.5. *Let $\epsilon_0 \in (0,1)$ and $N \in \mathbb{N}$. Let $d \geqslant 3$ and $1 < d_0 < d-1$. Assume that the time-independent coefficients σ and V either satisfy Assumption 1.2, or belong to the following class of singular interaction kernels: $\sigma(\rho) = \sqrt{a*\rho}, V = 2\,\mathrm{div} b$, where $a, b \in L^{\frac{d}{d-\gamma},\infty}(\mathbb{T}^d;\mathbb{R}^{d\times d}), 1 < \frac{d}{\gamma} < d_0' := \frac{d_0}{d_0-1}$. Then, for any smooth divergence-free vector field $\overline{v} \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, there exists a divergence-free vector field $v \in L^{d_0}(\mathbb{T}^d;\mathbb{R}^d)$ satisfying $\|v-\overline{v}\|_{L^1(\mathbb{T}^d)} \leqslant \epsilon_0$, such that the following assertions hold.*

(1) The stationary equation (1.6) admits at least N distinct non-constant probability density solutions ρ^i, $1 \leqslant i \leqslant N$. Moreover, for some $\epsilon > 0$ and every $1 \leqslant i \leqslant N$, one has $\rho^i \in L^{d_0'}(\mathbb{T}^d) \cap W^{1,1+\epsilon}(\mathbb{T}^d)$.
(2) The DDSDE (1.2) admits at least N distinct nontrivial stationary martingale solutions.

The proof of Theorem 1.5 is given in Section 8- Section 9.

Moreover, in the linear stationary case

$$-\Delta\rho + \mathrm{div}(v\rho) = 0,$$

with $v \in L^p$, several uniqueness and non-uniqueness results are known. For $p > d$ and $d \geqslant 2$, Bogachev, Röckner, and Stannat [BRS02] proved that equation admits a unique probability solution ρ in the class $v\rho - \nabla\rho \in L^1(\mathbb{T}^d;\mathbb{R}^d)$. In this paper, we prove that, for every $d \geqslant 3$, there exists a divergence-free drift $v \in L^{d-1-}$ such that the above linear stationary equation admits non-unique probability solutions, which moreover satisfy the conditions above. We also recall that, for divergence-free drifts $v \in L^p$, (non)-uniqueness in the class $\rho \in H^1$ was studied in [Zhi04, CO22].

1.2. **Main results in the whole space.** The results above can also be adapted to the whole-space setting. The periodic case is technically simpler, since the volume measure provides a natural stationary probability density for many equations under consideration. In contrast, there is no analogous canonical probability density on $\mathbb{R}^d$. As a consequence, the construction in the whole-space setting requires a more delicate analysis of the behavior at infinity.

We consider the following nonlinear Fokker–Planck equation on $\mathbb{R}^d$, $d \geqslant 2$:

$$\partial_t\rho - \Delta\beta(\rho) + \mathrm{div}(v\rho) + \mathrm{div}(Eb(\rho)\rho) = 0. \tag{1.7}$$

We assume that the coefficients satisfy the following conditions:

(a) $\beta \in C^2(\mathbb{R})$, $\beta(0) = 0$, and there exist constants $0 < \gamma < \gamma_1 < \infty$ and $0 < \gamma_2 < \infty$ such that $\gamma \leqslant \beta'(r) \leqslant \gamma_1, |\beta''(r)| \leqslant \gamma_2, r \in \mathbb{R}$.
(b) $b \in C_b^1(\mathbb{R})$ and $b(r) \geqslant b_0 > 0, r \geqslant 0$.
(c) $E = -\nabla\Phi$, where $\Phi \in C(\mathbb{R}^d) \cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies $\Phi \geqslant 1$, $\lim_{|x|\to\infty}\Phi(x) = +\infty$, and there exists $m \in [2,\infty)$ such that $\Phi^{-m} \in L^1(\mathbb{R}^d)$ and $|\nabla\Phi| + |\nabla^2\Phi| \leqslant \Phi^m$.

Such equations arise in statistical physics; see, for instance, [Fra05, FD01, SNC07, Tsa09]. They also play an important role in nonequilibrium statistical mechanics, where they describe the evolution of particle densities in disordered media.

The related DDSDE is formally given by

$$\mathrm{d}X_t = v(t,X_t)\,\mathrm{d}t + E(X_t)b(\rho_t(X_t))\,\mathrm{d}t + \sqrt{\frac{2\beta(\rho_t(X_t))}{\rho_t(X_t)}}\,\mathrm{d}W_t. \tag{1.8}$$

In the case $v=0$, the system is conservative and admits a stationary solution of the form

$$\rho^{\mathrm{st}}(x) := g^{-1}\big(-\Phi(x)+\mu^{\mathrm{st}}\big), \tag{1.9}$$

where $\mu^{\mathrm{st}} \in \mathbb{R}$ is uniquely determined by the mass constraint

$$\int_{\mathbb{R}^d} g^{-1}\big(-\Phi(x)+\mu^{\mathrm{st}}\big)\,\mathrm{d}x = 1,$$

and

$$g(r) := \int_1^r \frac{\beta'(s)}{sb(s)}\,\mathrm{d}s, \qquad r>0.$$

By assumptions (a) and (b), the function g is strictly increasing and has logarithmic growth at both 0 and ∞. Equivalently, g^{-1} has exponential-type growth. Together with the confining assumption on Φ, μ^{st} exists and is unique.

Recently, the well-posedness theory for this class of nonlinear Fokker–Planck equations has been further developed in a number of works. We refer the reader to [BR21, BR22, BR23a, BR23b, Gru25] and to the monograph [BKRS22] for a comprehensive account of the subject. For the reader's convenience, we recall below several results from [BR23a] that are particularly relevant to the present work. In addition to (a)–(c), some of these results require the following conditions:

(d) $E \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $\operatorname{div} E \in L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$.
(e) For a.e. $x\in\mathbb{R}^d$, $\gamma_1\Delta\Phi(x) - b_0|\nabla\Phi(x)|^2 \leqslant 0$.

The following properties are known when $v\equiv 0$:

- Under conditions (a)–(d), existence and uniqueness of mild solutions are established via a nonlinear contraction semigroup $S(t)$, $t>0$, acting on $L^1(\mathbb{R}^d)$.
- (H-Theorem) Under assumptions (a)–(e), the solution $u(t)$ converges, along any sequence $t_n\to\infty$, to the stationary state ρ^{st} in $L^1(\mathbb{R}^d)$.
- Under conditions (a)–(e), the stationary equation associated with (1.7) admits a unique stationary probability solution, given by (1.9).

Our main result is that the above properties may fail under suitable small but rough perturbations of the drift field.

First, we show that there exists a drift field v such that the equation admits infinitely many solutions starting from the stationary state ρ^{st}. Moreover, the perturbation v is compactly supported, and therefore it does not affect the behavior of E at infinity. Due to technical limitations, we additionally require the potential Φ to be constant on some domain.

Theorem 1.6. *Let $\epsilon_0\in(0,1)$. Let $d\geqslant 2$, $1<s<d$, and let $p,r\in[1,\infty]$ satisfy $\frac{d}{p}+\frac{1}{r}>1$. Assume that conditions* (a)–(c) *hold, and that Φ is constant on $[-\frac12,\frac12]^d$. Then there exists a divergence-free vector field $v(t,x)$ such that $\operatorname{supp}_{t,x} v\subset[\frac{1}{12},\frac{11}{12}]\times[-\frac12,\frac12]^d$, and $\|v\|_{L^r_tL^p(\mathbb{R}^d)}+\|v\|_{C_tL^s(\mathbb{R}^d)}\leqslant\epsilon_0$, such that the following assertions hold.*

(1) The nonlinear Fokker–Planck equation (1.7) admits infinitely many distinct probability density solutions $\rho^i\in C([0,1];L^1(\mathbb{R}^d))$, $i\in\mathbb{N}$, satisfying $\rho^i(t,\cdot)=\rho^{\mathrm{st}}$ for $t\in[0,\frac{1}{12}]\cup[\frac{11}{12},1]$.

(2) The DDSDE (1.8) *admits infinitely many distinct weak solutions starting from the stationary probability measure* $\rho^{\mathrm{st}}(x)\,\mathrm{d}x$.

We believe that the regularity of the drift field obtained here, $v \in C_t L^{d-}$, is essentially optimal. In the linear case $\beta'(r) = b(r) = 1$, uniqueness is known to hold for drift fields in $C_t L^{d+}$; see for example, [KR05, Zha11, RZ21]. Thus, at least in this particular setting, our result is sharp with respect to the integrability of the drift.

We next show that the full convergence-to-equilibrium conclusion associated with the H-theorem is not stable. Indeed, since $v = 0$ and $\rho^i = \rho^{\mathrm{st}}$ near the endpoints $t = 0$ and $t = 1$, we may extend the pair (v, ρ^i) periodically in time by setting $v(t+k) = v(t), \rho^i(t+k) = \rho^i(t), k \in \mathbb{N}, t \in [0,1)$. As a consequence, any non-stationary solution obtained in this way cannot converge to ρ^{st} or any other stationary solution as $t \to \infty$.

Finally, we show that there exists a drift field v such that the equation admits multiple stationary solutions.

Theorem 1.7. *Let* $\epsilon_0 \in (0,1)$ *and* $N \in \mathbb{N}$. *Let* $d \geqslant 3$ *and* $1 < d_0 < d-1$. *Assume that conditions* (a)–(c) *hold. Then there exists a compactly supported divergence-free vector field* $v \in L^{d_0}(\mathbb{R}^d;\mathbb{R}^d), \operatorname{supp} v \subset [-\frac{1}{2}, \frac{1}{2}]^d$, *satisfying* $\|v\|_{L^1(\mathbb{R}^d)} \leqslant \epsilon_0$, *such that the following assertions hold.*

(1) Equation (1.7) *admits at least* N *distinct stationary probability density solutions* ρ^i, $1 \leqslant i \leqslant N$. *Moreover, for some* $\epsilon > 0$ *and every* $1 \leqslant i \leqslant N$, $\rho^i \in L^{d_0'}(\mathbb{R}^d) \cap W^{1,1+\epsilon}(\mathbb{R}^d)$.
(2) The DDSDE (1.8) *admits at least* N *distinct stationary weak solutions.*

The proofs of the above two theorems follow essentially the same strategy as in the torus setting. The only new issue arises from the treatment of the inverse divergence operator in the whole space. This difficulty is overcome by introducing the Bogovskii operator in Section 10.1. We shall omit a number of routine computations and highlight only the necessary changes. The complete proofs are deferred to Section 10.

1.2.1. *Further discussion on the double-well model.* We further discuss a possible connection with the classical double-well model. Consider, for instance, an even double-well potential $\Phi(x) = \frac{1}{4}|x|^4 - |x|^2$. Then the stationary density ρ^{st} defined in (1.9) is also even. In particular, one has $\int_{\mathbb{R}^d} x_i \rho^{\mathrm{st}}(x)\,\mathrm{d}x = 0, 1 \leqslant i \leqslant d$. Here $x = (x_1, \ldots, x_d)$. This observation is relevant to the mean-field model considered by Dawson [Daw83]:

$$-\sigma_0 \Delta\rho + \operatorname{div}\big((-\nabla\Phi + m_\rho)\rho\big) = 0, \qquad m_\rho := \int_{\mathbb{R}^d} x\rho(x)\,\mathrm{d}x,$$

which also admits ρ^{st} as a stationary solution. Dawson [Daw83] showed that in 1D case, depending on the parameters of the model, the system may exhibit either a unique stationary state or 3 stationary states.

If, in Theorem 1.7, one could additionally ensure that the convex integration construction preserves the even symmetry of the density, then the theorem would also apply to the double-well model. A natural strategy would be to impose the symmetry conditions that v is odd and ρ^i is even. Accordingly, throughout the convex integration scheme, one would require v_q to be odd, ρ^i_q to be even, and M^i_q to be odd. Preserving these symmetries at each iteration would be an interesting problem, but it requires a more delicate analysis of the construction, in particular of the choice of the coefficients in the building blocks.

1.3. **Related literature and main new ideas.** The main objective of this article is to establish the first statement in Theorem 1.3 (and in Theorem 1.5), which will be proved via the convex integration method. This technique was first introduced to fluid dynamics by De Lellis and Székelyhidi Jr. [DLS09, DLS10, DLS13] and has led to numerous groundbreaking results for determined and stochastic fluid dynamics on the torus. For the incompressible Euler equations, the famous Onsager conjecture was proved in [Ise18, BDLSV19]. We refer to [BDLIS15, DSJ17, BFH20, NV23, GKN23, GR24] for further literature on the Euler equation. For the Navier-Stokes equations, the sharp non-uniqueness of weak solutions has been shown in [BV19b, BCV21, CL22a, CL23]. We refer to [BV19a, HZZ23b, HZZ24, HZZ25, LZ25a, LZ25b] for further results on the Navier-Stokes equations. For the ODE or transport equations with Sobolev vector fields, we refer to [CGSW15, MS18, MS19, MS20, BCDL21, CL21, PS23]. For the SDE or the Fokker-Planck equation, we refer to [LRZ25, LR25] for non-uniqueness results. Regarding the non-unique stationary solution to NS equation, we refer to [Luo19]. Recently, this method has been applied to fluid dynamics on the whole space, see [MNY24a, MNY24b, LR25].

Existing convex integration results are largely restricted to systems involving only finitely many equations, and one cannot directly apply it simultaneously to infinitely many equations. Indeed, doing so would require constructing the total perturbation as an infinite sum of sub-perturbations, each of which is designed to eliminate the corresponding stress term. The convergence of such a total perturbation would then become highly problematic.

Our approach is to introduce an increasing sequence of integers $\{N_q\}_{q\in\mathbb{N}}$ such that $2^{N_q} \sim \lambda_q^{\alpha}$ for some sufficiently small $\alpha > 0$. At step $q+1$, we apply the standard convex integration scheme only to the first N_{q+1} equations. To ensure convergence of the drift perturbation, we impose an additional decay factor 2^{-i} in the estimate of the i-th stress term. A key observation is that this extra decay does not interfere with the convex integration construction. Indeed, by the definition of N_{q+1}, for every $i \leqslant N_{q+1}$ one has $2^i \lesssim \lambda_{q+1}^{\alpha}$, which can be absorbed as a low-frequency loss. For the remaining equations, namely those with $i > N_{q+1}$, the stress term M_q^i is not treated by the convex integration perturbation at step $q+1$. Nevertheless, it must still satisfy the more restrictive estimates required at the next level. The key point is that M_q^i can be rewritten in a form that enjoys the stronger decay bound 4^{-i}. Since $i > N_{q+1}$, we have $2^{-i} \lesssim 2^{-N_{q+1}} \lesssim \lambda_{q+1}^{-\alpha}$, and hence $4^{-i} \lesssim 2^{-i}\lambda_{q+1}^{-\alpha}$. This provides precisely the additional smallness needed to close the iteration.

In the construction of stationary solutions, we can no longer exploit time intermittency to improve the estimates. Moreover, the intermittent flows used in the time-dependent construction, which have intermittency dimension $D = 0$, are no longer applicable. Instead, we employ Mikado-type flows with intermittency dimension $D = 1$. Consequently, the construction yields drift regularity only up to $L^{d-D-} = L^{d-1-}$.

2. Examples covered by the main results

In this section, we present several examples that fall within the scope of our framework. We restrict our discussion to the periodic setting, corresponding to Theorem 1.3–Theorem 1.5. These examples illustrate that the non-uniqueness mechanism developed in this paper applies to a broad class of nonlinear and nonlocal Fokker–Planck-type equations arising in probability theory, statistical physics, and continuum mechanics.

2.1. **The distribution-independent case.** We first consider a distribution-independent SDE on $\mathbb{T}^d$:

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + \sqrt{2}\,\sigma(t, X_t)\,\mathrm{d}W_t,$$

where $v : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{T}^d \to \mathbb{R}^{d\times d}$ are measurable functions. The corresponding Fokker–Planck equation reads as

$$\partial_t \rho - \operatorname{div}\operatorname{div}\big(\sigma\sigma^T \rho\big) + \operatorname{div}(v\rho) = 0.$$

For simplicity, assume that σ is uniformly elliptic and sufficiently regular. We also assume that $\operatorname{div}\operatorname{div}\big(\sigma\sigma^T\big) \equiv 0$. Then the constant density $\rho \equiv 1$ is a stationary probability solution when v is divergence-free. In this case, the model is covered by Assumption 1.2(1) with $V = 0$. Therefore, by Theorem 1.3, there exists a distribution-independent, divergence-free drift field $v \in C_t L^{d-}(\mathbb{T}^d)$ such that the Fokker–Planck equation and the corresponding SDE admit infinitely many weak solutions starting from the stationary density $\rho_0 \equiv 1$.

Moreover, this result is essentially optimal with respect to the integrability of the drift. Indeed, for every $v \in C_t L^p, p > d$, the SDE is well-posed and the associated Fokker–Planck equation is also well posed in the corresponding probability solution class. We refer to Zhang [Zha11] for the corresponding result in the whole-space setting; the same argument applies to the torus case by periodic lifting.

2.2. **The convolution-type equation.** We next consider the convolution-type coefficients covered by Assumption 1.2(2). Let $V \in L^p(\mathbb{T}^d; \mathbb{R}^d)$ for some $p > d$, and let $\Sigma \in C^2([0, T] \times \mathbb{T}^d; \mathbb{R}^{d\times d})$. This gives rise to the following McKean–Vlasov SDE:

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + (V * \rho_t)(X_t)\,\mathrm{d}t + \sqrt{2}\,(\Sigma * \rho_t)(X_t)\,\mathrm{d}W_t.$$

We also assume that the covariance matrix $(\Sigma * \rho)(\Sigma * \rho)^T$ is uniformly elliptic for the densities under consideration. This equation can be interpreted as the mean-field limit of a weakly interacting particle system, as discussed in the Introduction. The term v represents an external transport field acting on the particles, while $V * \rho_t$ and $\Sigma * \rho_t$ describe the mean-field interaction in the drift and diffusion coefficients, respectively. In this interpretation, X_t denotes the trajectory of a typical particle in the large-population limit.

The above equation falls within the scope of Theorem 1.3. Consequently, the introduction of an arbitrarily small but sufficiently irregular external divergence-free drift field $v \in C_t L^{d-}(\mathbb{T}^d)$ may lead to the existence of infinitely many weak solutions starting from the stationary density $\rho_0 \equiv 1$. This regularity threshold is essentially sharp. Indeed, in the whole-space setting, with the regime $v \in C_t L^p$ with $p > d$, one expects well-posedness under the usual ellipticity and stability assumptions. In the additive-noise case $\Sigma \equiv \mathrm{Id}$, the well-posedness theory was developed in [RZ21, Theorem 4.3]. For general distribution-dependent diffusion coefficient, we refer to [Zha25, Theorem 4.1].

Finally, applying Theorem 1.5, one can also construct a divergence-free drift field $v \in L^{d-1-}$ for which the corresponding stationary nonlinear Fokker–Planck equation admits non-unique stationary probability solutions.

2.3. **The porous-medium-type equation.** We next consider a nonlinear diffusion of porous-medium type. In Assumption 1.2(1), let $\beta : \mathbb{R} \to \mathbb{R}$ satisfy $\beta(0) = 0$ and $0 < \gamma \leqslant \beta'(r) \leqslant \gamma_1 < \infty, r \in \mathbb{R}$. We then choose $\sigma(\rho) = \sqrt{\frac{\beta(\rho)}{\rho}}\,\mathrm{Id}, V \equiv 0$. With this choice, equation (1.3) reduces to the nonlinear Fokker–Planck equation on $\mathbb{T}^d$

$$\partial_t \rho - \Delta\beta(\rho) + \operatorname{div}(v\rho) = 0.$$

Here ρ denotes the probability density associated with the corresponding DDSDE

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + \sqrt{\frac{2\beta(\rho_t(X_t))}{\rho_t(X_t)}}\,\mathrm{d}W_t.$$

The whole-space analogue of this model has been discussed in Section 1.2. In the periodic setting, Theorems 1.3 and 1.5 apply to this equation. Consequently, one can construct divergence-free drift fields $v \in C_t L^{d-}$ for which the equation admits infinitely many weak solutions starting from the stationary density $\rho_0 \equiv 1$, as well as divergence-free drift fields for which the corresponding stationary problem admits non-unique stationary probability solutions.

2.4. **Keller–Segel equation.** We consider aggregation equations ($\nu = 0$) and Keller–Segel-type models ($\nu > 0$) of the form

$$\partial_t \rho - \nu \Delta \rho + \operatorname{div}\big((\nabla G * \rho)\rho\big) + \operatorname{div}(v\rho) = 0.$$

The corresponding DDSDE is formally given by

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + (\nabla G * \rho_t)(X_t)\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t,$$

where X_t represents the position of a moving biological cell at time t. Here G denotes the Green function of the Laplacian on the torus. Since the Laplacian is not invertible on constants on $\mathbb{T}^d$, the inverse Laplacian is understood on mean-zero functions. More precisely, G is defined by

$$-\Delta G = \delta_0 - 1, \qquad \int_{\mathbb{T}^d} G(x)\,\mathrm{d}x = 0.$$

Near the origin, G has the same singular behavior as the classical Euclidean potential; see, for instance, [Jos19]. In particular,

$$|\nabla G(x)| \lesssim |x|^{1-d}, \qquad d \geqslant 2.$$

The Keller–Segel model [KS70, KS71] is a fundamental equation in chemotaxis and collective behavior, describing the motion of biological cells or microorganisms attracted by chemical signals produced collectively by the population itself. In this context, probability solutions are particularly natural, since the density ρ represents the distribution of particles or cells and must therefore remain nonnegative with conserved total mass.

Keller–Segel equations with an additional incompressible transport field v have been extensively studied, mainly in connection with enhanced dissipation, suppression of blow-up, and coupled fluid models, see for example [BH17, IXZ21, HKY25]. In these works, the external drift is typically smooth or has a special mixing structure and therefore plays a stabilizing role. In contrast, our result focuses on the opposite mechanism: we show that an arbitrarily small but sufficiently rough divergence-free drift can destroy uniqueness of probability solutions.

In the case $d = 2$, one has $\nabla G \in L^{2,\infty}(\mathbb{T}^2; \mathbb{R}^2)$. Therefore, by Theorem 1.3, there exists a divergence-free drift field $v \in C_t L^{2-}$ such that the above equation admits infinitely many probability solutions starting from the stationary density $\rho_0 \equiv 1$.

In the case $d \geqslant 3$, since $G \in L^{\frac{d}{d-2},\infty}$, the assumptions of Theorem 1.4 and Theorem 1.5 are satisfied with $\gamma = 2$, provided $1 < \frac{d}{2} < d_0' := \frac{d_0}{d_0 - 1}$. Consequently, for $d_0 < \frac{d}{d-2}$, there exists a divergence-free drift field $v \in C_t L^{d_0}$ such that the corresponding equation with initial datum $\rho_0 \equiv 1$ admits non-unique probability solutions. Similarly, by Theorem 1.5, one can construct a time-independent divergence-free drift field for which the stationary equation admits non-unique stationary probability solutions.

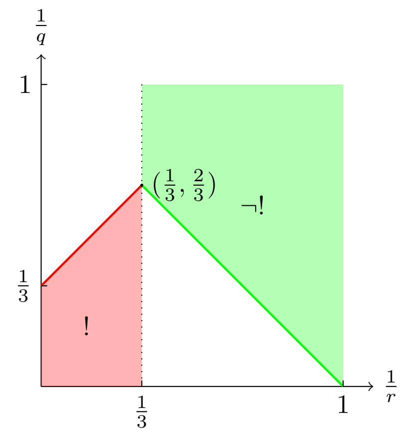

FIGURE 1. Uniqueness and non-uniqueness regimes for probability solutions in C_tL^q to the three-dimensional Keller–Segel equation with external drift $v \in C_tL^r$.
The red region corresponds to the expected well-posedness regime: for $r > 3$ and $\frac{1}{q} \leqslant \frac{1}{3} + \frac{1}{r}$, uniqueness is expected in the corresponding subcritical class.
The green region corresponds to the non-uniqueness regime obtained in this paper: for $1 < r < 3$ and $\frac{1}{q} \geqslant 1 - \frac{1}{r}$, there exists a divergence-free drift $v \in C_tL^r$ for which the equation admits multiple probability solutions.
The point $(\frac{1}{3}, \frac{2}{3})$ marks the sharp threshold separating these two regimes.

In the Keller–Segel case with $\nu > 0$, one expects well-posedness for sufficiently regular external drifts. For instance, when $v \in C_tL^r$ with $r > d$, the drift is subcritical with respect to the parabolic scaling. In this regime, local well-posedness and uniqueness of mild solutions in subcritical spaces of the form C_tL^q, $\frac{1}{q} = \frac{1}{d} + \frac{1}{r} < \frac{2}{d}$, are expected; see, for instance, [Kar99, BHN94]. In dimension $d = 3$, our construction reaches the sharp threshold suggested by this scaling. On the one hand, for $v \in C_tL^r$ with $r > 3$, one expects uniqueness in the class C_tL^q, $\frac{1}{q} = \frac{1}{3} + \frac{1}{r} < \frac{2}{3}$. On the other hand, for every $1 < r < 3$, our result provides a divergence-free drift $v \in C_tL^r$ such that the equation admits multiple probability solutions in the class C_tL^q, $\frac{1}{q} = 1 - \frac{1}{r} < \frac{2}{3}$. Thus the point $\left(\frac{1}{r}, \frac{1}{q}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$ marks the critical threshold separating the expected uniqueness regime from the non-uniqueness regime; see Figure 1 for an illustration.

2.5. **The 2D vorticity Navier–Stokes equation.** We next consider the two-dimensional case $d = 2$. Let $V \in L^{2,\infty}(\mathbb{T}^2; \mathbb{R}^2)$ and take $\sigma = \mathrm{Id}$. Then (1.3) becomes

$$\partial_t \rho - \Delta \rho + \operatorname{div}\big((V * \rho)\rho\big) + \operatorname{div}(v\rho) = 0.$$

The corresponding DDSDE is

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + (V * \rho_t)(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t.$$

In the case $v \equiv 0$, the existence and conditional uniqueness of the corresponding SDE and Fokker–Planck equation have been studied in [Kry20a, Kry20b, RZZ25].

In particular, if we choose the Biot–Savart kernel

$$V = \nabla^{\perp} G \in L^{2,\infty}(\mathbb{T}^2; \mathbb{R}^2),$$

where G is the Green function on the torus, then the equation reduces to the two-dimensional vorticity Navier–Stokes equation with an additional drift:

$$\partial_t \rho - \Delta\rho + \operatorname{div}\big((\nabla^{\perp} G * \rho)\rho\big) + \operatorname{div}(v\rho) = 0.$$

The two-dimensional vorticity Navier–Stokes equation is a fundamental model in fluid mechanics and turbulence theory, describing the evolution of the vorticity of an incompressible fluid. The well-posedness theory for the two-dimensional vorticity Navier–Stokes equation in the class $C_t L^1$ is now classical; see, for instance, [MB03, BA94, GG05]. The probabilistic formulation of the 2D vorticity Navier–Stokes equation in terms of nonlinear McKean–Vlasov SDEs has been studied, for instance, in [BRZ25, RZZ25].

On the other hand, by Theorem 1.3, there exists a divergence-free drift field $v \in C_t L^{2-}$ such that the above vorticity equation with the additional drift admits infinitely many probability solutions in the class $C_t L^1$, all starting from the stationary density $\rho_0 \equiv 1$.

2.6. **Landau-type equation on $\mathbb{T}^d$, $d \geqslant 3$.** We next consider a Landau-type equation on the torus. Let $\sigma(\rho) = \sqrt{a * \rho}, V = 2\operatorname{div} a$, where the matrix-valued kernel a has the Coulomb-type singularity

$$a(x) = \left(\mathrm{Id} - \frac{x \otimes x}{|x|^2}\right) |x|^{2-d}$$

near the origin. More precisely, a is understood as a periodic matrix-valued function on $\mathbb{T}^d$ which coincides with the above kernel in a neighborhood of the origin.

With this choice, equation (1.3) becomes the following Landau-type equation:

$$\partial_t \rho - \operatorname{div}\left[(a * \rho)\nabla\rho - (a * \nabla\rho)\rho\right] + \operatorname{div}(v\rho) = 0.$$

The corresponding DDSDE is formally given by

$$\mathrm{d}X_t = v(t, X_t)\,\mathrm{d}t + 2(a * \nabla\rho_t)(X_t)\,\mathrm{d}t + \sqrt{2(a * \rho_t)(X_t)}\,\mathrm{d}W_t.$$

The Landau equation is a fundamental kinetic model describing the evolution of dilute plasmas under Coulomb-type interactions, where the dynamics are dominated by the accumulation of grazing collisions. Classical references include [Lan36, Vil98, Vil02]. In the genuine kinetic setting, the variable x here should be interpreted as the velocity variable. From this viewpoint, it is not physically natural to impose periodicity in the velocity variable. However, due to the limitations of our current method, we restrict ourselves to the periodic setting on the torus.

Since $a \in L^{\frac{d}{d-2},\infty}$, the assumptions of Theorem 1.4 and Theorem 1.5 are satisfied with $\gamma = 2$, provided $1 < \frac{d}{2} < d_0'$, i.e. $d_0 < \frac{d}{d-2}$. Consequently, there exists a divergence-free drift field $v \in C_t L^{d_0}$ in the above range such that the corresponding Landau-type equation admits non-unique weak solutions. Moreover, by applying Theorem 1.5, one can also construct a time-independent divergence-free drift for which the equation admits non-unique stationary probability solutions.

2.7. **Further discussion.** Our approach is flexible and is expected to be adaptable to a much broader class of diffusion mechanisms than those treated in the present paper. With suitable modifications, one may hope to extend the construction to nonlinear diffusion operators such as the p-Laplacian $\operatorname{div}\big(|\nabla\rho|^{p-2}\nabla\rho\big)$, nonlocal diffusion operators such as the fractional Laplacian $(-\Delta)^{\alpha}$, and genuinely degenerate porous-medium diffusions such as $\Delta\rho^m$. These operators arise naturally in a variety of physical models, including non-Newtonian fluids, transport in heterogeneous media, and anomalous diffusion. We do not pursue these extensions here, since they would require additional estimates adapted to the specific structure of each diffusion operator.

In this paper, we restrict our attention to the torus and whole-space settings. Nevertheless, the method is expected to be adaptable to more general domains equipped with suitable boundary conditions. A main additional ingredient would be the use of Bogovskii operators [Bog79], which provide a convenient tool for solving divergence equations while taking boundary conditions into account. The presence of boundaries, however, also introduces new technical difficulties.

