# Probabilistic representation of solutions to the parabolic $p$-Laplace equation

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**Abstract**

This work is concerned with the probabilistic representation of solutions to the $p$-Laplace evolution equation $\frac{\partial u}{\partial t} = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in $(0,\infty) \times \mathbb{R}^d$, $u(0,x) = u_0(x)$, $x \in \mathbb{R}^d$. One proves that, if $p \geq 4$, and if $u_0$ is a probability density with compact support and $u_0 \in L^2$, $|\nabla u_0| \in L^\infty$, then $u$ can be represented as $u(t,x)dx = \mathcal{L}_{X(t)}(dx)$, where $\mathcal{L}_{X(t)}$ denotes the time marginal law of $X$ at time $t$ with $X$ being a probabilistically weak solution to a corresponding McKean–Vlasov stochastic differential equation. This result is based on a new second order global regularity result for the weak solutions to the parabolic $p$-Laplace equation.

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**Keywords:** Fokker–Planck equation, stochastic, semigroup, McKean–Vlasov equation, superposition principle.

## 1 Introduction

Consider herein the nonlinear parabolic Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \mathrm{div}(|\nabla u(t,x)|^{p-2}\nabla u(t,x)), && t > 0,\ x \in \mathbb{R}^d, \\ &u(0,x) = u_0(x), && x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $d \geq 1$ and $1 < p < \infty$.

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Following [5] (see, also, [6]), we associate with (1.1), rewritten as the Fokker–Planck equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta(|\nabla u|^{p-2}u) + \operatorname{div}(\nabla(|\nabla u|^{p-2})u) = 0, \quad t > 0,\ x \in \mathbb{R}^d, \\ &u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{R}^d, \end{aligned} \tag{1.2}$$

where $u_0$ is a probability density, the McKean-Vlasov SDE

$$\begin{aligned} dX(t) &= \nabla(|\nabla u(t,X(t))|^{p-2})dt + \sqrt{2}|\nabla u(t,X(t))|^{\frac{p-2}{2}}dW(t),\ t > 0, \\ \mathcal{L}_{X(t)} &= u(t,x)dx,\ t \geq 0, \end{aligned} \tag{1.3}$$

where $W(t)$ is a (standard) $d$-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We recall that $u \in L^1_{\rm loc}(0,\infty; W^{1,1}_{\rm loc}(\mathbb{R}^d))$ is a *distributional solution* to (1.2) if

$$\begin{aligned} |\nabla u|^{p-2} \in L^1_{\rm loc}((0,\infty); W^{1,1}_{\rm loc}(\mathbb{R}^d)),\ |\nabla u|^{p-2}u \in L^1_{\rm loc}(0,\infty;\mathbb{R}^d), \\ \nabla(|\nabla u|^{p-2})u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d;\mathbb{R}^d), \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} u \left( \frac{\partial \varphi}{\partial t} + |\nabla u|^{p-2}\Delta\varphi + \nabla(|\nabla u|^{p-2})\cdot\nabla\varphi \right) dxdt \\ + \int_{\mathbb{R}^d} \varphi(0,x)u_0(x)dx = 0, \end{aligned} \tag{1.5}$$

for all $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^d)$.

A *probabilistically weak solution* to (1.3) is a triple consisting of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a continuous $(\mathcal{F}_t)$-adapted $\mathbb{R}^d$-valued process $X = (X(t))_{t\geq 0}$ and an $(\mathcal{F}_t)$-Brownian motion $W = (W(t))_{t\geq 0}$ such that

$$\mathbb{E}\left[\int_0^T (|\nabla(|\nabla u(t,X(t))|^{p-2}) + |\nabla u(t,X(t))|^{p-2})dt\right] < \infty,\ \forall T > 0, \tag{1.6}$$

and $\mathbb{P}$-a.s.

$$X(t) = X_0 + \int_0^t \nabla(|\nabla u(s,X(s))|^{p-2})ds + \sqrt{2}\int_0^t |\nabla u(s,X(s))|^{\frac{p-2}{2}}dW(t), \quad \forall t \geq 0.$$

By [3, Section 2] (see also [4, Chapter 5]), for any $\mathcal{P}$ (= all probability measures on $\mathbb{R}^d$)-valued weakly continuous distributional solution $u$ to (1.2) satisfying the following global regularity result

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla(|\nabla u(t,x)|^{p-2} + |\nabla u(t,x)|^{p-2})u(t,x))dxdt < \infty, \ \forall t \geq 0, \quad (1.7)$$

(which is also a necessary requirement in view or (1.6)), there exists a (probabilistically) weak solution $X = (X(t))_{t\geq 0}$ to SDE (1.3) such that $\mathcal{L}_{X(t)}(dx) = u(t,x)dx$, $t \geq 0$. This means that we obtain a probabilistic representation of a solution $u$ to the parabolic $p$-Laplace equation, rewritten as (1.2), in the sense that $u$ is a time marginal law density of the solution $X$ to SDE (1.3). In [5], such a result was proved for the Barenblatt solutions to equation (1.1) and extended to Leibenson's equation in [6]. In these cases, the corresponding solution $X$ to (1.3) is even a probabilistically strong solution and the path laws of the solutions to (1.3) form a nonlinear Markov process in the sense of McKean. The extension of these probabilistic representation results to a more general class of distributional solutions to equation (1.1) is a challenging objective and this work is a step of this program. More precisely, we prove the global regularity result in Theorem 3.1 below, which is in fact stronger than (1.7), for bounded initial probability densities. Then, as just mentioned, by applying [3, Section 2] which, in turn, is based on the (linear) *superposition principle* from [16, Theorem 2.5], we obtain a weak solution to (1.3). This is formulated and proved in Section 4 (see Theorem 4.1) for $p \geq 4$ if $u(0) = u_0$ has compact support $u_0 \in L^2(\mathbb{R}^d)$, $|\nabla u_0| \in L^\infty(\mathbb{R}^d)$, and is a probability density. For this purpose, in Sections 2 and 3 we prove some sharp global second order estimates for weak solutions to equation (1.1) which to the best of our knowledge are new. We refer at this point to a related local estimate in the paper [12]. However, it is in time only for compact intervals in $(0,T)$ which is not sufficient for our purpose (see Remark 3.3).

Finally, we would like to emphasize that probabilistic representations of solutions to nonlinear parabolic equations of the type as in this paper was first suggested in McKean's seminal paper [15]. So, we realize his vision in this paper for the parabolic $p$-Laplace equation for a large class of initial data.

As already anticipated by McKean, having a solution $X = (X(t))_{t\geq 0}$ to the SDE (1.3) bears valuable information for its time marginal law $u$, i.e., the solution to (1.2), since one has all methods from stochastic analysis at hand to analyze $X$, as, e.g., applying Itô's formula, (semi) martingale theory, stochastic solutions to Dirichlet problems and, in particular, Malliavin calculus to prove further regularity of $u$. We refer to the recent work [7], where some parts of such methods have already been exploited for the generalized porous media equation. To do the same for the parabolic $p$-Laplace equation

(1.2) is our subject in a forthcoming work.

**Notations.** For $1 \le p \le \infty$, $L^p(\mathbb{R}^d)$ (also denoted $L^p$) stands for the usual real-valued $L^p$-spaces on $\mathbb{R}^d$ with the norm $|\cdot|_p$. For an open set $B \subset \mathbb{R}^d$, $W^{1,p}(B)$ is the Sobolev space $\{u \in L^p(B);\, D_i u \in L^p(B),\ i = 1,...,d\}$ with the standard norm $\|\cdot\|_{W^{1,p}(B)}$, where $D_i u = \frac{\partial}{\partial x_i} u$ is the distributional derivative of $u$. We denote by $W^{-1,p'}(B)$, $\frac{1}{p'} = 1 - \frac{1}{p}$, the dual space of $W^{1,p}(B)$. We set $W^{1,p} = W^{1,p}(\mathbb{R}^d)$, denote by $W^{1,p}_{\text{loc}}$ the corresponding local space and by $W^{2,p}$ the space $\{u \in L^p;\ D_j u \in L^p,\ D_i D_j u \in L^p;\ i,j = 1,...,d\}$. By $C_0^\infty((0,\infty) \times \mathbb{R}^d)$ we denote the space of infinitely differentiable real valued functions with compact support in $(0,T) \times \mathbb{R}^d$. By $\mathcal{D}'(\mathbb{R}^d)$, respectively $\mathcal{D}'((0,T) \times \mathbb{R}^d)$ we denote the space of Schwartz distributions on $\mathbb{R}^d$ and $(0,T) \times \mathbb{R}^d$, respectively. We use also the notations

$$W^{1,2}(B) = H^1(B),\ W^{1,2}_{\text{loc}} = H^1_{\text{loc}},\ W^{2,2} = H^2,\ W^{-1,2} = H^{-1}.$$

