SHARP NON-UNIQUENESS IN LAW FOR STOCHASTIC DIFFERENTIAL EQUATIONS ON THE WHOLE SPACE

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ABSTRACT. In this paper, we investigate the stochastic differential equation on $\mathbb{R}^d, d \geqslant 2$:
$$\mathrm{d}X_t = v(t, X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$
For any finite collection of initial probability measures $\{\mu_0^i\}_{1 \leqslant i \leqslant M}$ on $\mathbb{R}^d$ and $\frac{d}{p} + \frac{1}{r} > 1$, we construct a divergence-free drift field $v \in L_t^r L^p \cap C_t L^{d-}$ such that the associated SDE admits at least two distinct weak solutions originating from each initial measure μ_0^i. This result is sharp in view of the well-known uniqueness of strong solutions for drifts in $C_t L^{d+}$, as established in [KR05]. As a corollary, there exists a measurable set $A \subset \mathbb{R}^d$ with positive Lebesgue measure such that for any $x \in A$, the SDE with drift v admits at least two weak solutions when with start in $x \in A$. The proof proceeds by constructing two distinct probability solutions to the associated Fokker-Planck equation via a convex integration method adapted to all of $\mathbb{R}^d$ (instead of merely the torus), together with refined heat kernel estimate.

CONTENTS

1. Introduction

In this paper, we are concerned with stochastic differential equations (SDEs) on the whole space $\mathbb{R}^d, d \geqslant 2$, of type

$$\mathrm{d}X_t = v(t, X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \quad t \in [0, T], \tag{1.1}$$
$$\mathcal{L}_{X_0} = \mu_0,$$

where $T > 0$, $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Borel function, W_t is a standard $\mathbb{R}^d$-valued Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and μ_0 is a probability measure on $\mathbb{R}^d$.

The well-posedness of SDEs has been a central topic in both pure and applied mathematics, due to their fundamental role in modeling dynamical systems with random perturbations. To analyze the criticality of equation (1.1), we employ the following scaling transformation:

$$W_t^\epsilon = \epsilon^{-1} W_{\epsilon^2 t}, \ X_t^\epsilon = \epsilon^{-1} X_{\epsilon^2 t}, \ v^\epsilon(t, x) = \epsilon v(\epsilon^2 t, \epsilon x),$$

which preserve the law of the Brownian motion. Under this scaling, the behavior of the rescaled drift v^ϵ in the Lebesgue space $L_t^r L^p := L^r([0, T]; L^p(\mathbb{R}^d))$ is characterized by:

$$\|v^\epsilon\|_{L_t^r L^p} = \epsilon^{1 - \frac{d}{p} - \frac{2}{r}} \|v\|_{L_t^r L^p}.$$

As $\epsilon \to 0$, SDE (1.1) is classified as subcritical when $\frac{d}{p} + \frac{2}{r} < 1$, in which case the associated quantity vanishes. The critical regime corresponds to $\frac{d}{p} + \frac{2}{r} = 1$ and the supercritical regime corresponds to $\frac{d}{p} + \frac{2}{r} > 1$, where the quantity diverges, indicating that the Brownian noise is no longer sufficient to counteract the singularity of the drift.

For $p = r = \infty$, Veretennikov [Ver80] was the first to demonstrate the uniqueness of strong solutions using Zvonkin's transform [Zvo74]. In the subcritical case, Krylov and the second named author [KR05] established the existence and uniqueness of strong solutions to (1.1). Then the well-posedness problem with general multiplicative noise was studied in the foundational work [Zha05, Zha11]. Subsequent developments are contained in [BC03, FGP10, Zha11, FF13, Zha16, BFGM19, XXZZ20, RZ21].

The critical regime presents subtle challenges. Beck, Flandoli, Gubinelli and Maurelli [BFGM19] proved that for $\|v\|_{L^d}$ small enough, the SDE admits a strong solution starting from a diffusive random variable in a certain class. A significant progress was made by Krylov [Kry20a] who proved the strong well-posedness of SDEs in the case $v \in L^d(\mathbb{R}^d)$. Then, for the case $d \geqslant 3$, $v \in L_t^r L^p, d < p \leqslant \infty$, or $v \in C_t L^d$, or $v \in L_t^\infty L^d$ with divergence-free condition, Zhao and the second named author [RZ23] proved that (1.1) admits a unique weak solution within a class satisfying some Krylov-type estimate. We refer to [Nam20, Kry20b, Kry20c, RZ25, RZZ25] for further results.

However, in the supercritical case, the well-posedness problem is much more challenging and remains not fully understood. On the one hand, under the additional assumption that the drift is

divergence-free, weak well-posedness has been established in certain classes: Zhang and Zhao [ZZ21] established weak existence and uniqueness in the sense of approximation for $\frac{d}{p} + \frac{2}{r} \leqslant 2$; Galeati [Gal25] demonstrated strong existence and uniqueness for (1.1), under the assumption that v is a Leray solution to the 3D Navier-Stokes equations obtained through approximation. We also refer to [Gal24, HZ25, GP24, Grä24] for more results in the supercritical regime. On the other hand, notable counterexamples demonstrating non-uniqueness in law for (1.1) have also been constructed: For dimensions $d \geqslant 1$ and $p > d$, Galeati and Gerencsér [GG25] constructed an example (allowed to be not divergence-free), where non-uniqueness in law holds when starting from the origin. In $d \geqslant 3$, the second named author, Zhang and Zhao [RZZ25] exhibited a divergence-free drift in weak Lorentz space $L^{p,\infty}$, $\frac{d}{2} < p < d$, that admits two distinct weak solutions starting from the origin. When considering the torus case, sharp non-uniqueness in law in dimension 2 was shown in [LRZ25].

In particular, if we restrict to the case $r = \infty$, then for any vector field $v \in L_t^\infty L^p$, $p > d$, (1.1) admits a unique strong solution (see [KR05]). For the case $d \geqslant 3$, for any $v \in C_t L^d$, or divergence-free $v \in L_t^\infty L^d$, the SDE admits a unique weak solution within a class such that some Krylov-type estimate holds (see [RZ23]). However, for a given initial probability measure on $\mathbb{R}^d$, the question of whether uniqueness in law holds for SDEs with drifts $v \in L_t^\infty L^p$ in the supercritical regime $p < d$ remains open, especially for the two-dimensional case. Notably, in all known examples in the literature, non-uniqueness in law occurs only when starting from the origin.

It is also worth mentioning that, in the supercritical case, the divergence-free condition on the drift matters. Actually, if the drift is not divergence-free, there is an example in [BFGM19, Example 7.4] such that the SDE may not have weak solutions in $L^p(\mathbb{R}^d), p < d$. On the other hand, the divergence-free condition helps when considering the existence and conditional weak uniqueness, cf. [ZZ21, HZ25, RZZ25]. In summary, it is particularly intriguing to investigate divergence-free drift terms that may lead to a failure of weak uniqueness.

1.1. **Main result.** Regarding the aforementioned discussion, in this paper, we make a new contribution to the understanding of weak (non-)uniqueness in SDEs by constructing a divergence-free drift $v \in C_t L^p, p < d$ that leads to non-uniqueness in law, which covers the optimal range for the weak well-/ill-posedness of the SDE.

Let $\mathcal{P}$ denote the space of probability measures on $\mathbb{R}^d$. For any finite collection of initial probability measures $\{\mu_0^i\}_{1 \leqslant i \leqslant M} \subset \mathcal{P}$ and $\frac{d}{p} + \frac{1}{r} > 1$, our main result is to construct a divergence-free drift $v \in L_t^r L^p$ such that non-uniqueness in law holds for the respective SDE with drift v started from each of these initial measures:

$$\mathrm{d}X_t^i = v(t, X_t^i)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \tag{1.2}$$
$$\mathcal{L}_{X_0^i} = \mu_0^i.$$

Theorem 1.1. *Let $d \geqslant 2, \gamma \in (0,1), 1 < s < d$ and $p, r \in [1, \infty]$ satisfying $\frac{d}{p} + \frac{1}{r} > 1$. For a finite collection of initial distributions $\{\mu_0^i\}_{1 \leqslant i \leqslant M} \subset \mathcal{P}$, there exists a divergence-free vector field $v \in L^r([0,T]; L^p) \cap C([0,T]; L^s)$ such that each SDE (1.2) admits at least two distinct weak solutions $X^{i,1}, X^{i,2}$. Moreover, for $1 \leqslant i \leqslant M, j = 1, 2$, it holds that*

- $\mathbf{E}[\int_0^T |v(s, X_s^{i,j})|^{1+\epsilon}\mathrm{d}s] < \infty$ *for some $\epsilon > 0$;*
- *for $t > 0$, the solutions admit densities $\rho^{i,j}$ satisfying $\rho^{i,j}(t) \geqslant (1-\gamma)e^{t\Delta}\mu_0^i$.*

Our work also extends previous results in [GG25, RZZ25], which were limited to initial distributions concentrated at the origin. We demonstrate that weak non-uniqueness can occur for arbitrary initial probability measures. Through decomposition of probability measures, we further establish

that this non-uniqueness in law holds for a set of initial positions x with positive Lebesgue measure, thereby substantially broadening the scope of possible non-uniqueness scenarios.

Corollary 1.2. *Let $d \geqslant 2, 1 < s < d$ and $p, r \in [1, \infty]$ satisfying $\frac{d}{p} + \frac{1}{r} > 1$. Then there exists a divergence-free vector field $v \in L^r([0, T]; L^p) \cap C([0, T]; L^s)$ and a measurable set $A \subset \mathbb{R}^d$ with positive Lebesgue measure such that for every $x \in A$, there are at least two distinct weak solutions to* (1.1) *starting at δ_x.*

Unlike the previous counterexamples, in this paper we establish a stronger form of non-uniqueness, specifically, the non-uniqueness in the evolution of particle densities. These densities solve the associated Fokker-Planck equations, which read as

$$\partial_t \rho^i - \Delta \rho^i + \text{div}(v\rho^i) = 0, \quad \text{div} v = 0, \tag{1.3}$$
$$\rho^i(0)\mathrm{d}x = \mu_0^i(\mathrm{d}x),$$

where $\{\mu_0^i\}_{1 \leqslant i \leqslant M}$ is a collection of probability measures on $\mathbb{R}^d$. A solution to (1.3) is meant in the following weak sense:

Definition 1.3. *A non-negative function $\rho \in C((0, T]; L^1)$ is called a weak solution to* (1.3) *if*

$$\int_0^T \int_{\mathbb{R}^d} |v(t, x)| \rho(t, x) \mathrm{d}x \mathrm{d}t < \infty,$$

and for every test function $f \in C_c^2(\mathbb{R}^d)$, and every $t \in (0, T]$

$$\int_{\mathbb{R}^d} f(x)\rho(t, x)\mathrm{d}x - \int f(x)\mu_0^i(\mathrm{d}x) = \int_0^t \int_{\mathbb{R}^d} \rho(s, x)(\Delta f + v \cdot \nabla f)(s, x)\mathrm{d}x\mathrm{d}s. \tag{1.4}$$

With the help of the superposition principle introduced in [Tre16, Theorem 2.5], Theorem 1.1 is a direct result of the following non-uniqueness result for the Fokker-Planck equation (1.3), which we shall establish through the convex integration method, together with refined heat kernel estimates.

Theorem 1.4. *Let $d \geqslant 2, \gamma \in (0, 1), 1 < s < d$ and $p, r \in [1, \infty]$ satisfying $\frac{d}{p} + \frac{1}{r} > 1$. Then there exists a divergence-free vector field $v \in L^r([0, T]; L^p) \cap C([0, T]; L^s)$, such that each Fokker-Planck equation* (1.3) *admits at least two distinct solutions $\rho^{i,1}, \rho^{i,2} \in C((0, T]; L^1)$ in the sense of Definition 1.3.*

Moreover, for $1 \leqslant i \leqslant M, j = 1, 2$, it holds that $\int_{[0,T]} \int |v|^{1+\epsilon} \rho^{i,j} \mathrm{d}x \mathrm{d}s < \infty$ for some $\epsilon > 0$ and that $\rho^{i,j}(t) \geqslant (1 - \gamma)e^{t\Delta} \mu_0^i$ for $t > 0$.