3. Notations and Preliminaries

Let $T>0$, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. Throughout the manuscript, we write $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ for the d-dimensional flat torus. We employ the notation $a\lesssim b$ if there exists a constant $c>0$ such that $a\leqslant cb$. Given a Banach space E with a norm $\|\cdot\|_E$, we write $C_tE=C([0,T];E)$ for the space of continuous functions from $[0,T]$ to E, equipped with the supremum norm. For $p\in[1,\infty]$ we write $L^p_tE=L^p([0,T];E)$ for the space of L^p-integrable functions from $[0,T]$ to E, equipped with the usual L^p-norm. For $\alpha\in(0,1)$ we define C^α_tE as the space of α-Hölder continuous functions from $[0,T]$ to E, endowed with the norm $\|f\|_{C^\alpha_tE}=\sup_{s,t\in[0,T],s\neq t}\frac{\|f(s)-f(t)\|_E}{|t-s|^\alpha}+\sup_{t\in[0,T]}\|f(t)\|_E$, and write C^α_t in the case when $E=\mathbb{R}$. We use L^p to denote the set of standard L^p-integrable functions on $\mathbb{T}^d$. For $s>0$, $p>1$ we set $W^{s,p}:=\{f\in L^p;\|(I-\Delta)^{\frac{s}{2}}f\|_{L^p}<\infty\}$ with the norm $\|f\|_{W^{s,p}}=\|(I-\Delta)^{\frac{s}{2}}f\|_{L^p}$. We use $L^{p,\infty}$ to denote the Lorentz space

$$\|f\|_{L^{p,\infty}}:=\sup_{\lambda>0}\lambda\,|\{x\in\mathbb{T}^d:\ |f(x)|>\lambda\}|^{1/p}.$$

For $N\in\mathbb{N}_0$, C^N_x denotes the space of N-times differentiable functions equipped with the norm

$$\|f\|_{C^N_x}:=\sum_{|\alpha|\leqslant N,\alpha\in\mathbb{N}_0^d}\|D^\alpha f\|_{L^\infty_x}.$$

Similarly, if the norm is taken in space-time, we use $C^N_{t,x}$.

For a matrix-valued function $A=(A^{ij})_{1\leqslant i,j\leqslant d}$, we define $(\mathrm{div}A)_i:=\sum_{j=1}^d\partial_jA^{ij}$.

We define the projection $\mathbb{P}_{\neq0}f:=f-\int_{\mathbb{T}^d}f\mathrm{d}x$. We denote by $\mathrm{d}x$ the Lebesgue measure on the torus. We denote by $\mathcal{L}$ the law of random variable.

3.1. Some technical tools.

We collect some technical tools used in the construction of convex integration schemes.

We define $\mathcal{R}_1:=\nabla\Delta^{-1}$ as a right inverse of the div operator, i.e. $\mathrm{div}(\mathcal{R}_1v)=v$ for scalars v with $\int_{\mathbb{T}^d}v\mathrm{d}x=0$. Here we use the notation $\mathcal{R}_1v:=\mathcal{R}_1(v-\int v\mathrm{d}x)$ for a general scalar function v. Then since $\mathcal{R}_1$ is a Calderón-Zygmund operator, we have

Lemma 3.1. *Let $1\leqslant p\leqslant\infty$. For any vector field $f\in C^\infty_0(\mathbb{T}^d;\mathbb{R})$, $\sigma\in\mathbb{N}$,*

$$\|\mathcal{R}_1f(\sigma\cdot)\|_{L^p}\lesssim\sigma^{-1}\|f\|_{L^p}.$$

We introduce the bilinear version $\mathcal{B}_1:C^\infty(\mathbb{T}^d;\mathbb{R})\times C^\infty_0(\mathbb{T}^d;\mathbb{R})\to C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ by

$$\mathcal{B}_1(v,f)=v\mathcal{R}_1f-\mathcal{R}_1\Big(\nabla v\cdot\mathcal{R}_1f+\int vf\mathrm{d}x\Big).$$

Lemma 3.2. ([BCDL21, Lemma 3.3]) *Let $1\leqslant p\leqslant\infty$. For any $v\in C^\infty(\mathbb{T}^d;\mathbb{R})$ and $f\in C^\infty_0(\mathbb{T}^d;\mathbb{R})$, we have $\mathrm{div}(\mathcal{B}_1(v,f))=vf-\int_{\mathbb{T}^d}vf\mathrm{d}x$, and for $\sigma\in\mathbb{N}$,*

$$\|\mathcal{B}_1(v,f(\sigma\cdot))\|_{W^{k,p}}\lesssim\sigma^{k-1}\|v\|_{C^{k+1}}\|f\|_{W^{k,p}}.$$

Then we introduce the improved Hölder's inequality by using the additional decorrelation between frequencies.

Lemma 3.3. ([CL22a, Theorem B.1]) *Let $d \geqslant 2, p \in [1,\infty]$ and $a, f : \mathbb{T}^d \to \mathbb{R}$ be smooth functions. Then for any $\sigma \in \mathbb{N}$,*

$$|\|af(\sigma\cdot)\|_{L^p} - \|a\|_{L^p}\|f\|_{L^p}| \lesssim \sigma^{-1/p}\|a\|_{C^1}\|f\|_{L^p}.$$

Finally, we record the following weak Young inequality.

Lemma 3.4. *Let $d \geqslant 2$ and let $1 < p, q, r < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 > 0$. Then, for all $f \in L^p(\mathbb{T}^d)$ and $g \in L^{r,\infty}(\mathbb{T}^d)$, one has*

$$\|f * g\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}\|g\|_{L^{r,\infty}(\mathbb{T}^d)}. \tag{3.1}$$

4. Infinitely many evolutions from a stationary state

In this section, we prove our main result, Theorem 1.3. We first establish the first statement at the PDE level. For any fixed diffusion coefficient σ satisfying Assumption 1.2 and (1.4), we construct a distribution-independent divergence-free drift $v \in L^r_t L^p \cap C_t L^s$ such that the corresponding nonlinear Fokker–Planck equation (1.3) admits infinitely many distinct positive solutions $\rho^i \in C_t L^1$, satisfying moreover $v\rho^i \in L^1_t L^1$.

To achieve this, we apply a convex integration scheme to the Fokker–Planck equation (1.3). The overall iterative procedure follows the framework developed in previous works such as [LRZ25, LR25]. However, the present setting requires handling general diffusion operators together with the construction of infinitely many distinct solutions. Accordingly, in the sequel we mainly focus on the new ideas and techniques specific to this setting, while only briefly discussing those parts of the argument that are analogous to earlier works.

4.1. **Preparations.** From now on, we write $A(\rho) := \sigma\sigma^T(\rho)$. The following lemma collects several estimates that will be used later.

Lemma 4.1. *Under Assumption 1.2, it holds that*

$$\|A(\rho) - A(\tilde\rho)\|_{C^1_{t,x}} \lesssim \|\rho - \tilde\rho\|_{C^1_{t,x}} + \|\rho - \tilde\rho\|_{C^0_{t,x}}(\|\rho\|_{C^1_{t,x}} + \|\tilde\rho\|_{C^1_{t,x}}),$$
$$\|A(\rho)\|_{C^0_{t,x}} \lesssim 1, \quad \|A(\rho)\|_{C^1_{t,x}} \lesssim 1 + \|\rho\|_{C^1_{t,x}}, \quad \|A(\rho)\|_{C^2_{t,x}} \lesssim 1 + \|\rho\|^2_{C^1_{t,x}} + \|\rho\|_{C^2_{t,x}}.$$

Proof. We first consider case (1). Since $\sigma(t,x,\rho) = \sigma_1(t,x,\rho_t(x))$ and $\sigma_1 \in C^2_b([0,T]\times\mathbb{T}^d\times\mathbb{R})$, we have for $0 \leqslant i \leqslant d$ (∂_0 is regarded as the derivative to time t),

$$\begin{aligned}
&|\partial_i\sigma(t,x,\rho) - \partial_i\sigma(t,x,\tilde\rho)| \\
&\qquad \lesssim |(\partial_i\sigma_1)(t,x,\rho) - (\partial_i\sigma_1)(t,x,\tilde\rho)| + |(\partial_\rho\sigma_1)(t,x,\rho)\partial_i\rho - (\partial_\rho\sigma_1)(t,x,\tilde\rho)\partial_i\tilde\rho| \\
&\qquad \lesssim \|\rho - \tilde\rho\|_{C^1_{t,x}} + \|\rho - \tilde\rho\|_{C^0_{t,x}}\|\rho\|_{C^1_{t,x}}, \\
&\|\sigma(t,x,\rho)\|_{C^1_{t,x}} \lesssim 1 + \|\rho\|_{C^1_{t,x}}, \\
&\|\sigma(t,x,\rho)\|_{C^2_{t,x}} \lesssim 1 + \|\rho\|^2_{C^1_{t,x}} + \|\rho\|_{C^2_{t,x}}.
\end{aligned}$$

Next, in case (2), it suffices to consider the case that $\sigma(\rho) = \sigma_2 * \rho$. Young's inequality yields

$$\begin{aligned}
\|\sigma(t,x,\rho) - \sigma(t,x,\tilde\rho)\|_{C^1_{t,x}} &\lesssim \|\sigma_2\|_{C^1_t C^0_x}\|\rho_t - \tilde\rho_t\|_{C^1_{t,x}}, \\
\|\sigma(t,x,\rho)\|_{C^0_{t,x}} &\lesssim \|\sigma_2\|_{C^0_{t,x}}\|\rho_t\|_{L^1} \lesssim 1,
\end{aligned}$$

$$\|\sigma(t,x,\rho)\|_{C^i_{t,x}} \lesssim 1+\|\rho\|_{C^i_{t,x}},\ i=1,2.$$

Case (3) follows directly by combining the estimates from the previous two cases. The corresponding estimates for A then follow immediately. □

4.2. **The iteration procedure.** Let $0<\epsilon_0\leqslant 1$ be given. Without loss of generality, we may replace ϵ_0 by $\epsilon_0\wedge\frac{1}{4}$. The iteration is indexed by a parameter $q\in\mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}_0}\subset\mathbb{N}$ and a sequence $\{\delta_q\}_{q\in\mathbb{N}_0}\subset(0,1]$ defined by

$$\lambda_q=a^{b^q},\ q\geqslant 0,\ \delta_q=(\frac{\epsilon_0}{2})^{d+1}\lambda_1^{2\beta}\lambda_q^{-2\beta},\ q\geqslant 1,\ \delta_0=1,$$

where $\beta>0$ will be chosen sufficiently small, while a,b will be chosen sufficiently large. Moreover, by imposing

$$a^{2(b-1)\beta/(d+1)}>2, \tag{4.1}$$

we obtain

$$\sum_{q\geqslant 1}\delta_q^{1/(d+1)}\leqslant\frac{\epsilon_0}{2}\sum_{q\geqslant 1}a^{(1-q)2(b-1)\beta/(d+1)}\leqslant\frac{\epsilon_0}{2}\cdot\frac{1}{1-a^{-2(b-1)\beta/(d+1)}}<\epsilon_0.$$

Without loss of generality, we assume $T=1$ from now on. At each step q, we construct a family $(v_q,\rho_q^i,M_q^i)_{i\in\mathbb{N}}$ solving the system

$$\begin{aligned}\partial_t\rho_q^i-\mathrm{divdiv}\big(A(\rho_q^i)\rho_q^i\big)+\mathrm{div}(v_q\rho_q^i)+\mathrm{div}((V*\rho_q^i)\rho_q^i)&=-\mathrm{div}M_q^i,\\ \mathrm{div}v_q&=0.\end{aligned}\tag{4.2}$$

The sequence $\{\rho_q^i\}_{q\in\mathbb{N}_0}$ converges to a limit ρ^i, while the stress fields $\{M_q^i\}_{q\in\mathbb{N}_0}$ converge to 0 in a suitable topology as $q\to\infty$.

To incorporate the initial condition, we require that $\rho_q^i=1$ on $[0,T_q]$, where $T_q:=\frac{1}{3}-\sum_{1\leqslant r\leqslant q}\delta_r^{1/2}$. By (4.1), we have $0<T_q\leqslant\frac{1}{3}$. Here and throughout the paper, we use the convention $\sum_{1\leqslant r\leqslant 0}:=0$.

Let $\alpha\in(0,1)$ be fixed later. For each $q\in\mathbb{N}_0$, we define $N_q:=\min\{i\in\mathbb{N}:2^i\geqslant\lambda_q^\alpha\}$, which satisfies $\lambda_q^\alpha\leqslant 2^{N_q}<2\lambda_q^\alpha$. At the iteration step $q+1$, we construct new perturbations to eliminate the first N_{q+1} stress terms M_q^i, $1\leqslant i\leqslant N_{q+1}$, while ensuring that the remaining stress terms remain sufficiently small.

We initialize the iteration by setting

$$\begin{aligned}\rho_0^i(t,x)&=1+\frac{\sin 2\pi x_1}{4^i}\chi_0(t),\ v_0=\overline{v},\\ M_0^i(t,x)&=\partial_t\chi_0(t)\frac{\cos 2\pi x_1}{4^i\cdot 2\pi}(1,0,\dots,0)-\overline{v}(\rho_0^i-1)\\ &\quad+\mathrm{div}(A(t,x,\rho_0^i)\rho_0^i-A(t,x,1))-(V*\rho_0^i)\rho_0^i+V*1,\end{aligned}$$

where $x=(x_1,\dots,x_d)$ and χ_0 is a smooth function satisfying $\chi_0(t)=0$ on $[0,\frac{1}{3}]$, $\chi_0(t)=1$ on $[\frac{2}{3},1]$. It is easy to verify that $(v_0,\rho_0^i,M_0^i)_{i\in\mathbb{N}}$ is a solution to (4.2) by using the facts that $\mathrm{div}\overline{v}=0$, $\mathrm{divdiv}(A(t,x,1))=0$ and $\mathrm{div}(V*1)=0$. By definition, we know that $\rho_0^i=1$ on $[0,T_0]$, and then $M_0^i=0$ on $[0,T_0]$.

Then we have by weak Young's inequality

$$\|\mathrm{div}(A(t,x,\rho_0^i)\rho_0^i-A(t,x,1))\|_{C^0_{t,x}}\lesssim\|A(t,x,\rho_0^i)-A(t,x,1)\|_{C^1_{t,x}}+\|A(t,x,\rho_0^i)(\rho_0^i-1)\|_{C^1_{t,x}}$$

$$\lesssim \|\rho_0^i - 1\|_{C^1_{t,x}} \lesssim 4^{-i},$$
$$\|(V*\rho_0^i)\rho_0^i - V*1\|_{L^1_tL^1} \lesssim \|(V*\rho_0^i - V*1)\rho_0^i\|_{L^1_tL^1} + \|(V*1)(\rho_0^i-1)\|_{L^1_tL^1}$$
$$\lesssim \|\rho_0^i - 1\|_{C^0_{t,x}} \lesssim 4^{-i}.$$

By choosing $C_0 > 0$ sufficiently large (depending on ϵ_0), we obtain by Lemma 4.1

$$\|\rho_0^i - 1\|_{C^2_{t,x}} \leqslant 4^{-i}C_0^{1/d_0'}, \quad \|v_0\|_{C^1_{t,x}} \leqslant C_0^{1/d_0}, \quad \|M_0^i\|_{L^1_tL^1} \lesssim 4^{-i} \leqslant (\frac{\epsilon_0}{2})^{d+1}4^{-i}C_0. \tag{4.3}$$

With the above assumptions in hand, our main iteration relies on the first step of iteration and reads as follows:

Proposition 4.2. *Under the assumption of Theorem 1.3, there exist $d+1 > d_0 > 2 > d_0' > 1$ with $\frac{1}{d_0} + \frac{1}{d_0'} = 1$ and a choice of parameters a, b, α, β such that the following holds: Let $(v_q, \rho_q^i, M_q^i)_{i\in\mathbb{N}}$ be a solution to the system (4.2) satisfying $\int \rho_q^i \mathrm{d}x = 1$,*

$$\|v_q\|_{L^{d_0}_tL^{d_0}} \leqslant C_vC_0^{1/d_0}\sum_{m=0}^{q}\delta_m^{1/d_0}, \tag{4.4}$$

for some universal constant $C_v \geqslant 1$, and

$$\|v_q\|_{C^1_{t,x}} \leqslant C_0^{1/d_0}\lambda_q^{d+4}, \ \|\rho_q^i - 1\|_{C^1_{t,x}} + \lambda_q^{-2}\|\rho_q^i - 1\|_{C^2_{t,x}} \leqslant 4^{-i}C_0^{1/d_0'}\lambda_q^{d+4}, \tag{4.5}$$
$$\|M_q^i\|_{L^1_tL^1} \leqslant C_0 2^{-i}\delta_{q+1}, \tag{4.6}$$
$$M_q^i = M_0^i - (v_q - v_0)(\rho_0^i - 1), \ \rho_q^i = \rho_0^i, \ \text{for } i > N_q, \tag{4.7}$$
$$\rho_q^i(t) = 1, M_q^i(t) = 0 \text{ on } [0, T_q]. \tag{4.8}$$

Then there exists $(v_{q+1}, \rho_{q+1}^i, M_{q+1}^i)_{i\in\mathbb{N}}$ which solves (4.2) and satisfies (4.4)-(4.8) at the level $q+1$ and

$$\|v_{q+1} - v_q\|_{L^{d_0}_tL^{d_0}} \leqslant C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0}, \quad \|\rho_{q+1}^i - \rho_q^i\|_{L^{d_0'}_tL^{d_0'}} \leqslant C_vC_0^{1/d_0'}2^{-i/d_0'}\delta_{q+1}^{1/d_0'}. \tag{4.9}$$

Moreover, we have for some $\epsilon > 0$ small enough

$$\|v_{q+1} - v_q\|_{L^r_tL^p} + \|v_{q+1} - v_q\|_{C_tL^s} \leqslant \delta_{q+1}^{1/d_0}, \tag{4.10}$$
$$\|\rho_{q+1}^i - \rho_q^i\|_{C_tL^{1+\epsilon}} + \|\rho_{q+1}^i - \rho_q^i\|_{L^1_tW^{1,1+\epsilon}} \leqslant 4^{-i}\delta_{q+1}^{1/d_0'}, \quad \inf_{t\in[0,1]}(\rho_{q+1}^i - \rho_q^i) \geqslant -\delta_{q+1}^{1/d_0'}. \tag{4.11}$$

The proof of our main iteration Proposition 4.2 will be stated in the following section.

4.3. **Proof of Theorem 1.3.** We intend to start the iteration from $(v_0, \rho_0^i, M_0^i)_{i\in\mathbb{N}}$ which are defined as above. By (4.3), (4.4)-(4.8) are satisfied as $\delta_0 = 1, \delta_1 = (\frac{\epsilon_0}{2})^{d+1}$.

Next, we use Proposition 4.2 to build inductively $(v_q, \rho_q^i, M_q^i)_{i\in\mathbb{N}}$ for every $q \geqslant 1$. By (4.1) and (4.9)-(4.11), the sequence $\{(v_q, \rho_q^i)_{i\in\mathbb{N}}\}_{q\in\mathbb{N}}$ is Cauchy in

$$\Big(L^r([0,1];L^p) \cap L^{d_0}([0,1]\times\mathbb{T}^d) \cap C([0,1];L^s)\Big) \times \Big(L^{d_0'}([0,1]\times\mathbb{T}^d) \cap C([0,1];L^{1+\epsilon}) \cap L^1([0,1];W^{1,1+\epsilon})\Big)^{\mathbb{N}}$$

and we denote by (v, ρ^i) its limit. By (4.6), (4.8) and a similar calculation as (5.21), it is easy to verify (ρ^i, v) solves (1.3). Then we know that $\int \rho^i \mathrm{d}x = 1$.

By (4.1), (4.9) and (4.11) we have

$$\big|\|\rho^i-1\|_{C_tL^1}-\|\rho^i_0-1\|_{C_tL^1}\big|\leqslant\sum_{q=0}^{\infty}\|\rho^i_{q+1}-\rho^i_q\|_{C_tL^1}\leqslant 4^{-i}\sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'}\leqslant 4^{-i-1}, \tag{4.12}$$

$$\inf_{t\in[0,1]}\rho^i\geqslant\inf_{t\in[0,1]}\rho^i_0+\sum_{q=0}^{\infty}\inf_{t\in[0,1]}(\rho^i_{q+1}-\rho^i_q)\geqslant 1-\frac{1}{4}-\sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'}\geqslant\frac{1}{2},$$

at which point, together with the fact that $\|\rho^i_0-1\|_{C_tL^1}=\frac{2}{\pi 4^i}$, we obtain $\rho^i\to 1$ in C_tL^1 as $i\to\infty$, and ρ^i is nonnegative on $\mathbb{T}^d$. Moreover, for $i>j$, it holds that

$$\|\rho^i_0-\rho^j_0\|_{C_tL^1}\geqslant\frac{3}{2\pi 4^j},$$

which together with (4.12) implies that ρ^i do not coincide with each other.

By (4.1) and (4.10) we obtain that

$$\|v-\overline{v}\|_{L^r_tL^p}+\|v-\overline{v}\|_{C_tL^s}\leqslant\sum_{q\geqslant 0}(\|v_{q+1}-v_q\|_{L^r_tL^p}+\|v_{q+1}-v_q\|_{C_tL^s})\leqslant\sum_{q\geqslant 0}\delta_{q+1}^{1/d_0}\leqslant\epsilon_0.$$

We finish the proof of the first statement.

For the second statement, by (4.5), (4.9) and interpolation, we have $|v|\in L_t^{d_0(1+\epsilon)}L^{d_0(1+\epsilon)}$ for some $\epsilon>0$ sufficiently small. Since $\rho^i\in L_t^{d_0'}L^{d_0'}$, we deduce that $|v|^{1+\epsilon}\rho^i\in L^1_tL^1$. Moreover, since $t\mapsto\rho^i(t)$ is continuous on $[0,1]$, we may apply the superposition principle to the linear diffusion operator

$$L_{\rho^i}:=\sigma\sigma^T(\rho^i):\nabla^2+(v+V*\rho^i)\cdot\nabla,$$

where the density ρ^i is regarded as fixed.

More precisely, let $C([0,1];\mathbb{T}^d)$ denote the space of continuous paths, equipped with its Borel σ-algebra and the natural filtration generated by the canonical process $\Pi_t,\ t\in[0,1]$, defined by $\Pi_t(\omega):=\omega(t),\ \omega\in C([0,1];\mathbb{T}^d)$. By the superposition principle [Tre14, Section 7.2], there exists a probability measure $\mathbf{Q}^i$ on $C([0,1];\mathbb{T}^d)$ which is a martingale solution associated with L_{ρ^i} in the sense that, for every smooth function f on $\mathbb{T}^d$, the process

$$f(\Pi_t)-f(\Pi_0)-\int_0^t L_{\rho^i}f(\Pi_s)\,\mathrm{d}s$$

is a $\mathbf{Q}^i$-martingale. Moreover, $\mathrm{d}\mathbf{Q}^i\circ\Pi_t^{-1}=\rho^i(t)\,\mathrm{d}x,\ t\in[0,1]$.

Since $\{\rho^i\}_{i\in\mathbb{N}}$ is a family of distinct solutions to (1.3), it follows that $\{\mathbf{Q}^i\}_{i\in\mathbb{N}}$ is a family of distinct martingale solutions.

5. Proof of Proposition 4.2

In this section, we extend the convex integration scheme to a system of infinitely many equations in order to prove Proposition 4.2. At step $q+1$, we focus on the first N_{q+1} Fokker–Planck equations in (1.3), and construct new perturbations to eliminate the corresponding stress terms. At the same time, the remaining equations with indices larger than N_{q+1} are left unchanged and shown to remain sufficiently small so as to satisfy the required estimates at level $q+1$.

5.1. **Mollification.** For a sufficiently small parameter $\alpha \in (0,1)$ to be chosen later, we define $l := \lambda_{q+1}^{-7\alpha/4}\lambda_q^{-3d/2-6}$. Then

$$l^{-1} \leqslant \lambda_{q+1}^{2\alpha}, \quad l\lambda_q^{3d+12} \leqslant \lambda_{q+1}^{-3\alpha/2}, \quad \lambda_{q+1}^{-\alpha/2} \ll \delta_{q+2}, \tag{5.1}$$

provided that

$$\alpha b > 6d + 24, \quad \alpha > 4\beta b. \tag{5.2}$$

To avoid the loss of derivatives, we first mollify the first N_{q+1} equations. Let $\phi_l := \frac{1}{l^d}\phi\big(\frac{\cdot}{l}\big)$ be a family of standard radial mollifiers on $\mathbb{R}^d$, and let $\varphi_l := \frac{1}{l}\varphi\big(\frac{\cdot}{l}\big)$ be a family of standard mollifiers supported in $(0,1)$. We define

$$v_l = (v_q *_x \phi_l) *_t \varphi_l, \quad \rho_l^i = (\rho_q^i *_x \phi_l) *_t \varphi_l, \quad M_l^i = (M_q^i *_x \phi_l) *_t \varphi_l.$$

For the mollification around $t = 0$, since ρ_q^i and M_q are constants around $t = 0$, we can directly extend these definitions to $t \leqslant 0$ by their values at $t = 0$. For v_q, we also extend the definition to $t \leqslant 0$ by their values at $t = 0$. Since ρ_q^i and M_q^i are constant near $t = 0$, this causes no problem. Moreover, we know that $\rho_l^i = 1$ and $M_l^i = 0$ on $[0, T_{q+1}]$.

By straightforward calculations we obtain

$$\partial_t\rho_l^i - \mathrm{div}\mathrm{div}(A(\rho_l^i)\rho_l^i) + \mathrm{div}(v_l\rho_l^i) + \mathrm{div}((V*\rho_l^i)\rho_l^i) = -\mathrm{div}(M_l^i + M_{com}^i), \quad \mathrm{div} v_l = 0, \tag{5.3}$$

where

$$\begin{aligned} M_{com}^i :=\; & -v_l\rho_l^i + (v_q\rho_q^i) *_x \phi_l *_t \varphi_l + \mathrm{div}(A(\rho_l^i)\rho_l^i) - (\mathrm{div}(A(\rho_q^i)\rho_q^i)) *_x \phi_l *_t \varphi_l \\ & - (V*\rho_l^i)\rho_l^i + ((V*\rho_q^i)\rho_q^i) *_x \phi_l *_t \varphi_l. \end{aligned}$$

Moreover, we know that $M_{com}^i = 0$ on $[0, T_{q+1}]$.

Finally, by the standard mollification estimates, the space-time Sobolev embedding $W^{d+\frac{4}{3},1} \hookrightarrow L^\infty$, and (4.6), we obtain for every $N \geqslant 0$,

$$\|M_l^i\|_{C_{t,x}^N} \lesssim l^{-d-\frac{4}{3}-N}\|M_q^i\|_{L_t^1L^1} \lesssim C_0 2^{-i} l^{-d-\frac{4}{3}-N}. \tag{5.4}$$

5.2. **Construction of** v_{q+1}**.** As discussed above, we apply the convex integration scheme only to the first N_{q+1} equations. To construct the perturbations for v_q and ρ_q^i, $1 \leqslant i \leqslant N_{q+1}$, we employ the L^{d_0}-based building blocks introduced in Appendix A, for some exponent $d_0 > 1$ to be specified later. We also recall the parameters λ, $r_\perp$, $r_\parallel$, η, μ, σ introduced in Appendix A. Their precise choice will be specified subsequently.

Let $\chi \in C_c^\infty(-\frac{3}{4}, \frac{3}{4})$ be a nonnegative function such that $\sum_{n\in\mathbb{Z}} \chi(t-n) = 1$ for every $t \in \mathbb{R}$. Let $\tilde{\chi} \in C_c^\infty(-\frac{4}{5}, \frac{4}{5})$ be a nonnegative function satisfying $\tilde{\chi} = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$ and $\sum_{n\in\mathbb{Z}} \tilde{\chi}(t-n) \leqslant 2$.

For $1 \leqslant i \leqslant N_{q+1}$, we fix the parameters $\zeta^i := 20 \cdot 2^i \delta_{q+2}^{-1}$, and consider the two disjoint sets Λ^1, Λ^2 defined in Appendix A. We abuse the notation $\Lambda^i = \Lambda^1$ for odd i, and $\Lambda^i = \Lambda^2$ for even i.

We take $K = N_{q+1}$ in the construction of Appendix A, so that

$$\eta N_{q+1} \ll 1. \tag{5.5}$$

We then introduce a family of pairwise disjoint functions $g_{(\xi,i,d_0)}$ and $g_{(\xi,i,d_0')}$ for $\xi \in \Lambda$ and $1 \leqslant i \leqslant N_{q+1}$.

With these preparations, we define the rescaled building blocks by

$$W_{(\xi,n,i)}(x,t) := W_{(\xi,d_0)}\Big(x, \Big(\frac{n}{\zeta^i}\Big)^{1/d_0} H_{(\xi,i,d_0)}(t)\Big),$$

$$\Theta_{(\xi,n,i)}(x,t) := \Theta_{(\xi,d_0')}\Big(x, \Big(\frac{n}{\zeta^i}\Big)^{1/d_0} H_{(\xi,i,d_0)}(t)\Big).$$

Similarly, we define $V_{(\xi,n,i)}, \Phi_{(\xi,n,i)}$, and all the other quantities appearing in Appendix A. By (A.2) and (A.8), we have

$$\partial_t \Theta_{(\xi,n,i)} + \Big(\frac{n}{\zeta^i}\Big)^{1/d_0} g_{(\xi,i,d_0)} \mathrm{div}\big(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\big) = 0. \tag{5.6}$$

We next define the perturbations for the drift term. For $1 \leqslant i \leqslant N_{q+1}$, let

$$w_{q+1}^{(p,i)} := \sum_{n\geqslant 3} \tilde{\chi}(\zeta^i|M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} \sum_{\xi\in\Lambda^n} W_{(\xi,n,i)} g_{(\xi,i,d_0)},$$

and

$$\begin{aligned} w_{q+1}^{(c,i)} := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \Big(&- \tilde{\chi}(\zeta^i|M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} \frac{1}{(n_*\lambda_{q+1})^2} \nabla\Phi_{(\xi,n,i)}\,\xi\cdot\nabla\psi_{(\xi,n,i)} \\ &+ \nabla\big(\tilde{\chi}(\zeta^i|M_l^i| - n)\big)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} : V_{(\xi,n,i)}\Big) g_{(\xi,i,d_0)}. \end{aligned}$$

By (A.3), we obtain

$$w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)} = \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathrm{div}\Big(\tilde{\chi}(\zeta^i|M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} V_{(\xi,n,i)}\Big) g_{(\xi,i,d_0)}. \tag{5.7}$$

Since $V_{(\xi,n,i)}$ is skew-symmetric, it follows that $\mathrm{div}\big(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}\big) = 0$.

Finally, we define the total perturbation and the new velocity field by

$$w_{q+1} := \sum_{i=1}^{N_{q+1}} \big(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}\big), \qquad v_{q+1} := v_l + w_{q+1}.$$

Then v_{q+1} is mean-zero and divergence-free. Moreover, since $M_l^i(t) = 0$ on $[0, T_{q+1}]$, the perturbation w_{q+1} vanishes on $[0, T_{q+1}]$.