Given a Banach space $H$ we denote by $L^p(0,T;H)$, $0 < T \le \infty$, the space of Bochner measurable $p$-integrable functions $u : (0,T) \to H$. By $C([0,T];H)$ we denote the space of $H$-valued continuous functions $u : [0,T] \to H$. By $W^{1,p}([0,T];H)$ we denote the Sobolev space

$$\left\{ u \in L^p(0,T;H); \frac{du}{dt} \in L^p(0,T;H) \right\},$$

where $\frac{du}{dt}$ is the $H$-valued distributional derivative of $u$ (see, e.g., [2], p.23). We recall that each $u \in W^{1,p}([0,T];H)$ is absolutely continuous on $[0,T]$ and $\frac{du}{dt}(t)$ exists, a.e. on $(0,T)$. Denote by $\mathcal{D}'(0,T;H^{-1})$ the space of $H^{-1}$-valued distributions on $(0,T)$. By $\mathcal{M}_b$ we denote the space of bounded-Radon measures on $\mathbb{R}^d$. For each $0 < T \le \infty$, $L^p((0,T) \times \mathbb{R}^d)$ is the standard space of Lebesgue $p$-integrable real valued functions on $(0,T) \times \mathbb{R}^d$. The scalar product of $L^2$ is denoted by $(\cdot,\cdot)_2$.

# 2 General existence theory for the Cauchy problem (1.1)

Herein we shall briefly recall some standard existence results for the weak solutions to equation (1.1). Problem (1.1) can be treated as an infinite dimensional Cauchy problem in the space $H = L^2$, namely,

$$\begin{aligned} &\frac{du}{dt}(t) + Au(t) = 0,\ t > 0, \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

where the operator $A : D(A) \subset H \to H$ is defined by

$$\begin{aligned} Au &= -\text{div}(|\nabla u|^{p-2}\nabla u),\ \forall u \in D(A), \\ D(A) &= \{u \in L^2; \nabla u \in L^p, \text{div}(|\nabla u|^{p-2}\nabla u) \in L^2\}. \end{aligned} \tag{2.2}$$

Here, $\nabla$ and div are taken in the sense of Schwartz distributions on $\mathbb{R}^d$ and $\frac{du}{dt}$ is the distributional derivative of the function $u : [0, \infty) \to H$. More exactly, we look for solutions $u$ in the Sobolev space

$$W^{1,2}_{\text{loc}}([0, \infty); H) = \{u \in W^{1,2}([0, T]; H),\ \forall T > 0\},$$

which satisfy (1.2) in the weak or variational sense. To get the main existence result we shall prove first the proposition below.

**Proposition 2.1.** *The operator $A$ is m-accretive (maximal monotone) in $H \times H$ and*

$$A = \partial\Phi, \tag{2.3}$$

*where $\Phi : H \to ]-\infty, +\infty]$ is the function*

$$\Phi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathbb{R}^d} |\nabla u(x)|^p dx & \text{if } |\nabla u| \in L^p, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.4}$$

*which is convex and lower semicontinuous.*

*Herein, $\partial\Phi : H \to H$ is the subdifferential of the function $\Phi$, that is,*

$$\partial\Phi(u) = \{w \in H;\ \Phi(u) \le \Phi(v) + (w, u - v)_2,\ \forall v \in H\}. \tag{2.5}$$

This function is called the *potential* of the operator $A$.

*Proof.* By (2.2) we see that

$$(Au, u - v)_2 = \int_{\mathbb{R}^d} |\nabla u|^{p-2}\nabla u \cdot (\nabla u - \nabla v)dx,\ \forall u, v \in D(A), \tag{2.6}$$

and, therefore, $A$ is monotone (accretive), that is,

$$(Au - Av, u - v)_2 \ge 0,\ \forall u, v \in D(A). \tag{2.7}$$

We also have, by (2.4)–(2.6),

$$(Au, u - v) \geq \Phi(u) - \Phi(v), \ \forall u \in D(A), \ v \in D(\Phi) = \{v \in H; |\nabla v| \in L^p\}.$$

Hence,

$$Au \in \partial\Phi(u), \ \forall u \in D(A). \tag{2.8}$$

To complete the proof, it remains to be shown that $R(I + A) = H$. To this end, we fix $f \in H$ and consider the equation

$$u + Au = f. \tag{2.9}$$

Let $V = \{u \in L^2; \nabla u \in L^p\}$ with the norm

$$\|u\|_V = |u|_2 + |\nabla u|_p$$

and the dual $V'$. Clearly, we have

$$V \subset H \subset V'$$

algebraically and topologically.

Next, we consider the operator $A_V : V \to V'$ defined by

$${}_{V'}\langle A_V(u), v\rangle_V = \int_{\mathbb{R}^d} |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx, \ \forall u, v \in V,$$

and note that it is monotone, that is,

$${}_{V'}\langle A_V(u) - A_V(\bar{u}), u - \bar{u}\rangle_V \geq 0, \ \forall u, \bar{v} \in V,$$

and demicontinuous, that is strongly-weakly continuous. Then, by the Minty–Browder theorem (see, e.g., [2], p. 36), the operator $A_V$ is maximal monotone in $V \times V'$. Moreover, the operator $\widetilde{A}_V(u) = u + A_V(u)$ is coercive in $V \times V'$, because

$${}_{V'}(\widetilde{A}_V(u), u)_V = |u|_2^2 + |\nabla u|_p^p, \ \forall u \in V.$$

Since $\widetilde{A}_V$ is also maximal monotone in $V \times V'$, it follows that the range $R(I + \widetilde{A}_V)$ of $I + \widetilde{A}_V$ is all of $V'$ and, in particular, $H \subset R(I + \widetilde{A}_V)$, where $I$ is the identity operator in $H$. We have

$$Au = A_V(u), \ \forall u \in D(A) = \{u \in V; \ A_V(u) \in H\}.$$

Hence, $R(I + A) = H$ and so (2.9) has a solution $u \in D(A)$, as claimed.

Since $\Phi : H \to ]-\infty, +\infty]$ is convex and lower-semicontinuous, its subdifferential $\partial\Phi : H \to H$ is maximal monotone. As shown above, $A$ is maximal monotone and so, by (2.8), it follows that $A = \partial\Phi$, as claimed. □

By Proposition 2.1 and the general existence theory for the Cauchy problem in the Hilbert space $H$ associated with subgradient operators $A = \partial\Phi$, we have (see, e.g., [2], pp. 143 and 158),

**Proposition 2.2.** *Let $u_0 \in \overline{D(A)}$ (the closure of $D(A)$ in $H$). Then, for each $T > 0$ there is a unique function*

$$u \in C([0,T];H) \cap \bigcap_{0<\delta<T} W^{1,2}([\delta,T];H)$$

*such that $u(t) \in D(A)$, a.e. $t \in (0,T)$, and*

$$t^{\frac{1}{2}} \frac{du}{dt}, t^{\frac{1}{2}} Au \in L^2(0,T;H);\ \ \Phi(u) \in L^1(0,T) \tag{2.10}$$

$$\frac{du(t)}{dt} + Au(t) = 0,\ \ a.e.\ t \in (0,T);\ u(0) = u_0. \tag{2.11}$$

*If $\Phi(u_0) < \infty$, then*

$$\frac{du}{dt} \in L^2(0,T;H), \tag{2.12}$$

$$\Phi(u(t)) \le \Phi(u_0),\ \ \forall t \in [0,T]. \tag{2.13}$$

*Finally, if $u_0 \in D(A)$, then*

$$\frac{du}{dt} \in L^\infty(0,T;H),\ \ Au \in L^\infty(0,T;H), \tag{2.14}$$

$$\frac{d^+}{dt} u(t) + Au(t) = 0,\ \ \forall t \in [0,T]. \tag{2.15}$$

Applying Proposition 2.1 to the operator $A$ defined by (2.2) on the space $H = L^2$, we get the following existence and uniqueness result for problem (1.1).