Finally, we note that by linearity, (1.3) admits infinitely many distinct solutions for every $1 \leqslant i \leqslant M$. Consequently, the associated SDE (1.2) also has infinitely many distinct weak solutions.

The main result will be established using the convex integration technique. This technique was first introduced to fluid dynamics by De Lellis and Székelyhidi Jr. [DLS09, DLS10, DLS13]. This method has led to numerous groundbreaking results for fluid dynamics on the torus. For the incompressible Euler equations, the famous Onsager conjecture was proved in [Ise18, BDLSV19]. For the Navier-Stokes equations, sharp non-uniqueness of weak solutions has been shown in [BV19b, BCV21, CL22, CL23]. We refer to [BDLIS15, DSJ17, GKN23, NV23, GKN24, GR24, BCK24] for more progress on the Euler or Navier-Stokes equations. Recently, this method has been applied to fluid dynamics on the whole space, see [MNY24a, MNY24b]. For more details and references, interested readers are referred to the comprehensive reviews [BV19a, BV21].

Concerning transport equations on the tours, we refer to [CGSW15, MS18, MS19, MS20, BCDL21, CL21, PS23, CL24, MS24] for recent developments concerning non-uniqueness in transport equations and related ODEs. We also note that the convex integration method has been successfully applied to stochastic fluid dynamics, see [BFH20, HZZ22, Yam22, HZZ23a, HZZ23b, HLP24, HZZ24, Pap24, MS24, HZZ25, LZ25a, LZ25b, LRZ25].

1.2. **Idea of the proof.** As mentioned earlier, we will prove the main result using the convex integration method. This method is primarily developed in the torus setting, and for the convenience of readers who may not be so familiar with it, in Appendix A we provide an overview of how the convex integration method works on the torus.

Our work presents a novel extension of the convex integration method to establish non-uniqueness of positive solutions for Fokker-Planck equations on the whole space $\mathbb{R}^d$, going beyond previous periodic domain results. The transition from the torus to unbounded domains introduces substantial new challenges that require new probabilistic and analytical arguments. These technical innovations entail, to our knowledge, the first successful application of convex integration methods to prove non-uniqueness for positive solutions in non-periodic settings.

(1) When working in the whole space setting, maintaining compact support for the stress error remains crucial for implementing convex integration. However, a fundamental difficulty arises in the iteration process: we cannot use the inverse of the divergence operator directly, since even for a function with compact support, the inverse of its divergence may still have support in the whole space. To overcome this obstacle, we introduce a new decomposition of the stress term (detailed in Section 4.4.1) that separates it into two distinct components - a principal part, expressed as the divergence of a compactly supported function, and a residual global error term with small bounded $C_{t,x}^0$ norm. This decomposition enables us to proceed with the convex integration scheme by handling the principal part through standard techniques, while simultaneously controlling the global error via a specially constructed global perturbation.

(2) When constructing global perturbations, we face the fundamental challenge of preserving positivity of the solution, which differs significantly from those encountered in fluid equations. In addition, we do not have an invariant measure as a reference measure in contract to the torus case. The core of our approach lies in using the heat-propagated measure $e^{t\Delta}\mu_0$ as a reference solution, around which we build two distinct positive solutions through carefully designed perturbations. A critical technical requirement emerges from the spatial decay of $e^{t\Delta}\mu_0$: our perturbations must precisely match its exponential decay at infinity to maintain global positivity. This is achieved through a novel strategy by constructing drift terms with compact support, coupled with a careful analysis of the influence in exterior regions. The key innovation centers on our treatment of the global perturbation $\theta^{(\text{glo})}$ generated by the drift, which solves the heat equation:

$$\partial_t \theta^{(\text{glo})} - \Delta \theta^{(\text{glo})} = F,$$

with a specially designed forcing term F. Here F is constructed with compact spatial support and vanishes near the time origin $t = 0$, enabling the crucial proper exponential decay

$$\left| \int_0^t e^{(t-s)\Delta} F(s) \mathrm{d}s \right| \ll e^{t\Delta}\mu_0 \tag{1.5}$$

through Proposition 4.4 using heat kernel estimates. The required smallness condition is attained via an additional iteration of the inverse divergence operator developed in Section 4.4.1, combined with the periodic properties of our building blocks.

We further note that the key estimate (1.5) appears to be new and, to the best of our knowledge, has not been addressed in previous literature. This estimate plays a crucial role in our analysis, and its possible generalization to more general diffusion operators remains an open and particularly interesting problem.

(3) To achieve the supercritical regime $\frac{d}{p} + \frac{1}{r} > 1$, we introduce the L^m-based intermittent spatial-time jets, incorporating an additional degree of freedom through the parameter $m \in [1, \infty]$. This additional freedom stems from a feature of transport-type equations, contrasting with the L^2-based constraint in fluid equations.

Beyond those already mentioned, the proof also requires several additional modifications for the $\mathbb{R}^d$ setting, such as a carefully tuned mollification procedure and a refined construction of the principal density perturbations. We will detail these technical adaptations in the proof in the main text of this proof.

Organization of the paper. First, in Section 2, we introduce the L^m-based intermittent spatial-time jets, which play a crucial role in the convex integration construction. Subsequently, in Section 3, we establish Theorem 1.1 through a proof of the non-uniqueness of solutions to the Fokker-Planck equations, as stated in Theorem 1.4. The proof of the main convex integration iteration procedure is presented in Section 4. In Appendix A, we give an overview of the convex integration method on the torus.

Notations. Let $T > 0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout the manuscript, we write $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ for the d-dimensional flat torus, and identify $\mathbb{T}^d$-valued functions with periodic functions on $\mathbb{R}^d$.

- We employ the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$.
- Given a Banach space E with a norm $\|\cdot\|_E$, we write $C_t E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to E, equipped with the supremum norm. For $p \in [1, \infty]$ we write $L_t^p E = L^p([0, T]; E)$ for the space of L^p-integrable functions from $[0, T]$ to E, equipped with the usual L^p-norm. We also wirte $L_{[a,b]}^p E = L^p([a, b]; E)$.
- For $\alpha \in (0, 1)$ we define $C_t^\alpha E$ to be the space of α-Hölder continuous functions from $[0, T]$ to E, endowed with the norm $\|f\|_{C_t^\alpha E} = \sup_{s,t \in [0,T], s \neq t} \frac{\|f(s) - f(t)\|_E}{|t-s|^\alpha} + \sup_{t \in [0,T]} \|f(t)\|_E$, and write C_t^α in the case when $E = \mathbb{R}$.
- We use L^p to denote the set of standard L^p-integrable functions. For $s > 0$, $p > 1$ we set $W^{s,p} := \{f \in L^p; \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}$ with the norm $\|f\|_{W^{s,p}} = \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p}$.
- For $N \in \mathbb{N}_0$, C^N denotes the space of N-times differentiable functions equipped with the norm

$$\|f\|_{C^N} := \sum_{|\alpha| \leqslant N, \alpha \in \mathbb{N}_0^d} \|D^\alpha f\|_{L_x^\infty}.$$

  Similarly, if the norm is taken in space-time, we use the notation $C_{t,x}^N$.
- For any $\mathbb{T}^d$-valued smooth function f, we define the projections $\mathbb{P}_{=0} f := \int_{\mathbb{T}^d} f \, dx$, and $\mathbb{P}_{\neq 0} f := f - \int_{\mathbb{T}^d} f \, dx$.
- By $p(t, x) = (4\pi t)^{-\frac{d}{2}} e^{-|x|^2/4t}$ we denote the d-dimensional heat kernel for Δ on $\mathbb{R}^d$.
- We denote the Lebesgue measure on $\mathbb{R}^d$ by dx.
- We denote the law of a random variable X by $\mathcal{L}_X$.

2. L^m-BASED INTERMITTENT SPATIAL-TIME JETS

In this section we introduce the the notion of L^m-based intermittent spatial-time jets, where $m \in [1, \infty]$, which can be seen as a generalization of the L^2-based building blocks presented in [LRZ25, Appendix C.1]. In this section, the building blocks are defined on the torus and are regarded as periodic functions on $\mathbb{R}^d$.

First, we introduce the following geometrical lemma:

Lemma 2.1. [BCDL21, Lemma 3.1] *Let $d \geqslant 2$. There exists a finite set $\Lambda \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ and non-negative C^∞-function $\Gamma_\xi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that for every $R \in \mathbb{S}^{d-1}$*

$$R = \sum_{\xi \in \Lambda} \Gamma_\xi(R)\xi.$$

With Lemma 2.1 in hand, it is easy to generate 2 disjoint families Λ^1, Λ^2, where each one enjoys the property of Lemma 2.1 by taking suitable rational rotations of one fixed set. For simplicity, we denote $\Lambda := \Lambda^1 \cup \Lambda^2$. Moreover, we know that $\{\Gamma_\xi\}_{\xi \in \Lambda}$ are uniformly bounded.

For parameters $\lambda, r_\perp, r_\parallel > 0$, we assume

$$\lambda^{-1} \ll r_\perp \ll r_\parallel \ll 1, \quad \lambda r_\perp \in \mathbb{N}.$$

For each $\xi \in \Lambda$ let us define $A^i_\xi \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$, $i = 1, 2, ..., d-1$, such that $\{\xi, A^i_\xi, i = 1, ..., d-1\}$ form an orthonormal basis in $\mathbb{R}^d$. Let $n_* \in \mathbb{N}$ such that $\{n_*\xi, n_*A^i_\xi, i = 1, ..., d-1\} \subset \mathbb{Z}^d$ for every $\xi \in \Lambda$.

We define $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ to be a smooth function supported in the unit ball, such that $\phi \equiv 1$ on $B(0, \frac{1}{3})$ and has zero mean. We then define Φ by $\phi = -\Delta\Phi$. The existence of such functions can be found in [LRZ25, Appendix C.1]. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, mean-zero function with support in $B(0, 1)$ satisfying $\psi \equiv 1$ on $B(0, \frac{1}{3})$.

Define $\phi' : \mathbb{R}^{d-1} \to \mathbb{R}$ to be a smooth non-negative function with support in $B(0, \frac{1}{3})$ satisfying

$$\int_{\mathbb{R}^{d-1}} \phi'(x_1, x_2, ..., x_{d-1}) \mathrm{d}x_1 \mathrm{d}x_2 .. \mathrm{d}x_{d-1} = 1,$$

and let $\psi' : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with support in $B(0, \frac{1}{3})$ such that

$$\int_{\mathbb{R}} \psi'(x_d) \mathrm{d}x_d = 1.$$

Then, it is straightforward to verify that

$$\phi\phi' = \phi', \quad \psi\psi' = \psi'. \tag{2.1}$$

Let $m \geqslant 1$ be fixed, we define the rescaled cut-off functions

$$\phi_{r_\perp, m}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/m}} \phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}\right),$$

$$\Phi_{r_\perp, m}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/m}} \Phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}\right),$$

$$\psi_{r_\parallel, m}(x_d) = \frac{1}{r_\parallel^{1/m}} \psi\left(\frac{x_d}{r_\parallel}\right).$$

Similarly, for a conjugate exponent $m' \in [1, \infty]$ satisfying $\frac{1}{m'} + \frac{1}{m} = 1$, we define $\phi'_{r_\perp, m'}$ and $\psi'_{r_\parallel, m'}$ as the same manner. Then we periodize $\phi_{r_\perp, m}$, $\Phi_{r_\perp, m}$, $\psi_{r_\parallel, m}$, $\phi'_{r_\perp, m'}$ and $\psi'_{r_\parallel, m'}$ so that they can be

viewed as functions on $\mathbb{T}^{d-1}$ and $\mathbb{T}$ respectively. Consider a large time oscillation parameter $\mu > 0$. For every $\xi \in \Lambda$ we introduce

$$\psi_{(\xi,m)}(t,x) := \psi_{r_\parallel,m}(n_* r_\perp \lambda (x \cdot \xi - \mu t)),$$

$$\Phi_{(\xi,m)}(x) := \Phi_{r_\perp,m}(n_* r_\perp \lambda x \cdot A_\xi^1, ..., n_* r_\perp \lambda x \cdot A_\xi^{d-1}),$$

$$\phi_{(\xi,m)}(x) := \phi_{r_\perp,m}(n_* r_\perp \lambda x \cdot A_\xi^1, ..., n_* r_\perp \lambda x \cdot A_\xi^{d-1}).$$

Similarly we define $\phi'_{(\xi,m')}$ and $\psi'_{(\xi,m')}$.