5.3. **Construction of ρ_{q+1}^i.** We next define the perturbations for the density functions. For $1 \leqslant i \leqslant N_{q+1}$, we set

$$\begin{aligned} \theta_{q+1}^{(p,i)} &:= \sum_{n\geqslant 3} \chi(\zeta^i|M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'} \sum_{\xi\in\Lambda^n} \Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi,n,i)} g_{(\xi,i,d_0')}, \\ \theta_{q+1}^{(c,i)} &:= -\int_{\mathbb{T}^d} \theta_{q+1}^{(p,i)}\,\mathrm{d}x, \\ \theta_{q+1}^{(o,i)} &:= -\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi,i,d_0)} \mathrm{div}\Big(\chi(\zeta^i|M_l^i| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big). \end{aligned}$$

By an argument analogous to that in [LRZ25, (5.5)], we obtain

$$\begin{aligned} w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} = &\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \chi(\zeta^i|M_l^i| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\mathbb{P}_{\neq 0}\big(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\big) g_{(\xi,i,d_0)} g_{(\xi,i,d_0')} \\ &+ \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \chi(\zeta^i|M_l^i| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\big(g_{(\xi,i,d_0)} g_{(\xi,i,d_0')} - 1\big) \end{aligned}$$

$$+\chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\frac{M_l^i}{|M_l^i|}.$$

Combining this identity with (5.6), we deduce that

$$\begin{aligned}
&\partial_t\theta_{q+1}^{(p,i)}+\mathrm{div}\big(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)}-M_l^i\big)+\partial_t\theta_{q+1}^{(o,i)}\\
=&\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\partial_t\Big[\chi(\zeta^i|M_l^i|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)g_{(\xi,i,d_0')}\Big]\Theta_{(\xi,n,i)}\\
&+\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\nabla\Big[\chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big]g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}\mathbb{P}_{\neq 0}\big(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\big)\\
&+\mathrm{div}\Big(\sum_{n\geqslant 3}\chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\frac{M_l^i}{|M_l^i|}-M_l^i\Big)\\
&-\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}h_{(\xi,i,d_0)}\partial_t\mathrm{div}\Big(\chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big).
\end{aligned}\tag{5.8}$$

We now define, for every $1\leqslant i\leqslant N_{q+1}$,

$$\theta_{q+1}^i:=\theta_{q+1}^{(p,i)}+\theta_{q+1}^{(c,i)}+\theta_{q+1}^{(o,i)},\qquad \rho_{q+1}^i:=\rho_l^i+\theta_{q+1}^i.$$

For $i>N_{q+1}$, we simply set $\rho_{q+1}^i:=\rho_q^i$.

By construction, $\int_{\mathbb{T}^d}\rho_{q+1}^i\,\mathrm{d}x=1$ for every $i\in\mathbb{N}$. Since $M_l^i(t)=0$ on $[0,T_{q+1}]$, the perturbation $\theta_{q+1}^i(t)$ vanishes there for all $1\leqslant i\leqslant N_{q+1}$. Hence $\rho_{q+1}^i(t)=1$ on $[0,T_{q+1}]$ for all $1\leqslant i\leqslant N_{q+1}$. For $i>N_{q+1}$, we already have $\rho_{q+1}^i(t)=\rho_q^i(t)=1$ on $[0,T_{q+1}]$.

Moreover, since the functions $g_{(\xi,i,d_0)}$ have disjoint supports for distinct i, and since $\theta_{q+1}^{(c,i)}$ depends only on t, we obtain for $1\leqslant i\leqslant N_{q+1}$ that

$$\mathrm{div}\big(w_{q+1}\theta_{q+1}^i\big)=\mathrm{div}\Big((w_{q+1}^{(p,i)}+w_{q+1}^{(c,i)})\theta_{q+1}^{(p,i)}+w_{q+1}\theta_{q+1}^{(o,i)}\Big).\tag{5.9}$$

5.4. **Construction of the stress terms M_{q+1}^i.**

5.4.1. *The case* $1\leqslant i\leqslant N_{q+1}$. From (5.9), and the definition of the perturbations we obtain

$$\begin{aligned}
-\mathrm{div}M_{q+1}^i=&\,\partial_t\theta_{q+1}^i+\mathrm{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)}-M_l^i)(:=\mathrm{div}M_{osc}^i)\\
&-\mathrm{div}[\mathrm{div}(A(\rho_{q+1}^i)\rho_{q+1}^i-A(\rho_l^i)\rho_l^i)-(V*\rho_{q+1}^i)\rho_{q+1}^i+(V*\rho_l^i)\rho_l^i](:=\mathrm{div}M_{nonlin}^i)\\
&+\mathrm{div}(v_l\theta_{q+1}^i+w_{q+1}(\rho_l^i+\theta_{q+1}^{(o,i)})+w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)})(:=\mathrm{div}M_{lin}^i)\\
&-\mathrm{div}M_{com}^i,
\end{aligned}$$

where we define the nonlinear error and the linear error respectively, by

$$\begin{aligned}
M_{nonlin}^i:&=-\mathrm{div}(A(\rho_{q+1}^i)\rho_{q+1}^i-A(\rho_l^i)\rho_l^i)-(V*\rho_{q+1}^i)\rho_{q+1}^i+(V*\rho_l^i)\rho_l^i,\\
M_{lin}^i:&=v_l\theta_{q+1}^i+w_{q+1}(\rho_l^i+\theta_{q+1}^{(o,i)})+w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)}.
\end{aligned}$$

To define the oscillation error, using the inverse divergence operators $\mathcal{R}_1, \mathcal{B}_1$ introduced in Section 3.1 and (5.8), we define $M^i_{osc} := M^i_{osc,t} + M^i_{osc,x} + M^i_{osc,c} + M^i_{osc,o}$ as

$$M^i_{osc,t} := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathcal{R}_1\Big(\partial_t[\chi(\zeta^i|M^i_l| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)g_{(\xi,i,d_0')}]\Theta_{(\xi,n,i)}\Big),$$

$$M^i_{osc,x} := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathcal{B}_1\Big(\nabla[\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)], \mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})\Big)g_{(\xi,i,d_0)}g_{(\xi,i,d_0')},$$

$$M^i_{osc,c} := \sum_{n\geqslant 3}\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\frac{M^i_l}{|M^i_l|} - M^i_l,$$

$$M^i_{osc,o} := -\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi,i,d_0)}\partial_t\Big(\chi(\zeta|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\Big).$$

Then we define

$$-M^i_{q+1} := M^i_{osc} + M^i_{nonlin} + M^i_{lin} - M^i_{com}.$$

Since $M^i_l(t) = w_{q+1}(t) = \theta^i_{q+1}(t) = 0$ on $[0, T_{q+1}]$, we have $M^i_{q+1}(t) = 0$ on $[0, T_{q+1}]$, which implies (4.8) for $M^i_{q+1}, 1 \leqslant i \leqslant N_{q+1}$.

5.4.2. *The case $i > N_{q+1}$.* In this case, since $\operatorname{div}(v_{q+1} - v_q) = 0$ and $\rho^i_{q+1} = \rho^i_q = \rho^i_0$, we define

$$M^i_{q+1} := M^i_q - (v_{q+1} - v_q)(\rho^i_0 - 1) = M^i_0 - (v_{q+1} - v_0)(\rho^i_0 - 1).$$

It is straightforward to verify that $(v_{q+1}, \rho^i_{q+1}, M^i_{q+1})_{i>N_{q+1}}$ satisfies (4.2) at level $q+1$.

Moreover, since $\rho^i_0(t) = 1$ and $M^i_0(t) = 0$ on $[0, T_0]$, it follows that $M^i_{q+1}(t) = 0$ on $[0, T_{q+1}]$, which proves (4.8) for $M^i_{q+1},\ i > N_{q+1}$.

5.5. **Proof of Proposition 4.2.** In this section, we verify that the perturbations and the new stress terms constructed above satisfy the properties stated in Proposition 4.2. Most of the estimates are analogous to the basic convex integration estimates, so we mainly focus on the new features arising from the choice of N_{q+1} and on the estimates for M^i_{q+1} when $i > N_{q+1}$.

5.5.1. *Choice of parameters.* Regarding the parameters of the building blocks, we define

$$\lambda = \lambda_{q+1},\ r_\perp = \lambda_{q+1}^{-1+\frac{1}{N}},\ r_\parallel = \lambda_{q+1}^{-1+\frac{2}{N}},\ \eta = \lambda_{q+1}^{-1},\ \mu = r_\perp^{-\frac{d-1}{d_0}} r_\parallel^{-\frac{1}{d_0}},\ \sigma = \lambda_{q+1}^{\frac{1}{2N}},$$

where $N > 4d$ is a sufficiently large integer satisfying $\frac{d}{p} + \frac{1}{r} > 1 + \frac{4d}{N}, \frac{d}{s} > 1 + \frac{4d}{N}$. We further define $d_0 := d + 1 - \frac{4d}{N} \in (d, d+1), d_0' := \frac{d_0}{d_0 - 1} \in (1, 2)$. It is straightforward to verify (5.5). With these choices, we have

$$r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}}\eta^{-\frac{1}{d_0'}},\ r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{p}-\frac{1}{d_0}}\eta^{\frac{1}{r}-\frac{1}{d_0}},\ r_\perp^{\frac{d-1}{s}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{s}-\frac{1}{d_0}}\eta^{-\frac{1}{d_0}} \leqslant \lambda^{-\frac{1}{N}}. \tag{5.10}$$

In fact, by a direct calculation, it holds that

$$\begin{aligned}
\lambda r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}}\eta^{1-\frac{1}{d_0'}} &= r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}}\eta^{-\frac{1}{d_0'}} \leqslant r_\parallel^{\frac{d}{d_0}}\eta^{\frac{1}{d_0}-1} \\
&= \lambda^{\frac{2d}{Nd_0}+\frac{d_0-d-1}{d_0}} = \lambda^{-\frac{2d}{Nd_0}} \leqslant \lambda^{-\frac{1}{N}},
\end{aligned}$$

$$r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{p}-\frac{1}{d_0}}\eta^{\frac{1}{r}-\frac{1}{d_0}} \leqslant \lambda^{(\frac{d+1}{N}-d)(\frac{1}{p}-\frac{1}{d_0})-\frac{1}{r}+\frac{1}{d_0}} \leqslant \lambda^{-\frac{d}{p}-\frac{1}{r}+\frac{d+1}{d_0}+\frac{d+1}{N}}$$

$$
\begin{aligned}
&\leqslant \lambda^{-1-\frac{4d}{N}+1+\frac{2d}{N}+\frac{d+1}{N}} \leqslant \lambda^{-\frac{1}{N}},\\
r_{\perp}^{\frac{d-1}{s}-\frac{d-1}{d_0}} r_{\|}^{\frac{1}{s}-\frac{1}{d_0}} \eta^{-\frac{1}{d_0}} &\leqslant \lambda^{(\frac{d+1}{N}-d)(\frac{1}{s}-\frac{1}{d_0})+\frac{1}{d_0}} \leqslant \lambda^{-\frac{d}{s}+\frac{d+1}{d_0}+\frac{d+1}{N}}\\
&\leqslant \lambda^{-1-\frac{4d}{N}+1+\frac{2d}{N}+\frac{d+1}{N}} \leqslant \lambda^{-\frac{1}{N}}.
\end{aligned}
$$

In the sequel, we shall also require (5.2) together with

$$
\lambda_q^{d+4} \leqslant \lambda_{q+1}^{\alpha}, \quad (12d+44)\alpha < \frac{1}{2N}. \tag{5.11}
$$

These conditions can be achieved as follows: first choose $\alpha > 0$ sufficiently small such that $(12d+44)\alpha < \frac{1}{2N}$, then choose $b \in 2N\mathbb{N}$ sufficiently large so that $b > \frac{6d+24}{\alpha}$, and finally choose $\beta > 0$ sufficiently small such that $\alpha > 4\beta b$. At the end, we choose a sufficiently large in order to absorb all implicit and universal constants arising in the subsequent estimates, and to guarantee the validity of (4.1).

Finally, we record the following estimate for the amplitude functions.

Lemma 5.1. ([LRZ25, Proposition 5.2]). *For $M, N \in \mathbb{N}_0, 1 \leqslant i \leqslant N_{q+1}$ we have*

$$
\sum_{n\geqslant 3}\Big(\frac{n}{\zeta^i}\Big)^M\Big(\|\chi(\zeta^i|M_l^i|-n)\|_{C^N_{t,x}} + \|\tilde{\chi}(\zeta^i|M_l^i|-n)\|_{C^N_{t,x}}\Big) \lesssim l^{-(d+4)N-(d+2)(M+1)},
$$

$$
\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big(\frac{n}{\zeta^i}\Big)^M\Big\|\chi(\zeta^i|M_l^i|-n)\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big\|_{C^N_{t,x}} \lesssim l^{-(2d+8)N-(d+2)(M+1)}.
$$

5.5.2. *Proof of* (4.9) *for $v_{q+1}-v_q$.* From this section onward, we establish the desired estimates for the perturbation w_{q+1}.

We first estimate the principal perturbations $w_{q+1}^{(p,i)}$ for $1 \leqslant i \leqslant N_{q+1}$ in the $L^{d_0}_tL^{d_0}$-norm. By Cauchy's inequality and the fact that $\sum_{n\in\mathbb{Z}}\tilde{\chi}(t-n) \leqslant 2$ we have

$$
\begin{aligned}
|w_{q+1}^{(p,i)}|^{d_0} &\leqslant \Big(\sum_{n\geqslant 3}\tilde{\chi}(\zeta^i|M_l^i|-n)\Big)^{d_0-1}\sum_{n\geqslant 3}\tilde{\chi}(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\Big|\sum_{\xi\in\Lambda^n}W_{(\xi,n,i)}g_{(\xi,i,d_0)}\Big|^{d_0}\\
&\lesssim \sum_{n\geqslant 3}\tilde{\chi}(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i}\sum_{\xi\in\Lambda^n}\big|W_{(\xi,n,i)}g_{(\xi,i,d_0)}\big|^{d_0}.
\end{aligned}
$$

By applying the generalized Hölder inequality of Theorem 3.3 in spatial direction, together with the estimates for the building blocks in (A.6) and Lemma 5.1 we deduce

$$
\begin{aligned}
\|w_{q+1}^{(p,i)}(t)\|_{L^{d_0}}^{d_0} &\lesssim \sum_{n\geqslant 3}\Big\|\tilde{\chi}(\zeta^i|M_l^i(t)|-n)\frac{n}{\zeta^i}\Big\|_{L^1}\sum_{\xi\in\Lambda^n}\|W_{(\xi,n,i)}\|_{C_tL^{d_0}}^{d_0}g_{(\xi,i,d_0)}^{d_0}(t)\\
&\quad + (r_\perp\lambda_{q+1})^{-1}\Big\|\tilde{\chi}(\zeta^i|M_l^i(t)|-n)\frac{n}{\zeta^i}\Big\|_{C^1_{t,x}}\sum_{\xi\in\Lambda^n}\|W_{(\xi,n,i)}\|_{C_tL^{d_0}}^{d_0}g_{(\xi,i,d_0)}^{d_0}(t)\\
&\lesssim \left(\Big\|\sum_{n\geqslant 3}\tilde{\chi}(\zeta^i|M_l^i(t)|-n)(M_l^i(t)+|\zeta^i|^{-1})\Big\|_{L^1} + l^{-3d-8}\lambda_{q+1}^{-\frac{1}{N}}\right)\sum_{\xi\in\Lambda}g_{(\xi,i,d_0)}^{d_0}(t)\\
&\lesssim (\|M_l^i(t)\|_{L^1} + 2^{-i}\delta_{q+1})\sum_{\xi\in\Lambda}g_{(\xi,i,d_0)}^{d_0}(t),
\end{aligned}
$$

where we used the fact that $\sum_{n\in\mathbb{Z}}\tilde{\chi}(t-n)\leqslant 2$, and used conditions on the parameters to have $(6d+17)\alpha-\frac{1}{N}<-\alpha<-2\beta, 2^i\leqslant 2^{N_{q+1}}\leqslant\lambda_{q+1}^{\alpha}$. Here we recall the notation $\Lambda=\Lambda^1\cup\Lambda^2$. Then we apply the generalized Hölder inequality of Theorem 3.3 in time direction, the bounds (A.9) and (5.4) to deduce for some $C_v\geqslant 1$

$$\begin{aligned}\|w_{q+1}^{(p,i)}\|_{L_t^{d_0}L^{d_0}}^{d_0}&\lesssim(\|M_l^i\|_{L_t^1L^1}+2^{-i}\delta_{q+1}+\sigma^{-1}\|M_l^i\|_{C_{t,x}^1})\sum_{\xi\in\Lambda}\left\|g_{(\xi,i,d_0)}\right\|_{L_t^{d_0}}^{d_0}\\&\lesssim C_02^{-i}(\delta_{q+1}+\lambda_{q+1}^{(2d+6)\alpha-\frac{1}{2N}})\leqslant(\frac{1}{4}C_v(2^{\frac{1}{d+1}}-1))^{d_0}C_02^{-i}\delta_{q+1},\end{aligned}\tag{5.12}$$

where we used conditions on the parameters to have $(2d+6)\alpha-\frac{1}{2N}<-\alpha<-2\beta$.

For the general $L_t^uL^m$-norm with $u,m\in[1,\infty]$, by the estimates for the building blocks in (A.4)-(A.6), (A.9) and the estimate for the amplitude function in Lemma 5.1 we obtain

$$\begin{aligned}\|w_{q+1}^{(p,i)}\|_{L_t^uL^m}&\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\tilde{\chi}(\zeta^i|M_l^i|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0}\right\|_{C_{t,x}^0}\|W_{(\xi,n,i)}\|_{C_tL^m}\|g_{(\xi,i,d_0)}\|_{L_t^u}\\&\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}}r_\parallel^{\frac{1}{m}-\frac{1}{d_0}}\eta^{\frac{1}{u}-\frac{1}{d_0}},\end{aligned}\tag{5.13}$$

$$\begin{aligned}\|w_{q+1}^{(c,i)}\|_{L_t^uL^m}&\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\tilde{\chi}(\zeta^i|M_l^i|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0}\right\|_{C_{t,x}^1}\\&\qquad\times\Big(\frac{1}{\lambda_{q+1}^2}\|\nabla\Phi_{(\xi,n,i)}\xi\cdot\nabla\psi_{(\xi,n,i)}\|_{L^m}+\|V_{(\xi,n,i)}\|_{L^m}\Big)\|g_{(\xi,i,d_0)}\|_{L_t^u}\\&\lesssim l^{-3d-8}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}}r_\parallel^{\frac{1}{m}-\frac{1}{d_0}}\frac{r_\perp}{r_\parallel}\eta^{\frac{1}{u}-\frac{1}{d_0}}.\end{aligned}\tag{5.14}$$

With these estimates, combining with the choice of parameters in (5.1) and the bound (5.12) we obtain

$$\begin{aligned}\|w_{q+1}\|_{L_t^{d_0}L^{d_0}}&\leqslant\sum_{i=1}^{N_{q+1}}\frac{C_v}{4}2^{-i/d_0}C_0^{1/d_0}(2^{\frac{1}{d+1}}-1)\delta_{q+1}^{1/d_0}+N_{q+1}Cl^{-3d-8}\frac{r_\perp}{r_\parallel}\\&\leqslant\frac{C_v}{4}C_0^{1/d_0}\delta_{q+1}^{1/d_0}+C\lambda_{q+1}^{(6d+17)\alpha-\frac{1}{N}}\leqslant\frac{C_v}{2}C_0^{1/d_0}\delta_{q+1}^{1/d_0},\end{aligned}\tag{5.15}$$

where we used conditions on the parameters to have $(6d+18)\alpha<\frac{1}{N}, N_{q+1}\lesssim\lambda_{q+1}^{\alpha}$ and chose a large enough to absorb the universal constant. The above inequality yields that (4.9) holds for $v_{q+1}-v_q$ and then (4.4) holds for v_{q+1}:

$$\begin{aligned}\|v_{q+1}-v_q\|_{L_t^{d_0}L^{d_0}}&\leqslant\|w_{q+1}\|_{L_t^{d_0}L^{d_0}}+\|v_l-v_q\|_{L_t^{d_0}L^{d_0}}\\&\leqslant\frac{1}{2}C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0}+lC_0^{1/d_0}\lambda_q^{d+4}\leqslant C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0}.\end{aligned}$$

5.5.3. *Proof of* (4.10). Combining the bounds (5.13) and (5.14) above we obtain

$$\begin{aligned}&\|v_{q+1}-v_q\|_{L_t^rL^p}+\|v_{q+1}-v_q\|_{C_tL^s}\lesssim\|w_{q+1}\|_{C_tL^s}+\|w_{q+1}\|_{L_t^rL^p}+l\|v_q\|_{C_{t,x}^1}\\&\lesssim N_{q+1}l^{-3d-8}(r_\perp^{\frac{d-1}{s}-\frac{d-1}{d_0}}r_\parallel^{\frac{1}{s}-\frac{1}{d_0}}\eta^{-\frac{1}{d_0}}+r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}}r_\parallel^{\frac{1}{p}-\frac{1}{d_0}}\eta^{\frac{1}{r}-\frac{1}{d_0}})+l\lambda_q^{d+4}\\&\lesssim\lambda_{q+1}^{(6d+17)\alpha-\frac{1}{N}}+\lambda_{q+1}^{-\alpha}\leqslant\delta_{q+1}^{1/d_0}.\end{aligned}\tag{5.16}$$

Here we used (5.1), (5.10) and conditions on the parameters to have $(6d+18)\alpha < \frac{1}{N}$, $N_{q+1} \lesssim \lambda_{q+1}^{\alpha}$. Then we chose a large enough to absorb the universal constant.

5.5.4. *Proof of* (4.5) *for* v_{q+1}. By the estimates for the building blocks in (A.6), (A.9), (5.7) and the estimates for the amplitude functions in Lemma 5.1 we have for $d_0 \geqslant 2$

$$\begin{aligned}\|w_{q+1}\|_{C^1_{t,x}} &\lesssim \sum_{i=1}^{N_{q+1}} \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^n} \left\|\tilde{\chi}(\zeta^i|M_l^i|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0}\right\|_{C^2_{t,x}} \|\nabla V_{(\xi,n,i)}\|_{C^1_{t,x}} \|g_{(\xi,i,d_0)}\|_{C^1_t} \\ &\lesssim N_{q+1} l^{-4d-12} \lambda_{q+1} \mu r_{\parallel}^{-\frac{1}{d_0}} r_{\perp}^{-\frac{d-1}{d_0}} \sigma \eta^{-1-\frac{2}{d_0}} \lesssim N_{q+1} \lambda_{q+1}^{(8d+24)\alpha+d+\frac{7}{2}}.\end{aligned}$$

Thus, by $N_{q+1} \lesssim \lambda_{q+1}^{\alpha}, (8d+25)\alpha < \frac{1}{2}$ we obtain the following

$$\|v_{q+1}\|_{C^1_{t,x}} \leqslant \|v_l\|_{C^1_{t,x}} + \|w_{q+1}\|_{C^1_{t,x}} \leqslant C_0^{1/d_0} \lambda_q^{d+4} + \frac{1}{2}\lambda_{q+1}^{d+4} \leqslant C_0^{1/d_0} \lambda_{q+1}^{d+4},$$

where we chose a large enough to absorb the universal constant.

5.5.5. *Proof of* (4.9) *for* $\rho_{q+1}^i - \rho_q^i$. Similarly as before, we first estimate the principal perturbations $\theta_{q+1}^{(p,i)}$ in $L_t^{d_0'} L^{d_0'}$. By Hölder inequality and the fact that $\sum_{n\in\mathbb{Z}} \chi(t-n) = 1$, and Γ_ξ are uniformly bounded we have

$$\begin{aligned}|\theta_{q+1}^{(p,i)}|^{d_0'} &\lesssim \Big(\sum_{n\geqslant 3} \chi(\zeta^i|M_l^i|-n)\Big)^{d_0'-1} \sum_{n\geqslant 3} \chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i} \sum_{\xi\in\Lambda^n} \left|\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi,n,i)} g_{(\xi,i,d_0')}\right|^{d_0'} \\ &\lesssim \sum_{n\geqslant 3} \chi(\zeta^i|M_l^i|-n)\frac{n}{\zeta^i} \sum_{\xi\in\Lambda^n} \left|\Theta_{(\xi,n,i)} g_{(\xi,i,d_0')}\right|^{d_0'}.\end{aligned}$$

By the same argument as in (5.12), we have

$$\begin{aligned}\|\theta_{q+1}^{(p,i)}\|^{d_0'}_{L_t^{d_0'} L^{d_0'}} &\lesssim \Big(\|M_l^i\|_{L_t^1 L^1} + 2^{-i}\delta_{q+1} + \sigma^{-1}\|M_l^i\|_{C^1_{t,x}}\Big) \sum_{\xi\in\Lambda} \left\|g_{(\xi,i,d_0')}\right\|^{d_0'}_{L_t^{d_0'}} \\ &\lesssim 2^{-i} C_0(\delta_{q+1} + \lambda_{q+1}^{(2d+6)\alpha-\frac{1}{2N}}) \leqslant 2^{-i}(\frac{1}{2}C_v)^{d_0'} C_0 \delta_{q+1},\end{aligned} \tag{5.17}$$

where we used conditions on the parameters to have $(2d+6)\alpha - \frac{1}{2N} < -\alpha < -2\beta$.

For general $L_t^u L^m$-norm with $u, m \in [1,\infty]$, by the estimates for the building blocks in (A.7), (A.9) and Lemma 5.1 we obtain

$$\begin{aligned}\|\theta_{q+1}^{(p,i)}\|_{L_t^u L^m} &\lesssim \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^n} \left\|\chi(\zeta^i|M_l^i|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'} \Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\right\|_{C^0_{t,x}} \|\Theta_{(\xi,n,i)}\|_{C_t L^m} \|g_{(\xi,i,d_0')}\|_{L_t^u} \\ &\lesssim l^{-2d-4} r_{\perp}^{\frac{d-1}{m}-\frac{d-1}{d_0'}} r_{\parallel}^{\frac{1}{m}-\frac{1}{d_0'}} \eta^{\frac{1}{u}-\frac{1}{d_0'}}.\end{aligned} \tag{5.18}$$

Moreover, by the bounds (A.9) and (5.10) we have for some $\epsilon > 0$ small enough

$$\|\theta_{q+1}^{(c,i)}\|_{C_t} \lesssim \|\theta_{q+1}^{(p,i)}\|_{C_t L^{1+\epsilon}} \lesssim l^{-2d-4} r_{\perp}^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}} r_{\parallel}^{\frac{1}{1+\epsilon}-\frac{1}{d_0'}} \eta^{-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(4d+8)\alpha-\frac{1}{N}+d\epsilon} \lesssim 4^{-i}\lambda_{q+1}^{-2\alpha}, \tag{5.19}$$

$$\|\theta_{q+1}^{(o,i)}\|_{C_t C^1} \lesssim \sigma^{-1} l^{-6d-20} \lesssim \lambda_{q+1}^{(12d+40)\alpha-\frac{1}{2N}} \lesssim 4^{-i}\lambda_{q+1}^{-2\alpha}, \tag{5.20}$$

where we used conditions on the parameters to have $(12d+44)\alpha < \frac{1}{2N}, 2^i \leqslant 2^{N_{q+1}} \lesssim \lambda_{q+1}^{\alpha}$ and chose a large enough to absorb the universal constant. We also chose $\epsilon > 0$ small such that $d\epsilon < \alpha$. Then, combining the above estimates together, we obtain by (4.5)

$$\begin{aligned}\|\rho^i_{q+1}-\rho^i_q\|_{L_t^{d_0'}L^{d_0'}} &\leqslant \|\theta^i_{q+1}\|_{L_t^{d_0'}L^{d_0'}} + l\|\rho^i_q - 1\|_{C^1_{t,x}}\\ &\leqslant \frac{1}{2}C_v C_0^{1/d_0'}2^{-i/d_0'}\delta_{q+1}^{1/d_0'} + 4^{-i}\lambda_{q+1}^{-2\alpha} + l2^{-i}C_0^{1/d_0'}\lambda_q^{d+4} \leqslant C_v C_0^{1/d_0'}2^{-i/d_0'}\delta_{q+1}^{1/d_0'},\end{aligned}$$

which implies (4.9) for $\rho^i_{q+1}-\rho^i_q$, $1\leqslant i\leqslant N_{q+1}$. Here we chose a large enough to absorb the universal constant. For $i > N_{q+1}$, we have $\rho^i_{q+1}-\rho^i_q = 0$. Hence, the second estimate in (4.9) also holds for $i > N_{q+1}$.

5.5.6. *Proof of* (4.11). We first estimate $\theta^{(p,i)}_{q+1}$ in $W^{1,1+\epsilon}$-norm for some $\epsilon > 0$ small enough. By the bounds for the building blocks in (A.7), (A.9) and the bounds for the amplitude functions in Lemma 5.1 we have

$$\begin{aligned}\|\theta^{(p,i)}_{q+1}\|_{L^1_tW^{1,1+\epsilon}} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C_tW^{1,1+\epsilon}}\|g_{(\xi,i,d_0')}\|_{L^1_t}\\ &\lesssim l^{-4d-12}\lambda_{q+1}r_{\|}^{\frac{1}{1+\epsilon}-\frac{1}{d_0'}}r_{\perp}^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}}\eta^{1-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(8d+24)\alpha-\frac{1}{N}+d\epsilon} \lesssim 2^{-i}\lambda_{q+1}^{-2\alpha},\end{aligned}$$

where we used the choice of parameters in (5.10) and chose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used conditions on the parameters to have $(8d+28)\alpha < \frac{1}{N}, 2^i \leqslant 2^{N_{q+1}} \lesssim \lambda_{q+1}^{\alpha}$.

Similarly, by the fact that $\theta^{(p,i)}_{q+1}$ is non-negative, (4.5) and the choice of parameters in (5.1), (5.10) we have

$$\begin{aligned}\|\rho^i_{q+1}-\rho^i_q\|_{C_tL^{1+\epsilon}} + \|\rho^i_{q+1}-\rho^i_q\|_{L^1_tW^{1,1+\epsilon}} &\leqslant \|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}} + \|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}} + l\|\rho^i_q-1\|_{C^2_{t,x}}\\ &\lesssim 4^{-i}\lambda_{q+1}^{-\alpha} \leqslant 4^{-i}\delta_{q+1}^{1/d_0'},\\ \inf_{t\in[0,1]}(\rho^i_{q+1}-\rho^i_q) \geqslant -\|\theta^{(c,i)}_{q+1}+\theta^{(o,i)}_{q+1}\|_{C^0_{t,x}} - l\|\rho^i_q-1\|_{C^1_{t,x}} &\geqslant -C\lambda_{q+1}^{-\alpha} \geqslant -\delta_{q+1}^{1/d_0'},\end{aligned}$$

which yields (4.11) for $1\leqslant i\leqslant N_{q+1}$. Here we chose a large enough to absorb the universal constant. Since $\rho^i_{q+1}-\rho^i_q = 0$ for $i > N_{q+1}$, the estimate (4.11) also holds in this case.