**Theorem 2.3.** *Let $u_0 \in L^2$. Then, for each $T > 0$, there is a unique function*

$$u \in C([0,T];L^2) \cap \cap \bigcap_{0<\delta<T} W^{1,2}([\delta,T];H) \tag{2.16}$$

*such that $u(t) \in D(A)$, a.e. $t \in (0,T)$, and*

$$t^{\frac{1}{2}}\,\frac{du}{dt} \in L^2(0,T;L^2),\ t^{\frac{1}{2}}Au \in L^2(0,T;L^2),\ |\nabla u| \in L^p(0,T;L^p), \tag{2.17}$$

$$\frac{du}{dt}(t) = \mathrm{div}(|\nabla u(t)|^{p-2}\nabla u(t)),\ \ a.e.\ t \in (0,T), \tag{2.18}$$

$$\frac{1}{2}|u(t)|_2^2 + \int_0^t \int_{\mathbb{R}^d} |\nabla u(s,x)|^p dxds = \frac{1}{2}|u_0|_2^2,\ \forall t \geq 0. \tag{2.19}$$

*If* $|\nabla u_0| \in L^p$*, then*

$$\frac{du}{dt} \in L^2(0,T;L^2),\ \mathrm{div}(|\nabla u|^{p-2}\nabla u) \in L^2((0,T) \times \mathbb{R}^d), \tag{2.20}$$

$$|\nabla u(t)|_p^p \leq |\nabla u_0|_p^p,\ \ a.e.\ t \in [0,T]. \tag{2.21}$$

*Finally, if* $u_0 \in L^2,\ |\nabla u_0| \in L^p,\ \mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0) \in L^2$*, then*

$$\frac{du}{dt} \in L^\infty(0,T;L^2),\ \mathrm{div}(|\nabla u|^{p-2}\nabla u) \in L^\infty(0,T;L^2), \tag{2.22}$$

$$\frac{d^+}{dt}\,u(t) = \mathrm{div}(|\nabla u(t)|^{p-2}\nabla u(t)),\ \forall t \in [0,T). \tag{2.23}$$

(*Herein,* $\frac{du}{dt}(t)$ *is the strong derivative of the function* $u : [0,T] \to H$*.*)

In the following, we shall call such a function $u$ the *weak solution* to the parabolic $p$-Laplace equation (1.1).

We also have

**Theorem 2.4.** *If* $u_0 \in L^\infty \cap L^2$*, then* $u \in L^\infty((0,T) \times \mathbb{R}^d)$*,* $|u(t)|_\infty \leq |u_0|_\infty$*, a.e.* $t \in (0,T)$*. Moreover, if* $u_0 \in L^1 \cap L^2$*, then* $u(t) \in L^\infty(0,T;L^1)$ *and, if* $u_0 \geq 0$*, a.e. on* $\mathbb{R}^d$*, then*

$$u \geq 0,\ \ a.e.\ \ on\ (0,T) \times \mathbb{R}^d, \tag{2.24}$$

*and, if* $p \geq d$*, we have*

$$\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx,\ \forall t \in [0,T]. \tag{2.25}$$

*Proof.* We first prove that, if $u_0 \in L^1 \cap L^2$, then $u \in L^\infty(0,T;L^1)$. Let $\mathcal{X}$ be the function

$$\mathcal{X}_\delta(r) = \begin{cases} 1 & \text{for } r \geq \delta, \\ \dfrac{r}{\delta} & \text{for } |r| < \delta, \\ -1 & \text{for } r \leq -\delta. \end{cases} \tag{2.26}$$

Taking into account that $\mathcal{X}_\delta(u(t)) \in L^2$, $\forall t \in (0,T)$, we have (see, e.g., [2], p. 158, Lemma 4.1)

$$\left(\frac{d}{dt}u(t), \mathcal{X}_\delta(u(t))\right)_2 = \frac{d}{dt}\int_{\mathbb{R}^d} j_\delta(u(t,x))dx, \text{ a.e. } t \in (0,T),$$

where $j_\delta(v) = \int_0^v \mathcal{X}_\delta(s)ds$, $\forall v \in \mathbb{R}$. We get by (2.18)

$$\frac{d}{dt}\int_{\mathbb{R}^d} j_\delta(u(t,x))dx = -\int_{\mathbb{R}^d} |\nabla u(t,x)|^p \mathcal{X}'_\delta(u(t,x))dx \leq 0, \text{ a.e. } t > 0.$$

For $\delta \to 0$, this yields, $\forall t \in [0,T]$,

$$\lim_{\delta \to 0} \int_{\mathbb{R}^d} j_\delta(u(t,x))dx \leq \int_{\mathbb{R}^d} |u_0(x)|dx, \tag{2.27}$$

and so we get

$$|u(t)|_1 \leq |u_0|_1, \quad \text{a.e. } \forall t \in [0,T],$$

as claimed. Next, if $u_0 \geq 0$, a.e. on $\mathbb{R}^d$, we get in a similar way that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^-(t,x)dx \leq 0, \text{ a.e. } t \in (0,T),$$

and, therefore, $u \geq 0$, a.e. on $(0,T) \times \mathbb{R}^d$.

Assume now that $u_0 \in L^\infty$. Then, by (2.18) we see that

$$\frac{d}{dt}(u(t,x) - |u_0|_\infty) - \mathrm{div}(|\nabla(u - |u_0|_\infty)|^{p-2}\nabla(u - |u_0|_\infty)) = 0, \qquad \text{a.e. on } (0,T) \times \mathbb{R}^d,$$

and multiplying by $(u - |u_0|_\infty)^+$, we get after integration over $(0,t) \times \mathbb{R}^d$

$$(u(t,x) - |u_0|_\infty)^+ = 0, \text{ a.e. } (t,x) \in (0,T) \times \mathbb{R}^d.$$

Hence, $u \leq |u_0|_\infty$, a.e. on $(0,T) \times \mathbb{R}^d$ and in a similar way it follows $u \geq -|u_0|_\infty$, a.e. on $(0,T) \times \mathbb{R}^d$. Hence, $|u(t,x)| \leq |u_0|_\infty$, a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$, as claimed.

To get (2.25), we note that by (2.18)

$$\frac{d}{dt}(u(t),\varphi_n) = -\int_{\mathbb{R}^d} |\nabla u(t,x)|^{p-2}(\nabla u(t,x)\cdot\nabla\varphi_n(x))dx, \text{ a.e. } t>0,$$

where $\varphi_n(x) = \eta\left(\frac{|x|^2}{n}\right)$, $\eta \in C^2([0,\infty))$, $\eta(r)=1$, $\forall r\in[0,1]$, $\eta(r)=0$, $\forall r\in[2,\infty)$. This yields

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi_n(x)u(t,x)dx = -&\int_0^t\int_{\mathbb{R}^d}|\nabla u(s,x)|^{p-2}\nabla u(s,x)\cdot\nabla\varphi_n(x)dxds\\ &+\int_{\mathbb{R}^d}\varphi_n(x)u_0(x)dx.\end{aligned} \tag{2.28}$$

Taking into account that, for $B_n=\{x;\sqrt{n}\le|x|\le\sqrt{2n}\}$,

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^d}|\nabla u(s,x)|^{p-2}\nabla u(s,x)\cdot\nabla\varphi_n(x)dxds\right|\\ &\qquad\le|\eta'|_\infty\frac{2\sqrt{2}}{\sqrt{n}}\int_0^t ds\left(\int_{[|x|\ge\sqrt{n}]}|\nabla u(s,x)|^p dx\right)^{\frac{p-1}{p}}(Vol(B_n))^{\frac{1}{p}}\\ &\qquad\le Cn^{\frac{1}{2}\left(\frac{d}{p}-1\right)}\int_0^t ds\left(\int_{[|x|\ge\sqrt{n}]}|\nabla u(s,x)|^p dx\right)^{\frac{p-1}{p}}\end{aligned}$$

and so, letting $n\to\infty$ in (2.28), we get (2.25). □

An important property of equation (1.1) is the finite speed of propagation of the solution s. Namely, we have (see, e.g., Theorem 3.4 in [8], and also [11])

**Proposition 2.5.** *Let $p>2$, $u_0\in L^2$, and let $u$ be the solution given by Theorem 2.3. Assume that*

$$\textit{support } u_0 \subset B_R = \{x\in\mathbb{R}^d;\ |x|\le R\}. \tag{2.29}$$

*Then,*

$$\textit{support } u(t,\cdot)\subset B_{2R+R(t)},\ \forall t\ge 0, \tag{2.30}$$

*where*

$$R(t) = C\,t^{\frac{1}{d(p-2)+p}}\,|u_0|_1^{\frac{p-2}{d(p-2)+p}},\ t\ge 0, \tag{2.31}$$

*and $C>0$ is independent of $t$ and $u_0$.*

**Remark 2.6.** Under the hypotheses of Proposition 2.5 it follows by (2.28) that the condition $d \le p$ for the proof of (2.25) is no longer necessary.