The building blokes $W_{(\xi,m)} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d$ and $\Theta_{(\xi,m')} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ are defined as

$$W_{(\xi,m)}(t,x) := \xi \psi_{(\xi,m)}(t,x) \phi_{(\xi,m)}(x),$$

$$\Theta_{(\xi,m')}(t,x) := \psi'_{(\xi,m')}(t,x) \phi'_{(\xi,m')}(x).$$

By the definition and (2.1) we have that

$$\int_{\mathbb{T}^d} W_{(\xi,m)} \Theta_{(\xi,m')} \mathrm{d}x = \xi, \tag{2.2}$$

$$\partial_t \Theta_{(\xi,m')} + \mu r_\perp^{\frac{d-1}{m}} r_\parallel^{\frac{1}{m}} \mathrm{div}(W_{(\xi,m)} \Theta_{(\xi,m')}) = 0. \tag{2.3}$$

Since $W_{(\xi,m)}$ is not divergence-free, we introduce the skew-symmetric corrector term

$$V_{(\xi,m)} := \frac{1}{(n_* \lambda)^2}(\xi \otimes \nabla \Phi_{(\xi,m)} - \nabla \Phi_{(\xi,m)} \otimes \xi)\psi_{(\xi,m)}$$

satisfying

$$\mathrm{div}V_{(\xi,m)} = W_{(\xi,m)} - \frac{1}{(n_* \lambda)^2}\nabla \Phi_{(\xi,m)} \xi \cdot \nabla \psi_{(\xi,m)}. \tag{2.4}$$

Finally, we obtain that for $N, M \geqslant 0$ and $p \in [1, \infty]$ the following holds

$$\|\nabla^N \partial_t^M \psi_{(\xi,m)}\|_{C_t L^p(\mathbb{T}^d)} \lesssim r_\parallel^{\frac{1}{p}-\frac{1}{m}}(\frac{r_\perp \lambda}{r_\parallel})^N(\frac{r_\perp \lambda \mu}{r_\parallel})^M, \tag{2.5}$$

$$\|\nabla^N \phi_{(\xi,m)}\|_{L^p(\mathbb{T}^d)} + \|\nabla^N \Phi_{(\xi,m)}\|_{L^p(\mathbb{T}^d)} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{m}}\lambda^N, \tag{2.6}$$

$$\|\nabla^N \partial_t^M W_{(\xi,m)}\|_{C_t L^p(\mathbb{T}^d)} + \lambda\|\nabla^N \partial_t^M V_{(\xi,m)}\|_{C_t L^p(\mathbb{T}^d)} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{m}} r_\parallel^{\frac{1}{p}-\frac{1}{m}} \lambda^N(\frac{r_\perp \lambda \mu}{r_\parallel})^M, \tag{2.7}$$

$$\|\nabla^N \partial_t^M \Theta_{(\xi,m')}\|_{C_t L^p(\mathbb{T}^d)} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{m'}} r_\parallel^{\frac{1}{p}-\frac{1}{m'}} \lambda^N(\frac{r_\perp \lambda \mu}{r_\parallel})^M, \tag{2.8}$$

where the implicit constants may depend on p, m, N and M, but are independent of $\lambda, r_\perp, r_\parallel, \mu$. These estimates can be easily deduced from the definitions.

Then let us introduce a family of temporal functions to oscillate the building blocks intermittently in time. Let $K \in \mathbb{N}$ be fixed, and $G \in C_c^\infty(0,1)$ be non-negative and

$$\int_0^1 G^2(t)\mathrm{d}t = 1.$$

Let $\eta > 0$ be a small constant satisfying $\eta K \ll 1$. For $\xi \in \Lambda$ as defined above, and $1 \leqslant i \leqslant K$, we define $\tilde{g}_{(\xi,i,m)} : \mathbb{T} \to \mathbb{R}$ as the 1-periodic extension of $\eta^{-1/m}G(\frac{t-t_{(\xi,i)}}{\eta})$, where $t_{(\xi,i)}$ are chosen so that $\tilde{g}_{(\xi,i,m)}$ have disjoint supports for distinct (ξ,i). We will also oscillate the perturbations at a large frequency $\sigma \in \mathbb{N}$. So, we define

$$g_{(\xi,i,m)}(t) = \tilde{g}_{(\xi,i,m)}(\sigma t).$$

For the corrector term we define $H_{(\xi,i,m)}, h_{(\xi,i,m)} : \mathbb{T} \to \mathbb{R}$ by

$$H_{(\xi,i,m)}(t) = \int_0^t g_{(\xi,i,m)}(s)\mathrm{d}s, \quad h_{(\xi,i,m)}(t) = \int_0^{\sigma t} (\tilde{g}_{(\xi,i,m)}(s)\tilde{g}_{(\xi,i,m')}(s) - 1)\mathrm{d}s, \tag{2.9}$$

where we recall that $\frac{1}{m'} + \frac{1}{m} = 1$, In view of the zero-mean condition for $\tilde{g}_{(\xi,i,m)}(t)\tilde{g}_{(\xi,i,m')}(t) - 1$, we see that $h_{(\xi,i,m)}$ is $\mathbb{T}/\sigma$-periodic, and for any $N \geqslant 0, p \geqslant 1$

$$\|g_{(\xi,i,m)}\|_{W^{N,p}} \lesssim (\frac{\sigma}{\eta})^N \eta^{1/p-1/m}, \quad \|h_{(\xi,i,m)}\|_{L^\infty} \leqslant 1, \tag{2.10}$$

where the universal constant is independent of the choices of i and ξ.

3. Proof of Main result

In this section, we prove our main result, Theorem 1.1. Without loss of generality, we set $T = 1$ from now on. The proof begins by establishing Theorem 1.4 at the PDE level, where we construct non-unique solutions to the Fokker-Planck equation (1.3). To this end, we consider a system of $N_0 = 2M$ equations given by

$$\partial_t \rho^i - \Delta \rho^i + \mathrm{div}(v\rho^{in,i}) + \mathrm{div}(v\rho^i) = 0, \quad \mathrm{div}v = 0, \tag{3.1}$$

$$\rho^i(0) = 0,$$

where the initial densities are defined for $1 \leqslant k \leqslant M$ by

$$\rho^{in,2k-1}(t) = \rho^{in,2k}(t) := e^{t\Delta}\mu_0^i.$$

In what follows, the drift v will be constructed so that it vanishes near the origin $t = 0$. It is then straightforward to verify that $\rho^{in,i} + \rho^i$ satisfies equation (1.3).

Regarding equation (3.1), for parameters satisfying $1 < s < d$ and $\frac{d}{p} + \frac{1}{r} > 1$, our goal is to prove Theorem 1.4 by constructing a divergence-free drift $v \in L_t^r L^p \cap L_t^{d_0} L^{d_0} \cap C_t L^s$ such that for each i, equation (3.1) admits a solution $\rho^i \in L_t^{d_0'} L^{d_0'} \cap C_t L^1$, where d_0 and d_0' are conjugate exponents satisfying $\frac{1}{d_0} + \frac{1}{d_0'} = 1$. Moreover, we show that $\rho^{2k-1} \neq \rho^{2k}$ for $1 \leqslant k \leqslant M$, thereby demonstrating the non-uniqueness of solutions.

The proof employs a convex integration scheme specifically adapted to construct positive solutions to the Fokker–Planck equations on the whole space $\mathbb{R}^d$. In Section 3.1, we establish several key estimates for the initial components $\rho^{in,i}$. Subsequently, the framework of the convex integration iteration and the corresponding iterative procedure are developed in Proposition 3.1 within Section 3.2. The proof of Theorem 1.4 is then presented in Section 3.3, followed by the derivation of Theorem 1.1 through an application of the superposition principle.

3.1. **Estimate on $\rho^{in,i}$.** We begin by deriving several fundamental estimates for the initial densities $\rho^{in,i}$, which will play a crucial role in the subsequent analysis. Recall that $p(t,x)$ denotes the d-dimensional heat kernel associated with the Laplacian Δ. We define the averaged heat kernel

$$\bar{p}(t,x) := \int_{[-\frac{1}{2},\frac{1}{2}]^d} p(t,x-y)\mathrm{d}y.$$

By definition and the positivity of the heat kernel $p(t,x)$, it is straightforward to verify that $\rho^{in,i}(\frac{1}{12}, x) > 0$ for any $1 \leqslant i \leqslant N_0, x \in \mathbb{R}^d$. Moreover, since $[-1/2, 1/2]^d$ is a compact domain, there exists $c_{in} > 0$ such that for any $1 \leqslant i \leqslant N_0, x \in [-1/2, 1/2]^d$,

$$\rho^{in,i}(\frac{1}{12}, x) \geqslant c_{in}.$$

Then, it holds that for $t \in [\frac{1}{12}, 1]$,

$$\rho^{in,i}(t,x) = \int_{\mathbb{R}^d} p(t - \frac{1}{12}, x - y)\rho^{in,i}(\frac{1}{12}, y)\mathrm{d}y$$

$$\geqslant c_{in} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(t - \frac{1}{12}, x - y)\mathrm{d}y = c_{in}\overline{p}(t - \frac{1}{12}, x).$$

In particular, for $(t,x) \in [\frac{1}{6}, 1] \times [-1/2, 1/2]^d$, by definition, it holds that

$$\overline{p}(t - \frac{1}{12}, x) \geqslant \int_{[-\frac{1}{2}, \frac{1}{2}]^d} (4\pi)^{-d/2}e^{-3d}\mathrm{d}y =: c_d,$$

which implies that for $(t,x) \in [\frac{1}{6}, 1] \times [-1/2, 1/2]^d$, it holds that

$$\rho^{in,i}(t,x) \geqslant c_{in}\overline{p}(t - \frac{1}{12}, x) \geqslant c_{in}c_d. \tag{3.2}$$

Moreover, by the smoothing property of the heat kernel, there exists a constant $0 < C_{in} < \infty$ such that, for all $1 \leqslant i \leqslant N_0$, the following estimate holds:

$$\|\rho^{in,i}\|_{C^1_{[\frac{1}{6}, 1], x}} \leqslant C_{in}. \tag{3.3}$$

3.2. **Convex integration set up.** We now apply the convex integration iteration to the system (3.1), with the iteration indexed by $q \in \mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q \in \mathbb{N}_0} \subset \mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q \in \mathbb{N}_0} \subset (0, 1]$ which is decreasing to 0. We choose $a \in \mathbb{N}_0, \beta \in (0, 1)$ and $b \in \mathbb{N}$. Let

$$\lambda_q = a^{(b^q)}, q \geqslant 0, \quad \delta_q = \epsilon_0^{d+1}\lambda_1^{2\beta}\lambda_q^{-2\beta}, q \geqslant 1, \quad \delta_0 = 1,$$

where $\beta > 0$ will be chosen sufficiently small and a, b will be chosen sufficiently large. Here $\epsilon_0 \in (0, \frac{1}{12}]$ is a small universal constant to be determined later. Moreover, we use the estimate

$$\sum_{q \geqslant 1} \delta_q^{1/(d+1)} \leqslant \epsilon_0 \sum_{q \geqslant 1} a^{(1-q)2b\beta/(d+1)} \leqslant \frac{\epsilon_0}{1 - a^{-2b\beta/(d+1)}} < 2\epsilon_0,$$

which boils down to

$$a^{2b\beta/(d+1)} > 2, \tag{3.4}$$

assumed from now on.