5.5.7. *Proof of* (4.5) *for* ρ^i_{q+1}. By (A.7), (A.9) and Lemma 5.1 we have

$$\begin{aligned}\|\theta^{(p,i)}_{q+1}\|_{C^1_{t,x}} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C^1_{t,x}}\|g_{(\xi,i,d_0')}\|_{C^1_t}\\ &\lesssim l^{-4d-12}\lambda_{q+1}\mu r_{\|}^{-\frac{1}{d_0'}}r_{\perp}^{-\frac{d-1}{d_0'}}\sigma\eta^{-1-\frac{1}{d_0}-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(8d+24)\alpha+d+\frac{7}{2}},\\ \|\theta^{(c,i)}_{q+1}\|_{C^1_t} &\lesssim \|\theta^{(p,i)}_{q+1}\|_{C^1_{t,x}} \lesssim \lambda_{q+1}^{(8d+24)\alpha+d+\frac{7}{2}},\\ \|\theta^{(o,i)}_{q+1}\|_{C^1_{t,x}} &\lesssim \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|h_{(\xi,i,d_0)}\|_{C^1_t}\Big\|\mathrm{div}\Big(\chi(\zeta^i|M_l|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\Big)\Big\|_{C^1_{t,x}}\\ &\lesssim \sigma^{-1}\eta^{-1}l^{-6d-20} \lesssim \lambda_{q+1}^{(12d+40)\alpha+2}.\end{aligned}$$

Moreover,

$$\|\theta^{(p,i)}_{q+1}\|_{C^2_{t,x}} \lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^2_{t,x}}\|\Theta_{(\xi,n,i)}g_{(\xi,i,d_0')}\|_{C^2_{t,x}}$$
$$\lesssim l^{-6d-20}\lambda_{q+1}\mu r_{\|}^{-\frac{1}{d_0'}}r_\perp^{-\frac{d-1}{d_0'}}\sigma\eta^{-2}(\lambda_{q+1}\mu+\sigma\eta^{-1})\lesssim\lambda_{q+1}^{(12d+40)\alpha+d+\frac{11}{2}},$$
$$\|\theta^{(o,i)}_{q+1}\|_{C^2_{t,x}} \lesssim \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|h_{(\xi,i,d_0)}\|_{C^2_t}\left\|\chi(\zeta^i|M_l|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\right\|_{C^3_{t,x}}\lesssim\lambda_{q+1}^{(16d+56)\alpha+4}.$$

By choosing $(12d+42)\alpha<\frac{1}{2}$, $2^i\leqslant 2^{N_{q+1}}\lesssim\lambda^\alpha_{q+1}$ we deduce

$$4^i\|\rho^i_{q+1}-1\|_{C^1_{t,x}}+\lambda^{-2}_{q+1}\|\rho^i_{q+1}\|_{C^2_{t,x}}\leqslant C_0^{1/d_0'}\lambda_q^{d+4}+\frac{1}{2}\lambda^{d+4}_{q+1}\leqslant C_0^{1/d_0'}\lambda^{d+4}_{q+1},$$

which implies (4.5) for $\rho^i_{q+1}, 1\leqslant i\leqslant N_{q+1}$. For $i>N_{q+1}$, the estimate (4.5) also holds, since $\rho^i_{q+1}-\rho^i_q=0$ and $\lambda_q\leqslant\lambda_{q+1}$.

5.5.8. *Proof of* (4.6) *for* $1\leqslant i\leqslant N_{q+1}$. We now estimate each term in the definition of M^i_{q+1} separately.

Oscillation error M^i_{osc}. By Lemma 3.1, the estimates for the amplitude functions and for the building blocks in Lemma 5.1, (A.7) and (A.9) respectively we obtain

$$\|M^i_{osc,t}\|_{L^1_tL^1}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^1_{t,x}}\|g_{(\xi,i,d_0')}\|_{W^{1,1}_t}\|\Theta_{(\xi,n,i)}\|_{C_tL^1}$$
$$\lesssim l^{-4d-12}r_\perp^{d-1-\frac{d-1}{d_0'}}r_{\|}^{1-\frac{1}{d_0'}}\sigma\eta^{-\frac{1}{d_0'}}\lesssim\lambda_{q+1}^{(8d+24)\alpha-\frac{1}{2N}}\lesssim 2^{-i}\lambda^{-\alpha}_{q+1},$$

where we used the choice of parameters in (5.1), (5.10) and conditions for the parameters to have $(8d+26)\alpha<\frac{1}{2N}, 2^i\leqslant 2^{N_{q+1}}\lesssim\lambda^\alpha_{q+1}$.

We observe that $W_{(\xi,n,i)}\Theta_{(\xi,n,i)}$ is $(\mathbb{T}/r_\perp\lambda_{q+1})^d$-periodic. So by Theorem 3.2, the estimates for the amplitude functions and for the building blocks in Lemma 5.1, (A.6), (A.9) and (A.7) respectively we have

$$\|M^i_{osc,x}\|_{L^1_tL^1}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta^i|M^i_l|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^2_{t,x}}$$
$$\times(r_\perp\lambda_{q+1})^{-1}\|W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\|_{C_tL^1}\|g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}\|_{L^1_t}$$
$$\lesssim l^{-6d-20}(r_\perp\lambda_{q+1})^{-1}\lesssim\lambda_{q+1}^{(12d+40)\alpha-\frac{1}{N}}\lesssim 2^{-i}\lambda^{-\alpha}_{q+1},$$

where we used the choice of parameters in (5.1), (5.10) and conditions on the parameters to have $(12d+42)\alpha<\frac{1}{N}, 2^i\leqslant 2^{N_{q+1}}\lesssim\lambda^\alpha_{q+1}$.

For the stress term $M^i_{osc,c}$, it holds

$$\big|M^i_{osc,c}\big|\leqslant\left|\sum_{n=-1}^{2}\chi(\zeta^i|M^i_l|-n)M^i_l\right|+\left|\sum_{n\geqslant 3}\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\frac{M^i_l}{|M^i_l|}-M^i_l\Big)\right|$$
$$\leqslant\frac{3}{\zeta^i}+\sum_{n\geqslant 3}\chi(\zeta^i|M^i_l|-n)\left|\frac{n}{\zeta^i}-|M^i_l|\right|\leqslant\frac{3}{20}2^{-i}\delta_{q+2}+\frac{1}{20}2^{-i}\delta_{q+2}\leqslant\frac{1}{5}C_02^{-i}\delta_{q+2}.$$

By the bounds (A.9), (4.6) and (5.1) we have

$$\begin{aligned}\|M^i_{osc,o}\|_{L^1_tL^1} &\lesssim \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|h_{(\xi,d_0)}\|_{L^\infty_t}\left\|\partial_t[\chi(\zeta|M^i_l|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi]\right\|_{C^0_{t,x}}\\ &\lesssim \sigma^{-1}l^{-4d-12}\lesssim \lambda_{q+1}^{(8d+24)\alpha-\frac{1}{2N}}\lesssim 2^{-i}\lambda_{q+1}^{-\alpha},\end{aligned}$$

where we used conditions on the parameters to have $(8d+26)\alpha<\frac{1}{2N}, 2^i\leqslant 2^{N_{q+1}}\lesssim\lambda_{q+1}^\alpha$.

In summary, we have

$$\|M^i_{osc}\|_{L^1_tL^1}\leqslant C2^{-i}\lambda_{q+1}^{-\alpha}+\frac{1}{5}C_02^{-i}\delta_{q+2}\leqslant\frac{2}{5}C_02^{-i}\delta_{q+2},$$

where we chose a large to absorb the universal constant.

Nonlinear error M^i_{nonlin}. We first consider case Assumption 1.2 (1). Using the notation $\overline{\sigma}(\rho)=\sigma(\rho)\sigma^T(\rho)\rho$, we have

$$\begin{aligned}|\partial_i[\overline{\sigma}(\rho^i_{q+1})-\overline{\sigma}(\rho^i_l)]| &\lesssim |(\partial_i\overline{\sigma})(\rho^i_{q+1})-(\partial_i\overline{\sigma})(\rho^i_l)|+|(\partial_\rho\overline{\sigma})(\rho^i_{q+1})\partial_i\rho^i_{q+1}-(\partial_\rho\overline{\sigma})(\rho^i_l)\partial_i\rho^i_l]|\\ &\lesssim |\rho^i_{q+1}-\rho^i_l|(1+|\rho^i_l|+|\partial_i\rho^i_l|)+|\partial_i\rho^i_{q+1}-\partial_i\rho^i_l|,\end{aligned}$$

which implies

$$\|\mathrm{div}[A(\rho^i_{q+1})\rho^i_{q+1}-A(\rho^i_l)\rho^i_l]\|_{L^1_tL^1}\lesssim\|\theta^i_{q+1}\|_{L^1_tL^1}(1+\|\rho^i_q\|_{C^1_{t,x}})+\|\theta^i_{q+1}\|_{L^1_tW^{1,1}}.$$

Next, in case Assumption 1.2 (2), Young's inequality yields

$$\begin{aligned}&\|\mathrm{div}[A(\rho^i_{q+1})\rho^i_{q+1}-A(\rho^i_l)\rho^i_l]\|_{L^1_tL^1}\\ &\quad\lesssim\|A(\rho^i_{q+1})-A(\rho^i_l)\|_{C^0_{t,x}}\|\nabla\rho^i_{q+1}\|_{L^1_tL^1}+\|A(\rho^i_l)\|_{C^0_{t,x}}\|\nabla\theta^i_{q+1}\|_{L^1_tL^1}\\ &\qquad+\|(\mathrm{div}A(\rho^i_{q+1}-\rho^i_l))\rho^i_l\|_{L^1_tL^1}+\|\mathrm{div}A(\rho^i_{q+1})\|_{L^1_tC^0_x}\|\rho^i_{q+1}-\rho^i_l\|_{C_tL^1}\\ &\quad\lesssim\|\theta^i_{q+1}\|_{C_tL^1}(\|\rho^i_q\|_{C^1_{t,x}}+\|\nabla\theta^i_{q+1}\|_{L^1_tL^1})+\|\theta^i_{q+1}\|_{L^1_tW^{1,1}}\|\rho^i_q\|_{C^0_{t,x}}.\end{aligned}$$

Case Assumption 1.2 (3) follows directly by combining the estimates from the previous two cases.

By weak Young's inequality and Sobolev embedding, we obtain

$$\begin{aligned}&\|(V*\theta^i_{q+1})\rho^i_l\|_{L^1_tL^1}+\|(V*\rho^i_{q+1})\theta^i_{q+1}\|_{L^1_tL^1}\\ &\quad\lesssim\|V*\theta^i_{q+1}\|_{L^1_tL^{1+1/\epsilon}}\|\rho^i_l\|_{C_tL^{1+\epsilon}}+\|V*\rho^i_{q+1}\|_{L^1_tL^{1+1/\epsilon}}\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}\\ &\quad\lesssim\|\theta^i_{q+1}\|_{L^1_tL^{d/d-1}}\|\rho^i_l\|_{C_tL^{1+\epsilon}}+\|\rho^i_{q+1}\|_{L^1_tL^{d/d-1}}\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}\\ &\quad\lesssim(\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\theta^i_{q+1}\|_{L^1_tL^{d/d-1}})(\|\rho^i_l\|_{C^1_{t,x}}+\|\theta^i_{q+1}\|_{L^1_tL^{d/d-1}})\\ &\quad\lesssim(\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}})(\|\rho^i_l\|_{C^1_{t,x}}+\|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}}).\end{aligned}$$

Then, by the bounds in (5.19) and (5.20) we have

$$\begin{aligned}\|M^i_{nonlin}\|_{L^1_tL^1}&\lesssim(\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}})(1+\|\rho^i_q\|_{C^1_{t,x}}+\|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}})\\ &\lesssim(\lambda_q^{d+4}+1)2^{-i}\lambda_{q+1}^{-2\alpha}\lesssim 2^{-i}C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_02^{-i}\delta_{q+2},\end{aligned}\tag{5.21}$$

where we used (5.1). We also chose a large to absorb the universal constant.

Linear error M^i_{lin}. By the estimates in (4.5), (5.14), (5.16), (5.17) and (5.20) respectively we obtain

$$\|M^i_{lin}\|_{L^1_tL^1}\lesssim\|v_l\|_{C^0_{t,x}}\|\theta^i_{q+1}\|_{C_tL^1}+\|\rho^i_l+\theta^{(o,i)}_{q+1}\|_{C^0_{t,x}}\|w_{q+1}\|_{L^r_tL^p}+\|\theta^{(p,i)}_{q+1}\|_{L^{d_0'}_tL^{d_0'}}\|w^{(c,i)}_{q+1}\|_{L^{d_0}_tL^{d_0}}$$

$$\lesssim C_vC_0(\lambda_q^{d+4}+1)\lambda_{q+1}^{(12d+40)\alpha-\frac{1}{2N}} \lesssim 2^{-i}C_0\lambda_{q+1}^{-\alpha} \leqslant \frac{1}{5}C_0 2^{-i}\delta_{q+2},$$

where we used (5.1) and conditions on the parameters to have $(12d+43)\alpha < \frac{1}{2N}, 2^i \leqslant 2^{N_{q+1}} \lesssim \lambda_{q+1}^{\alpha}$.

Commutator error M^i_{com}**.** By the bounds in (4.5), (5.1) and Lemma 4.1 we obtain

$$\begin{aligned}
\|M^i_{com}\|_{L^1_tL^1} &\lesssim l\|v_q\|_{C^1_{t,x}}\|\rho^i_q\|_{C^1_{t,x}} + \|(A(\rho^i_l)-A(\rho^i_q))\rho^i_l\|_{C^1_{t,x}}\\
&\quad + \|A(\rho^i_q)(\rho^i_l-\rho^i_q)\|_{C^1_{t,x}} + l\|A(\rho^i_q)\rho^i_q\|_{C^2_{t,x}}\\
&\quad + \|(V*(\rho^i_l-\rho^i_q))\rho^i_l\|_{L^1_tL^1} + \|(V*\rho^i_q)(\rho^i_l-\rho^i_q)\|_{L^1_tL^1} + l\|(\partial_t+\nabla)(V*\rho^i_q)\rho^i_q\|_{L^1_tL^1}\\
&\lesssim l\|v_q\|_{C^1_{t,x}}\|\rho^i_q\|_{C^1_{t,x}} + l(1+\|\rho^i_q\|_{C^2_{t,x}}+\|\rho^i_q\|^2_{C^1_{t,x}})\|\rho^i_q\|_{C^1_{t,x}}\\
&\lesssim C_0 l\lambda_q^{3d+12} \lesssim 2^{-i}C_0\lambda_{q+1}^{-\alpha/2} \leqslant \frac{1}{5}C_0 2^{-i}\delta_{q+2},
\end{aligned} \tag{5.22}$$

where we used $2^i \leqslant 2^{N_{q+1}} \lesssim \lambda_{q+1}^{\alpha}$, and chose a large to absorb the universal constant.

Summarizing all the estimates above we obtain (4.6).

5.5.9. *Proof of* (4.6) *for* $i > N_{q+1}$. By the corresponding estimates in (4.4) and (4.3) we have

$$\begin{aligned}
\|M^i_{q+1}\|_{L^1_tL^1} &\leqslant \|M^i_0\|_{L^1_tL^1} + \|v_{q+1}-\overline{v}\|_{L^{d_0}_tL^{d_0}}\|\rho^i_0-1\|_{L^{d_0'}_tL^{d_0'}}\\
&\lesssim C_0C_v4^{-i} \lesssim C_0C_v2^{-i}2^{-N_{q+1}} \lesssim C_0C_v2^{-i}\lambda_{q+1}^{-\alpha} \leqslant C_0 2^{-i}\delta_{q+2},
\end{aligned}$$

where we used (5.1) and the conditions on the parameters to deduce $\lambda_{q+1}^{\alpha} \leqslant 2^{N_{q+1}}$ and chose a large enough to absorb the universal constants.

6. More singular interaction case

In this section, we follow the strategy developed in Section 4 to treat the singular interaction case. The main idea is to remove the temporal building blocks in order to obtain stronger regularity estimates for ρ, which in turn allows us to justify the well-posedness of the convolutions involving singular interaction kernels. The price to pay is that we only obtain estimates in the supremum-in-time norm, and spatial integrability estimates up to L^{d-}.

Let $N > 0, 0 < \epsilon_0 \leqslant \frac{1}{4}$ and $1 < d_0 < d$ be given. We recall $d_0' = \frac{d_0}{d_0-1}$. The iteration is again indexed by a parameter $q \in \mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}_0} \subset \mathbb{N}$ and a sequence $\{\delta_q\}_{q\in\mathbb{N}_0} \subset (0,1]$ defined by

$$\lambda_q = a^{b^q},\ q \geqslant 0,\ \delta_q = (\frac{\epsilon_0}{2})^{d_0+d_0'}\lambda_1^{2\beta}\lambda_q^{-2\beta},\ q \geqslant 1,\ \delta_0 = 1.$$

By imposing

$$a^{2(b-1)\beta/(d_0+d_0')} > 2, \tag{6.1}$$

we obtain

$$\sum_{q\geqslant 1}\delta_q^{1/(d_0+d_0')} \leqslant \frac{\epsilon_0}{2}\sum_{q\geqslant 1}a^{(1-q)2(b-1)\beta/(d_0+d_0')} \leqslant \frac{\epsilon_0}{2}\cdot\frac{1}{1-a^{-2(b-1)\beta/(d_0+d_0')}} < \epsilon_0.$$

Without loss of generality, we assume $T = 1$ from now on. At each step q, we construct a family $(v_q, \rho^i_q, M^i_q)_{1\leqslant i\leqslant N}$ solving the system

$$\begin{aligned}
\partial_t\rho^i_q - \mathrm{div}\mathrm{div}\big(A(\rho^i_q)\rho^i_q\big) + \mathrm{div}(v_q\rho^i_q) + \mathrm{div}((V*\rho^i_q)\rho^i_q) &= -\mathrm{div}M^i_q,\\
\mathrm{div}v_q &= 0.
\end{aligned} \tag{6.2}$$

The sequence $\{\rho_q^i\}_{q\in\mathbb{N}_0}$ converges to a limit ρ^i, while the stress fields $\{M_q^i\}_{q\in\mathbb{N}_0}$ converge to 0 in a suitable topology as $q\to\infty$.

We define $T_q := \frac{1}{3} - \sum_{1\leqslant r\leqslant q}\delta_r^{1/2}$. We initialize the iteration by setting

$$\rho_0^i(t,x) = 1 + \frac{\sin 2\pi x_1}{4^i}\chi_0(t),\ v_0 = \overline{v},$$
$$M_0^i(t,x) = \partial_t\chi_0(t)\frac{\cos 2\pi x_1}{4^i\cdot 2\pi}(1,0,\dots,0) - \overline{v}(\rho_0^i - 1) + \mathrm{div}(A(t,x,\rho_0^i)\rho_0^i) - (V*\rho_0^i)\rho_0^i,$$

where $x=(x_1,\dots,x_d)$ and χ_0 is a smooth function satisfying $\chi_0(t)=0$ on $[0,\frac{1}{3}]$, $\chi_0(t)=1$ on $[\frac{2}{3},1]$. It is easy to verify that $(v_0,\rho_0^i,M_0^i)_{1\leqslant i\leqslant N}$ is a solution to (4.2). By definition, we know that $\rho_0^i=1$ on $[0,T_0]$, and then $M_0^i=0$ on $[0,T_0]$ by using the facts that $\mathrm{div}\overline{v}=0$, $\mathrm{div}(a*1)=0$.

Then by (3.1) we have

$$\begin{aligned}
\|M_0^i\|_{C_tL^1} &\lesssim 1 + \|(a*\nabla\rho_0^i)\rho_0^i\|_{C_tL^1} + \|(a*\rho_0^i)\nabla\rho_0^i\|_{C_tL^1} + \|(b*\nabla\rho_0^i)\rho_0^i\|_{C_tL^1}\\
&\lesssim 1 + \|a*\nabla\rho_0^i\|_{C_tL^1} + \|a*\rho_0^i\|_{C_tL^1} + \|b*\nabla\rho_0^i\|_{C_tL^1}\\
&\lesssim 1 + \|\rho_0^i\|_{C^1_{t,x}} \leqslant (\frac{\epsilon_0}{2})^{d_0+d_0'}C_0.
\end{aligned}\tag{6.3}$$

With the above assumptions in hand, our main iteration relies on the first step of iteration and reads as follows:

Proposition 6.1. *Under the assumption of Theorem 1.4, there exists a choice of parameters a,b,β such that the following holds: Let $(v_q,\rho_q^i,M_q^i)_{1\leqslant i\leqslant N}$ be a solution to the system (6.2) satisfying $\int\rho_q^i\mathrm{d}x=1$, and*

$$\|v_q\|_{C^1_{t,x}} \leqslant C_0^{1/d_0}\lambda_q^{2d+2},\ \|\rho_q^i\|_{C^1_{t,x}} + \lambda_q^{-d-1}\|\rho_q^i\|_{C^2_{t,x}} \leqslant C_0^{1/d_0'}\lambda_q^{2d+2},\tag{6.4}$$
$$\|M_q^i\|_{C_tL^1} \leqslant C_0\delta_{q+1},\tag{6.5}$$
$$\rho_q^i(t)=1, M_q^i(t)=0 \text{ on } [0,T_q].\tag{6.6}$$

Then there exists $(v_{q+1},\rho_{q+1}^i,M_{q+1}^i)_{1\leqslant i\leqslant N}$ which solves (6.2) and satisfies (6.4)-(6.6) at the level $q+1$ and for some universal constant $C_v\geqslant 1$

$$\|v_{q+1}-v_q\|_{C_tL^{d_0}} \leqslant C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0},\quad \|\rho_{q+1}^i-\rho_q^i\|_{C_tL^{d_0'}} \leqslant C_vC_0^{1/d_0'}\delta_{q+1}^{1/d_0'}.\tag{6.7}$$

Moreover, we have for some $\epsilon>0$

$$\|v_{q+1}-v_q\|_{C_tL^1} \leqslant \delta_{q+1}^{1/d_0},\tag{6.8}$$
$$\|\rho_{q+1}^i-\rho_q^i\|_{C_tW^{1,1+\epsilon}} \leqslant 4^{-i}\delta_{q+1}^{1/d_0'},\quad \inf_{t\in[0,1]}(\rho_{q+1}^i-\rho_q^i) \geqslant -\delta_{q+1}^{1/d_0'}.\tag{6.9}$$

Proof of Theorem 1.4. We intend to start the iteration from $(v_0,\rho_0^i,M_0^i)_{1\leqslant i\leqslant N}$ which are defined as above. By (6.3), (6.4)-(6.6) are satisfied as $\delta_1=(\frac{\epsilon_0}{2})^{d_0+d_0'}$. Next, we use Proposition 6.1 to build inductively $(v_q,\rho_q^i,M_q^i)_{1\leqslant i\leqslant N}$. By (6.1) and (6.7)-(6.9), the sequence $\{(v_q,\rho_q^i)_{1\leqslant i\leqslant N}\}_{q\in\mathbb{N}}$ is Cauchy in $C([0,1];L^{d_0})\times(C([0,1];L^{d_0'}\cap W^{1,1+\epsilon}))^N$ and we denote by (v,ρ^i) its limit.

Now we need to verify (ρ^i,v) solves (1.3). By (3.1), we have for $\epsilon>0$ small enough, $q'=1+\epsilon$, $q=\frac{q'}{q'-1}$, $\frac{1}{q}=\frac{1}{p}-\frac{\gamma}{d}$, $p_1=1+\epsilon$, $\frac{1}{q_1}=\frac{1}{p_1}-\frac{\gamma}{d}$, $q_1'=\frac{q_1}{q_1-1}$, we have

$$\begin{aligned}
&\|(a*\rho)\nabla\rho-(a*\tilde\rho)\nabla\tilde\rho\|_{L^1} + \|(a*\nabla\rho)\rho-(a*\nabla\tilde\rho)\tilde\rho\|_{L^1}\\
&\qquad\lesssim \|a*(\rho-\tilde\rho)\|_{L^q}\|\nabla\rho\|_{L^{q'}} + \|a*\tilde\rho\|_{L^q}\|\nabla(\rho-\tilde\rho)\|_{L^{q'}}
\end{aligned}$$

$$
\begin{aligned}
&+\|a*\nabla(\rho-\tilde\rho)\|_{L^{q_1}}\|\rho\|_{L^{q_1'}}+\|a*\nabla\tilde\rho\|_{L^{q_1}}\|\rho-\tilde\rho\|_{L^{q_1'}}\\
&\lesssim\|\rho-\tilde\rho\|_{L^p}\|\nabla\rho\|_{L^{q'}}+\|\tilde\rho\|_{L^p}\|\nabla(\rho-\tilde\rho)\|_{L^{q'}}\\
&+\|\nabla(\rho-\tilde\rho)\|_{L^{p_1}}\|\rho\|_{L^{q_1'}}+\|\nabla\tilde\rho\|_{L^{p_1}}\|\rho-\tilde\rho\|_{L^{q_1'}}.
\end{aligned}\tag{6.10}
$$

The similar calculation also holds for b. Then we know that

$$
\|(a*\rho_q^i)\nabla\rho_q^i-(a*\rho^i)\nabla\rho^i\|_{L^1}+\|(a*\nabla\rho_q^i)\rho_q^i-(a*\nabla\rho^i)\rho^i\|_{L^1}\to 0
$$

By (6.5), (6.6) we then obtain that (ρ^i,v) solves (1.3).

By (6.1), (6.7) and (6.9) we have

$$
\big|\|\rho^i-1\|_{C_tL^1}-\|\rho_0^i-1\|_{C_tL^1}\big|\leqslant\sum_{q=0}^\infty\|\rho_{q+1}^i-\rho_q^i\|_{C_tL^1}\leqslant 4^{-i}\sum_{q=0}^\infty\delta_{q+1}^{1/d_0'}\leqslant 4^{-i-1},\tag{6.11}
$$

$$
\inf_{t\in[0,1]}\rho^i\geqslant\inf_{t\in[0,1]}\rho_0^i+\sum_{q=0}^\infty\inf_{t\in[0,1]}(\rho_{q+1}^i-\rho_q^i)\geqslant\frac34-\sum_{q=0}^\infty\delta_{q+1}^{1/d_0'}\geqslant\frac12,
$$

at which point, by a similar argument as before, we obtain ρ^i is nonnegative on $\mathbb{T}^d$, and do not coincide with each other.

By (6.1) and (6.8) we obtain that $\|v-\overline{v}\|_{C_tL^1}\leqslant\sum_{q\geqslant0}\|v_{q+1}-v_q\|_{C_tL^1}\leqslant\sum_{q\geqslant0}\delta_{q+1}^{1/d_0}\leqslant\epsilon_0$.

By (6.4), (6.7) and interpolation, we have $|v|\in C_tL^{d_0(1+\epsilon)}$ for some $\epsilon>0$ sufficiently small. Since $\rho^i\in C_tL^{d_0'}$, we deduce that $|v|^{1+\epsilon}\rho^i\in C_tL^1$. We conclude the proof by using the superposition principle. □

7. Proof of Proposition 6.1

The construction and the corresponding estimates are essentially the same as those in Section 5, after formally regarding the temporal building blocks $g_{(\xi)}$ as identically equal to 1. We shall choose different parameters in order to obtain higher regularity for ρ.

7.1. **Mollification.** For a sufficiently small parameter $\alpha\in(0,1)$ to be chosen later, we define $l:=\lambda_{q+1}^{-3\alpha/2}\lambda_q^{-5d/2-5/2}$. Then

$$
l^{-1}\leqslant\lambda_{q+1}^{2\alpha},\quad l\lambda_q^{5d+5}\leqslant\lambda_{q+1}^{-\alpha}\ll\delta_{q+2},\tag{7.1}
$$

provided that

$$
\alpha b>5d+5,\quad \alpha>2\beta b.\tag{7.2}
$$

To avoid the loss of derivatives, we first mollify the stress term. As before, we define

$$
v_l=(v_q*_x\phi_l)*_t\varphi_l,\quad \rho_l^i=(\rho_q^i*_x\phi_l)*_t\varphi_l,\quad M_l^i=(M_q^i*_x\phi_l)*_t\varphi_l.
$$

Moreover, we know that $\rho_l^i=1$ and $M_l^i=0$ on $[0,T_{q+1}]$.

By straightforward calculations we obtain

$$
\partial_t\rho_l^i-\mathrm{div}\mathrm{div}(A(\rho_l^i)\rho_l^i)+\mathrm{div}(v_l\rho_l^i)+\mathrm{div}((V*\rho_l^i)\rho_l^i)=-\mathrm{div}(M_l^i+M_{com}^i),\quad \mathrm{div}v_l=0,\tag{7.3}
$$

where

$$
\begin{aligned}
M_{com}^i:=&-v_l\rho_l^i+(v_q\rho_q^i)*_x\phi_l*_t\varphi_l+\mathrm{div}(A(\rho_l^i)\rho_l^i)-(\mathrm{div}(A(\rho_q^i)\rho_q^i))*_x\phi_l*_t\varphi_l\\
&-(V*\rho_l^i)\rho_l^i+((V*\rho_q^i)\rho_q^i)*_x\phi_l*_t\varphi_l.
\end{aligned}
$$

Finally, by Sobolev embedding, we still have for every $N \geqslant 0$,

$$\|M_l^i\|_{C^N_{t,x}} \lesssim l^{-d-\frac{1}{3}-N}\|M_q^i\|_{C_tL^1} \lesssim C_0 l^{-d-\frac{4}{3}-N}.$$

Here we still write $l^{-\frac{4}{3}}$ so that the estimate remains formally the same as before.

7.2. **Construction of** (v_{q+1},ρ_{q+1})**.** To construct the perturbations of v_l and ρ_l^i, we still consider the space-building blocks in Appendix A. We consider $2N$ disjoint sets $\Lambda^{1,n},\Lambda^{2,n}$ defined in Appendix A by taking suitable rational rotations of one fixed set. We use the notation $\Lambda^{j,n}=\Lambda^{1,n}$ for j odd, and $\Lambda^{j,n}=\Lambda^{2,n}$ for j even. For a fixed exponent $d_0>1$, and $\xi\in\Lambda:=\cup_{i=1}^N(\Lambda^{1,i}\cup\Lambda^{2,i})$ we recall the notation $\Phi_{(\xi,d_0)},\phi_{(\xi,d_0)},\phi'_{(\xi,d_0')}$ introduced in Appendix A. We recall the parameters $\lambda,\ r_\perp,\ r_\parallel,\ \mu$, introduced in Appendix A. Their precise choice will be specified subsequently.

Remark 7.1. Here constructing infinitely many solutions appears to be nontrivial. Indeed, if one attempts to treat N equations simultaneously, many parameters arising in the spatial building blocks (such as n_*) necessarily depend on N. It then becomes necessary to carefully control the growth of these parameters, which should increase at most polynomially in N. In the former case, this difficulty can be overcome by introducing suitable time building blocks.

To ensure that the building blocks have mutually disjoint supports, we introduce the following lemma.

Lemma 7.2. ([BMS21, Lemma 3]) *Let $d\geqslant 3$. Then there exist $\{\alpha_\xi\}_{\xi\in\Lambda}$ and $a>0$ such that*

$$(B_a(\alpha_\xi)+\{s\xi\}_{s\in\mathbb{R}}+(\mathbb{Z}/r_\perp\lambda)^d)\cap(B_a(\alpha_{\xi'})+\{s'\xi'\}_{s'\in\mathbb{R}}+(\mathbb{Z}/r_\perp\lambda)^d)=\varnothing,$$

for all $\xi,\xi'\in\Lambda,\xi\neq\xi'$.

Let $\chi\in C_c^\infty(-\frac{3}{4},\frac{3}{4})$ be a nonnegative function such that $\sum_{n\in\mathbb{Z}}\chi(t-n)=1$ for every $t\in\mathbb{R}$. Let $\tilde\chi\in C_c^\infty(-\frac{4}{5},\frac{4}{5})$ be a nonnegative function satisfying $\tilde\chi=1$ on $[-\frac{3}{4},\frac{3}{4}]$ and $\sum_{n\in\mathbb{Z}}\tilde\chi(t-n)\leqslant 2$.