**Remark 2.7.** By Proposition 2.1 it follows also that the operator $A$ generates a continuous semigroup of contractions $S(t) = e^{-tA}$ in $H = L^2$ given by

$$S(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0 \ \text{ strongly in } H,\ \forall t \ge 0. \tag{2.32}$$

For $u_0 \in L^2$ we have, therefore,

$$S(t)u_0 = u(t),\ t \ge 0,\ x \in \mathbb{R}^d, \tag{2.33}$$

where $u$ is the solution $u$ to (1.1) given by Theorem 2.3. By (2.17)–(2.22) it follows that $S(t)$ has a *smoothing effect* on initial data. Moreover, as seen in Theorem 2.4, $S(t)$ leaves invariant the space $L^1 \cap L^2$.

# 3 Second order estimates for solutions to (2.18)

Everywhere in the following $u$ is the weak solution to equation (1.1) (equivalently, (2.11)), given by Theorem 2.3.

**Theorem 3.1.** *Let $p \ge 4$, and let $u_0 \in L^2$ with compact support in $\mathbb{R}^d$ and $|\nabla u_0| \in L^\infty$. Assume that* (2.29) *holds. Then, we have $\nabla u \in L^\infty(0,T;L^\infty)$, $\forall T > 0$, and*

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u(t,x)|^{p-2} dxdt \le (T(\mu(T,R)))^{\frac{2}{p}} |u_0|_2^{\frac{2(p-2)}{p}}, \tag{3.1}$$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} &|\nabla(|\nabla u(t,x)|^{p-2})| dxdt \\ &\le \frac{\sqrt{d}\,(p-2)}{2} (2T\mu(T,R))^{\frac{2}{p}} |u_0|_2^{\frac{p-4}{p}} |\nabla u_0|_2,\end{aligned} \tag{3.2}$$

*where $\mu(T,R) = Vol(B_{2R+R(T)})$ and $R(t)$ is given by* (2.31).

*Proof.* By (2.19) and (2.30) we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d} |\nabla u(t,x)|^{p-2} dxdt \le \int_0^T dt \left(\int_{\mathbb{R}^d} |\nabla u(t,x)|^p dx\right)^{\frac{p-2}{p}} (Vol(B_{2R+R(T)}))^{\frac{2}{p}} \\ &\le T^{\frac{2}{p}} (\mu(T,R))^{\frac{2}{p}} \left(\int_0^T \int_{\mathbb{R}^d} |\nabla u(t,x)|^p dxdt\right)^{\frac{p-2}{p}} \le T^{\frac{2}{p}} (\mu(T,R))^{\frac{2}{p}} |u_0|_2^{\frac{2(p-2)}{p}}\end{aligned}$$

and so (3.1) follows.

*Proof of* (3.2). We note first that, since $u_0 \in L^2$ and $|\nabla u_0| \in L^\infty$, then by the Sobolev–Gagliardo–Nirenberg theorem combined with the Morrey theorem (see, e.g., [9], pp. 278, 282) it follows that $u_0 \in L^\infty$. We approximate (2.18) by

$$\begin{aligned} &\frac{du}{dt} = \varepsilon\Delta u + \operatorname{div}(|\nabla u|^{p-2}\nabla u), \ \text{a.e. } t \ge 0, \ x \in \mathbb{R}^d, \\ &u(0) = u_0, \end{aligned} \tag{3.3}$$

where $\varepsilon > 0$. We may rewrite (3.3) as

$$\frac{du}{dt} + A_\varepsilon u = 0, \ \text{a.e. } t \in (0,\infty); \ u(0) = u_0, \tag{3.4}$$

where $A_\varepsilon : D(A) \subset L^2 \to L^2$ is given by

$$\begin{aligned} A_\varepsilon u &= -\varepsilon\Delta u - \operatorname{div}(|\nabla u|^{p-2}\nabla u), \ \forall u \in D(A_\varepsilon), \\ D(A_\varepsilon) &= \{u \in H^1; \ \nabla u \in L^p, \ \varepsilon\Delta u + \operatorname{div}(|\nabla u|^{p-2}\nabla u) \in L^2\}. \end{aligned}$$

Arguing as in the proof of Proposition 2.1, it follows that $A_\varepsilon$ is maximal monotone in $L^2 \times L^2$ and $A_\varepsilon = \partial\Phi_\varepsilon$, where $\Phi_\varepsilon : L^2 \to ]-\infty, +\infty]$ is the potential function of the operator given by

$$\Phi_\varepsilon(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{p}|\nabla u|^p\right) dx & \text{if } \nabla u \in L^2 \cap L^p, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, by Proposition 2.2 it follows that (3.3) has a unique solution $u_\varepsilon \in C([0,\infty); L^2) \cap L^\infty(0,\infty; W^{1,2})$ satisfying (see (2.20)–(2.21))

$$\frac{\varepsilon}{2}|\nabla u_\varepsilon(t)|_2^2 + \frac{1}{p}|\nabla u_\varepsilon(t)|_p^p \le \frac{\varepsilon}{2}|\nabla u_0|_2^2 + \frac{1}{p}|\nabla u_0|_p^p, \ \forall t \ge 0, \tag{3.5}$$

$$|u_\varepsilon(t)|_2 \le |u_0|_2, \ \forall t \ge 0, \tag{3.6}$$

$$\frac{du_\varepsilon}{dt} \in L^2(0,T; L^2), \ T > 0, \tag{3.7}$$

$$\varepsilon\Delta u_\varepsilon + \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) \in L^2(0,T; L^2), \ \forall T > 0. \tag{3.8}$$

**Claim 1.** *We have $u_\varepsilon(t) \in H^2$, a.e. $t \in (0,\infty)$, and*

$$\varepsilon|\Delta u_\varepsilon(t)|_2 \le |f(t)|_2, \ \text{a.e. } t \in (0,\infty), \tag{3.9}$$

*where $f(t) = \varepsilon\Delta u_\varepsilon(t) + \operatorname{div}(|\nabla u_\varepsilon(t)|^{p-2}\nabla u_\varepsilon(t))$.*

*Proof.* We set

$$B_\lambda(u) = -\Delta(I-\lambda\Delta)^{-1}u = \frac{1}{\lambda}(u-(I-\lambda\Delta)^{-1}u),\ \forall u \in L^2.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^d} &|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla(B_\lambda(u_\varepsilon))dx\\ &= \frac{1}{\lambda}\int_{\mathbb{R}^d}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot(\nabla u_\varepsilon - \nabla(I-\lambda\Delta)^{-1}u_\varepsilon)dx \ge 0,\end{aligned}$$

because

$$|\nabla(I-\lambda\Delta)^{-1}u_\varepsilon|_p \le |\nabla u_\varepsilon|_p,\ \forall \lambda>0.$$

This yields

$$\varepsilon\left\langle \Delta u_\varepsilon, \Delta(I-\lambda\Delta)^{-1}u_\varepsilon\right\rangle \le -(B_\lambda(u_\varepsilon),f)_2 \le |B_\lambda(u_\varepsilon)|_2|f|_2.$$

Since

$$\begin{aligned}{}_{H^{-1}}\left\langle \Delta u_\varepsilon, \Delta(I-\lambda\Delta)^{-1}u_\varepsilon\right\rangle_{H^1} &= {}_{H^{-1}}\left\langle \Delta u_\varepsilon, (I-\lambda\Delta)^{-1}\Delta u_\varepsilon\right\rangle_{H^1}\\ &\ge |\Delta(I-\lambda\Delta)^{-1}u_\varepsilon|_2^2,\end{aligned}$$

we have

$$\varepsilon|\Delta(I-\lambda\Delta)^{-1}u_\varepsilon|_2 \le |f|_2,\ \forall\lambda>0,$$

and so, letting $\lambda\to 0$, we get (3.9), as claimed. □

**Claim 2.** *For $\varepsilon\to 0$, we have, for all $T>0$,*

$$u_\varepsilon(t)\to u(t) \text{ in } L^2 \text{ uniformly on } [0,T], \tag{3.10}$$

*where $u$ is the solution given by Theorem 2.3.*

The latter follows by the Kato–Trotter theorem for nonlinear semigroups of contractions (see, e.g., [2], pp. 168, 169) because we have

**Lemma 3.2.** *For all $\lambda>0$, we have*

$$(I+\lambda A_\varepsilon)^{-1}u_0 \overset{\varepsilon\to 0}{\longrightarrow} (I+\lambda A)^{-1}u_0 \text{ strongly in } L^2. \tag{3.11}$$

*Proof.* We set $v_\varepsilon=(I+\lambda A_\varepsilon)^{-1}u_0$, that is, $v_\varepsilon\in H^2$, and

$$v_\varepsilon - \lambda\varepsilon\Delta v_\varepsilon - \lambda\,\mathrm{div}(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon) = u_0 \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{3.12}$$