At each step q, a pair $(v_q, \rho_q^i, M_q^i)_{1 \leqslant i \leqslant N_0}$ is constructed solving the following system on $[0, 1]$:

$$\partial_t\rho_q^i - \Delta\rho_q^i + \mathrm{div}(v_q\rho^{in,i}) + \mathrm{div}(v_q\rho_q^i) = -\mathrm{div}M_q^i, \quad \mathrm{div}v_q = 0, \tag{3.5}$$

where M_q^i are some vector fields.

To handle the initial condition, we require that $\rho_q^i = 0$ on $[0, T_q]$, where

$$T_q := \frac{1}{3} - \sum_{1 \leqslant r \leqslant q} \delta_r^{1/2}.$$

By applying (3.4) we obtain $\frac{1}{6} < T_q \leqslant \frac{1}{3}$. Here and in the following we define $\sum_{1 \leqslant r \leqslant 0} := 0$. To control the spatial support of the drift and the stress term, we further define

$$\Omega_q := \Big[-\frac{1}{3} - \sum_{1 \leqslant r \leqslant q} \delta_r^{1/2}, \frac{1}{3} + \sum_{1 \leqslant r \leqslant q} \delta_r^{1/2}\Big]^d \subset [-\frac{1}{2}, \frac{1}{2}]^d.$$

With the above assumptions in place, the main iteration scheme is formulated as follows:

Proposition 3.1. *Under the assumption of Theorem 1.1, there exist $d + 1 > d_0 > 2 > d_0' > 1$ with $\frac{1}{d_0} + \frac{1}{d_0'} = 1$ and a choice of parameters a, b, β such that the following holds: Let $(v_q, \rho_q^i, M_q^i)_{1 \leqslant i \leqslant N_0}$ be a solution to the system (3.5) satisfying $\int_{\mathbb{R}^d} \rho_q^i \mathrm{d}x = 0$,*

$$\|v_q\|_{L_t^{d_0} L^{d_0}} \leqslant C_v C_0^{1/d_0} \sum_{m=0}^{q} \delta_m^{1/d_0}, \quad \|\rho_q^i\|_{L_t^{d_0'} L^{d_0'}} \leqslant C_\rho C_0^{1/d_0'} \sum_{m=0}^{q} \delta_m^{1/d_0'} \tag{3.6}$$

for some universal constants $C_0, C_v, C_\rho \geqslant 1$, and

$$\|v_q\|_{C_{t,x}^1} \leqslant C_0^{1/d_0} \lambda_q^{d+4}, \ \|\rho_q^i\|_{C_{t,x}^1} \leqslant C_0^{1/d_0'} \lambda_q^{d+4}, \tag{3.7}$$

$$\|M_q^i\|_{L_t^1 L^1} \leqslant C_0 \delta_{q+1}, \quad \|\partial_t M_q^i\|_{L_t^1 L^1} + \|\nabla M_q^i\|_{L_t^1 L^1} \leqslant \lambda_q^{2d+8}, \tag{3.8}$$

$$\rho_q^i(t) = v_q(t) = M_q^i(t) = 0 \text{ on } [0, T_q], \tag{3.9}$$

$$\operatorname{supp} v_q, \operatorname{supp} M_q^i \subset \Omega_q. \tag{3.10}$$

Then there exists $(v_{q+1}, \rho_{q+1}^i, M_{q+1}^i)_{1 \leqslant i \leqslant N_0}$ which solves (3.5) and satisfies (3.6)-(3.10) at the level $q + 1$ and

$$\|v_{q+1} - v_q\|_{L_t^{d_0} L^{d_0}} \leqslant C_v C_0^{1/d_0} \delta_{q+1}^{1/d_0}, \quad \|\rho_{q+1}^i - \rho_q^i\|_{L_t^{d_0'} L^{d_0'}} \leqslant C_\rho C_0^{1/d_0'} \delta_{q+1}^{1/d_0'}. \tag{3.11}$$

Moreover, we have

$$\|v_{q+1} - v_q\|_{L_t^r L^p} \leqslant \delta_{q+1}^{1/d_0}, \quad \|v_{q+1} - v_q\|_{C_t L^s} \leqslant \delta_{q+1}^{1/d_0}, \tag{3.12}$$

$$\|\rho_{q+1}^i - \rho_q^i\|_{C_t L^1} \leqslant \delta_{q+1}^{1/d_0'}, \tag{3.13}$$

$$(\rho_{q+1}^i - \rho_q^i)(t, x) \geqslant -\delta_{q+1}^{1/d_0'} \overline{p}(t - \frac{1}{12}, x) \text{ for } t \in [\frac{1}{6}, 1]. \tag{3.14}$$

Here C_0 is determined by the choice of the starting iterations, and C_v, C_ρ are two constants determined by the improved Hölder's inequality for v_q, $\{\rho_q^i\}_{1 \leqslant i \leqslant N_0}$ respectively, and other implicit constants in the proof. We further note that the term $v_q \rho^{in,i}$ is always well-defined by noticing (3.3), (3.7) and the fact that $v_q = 0$ on $[0, T_q]$.

Proposition 3.1 constitutes the central technical component of this paper. A detailed proof of the proposition will be presented in Section 4 below.

3.3. **Proof of main result.** In this section, we first present the proof of Theorem 1.4, assuming the validity of Proposition 3.1.

Proof of Theorem 1.4. Let $\gamma \in (0, 1)$ be fixed. Let $F(t, x)$ be a smooth (not divergence-free) bounded $\mathbb{R}^d$-valued function with support in $[\frac{1}{3}, 1] \times \Omega_0$, and satisfying

$$\|\operatorname{div} F\|_{C_{t,x}^0} \leqslant \frac{\gamma}{2} c_{in} c_d.$$

Then we denote

$$\|\operatorname{div} F\|_{C_t L^1} =: c_F > 0.$$

Let $\epsilon_0 = \min\{\frac{\gamma c_{in}}{4}, \frac{c_F}{2}, \frac{1}{12}\}$. We intend to start the iteration from $(v_0, \rho_0^i, M_0^i)_{1 \leqslant i \leqslant N_0}$ which are defined as

$$\rho_0^i = (-1)^i \operatorname{div} F, \quad v_0 = 0, \quad M_0^i = (-1)^{i+1} (\partial_t - \Delta) F.$$

By choosing C_0 large enough (depending on ϵ_0), we have

$$\|\rho_0^i\|_{L_t^{d_0'} L^{d_0'}} + \|\rho_0^i\|_{C_{t,x}^1} \leqslant C_0^{1/d_0'}, \quad \|M_0^i\|_{L_t^1 L^1} \leqslant \epsilon_0^{d+1} C_0.$$

Then (3.6)-(3.10) are satisfied as $\delta_0 = 1, \delta_1 = \epsilon_0^{d+1}$.

Moreover, by (3.2) we have

$$\frac{\gamma}{2}\rho^{in,i} + \rho_0^i \geqslant (\frac{\gamma}{2}\rho^{in,i} + \rho_0^i)1_{[\frac{1}{6},1]\times[-1/2,1/2]^d} \geqslant \frac{\gamma}{2}c_{in}c_d - \frac{\gamma}{2}c_{in}c_d = 0. \tag{3.15}$$

Next, we use Proposition 3.1 to build inductively (v_q, ρ_q^i, M_q^i) for every $q \geqslant 1$. By (3.4) and (3.11)-(3.13), the sequence $\{(v_q, \rho_q^i)\}_{q\in\mathbb{N}}$ is Cauchy in

$$\left(L^r([0,1];L^p) \cap L^{d_0}([0,1] \times \mathbb{R}^d) \cap C([0,1];L^s)\right) \times \left(L^{d_0'}([0,1] \times \mathbb{R}^d) \cap C([0,1];L^1)\right)^{N_0}$$

and we denote by (v, ρ^i) its limit, where v is also divergence-free with compact support. Since $\int \rho_q^i \mathrm{d}x = 0$, we deduce that $\int \rho^i \mathrm{d}x = 0$. Clearly by (3.8) and (3.9), ρ^i solves (3.1).

Now, we define $\overline{\rho}^i := \rho^i + \rho^{in,i}$, which satisfies (1.3) in the sense of Definition 1.3. Then we verify that each $\overline{\rho}^i$ is nonnegative and $\overline{\rho}^{2k-1}$ and $\overline{\rho}^{2k}$ are distinct from one another with the same initial distribution. In fact, by (3.2), (3.4), (3.13)-(3.15) we have

$$\rho^i + \gamma\rho^{in,i} \geqslant \frac{\gamma}{2}\rho^{in,i} + \sum_{q=0}^{\infty}(\rho_{q+1}^i - \rho_q^i) \geqslant (\frac{\gamma}{2}c_{in} - \sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'})\overline{p}(t - \frac{1}{12}, x) > 0, \text{ for } t > \frac{1}{6},$$

$$\|\overline{\rho}^{2k-1} - \overline{\rho}^{2k}\|_{C_{[\frac{2}{3},1]}L^1} \geqslant \|\rho_0^{2k-1} - \rho_0^{2k}\|_{C_{[\frac{2}{3},1]}L^1} - \|\rho^{2k-1} - \rho_0^{2k-1}\|_{C_{[\frac{2}{3},1]}L^1} - \|\rho_0^{2k} - \rho^{2k}\|_{C_{[\frac{2}{3},1]}L^1}$$

$$\geqslant 2(\|\mathrm{div}F\|_{C_tL^1} - \sum_{q=0}^{\infty}\delta_{q+1}^{1/d_0'}) > 0.$$

Then by the fact that $\rho^i(t) = 0$ for $t \in [0, \frac{1}{6}]$, we imply that $\overline{\rho}^i \geqslant (1-\gamma)\rho^{in,i}$.

By (3.7), (3.11) and interpolation, we have $|v| \in L_t^{d_0(1+\epsilon)}L^{d_0(1+\epsilon)}$ for some $\epsilon > 0$ sufficiently small. Since $|\rho^i| \in L_t^{d_0'}L^{d_0'}$, we deduce that $|v|^{1+\epsilon}\rho^i \in L_t^1L^1$. Furthermore, from (3.3) and the fact that v has compact support satisfying $v = 0$ on $[0, \frac{1}{6}]$, we conclude $|v|^{1+\epsilon}\rho^{in,i} \in L_t^1L^1$. Combining these results, we obtain $|v|^{1+\epsilon}\overline{\rho}^i \in L_t^1L^1$. $\square$

Having established Theorem 1.4, Theorem 1.1 follows directly through an application of the superposition principle.

Proof of Theorem 1.1. For $1 < s < d, \frac{d}{p} + \frac{1}{r} > 1$, by applying Theorem 1.4, there are $v \in L^r([0,1];L^p) \cap C([0,1];L^s)$ and a collection of probability densities $\overline{\rho}^i(t)$ satisfying (1.3). Then we define $\mu_t^i(\mathrm{d}x) := \overline{\rho}^i(t)\mathrm{d}x$, which forms a family of probability measures. Moreover, these measures satisfy

$$\int_0^1 \int |v(s,x)|\mu_s^i(\mathrm{d}x)\mathrm{d}s = \int_0^1 \int_{\mathbb{R}^d} |v(s,x)|\overline{\rho}^i(s,x)\mathrm{d}x\mathrm{d}s < \infty,$$

and $t \to \mu_t^i$ is weakly continuous on $[0,1]$, we are in position to apply the superposition principle (see [Tre16, Theorem 2.5]) for

$$\partial_t\mu^i - \Delta\mu^i + \mathrm{div}(v\mu^i) = 0, \quad \mu_{t=0}^i = \mu_0^i.$$

More precisely, let $C([0,1];\mathbb{R}^d)$ be the space of continuous functions, equipped with its Borel σ-algebra and its natural filtration generated by the canonical process $\Pi_t, t \in [0,1]$, defined by

$$\Pi_t(\omega) := \omega(t), \quad \omega \in C([0,1];\mathbb{R}^d).$$

There exists a group of probability measure $\mathbf{Q}^i$ on $C([0,1];\mathbb{R}^d)$ which is a martingale solution associated to diffusion operator

$$L := \Delta + v \cdot \nabla.$$

Then by a standard result (see [Str87, Theorem 2.6]), there exists a d-dimensional $(\mathcal{F}_t)_{t \geqslant 0}$-Brownian motion W_t, $t \in [0,1]$, on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbf{P})$ and a group of continuous $(\mathcal{F}_t)_{t \geqslant 0}$-progressively measurable maps $\{X_t^i\}_{1 \leqslant i \leqslant 2M} : [0,1] \times \Omega \to \mathbb{R}^d$ satisfying SDE (1.2). Moreover, we have $\mathbf{P} \circ (X_t^i)^{-1} = \mathbf{Q}^i \circ \Pi_t^{-1} = \mu_t^i$ for $t \in [0,1]$. Then it is easy to see that $\mathcal{L}_{X_0^{2k-1}} = \mathcal{L}_{X_0^{2k}} = \mu_0^i$, while $\mathcal{L}_{X_t^{2k-1}} \neq \mathcal{L}_{X_t^{2k}}$.