For $1\leqslant i\leqslant N$, we fix the parameters $\zeta:=20\delta_{q+2}^{-1}$. With these preparations, we define the rescaled building blocks by

$$W_{(\xi,n)}(x,t):=W_{(\xi,d_0)}\Big(x-\alpha_\xi,\Big(\frac{n}{\zeta}\Big)^{1/d_0}t\Big),$$

$$\Theta_{(\xi,n)}(x,t):=\Theta_{(\xi,d_0')}\Big(x-\alpha_\xi,\Big(\frac{n}{\zeta}\Big)^{1/d_0}t\Big).$$

By the choice of the shifts α_ξ in Lemma 7.2, the building blocks $W_{(\xi)}$ have mutually disjoint supports. Similarly, we define $V_{(\xi,n)},\Phi_{(\xi,n)}$, and all the other quantities appearing in Appendix A. By (A.2) and (A.8), we have

$$\partial_t\Theta_{(\xi,n)}+\Big(\frac{n}{\zeta}\Big)^{1/d_0}\mathrm{div}\big(W_{(\xi,n)}\Theta_{(\xi,n)}\big)=0.$$

We next define the perturbations for the drift term. For $1\leqslant i\leqslant N$, let

$$\begin{aligned}
w_{q+1}^{(p,i)}&:=\sum_{n\geqslant 3}\tilde\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\sum_{\xi\in\Lambda^{n,i}}W_{(\xi,n)},\\
w_{q+1}^{(c,i)}&:=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}-\tilde\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\frac{1}{(n_*\lambda_{q+1})^2}\nabla\Phi_{(\xi,n)}\,\xi\cdot\nabla\psi_{(\xi,n)}\\
&\quad+\nabla\big(\tilde\chi(\zeta|M_l^i|-n)\big)\Big(\frac{n}{\zeta}\Big)^{1/d_0}:V_{(\xi,n)}.
\end{aligned}$$

By (A.3), we obtain

$$w_{q+1}^{(p,i)}+w_{q+1}^{(c,i)}=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\operatorname{div}\Big(\tilde{\chi}(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}V_{(\xi,n)}\Big). \tag{7.4}$$

Since $V_{(\xi,n)}$ is skew-symmetric, it follows that $\operatorname{div}\big(w_{q+1}^{(p,i)}+w_{q+1}^{(c,i)}\big)=0$.

Finally, we define the total perturbation and the new velocity field by

$$w_{q+1}:=\sum_{i=1}^{N}\big(w_{q+1}^{(p,i)}+w_{q+1}^{(c,i)}\big),\qquad v_{q+1}:=v_l+w_{q+1}.$$

Then v_{q+1} is mean-zero and divergence-free. Moreover, since $M_l^i(t)=0$ on $[0,T_{q+1}]$, the perturbation w_{q+1} vanishes on $[0,T_{q+1}]$.

We next define the perturbations for the density functions. For $1\leqslant i\leqslant N$, we set

$$\begin{aligned}\theta_{q+1}^{(p,i)}&:=\sum_{n\geqslant 3}\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\sum_{\xi\in\Lambda^{n,i}}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi,n)},\\ \theta_{q+1}^{(c,i)}&:=-\fint_{\mathbb{T}^d}\theta_{q+1}^{(p,i)}\,\mathrm{d}x.\end{aligned}$$

By an argument analogous to (5.8), we obtain

$$\begin{aligned}&\partial_t\theta_{q+1}^{(p,i)}+\operatorname{div}\big(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)}-M_l^i\big)\\ =&\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\partial_t\Big[\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big]\Theta_{(\xi,n)}+\operatorname{div}\Big(\sum_{n\geqslant 3}\chi(\zeta|M_l^i|-n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|}-M_l^i\Big)\\ &+\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\nabla\Big[\chi(\zeta|M_l^i|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big]\mathbb{P}_{\neq 0}\big(W_{(\xi,n)}\Theta_{(\xi,n)}\big).\end{aligned}\tag{7.5}$$

We now define, for every $1\leqslant i\leqslant N$,

$$\theta_{q+1}^i:=\theta_{q+1}^{(p,i)}+\theta_{q+1}^{(c,i)},\qquad \rho_{q+1}^i:=\rho_l^i+\theta_{q+1}^i.$$

By construction, $\int_{\mathbb{T}^d}\rho_{q+1}^i\,\mathrm{d}x=1$. Since $M_l^i(t)=0$ on $[0,T_{q+1}]$, we know $\theta_{q+1}^i(t)=0,\rho_{q+1}^i(t)=1$ on $[0,T_{q+1}]$.

Moreover, by Lemma 7.2, we obtain for $1\leqslant i\leqslant N$ that

$$\operatorname{div}\big(w_{q+1}\theta_{q+1}^i\big)=\operatorname{div}\Big((w_{q+1}^{(p,i)}+w_{q+1}^{(c,i)})\theta_{q+1}^{(p,i)}\Big).$$

7.3. **Construction of the stress terms M_{q+1}^i.** From the definition of the perturbations, as in Section 5.4, we define

$$-M_{q+1}^i:=M_{osc}^i+M_{nonlin}^i+M_{lin}^i-M_{com}^i.$$

Here, we define the nonlinear error and the linear error by

$$\begin{aligned}M_{nonlin}^i:&=-\operatorname{div}((a*\rho_{q+1}^i)\rho_{q+1}^i-(a*\rho_l^i)\rho_l^i)-(b*\nabla\rho_{q+1}^i)\rho_{q+1}^i+(b*\nabla\rho_l^i)\rho_l^i,\\ M_{lin}^i:&=v_l\theta_{q+1}^i+w_{q+1}\rho_l^i+w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)}.\end{aligned}$$

We define the oscillation error $M_{osc}^i:=M_{osc,t}^i+M_{osc,x}^i+M_{osc,c}^i$ as

$$M_{osc,t}^i:=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\mathcal{R}_1\Big(\partial_t[\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi,n)})\Big),$$

$$
\begin{aligned}
M^i_{osc,x} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \mathcal{B}_1\Big(\nabla[\chi(\zeta|M^i_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)], \mathbb{P}_{\neq 0}(W_{(\xi,n)}\Theta_{(\xi,n)})\Big),\\
M^i_{osc,c} &:= \sum_{n\geqslant 3}\chi(\zeta|M^i_l|-n)\frac{n}{\zeta}\frac{M^i_l}{|M^i_l|} - M^i_l.
\end{aligned}
$$

Since $M^i_l(t)=0, w_{q+1}(t)=0, \theta^i_{q+1}(t)=0$ on $[0,T_{q+1}]$, we have $M^i_{q+1}(t)=0$ on $[0,T_{q+1}]$, which implies (6.6) for M^i_{q+1}.

7.4. **Proof of Proposition 6.1.**

7.4.1. *Choice of parameters.* Regarding the parameters of the building blocks, we define

$$
\lambda=\lambda_{q+1},\ r_\perp=\lambda_{q+1}^{-1+\frac{1}{M}},\ r_\parallel=\lambda_{q+1}^{-1+\frac{2}{M}},\ \mu=r_\perp^{-\frac{d-1}{d_0}}r_\parallel^{-\frac{1}{d_0}},
$$

where $M>0$ is a sufficiently large integer satisfying $(-1+\frac{2}{M})d(\frac{\gamma}{d}-\frac{1}{d_0'})<-\frac{1}{M}, (-1+\frac{2}{M})\frac{d}{d_0'}<-\frac{1}{M}, 1+(-1+\frac{2}{M})\frac{d}{d_0}<-\frac{1}{M}$. With these choices, we obtain

$$
r_\perp^{\frac{d-1}{d/\gamma}-\frac{d-1}{d_0'}}r_\parallel^{\frac{1}{d/\gamma}-\frac{1}{d_0'}},\ r_\perp^{d-1-\frac{d-1}{d_0}}r_\parallel^{1-\frac{1}{d_0}},\ \lambda r_\perp^{\frac{d-1}{1}-\frac{d-1}{d_0'}}r_\parallel^{\frac{1}{1}-\frac{1}{d_0'}}\leqslant\lambda^{-\frac{1}{M}}. \tag{7.6}
$$

In the sequel, we shall also require (7.2) together with

$$
\lambda_q^{2d+2}\leqslant\lambda_{q+1}^{\alpha},\quad (12d+41)\alpha<\frac{1}{M}.
$$

These conditions can be achieved as follows: first choose $\alpha>0$ sufficiently small such that $(12d+41)\alpha<\frac{1}{M}$, then choose $b\in 2N\mathbb{N}$ sufficiently large so that $b>\frac{5d+5}{\alpha}$, and finally choose $\beta>0$ sufficiently small such that $\alpha>2\beta b$. At the end, we choose a sufficiently large in order to absorb all implicit and universal constants arising in the subsequent estimates, and to guarantee the validity of (6.1).

Finally, we record the following estimate for the amplitude functions.

Lemma 7.3. ([LRZ25, Proposition 5.2]). *For $M,K\in\mathbb{N}_0, 1\leqslant i\leqslant N$ we have*

$$
\sum_{n\geqslant 3}\Big(\frac{n}{\zeta}\Big)^M\Big(\|\chi(\zeta|M^i_l|-n)\|_{C^K_{t,x}}+\|\tilde\chi(\zeta|M^i_l|-n)\|_{C^K_{t,x}}\Big)\lesssim l^{-(d+4)K-(d+2)(M+1)},
$$

$$
\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big(\frac{n}{\zeta}\Big)^M\Big\|\chi(\zeta|M^i_l|-n)\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big\|_{C^K_{t,x}}\lesssim l^{-(2d+8)K-(d+2)(M+1)}.
$$

7.4.2. *Proof of* (6.7) *for $v_{q+1}-v_q$.* We first estimate the principal perturbations $w^{(p,i)}_{q+1}$ for $1\leqslant i\leqslant N$ in the $C_tL^{d_0}$-norm. By Cauchy's inequality we have

$$
|w^{(p,i)}_{q+1}|^{d_0}\lesssim\sum_{n\geqslant 3}\tilde\chi(\zeta|M^i_l|-n)\frac{n}{\zeta}\sum_{\xi\in\Lambda^{n,i}}\left|W_{(\xi,n,i)}\right|^{d_0}.
$$

By applying the generalized Hölder inequality of Theorem 3.3 in spatial direction, together with the estimates for the building blocks in (A.6) and Lemma 7.3 we deduce

$$
\|w^{(p,i)}_{q+1}(t)\|^{d_0}_{L^{d_0}}\lesssim\sum_{n\geqslant 3}\Big\|\tilde\chi(\zeta|M^i_l(t)|-n)\frac{n}{\zeta}\Big\|_{L^1}\sum_{\xi\in\Lambda^{n,i}}\|W_{(\xi,n,i)}\|^{d_0}_{C_tL^{d_0}}
$$

$$+ (r_\perp \lambda_{q+1})^{-1} \left\| \tilde{\chi}(\zeta |M_l^i(t)| - n) \frac{n}{\zeta} \right\|_{C^1_{t,x}} \sum_{\xi \in \Lambda^{n,i}} \|W_{(\xi,n,i)}\|^{d_0}_{C_t L^{d_0}}$$

$$\lesssim \left\| M_l^i(t) + |\zeta|^{-1} \right\|_{L^1} + l^{-3d-8} \lambda_{q+1}^{-\frac{1}{M}} \lesssim \|M_l^i(t)\|_{L^1} + C_0 \delta_{q+1} \lesssim C_0 \delta_{q+1},$$

where we used the fact that $\sum_{n\in\mathbb{Z}} \tilde{\chi}(t-n) \leqslant 2$, and used the conditions on the parameters to have $(6d+16)\alpha - \frac{1}{M} < -\alpha < -2\beta$.

For the general $C_t L^m$-norm with $m \in [1,\infty]$, by the estimates for the building blocks in (A.4)-(A.6) and the estimate for the amplitude function in Lemma 7.3 we obtain

$$\begin{aligned} \|w_{q+1}^{(p,i)}\|_{C_t L^m} &\lesssim \sum_{n\geqslant 3} \sum_{\xi \in \Lambda^{n,i}} \left\| \tilde{\chi}(\zeta |M_l^i| - n) \Big(\frac{n}{\zeta}\Big)^{1/d_0} \right\|_{C^0_{t,x}} \|W_{(\xi,n,i)}\|_{C_t L^m} \\ &\lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m} - \frac{d-1}{d_0}} r_\parallel^{\frac{1}{m} - \frac{1}{d_0}}, \end{aligned} \tag{7.7}$$

$$\begin{aligned} \|w_{q+1}^{(c,i)}\|_{C_t L^m} &\lesssim \sum_{n\geqslant 3} \sum_{\xi \in \Lambda^{n,i}} \left\| \tilde{\chi}(\zeta |M_l^i| - n) \Big(\frac{n}{\zeta}\Big)^{1/d_0} \right\|_{C^1_{t,x}} \\ &\quad \times \Big(\frac{1}{\lambda_{q+1}^2} \|\nabla \Phi_{(\xi,n,i)} \xi \cdot \nabla \psi_{(\xi,n,i)}\|_{C_t L^m} + \|V_{(\xi,n,i)}\|_{C_t L^m} \Big) \\ &\lesssim l^{-3d-8} r_\perp^{\frac{d-1}{m} - \frac{d-1}{d_0}} r_\parallel^{\frac{1}{m} - \frac{1}{d_0}} \frac{r_\perp}{r_\parallel}. \end{aligned} \tag{7.8}$$

With these estimates, combining with the choice of parameters in (7.1) we obtain

$$\|w_{q+1}\|_{C_t L^{d_0}} \leqslant \frac{C_v}{4} C_0^{1/d_0} \delta_{q+1}^{1/d_0} + C \lambda_{q+1}^{(6d+17)\alpha - \frac{1}{M}} \leqslant \frac{C_v}{2} C_0^{1/d_0} \delta_{q+1}^{1/d_0}, \tag{7.9}$$

where we used conditions on the parameters to have $(6d+18)\alpha < \frac{1}{M}, N \lesssim \lambda_{q+1}^\alpha$ and chose a large enough to absorb the universal constant. The above inequality yields that (6.7) holds for $v_{q+1} - v_q$:

$$\begin{aligned} \|v_{q+1} - v_q\|_{C_t L^{d_0}} &\leqslant \|w_{q+1}\|_{C_t L^{d_0}} + \|v_l - v_q\|_{C_t L^{d_0}} \\ &\leqslant \frac{1}{2} C_v C_0^{1/d_0} \delta_{q+1}^{1/d_0} + l C_0^{1/d_0} \lambda_q^{d+4} \leqslant C_v C_0^{1/d_0} \delta_{q+1}^{1/d_0}. \end{aligned}$$

7.4.3. *Proof of* (6.8). Combining the bounds (7.7) and (7.8) above we obtain

$$\begin{aligned} \|v_{q+1} - v_q\|_{C_t L^1} &\lesssim \|w_{q+1}\|_{C_t L^1} + l \|v_q\|_{C^1_{t,x}} \\ &\lesssim N l^{-3d-8} r_\perp^{d-1-\frac{d-1}{d_0}} r_\parallel^{1-\frac{1}{d_0}} + l \lambda_q^{d+4} \lesssim \lambda_{q+1}^{(6d+17)\alpha - \frac{1}{M}} + \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/d_0}. \end{aligned}$$

Here we used (7.1), (7.6) and conditions on the parameters to have $(6d+18)\alpha < \frac{1}{M}, N \lesssim \lambda_{q+1}^\alpha$. Then we chose a large enough to absorb the universal constant.

7.4.4. *Proof of* (6.4) *for* v_{q+1}. By the estimates for the building blocks in (A.6), (7.4) and the estimates for the amplitude functions in Lemma 7.3 we have

$$\begin{aligned} \|w_{q+1}\|_{C^1_{t,x}} &\lesssim \sum_{i=1}^N \sum_{n\geqslant 3} \sum_{\xi \in \Lambda^{n,i}} \left\| \tilde{\chi}(\zeta |M_l^i| - n) \Big(\frac{n}{\zeta}\Big)^{1/d_0} \right\|_{C^2_{t,x}} \|\nabla V_{(\xi,n,i)}\|_{C^1_{t,x}} \\ &\lesssim N l^{-4d-12} \lambda_{q+1} \mu r_\parallel^{-\frac{1}{d_0}} r_\perp^{-\frac{d-1}{d_0}} \lesssim N_{q+1} \lambda_{q+1}^{(8d+24)\alpha + 2d+1}. \end{aligned}$$

Thus, by $(8d+25)\alpha < \frac{1}{2}$ we obtain the following

$$\|v_{q+1}\|_{C^1_{t,x}} \leqslant \|v_l\|_{C^1_{t,x}} + \|w_{q+1}\|_{C^1_{t,x}} \leqslant C_0^{1/d_0}\lambda_q^{2d+2} + \frac{1}{2}\lambda_{q+1}^{2d+2} \leqslant C_0^{1/d_0}\lambda_{q+1}^{2d+2},$$

where we chose a large enough to absorb the universal constant.

7.4.5. *Proof of* (6.7) *for* $\rho^i_{q+1}-\rho^i_q$. Similarly as before, we first estimate the principal perturbations $\theta^{(p,i)}_{q+1}$ in $C_tL^{d_0'}$. By the same argument as in (5.12), we have

$$\|\theta^{(p,i)}_{q+1}\|^{d_0'}_{C_tL^{d_0'}} \lesssim \|M^i_l\|_{C_tL^1} + C_0\delta_{q+1} \lesssim C_0\delta_{q+1}.$$

For general C_tL^m-norm with $m\in[1,\infty]$, by the estimates for the building blocks in (A.7) and Lemma 7.3 we obtain

$$\begin{aligned}\|\theta^{(p,i)}_{q+1}\|_{C_tL^m} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big\|_{C^0_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C_tL^m}\\ &\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0'}}r_\parallel^{\frac{1}{m}-\frac{1}{d_0'}}. \end{aligned}\tag{7.10}$$

Moreover, by (7.6) we have

$$\|\theta^{(c,i)}_{q+1}\|_{C_t} \lesssim \|\theta^{(p,i)}_{q+1}\|_{C_tL^{d/\gamma}} \lesssim l^{-2d-4}r_\perp^{\frac{d-1}{d/\gamma}-\frac{d-1}{d_0'}}r_\parallel^{\frac{1}{d/\gamma}-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(4d+8)\alpha-\frac{1}{M}} \lesssim 4^{-i}\lambda_{q+1}^{-2\alpha},\tag{7.11}$$

where we used conditions on the parameters to have $(4d+11)\alpha<\frac{1}{M}$, $4^i\leqslant 4^N\lesssim\lambda^\alpha_{q+1}$ and chose a large enough to absorb the universal constant. Then, combining the above estimates together, we obtain

$$\begin{aligned}\|\rho^i_{q+1}-\rho^i_q\|_{C_tL^{d_0'}} &\leqslant \|\theta^i_{q+1}\|_{C_tL^{d_0'}} + \|\rho^i_l-\rho^i_q\|_{C_tL^{d_0'}}\\ &\leqslant \frac{1}{2}C_vC_0^{1/d_0'}\delta_{q+1}^{1/d_0'} + C\lambda_{q+1}^{-\alpha} + lC_0^{1/d_0'}\lambda_q^{d+4} \leqslant C_vC_0^{1/d_0'}\delta_{q+1}^{1/d_0'},\end{aligned}$$

which implies (6.7) for $\rho^i_{q+1}-\rho^i_q$. Here we chose a large enough to absorb the universal constant.

7.4.6. *Proof of* (6.9). We first estimate $\theta^{(p,i)}_{q+1}$ in $W^{1,1+\epsilon}$-norm for some $\epsilon>0$ small enough. By the bounds for the building blocks in (A.7) and the bounds for the amplitude functions in Lemma 7.3 we have

$$\begin{aligned}\|\theta^{(p,i)}_{q+1}\|_{C_tW^{1,1+\epsilon}} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C_tW^{1,1+\epsilon}}\\ &\lesssim l^{-4d-12}\lambda_{q+1}r_\parallel^{\frac{1}{1+\epsilon}-\frac{1}{d_0'}}r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}} \lesssim \lambda_{q+1}^{(8d+24)\alpha-\frac{1}{M}+d\epsilon} \lesssim \lambda_{q+1}^{-2\alpha},\end{aligned}\tag{7.12}$$

where we chose $\epsilon>0$ small enough such that $d\epsilon<\alpha$. We also used conditions on the parameters to have $(8d+27)\alpha<\frac{1}{M}$.

By the fact that $\theta^{(p,i)}_{q+1}$ is non-negative, and the choice of parameters in (7.1), (7.6) we have

$$\begin{aligned}\|\rho^i_{q+1}-\rho^i_q\|_{C_tW^{1+\epsilon}} &\lesssim \|\theta^{(p,i)}_{q+1}\|_{C_tW^{1,1+\epsilon}} + l\|\rho^i_q\|_{C^2_{t,x}} \lesssim 4^{-i}\lambda_{q+1}^{-\alpha} \leqslant 4^{-i}\delta_{q+1}^{1/d_0'},\\ \inf_{t\in[0,1]}(\rho^i_{q+1}-\rho^i_q) &\geqslant -\|\theta^{(c,i)}_{q+1}\|_{C^0_{t,x}} - l\|\rho^i_q\|_{C^1_{t,x}} \geqslant -C\lambda_{q+1}^{-\alpha} \geqslant -\delta_{q+1}^{1/d_0'},\end{aligned}\tag{7.13}$$

which yields (6.9). Here we chose a large enough to absorb the universal constant, including $4^i\leqslant 4^N$.

7.4.7. *Proof of* (6.4) *for* ρ^i_{q+1}. By (A.7) and Lemma 7.3 we have for $j=1,2$

$$\|\theta^{(c,i)}_{q+1}\|_{C^j_t}\lesssim\|\theta^{(p,i)}_{q+1}\|_{C^j_{t,x}}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\left\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^j_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C^j_{t,x}}$$
$$\lesssim l^{-6d-20}(\lambda_{q+1}\mu)^j r_\|^{-\frac{1}{d_0'}}r_\perp^{-\frac{d-1}{d_0'}}\lesssim\lambda_{q+1}^{(12d+40)\alpha+d+j(d+1)}.$$

By choosing $(12d+42)\alpha<\frac12$ we deduce

$$\|\rho^i_{q+1}\|_{C^1_{t,x}}+\lambda_{q+1}^{-d-1}\|\rho^i_{q+1}\|_{C^2_{t,x}}\leqslant C_0^{1/d_0'}\lambda_q^{2d+2}+\frac12\lambda_{q+1}^{2d+2}\leqslant C_0^{1/d_0'}\lambda_{q+1}^{2d+2},$$

which implies (6.4) for $\rho^i_{q+1}, 1\leqslant i\leqslant N$.

7.4.8. *Proof of* (6.5). We now estimate each term in the definition of M^i_{q+1} separately.

Oscillation error M^i_{osc}. By Lemma 3.1, the estimates for the amplitude functions and for the building blocks in Lemma 7.3, (A.7) we obtain

$$\|M^i_{osc,t}\|_{C_tL^1}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\left\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^1_{t,x}}\|\Theta_{(\xi,n,i)}\|_{C_tL^1}$$
$$\lesssim l^{-4d-12}r_\perp^{d-1-\frac{d-1}{d_0'}}r_\|^{1-\frac{1}{d_0'}}\lesssim\lambda_{q+1}^{(8d+24)\alpha-\frac{1}{M}}\lesssim\lambda_{q+1}^{-\alpha},$$

where we used the choice of parameters in (7.1), (7.6) and conditions for the parameters to have $(8d+25)\alpha<\frac{1}{M}$.

We observe that $W_{(\xi,n,i)}\Theta_{(\xi,n,i)}$ is $(\mathbb{T}/r_\perp\lambda_{q+1})^d$-periodic. So by Theorem 3.2, the estimates for the amplitude functions and for the building blocks in Lemma 7.3, (A.6) and (A.7) respectively we have

$$\|M^i_{osc,x}\|_{C_tL^1}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\left\|\chi(\zeta|M^i_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^2_{t,x}}(r_\perp\lambda_{q+1})^{-1}\|W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\|_{C_tL^1}$$
$$\lesssim l^{-6d-20}(r_\perp\lambda_{q+1})^{-1}\lesssim\lambda_{q+1}^{(12d+40)\alpha-\frac{1}{M}}\lesssim\lambda_{q+1}^{-\alpha},$$

where we used the choice of parameters in (7.1), (7.6) and conditions on the parameters to have $(12d+41)\alpha<\frac{1}{M}$.

For the stress term $M^i_{osc,c}$, it holds

$$\left|M^i_{osc,c}\right|\leqslant\frac{3}{\zeta}+\sum_{n\geqslant 3}\chi(\zeta|M^i_l|-n)\left|\frac{n}{\zeta}-|M^i_l|\right|\leqslant\frac{3}{20}\delta_{q+2}+\frac{1}{20}\delta_{q+2}\leqslant\frac15 C_0\delta_{q+2}.$$

In summary, we have

$$\|M^i_{osc}\|_{C_tL^1}\leqslant C\lambda_{q+1}^{-\alpha}+\frac15 C_0\delta_{q+2}\leqslant\frac25 C_0\delta_{q+2},$$

where we chose a large to absorb the universal constant.

Nonlinear error M^i_{nonlin}. By (6.10), (7.11) and (7.12) we have

$$\|M^i_{nonlin}\|_{C_tL^1}\lesssim\|\theta^i_{q+1}\|_{C_tL^{d/\gamma}}\|\nabla\rho^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\rho^i_l\|_{C_tL^{d/\gamma}}\|\nabla\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}$$
$$+\|\nabla\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}\|\rho^i_{q+1}\|_{C_tL^{d/\gamma}}+\|\nabla\rho^i_l\|_{C_tL^{1+\epsilon}}\|\theta^i_{q+1}\|_{C_tL^{d/\gamma}}$$
$$\lesssim(\|\theta^i_{q+1}\|_{C_tL^{d/\gamma}}+\|\nabla\theta^i_{q+1}\|_{C_tL^{1+\epsilon}})(\|\theta^i_{q+1}\|_{C_tL^{d/\gamma}}+\|\nabla\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\rho^i_q\|_{C^1_{t,x}})$$

$$\lesssim C_0\lambda_{q+1}^{-2\alpha}(\lambda_{q+1}^{-\alpha}+\lambda_q^{2d+2})\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2}.$$

Linear error M^i_{lin}. By the estimates in (6.4), (7.7), (7.8), (7.10) respectively we obtain

$$\begin{aligned}\|M^i_{lin}\|_{C_tL^1}&\lesssim\|v_l\|_{C^0_{t,x}}\|\theta^i_{q+1}\|_{C_tL^1}+\|\rho^i_l\|_{C^0_{t,x}}\|w_{q+1}\|_{C_tL^1}+\|\theta^{(p,i)}_{q+1}\|_{C_tL^{d_0'}}\|w^{(c,i)}_{q+1}\|_{C_tL^{d_0}}\\&\lesssim C_vC_0(\lambda_q^{2d+2}+1)\lambda_{q+1}^{(6d+16)\alpha-\frac{1}{N}}\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2},\end{aligned}$$

where we used (7.1) and conditions on the parameters to have $(6d+18)\alpha<\frac{1}{N}$.

Commutator error M^i_{com}. By the bounds in (6.4), (6.10) and (7.1) we obtain

$$\|M^i_{com}\|_{C_tL^1}\lesssim l\|v_q\|_{C^1_{t,x}}\|\rho^i_q\|_{C^1_{t,x}}+l\|\rho^i_q\|_{C^2_{t,x}}\|\rho^i_q\|_{C^1_{t,x}}\lesssim C_0l\lambda_q^{5d+5}\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2}. \tag{7.14}$$

Summarizing all the estimates above we obtain (6.5).

8. Non-uniqueness of stationary solutions

In this section, we prove Theorem 1.5 by constructing non-unique stationary solutions. For any time-independent diffusion coefficient σ satisfying Assumption 1.2, or the singular case, we apply the convex integration method to the stationary Fokker–Planck equation (1.6). Compared with previous works, the main novelty of the argument lies in the treatment of the time-independent setting. Accordingly, we mainly emphasize the ideas and estimates specific to the stationary case.

Let $N>0, 0<\epsilon_0\leqslant\frac{1}{4}$ and $1<d_0<d-1$ be given. We recall $d_0'=\frac{d_0}{d_0-1}$. The iteration is again indexed by a parameter $q\in\mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}_0}\subset\mathbb{N}$ and a sequence $\{\delta_q\}_{q\in\mathbb{N}_0}\subset(0,1]$ defined by

$$\lambda_q=a^{b^q},\ q\geqslant0,\ \delta_q=(\frac{\epsilon_0}{2})^{d_0+d_0'}\lambda_1^{2\beta}\lambda_q^{-2\beta},\ q\geqslant1,\ \delta_0=1.$$

By imposing

$$a^{2(b-1)\beta/(d_0+d_0')}>2, \tag{8.1}$$

we obtain

$$\sum_{q\geqslant1}\delta_q^{1/(d_0+d_0')}\leqslant\frac{\epsilon_0}{2}\sum_{q\geqslant1}a^{(1-q)2(b-1)\beta/(d_0+d_0')}\leqslant\frac{\epsilon_0}{2}\cdot\frac{1}{1-a^{-2(b-1)\beta/(d_0+d_0')}}<\epsilon_0.$$

At each step q, we construct $(v_q,\rho^i_q,M^i_q)_{1\leqslant i\leqslant N}$ solving the stationary system

$$-\mathrm{div}\mathrm{div}(A(\rho^i_q)\rho^i_q)+\mathrm{div}(v_q\rho^i_q)+\mathrm{div}((V*\rho^i_q)\rho^i_q)=-\mathrm{div}M^i_q,\qquad \mathrm{div}v_q=0, \tag{8.2}$$

where we recall the notation $A(\rho)=(\sigma\sigma^T)(\rho)$. For each $1\leqslant i\leqslant N$, the sequence $\{\rho^i_q\}_{q\in\mathbb{N}_0}$ will be constructed to converge to a function ρ^i, while $\{M^i_q\}_{q\in\mathbb{N}_0}$ converges to 0 in a suitable topology as $q\to\infty$.

We initialize the iteration by defining (v_0,ρ^i_0,M^i_0) through

$$\rho^i_0(x)=1+\frac{\sin2\pi x_1}{4^i},\ v_0(x)=\overline{v}(x),\ M^i_0(x)=-(\overline{v}+V*\rho^i_0)\rho^i_0(x)+\mathrm{div}(A(\rho^i_0)\rho^i_0)(x),$$

where $x=(x_1,\dots,x_d)$. Then, by Assumption 1.2, there exists a sufficiently large constant $C_0>0$ such that

$$\|\rho^i_0\|_{C^2_x}\leqslant C_0^{1/d_0'},\ \|v_0\|_{C^1_x}\leqslant C_0^{1/d_0},\ \|M^i_0\|_{L^1}\leqslant(\frac{\epsilon_0}{2})^{d_0+d_0'}C_0. \tag{8.3}$$

In the singular interaction case, by (3.1) it holds that

$$\begin{aligned}\|M_0^i\|_{L^1} &\lesssim 1+\|(a*\nabla\rho_0^i)\rho_0^i\|_{L^1}+\|(a*\rho_0^i)\nabla\rho_0^i\|_{L^1}+\|(b*\nabla\rho_0^i)\rho_0^i\|_{L^1}\\ &\lesssim 1+\|a*\nabla\rho_0^i\|_{L^1}+\|a*\rho_0^i\|_{L^1}+\|b*\nabla\rho_0^i\|_{L^1}\lesssim 1+\|\rho_0^i\|_{C^1_x}\leqslant (\frac{\epsilon_0}{2})^{d_0+d_0'}C_0. \end{aligned}\tag{8.4}$$

With these preparations, we are ready to state the iteration proposition, which forms the basis of the whole scheme.