Since $L^1 \cap L^\infty$ is dense in $L^2$ and $(I+\lambda A_\varepsilon)^{-1}$, $\varepsilon>0$, are contractions (hence equicontinuous) on $L^2$, it suffices to prove (3.12) for $u_0 \in L^2 \cap L^\infty$. But then, as in the proof of Theorem 2.4, it follows that

$$|v_\varepsilon|_\infty \le |u_0|_\infty,\ \forall \varepsilon>0. \tag{3.13}$$

Hence, by (3.13) and an approximation argument, we have the estimate

$$\frac{1}{2}|v_\varepsilon|_2^2 + \lambda\varepsilon|\nabla v_\varepsilon|_2^2 + \lambda|\nabla v_\varepsilon|_p^p \le \frac{1}{2}|u_0|_2^2,\ \forall \varepsilon>0, \tag{3.14}$$

and

$$\begin{aligned}\frac{1}{p}\,|v_\varepsilon|_p^p\ &+\lambda\varepsilon(p-1)\int_{\mathbb{R}^d}|\nabla v_\varepsilon|^2|v_\varepsilon|^{p-2}dx\\ &+\lambda(p-1)\int_{\mathbb{R}^d}|\nabla v_\varepsilon|^p|v_\varepsilon|^{p-2}dx \le \frac{1}{p}\,|u_0|_p.\end{aligned}$$

Hence,

$$|v_\varepsilon|_p \le |u_0|_p,\ \forall u_0 \in L^2\cap L^p, \tag{3.15}$$

and this implies that along a subsequence, again denoted $\{\varepsilon\}\to 0$, we have

$$\begin{array}{rcll} v_\varepsilon & \to & v & \text{strongly in } L^2_{\rm loc} \text{ and weakly in } L^2\\ \nabla v_\varepsilon & \to & \nabla v & \text{weakly in } (L^p)^d\\ \varepsilon\nabla v_\varepsilon & \to & 0 & \text{strongly in } (L^2)^d\\ |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon & \to & \eta & \text{weakly in } (L^{p'})^d. \end{array} \tag{3.16}$$

By (3.12)–(3.16) it follows that

$$v-\lambda\,{\rm div}\,\eta = u_0 \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{3.17}$$

On the other hand, we have

$$\begin{aligned}-\int_{\mathbb{R}^d}{\rm div}(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon)(v_\varepsilon-w)dx &\ge \frac{1}{p}\int_{\mathbb{R}^d}|\nabla v_\varepsilon|^p dx\\ &-\frac{1}{p}\int_{\mathbb{R}^d}|\nabla w|^p dx,\ \forall w\in L^2,\ |\nabla w|\in L^p.\end{aligned}$$

This yields

$$\frac{\lambda}{p}\int_{\mathbb{R}^d}|\nabla v_\varepsilon|^p dx - \int_{\mathbb{R}^d}(u_0-v_\varepsilon+\varepsilon\lambda\Delta v_\varepsilon)(v_\varepsilon-w)dx \le \frac{\lambda}{p}\int_{\mathbb{R}^d}|\nabla w|^p dx$$

and for $\varepsilon\to 0$ it follows by (3.16) that

$$\frac{\lambda}{p}\int_{\mathbb{R}^d}|\nabla v|^p dx \le \frac{\lambda}{p}\int_{\mathbb{R}^d}|\nabla w|^p dx+\int_{\mathbb{R}^d}(u_0-v)(v-w)dx,\ \forall w\in L^2,\ |\nabla w|\in L^p,$$

which means that $\operatorname{div}\eta \in \partial\Phi(v)$, where $\Phi$ is the function (2.4). Hence, by Proposition 2.1, $\operatorname{div}\eta = -\operatorname{div}|\nabla v|^{p-2}\nabla v$, a.e. in $\mathbb{R}^d$, and thus $v\in D(A)$ and $v=(I+\lambda A)^{-1}u_0$.

To conclude the proof, it remains to be shown that

$$v_\varepsilon \to v \text{ strongly in } L^2 \text{ as } \varepsilon\to 0. \tag{3.18}$$

By the density of $L^p\cap L^2$ in $L^2$ and since $|(I+\lambda A_\varepsilon)^{-1}u-(I+\lambda A_\varepsilon)^{-1}v|_2 \le |u-v|_2$, $\forall u,v\in L^2$, it suffices to prove (3.18) for $u_0\in L^2\cap L^p$.

Let $\eta\in C^2([0,\infty))$ be such that $\eta\ge 0$, $\eta(r)=0$, $\forall r\in[0,1]$, $\eta(r)=1$, $\forall r\ge 2$. If we multiply (3.12) by $\eta_n v_\varepsilon$, where $\eta_n(x)=\eta\left(\frac{|x|}{n}\right)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} v_\varepsilon^2(x)\eta_n(x)dx+\lambda\varepsilon\int_{\mathbb{R}^d}\nabla v_\varepsilon(x)\cdot\left(\eta_n(x)\nabla v_\varepsilon(x)+\frac{1}{n}\eta'\left(\frac{|x|}{n}\right)\frac{x}{|x|}v_\varepsilon(x)\right)dx\\
&\quad+\lambda\int_{\mathbb{R}^d}|\nabla v_\varepsilon(x)|^{p-2}\nabla v_\varepsilon(x)\cdot\left(\eta_n(x)\nabla v_\varepsilon(x)+\frac{1}{n}\eta'\left(\frac{|x|}{n}\right)\frac{x}{|x|}v_\varepsilon(x)\right)dx\\
&\quad=\int_{\mathbb{R}^d}u_0(x)v_\varepsilon(x)\eta_n(x)dx.
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}v_\varepsilon^2(x)\eta_n(x)dx+\frac{\lambda\varepsilon}{n}\int_{\mathbb{R}^d}\left(\nabla v_\varepsilon(x)\cdot\frac{x}{|x|}\right)\eta'\left(\frac{|x|}{n}\right)v_\varepsilon(x)dx\\
&\quad+\frac{\lambda}{n}\int_{\mathbb{R}^d}|\nabla v_\varepsilon(x)|^{p-2}\left(\nabla v_\varepsilon(x)\cdot\frac{x}{|x|}\right)\eta'\left(\frac{|x|}{n}\right)v_\varepsilon(x)dx\\
&\quad\le\frac{1}{2}\int_{\mathbb{R}^d}u_0^2(x)\eta_n(x)dx.
\end{aligned}$$

Taking into account that by (3.14)–(3.15)

$$\int_{\mathbb{R}^d}(|\nabla v_\varepsilon|\,|v_\varepsilon|dx+|\nabla v_\varepsilon|^{p-1}|v_\varepsilon|)dx\le|\nabla v_\varepsilon|_2|v_\varepsilon|_2+|\nabla v_\varepsilon|_p^{p-1}|v_\varepsilon|_p\le C,\quad \forall\varepsilon>0,$$

we get $$\int_{[|x|\ge 2n]}v_\varepsilon^2(x)dx\le\frac{C}{n}+\int_{[|x|\ge n]}u_0^2(x)dx,\ \forall\varepsilon>(0,1),\ n\in\mathbb{N},$$

where $C$ is independent of $\varepsilon$. Combined with (3.16), the latter yields (3.18). □

*Proof of* (3.2) (*continued*). For each $\varepsilon > 0$ and $i = 1, ..., d$ we set

$$w_i^\varepsilon = D_i u_\varepsilon,\ w^\varepsilon = (w_i^\varepsilon)_{i=1}^d = \nabla u_\varepsilon.$$

Taking into account that by (3.9), $u_\varepsilon \in L^2(0,T;H^2)$, $\forall T > 0$, we have $w_i^\varepsilon \in L^2(0,T;H^1)$. Differentiating formally (3.3) with respect to each $i$, we get

$$\begin{aligned} \frac{\partial w_i^\varepsilon}{\partial t} &= \mathrm{div}(|w^\varepsilon|^{p-2}\nabla w_i^\varepsilon + (p-2)|w^\varepsilon|^{p-4}w^\varepsilon(\nabla w_i^\varepsilon \cdot w^\varepsilon)) \\ &\qquad + \varepsilon\Delta w_i^\varepsilon \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d), \\ w_i^\varepsilon(0,x) &= D_i u_0(x), \quad i = 1,2,...,d. \end{aligned} \tag{3.19}$$