$\square$

Corollary 1.2 follows by an argument analogous to the proof of [LRZ25, Theorem 1.7].

4. Proof of Proposition 3.1

In this section, we extend the convex integration method to construct solutions for a system defined on the whole space. The proof follows the convex integration approach while incorporating necessary adaptations for the unbounded domain. We begin in Section 4.1 by determining the choice of parameters. Section 4.2 then details the mollification procedure. The core construction appears in Section 4.3, where we define the new iteration pair (v_{q+1}, ρ_{q+1}^i) with a carefully designed global perturbation $\theta_{q+1}^{(g,i)}$ that ensures that we can define a suitable new stress error supported in the domain Ω_{q+1}. Subsequently, Section 4.4 introduces the crucial stress terms M_{q+1}^i and the global stress error F_{q+1}^i. At this stage, it is essential to apply the inverse divergence operator iteration, as introduced in Section 4.4.1, to ensure that the support of M_{q+1}^i is located in Ω_{q+1}. This condition is crucial for maintaining the consistency of the iterative process. In the end, in Sections 4.5-4.9, we conclude the proof with the verification of all required inductive estimates. The most technically involved step consists in estimating the global perturbation $\theta_{q+1}^{(g,i)}$ to ensure that it exhibits the decay behavior required by (3.14). This analysis relies crucially on the heat kernel estimates established in Proposition 4.4.

4.1. **Choice of parameters.** In our analysis, several carefully chosen parameters will play crucial roles, with their values being precisely calibrated to satisfy the intricate network of compatibility conditions required for our estimates. Let $d \geqslant 2, 1 < s < d, \frac{d}{p} + \frac{1}{r} > 1$ be fixed. First we introduce a integer $N > 4d$ large enough satisfying

$$\frac{d}{p} + \frac{1}{r} > 1 + \frac{4d}{N}, \quad \frac{d}{s} > 1 + \frac{4d}{N},$$

and define

$$d_0 := d + 1 - \frac{4d}{N} \in (d, d+1), \quad d_0' := \frac{d_0}{d_0 - 1} \in (1, 2).$$

Then for the sufficiently small $\alpha \in (0,1)$ to be chosen, we take $l := \lambda_{q+1}^{-\frac{3\alpha}{2}} \lambda_q^{-d-4}$ and have

$$l^{-1} \leqslant \lambda_{q+1}^{2\alpha}, \quad l\lambda_q^{2d+8} \leqslant \lambda_{q+1}^{-\alpha} \ll \delta_{q+2}, \quad \lambda_q^{d+4} \ll \lambda_{q+1}^{\alpha}, \tag{4.1}$$

provided $\alpha b > 2d + 8, \alpha > 2\beta b$. In the sequel, we also need $(12d + 42)\alpha < \frac{1}{2N}$.

The above can be obtained by choosing $\alpha > 0$ small such that $(12d + 42)\alpha < \frac{1}{2N}$, and choosing $b \in 2N\mathbb{N}$ and large enough such that $\alpha b > 2d + 8$, and finally choosing $0 < \beta < \frac{\alpha}{2b}$.

In the end, we increase a such that (3.4) holds. In the sequel, we also increase a to absorb various implicit and universal constants in subsequent estimates.

4.2. Mollification.

To avoid the loss of derivative, we first need to mollify the stress term. Let $\phi_l := \frac{1}{l^d}\phi(\frac{\cdot}{l})$ be a family of standard radial mollifiers on $\mathbb{R}^d$, and $\varphi_l := \frac{1}{l}\varphi(\frac{\cdot}{l})$ be a family of standard mollifiers with support in $(0,1)$. We define

$$M_l^i := (M_q^i *_x \phi_l) *_t \varphi_l.$$

For the mollification around $t = 0$, since M_q^i vanishes around $t = 0$, we can directly extend its value to $t \leqslant 0$ by 0. Here we deliberately avoid mollifying the density terms to avoid the challenge in controlling the difference $\rho_q^i - \rho_l^i$, which must maintain exponential decay crucial to the argument.

Since $l \leqslant \delta_{q+1}^{1/2}$, we know that $M_l^i(t) = 0$ for $t \in [0, T_{q+1}]$, and $\operatorname{supp} M_l^i \subset \Omega_{q+1}$.

To end this section, by the mollification estimates, the spatial-time embedding $W^{d+\frac{4}{3},1} \subset L^\infty$ and (3.8) we obtain for $N \geqslant 0$,

$$\|M_l^i\|_{C_{t,x}^N} \lesssim l^{-d-\frac{4}{3}-N}\|M_q^i\|_{L_t^1 L^1} \lesssim C_0 l^{-d-\frac{4}{3}-N}. \tag{4.2}$$

4.3. Construction of v_{q+1} and ρ_{q+1}^i.

In this section, we proceed with the construction on the perturbations on v_q and ρ_q^i employing the L^{d_0}-based building blocks introduced in Section 2. First, we define the parameters

$$\lambda = \lambda_{q+1}, \ r_\perp = \lambda^{-1+\frac{1}{N}}, \ r_\parallel = \lambda^{-1+\frac{2}{N}}, \ \eta = \lambda^{-1}, \ \mu = r_\perp^{-\frac{d-1}{d_0}} r_\parallel^{-\frac{1}{d_0}}, \ \sigma = \lambda^{\frac{1}{2N}}, \tag{4.3}$$

where we recall that d_0, d_0' and $N > 4d$ are defined in Section 4.1. Then we have

$$r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{p}-\frac{1}{d_0}} \eta^{\frac{1}{r}-\frac{1}{d_0}}, \ r_\perp^{\frac{d-1}{s}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{s}-\frac{1}{d_0}} \eta^{-\frac{1}{d_0}} \leqslant \lambda^{-\frac{1}{N}},$$

$$r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \eta^{-\frac{1}{d_0'}}, \ \lambda r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \eta^{1-\frac{1}{d_0'}} \leqslant \lambda^{-\frac{1}{N}}. \tag{4.4}$$

In fact, by a direct calculation we have

$$\lambda r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \eta^{1-\frac{1}{d_0'}} = r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \eta^{-\frac{1}{d_0'}} \leqslant r_\parallel^{\frac{d}{d_0}} \eta^{\frac{1}{d_0}-1} = \lambda^{\frac{2d}{Nd_0}+\frac{d_0-d-1}{d_0}} = \lambda^{-\frac{2d}{Nd_0}} \leqslant \lambda^{-\frac{1}{N}},$$

$$r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{p}-\frac{1}{d_0}} \eta^{\frac{1}{r}-\frac{1}{d_0}} \leqslant \lambda^{(\frac{d+1}{N}-d)(\frac{1}{p}-\frac{1}{d_0})-\frac{1}{r}+\frac{1}{d_0}} \leqslant \lambda^{-\frac{d}{p}-\frac{1}{r}+\frac{d+1}{d_0}+\frac{d+1}{N}} \leqslant \lambda^{-1-\frac{4d}{N}+1+\frac{2d}{N}+\frac{d+1}{N}} \leqslant \lambda^{-\frac{1}{N}},$$

$$r_\perp^{\frac{d-1}{s}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{s}-\frac{1}{d_0}} \eta^{-\frac{1}{d_0}} \leqslant \lambda^{(\frac{d+1}{N}-d)(\frac{1}{s}-\frac{1}{d_0})+\frac{1}{d_0}} \leqslant \lambda^{-\frac{d}{s}+\frac{d+1}{d_0}+\frac{d+1}{N}} \leqslant \lambda^{-1-\frac{4d}{N}+1+\frac{2d}{N}+\frac{d+1}{N}} \leqslant \lambda^{-\frac{1}{N}}.$$

Here it is required that b is a multiple of $2N$ to ensure that $\lambda_{q+1} r_\perp = a^{(b^{q+1})/N} \in \mathbb{N}$ and $\sigma = a^{(b^{q+1})/2N} \in \mathbb{N}$.

Let $\chi \in C_c^\infty(-\frac{3}{4}, \frac{3}{4})$ be a non-negative function such that $\sum_{n \in \mathbb{Z}} \chi(t-n) = 1$ for every $t \in \mathbb{R}$. Let $\tilde{\chi} \in C_c^\infty(-\frac{4}{5}, \frac{4}{5})$ be a non-negative function satisfying $\tilde{\chi} = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$ and $\sum_{n \in \mathbb{Z}} \tilde{\chi}(t-n) \leqslant 2$.

Then, we fix a family parameters $\zeta = 20/\delta_{q+2}$ and consider two disjoint sets Λ^1, Λ^2 defined in Section 2. We use the notation $\Lambda^i = \Lambda^1$ for i odd, and $\Lambda^i = \Lambda^2$ for i even. Taking $K = N_0$ in the construction of Section 2, we obtain a family of pairwise disjoint functions $g_{(\xi,i,d_0)}$ and $g_{(\xi,i,d_0')}$ for $\xi \in \Lambda$ and $1 \leqslant i \leqslant N_0$. In the following we apply the building blocks $W_{(\xi,d_0)}, \Theta_{(\xi,d_0')}$ and $H_{(\xi,i,d_0)}$ as defined in Section 2 to define

$$W_{(\xi,n,i)}(x,t) := W_{(\xi,d_0)}\left(x, \left(\frac{n}{\zeta}\right)^{1/d_0} H_{(\xi,i,d_0)}(t)\right),$$

$$\Theta_{(\xi,n,i)}(x,t) := \Theta_{(\xi,d_0')}\Big(x, \Big(\frac{n}{\zeta}\Big)^{1/d_0} H_{(\xi,i,d_0)}(t)\Big).$$

Similarly, we define $V_{(\xi,n,i)}, \Phi_{(\xi,n,i)}$ and all other terms appearing in Section 2. Now by (2.3), (2.9) and the choice of μ we have

$$\partial_t \Theta_{(\xi,n,i)} + \Big(\frac{n}{\zeta}\Big)^{1/d_0} g_{(\xi,i,d_0)} \operatorname{div}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}) = 0. \tag{4.5}$$

As the next step, we first define the perturbations for the drift term. For every $1 \leqslant i \leqslant N_0$, we define the principle perturbation and the correct perturbation as

$$w_{q+1}^{(p,i)} := \sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} \tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} W_{(\xi,n,i)} g_{(\xi,i,d_0)},$$

$$w_{q+1}^{(c,i)} := \sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} \Big(-\tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} \frac{1}{(n_*\lambda_{q+1})^2}\nabla\Phi_{(\xi,n,i)}\xi \cdot \nabla\psi_{(\xi,n,i)}$$

$$+ \nabla(\tilde{\chi}(\zeta|M_l^i| - n))\Big(\frac{n}{\zeta}\Big)^{1/d_0} : V_{(\xi,n,i)}\Big)g_{(\xi,i,d_0)}.$$

Here we use the notation $(\nabla(\tilde{\chi}(\zeta|M_l^i| - n)) : V_{(\xi,n,i)})^i := \sum_{j=1}^d \partial_j(\tilde{\chi}(\zeta|M_l^i| - n))V_{(\xi,n,i)}^{ij}$ for $i = 1, 2, ..., d$. We remark that here and in the following the first sum runs for n in the range

$$3 \leqslant n \leqslant 1 + \zeta|M_l^i| \leqslant 1 + l^{-2/3}\|M_l^i\|_{C_{t,x}^0} \leqslant 1 + Cl^{-d-2},$$

where we used (4.2) and C is a universal constant.