Proposition 8.1. *Under the assumption of Theorem 1.5, there exists a choice of parameters a, b, β such that the following holds: Let $(v_q, \rho_q^i, M_q^i)_{1\leqslant i\leqslant N}$ be a solution to the system (8.2) satisfying $\int \rho_q^i \mathrm{d}x = 1$,*

$$\|v_q\|_{C^1_x}\leqslant C_0^{1/d_0}\lambda_q^{d+1},\ \ \|\rho_q^i\|_{C^1_x}+\lambda_q^{-1}\|\rho_q^i\|_{C^2_x}\leqslant C_0^{1/d_0'}\lambda_q^{d+1},\tag{8.5}$$

$$\|M_q^i\|_{L^1}\leqslant C_0\delta_{q+1}.\tag{8.6}$$

Then there exists $(v_{q+1}, \rho_{q+1}^i, M_{q+1}^i)_{1\leqslant i\leqslant N}$ which solves (8.2) and satisfies (8.5)-(8.6) at the level $q+1$ and

$$\|v_{q+1}-v_q\|_{L^{d_0}}\leqslant C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0},\quad \|\rho_{q+1}^i-\rho_q^i\|_{L^{d_0'}}\leqslant C_vC_0^{1/d_0'}\delta_{q+1}^{1/d_0'},\tag{8.7}$$

for some universal constant $C_v\geqslant 1$. Moreover, we have for some $\epsilon>0$ small enough,

$$\|v_{q+1}-v_q\|_{L^1}\leqslant \delta_{q+1}^{1/d_0},\tag{8.8}$$

$$\|\rho_{q+1}^i-\rho_q^i\|_{W^{1,1+\epsilon}}\leqslant 4^{-i}\delta_{q+1}^{1/d_0'},\ \ \rho_{q+1}^i-\rho_q^i\geqslant -\delta_{q+1}^{1/d_0'}.\tag{8.9}$$

The proof of the main iteration, Proposition 8.1, will be given in the next section. Assuming Proposition 8.1, we now complete the proof of Theorem 1.5.

Proof of Theorem 1.5. We initialize the iteration from the triple (v_0, ρ_0^i, M_0^i) defined above. By (8.3) and (8.4), the required estimates hold at level $q=0$, since $\delta_1=(\frac{\epsilon_0}{2})^{d_0+d_0'}$. Next, applying Proposition 8.1 inductively, we construct (v_q, ρ_q^i, M_q^i) for every $q\geqslant 1$. By (8.1) together with (8.7)–(8.9), the sequence (v_q, ρ_q^i) is Cauchy in $L^{d_0}\times(L^{d_0'}\cap W^{1,1+\epsilon})^N$, and we denote its limit by (v, ρ^i). Moreover, by (8.6), it is straightforward to verify that (ρ^i, v) solves (1.6).

Furthermore, by (8.1), (8.7), and (8.9), we obtain

$$||\|\rho^i-1\|_{L^1}-\|\rho_0^i-1\|_{L^1}|\leqslant\sum_{q=0}^{\infty}\|\rho_{q+1}^i-\rho_q^i\|_{L^1}\leqslant 4^{-i}\sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'}\leqslant 4^{-i-1},$$

$$\rho^i\geqslant\rho_0^i+\sum_{q=0}^{\infty}(\rho_{q+1}^i-\rho_q^i)\geqslant\frac{3}{4}-\sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'}>0.$$

Hence, by the same calculation as before, ρ^i is nonnegative, nontrivial, and ρ^i do not coincide with each other. Since $\int_{\mathbb{T}^d}\rho_q^i\,\mathrm{d}x=1$, it follows that ρ^i is a probability density.

Moreover, by (8.8) we have $\|v-\overline{v}\|_{L^1}\leqslant\sum_{q=0}\|v_{q+1}-v_q\|_{L^1}\leqslant\sum_{q=0}\delta_{q+1}^{1/d_0}\leqslant\epsilon_0$.

By (8.5), (8.7), and interpolation, we obtain $v\in L^{d_0(1+\epsilon)}$ for some sufficiently small $\epsilon>0$. Since $\rho^i\in L^{d_0'}$, it follows that $|v|^{1+\epsilon}\rho^i\in L^1$. Therefore, we apply the superposition principle [Tre14, Section 7.2] and there exists a probability measure $\mathbf{Q}^i$ on $C([0,\infty);\mathbb{T}^d)$ which is a martingale solution associated with L_{ρ^i} and satisfies

$$\mathrm{d}\mathbf{Q}^i\circ\Pi_t^{-1}=\rho^i\,\mathrm{d}x,\ \ t\geqslant 0.$$

The resulting solution is stationary, since the density ρ^i is independent of time. $\square$

9. Proof of Proposition 8.1

In this section, we apply the convex integration method to prove Proposition 8.1. The overall strategy is similar to that in Section 5.4. However, in the present stationary setting, we introduce new building blocks adapted to the time-independent framework, which enable us to implement the convex integration scheme and derive the desired estimates.

9.1. **Mollification.** For given sufficiently small $\alpha \in (0,1)$ to be determined, we take $l := \lambda_{q+1}^{-3\alpha/2}\lambda_q^{-3d/2-2}$ and have

$$l^{-1} \leqslant \lambda_{q+1}^{2\alpha}, \quad l\lambda_q^{3d+4} \leqslant \lambda_{q+1}^{-\alpha} \ll \delta_{q+2}. \tag{9.1}$$

Then, in this section, we also replace (v_q, ρ_q^i, M_q^i) by a space-direction mollified field

$$v_l = v_q * \phi_l, \quad \rho_l^i = \rho_q^i * \phi_l, \;\; M_l^i = M_q^i * \phi_l,$$

where we recall that ϕ_l is a standard radial mollifiers on $\mathbb{R}^d$. By calculation we obtain that

$$-\mathrm{divdiv}(A(\rho_l^i)\rho_l^i) + \mathrm{div}(v_l\rho_l^i) + \mathrm{div}((V * \rho_l^i)\rho_l^i) = -\mathrm{div}(M_l^i + M_{com}^i), \quad \mathrm{div}v_l = 0,$$

where

$$\begin{aligned} M_{com}^i :&= -v_l\rho_l^i + (v_q\rho_q^i) * \phi_l + \mathrm{div}(A(\rho_l^i)\rho_l^i) - (\mathrm{div}(A(\rho_q^i)\rho_q^i)) * \phi_l \\ &\quad - (V * \rho_l^i)\rho_l^i + ((V * \rho_q^i)\rho_q^i) *_x \phi_l. \end{aligned}$$

By the mollification estimate (8.6) and (9.1) we have for $N \geqslant 0$,

$$\|M_l^i\|_{C_x^N} \lesssim l^{-d-\frac{1}{3}-N}\|M_q^i\|_{L^1} \lesssim C_0 l^{-d-\frac{1}{3}-N}.$$

9.2. **Construction of the perturbations.** To construct the perturbations of v_l and ρ_l^i, we introduce in this section a family of Mikado-type building blocks, which may be regarded as a generalization of those in Appendix A. We consider $2N$ disjoint sets $\Lambda^{1,n}, \Lambda^{2,n}$ defined in Appendix A by taking suitable rational rotations of one fixed set. We use the notation $\Lambda^{j,n} = \Lambda^{1,n}$ for j odd, and $\Lambda^{j,n} = \Lambda^{2,n}$ for j even. For a fixed exponent $d_0 > 1$, and $\xi \in \Lambda := \cup_{i=1}^N (\Lambda^{1,i} \cup \Lambda^{2,i})$ we recall the notation $\Phi_{(\xi,d_0)}, \phi_{(\xi,d_0)}, \phi'_{(\xi,d_0')}$ introduced in Appendix A.

Remark 9.1. We recall that in [Luo19], the construction of stationary solutions requires the spatial dimension to be larger than 3. In contrast, in the present work, we are able to employ more general L^{d_0}-based building blocks, which allows us to construct stationary solutions also in dimension three.

We recall α_ξ from Lemma 7.2, and define

$$W_{(\xi)}(x) := \xi\phi_{(\xi,d_0)}(x - \alpha_\xi), \;\; \Theta_{(\xi)}(x) := \phi'_{(\xi,d_0')}(x - \alpha_\xi),$$

and

$$V_{(\xi)} := \frac{1}{(n_*\lambda)^2}(\xi \otimes \nabla\Phi_{(\xi,d_0)} - \nabla\Phi_{(\xi,d_0)} \otimes \xi)(x - \alpha_\xi)$$

with

$$\mathrm{div}V_{(\xi)} = W_{(\xi)}. \tag{9.2}$$

In this case, by choosing λ large enough, by Lemma 7.2 we have

$$W_{(\xi)}\Theta_{(\xi')} \equiv 0, \;\text{ for } \xi \neq \xi' \in \Lambda, \tag{9.3}$$

$$\int_{\mathbb{T}^d} W_{(\xi)}\Theta_{(\xi)}\mathrm{d}x = \xi, \quad \mathrm{div}(W_{(\xi)}\Theta_{(\xi)}) = 0.$$

Finally, we obtain that for $N \geqslant 0$ and $p \in [1, \infty]$, it holds

$$\|\nabla^N W_{(\xi)}\|_{L^p} + \lambda \|\nabla^N V_{(\xi)}\|_{L^p} \lesssim r_\perp^{\frac{d-1}{p} - \frac{d-1}{d_0}} \lambda^n, \tag{9.4}$$

$$\|\nabla^N \Theta_{(\xi)}\|_{L^p} \lesssim r_\perp^{\frac{d-1}{p} - \frac{d-1}{d_0'}} \lambda^n, \tag{9.5}$$

where the implicit constants may depend on p, N, but are independent of $\lambda, r_\perp$.

With these building blocks in hand, then, we fix a parameter $\zeta = 20\delta_{q+2}^{-1}$. Let $\chi \in C_c^\infty(-\frac{3}{4}, \frac{3}{4})$ be a nonnegative function such that $\sum_{n\in\mathbb{Z}} \chi(t-n) = 1$ for every $t \in \mathbb{R}$. Let $\tilde{\chi} \in C_c^\infty(-\frac{4}{5}, \frac{4}{5})$ be a nonnegative function satisfying $\tilde{\chi} = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$ and $\sum_{n\in\mathbb{Z}} \tilde{\chi}(t-n) \leqslant 2$.

Then we define the perturbations

$$\begin{aligned} w_{q+1}^{(p,i)} &:= \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^{n,i}} \tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} W_{(\xi)}, \\ w_{q+1}^{(c,i)} &:= \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^{n,i}} \nabla(\tilde{\chi}(\zeta|M_l^i| - n))\Big(\frac{n}{\zeta}\Big)^{1/d_0} : V_{(\xi)}, \end{aligned}$$

which implies that by (9.2)

$$w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)} = \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^{n,i}} \mathrm{div}\Big(\tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} V_{(\xi)}\Big). \tag{9.6}$$

It then follows that $\mathrm{div}(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}) = 0$ and $w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}$ is mean-zero.

Finally, the total perturbation and new iteration are defined by

$$w_{q+1} := \sum_{i=1}^{N} \Big(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}\Big), \quad v_{q+1} := v_l + w_{q+1}.$$

Then w_{q+1}, v_{q+1} are both mean-zero and divergence-free functions.

Now, we define the perturbations for the stationary densities functions as

$$\begin{aligned} \theta_{q+1}^{(p,i)} &:= \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^{n,i}} \chi(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'} \Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi)}, \\ \theta_{q+1}^{(c,i)} &:= -\int_{\mathbb{T}^d} \theta_{q+1}^{(p,i)}\mathrm{d}x. \end{aligned}$$

Then, by (9.3) and a similar calculation as before, we obtain

$$\begin{aligned} \mathrm{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} - M_l^i) &= \sum_{n\geqslant 3} \sum_{\xi\in\Lambda^{n,i}} \nabla\Big[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big]\mathbb{P}_{\neq 0}(W_{(\xi)}\Theta_{(\xi)}) \\ &\quad + \mathrm{div}\Big(\sum_{n\geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i\Big). \end{aligned} \tag{9.7}$$

Finally, we define

$$\theta_{q+1}^i := \theta_{q+1}^{(p,i)} + \theta_{q+1}^{(c,i)}, \quad \rho_{q+1}^i := \rho_l^i + \theta_{q+1}^i.$$

By definition, it holds that $\int \rho_{q+1}^i \mathrm{d}x = 1$.

Moreover, by Lemma 7.2, we obtain for $1 \leqslant i \leqslant N$ that

$$\operatorname{div}\big(w_{q+1}\theta^i_{q+1}\big) = \operatorname{div}\Big((w^{(p,i)}_{q+1} + w^{(c,i)}_{q+1})\theta^{(p,i)}_{q+1}\Big). \tag{9.8}$$

9.3. **Construction of the stress term M^i_{q+1}.** From (9.8) and the definition of the perturbations we obtain

$$\begin{aligned}
-\operatorname{div}M^i_{q+1} &= \operatorname{div}(w^{(p,i)}_{q+1}\theta^{(p,i)}_{q+1} - M^i_l)(:= \operatorname{div}M^i_{osc})\\
&\quad - \operatorname{div}[\operatorname{div}(A(\rho^i_{q+1})\rho^i_{q+1} - A(\rho^i_l)\rho^i_l) - (V * \rho^i_{q+1})\rho^i_{q+1} + (V * \rho^i_l)\rho^i_l](:= \operatorname{div}M^i_{nonlin})\\
&\quad + \operatorname{div}(v_l\theta^i_{q+1} + w_{q+1}\rho^i_l + w^{(c,i)}_{q+1}\theta^{(p,i)}_{q+1})(:= \operatorname{div}M^i_{lin})\\
&\quad - \operatorname{div}M^i_{com},
\end{aligned}$$

where we define the nonlinear error M^i_{nonlin} and linear error M^i_{lin} respectively by

$$\begin{aligned}
M^i_{nonlin} &:= -\operatorname{div}(A(\rho^i_{q+1})\rho^i_{q+1} - A(\rho^i_l)\rho^i_l) + (V * \rho^i_{q+1})\rho^i_{q+1} - (V * \rho^i_l)\rho^i_l,\\
M^i_{lin} &:= v_l\theta^i_{q+1} + w_{q+1}\rho^i_l + w^{(c,i)}_{q+1}\theta^{(p,i)}_{q+1}.
\end{aligned}$$

Using (9.7) and the inverse divergence operator $\mathcal{B}_1$, we define the oscillation error $M^i_{osc} := M^i_{osc,x} + M^i_{osc,c}$ by

$$\begin{aligned}
M^i_{osc,x} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \mathcal{B}_1\Big(\nabla\Big[\chi(\zeta|M^i_l| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big], \mathbb{P}_{\neq 0}(W_{(\xi)}\Theta_{(\xi)})\Big),\\
M^i_{osc,c} &:= \sum_{n\geqslant 3}\chi(\zeta|M^i_l| - n)\frac{n}{\zeta}\frac{M^i_l}{|M^i_l|} - M^i_l.
\end{aligned}$$

Then we define

$$-M^i_{q+1} := M^i_{osc} + M^i_{nonlin} + M^i_{lin} - M^i_{com}.$$

9.4. **Proof of Proposition 8.1.**

9.4.1. *Choice of parameters.* Regarding the parameters of the building blocks, we define

$$\lambda = \lambda_{q+1},\ r_\perp = \lambda_{q+1}^{-1+\frac{1}{M}},$$

where we introduce a integer $M > 0$ large enough satisfying $0 < \frac{1}{M}(\frac{d-1}{d_0}+1) < \frac{d-1}{d_0}-1, (1-\frac{1}{M})\frac{d-1}{d_0'} > \frac{1}{M}$ and $(1-\frac{1}{M})(d-1)(\frac{\gamma}{d} - \frac{1}{d_0'}) > \frac{1}{M}$. Then, it holds that

$$\lambda_{q+1} r_\perp^{d-1-\frac{d-1}{d_0'}} \lesssim \lambda_{q+1}^{-\frac{1}{M}},\quad r_\perp^{d-1-\frac{d-1}{d_0}} \lesssim \lambda_{q+1}^{-\frac{1}{M}},\quad r_\perp^{\frac{d-1}{d/\gamma} - \frac{d-1}{d_0'}} \lesssim \lambda_{q+1}^{-\frac{1}{M}}. \tag{9.9}$$

In the sequel, we also need (9.1) and

$$\lambda_q^{d+1} \leqslant \lambda_{q+1}^\alpha,\quad (12d+41)\alpha < \frac{1}{M}. \tag{9.10}$$

The above conditions can be achieved as follows. First, choose $\alpha > 0$ sufficiently small such that $(12d+41)\alpha < \frac{1}{M}$. Next, choose $b \in M\mathbb{N}$ sufficiently large so that $b > \frac{3d+4}{\alpha}$. We then select $\beta > 0$ sufficiently small such that $\alpha > 2\beta b$. Finally, we choose a sufficiently large in order to absorb all implicit and universal constants arising in the subsequent estimates and to guarantee the validity of (8.1).

In the end, we establish the estimate of the amplitude functions as in Lemma 5.1.

Lemma 9.2. *For $M, K \in \mathbb{N}_0$ and $1 \leqslant i \leqslant N$, we have*

$$\sum_{n\geqslant 3}\Big(\frac{n}{\zeta}\Big)^M\Big(\|\chi(\zeta|M_l^i|-n)\|_{C_x^K}+\|\tilde\chi(\zeta|M_l^i|-n)\|_{C_x^K}\Big)\lesssim l^{-(d+4)K-(d+2)(M+1)},$$

$$\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big(\frac{n}{\zeta}\Big)^M\Big\|\chi(\zeta|M_l^i|-n)\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big\|_{C_x^K}\lesssim l^{-(2d+8)K-(d+2)(M+1)}.$$

9.4.2. *Proof of* (8.7) *for* $v_{q+1}-v_q$. We first estimate the principal perturbations $w_{q+1}^{(p,i)}$. By Cauchy's inequality we have

$$|w_{q+1}^{(p,i)}|^{d_0}\lesssim\sum_{n\geqslant 3}\tilde\chi(\zeta|M_l^i|-n)\frac{n}{\zeta}\sum_{\xi\in\Lambda^{n,i}}\big|W_{(\xi)}\big|^{d_0}.$$

By applying the generalized Hölder inequality of Theorem 3.3 in spatial direction, together with the estimates for the building blocks in (9.4) and Lemma 9.2 we deduce

$$\begin{aligned}\|w_{q+1}^{(p,i)}\|_{L^{d_0}}^{d_0}&\lesssim\sum_{n\geqslant 3}\Big\|\tilde\chi(\zeta|M_l^i|-n)\frac{n}{\zeta}\Big\|_{L^1}\sum_{\xi\in\Lambda^{n,i}}\|W_{(\xi)}\|_{L^{d_0}}^{d_0}\\&\quad+(r_\perp\lambda_{q+1})^{-1}\Big\|\tilde\chi(\zeta|M_l^i|-n)\frac{n}{\zeta}\Big\|_{C_x^1}\sum_{\xi\in\Lambda^{n,i}}\|W_{(\xi)}\|_{L^{d_0}}^{d_0}\\&\lesssim\big\|M_l^i+\zeta^{-1}\big\|_{L^1}+l^{-3d-8}\lambda_{q+1}^{-\frac{1}{M}}\lesssim\|M_l^i\|_{L^1}+\delta_{q+1}\lesssim C_0\delta_{q+1},\end{aligned}$$

where we used the condition on the parameters to have $(6d+16)\alpha-\frac{1}{M}<-\alpha<-2\beta$.

In the next step, we estimate in general L^m-norm with $m\in[1,\infty]$. By (9.4) and Lemma 9.2 we obtain

$$\|w_{q+1}^{(p,i)}\|_{L^m}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C_x^0}\|W_{(\xi)}\|_{L^m}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}},\tag{9.11}$$

$$\|w_{q+1}^{(c,i)}\|_{L^m}\lesssim\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C_x^1}\|V_{(\xi)}\|_{L^m}\lesssim l^{-3d-8}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}}\lambda_{q+1}^{-1}.\tag{9.12}$$

We then have

$$\|w_{q+1}\|_{L^{d_0}}\lesssim NC_0^{1/d_0}\delta_{q+1}^{1/d_0}+N\lambda_{q+1}^{(6d+16)\alpha-1}\leqslant\frac{C_v}{2}C_0^{1/d_0}\delta_{q+1}^{1/d_0},\tag{9.13}$$

where we used conditions on the parameters to have $(6d+16)\alpha-1<-\alpha<-2\beta$. The above inequality together with (8.5) and (9.1) implies (8.7) for v_{q+1}:

$$\|v_{q+1}-v_q\|_{L^{d_0}}\leqslant\|w_{q+1}\|_{L^{d_0}}+l\|v_q\|_{C_x^1}\leqslant\frac{C_v}{2}C_0^{1/d_0}\delta_{q+1}^{1/d_0}+lC_0^{1/d_0}\lambda_q^{d+1}\leqslant C_vC_0^{1/d_0}\delta_{q+1}^{1/d_0}.$$

9.4.3. *Proof of* (8.5) *for* v_{q+1}. By the estimates for the building blocks in (9.4), (9.6) and Lemma 9.2 we have

$$\begin{aligned}\|w_{q+1}\|_{C_x^1}&\lesssim\sum_{i=1}^N\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M_l^i|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C_x^2}\|V_{(\xi)}\|_{C_x^2}\\&\lesssim Nl^{-4d-12}\lambda_{q+1}r_\perp^{-\frac{d-1}{d_0}}\lesssim\lambda_{q+1}^{(8d+24)\alpha+d}.\end{aligned}$$

Thus, by $(8d+24)\alpha<1$ we obtain

$$\|v_{q+1}\|_{C^1_x}\leqslant\|v_l\|_{C^1_x}+\|w_{q+1}\|_{C^1_x}\leqslant C_0^{1/d_0}\lambda_q^{d+1}+\lambda_{q+1}^{d+1}\leqslant C_0^{1/d_0}\lambda_{q+1}^{d+1},$$

where we chose a large enough to absorb the universal constant.

9.4.4. *Proof of* (8.8). By (9.9), (9.11) and (9.12), it holds that

$$\|w_{q+1}\|_{L^1}\lesssim Nl^{-2d-4}r_\perp^{d-1-\frac{d-1}{d_0}}\lesssim C_0\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{M}},$$

which together with (8.1), (8.5) implies

$$\|v_{q+1}-v_q\|_{L^1}\lesssim C_0\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{M}}+lC_0\lambda_q^{d+1}\lesssim\lambda_{q+1}^{-\alpha}\leqslant\delta_{q+1}^{1/d_0},$$

where we used the conditions on the parameters to have $(4d+9)\alpha<\frac{1}{M}$. We also chose a large to absorb the universal constant.

9.4.5. *Proof of* (8.7) *for* $\rho^i_{q+1}-\rho^i_q$. We first estimate the principal perturbations $\theta^{(p,i)}_{q+1}$. By the same argument as in (5.12),

$$\|\theta^{(p,i)}_{q+1}\|^{d_0'}_{L^{d_0'}}\lesssim\|M^i_l\|_{L^1}+\delta_{q+1}\lesssim C_0\delta_{q+1}.$$

Then we estimate in general L^m-norm with $m\in[1,\infty]$, by the estimates for the building blocks in (9.5) and Lemma 9.2 we obtain

$$\|\theta^{(p,i)}_{q+1}\|_{L^m}\lesssim\sum_{n\geqslant3}\sum_{\xi\in\Lambda^{n,i}}\left\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^0_x}\|\Theta_{(\xi)}\|_{L^m}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0'}},\quad(9.14)$$

which implies that by (9.9)

$$|\theta^{(c,i)}_{q+1}|\lesssim\|\theta^{(p,i)}_{q+1}\|_{L^{d/\gamma}}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{d/\gamma}-\frac{d-1}{d_0'}}\lesssim\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{M}}\lesssim\lambda_{q+1}^{-2\alpha},\tag{9.15}$$

where we used conditions on the parameters to have $(4d+10)\alpha<\frac{1}{M}$ and chose a large enough to absorb the universal constant. Then together with (9.1) we imply

$$\|\rho^i_{q+1}-\rho^i_q\|_{L^{d_0'}}\leqslant\|\theta^i_{q+1}\|_{L^{d_0'}}+l\|\rho^i_q\|_{C^1_x}\lesssim C_0^{1/d_0'}(\delta_{q+1}^{1/d_0'}+\lambda_{q+1}^{-\alpha})\leqslant C_vC_0^{1/d_0'}\delta_{q+1}^{1/d_0'},$$

which implies (8.7) for $\rho^i_{q+1}-\rho^i_q$.

9.4.6. *Proof of* (8.9). We first estimate $\theta^{(p,i)}_{q+1}$ in $W^{1,1+\epsilon}$-norm for some $\epsilon>0$. By (9.5) and Lemma 9.2 we have

$$\begin{aligned}\|\theta^{(p,i)}_{q+1}\|_{W^{1,1+\epsilon}}&\lesssim\sum_{n\geqslant3}\sum_{\xi\in\Lambda^{n,i}}\left\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^1_x}\|\Theta_{(\xi)}\|_{W^{1,1+\epsilon}}\\&\lesssim l^{-4d-12}\lambda_{q+1}r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}}\lesssim\lambda_{q+1}^{(8d+24)\alpha-\frac{1}{M}+d\epsilon}\lesssim\lambda_{q+1}^{-2\alpha},\end{aligned}\tag{9.16}$$

where we chose $\epsilon>0$ small enough to ensure $d\epsilon<\alpha$, and used (9.9) and (9.10). We also used conditions on the parameters to have $(8d+27)\alpha<\frac{1}{M}$.

Since $\theta^{(p,i)}_{q+1}$ is non-negative, by the choice of parameters in (9.1) and (9.15) we have

$$\begin{aligned}\|\rho^i_{q+1}-\rho^i_q\|_{W^{1,1+\epsilon}}&\leqslant\|\theta^i_{q+1}\|_{W^{1,1+\epsilon}}+l\|\rho^i_q\|_{C^2_x}\lesssim\lambda_{q+1}^{-\alpha}+l\lambda_q^{d+1}\leqslant4^{-i}\delta_{q+1}^{1/d_0'},\\\rho^i_{q+1}-\rho^i_q&\geqslant-|\theta^{(c,i)}_{q+1}|-l\|\rho^i_q\|_{C^1_x}\geqslant-C\lambda_{q+1}^{-\alpha}\geqslant-\delta_{q+1}^{1/d_0'},\end{aligned}$$

which yields (8.9). Here we chose a large enough to absorb the universal constant.

9.4.7. *Proof of* (8.5) *for* ρ^i_{q+1}. By (9.5) and Lemma 9.2 we have for $j=1,2$

$$\|\theta^{(p,i)}_{q+1}\|_{C^j_x} \lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \left\|\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^j_x}\|\Theta_{(\xi)}\|_{C^j_x}$$
$$\lesssim l^{-6d-20}\lambda^j_{q+1} r_\perp^{-\frac{d-1}{d_0'}} \lesssim \lambda_{q+1}^{(12d+40)\alpha+d+j-1}.$$

By choosing $(12d+40)\alpha<\frac{1}{2}$ we deduce

$$\|\rho^i_{q+1}\|_{C^1_x}+\lambda^{-1}_{q+1}\|\rho^i_{q+1}\|_{C^2_x} \leqslant C_0^{1/d_0'}\lambda_q^{d+1}+\lambda_{q+1}^{d+1}\leqslant C_0^{1/d_0'}\lambda_{q+1}^{d+1},$$

which implies (8.5). Here we chose a large enough to absorb the universal constant.

9.4.8. *Proof of* (8.6). We estimate each terms in the definition of M^i_{q+1} separately.

Oscillation error M^i_{osc}. We observe that $W_{(\xi)}\Theta_{(\xi)}$ is $(\mathbb{T}/r_\perp\lambda_{q+1})^d$-periodic, so by Theorem 3.2, Lemma 9.2, (9.4) and (9.5) we have

$$\|M^i_{osc,x}\|_{L^1} \lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \left\|\chi(\zeta|M^i_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^2_x}(r_\perp\lambda_{q+1})^{-1}\|W_{(\xi)}\Theta_{(\xi)}\|_{L^1}$$
$$\lesssim l^{-6d-20}(r_\perp\lambda_{q+1})^{-1}\lesssim \lambda_{q+1}^{(12d+40)\alpha-\frac{1}{M}}\lesssim\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2},$$

where we used the conditions on the parameters to have $(12d+41)\alpha<\frac{1}{M}$.

The estimate on the stress term $M^i_{osc,c}$ is directly from Section 9.4.8:

$$\big|M^i_{osc,c}\big|\leqslant\frac{3}{\zeta}+\sum_{n\geqslant 3}\chi(\zeta|M^i_l|-n)\left|\frac{n}{\zeta}-|M^i_l|\right|\leqslant\frac{1}{5}C_0\delta_{q+2}.$$

Nonlinear error M^i_{nonlin}. Under Assumption 1.2, by the calculation in (5.21), the bounds in (8.5), (9.1) and (9.16) we have

$$\|M^i_{nonlin}\|_{L^1}\lesssim\|\theta^i_{q+1}\|_{W^{1,1+\epsilon}}(1+\|\rho^i_q\|_{C^1_x}+\|\theta^i_{q+1}\|_{W^{1,1+\epsilon}})$$
$$\lesssim(\lambda_q^{d+1}+1)\lambda_{q+1}^{-2\alpha}\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2}.$$

For the singular interaction case, by (6.10), we have

$$\|M^i_{nonlin}\|_{L^1}\lesssim(\|\theta^i_{q+1}\|_{L^{d/\gamma}}+\|\nabla\theta^i_{q+1}\|_{L^{1+\epsilon}})(\|\theta^i_{q+1}\|_{L^{d/\gamma}}+\|\nabla\theta^i_{q+1}\|_{L^{1+\epsilon}}+\|\rho^i_q\|_{C^1_x})$$
$$\lesssim C_0\lambda_{q+1}^{-2\alpha}(1+\lambda_q^{d+1})\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2}.$$

Linear error M^i_{lin}. Together with the estimates in (8.5), (9.1), (9.11), (9.12) and (9.15) we obtain

$$\|M^i_{lin}\|_{L^1}\lesssim\|v_l\|_{C^0_x}\|\theta^i_{q+1}\|_{L^1}+\|\rho^i_l\|_{C^0_x}\|w_{q+1}\|_{L^1}+\|\theta^{(p,i)}_{q+1}\|_{L^{d_0'}}\|w^{(c,i)}_{q+1}\|_{L^{d_0}}$$
$$\lesssim C_vC_0(\lambda_q^{d+1}+1)\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{M}}\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant\frac{1}{5}C_0\delta_{q+2},$$

where we used the conditions on the parameters to have $(4d+10)\alpha<\frac{1}{M}$.

Commutator error M^i_{com}. By the calculation as in (5.22) or (7.14), the bounds in (8.5), (9.1) and Lemma 4.1 we obtain

$$\begin{aligned}\|M^i_{com}\|_{L^1} &\lesssim l\|v_q\|_{C^1_x}\|\rho^i_q\|_{C^1_x} + l(1+\|\rho^i_q\|_{C^2_x} + \|\rho^i_q\|^2_{C^1_x})\|\rho^i_q\|_{C^1_x}\\ &\lesssim C_0 l\lambda_q^{3d+4} \lesssim C_0\lambda_{q+1}^{-\alpha} \leqslant \frac{1}{5}C_0\delta_{q+2},\end{aligned}$$

where we also chose a large to absorb the universal constant.