One might suspect that $w_i^\varepsilon$ is the strong solution to (3.19) in the space $L^2(0,\infty;H^1)$. To prove rigorously the exact significance of (3.19) we consider the finite differences

$$\begin{aligned} u_{\varepsilon,i}^h(t,x) = \frac{1}{h}(u_\varepsilon(t,x+he_i) - u_\varepsilon(t,x)),\ e_i = (0,0,...,1,0,...,0), \\ (t,x) \in (0,\infty)\times\mathbb{R}^d, \end{aligned}$$

and get

$$\begin{aligned} \frac{\partial}{\partial t}u_{\varepsilon,i}^h(t,x) &= \mathrm{div}(|\nabla u_\varepsilon(t,x+he_i)|^{p-2}\nabla u_\varepsilon(t,x+he_i) \\ &\quad -|\nabla u_\varepsilon(t,x)|^{p-2}\nabla u_\varepsilon(t,x)) + \varepsilon\Delta u_{\varepsilon,i}^h(t,x) \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d, \\ u_{\varepsilon,i}^h(0,x) &= \frac{1}{h}(u_0(x+he_i) - u_0(x)),\ x \in \mathbb{R}^d. \end{aligned} \tag{3.20}$$

Define the function $F : \mathbb{R}^d \to \mathbb{R}^d$ by

$$F(x) = |x|^{p-2}x,\ \forall x \in \mathbb{R}^d.$$

We have

$$\begin{aligned} F(y) - F(x) &= \int_0^1 \frac{d}{d\tau}F(\tau y - (1-\tau)x)d\tau \\ &= \int_0^1 DF(\tau y - (1-\tau)x)\cdot(y-x)d\tau,\ \forall x,y \in \mathbb{R}^d, \end{aligned}$$

where $DF = (D_j F_i)_{i,j=1}$ and

$$D_j F_i(x) = (p-2)|x|^{p-4}x_i x_j + \delta_{ij}|x|^{p-2},\ i,j = 1,...,d.$$

We set

$$f^h_{\varepsilon,i}(t,x,\tau)=\nabla(\tau u_\varepsilon(t,x+he_i)+(1-\tau)u_\varepsilon(t,x)),\ \tau\in(0,1),\ (t,x)\in(0,\infty)\times\mathbb{R}^d.$$

Then, (3.20) turns into

$$\begin{aligned}&\frac{\partial}{\partial t}u^h_{\varepsilon,i}(t,x) \qquad (3.21)\\ &=\operatorname{div}\left((p-2)\int_0^1|f^h_{\varepsilon,i}(t,x,\tau)|^{p-4}f^h_{\varepsilon,i}(t,x,\tau)(f^h_{\varepsilon,i}(t,x,h)\cdot\nabla u^h_{\varepsilon,i}(t,x))d\tau\right.\\ &\left.+\int_0^1|f^h_{\varepsilon,i}(t,x,\tau)|^{p-2}d\tau\,\nabla u^h_{\varepsilon,i}(t,x)\right)+\varepsilon\Delta u^h_{\varepsilon,i}(t,x).\end{aligned}$$

Taking into account that $u^h_{\varepsilon,i}\in L^2(0,\infty;H^2)$, we get by (3.21)

$$\begin{aligned}&\frac12|u^h_{\varepsilon,i}(t)|_2^2+\int_0^t ds\int_{\mathbb{R}^d}\left(\int_0^1|f^h_{\varepsilon,i}(s,x,\tau)|^{p-2}d\tau\right)|\nabla u^h_{\varepsilon,i}(s,x)|^2dx \qquad (3.22)\\ &+(p-2)\int_0^t ds\int_{\mathbb{R}^d}\left(\int_0^1|f^h_{\varepsilon,i}(s,x,\tau)|^{p-4}(f^h_{\varepsilon,i}(s,x,h)\cdot\nabla u^h_{\varepsilon,i}(s,x))^2dxd\tau\right)\\ &+\varepsilon\int_0^t ds\int_{\mathbb{R}^d}|\nabla u^h_{\varepsilon,i}(s,x)|^2dx=\frac12|u^h_{\varepsilon,i}(0)|_2^2,\ \forall t\ge0.\end{aligned}$$

On the other hand, we know that for $h\to0$,

$$\begin{aligned}\nabla u^h_{\varepsilon,i}&\to\nabla w^\varepsilon_i &&\text{in } L^2(0,T;L^2),\\ |\nabla u_\varepsilon(t,x+he_i)|^{p-2}&\to|\nabla u_\varepsilon(t,x)|^{p-2}&&\text{in } L^{\frac{p}{p-2}}((0,T)\times\mathbb{R}^d).\end{aligned}$$

Hence, for $h\to0$, along a subsequence, again denoted $\{h\}\to0$,

$$\nabla u^h_{\varepsilon,i}(t,x)\to\nabla w^\varepsilon_i(t,x),\ |\nabla u_{\varepsilon,i}(t,x+he_i)|\to|\nabla u_\varepsilon(t,x)|,\ \text{a.e. } (t,x)\in(0,T)\times\mathbb{R}^d,$$

and so, by (3.22) and by Fatou's lemma, we get the estimate

$$\begin{aligned}&\frac12|w^\varepsilon_i(t)|_2^2+\int_0^t ds\int_{\mathbb{R}^d}(|w^\varepsilon(s,x)|^{p-2}|\nabla w^\varepsilon_i(s,x)|^2\\ &\qquad+(p-2)|w^\varepsilon(s,x)|^{p-4}(w^\varepsilon(s,x)\cdot\nabla w^\varepsilon_i(s,x))^2)dx \qquad (3.23)\\ &\qquad+\varepsilon\int_0^t ds\int_{\mathbb{R}^d}|\nabla w^\varepsilon_i(s,x)|^2dx\le\frac12|D_iu_0|_2^2,\ i=1,2,...,d.\end{aligned}$$

Moreover, we have

$$w_i^\varepsilon \in L^\infty(0,\infty;L^1\cap L^\infty)),\quad \forall i=1,...,d, \tag{3.24}$$

$$|w_i^\varepsilon(t)|_\infty \le |w_i^\varepsilon(0)|_\infty = |D_iu_0|_\infty,\quad \forall t\ge 0,\ i=1,...,d, \tag{3.25}$$

$$|w_i^\varepsilon(t)|_1 \le |w_i^\varepsilon(0)|_1 = |D_iu_0|_1,\quad \forall t\ge 0,\ i=1,...,d. \tag{3.26}$$

Indeed, if $M_i=|u^h_{\varepsilon,i}(0)|_\infty$, we have by (3.21)

$$\begin{aligned}&\frac{\partial}{\partial t}(u^h_{\varepsilon,i}-M_i)\\ &=\mathrm{div}\Bigg((p-2)\int_0^1|f^h_{\varepsilon,i}(t,x,\tau)|^{p-4}f^h_{\varepsilon,i}(t,x,\tau)(f^h_{\varepsilon,i}(t,x,h)\cdot\nabla(u^h_{\varepsilon,i}(t,x)-M_i))d\tau\\ &+\int_0^1|f^h_{\varepsilon,i}(t,x,\tau)|^{p-2}d\tau\nabla(u^h_{\varepsilon,i}(t,x)-M_i)\Bigg)+\varepsilon\Delta(u^h_{\varepsilon,i}(t,x)-M_i).\end{aligned}$$

Multiplying by $(u^h_{\varepsilon,i}-M_i)^+$ and integrating over $\mathbb{R}^d$ yields

$$\frac{d}{dt}|(u^h_{\varepsilon,i}(t)-M_i)^+|_2^2\le 0,\ \text{a.e. } t>0,$$

and, therefore, $u^h_{\varepsilon,i}(t,x)\le M_i$, a.e. $(t,x)\in(0,\infty)\times\mathbb{R}^d$. Similarly, it follows that $u^h_{\varepsilon,i}(t,x)\ge -M_i$, a.e. $(t,x)\in(0,\infty)\times\mathbb{R}^d$. Hence, (3.25) follows.