By (2.4) we have $\operatorname{div}(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}) = 0$ since

$$w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)} = \sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} \operatorname{div}\Big(\tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} V_{(\xi,n,i)}\Big)g_{(\xi,i,d_0)}, \tag{4.6}$$

and $V_{(\xi,n,i)}$ is a skew-symmetric matrix.

Then we define the perturbations for the densities. For $1 \leqslant i \leqslant N_0$, we define the principle perturbation and the correct perturbation as

$$\theta_{q+1}^{(p,i)} := \sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'}\sum_{\xi \in \Lambda^n}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\Big(\Theta_{(\xi,n,i)} - \mathbb{P}_0\Theta_{(\xi,n,i)}\Big)g_{(\xi,i,d_0')},$$

$$\theta_{q+1}^{(c,i)} := -\hat{\rho}_q^i \cdot \int_{\mathbb{R}^d}\theta_{q+1}^{(p,i)}\mathrm{d}x, \tag{4.7}$$

where we recall $\mathbb{P}_0 f = \int_{\mathbb{T}^d} f\mathrm{d}x$. Here $\hat{\rho}_q^i$ is some probability density with support in Ω_{q+1} satisfying $\|\hat{\rho}_q^i\|_{C_x^2} \lesssim 1$.

Here, we use of the projected term $\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)}$ rather than $\Theta_{(\xi,n,i)}$ in constructing the principal perturbation. This projection guarantees the applicability of the inverse divergence operation during subsequent stress term construction.

Then by the identity (2.2), Lemma 2.1, the fact that $g_{(\xi,i,d_0)}$ have disjoint supports for $\xi \neq \xi'$, and the fact that $\chi\tilde{\chi} = \chi$ we obtain

$$w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} = \sum_{n \geqslant 3}\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\sum_{\xi \in \Lambda^n}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)W_{(\xi,n,i)}\Big(\Theta_{(\xi,n,i)} - \mathbb{P}_0\Theta_{(\xi,n,i)}\Big)g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}$$

$$
\begin{aligned}
&= \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})g_{(\xi,i,d_0)}g_{(\xi,i,d_0')} \\
&\quad - \sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\sum_{\xi \in \Lambda^n}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)W_{(\xi,n,i)}\mathbb{P}_0\Theta_{(\xi,n,i)}g_{(\xi,i,d_0)}g_{(\xi,i,d_0')} \\
&\quad + \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n}\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi(g_{(\xi,i,d_0)}g_{(\xi,i,d_0')} - 1) \\
&\quad + \sum_{n \geqslant 3}\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|}. \tag{4.8}
\end{aligned}
$$

Now to deal with the undesired term in the last second line in (4.8), we define the oscillation perturbation as

$$
\theta_{q+1}^{(o,i)} := -\sigma^{-1}\sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} h_{(\xi,i,d_0)}\mathrm{div}\Big(\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big).
$$

By (2.9), we have

$$
\begin{aligned}
&\partial_t\theta_{q+1}^{(o,i)} + \sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n}(g_{(\xi,i,d_0)}g_{(\xi,i,d_0')} - 1)\mathrm{div}\Big(\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big) \\
&\qquad = -\sigma^{-1}\sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} h_{(\xi,i,d_0)}\partial_t\mathrm{div}\Big(\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\xi\Big). \tag{4.9}
\end{aligned}
$$

Moreover, since M_l^i is compact supported and bounded, we know that $\theta_{q+1}^{(o,i)}$ is mean-zero.

Furthermore, we define the global perturbation as a solution to

$$
\begin{aligned}
\partial_t\theta_{q+1}^{(g,i)} - \Delta\theta_{q+1}^{(g,i)} &= F_{q+1}^i, \\
\theta_{q+1}^{(g,i)}(0) &= 0, \tag{4.10}
\end{aligned}
$$

where the force term F_{q+1}^i is determined by the "inverse divergence" of stress term at step $q+1$ with support outside Ω_{q+1}, see Proposition 4.3 below for the precise definition. In order not to cause any confusion, we should remark that F_{q+1}^i in fact depends only on $w_{q+1}^{(p,i)}$ and $\theta_{q+1}^{(p,i)}$. By Proposition 4.3, F_{q+1}^i is mean-zero, and satisfies $F_{q+1}^i(t) = 0$ on $[0, T_{q+1}]$. Then we have that $\theta_{q+1}^{(g,i)}$ is mean-zero, and satisfies $\theta_{q+1}^{(g,i)}(t) = 0$ on $[0, T_{q+1}]$.

Finally, the total perturbation and new iteration are defined by

$$
w_{q+1} := \sum_{i=1}^{N_0}\Big(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)}\Big), \quad v_{q+1} := v_q + w_{q+1}.
$$

Moreover, for every $1 \leqslant i \leqslant N_0$, we define

$$
\theta_{q+1}^{(loc,i)} := \theta_{q+1}^{(p,i)} + \theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)}, \quad \theta_{q+1}^i := \theta_{q+1}^{(loc,i)} + \theta_{q+1}^{(g,i)}, \quad \rho_{q+1}^i := \rho_q^i + \theta_{q+1}^i.
$$

Then v_{q+1} is mean-zero and divergence-free. By the definition, it is easy to see that $\int_{\mathbb{R}^d}(\theta_{q+1}^{(p,i)} + \theta_{q+1}^{(c,i)})\mathrm{d}x = 0$, which implies that $\int_{\mathbb{R}^d}\rho_{q+1}^i\mathrm{d}x = 0$.

Since $M_l^i(t) = 0$ on $[0, T_{q+1}]$, we have $w_{q+1}(t) = \theta_{q+1}^i(t) = 0$ on $[0, T_{q+1}]$. Then we have $v_{q+1}(t) = \rho_{q+1}^i(t) = 0$ on $[0, T_{q+1}]$. In summary we imply (3.9) for v_{q+1} and ρ_{q+1}^i. Similarly, since $\text{supp}\, M_l^i \subset \Omega_{q+1}$, we have that $\text{supp}\, w_{q+1}, \text{supp}\, \theta_{q+1}^{(loc,i)} \subset \Omega_{q+1}$, which implies (3.10) for v_{q+1}.

Moreover, by the fact that $g_{(\xi,i,d_0)}$ have disjoint supports for distinct i, we have

$$w_{q+1}\theta_{q+1}^i = (w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)})\theta_{q+1}^{(p,i)} + w_{q+1}(\theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)} + \theta_{q+1}^{(g,i)}). \tag{4.11}$$

4.4. **Construction of the stress terms M_{q+1}^i.** In this section, we present the exact expression for the stress term M_{q+1}^i. Unlike the torus setting, our construction cannot directly apply inverse divergence operators, since even when the stress term M_l^i has compact support, its inverse divergence may still remain support in the whole space.

4.4.1. *Inverse divergence iteration.* First, we introduce the following inverse divergence iteration. A similar procedure for vector-valued functions was introduced in [BMNV21, Proposition A.17].

Let $\{\rho_{(n)}\}_{0 \leqslant n \leqslant N+1}$ be the zero-mean smooth $\mathbb{T}^d$-periodic functions such that $\rho_{(n)} = \Delta\rho_{(n+1)}$. Then for any given function G on $\mathbb{R}^d$, we have

$$G\rho_{(0)} = \text{div}M_{(0)} + F_{(0)},$$

where $M_{(0)}^i = G\partial_i\rho_{(1)}$, $F_{(0)} := -\sum_i \partial_iG\partial_i\rho_{(1)}$.

Then, applying the decomposition to the error term $F_{(0)}$ at each step for N times, we have

$$G\rho_{(0)} = \text{div}M_{(N)} + F_{(N)}, \tag{4.12}$$

where for $1 \leqslant i \leqslant d$

$$M_{(N)}^i = \sum_{m=0}^N \sum_{|\alpha_m^i|=m+1} M_{\alpha_m^i}^{(m+1)}(G)\partial_{\alpha_m^i}\rho_{(m+1)}, \quad F_{(N)} = \sum_{|\alpha_N|=N+1} F_{\alpha_N}^{(N+1)}(G)\partial_{\alpha_N}\rho_{(N+1)}.$$

Here $M_{\alpha_m}^{(m+1)}(G)$ is composed of the m-th derivative of G and $F_{\alpha_m}^{(m)}(G)$ the m-th derivative of G.

Particularly, if $\text{supp}\, G \subset [-\frac{1}{2}, \frac{1}{2}]^d$, we have $\text{supp}\, M_{(N)}, \text{supp}\, F_{(N)} \subset \text{supp}\, G$.

4.4.2. *Decomposition of M_{q+1}^i.* From now on, we will write $\Gamma_\xi(\frac{M_l^i}{|M_l^i|}) = \Gamma_\xi$ if there is no ambiguity. From (4.11), the definition of the perturbations we obtain

$$-\text{div}M_{q+1}^i = \partial_t\theta_{q+1}^i + \text{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} - M_l^i) - \text{div}(M_q^i - M_l^i)$$
$$- \text{div}(\nabla\theta_{q+1}^i) + \text{div}(v_q\theta_{q+1}^i + w_{q+1}(\rho_q^i + \theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)} + \theta_{q+1}^{(g,i)} + \rho^{in,i}) + w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)}),$$

where together with (4.10) we define the linear error and commutator error by

$$M_{lin}^i := -\nabla\theta_{q+1}^{(loc,i)} + v_q\theta_{q+1}^i + w_{q+1}(\rho_q^i + \theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)} + \theta_{q+1}^{(g,i)} + \rho^{in,i}) + w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)},$$
$$M_{com}^i := M_q^i - M_l^i.$$

Then it holds that

$$-\text{div}M_{q+1}^i = \partial_t\theta_{q+1}^{(loc,i)} + \text{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} - M_l^i) + F_{q+1}^i + \text{div}M_{lin}^i - \text{div}M_{com}^i.$$

For simplicity, we define $\mathbb{P}_{\neq 0,\hat{\rho}_q^i}f := f - \hat{\rho}_q^i \cdot \int_{\mathbb{R}^d} f\mathrm{d}x$, where $\hat{\rho}_q^i$ is defined in (4.7). In particular, for any mean-zero function f, we have $\mathbb{P}_{\neq 0,\hat{\rho}_q^i}f = f$. To define the oscillation error, by the identities (4.8) and (4.9) we have

$$\partial_t\theta_{q+1}^{(loc,i)} + \text{div}(w_{q+1}^{(p,i)}\theta_{q+1}^{(p,i)} - M_l^i)$$

$$= \mathbb{P}_{\neq 0, \hat{\rho}_q^i}(\partial_t \theta_{q+1}^{(p,i)} + \mathrm{div}(w_{q+1}^{(p,i)} \theta_{q+1}^{(p,i)} - M_l^i) + \partial_t \theta_{q+1}^{(o,i)})$$

$$= \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \mathbb{P}_{\neq 0, \hat{\rho}_q^i}\Big(\partial_t[\chi(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'} \Gamma_\xi g_{(\xi,i,d_0')}]\mathbb{P}_{\neq 0} \Theta_{(\xi,n,i)}\Big)(:= \mathrm{div}M_{osc,t}^i + F_{osc,t}^i)$$

$$+ \mathbb{P}_{\neq 0, \hat{\rho}_q^i}\Big(\nabla[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi]g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})\Big)(:= \mathrm{div}M_{osc,x}^i + F_{osc,x}^i)$$

$$+ \mathbb{P}_{\neq 0, \hat{\rho}_q^i}\Big(\chi(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0'} g_{(\xi,i,d_0')}\Gamma_\xi$$

$$\times \Big(\partial_t \Theta_{(\xi,n,i)} + \Big(\frac{n}{\zeta}\Big)^{1/d_0} g_{(\xi,i,d_0)}\mathrm{div}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})\Big)\Big)$$

$$- \mathrm{div}\Big(\sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\sum_{\xi \in \Lambda^n} \Gamma_\xi W_{(\xi,n,i)}\mathbb{P}_0 \Theta_{(\xi,n,i)}g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}\Big)(:= \mathrm{div}M_{osc,a}^i)$$

$$+ \mathrm{div}\Big(\sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i\Big)(:= \mathrm{div}M_{osc,c}^i)$$

$$- \mathrm{div}\Big(\sigma^{-1}\sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} h_{(\xi,i,d_0)}\partial_t\Big(\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi \xi\Big)\Big)(:= \mathrm{div}M_{osc,o}^i),$$

where the last forth term equals to 0 by (4.5). Here we define

$$M_{osc,a}^i := -\sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\sum_{\xi \in \Lambda^n} \Gamma_\xi W_{(\xi,n,i)}\mathbb{P}_0 \Theta_{(\xi,n,i)}g_{(\xi,i,d_0)}g_{(\xi,i,d_0')},$$

$$M_{osc,c}^i := \sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i,$$

$$M_{osc,o}^i := -\sigma^{-1}\sum_{n \geqslant 3}\sum_{\xi \in \Lambda^n} h_{(\xi,i,d_0)}\partial_t\Big(\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi \xi\Big).$$