Summarizing all the estimates above we obtain (8.6) at the level $q+1$.

10. Application to the whole space case

In this section, we consider the whole-space setting and prove Theorem 1.7 and Theorem 1.6. The non-uniqueness solutions to linear Fokker–Planck equations on $\mathbb{R}^d$ was established in [LR25]. A key ingredient in the whole-space construction is the Bogovskii operator, which replaces the inverse divergence operator used in the periodic setting. As a inverse of the divergence operator, the Bogovskii operator preserves compact support: when applied to a compactly supported function, the resulting vector field remains compactly supported. This property allows us to localize the entire convex integration construction to a fixed domain, say $[-\frac{1}{2},\frac{1}{2}]^d$.

The overall construction and the corresponding estimates are largely analogous to those developed in the torus setting. For this reason, we omit many repetitive computations and focus only on the modifications required in the whole-space case. Moreover, the solutions constructed here may be viewed as small perturbations of a stationary solution ρ^{st}, which plays a role analogous to that of the uniform distribution in the periodic setting.

10.1. **Bogovskii operator.** Let $K\subset\mathbb{R}^d, d\geqslant 2$ be a bounded and star shaped domain with respect to a ball $K'\subset\subset K$. We choose a non-negative $\omega\in C_c^\infty(K')$ with $\int_{K'}\omega \mathrm{d}x = 1$ and define for $g\in C_c^\infty(K)$

$$\mathcal{B}_\omega g(x) := \int_K g(y)\,\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x + r\,\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}\,\mathrm{d}r\,\mathrm{d}y.$$

By rewriting

$$\mathcal{B}_\omega g(x) := \int_K g(y)(x-y)\int_1^\infty \omega(y+r(x-y))\,r^{n-1}\,\mathrm{d}r\,\mathrm{d}y,$$

the following Lemma is well-known (see c.f. [Bog79, Theorem 1]).

Proposition 10.1. *Let $k\geqslant 0, 1<p<\infty$. Let $f\in W^{k,p}_0(K)$. Then it holds that*

$$\mathcal{B}_\omega(C_c^\infty(K))\subset C_c^\infty(K)^d,$$

and

$$\mathrm{div}\mathcal{B}_\omega f = f - \omega\int_K f\mathrm{d}x.$$

Moreover, we have

$$\|\mathcal{B}_\omega f\|_{W^{k+1,p}(K)} \lesssim_\omega \|f\|_{W^{k,p}(K)}.$$

Then, we introduce the the bilinear version $\mathcal{R}_\omega : C_c^\infty(K;\mathbb{R})\times C_0^\infty(\mathbb{T}^d;\mathbb{R})\to C_c^\infty(K;\mathbb{R}^d)$ by

$$\mathcal{R}_\omega(v,f) := v\nabla\Delta^{-1}f - \mathcal{B}_\omega\Big(\nabla v\cdot\nabla\Delta^{-1}f\Big).$$

Here $C_0^\infty(\mathbb{T}^d;\mathbb{R})$ denotes the space of smooth $\mathbb{T}^d$-periodic functions on $\mathbb{R}^d$ with zero-mean.

Proposition 10.2. *Let $k \geqslant 0, 1 < p < \infty$. For any $v \in C_c^\infty(K;\mathbb{R})$ and $f \in C_0^\infty(\mathbb{T}^d;\mathbb{R})$, we have*

$$\operatorname{div}(\mathcal{R}_\omega(v,f)) = vf - \omega \int_K vf \mathrm{d}x,$$

and for $\sigma \in \mathbb{N}$,

$$\|\mathcal{R}_\omega(v, f(\sigma\cdot))\|_{W^{k,p}(K)} \lesssim \sigma^{k-1}\|v\|_{C^{k+1}(K)}\|f\|_{W^{k,p}(\mathbb{T}^d)}.$$

10.2. **Some estimates on ρ^{st}.** We collect below several estimates that will be used throughout the remainder of this section.

Lemma 10.3. *The stationary solution satisfies that*

$$\|\rho^{st}\|_{W^{2,\infty}} < \infty. \tag{10.1}$$

Proof. By rewriting $g(\rho^{st}(x)) = -\Phi(x) + \mu^{st}$, taking derivative and using the monotony of g^{-1}, it holds that

$$|\rho^{st}(x)| \leqslant g^{-1}(\mu^{st}) < \infty.$$
$$|\nabla\rho^{st}(x)| \leqslant |g'(\rho^{st}(x))|^{-1}|\nabla\Phi| \lesssim |\rho^{st}(x)|\Phi^m \lesssim e^{-c\Phi}\Phi^m < \infty,$$

where we also used that $\rho^{st} = g^{-1}(-\Phi + \mu^{st}) \lesssim e^{c(-\Phi+\mu^{st})}$.

By calculation, we know that

$$|g''(x)| \leqslant \frac{|\beta''(x)xb(x)| + |\beta'(x)(b(x) + xb'(x))|}{(xb(x))^2} \lesssim \frac{1}{x} + \frac{1}{x^2},$$

which implies that

$$|g''(\rho^{st}(x))(\nabla\rho^{st}(x))^2| \lesssim |g''(\rho^{st}(x))||\rho^{st}(x)|^2\Phi^{2m} \lesssim (1 + |\rho^{st}(x)|)\Phi^{2m} \lesssim \Phi^{2m}.$$

Then

$$|\nabla^2\rho^{st}| \leqslant |g'(\rho^{st}(x))|^{-1}(|g''(\rho^{st}(x))(\nabla\rho^{st}(x))^2| + |\nabla^2\Phi|) \lesssim \rho^{st}\Phi^{2m} \lesssim e^{-c\Phi}\Phi^{2m} < \infty.$$

□

Moreover, by definition, ρ^{st} is a strictly positive function. Since $[-\frac{1}{2}, \frac{1}{2}]^d$ is compact and ρ^{st} is continuous, there exists a constant $c_{\text{in}} > 0$ such that

$$\rho^{st} \geqslant 1_{[-\frac{1}{2},\frac{1}{2}]^d} c_{in}. \tag{10.2}$$

10.3. **The main iteration used in the proof of Theorem 1.7.** Let $N > 0, 0 < \epsilon_0 \leqslant \min\{\frac{1}{8}, \frac{1}{2^{d+10}} c_{in}\}$ and $1 < d_0 < d - 1$ be given. The iteration is again indexed by a parameter $q \in \mathbb{N}_0$. We consider the sequences

$$\lambda_q = a^{b^q},\ q \geqslant 0,\ \delta_q = (\frac{\epsilon_0}{2})^{d_0+d_0'}\lambda_1^{2\beta}\lambda_q^{-2\beta},\ q \geqslant 1,\ \delta_0 = 1.$$

By (8.1) we obtain $\sum_{q\geqslant 1}\delta_q^{1/(d_0+d_0')} < \epsilon_0$.

At each step q, we construct $(v_q, \rho_q^i, M_q^i)_{1\leqslant i\leqslant N}$ solving the stationary system on the whole space:

$$\begin{aligned} -\operatorname{div}(\beta'(\rho_q^i + \rho^{st})\nabla(\rho_q^i + \rho^{st})) + \operatorname{div}(v_q(\rho_q^i + \rho^{st})) + \operatorname{div}(B(\rho_q^i + \rho^{st})) &= -\operatorname{div}M_q^i, \\ \operatorname{div}v_q &= 0, \end{aligned} \tag{10.3}$$

where we rewrite the notation $B(\rho) = Eb(\rho)\rho$. In particular, we also need the family $(v_q, \rho_q^i, M_q^i)_{1\leqslant i\leqslant N}$ has compact support on $\Omega_q \subset [-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$, where

$$\Omega_q := \Big[-\frac{1}{3} - \sum_{1\leqslant r\leqslant q} \delta_r^{1/2}, \frac{1}{3} + \sum_{1\leqslant r\leqslant q} \delta_r^{1/2} \Big]^d.$$

In fact, we will apply the convex integration on the $[-\frac{1}{2}, \frac{1}{2}]^d$ with zero boundary condition.

We initialize the iteration by defining $(v_0, \rho_0^i, M_0^i)_{1\leqslant i\leqslant N}$ through

$$\begin{aligned}
&\rho_0^i = 4^{-i}F, \quad v_0 = 0,\\
&M_0^i = \beta'(\rho_0^i + \rho^{st})\nabla(\rho_0^i + \rho^{st}) - \beta'(\rho^{st})\nabla(\rho^{st}) - B(\rho_0^i + \rho^{st}) + B(\rho^{st}),
\end{aligned}$$

where $F(x)$ is a smooth mean-zero bounded function with support in Ω_0, and satisfying $\|F\|_{C_x^0} = \frac{1}{2}c_{in}, \|F\|_{L^1} \geqslant \frac{1}{2^{d+1}}c_{in}$ by multiplying by a suitable constant. In particular, since ρ^{st} is a stationary solution, we know that (v_0, ρ_0^i, M_0^i) is a solution to (10.3), and $\operatorname{supp}\rho_0^i, \operatorname{supp} v_0, \operatorname{supp} M_0^i \subset \Omega_0$. Also, there exists a sufficiently large constant $C_0 > 0$ such that

$$\begin{aligned}
&\|\rho_0^i\|_{C_x^2(\mathbb{R}^d)} \leqslant C_0^{1/d_0'}, \ \|v_0\|_{C_x^1(\mathbb{R}^d)} \leqslant C_0^{1/d_0},\\
&\|M_0^i\|_{L^1(\mathbb{R}^d)} \lesssim \|\beta'\|_{C^1} + \|E\|_{C_x^1([-\frac{1}{2},\frac{1}{2}]^d)}\|b\|_{C^1} \leqslant (\frac{\epsilon_0}{2})^{d_0+d_0'}C_0.
\end{aligned} \tag{10.4}$$

Our main iteration procedure reads as follows:

Proposition 10.4. *Under the assumption of Theorem 1.7, there exists a choice of parameters a, b, β such that the following holds: Let $(v_q, \rho_q^i, M_q^i)_{1\leqslant i\leqslant N}$ be a solution to the system (10.3) satisfying*

$$\|v_q\|_{C_x^1(\mathbb{R}^d)} \leqslant C_0^{1/d_0}\lambda_q^{d+1}, \ \|\rho_q^i\|_{C_x^1(\mathbb{R}^d)} + \lambda_q^{-1}\|\rho_q^i\|_{C_x^2(\mathbb{R}^d)} \leqslant C_0^{1/d_0'}\lambda_q^{d+1}, \tag{10.5}$$

$$\|M_q^i\|_{L^1(\mathbb{R}^d)} \leqslant C_0\delta_{q+1}, \tag{10.6}$$

$$\operatorname{supp}\rho_q^i, \operatorname{supp} v_q, \operatorname{supp} M_q^i \subset \Omega_q. \tag{10.7}$$

Then there exists $(v_{q+1}, \rho_{q+1}^i, M_{q+1}^i)_{1\leqslant i\leqslant N}$ which solves (10.3) and satisfies (10.5)-(10.7) at the level $q+1$ and

$$\|v_{q+1} - v_q\|_{L^{d_0}(\mathbb{R}^d)} \leqslant C_v C_0^{1/d_0}\delta_{q+1}^{1/d_0}, \quad \|\rho_{q+1}^i - \rho_q^i\|_{L^{d_0'}(\mathbb{R}^d)} \leqslant C_v C_0^{1/d_0'}\delta_{q+1}^{1/d_0'}, \tag{10.8}$$

$$\|v_{q+1} - v_q\|_{L^1(\mathbb{R}^d)} + 4^i\|\rho_{q+1}^i - \rho_q^i\|_{W^{1,1+\epsilon}(\mathbb{R}^d)} \leqslant \delta_{q+1}^{1/d_0'}, \quad \rho_{q+1}^i - \rho_q^i \geqslant -\delta_{q+1}^{1/d_0'}. \tag{10.9}$$

for some universal constant $C_v \geqslant 1$.

By (10.7), we know that all the bounds on $\mathbb{R}^d$ are in fact restricted on the domain $[-\frac{1}{2}, \frac{1}{2}]^d$, so the estimates are similar to Section 9. The proof of the main iteration will be given below. Assuming Proposition 10.4, we now complete the proof of Theorem 1.7.

Proof of Theorem 1.7. We initialize the iteration from the triple (v_0, ρ_0^i, M_0^i) defined above. By (10.4), the required estimates hold at level $q = 0$. Next, applying Proposition 10.4 inductively, we construct (v_q, ρ_q^i, M_q^i) for every $q \geqslant 1$. By (10.8)–(10.9), the sequence (v_q, ρ_q^i) is Cauchy in $L^{d_0} \times (L^{d_0'} \cap W^{1,1+\epsilon})^N$, and we denote its limit by (v, ρ^i). We then define $\overline{\rho}^i := \rho^i + \rho^{st}$. By (10.6), it is straightforward to verify that $(\overline{\rho}^i, v)$ solves (1.7).

Furthermore, by (10.2), (10.8), and (10.9), we obtain

$$\|v\|_{L^1(\mathbb{R}^d)} \leqslant \sum_{q=0} \|v_{q+1} - v_q\|_{L^1(\mathbb{R}^d)} \leqslant \sum_{q=0} \delta_{q+1}^{1/d_0} \leqslant \epsilon_0.$$

$$\|\rho^i-\rho^i_0\|_{L^1(\mathbb{R}^d)}\leqslant\sum_{q=0}^\infty\|\rho^i_{q+1}-\rho^i_q\|_{L^1(\mathbb{R}^d)}\leqslant 4^{-i}\sum_{q=0}^\infty\delta_{q+1}^{1/d_0'}\leqslant 4^{-i}\frac{c_{in}}{2^{d+10}},$$

$$\overline{\rho}^i\geqslant\rho^{st}+\rho^i_0+\sum_{q=0}^\infty(\rho^i_{q+1}-\rho^i_q)1_{[-\frac12,\frac12]^d}\geqslant 1_{[-\frac12,\frac12]^d}(c_{in}-\frac12 c_{in}-\sum_{q=0}^\infty\delta_{q+1}^{1/d_0'})>0.$$

Hence, by noticing that $\|\rho^i_0-\rho^j_0\|_{L^1}\geqslant|4^{-i}-4^{-j}|\cdot\frac{c_{in}}{2^{d+1}}$, it holds that $\overline{\rho}^i$ is nonnegative, and ρ^i do not coincide with each other. Since $\int_{\mathbb{R}^d}\rho^i_q\,\mathrm{d}x=0$, it follows that $\overline{\rho}^i$ is a probability density.

Since $v\in L^{d_0}$, $\overline{\rho}^i\in L^{d_0'}$, we apply the superposition principle [Tre16] and there exists a probability measure $\mathbf{Q}^i$ on $C([0,\infty);\mathbb{R}^d)$ which is a stationary solution and satisfies $\mathrm{d}\mathbf{Q}^i\circ\Pi_t^{-1}=\overline{\rho}^i\,\mathrm{d}x,\ t\geqslant 0$. Then, a standard result implies that there exists a d-dimensional Brownian motion $W_t,t\geqslant 0$, on a stochastic basis and continuous measurable maps $X^i_t,1\leqslant i\leqslant N$satisfying the corresponding (DD)SDE. □

10.4. **Proof of Proposition 10.4.** We choose the all same parameter as in Section 9.4.1.

First, we also replace (v_q,ρ^i_q,M^i_q) by a space-direction mollified field

$$v_l=v_q*\phi_l,\quad \rho^i_l=\rho^i_q*\phi_l,\ M^i_l=M^i_q*\phi_l,$$

where we recall that ϕ_l is a standard radial mollifiers supported on $[-1,1]^d$. By calculation we obtain that

$$-\mathrm{div}(\beta'(\rho^i_l+\rho^{st})\nabla(\rho^i_l+\rho^{st}))+\mathrm{div}(v_l(\rho^i_l+\rho^{st}))+\mathrm{div}(B(\rho^i_l+\rho^{st}))=-\mathrm{div}(M^i_l+M^i_{com}),$$
$$\mathrm{div}v_l=0,$$

where

$$\begin{aligned}M^i_{com}:=&-v_l(\rho^i_l+\rho^{st})+(v_q(\rho^i_q+\rho^{st}))*\phi_l\\&+\beta'(\rho^i_l+\rho^{st})\nabla(\rho^i_l+\rho^{st})-(\beta'(\rho^i_q+\rho^{st})\nabla(\rho^i_q+\rho^{st}))*_x\phi_l\\&-\beta'(\rho^{st})\nabla\rho^{st}+(\beta'(\rho^{st})\nabla\rho^{st})*_x\phi_l\\&-B(\rho^i_l+\rho^{st})+B(\rho^i_q+\rho^{st})*_x\phi_l+B(\rho^{st})-B(\rho^{st})*_x\phi_l.\end{aligned}$$

Here we used the fact that ρ^{st} is a stationary solution. Since $\operatorname{supp}\rho^i_q,\operatorname{supp}v_q,\operatorname{supp}M^i_q\subset\Omega_q$ by (10.7), we know that $\operatorname{supp}\rho^i_l,\operatorname{supp}v_l,\operatorname{supp}M^i_l\subset\Omega_{q+1}$, and then $\operatorname{supp}M^i_{com}\subset\Omega_{q+1}$.

Then, we define the perturbations

$$w^{(p,i)}_{q+1}:=\sum_{n\geqslant3}\sum_{\xi\in\Lambda^{n,i}}\tilde{\chi}(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}W_{(\xi)},$$
$$w^{(c,i)}_{q+1}:=\sum_{n\geqslant3}\sum_{\xi\in\Lambda^{n,i}}\nabla\Big(\tilde{\chi}(\zeta|M^i_l|-n)\Big)\Big(\frac{n}{\zeta}\Big)^{1/d_0}:V_{(\xi)},$$

and the total perturbation and new iteration

$$w_{q+1}:=\sum_{i=1}^N\Big(w^{(p,i)}_{q+1}+w^{(c,i)}_{q+1}\Big),\quad v_{q+1}:=v_l+w_{q+1}.$$

Here we note that $W_{(\xi)},V_{(\xi)}$ are introduced in Section 9.2, and are seen as a periodic function on $\mathbb{R}^d$. Since $\operatorname{supp}M^i_l\subset\Omega_{q+1}$, we know that $\operatorname{supp}(w^{(p,i)}_{q+1}+w^{(c,i)}_{q+1})\subset\Omega_{q+1}$ and so is v_{q+1}. Here we note that the building blocks $W_{(\xi)}$ and $V_{(\xi)}$ are not globally integrable. However, since the amplitude functions are supported in Ω_{q+1}, all estimates involving the building blocks are restricted to the interval $[-\frac{1}{2},\frac{1}{2}]^d$. On this domain, the relevant norms coincide with those on the torus, and therefore the same estimates remain valid.

We define the perturbations for the stationary densities functions as

$$\begin{aligned}\theta^{(p,i)}_{q+1} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Theta_{(\xi)},\\ \theta^{(c,i)}_{q+1} &:= -\hat\rho^i_q\int_{\mathbb{R}^d}\theta^{(p,i)}_{q+1}\mathrm{d}x.\end{aligned}$$

and then define

$$\theta^i_{q+1}:=\theta^{(p,i)}_{q+1}+\theta^{(c,i)}_{q+1},\quad \rho^i_{q+1}:=\rho^i_l+\theta^i_{q+1}.$$

Here we note that $\Theta_{(\xi)}$ is seen as a periodic function on $\mathbb{R}^d$, $\hat\rho^i_q$ is a probability density with support in Ω_q satisfying $\|\hat\rho^i_q\|_{C^2_x}\lesssim 1$. Since $\operatorname{supp} M^i_l\subset\Omega_{q+1}$, we know that $\operatorname{supp}(\theta^{(p,i)}_{q+1}+\theta^{(c,i)}_{q+1})\subset\Omega_{q+1}$ and so is ρ_{q+1}. By the definition, it is easy to see that $\int_{\mathbb{R}^d}(\theta^{(p,i)}_{q+1}+\theta^{(c,i)}_{q+1})\mathrm{d}x=0$, which implies that $\int_{\mathbb{R}^d}\rho^i_{q+1}\mathrm{d}x=0$.

Now, we need to establish the corresponding estimates on the perturbations. The definition of the perturbations are the same as before, except for $\theta^{(c,i)}_{q+1}$. However, it could be bounded as the same manner, since there is only an extra $\hat\rho_q$. For example,

$$\begin{aligned}\|w^{(p,i)}_{q+1}\|_{L^m(\mathbb{R}^d)} &= \|w^{(p,i)}_{q+1}\|_{L^m([-\frac12,\frac12]^d)}\\ &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C^0_x([-\frac12,\frac12]^d)}\|W_{(\xi)}\|_{L^m(\mathbb{T}^d)}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}},\\ \|w^{(c,i)}_{q+1}\|_{L^m(\mathbb{R}^d)} &= \|w^{(c,i)}_{q+1}\|_{L^m([-\frac12,\frac12]^d)}\\ &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C^1_x([-\frac12,\frac12]^d)}\|V_{(\xi)}\|_{L^m(\mathbb{T}^d)}\lesssim l^{-3d-8}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}}\lambda_{q+1}^{-1},\end{aligned}$$

which are the same as before. All estimates for the perturbations can be obtained by an almost word-for-word repetition of the arguments in Section 9.4: for $A_{i,n,\xi}:=\chi(\zeta|M^i_l|-n)\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}$,

$$\begin{aligned}\|w^i_{q+1}\|_{C^1_x(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\Big\|\tilde\chi(\zeta|M^i_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/d_0}\Big\|_{C^2_x([-\frac12,\frac12]^d)}\|V_{(\xi)}\|_{C^2_x(\mathbb{T}^d)}\lesssim l^{-4d-12}\lambda_{q+1}r_\perp^{-\frac{d-1}{d_0}},\\ \|\theta^{(p,i)}_{q+1}\|_{L^m(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\|A_{i,n,\xi}\|_{C^0_x([-\frac12,\frac12]^d)}\|\Theta_{(\xi)}\|_{L^m(\mathbb{T}^d)}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0'}},\\ \|\theta^{(p,i)}_{q+1}\|_{W^{1,1+\epsilon}(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\|A_{i,n,\xi}\|_{C^1_x([-\frac12,\frac12]^d)}\|\Theta_{(\xi)}\|_{W^{1,1+\epsilon}(\mathbb{T}^d)}\lesssim l^{-4d-12}\lambda_{q+1}r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}},\\ \|\theta^{(p,i)}_{q+1}\|_{C^i_x(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}}\|A_{i,n,\xi}\|_{C^i_x([-\frac12,\frac12]^d)}\|\Theta_{(\xi)}\|_{C^i_x(\mathbb{T}^d)}\lesssim l^{-4d-12}\lambda^i_{q+1}r_\perp^{-\frac{d-1}{d_0'}},\ i=1,2.\end{aligned}$$

Then the desired estimates on the perturbations are obtained by the same calculations.

As for the stress term, by the same calculation as Section 9.3, we could define

$$-M_{q+1}^i := M_{osc}^i + M_{nonlin}^i + M_{lin}^i - M_{com}^i,$$

where

$$\begin{aligned} M_{nonlin}^i : &= -\beta'(\rho_{q+1}^i + \rho^{st})\nabla(\rho_{q+1}^i + \rho^{st}) + \beta'(\rho_l^i + \rho^{st})\nabla(\rho_l^i + \rho^{st}) \\ &\quad + Eb(\rho_{q+1}^i + \rho^{st})(\rho_{q+1}^i + \rho^{st}) - Eb(\rho_l^i + \rho^{st})(\rho_l^i + \rho^{st}), \\ M_{lin}^i : &= v_l\theta_{q+1}^i + w_{q+1}(\rho_l^i + \rho^{st} + \theta_{q+1}^{(c,i)}) + w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} M_{osc}^i &:= \operatorname{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} - M_l^i) \\ &= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \nabla\Big[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big]\mathbb{P}_{\neq 0}(W_{(\xi)}\Theta_{(\xi)}) \\ &\quad + \operatorname{div}\Big(\sum_{n\geqslant 3}\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i\Big). \end{aligned}$$

Using the inverse divergence operator $\mathcal{B}_{\hat{\rho}_q^i}$ in Proposition 10.2, we define the oscillation error $M_{osc}^i := M_{osc,x}^i + M_{osc,c}^i$ by

$$\begin{aligned} M_{osc,x}^i : &= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \mathcal{B}_{\hat{\rho}_q^i}\Big(\nabla\Big[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big], \mathbb{P}_{\neq 0}(W_{(\xi)}\Theta_{(\xi)})\Big), \\ M_{osc,c}^i : &= \sum_{n\geqslant 3}\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i. \end{aligned}$$

Since $\operatorname{supp}\rho_l^i, \operatorname{supp}\theta_{q+1}^i, \operatorname{supp} M_l^i \subset \Omega_{q+1}$, we know that $\operatorname{supp} M_{q+1}^i \subset \Omega_{q+1}$.

We estimate the terms appearing in the definition of M_{q+1}^i separately. For the oscillation error M_{osc}^i, the operator $\mathcal{B}_{\hat{\rho}_q^i}$ satisfies the same estimates as in Theorem 3.2, so the underlying calculations remain unchanged.

$$\begin{aligned} \|M_{osc,x}^i\|_{L^1(\mathbb{R}^d)} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^{n,i}} \Big\|\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big\|_{C_x^2([-\frac12,\frac12]^d)} (r_\perp\lambda_{q+1})^{-1}\|W_{(\xi)}\Theta_{(\xi)}\|_{L^1(\mathbb{T}^d)} \\ &\lesssim l^{-6d-20}(r_\perp\lambda_{q+1})^{-1} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \frac{1}{5}C_0\delta_{q+2}. \end{aligned}$$

The treatment of $M_{osc,c}^i$ and the linear error M_{lin}^i is identical as before.

For the nonlinear error M_{nonlin}^i, by the same calculation as in (5.21), together with (10.1) we have

$$\begin{aligned} \|M_{nonlin}^i\|_{L^1} &\lesssim (\|\beta'\|_{C^1} + \|E\|_{C_x^0([-\frac12,\frac12]^d)}\|b\|_{C^1})\|\theta_{q+1}^i\|_{W^{1,1+\epsilon}}(1 + \|\rho_l^i + \rho^{st}\|_{C_x^1}) \\ &\lesssim (\lambda_q^{d+1} + 1)\lambda_{q+1}^{-2\alpha} \lesssim C_0\lambda_{q+1}^{-\alpha} \leqslant \frac{1}{5}C_0\delta_{q+2}. \end{aligned}$$

For the commutator error M_{com}^i, By the calculation as in (5.22), together with (10.1) we obtain

$$\begin{aligned} \|M_{com}^i\|_{L^1} &\lesssim l\|v_q\|_{C_x^1}(\|\rho_q^i\|_{C_x^1} + \|\rho^{st}\|_{C_x^1}) \\ &\quad + l(\|\beta'\|_{C^1} + \|E\|_{C_x^1([-\frac12,\frac12]^d)}\|b\|_{C^1}) \end{aligned}$$

$$
\begin{aligned}
&\times (1+\|\rho_q^i\|_{C_x^2}+\|\rho^{st}\|_{W^{2,\infty}}^2+\|\rho_q^i\|_{C_x^1}^2)(\|\rho_q^i\|_{C_x^1}+\|\rho^{st}\|_{W^{1,\infty}})\\
&\lesssim C_0 l\lambda_q^{3d+4}\lesssim C_0\lambda_{q+1}^{-\alpha}\leqslant \frac{1}{5}C_0\delta_{q+2}.
\end{aligned}
$$

Then the proof is finished.

10.5. **The main iteration used in the proof of Theorem 1.6.** Let $0<\epsilon_0\leqslant \min\{\frac{1}{8},\frac{1}{2^{d+10}}c_{in}\}$ be given. The iteration is indexed by a parameter $q\in\mathbb{N}_0$. We consider sequences of parameters

$$
\lambda_q=a^{b^q},\ q\geqslant 0,\ \delta_q=(\frac{\epsilon_0}{2})^{d+1}\lambda_1^{2\beta}\lambda_q^{-2\beta},\ q\geqslant 1,\ \delta_0=1,
$$

by (4.1) we obtain $\sum_{q\geqslant 1}\delta_q^{1/(d+1)}<\epsilon_0$.

At each step q, we construct a family $(v_q,\rho_q^i,M_q^i)_{i\in\mathbb{N}}$ solving the system

$$
\begin{aligned}
\partial_t\rho_q^i-\mathrm{div}\big(\beta'(\rho_q^i+\rho^{st})\nabla(\rho_q^i+\rho^{st})\big)+\mathrm{div}(v_q(\rho_q^i+\rho^{st}))+\mathrm{div}(B(\rho_q^i+\rho^{st}))&=-\mathrm{div}M_q^i,\\
\mathrm{div}v_q&=0.
\end{aligned}\tag{10.10}
$$

We recall the notation $T_q:=\frac{1}{3}-\sum_{1\leqslant r\leqslant q}\delta_r^{1/2}$ and initialize the iteration by defining (v_0,ρ_0^i,M_0^i) through

$$
\begin{aligned}
&\rho_0^i(x)=4^{-i}\mathrm{div}F,\quad v_0(x)=0,\\
&M_0^i(x)=-4^{-i}\partial_tF+\beta'(\rho_0^i+\rho^{st})\nabla(\rho_0^i+\rho^{st})-\beta'(\rho^{st})\nabla\rho^{st}-B(\rho_0^i+\rho^{st})+B(\rho^{st}),
\end{aligned}
$$

where $F(t,x)$ is a smooth non-divergence-free bounded $\mathbb{R}^d$-valued function with support in $[\frac{1}{3},\frac{2}{3}]\times\Omega_0$, and satisfying $\|\mathrm{div}F\|_{C^0_{t,x}}=\frac{1}{2}c_{in}$, $\|\mathrm{div}F\|_{C_tL^1}\geqslant\frac{1}{2^{d+1}}c_{in}$. In particular, $\mathrm{supp}\,\rho_0^i,\mathrm{supp}\,v_0,\mathrm{supp}\,M_0^i\subset[T_0,1-T_0]\times\Omega_0$.