In particular, (3.23) implies that

$$|w^\varepsilon|^{\frac{p-2}{2}}|\nabla w_i^\varepsilon|,\ |w^\varepsilon|^{\frac{p-4}{2}}(w^\varepsilon\cdot\nabla w_i^\varepsilon)\in L^2(0,\infty;L^2),\ i=1,...,d,$$

and so, by (3.20) and (3.23) it follows that $w_i^\varepsilon$ is the solution to (3.19) in the following strong sense

$$w_i^\varepsilon\in L^2(0,T;H^1),\ \frac{dw_i^\varepsilon}{dt}\in L^2(0,T;H^{-1}),\ T>0, \tag{3.27}$$

$$\begin{aligned}&\frac{dw_i^\varepsilon}{dt}(t)=\mathrm{div}(|w^\varepsilon(t)|^{p-2}\nabla w_i^\varepsilon(t)+(p-2)|w^\varepsilon(t)|^{p-4}w^\varepsilon(t)(\nabla w^\varepsilon(t)\cdot\nabla w_i^\varepsilon(t)))\\ &\qquad\qquad+\varepsilon\Delta w_i^\varepsilon(t),\ \text{a.e. } t\in(0,\infty),\ \text{in } H^{-1}, \qquad (3.28)\\ &w_i^\varepsilon(0)=D_iu_0.\end{aligned}$$

Next, to get (3.26) we multiply (3.28) by $\mathcal{X}_\delta(w_i^\varepsilon)$, where $\mathcal{X}_\delta$ is the function (2.26), and let $\delta\to 0$. We omit the details (see the proof of (2.25)).

In particular, it follows by (3.23) that

$$\int_0^\infty dt \int_{\mathbb{R}^d} \left(|w^\varepsilon(t,x)|^{p-2}|\nabla w_i^\varepsilon(t,x)|^2 + \varepsilon|\nabla w_i^\varepsilon(t,x)|^2\right) dx \le \frac{1}{2}|D_i u_0|_2^2, \quad \forall i = 1,...,d. \tag{3.29}$$

This yields

$$\int_0^\infty dt \int_{\mathbb{R}^d} |\nabla(|w^\varepsilon(t,x)|^{\frac{p}{2}})|^2 dxdt \le \frac{dp^2}{8}|\nabla u_0|_2^2. \tag{3.30}$$

Since by (3.24) $\{w_i^\varepsilon\}_\varepsilon$ is bounded in $L^2((0,T)\times\mathbb{R}^d)$, $\forall T>0$, it follows in particular that along a subsequence, again denoted $\{\varepsilon\}\to 0$,

$$w_i^\varepsilon \;\to\; w_i \;\text{ weakly in } L^2((0,T)\times\mathbb{R}^d),\; \forall T>0, \tag{3.31}$$

where by (3.10) and the closedness of the operator $D_i$ on $L^2(0,T;L^2)$, $w_i = D_i u$.

By (3.25) it follows also that

$$|\nabla u| \in L^\infty((0,T)\times L^\infty). \tag{3.32}$$

We denote by $\frac{d}{dt}(|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t))$ the distributional derivative of the function $t\to|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t)$, that is,

$$_{\mathcal{D}'}\left\langle \frac{d}{dt}|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t),\varphi\right\rangle_{\mathcal{D}} = -\int_0^T\int_{\mathbb{R}^d}\frac{d}{dt}|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t)\frac{d\varphi}{dt}(t,x)dxdt, \quad \forall\varphi\in C_0^\infty((0,T)\times\mathbb{R}^d) =: \mathcal{D}.$$

**Claim 3.** *Let* $1<q<\min\left(\frac{d}{d-1},2\right)$ *if* $d\ge 3$, $q\in(1,2)$ *if* $d=2$, *and* $q=2$ *if* $d=1$. *Then,*

$$\sup_{0<\varepsilon\le 1}\int_0^T \left\|\frac{d}{dt}(|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t))\right\|_{W^{-1,q}} dt < \infty. \tag{3.33}$$

*Proof.* To prove (3.33), it suffices to mention that by (3.27) the function $t\to w_i^\varepsilon(t)$ is $H^{-1}$-valued absolutely continuous on $[0,T]$ and a.e. $t\in(0,T)$,

$$\frac{d}{dt}w_i^\varepsilon(t) = \lim_{h\to 0}\frac{w_i^\varepsilon(t+h)-w_i^\varepsilon(t)}{h} \;\text{ strongly in } H^{-1},\;\text{hence in } W^{-1,q}.$$

This yields

$$\int_0^T\int_{\mathbb{R}^d}|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t)\frac{d\varphi}{dt}dxdt$$
$$=\lim_{h\to 0}\frac{1}{h}\int_0^{T-h}\int_{\mathbb{R}^d}|w_i^\varepsilon(t)|^{\frac{p}{2}-1}w_i^\varepsilon(t)(\varphi(t+h,x)-\varphi(t,x))dxdt$$
$$=\lim_{h\to 0}\frac{1}{h}\int_h^T\int_{\mathbb{R}^d}(|w_i^\varepsilon(t-h,x)|^{\frac{p}{2}-1}w_i^\varepsilon(t-h,x)-|w_i^\varepsilon(t,x)|^{\frac{p}{2}-1}w_i^\varepsilon(t,x))\varphi(t,x)dxdt$$
$$=-\lim_{h\to 0}\int_h^T dt\int_{\mathbb{R}^d}\int_0^1\frac{p}{2}|\tau w_i^\varepsilon(t,x)+(1-\tau)w_i^\varepsilon(t-h,x)|^{\frac{p-2}{2}}d\tau$$
$$\frac{1}{h}(w_i^\varepsilon(t,x)-w_i^\varepsilon(t-h,x))\varphi(t,x)dx$$
$$=-\lim_{h\to 0}\frac{p}{2}\int_h^T dt\ {}_{H^{-1}}\left\langle\frac{1}{h}(w_i^\varepsilon(t)-w_i^\varepsilon(t-h)),\int_0^1|\tau w_i^\varepsilon(t)+(1-\tau)w_i^\varepsilon(t-h)|^{\frac{p-2}{2}}d\tau\,\varphi(t)\right\rangle_{H^1}$$
$$=-\frac{p}{2}\int_0^T {}_{H^{-1}}\left\langle\frac{dw_i^\varepsilon(t)}{dt}|w_i^\varepsilon(t)|^{\frac{p-2}{2}},\varphi(t)\right\rangle_{H^1}dt,\ \forall\varphi\in C_0^\infty((0,T)\times\mathbb{R}^d),\tag{3.34}$$

where we used that the functions in the second slot of the dualization above are bounded in $L^2(0,T;H^1)$, uniformly in $h$, hence (selecting a subsequence, if necessary) weakly converge in $L^2(0,T;H^1)$ to $|w_i^\varepsilon|^{\frac{p-2}{2}}\varphi$, as $h\to 0$. We note that we have also used above that (3.27) implies $w_\varepsilon\in C([0,T];L^2)$. So, to prove (3.33), it suffices to show that the last expression in (3.34) is a continuous (linear) function in $\varphi$ with respect to the norm of $L^2(0,T;W^{1,\frac{q}{q-1}})$, whose norm is uniformly bounded in $\varepsilon\in(0,1]$. To this end, we first note that by (3.28) the last expression in (3.34) is equal to

$$\frac{p}{2}\int_0^T\int_{\mathbb{R}^d}\left(f(t,x)\cdot(\varphi(t,x)\nabla g(t,x)+g(t,x)\nabla\varphi(t,x))\right)dx\,dt,\tag{3.35}$$

where

$$f=|w^\varepsilon|^{p-2}\nabla w_i^\varepsilon+(p-2)w^\varepsilon(w^\varepsilon\cdot\nabla w_i^\varepsilon)|w^\varepsilon|^{p-4}+\varepsilon\nabla w_i^\varepsilon,$$
$$g=|w_i^\varepsilon|^{\frac{p-2}{2}}.$$

But, since $\frac{q}{q-1}>d$, the absolute value of the term in (3.35) up to a constant is dominated by

$$\int_0^T (|f(t)\nabla g(t)|_1 \|\varphi(t)\|_{W^{1,\frac{q}{q-1}}} + |f(t)|_2 |g(t)|_{\frac{2q}{2-q}} |\nabla\varphi(t)|_{\frac{q}{q-1}})dt$$
$$\leq \int_0^T (|f(t)\nabla g(t)|_1 + |f(t)|_2^2 + |g(t)|^2_{\frac{2q}{q-2}})dt \|\varphi\|_{L^\infty(0,T;W^{1,\frac{q}{q-1}})}$$
$$\leq C_p \Big( |D_i u_0|_\infty^{\frac{p-4}{2}} |D_i u_0|_2^2$$
$$+ (|\nabla u_0|_\infty^{p-2} + 1)|D_i u_0|_2^2 + T|D_i u_0|_2^{\frac{2(2-q)}{q}} |D_i u_0|_\infty^{\frac{4(p-1)}{q}} \Big) \|\varphi\|_{L^\infty(0,T;W^{1,\frac{q}{q-1}})},$$

where we have used (3.23) and (3.25). So, (3.33) is proved.