Now, we apply the inverse divergence iteration introduced above to define $M_{osc,x}$ and $M_{osc,t}$. We first write

$$A_{q+1}^{1,(\xi,n,i)} := \partial_t[\chi(\zeta|M_l^i| - n)(\frac{n}{\zeta})^{1/d_0'}\Gamma_\xi g_{(\xi,i,d_0')}],$$

$$A_{q+1}^{2,(\xi,n,i)} := \nabla[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi]g_{(\xi,i,d_0)}g_{(\xi,i,d_0')}.$$

Then, we obverse that $\Theta_{(\xi,n,i)}$ and $W_{(\xi,n,i)}\Theta_{(\xi,n,i)}$ are both $(\mathbb{T}/r_\perp \lambda_{q+1})^d$-periodic, so it is possible to define $\Delta^{-n}\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)}$ and $\Delta^{-n}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})$ for every $n \in \mathbb{N}$. By using the inverse divergence iterations in Section 4.4.1 to $\rho_{(n)} = \Delta^{-n}\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)}$ and to $\rho_{(n)} = \Delta^{-n}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)})$ respectively we obtain

$$A_{q+1}^{1,(\xi,n,i)}\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)} = \mathrm{div}M_{(N)}^{1,(\xi,n,i)} + F_{(N)}^{1,(\xi,n,i)},$$

$$A_{q+1}^{2,(\xi,n,i)}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}) = \mathrm{div}M_{(N)}^{2,(\xi,n,i)} + F_{(N)}^{2,(\xi,n,i)}.$$

Here we recall that $N > 4d$ is determined in Section 4.1, and we abuse the notation to define

$$(M_{(N)}^{1,(\xi,n,i)})^j := \sum_{m=0}^{N^2-1} \sum_{|\alpha_m^j|=m+1} M_{\alpha_m^j}^{(m+1)}(A_{q+1}^{1,(\xi,n,i)})\partial_{\alpha_m^j}\Delta^{-m-1}\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)},$$

$$F_{(N)}^{1,(\xi,n,i)} := \sum_{|\alpha_N|=N^2} F_{\alpha_N}^{(N^2)}(A_{q+1}^{1,(\xi,n,i)})\partial_{\alpha_N}\Delta^{-N^2}\mathbb{P}_{\neq 0}\Theta_{(\xi,n,i)},$$

$$(M_{(N)}^{2,(\xi,n,i)})^j := \sum_{m=0}^{N^2-1}\sum_{|\alpha_m^j|=m+1} M_{\alpha_m^j}^{(m+1)}(A_{q+1}^{2,(\xi,n,i)})\partial_{\alpha_m^j}\Delta^{-m-1}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}),$$

$$F_{(N)}^{2,(\xi,n,i)} := \sum_{|\alpha_N|=N^2} F_{\alpha_N}^{(N^2)}(A_{q+1}^{2,(\xi,n,i)})\partial_{\alpha_N}\Delta^{-N^2}\mathbb{P}_{\neq 0}(W_{(\xi,n,i)}\Theta_{(\xi,n,i)}).$$

Moreover, we have supp $M_{(N)}^{1,(\xi,n,i)}$, supp $M_{(N)}^{2,(\xi,n,i)}$, supp $F_{(N)}^{1,(\xi,n,i)}$, supp $F_{(N)}^{2,(\xi,n,i)} \subset \Omega_{q+1}$, and vanish on $[0, T_{q+1}]$. Then, we could define

$$M_{osc,t}^i := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} M_{(N)}^{1,(\xi,n,i)}, \quad M_{osc,x}^i := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} M_{(N)}^{2,(\xi,n,i)},$$

and

$$F_{osc,t}^i := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathbb{P}_{\neq 0,\hat{\rho}_q^i} F_{(N)}^{1,(\xi,n,i)}, \quad F_{osc,x}^i := \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \mathbb{P}_{\neq 0,\hat{\rho}_q^i} F_{(N)}^{2,(\xi,n,i)}.$$

Finally, we define $M_{osc}^i := M_{osc,t}^i + M_{osc,x}^i + M_{osc,a}^i + M_{osc,c}^i + M_{osc,o}^i$ and

$$-M_{q+1}^i := M_{osc}^i + M_{lin}^i - M_{com}^i, \quad F_{q+1}^i := -F_{osc,t}^i - F_{osc,x}^i. \tag{4.13}$$

Since $M_l^i(t) = w_{q+1}(t) = \theta_{q+1}^i(t) = 0$ on $[0, T_{q+1}]$, we have $M_{q+1}^i(t) = 0$ on $[0, T_{q+1}]$, which implies (3.9) for M_{q+1}^i. Similarly, we have supp $M_{q+1}^i \subset \Omega_{q+1}$, which implies (3.10) for M_{q+1}^i.

4.5. **Estimates of w_{q+1}.** In this section, we aim to establish the desired estimates for the perturbation w_{q+1}. First, we establish the estimate of the amplitude functions which could be obtained by the same calculation as [LRZ25, Proposition 5.2].

Lemma 4.1. *For $N \in \mathbb{N}_0, 1 \leqslant i \leqslant N_0$ we have*

$$\sum_{n\geqslant 3}\|\chi(\zeta|M_l^i|-n)\|_{C_{t,x}^N} + \sum_{n\geqslant 3}\|\tilde{\chi}(\zeta|M_l^i|-n)\|_{C_{t,x}^N} \lesssim l^{-(d+4)N-(d+2)},$$

$$\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\chi(\zeta|M_l^i|-n)\Gamma_\xi\Big(\frac{M_l^i}{|M_l^i|}\Big)\right\|_{C_{t,x}^N} \lesssim l^{-(2d+8)N-(d+2)},$$

$$\Big(\frac{n}{\zeta}\Big)^N 1_{\{\tilde{\chi}(\zeta|M_l^i|-n)>0\}} + \Big(\frac{n}{\zeta}\Big)^N 1_{\{\chi(\zeta|M_l^i|-n)>0\}} \lesssim l^{-N(d+2)},$$

$$\sum_{n\geqslant 3}\left\|\chi(\zeta|M_l^i|-n)\frac{M_l^i}{|M_l^i|}\right\|_{C_{t,x}^N} \lesssim l^{-(d+5)N-(2d+5)}.$$

Then we introduce the improved Hölder's inequality by using the additional decorrelation between frequencies.

Lemma 4.2. ([MNY24a, Lemma A.4]) *Let $d \geqslant 2, p \in [1,\infty]$. Let $f : \mathbb{T}^d \to \mathbb{R}$ be a smooth function and a be a smooth function on $\mathbb{R}^d$ such that* supp $a \subset [-\frac{1}{2}, \frac{1}{2}]^d$. *Then for any $\sigma \in \mathbb{N}$,*

$$\big|\|af(\sigma\cdot)\|_{L^p([-\frac{1}{2},\frac{1}{2}]^d)} - \|a\|_{L^p([-\frac{1}{2},\frac{1}{2}]^d)}\|f\|_{L^p(\mathbb{T}^d)}\big| \lesssim \sigma^{-1/p}\|a\|_{C^1([-\frac{1}{2},\frac{1}{2}]^d)}\|f\|_{L^p(\mathbb{T}^d)}.$$

Recall from Section 4.3 that the perturbation w_{q+1} decomposes as $w_{q+1} = \sum_{i=1}^{N_0}(w_{q+1}^{(p,i)} + w_{q+1}^{(c,i)})$. We first estimate the principle perturbations $w_{q+1}^{(p,i)}$ for $1 \leqslant i \leqslant N_0$ in $L_t^{d_0} L^{d_0}$-norm. By Cauchy's inequality we have

$$|w_{q+1}^{(p,i)}|^{d_0} \leqslant \Big(\sum_{n \geqslant 3} \tilde{\chi}(\zeta|M_l^i| - n) \Big)^{d_0-1} \sum_{n \geqslant 3} \tilde{\chi}(\zeta|M_l^i| - n)\frac{n}{\zeta} \Big| \sum_{\xi \in \Lambda^n} W_{(\xi,n,i)} g_{(\xi,i,d_0)} \Big|^{d_0}$$

$$\lesssim \sum_{n \geqslant 3} \tilde{\chi}(\zeta|M_l^i| - n)\frac{n}{\zeta} \sum_{\xi \in \Lambda^n} \big| W_{(\xi,n,i)} g_{(\xi,i,d_0)} \big|^{d_0},$$

where we used the fact that $\sum_{n \in \mathbb{Z}} \tilde{\chi}(t - n) \leqslant 2$.

By applying the generalized Hölder inequality of Theorem 4.2 in spatial direction, together with the estimates for the building blocks in (2.7) and Lemma 4.1 we deduce

$$\|w_{q+1}^{(p,i)}(t)\|_{L^{d_0}}^{d_0} \lesssim \sum_{n \geqslant 3} \Big\| \tilde{\chi}(\zeta|M_l^i(t)| - n)\frac{n}{\zeta} \Big\|_{L^1} \sum_{\xi \in \Lambda^n} \|W_{(\xi,n,i)}\|_{C_t L^{d_0}(\mathbb{T}^d)}^{d_0} g_{(\xi,i,d_0)}^{d_0}(t)$$

$$+ (r_\perp \lambda_{q+1})^{-1} \Big\| \tilde{\chi}(\zeta|M_l^i(t)| - n)\frac{n}{\zeta} \Big\|_{C_{t,x}^1} \sum_{\xi \in \Lambda^n} \|W_{(\xi,n,i)}\|_{C_t L^{d_0}(\mathbb{T}^d)}^{d_0} g_{(\xi,i,d_0)}^{d_0}(t)$$

$$\lesssim \Big(\Big\| \sum_{n \geqslant 3} \tilde{\chi}(\zeta|M_l^i(t)| - n)(M_l^i(t) + \zeta^{-1}) \Big\|_{L^1} + l^{-3d-8} \lambda_{q+1}^{-\frac{1}{N}} \Big) \sum_{\xi \in \Lambda} g_{(\xi,i,d_0)}^{d_0}(t)$$

$$\lesssim (\|M_l^i(t)\|_{L^1} + \delta_{q+1}) \sum_{\xi \in \Lambda} g_{(\xi,i,d_0)}^{d_0}(t),$$

where we used the fact that $\sum_{n \in \mathbb{Z}} \tilde{\chi}(t - n) \leqslant 2$, and used conditions on the parameters to have $(6d+1)\alpha - \frac{1}{N} < -\alpha < -2\beta$. Here we recall the notation $\Lambda = \Lambda^1 \cup \Lambda^2$. Then we apply the generalized Hölder inequality of Theorem 4.2 in time direction, the bounds (2.10) and (4.2) to deduce for some $C_v \geqslant 1$

$$\|w_{q+1}^{(p,i)}\|_{L_t^{d_0} L^{d_0}}^{d_0} \lesssim (\|M_l^i\|_{L_t^1 L^1} + \delta_{q+1} + \sigma^{-1}\|M_l^i\|_{C_{t,x}^1}) \sum_{\xi \in \Lambda} \big\| g_{(\xi,i,d_0)} \big\|_{L_t^{d_0}}^{d_0}$$

$$\lesssim C_0(\delta_{q+1} + \lambda_{q+1}^{(2d+6)\alpha - \frac{1}{2N}}) \leqslant (\frac{1}{4N_0} C_v)^{d_0} C_0 \delta_{q+1}, \tag{4.14}$$

where we used conditions on the parameters to have $(2d + 6)\alpha - \frac{1}{2N} < -\alpha < -2\beta$.