By choosing $C_0>0$ sufficiently large we obtain

$$
\begin{aligned}
&\|\rho_0^i\|_{C^2_{t,x}}\leqslant 4^{-i}C_0^{1/d_0'},\quad \|v_0\|_{C^1_{t,x}}\leqslant C_0^{1/d_0},\\
&\|M_0^i\|_{L^1_tL^1}\lesssim(\|\beta'\|_{C^1}+\|E\|_{C^1_x([-\frac{1}{2},\frac{1}{2}]^d)}\|b\|_{C^1})\|\rho_0^i\|_{C^1_{t,x}}\lesssim 4^{-i}\leqslant(\frac{\epsilon_0}{2})^{d+1}4^{-i}C_0.
\end{aligned}\tag{10.11}
$$

With the above assumptions in hand, our main iteration relies on the first step of iteration and reads as follows:

Proposition 10.5. *Under the assumption of Theorem 1.6, there exist $d+1>d_0>2>d_0'>1$ with $\frac{1}{d_0}+\frac{1}{d_0'}=1$ and a choice of parameters a,b,α,β such that the following holds: Let $(v_q,\rho_q^i,M_q^i)_{i\in\mathbb{N}}$ be a solution to the system (10.10) satisfying*

$$
\|v_q\|_{L_t^{d_0}L^{d_0}(\mathbb{R}^d)}\leqslant C_vC_0^{1/d_0}\sum_{m=0}^{q}\delta_m^{1/d_0},\tag{10.12}
$$

for some universal constant $C_v\geqslant 1$, and

$$
\|v_q\|_{C^1_{t,x}(\mathbb{R}^d)}\leqslant C_0^{1/d_0}\lambda_q^{d+4},\ 4^i\|\rho_q^i\|_{C^1_{t,x}(\mathbb{R}^d)}+\lambda_q^{-2}\|\rho_q^i\|_{C^2_{t,x}(\mathbb{R}^d)}\leqslant C_0^{1/d_0'}\lambda_q^{d+4},\tag{10.13}
$$

$$
\|M_q^i\|_{L^1_tL^1(\mathbb{R}^d)}\leqslant C_02^{-i}\delta_{q+1},\tag{10.14}
$$

$$
M_q^i=M_0^i-v_q\rho_0^i,\ \rho_q^i=\rho_0^i,\ \text{for } i>N_q,\tag{10.15}
$$

$$
\mathrm{supp}\,\rho_q^i,\mathrm{supp}\,v_q,\mathrm{supp}\,M_q^i\subset[T_q,1-T_q]\times\Omega_q.\tag{10.16}
$$

Then there exists $(v_{q+1}, \rho^i_{q+1}, M^i_{q+1})_{i\in\mathbb{N}}$ *which solves* (10.10) *and satisfies* (10.12)-(10.16) *at the level* $q+1$ *and*

$$\|v_{q+1}-v_q\|_{L^{d_0}_t L^{d_0}(\mathbb{R}^d)} \leqslant C_v C_0^{1/d_0}\delta_{q+1}^{1/d_0}, \quad \|\rho^i_{q+1}-\rho^i_q\|_{L^{d_0'}_t L^{d_0'}(\mathbb{R}^d)} \leqslant C_v C_0^{1/d_0'} 2^{-i/d_0'}\delta_{q+1}^{1/d_0'}, \tag{10.17}$$

$$\|v_{q+1}-v_q\|_{L^r_t L^p(\mathbb{R}^d)} + \|v_{q+1}-v_q\|_{C_t L^s(\mathbb{R}^d)} \leqslant \delta_{q+1}^{1/d_0}, \tag{10.18}$$

$$\|\rho^i_{q+1}-\rho^i_q\|_{C_t L^1(\mathbb{R}^d)} \leqslant 4^{-i}\delta_{q+1}^{1/d_0'}, \quad \inf_{t\in[0,1]}(\rho^i_{q+1}-\rho^i_q) \geqslant -\delta_{q+1}^{1/d_0'}. \tag{10.19}$$

Proof of Theorem 1.6. We intend to start the iteration from $(v_0, \rho^i_0, M^i_0)_{i\in\mathbb{N}}$ which are defined as above. Next, we use Proposition 10.5 to build inductively $(v_q, \rho^i_q, M^i_q)_{i\in\mathbb{N}}$ for every $q \geqslant 1$. By (10.17)-(10.19), the sequence $\{(v_q, \rho^i_q)_{i\in\mathbb{N}}\}_{q\in\mathbb{N}}$ is Cauchy in $\Big(L^r([0,1];L^p) \cap L^{d_0}([0,1]\times\mathbb{R}^d) \cap C([0,1];L^s)\Big) \times \Big(L^{d_0'}([0,1]\times\mathbb{R}^d) \cap C([0,1];L^1)\Big)^{\mathbb{N}}$ and we denote by (v, ρ^i) its limit. We then define $\overline{\rho}^i := \rho^i + \rho^{st}$.

By (10.17)-(10.19) we have

$$\|\rho^i - \rho^i_0\|_{C_t L^1} \leqslant 4^{-i}\sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'} \leqslant 4^{-i}\frac{c_{in}}{2^{d+10}}, \quad \|v\|_{L^r_t L^p} + \|v\|_{C_t L^s} \leqslant \sum_{q\geqslant 0}\delta_{q+1}^{1/d_0} \leqslant \epsilon_0,$$

$$\inf_{t\in[0,1]}\overline{\rho}^i \geqslant \rho^{st} + \inf_{t\in[0,1]}\rho^i_0 + \sum_{q=0}^{\infty}\inf_{t\in[0,1]}(\rho^i_{q+1}-\rho^i_q) \geqslant (c_{in} - \frac{1}{2}c_{in} - \sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'})1_{[-\frac12,\frac12]} > 0,$$

at which point, together with the fact that $\|\rho^i_0 - \rho^j_0\|_{C_t L^1} \geqslant |4^{-i}-4^{-j}| \cdot \frac{c_{in}}{2^{d+1}}$, we obtain $\overline{\rho}^i \to \rho^{st}$ in $C_t L^1$ as $i\to\infty$, $\overline{\rho}^i$ is a probability density function, and $\overline{\rho}^i$ do not coincide with each other. We finish the proof of the first statement.

For the second statement, Since $|v| \in L^{d_0}_t L^{d_0}, \overline{\rho}^i \in L^{d_0'}_t L^{d_0'}$, and $t \mapsto \overline{\rho}^i(t)$ is continuous on $[0,1]$, we finish the proof using the superposition principle. □

10.6. **Proof of Proposition 10.5.** We choose the all same parameter as in Section 5.

First, we mollify the first N_{q+1} equations by

$$v_l = (v_q *_x \phi_l) *_t \varphi_l, \quad \rho^i_l = (\rho^i_q *_x \phi_l) *_t \varphi_l, \quad M^i_l = (M^i_q *_x \phi_l) *_t \varphi_l,$$

where $\phi_l := \frac{1}{l^d}\phi\big(\frac{\cdot}{l}\big)$ is a family of standard radial mollifiers on $[-1,1]^d$, and $\varphi_l := \frac{1}{l}\varphi\big(\frac{\cdot}{l}\big)$ is a family of standard mollifiers supported in $(0,1)$. By straightforward calculations we obtain

$$\begin{aligned}\partial_t\rho^i_l - \mathrm{div}(\beta'(\rho^i_l+\rho^{st})\nabla(\rho^i_l+\rho^{st})) + \mathrm{div}(v_l(\rho^i_l+\rho^{st})) + \mathrm{div}(B(\rho^i_l+\rho^{st})) &= -\mathrm{div}(M^i_l + M^i_{com}),\\ \mathrm{div} v_l &= 0,\end{aligned}$$

where

$$\begin{aligned} M^i_{com} := &- v_l(\rho^i_l+\rho^{st}) + (v_q(\rho^i_q+\rho^{st})) * \phi_l *_t \varphi_l\\ &+ \beta'(\rho^i_l+\rho^{st})\nabla(\rho^i_l+\rho^{st}) - (\beta'(\rho^i_q+\rho^{st})\nabla(\rho^i_q+\rho^{st})) *_x \phi_l *_t \varphi_l\\ &- \beta'(\rho^{st})\nabla\rho^{st} + (\beta'(\rho^{st})\nabla\rho^{st}) *_x \phi_l *_t \varphi_l\\ &- B(\rho^i_l+\rho^{st}) + B(\rho^i_q+\rho^{st}) *_x \phi_l *_t \varphi_l + B(\rho^{st}) - B(\rho^{st}) *_x \phi_l *_t \varphi_l.\end{aligned}$$

Since $\mathrm{supp}\,\rho^i_q, \mathrm{supp}\,v_q, \mathrm{supp}\,M^i_q \subset [T_q, 1-T_q]\times\Omega_q$, we know that $\mathrm{supp}\,\rho^i_l, \mathrm{supp}\,v_l, \mathrm{supp}\,M^i_l \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$, and then $\mathrm{supp}\,M^i_{com} \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$.

We next define the perturbations for the drift term. For $1 \leqslant i \leqslant N_{q+1}$, let

$$w_{q+1}^{(p,i)} := \sum_{n\geqslant 3} \tilde{\chi}(\zeta^i |M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} \sum_{\xi\in\Lambda^n} W_{(\xi,n,i)} g_{(\xi,i,d_0)},$$

$$w_{q+1}^{(c,i)} := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big(- \tilde{\chi}(\zeta^i |M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} \frac{1}{(n_*\lambda_{q+1})^2}\nabla\Phi_{(\xi,n,i)}\, \xi\cdot\nabla\psi_{(\xi,n,i)}$$
$$+ \nabla\big(\tilde{\chi}(\zeta^i |M_l^i| - n)\big)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0} : V_{(\xi,n,i)}\Big) g_{(\xi,i,d_0)},$$

and the total perturbation, the new velocity field by

$$w_{q+1} := \sum_{i=1}^{N_{q+1}}\big(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}\big), \qquad v_{q+1} := v_l + w_{q+1}.$$

Here we note that $W_{(\xi,n,i)}, V_{(\xi,n,i)}$ are introduced in Section 5.2, and are seen as a periodic function on $\mathbb{R}^d$. Since $\operatorname{supp} M_l^i \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$, we know that $\operatorname{supp}(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}) \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$ and so is v_{q+1}.

We next define the perturbations for the density functions. For $1 \leqslant i \leqslant N_{q+1}$, we set

$$\theta_{q+1}^{(p,i)} := \sum_{n\geqslant 3} \chi(\zeta^i |M_l^i| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'} \sum_{\xi\in\Lambda^n} \Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Theta_{(\xi,n,i)} g_{(\xi,i,d_0')},$$

$$\theta_{q+1}^{(c,i)} := -\hat{\rho}_q^i \int_{\mathbb{R}^d} \theta_{q+1}^{(p,i)}\,\mathrm{d}x,$$

$$\theta_{q+1}^{(o,i)} := -\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi,i,d_0)}\operatorname{div}\Big(\chi(\zeta^i |M_l^i| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big).$$

We now define, for every $1 \leqslant i \leqslant N_{q+1}$,

$$\theta_{q+1}^i := \theta_{q+1}^{(p,i)} + \theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)}, \qquad \rho_{q+1}^i := \rho_l^i + \theta_{q+1}^i.$$

For $i > N_{q+1}$, we simply set $\rho_{q+1}^i := \rho_q^i$.

By construction, $\int_{\mathbb{R}^d}\rho_{q+1}^i\,\mathrm{d}x = 0$ for every $i\in\mathbb{N}$. Since $\operatorname{supp} M_l^i \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$, we know that $\operatorname{supp}(\theta_{q+1}^{(p,i)} + \theta_{q+1}^{(c,i)}) \subset [T_{q+1}, 1-T_{q+1}]\times\Omega_{q+1}$ and so is ρ_{q+1}.

Now we establish the corresponding estimates on the perturbations. The definition of the perturbations are also the same as before, except for $\theta_{q+1}^{(c,i)}$. Here we note that since $\operatorname{supp}_x M_l^i \subset [-\frac{1}{2},\frac{1}{2}]^d$, all estimates are restricted to the domain $[-\frac{1}{2},\frac{1}{2}]^d$. We now collect several key estimates in the sequel:

$$\|w_{q+1}^{(p,i)}\|_{L_t^u L^m(\mathbb{R}^d)} \lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{m}-\frac{1}{d_0}} \eta^{\frac{1}{u}-\frac{1}{d_0}},$$

$$\|w_{q+1}^{(c,i)}\|_{L_t^u L^m(\mathbb{R}^d)} \lesssim l^{-3d-8} r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{m}-\frac{1}{d_0}} \frac{r_\perp}{r_\parallel}\eta^{\frac{1}{u}-\frac{1}{d_0}},$$

$$\|\theta_{q+1}^{(p,i)}\|_{L_t^u L^m(\mathbb{R}^d)} \lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0'}} r_\parallel^{\frac{1}{m}-\frac{1}{d_0'}} \eta^{\frac{1}{u}-\frac{1}{d_0'}},$$

$$\|\theta_{q+1}^{(o,i)}\|_{C_t C^1(\mathbb{R}^d)} \lesssim \sigma^{-1} l^{-6d-20},$$

$$\|\theta_{q+1}^{(p,i)}\|_{L_t^1 W^{1,1+\epsilon}(\mathbb{R}^d)} \lesssim l^{-4d-12}\lambda_{q+1} r_\parallel^{\frac{1}{1+\epsilon}-\frac{1}{d_0'}} r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{d_0'}} \eta^{1-\frac{1}{d_0'}}.$$

Then the desired estimates on the perturbations are obtained by the same calculations.

As for the stress term, by the same calculation as Section 5.4, we could define for $1 \leqslant i \leqslant N_{q+1}$,

$$-M^i_{q+1} := M^i_{osc} + M^i_{nonlin} + M^i_{lin} - M^i_{com},$$

where

$$\begin{aligned} M^i_{nonlin} : &= -\beta'(\rho^i_{q+1} + \rho^{st})\nabla(\rho^i_{q+1} + \rho^{st}) + \beta'(\rho^i_l + \rho^{st})\nabla(\rho^i_l + \rho^{st}) \\ &\quad - Eb(\rho^i_{q+1} + \rho^{st})(\rho^i_{q+1} + \rho^{st}) + Eb(\rho^i_l + \rho^{st})(\rho^i_l + \rho^{st}), \\ M^i_{lin} : &= v_l\theta^i_{q+1} + w_{q+1}(\rho^i_l + \rho^{st} + \theta^{(c,i)}_{q+1} + \theta^{(o,i)}_{q+1}) + w^{(c,i)}_{q+1}\theta^{(p,i)}_{q+1}. \end{aligned}$$

To define the oscillation error, we define $\mathbb{P}_{\neq 0,\hat{\rho}^i_q} f := f - \hat{\rho}^i_q \cdot \int_{\mathbb{R}^d} f \mathrm{d}x$. In particular, for any mean-zero function f, we have $\mathbb{P}_{\neq 0,\hat{\rho}^i_q} f = f$. Then, we have

$$\begin{aligned} M^i_{osc} :=& \partial_t\theta^i_{q+1} + \operatorname{div}(w^{(p,i)}_{q+1}\theta^{(p,i)}_{q+1} - M^i_l) \\ =& \mathbb{P}_{\neq 0,\hat{\rho}^i_q}(\partial_t\theta^{(p,i)}_{q+1} + \operatorname{div}(w^{(p,i)}_{q+1}\theta^{(p,i)}_{q+1} - M^i_l) + \partial_t\theta^{(o,i)}_{q+1}) \\ =& \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathbb{P}_{\neq 0,\hat{\rho}^i_q}\Big(\partial_t\Big[\chi(\zeta^i|M^i_l| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)g_{(\xi,i,d_0')}\Big]\Theta_{(\xi,n,i)}\Big) \\ &+ \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathbb{P}_{\neq 0,\hat{\rho}^i_q}\Big(\nabla\Big[\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\Big]g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}\mathbb{P}_{\neq 0}\big(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\big)\Big) \\ &+ \operatorname{div}\Big(\sum_{n\geqslant 3}\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\frac{M^i_l}{|M^i_l|} - M^i_l\Big) \\ &- \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi,i,d_0)}\partial_t\operatorname{div}\Big(\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\Big). \end{aligned}$$

Now, we apply the Bogovskii operators $\mathcal{B}_{\hat{\rho}^i_q}, \mathcal{R}_{\hat{\rho}^i_q}$ in Section 10.1 to define $M^i_{osc} := M^i_{osc,t} + M^i_{osc,x} + M^i_{osc,c} + M^i_{osc,o}$ as

$$\begin{aligned} M^i_{osc,t} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathcal{B}_{\hat{\rho}^i_q}\Big(\partial_t[\chi(\zeta^i|M^i_l| - n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)g_{(\xi,i,d_0')}]\Theta_{(\xi,n,i)}\Big), \\ M^i_{osc,x} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathcal{R}_{\hat{\rho}^i_q}\Big(\nabla[\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)], \mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})\Big)g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}, \\ M^i_{osc,c} &:= \sum_{n\geqslant 3}\chi(\zeta^i|M^i_l| - n)\frac{n}{\zeta^i}\frac{M^i_l}{|M^i_l|} - M^i_l, \\ M^i_{osc,o} &:= -\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi,i,d_0)}\partial_t\Big(\chi(\zeta|M^i_l| - n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\Big). \end{aligned}$$

Since $\operatorname{supp}\rho^i_l, \operatorname{supp}\theta^i_{q+1}, \operatorname{supp} M^i_l \subset [T_{q+1}, 1 - T_{q+1}] \times \Omega_{q+1}$, we know that $\operatorname{supp} M^i_{q+1} \subset [T_{q+1}, 1 - T_{q+1}] \times \Omega_{q+1}$.

We now estimate each term in the definition of M^i_{q+1} separately for $1 \leqslant i \leqslant N_{q+1}$.

For the oscillation error $M^i_{osc,t}$, the operator $\mathcal{R}_{\hat{\rho}^i_q}$ satisfies the same estimates as in Theorem 3.2, so the underlying calculations remain unchanged.

$$\begin{aligned}\|M^i_{osc,t}\|_{L^1_tL^1} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta^i|M^i_l|-n)\Big(\frac{n}{\zeta^i}\Big)^{1/d_0'}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\right\|_{C^1_{t,x}}\|g_{(\xi,i,d_0')}\|_{W^{1,1}_t}\|\Theta_{(\xi,n,i)}\|_{C_tL^1}\\ &\lesssim l^{-4d-12}r_\perp^{d-1-\frac{d-1}{d_0'}}r_\parallel^{1-\frac{1}{d_0'}}\sigma\eta^{-\frac{1}{d_0'}}\lesssim 2^{-i}\lambda_{q+1}^{-\alpha}.\end{aligned}$$

By (A.9) and the bounds above we have

$$\|M^i_{osc,o}\|_{L^1_tL^1}\lesssim\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta|M^i_l|-n)\frac{n}{\zeta^i}\Gamma_\xi\Big(\frac{M^i_l}{|M^i_l|}\Big)\xi\right\|_{C^1_{t,x}}\lesssim\sigma^{-1}l^{-4d-12}\lesssim 2^{-i}\lambda_{q+1}^{-\alpha}.$$

For the nonlinear error M^i_{nonlin}, similar to (5.21), we have

$$\begin{aligned}\|M^i_{nonlin}\|_{L^1_tL^1}&\lesssim(\|\beta'\|_{C^1}+\|E\|_{C^0_x([-\frac12,\frac12]^d)}\|b\|_{C^1})\\ &\qquad\times(\|\theta^i_{q+1}\|_{C_tL^{1+\epsilon}}+\|\theta^i_{q+1}\|_{L^1_tW^{1,1+\epsilon}})(1+\|\rho^i_q+\rho^{st}\|_{C^1_{t,x}})\\ &\lesssim(\lambda_q^{d+4}+1)2^{-i}\lambda_{q+1}^{-2\alpha}\lesssim 2^{-i}C_0\lambda_{q+1}^{-\alpha}\leqslant\frac15 C_0 2^{-i}\delta_{q+2}.\end{aligned}$$

For the commutator error M^i_{com}, we obtain

$$\begin{aligned}\|M^i_{com}\|_{L^1_tL^1}&\lesssim l\|v_q\|_{C^1_x}(\|\rho^i_q\|_{C^1_{t,x}}+\|\rho^{st}\|_{C^1_x})\\ &\quad+l(\|\beta'\|_{C^1}+\|E\|_{C^1_{t,x}([-\frac12,\frac12]^d)}\|b\|_{C^1})\\ &\qquad\times(1+\|\rho^i_q\|_{C^2_{t,x}}+\|\rho^i_q\|^2_{C^1_{t,x}}+\|\rho^{st}\|^2_{W^{2,\infty}})(\|\rho^i_q\|_{C^1_{t,x}}+\|\rho^{st}\|_{W^{1,\infty}})\\ &\lesssim C_0l\lambda_q^{3d+12}\lesssim 2^{-i}C_0\lambda_{q+1}^{-\alpha/2}\leqslant\frac15 C_0 2^{-i}\delta_{q+2}.\end{aligned}$$

For the case $i>N_{q+1}$, since $\mathrm{supp}_x\, v_{q+1},\mathrm{supp}_x\, v_q\subset[-\frac12,\frac12]^d$ and $\rho^{st}=g^{-1}(-\Phi(x)+\mu^{st})$ is constant on $[-\frac12,\frac12]^d$, we obtain that $\mathrm{div}((v_{q+1}-v_q)\rho^{st})=0$. Together with $\rho^i_{q+1}=\rho^i_q=\rho^i_0$, we define

$$M^i_{q+1}:=M^i_q-(v_{q+1}-v_q)\rho^i_0=M^i_0-v_{q+1}\rho^i_0.$$

Since $\mathrm{supp}\,\rho^i_0,\mathrm{supp}\,M^i_0\subset[T_{q+1},1-T_{q+1}]\times\Omega_{q+1}$, we know that $\mathrm{supp}\,M^i_{q+1}\subset[T_{q+1},1-T_{q+1}]\times\Omega_{q+1}$.

In the end, by (10.11) and (10.12) we have

$$\|M^i_{q+1}\|_{L^1_tL^1}\leqslant\|M^i_0\|_{L^1_tL^1}+\|v_{q+1}\|_{L^{d_0}_tL^{d_0}}\|\rho^i_0\|_{L^{d_0'}_tL^{d_0'}}\lesssim C_0C_v4^{-i}\leqslant C_02^{-i}\delta_{q+2}.$$

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Appendix A. L^{d_0}-based intermittent spatial-time jets

In this section we recall the L^{d_0}-based intermittent spatial-time jets as presented in [LRZ25, Appendix C.1].

First we introduce the following geometrical lemma:

Lemma A.1. [BCDL21, Lemma 3.1] *Let $d \geqslant 2$. There exists a finite set $\Lambda \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ and non-negative C^∞-function $\Gamma_\xi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that for every $R \in \mathbb{S}^{d-1}$*

$$R = \sum_{\xi \in \Lambda} \Gamma_\xi(R)\xi.$$

With Lemma A.1 in hand, it is easy to generate 2 disjoint families Λ^1, Λ^2, where each one enjoys the property of Lemma A.1 by taking suitable rational rotations of one fixed set. For simplicity, we denote $\Lambda := \Lambda^1 \cup \Lambda^2$. Moreover, we know that $\{\Gamma_\xi\}_{\xi \in \Lambda}$ are uniformly bounded.

For parameters $\lambda, r_\perp, r_\parallel > 0$, we assume

$$\lambda^{-1} \ll r_\perp \ll r_\parallel \ll 1, \quad \lambda r_\perp \in \mathbb{N}.$$

For each $\xi \in \Lambda$ let us define $A^i_\xi \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d,\ i = 1, 2, ..., d-1$, such that $\{\xi, A^i_\xi, i = 1, ..., d-1\}$ form an orthonormal basis in $\mathbb{R}^d$. Let $n_* \in \mathbb{N}$ such that$\{n_*\xi, n_*A^i_\xi, i = 1, ..., d-1\} \subset \mathbb{Z}^d$ for every $\xi \in \Lambda$.

We define $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a smooth function with support in a ball of radius 1, $\phi \equiv 1$ on $B(0, \frac{1}{3})$ and mean-zero. We define Φ such that $\phi = -\Delta\Phi$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, mean-zero function with support in $B(0, 1)$ satisfying $\psi \equiv 1$ on $B(0, \frac{1}{3})$.

Define $\phi' : \mathbb{R}^{d-1} \to \mathbb{R}$ to be a smooth non-negative function with support in $B(0, \frac{1}{3})$ satisfying

$$\int_{\mathbb{R}^{d-1}} \phi'(x_1, x_2, ..., x_{d-1}) \mathrm{d}x_1 \mathrm{d}x_2..\mathrm{d}x_{d-1} = 1,$$

and let $\psi' : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with support in $B(0, \frac{1}{3})$ such that

$$\int_{\mathbb{R}} \psi'(x_d) \mathrm{d}x_d = 1.$$

Let $d_0 \geqslant 1$ be fixed, we define the rescaled cut-off functions

$$\phi_{r_\perp, d_0}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/d_0}} \phi(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}),$$

$$\Phi_{r_\perp, d_0}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/d_0}} \Phi(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}),$$

$$\psi_{r_\parallel, d_0}(x_d) = \frac{1}{r_\parallel^{1/d_0}} \psi(\frac{x_d}{r_\parallel}).$$

Similarly, for a conjugate exponent $d_0' \in [1, \infty]$, we define $\phi'_{r_\perp, d_0'}$ and $\psi'_{r_\parallel, d_0'}$ as the same manner. Then we periodize $\phi_{r_\perp, v}$, $\Phi_{r_\perp, d_0}$, $\psi_{r_\parallel, d_0}$, $\phi'_{r_\perp, d_0'}$ and $\psi'_{r_\parallel, d_0'}$ so that they can be viewed as functions on $\mathbb{T}^{d-1}$ and $\mathbb{T}$ respectively. Consider a large time oscillation parameter $\mu = r_\perp^{-\frac{d-1}{d_0}} r_\parallel^{-\frac{1}{d_0}} > 0$. For every $\xi \in \Lambda$ we introduce

$$\psi_{(\xi, d_0)}(t, x) := \psi_{r_\parallel, d_0}(n_* r_\perp \lambda(x \cdot \xi - \mu t)),$$

$$\Phi_{(\xi, d_0)}(x) := \Phi_{r_\perp, d_0}(n_* r_\perp \lambda x \cdot A^1_\xi, ..., n_* r_\perp \lambda x \cdot A^{d-1}_\xi),$$

$$\phi_{(\xi, d_0)}(x) := \phi_{r_\perp, d_0}(n_* r_\perp \lambda x \cdot A^1_\xi, ..., n_* r_\perp \lambda x \cdot A^{d-1}_\xi).$$

Similarly we define $\phi'_{(\xi, d_0')}$ and $\psi'_{(\xi, d_0')}$.

The building blocks $W_{(\xi,d_0)}:\mathbb{R}\times\mathbb{T}^d\to\mathbb{R}^d, \Theta_{(\xi,d_0')}:\mathbb{R}\times\mathbb{T}^d\to\mathbb{R}$ are defined as

$$W_{(\xi,d_0)}(t,x):=\xi\psi_{(\xi,d_0)}(t,x)\phi_{(\xi,d_0)}(x),$$
$$\Theta_{(\xi,d_0')}(t,x):=\psi'_{(\xi,d_0')}(t,x)\phi'_{(\xi,d_0')}(x).$$

By the definition and the choice of μ we have that

$$\int_{\mathbb{T}^d}W_{(\xi,d_0)}\Theta_{(\xi,d_0')}\mathrm{d}x=\xi, \tag{A.1}$$
$$\partial_t\Theta_{(\xi,d_0')}+\mathrm{div}(W_{(\xi,d_0)}\Theta_{(\xi,d_0')})=0. \tag{A.2}$$

Since $W_{(\xi,d_0)}$ is not divergence-free, we introduce the skew-symmetric corrector term

$$V_{(\xi,d_0)}:=\frac{1}{(n_*\lambda)^2}(\xi\otimes\nabla\Phi_{(\xi,d_0)}-\nabla\Phi_{(\xi,d_0)}\otimes\xi)\psi_{(\xi,d_0)}$$

satisfying

$$\mathrm{div}V_{(\xi,d_0)}=W_{(\xi,d_0)}-\frac{1}{(n_*\lambda)^2}\nabla\Phi_{(\xi,d_0)}\xi\cdot\nabla\psi_{(\xi,d_0)}. \tag{A.3}$$

Finally, we obtain that for $N,M\geqslant 0$ and $p\in[1,\infty]$ the following holds

$$\|\nabla^N\partial_t^M\psi_{(\xi,d_0)}\|_{C_tL^p}\lesssim r_{\|}^{\frac{1}{p}-\frac{1}{d_0}}(\frac{r_\perp\lambda}{r_{\|}})^N(\frac{r_\perp\lambda\mu}{r_{\|}})^M, \tag{A.4}$$

$$\|\nabla^N\phi_{(\xi,d_0)}\|_{L^p}+\|\nabla^N\Phi_{(\xi,d_0)}\|_{L^p}\lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}}\lambda^N, \tag{A.5}$$

$$\|\nabla^N\partial_t^M W_{(\xi,d_0)}\|_{C_tL^p}+\lambda\|\nabla^N\partial_t^M V_{(\xi,d_0)}\|_{C_tL^p}\lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}}r_{\|}^{\frac{1}{p}-\frac{1}{d_0}}\lambda^N(\frac{r_\perp\lambda\mu}{r_{\|}})^M, \tag{A.6}$$

$$\|\nabla^N\partial_t^M\Theta_{(\xi,d_0')}\|_{C_tL^p}\lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0'}}r_{\|}^{\frac{1}{p}-\frac{1}{d_0'}}\lambda^N(\frac{r_\perp\lambda\mu}{r_{\|}})^M, \tag{A.7}$$

where the implicit constants may depend on p,N and M, but are independent of $\lambda,r_\perp,r_{\|},\mu$. These estimates can be easily deduced from the definitions.

Then let us introduce a family of temporal functions to oscillate the building blocks intermittently in time. Let $K\in\mathbb{N}$ be fixed, and $G\in C_c^\infty(0,1)$ be non-negative and $\int_0^1 G^2(t)\mathrm{d}t=1$. Let $\eta>0$ be a small constant satisfying $\eta K\ll 1$. For $\xi\in\Lambda$ as defined above, and $1\leqslant i\leqslant K$, we define $\tilde{g}_{(\xi,i,d_0)}:\mathbb{T}\to\mathbb{R}$ as the 1-periodic extension of $\eta^{-1/d_0}G(\frac{t-t_{(\xi,i)}}{\eta})$, where $t_{(\xi,i)}$ are chosen so that $\tilde{g}_{(\xi,i,d_0)}$ have disjoint supports for distinct (ξ,i). We will also oscillate the perturbations at a large frequency $\sigma\in\mathbb{N}$. So, we define

$$g_{(\xi,i,d_0)}(t)=\tilde{g}_{(\xi,i,d_0)}(\sigma t).$$

Similarly we define $\tilde{g}_{(\xi,i,d_0')}$ and $g_{(\xi,i,d_0')}$.

For the corrector term we define $H_{(\xi,i,d_0)},h_{(\xi,i,d_0)}:\mathbb{T}\to\mathbb{R}$ by

$$H_{(\xi,i,d_0)}(t)=\int_0^t g_{(\xi,i,d_0)}(s)\mathrm{d}s,\quad h_{(\xi,i,d_0)}(t)=\int_0^{\sigma t}(\tilde{g}_{(\xi,i,d_0)}(s)\tilde{g}_{(\xi,i,d_0')}(s)-1)\mathrm{d}s, \tag{A.8}$$

where we recall that $\frac{1}{d_0'}+\frac{1}{d_0}=1$, In view of the zero-mean condition for $\tilde{g}_{(\xi,i,d_0)}(t)\tilde{g}_{(\xi,i,d_0')}(t)-1$, we see that $h_{(\xi,i,d_0)}$ is $\mathbb{T}/\sigma$-periodic, and for any $N\geqslant 0,p\geqslant 1$

$$\|g_{(\xi,i,d_0)}\|_{W^{N,p}}\lesssim(\frac{\sigma}{\eta})^N\eta^{1/p-1/d_0},\quad \|h_{(\xi,i,d_0)}\|_{L^\infty}\leqslant 1, \tag{A.9}$$

where the universal constant is independent of the choices of i and ξ.

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