We note that

$$|\nabla(|w_i^\varepsilon|^{\frac{p}{2}})| = |\nabla(w_i^\varepsilon|^{\frac{p-2}{2}} w_i^\varepsilon)|, \text{ a.e. on } (0,T)\times\mathbb{R}^d,$$

and so, by (3.30) and (3.33) we have

$$\int_0^T \int_{\mathbb{R}^d} |\nabla(\eta_i^\varepsilon(t,x))|^2 dx\, dt + \int_0^T \left\| \frac{d}{dt}(\eta_i^\varepsilon(t)) \right\|_{W^{-1,q}} dt \leq C,$$

where

$$\eta_i^\varepsilon(t,x) = |w_i^\varepsilon(t,x)|^{\frac{p-2}{2}} w_i^\varepsilon(t,x), \text{ a.e. } (t,x)\in(0,T)\times\mathbb{R}^d,$$

and $C$ is independet of $\varepsilon$. In particular, it follows that

$$\int_0^T \left( \int_{B_R} |\nabla\eta_i^\varepsilon(t,x)|^2 dx + \left\| \frac{d}{dt}\eta_i(t) \right\|_{W^{-1,q}(B_R)} \right) dt \leq C,$$

for each ball $B_R = \{x; |x| < R\}$.

Then, by the Aubin-Lions–Simon compactness theorem ([14]) applied to the spaces $H^1(B_R) \subset L^2(B_R) \subset W^{-1,q}(B_R)$, where $R>0$ is arbitrary, we infer that, for each $i=1,...,d$, the sequence $\{\eta_i^{\varepsilon_n}\}_n$ is compact in $L^2(0,T;L^2_{\rm loc})$. Hence, along a subsequence $\{\varepsilon_n\}\to 0$, as $n\to\infty$, we have

$$\eta_i^{\varepsilon_n} \to \eta_i \text{ strongly in } L^2(0,T;L^2_{\rm loc})$$

and so, selecting the further subsequence,

$$\eta_i^{\varepsilon_n}(t,x) \to \eta_i(t,x), \text{ a.e. } (t,x)\times\mathbb{R}^d. \tag{3.36}$$

Taking into account that the function $z \to |z|^{\frac{p-2}{2}} z$ is monotonically increasing on $\mathbb{R}$, we have

$$w_i^{\varepsilon_n} \to \widetilde{w}_i, \text{ a.e. in } (0,T) \times \mathbb{R}^d$$

and, recalling (3.31), we infer that $\widetilde{w}_i = w_i$ and so, by (3.36) it follows that $\eta_i = |w_i|^{\frac{p-2}{2}} w_i$, a.e. on $(0,T) \times \mathbb{R}^d$ and, therefore, it follows that, for $n \to \infty$,

$$|w_i^{\varepsilon_n}|^{\frac{p}{2}} \to |w_i|^{\frac{p}{2}} \text{ strongly in } L^2(0,T;L^2_{\text{loc}}),\ i = 1, ..., d.$$

Since, by (3.31), $\{|\nabla(w^{\varepsilon_n})|\}_n$ is bounded in $L^2(0,T;L^2)$, we also have

$$\nabla(|w^{\varepsilon_n}|^{\frac{p}{2}}) \to \nabla(|w|^{\frac{p}{2}}) \text{ weakly in } L^2(0,T;(L^2)^d),$$

and so, by weak lower-semicontinuity it follows by (3.30) that

$$\int_0^\infty \int_{\mathbb{R}^d} |\nabla(|w|^{\frac{p}{2}})|^2 dxdt \le \frac{dp^2}{8} |\nabla u_0|_2^2.$$

Hence, we have

$$\int_0^T \int_{\mathbb{R}^d} |\nabla |\nabla u|^{\frac{p}{2}}|^2 dxdt \le \frac{dp^2}{8} |\nabla u_0|_2^2. \tag{3.37}$$

This yields

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} |\nabla(|\nabla u|^{p-2})| dxdt &= \frac{2(p-2)}{p} \int_0^T \int_{\mathbb{R}^d} |\nabla(|\nabla u|^{\frac{p}{2}})|\, |\nabla u|^{\frac{p-4}{2}} dxdt \\
&\le \sqrt{\frac{d}{2}}\,(p-2)|\nabla u_0|_2 \left( \int_0^T |\nabla u(t)|_{p-4}^{p-4} dt \right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{d}{2}}\,(p-2)|\nabla u_0|_2 (Vol(B_{R(T)}))^{\frac{2}{p}} \left( \int_0^T |\nabla u(t)|_p^p dt \right)^{\frac{p-4}{2p}} \\
&\le 2^{\frac{2}{p}} \sqrt{d}\, \frac{p-2}{2} |\nabla u_0|_2 |u_0|_2^{\frac{p-4}{p}}\ T^{\frac{2}{p}} (Vol(B_{R(T)}))^{\frac{2}{p}},\ \forall T > 0,
\end{aligned} \tag{3.38}$$

where we used (2.19). □

**Remark 3.3.** There are several earlier works concerning the first order regularity for weak solutions to the parabolic $p$-Laplace equation (1.1) (see, e.g., [1], [8], [10], [11]). However, it seems that the estimate (3.3) is new. As mentioned earlier, a related local second-order result was established in [12] (see also [13]). But, therein, integrability with respect to time is only proved over time intervals $(\delta, T]$ with $\delta > 0$. We, however, crucially need integrability over $[0,T]$, as will be seen in the next section.

# 4 The probabilistic representation

We shall assume here, as in Section 4, that $p \geq 4$ and

$$u_0 \in W^{1,\infty}, \quad u_0 \geq 0, \text{ a.e. in } \mathbb{R}^d \tag{4.1}$$

$$\textit{support } u_0 \subset B_R, \ \int_{\mathbb{R}^d} u_0(x)dx = 1. \tag{4.2}$$

Then, as seen earlier in Theorem 2.3 and Theorem 2.4, equation (1.1) has a unique weak solution $u$ on $(0,\infty) \times \mathbb{R}^d$, satisfying (2.17)–(2.21) and

$$u \in C([0,\infty); L^2) \cap L^\infty(0,\infty; L^2), \ u \geq 0, \text{ a.e. on } (0,\infty) \times \mathbb{R}, \tag{4.3}$$

$$u \in L^2(0,\infty; H^1) \cap L^\infty((0,\infty) \times \mathbb{R}^d), \tag{4.4}$$

$$\int_{\mathbb{R}^d} u(t,x)dx = 1, \ \forall t \geq 0. \tag{4.5}$$

Moreover, the estimates (3.1)–(3.2) hold.

As seen earlier in (1.2), we may rewrite equation (1.1) as the Fokker–Planck equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta(|\nabla u|^{p-2}u) + \operatorname{div}(\nabla(|\nabla u|^{p-2})u) = 0 \text{ in } \mathcal{D}'((0,\infty) \times \mathbb{R}^d), \\ &u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \end{aligned} \tag{4.6}$$

and taking into account that $u \in L^\infty((0,T) \times \mathbb{R}^d)$, by (3.1), (3.2), we have the estimates

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u(t,x)|^{p-2} |u(t,x)| dxdt \leq C_T(R) |u_0|_2^{\frac{2(p-2)}{2}}, \tag{4.7}$$

$$\int_0^T \int_{\mathbb{R}^d} |\nabla(|\nabla u(t,x)|^{p-2})||u(t,x)| dxdt \leq C_T^1(R) T |u_0|_2^{\frac{p-4}{p}} |\nabla u_0|_2, \tag{4.8}$$

for all $T > 0$. Moreover, by Proposition 2.5, we have that

$$\textit{support } u(t) \subset B_{2R+R(t)}, \ \forall t \geq 0. \tag{4.9}$$

Also, by (2.16) it follows that $u : [0,\infty) \to L^2$ is continuous. Then, we finally obtain:

**Theorem 4.1.** *Let $u$ be the solution to* (1.1) *under hypotheses* (4.1)–(4.2) *on the initial data $u_0$. Then, there exists a $d$-dimensional $(\mathcal{F}_t)$-Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and an $(\mathcal{F}_t)$-progressively measurable map $X : [0,\infty) \times \Omega \to \mathbb{R}^d$, continuous in $t$, solving the McKean–Vlasov stochastic differential equation*

$$dX(t) = \nabla(|\nabla u(t, X(t))|^{p-2})dt + \sqrt{2}|\nabla u(t, X(t))|^{\frac{p-2}{2}} dW(t) \qquad (4.10)$$

*such that*

$$u(t, x)dx = \mathbb{P} \circ X(t)^{-1}(dx), \ \ t \geq 0. \qquad (4.11)$$

*Proof.* This is a direct consequence of [3, Section 2] (see also [4, Chapter 5] and the proof of [5, Theorem 3.3]). □

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