For the general $L_t^u L^m$-norm with $u, m \in [1,\infty]$, by the estimates for the building blocks in (2.5)-(2.7), (2.10) and the estimate for the amplitude function in Lemma 4.1 we obtain

$$\|w_{q+1}^{(p,i)}\|_{L_t^u L^m} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \Big\| \tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} \Big\|_{C_{t,x}^0} \|W_{(\xi,n,i)}\|_{C_t L^m(\mathbb{T}^d)} \|g_{(\xi,i,d_0)}\|_{L_t^u}$$

$$\lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m} - \frac{d-1}{d_0}} r_\parallel^{\frac{1}{m} - \frac{1}{d_0}} \eta^{\frac{1}{u} - \frac{1}{d_0}}, \tag{4.15}$$

$$\|w_{q+1}^{(c,i)}\|_{L_t^u L^m} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \Big\| \tilde{\chi}(\zeta|M_l^i| - n)\Big(\frac{n}{\zeta}\Big)^{1/d_0} \Big\|_{C_{t,x}^1}$$

$$\times \Big(\frac{1}{\lambda_{q+1}^2} \|\nabla\Phi_{(\xi,n,i)}\xi \cdot \nabla\psi_{(\xi,n,i)}\|_{C_t L^m(\mathbb{T}^d)} + \|V_{(\xi,n,i)}\|_{C_t L^m(\mathbb{T}^d)} \Big) \|g_{(\xi,i,d_0)}\|_{L_t^u}$$

$$\lesssim l^{-3d-8} r_\perp^{\frac{d-1}{m}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{m}-\frac{1}{d_0}} \frac{r_\perp}{r_\parallel} \eta^{\frac{1}{u}-\frac{1}{d_0}}. \tag{4.16}$$

Here and in the following, we should remark that since supp $M_l^i \subset \Omega_{q+1}$, all integrations involving the building blocks reduce to computations on $[-\frac{1}{2}, \frac{1}{2}]^d$, equivalent to integration on the torus.

With these estimates, combining with the choice of parameters in (4.1) and the bound (4.14) we obtain

$$\|w_{q+1}\|_{L_t^{d_0} L^{d_0}} \leqslant \frac{C_v}{4} C_0^{1/d_0} \delta_{q+1}^{1/d_0} + N_0 C l^{-3d-8} \frac{r_\perp}{r_\parallel} \leqslant \frac{C_v}{4} C_0^{1/d_0} \delta_{q+1}^{1/d_0} + C \lambda_{q+1}^{(6d+17)\alpha - \frac{1}{N}}$$

$$\leqslant \frac{C_v}{4} C_0^{1/d_0} \delta_{q+1}^{1/d_0} + C \lambda_{q+1}^{-\alpha} \leqslant C_v C_0^{1/d_0} \delta_{q+1}^{1/d_0},$$

where we used conditions on the parameters to have $(6d+18)\alpha < \frac{1}{N}$, $N_0 \lesssim \lambda_{q+1}^{\alpha}$ and chose a large enough to absorb the universal constant. The above inequality yields (3.11) for v_{q+1}. Then (3.6) holds for v_{q+1}.

Then we turn to estimate the $L_t^r L^p$-norm and the $C_t L^s$-norm. Combining with the bounds (4.15) and (4.16) above we obtain

$$\|w_{q+1}\|_{L_t^r L^p} \lesssim N_0 l^{-3d-8} r_\perp^{\frac{d-1}{p}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{p}-\frac{1}{d_0}} \eta^{\frac{1}{r}-\frac{1}{d_0}} \lesssim \lambda_{q+1}^{(6d+17)\alpha - \frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/d_0}, \tag{4.17}$$

$$\|w_{q+1}\|_{C_t L^s} \lesssim N_0 l^{-3d-8} r_\perp^{\frac{d-1}{s}-\frac{d-1}{d_0}} r_\parallel^{\frac{1}{s}-\frac{1}{d_0}} \eta^{-\frac{1}{d_0}} \lesssim \lambda_{q+1}^{(6d+17)\alpha - \frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/d_0},$$

which leads to (3.12). Here we used (4.1), (4.4) and $(6d+18)\alpha < \frac{1}{N}$, $N_0 \lesssim \lambda_{q+1}^{\alpha}$. Then we chose a large enough to absorb the universal constant.

Next we estimate the $C_{t,x}^1$-norm. By the fact that

$$\partial_t (V_{(\xi,n,i)}(t)) = \left(\frac{n}{\zeta}\right)^{1/d_0} g_{(\xi,i,d_0)} \left(\partial_t V_{(\xi)}\right) \left(\left(\frac{n}{\zeta}\right)^{1/d_0} H_{(\xi,i,d_0)}(t)\right),$$

the estimates for the building blocks in (2.7), (2.10), (4.6) and the estimates for the amplitude functions in Lemma 4.1 we have for $d_0 \geqslant 2$

$$\|w_{q+1}\|_{C_{t,x}^1} \lesssim \sum_{i=1}^{N_0} \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\| \tilde{\chi}(\zeta|M_l^i| - n) \left(\frac{n}{\zeta}\right)^{1/d_0} \right\|_{C_{t,x}^2} \|\nabla V_{(\xi,n,i)}\|_{C_{t,x}^1} \|g_{(\xi,i,d_0)}\|_{C_t^1}$$

$$\lesssim \sum_{i=1}^{N_0} \sum_{n \geqslant 3} \left\| \tilde{\chi}(\zeta|M_l^i| - n) \frac{n}{\zeta} \right\|_{C_{t,x}^2} \lambda_{q+1} \mu r_\parallel^{-\frac{1}{d_0}} r_\perp^{-\frac{d-1}{d_0}} \sigma \eta^{-1-\frac{2}{d_0}} \lesssim N_0 \lambda_{q+1}^{(8d+24)\alpha+d+\frac{7}{2}}.$$

Thus by $N_0 \lesssim \lambda_{q+1}^{\alpha}$, $(8d+25)\alpha < \frac{1}{2}$ we obtain

$$\|v_{q+1}\|_{C_{t,x}^1} \leqslant \|v_q\|_{C_{t,x}^1} + \|w_{q+1}\|_{C_{t,x}^1} \leqslant C_0^{1/d_0} \lambda_q^{d+4} + \frac{1}{2} \lambda_{q+1}^{d+4} \leqslant C_0^{1/d_0} \lambda_{q+1}^{d+4},$$

which implies (3.7) for v_{q+1}. Here we chose a large enough to absorb the universal constant.

4.6. **Estimates of $\theta_{q+1}^{(loc,i)}$.** Similarly as before, we first estimate the principle perturbations $\theta_{q+1}^{(p,i)}$ in $L_t^{d_0'} L^{d_0'}$. By the fact that Γ_ξ are uniformly bounded we have

$$|\theta_{q+1}^{(p,i)}|^{d_0'} \lesssim \sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n) \frac{n}{\zeta} \sum_{\xi \in \Lambda^n} \left| \mathbb{P}_{\neq 0} \Theta_{(\xi,n,i)} g_{(\xi,i,d_0')} \right|^{d_0'}.$$

Then by the same argument as in (4.14), we have for some $C_\rho \geqslant 1$

$$\|\theta_{q+1}^{(p,i)}\|_{L_t^{d_0'} L^{d_0'}}^{d_0'} \lesssim \left(\|M_l^i\|_{L_t^1 L^1} + \delta_{q+1} + \sigma^{-1} \|M_l^i\|_{C_{t,x}^1} \right) \sum_{\xi \in \Lambda} \|g_{(\xi,i,d_0')}\|_{L_t^{d_0'}}^{d_0'}$$

$$\lesssim C_0 (\delta_{q+1} + \lambda_{q+1}^{(2d+6)\alpha - \frac{1}{2N}}) \leqslant (\frac{1}{2} C_\rho)^{d_0'} C_0 \delta_{q+1}, \qquad (4.18)$$

where we used conditions on the parameters to have $(2d+6)\alpha - \frac{1}{2N} < -\alpha < -2\beta$.

For general $L_t^u L^m$-norm with $u, m \in [1, \infty]$, by the estimates for the building blocks in (2.8), (2.10) and Lemma 4.1 we obtain

$$\|\theta_{q+1}^{(p,i)}\|_{L_t^u L^m} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\| \chi(\zeta|M_l^i| - n) \left(\frac{n}{\zeta}\right)^{1/d_0'} \Gamma_\xi \right\|_{C_{t,x}^0} \|\Theta_{(\xi,n,i)}\|_{C_t L^m(\mathbb{T}^d)} \|g_{(\xi,i,d_0')}\|_{L_t^u}$$

$$\lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m} - \frac{d-1}{d_0'}} r_\parallel^{\frac{1}{m} - \frac{1}{d_0'}} \eta^{\frac{1}{u} - \frac{1}{d_0'}}. \qquad (4.19)$$

Now we estimate the $C_{t,x}^1$-norm. By (2.8), (2.10), (4.3), (4.4) and Lemma 4.1 we have

$$\|\theta_{q+1}^{(p,i)}\|_{C_{t,x}^1} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\| \chi(\zeta|M_l^i| - n) \left(\frac{n}{\zeta}\right)^{1/d_0'} \Gamma_\xi \right\|_{C_{t,x}^1} \|\Theta_{(\xi,n,i)}\|_{C_{t,x}^1} \|g_{(\xi,i,d_0')}\|_{C_t^1}$$

$$\lesssim \sum_{n \geqslant 3} \left\| \chi(\zeta|M_l^i| - n) \frac{n}{\zeta} \Gamma_\xi \right\|_{C_{t,x}^1} \lambda_{q+1} \mu r_\parallel^{-\frac{1}{d_0'}} r_\perp^{-\frac{d-1}{d_0'}} \sigma \eta^{-1-\frac{1}{d_0}-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(8d+24)\alpha+d+\frac{7}{2}},$$

$$\|\theta_{q+1}^{(c,i)}\|_{C_{t,x}^1} \lesssim \|\theta_{q+1}^{(p,i)}\|_{C_t L^1} \lesssim l^{-2d-4} r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \eta^{-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(4d+8)\alpha-\frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha}, \qquad (4.20)$$

$$\|\theta_{q+1}^{(o,i)}\|_{C_{t,x}^1} \lesssim \sigma^{-1} \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|h_{(\xi,i,d_0)}\|_{C_t^1} \left\| \mathrm{div}\left(\chi(\zeta|M_l| - n) \frac{n}{\zeta} \Gamma_\xi \xi \right) \right\|_{C_{t,x}^1}$$

$$\lesssim \sigma^{-1} \eta^{-1} l^{-6d-20} \lesssim \lambda_{q+1}^{(12d+40)\alpha+2},$$

$$\|\theta_{q+1}^{(o,i)}\|_{C_t C^1} \lesssim \sigma^{-1} \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|h_{(\xi,i,d_0)}\|_{L_t^\infty} \left\| \chi(\zeta|M_l| - n) \frac{n}{\zeta} \Gamma_\xi \xi \right\|_{C_t C^2}$$

$$\lesssim \sigma^{-1} l^{-6d-20} \lesssim \lambda_{q+1}^{(12d+40)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha}, \qquad (4.21)$$

where we used conditions on the parameters to have $(12d+41)\alpha < \frac{1}{2N}$ and chose a large enough to absorb the universal constant.

Moreover, once we take a space derivative of $\theta_{q+1}^{(p,i)}$ or $\theta_{q+1}^{(o,i)}$, by Lemma 4.1, (2.8) and (2.10), we will obtain an extra power of λ_{q+1}, which implies that

$$\|\theta_{q+1}^{(loc,i)}\|_{C_{t,x}^1} + \lambda_{q+1}^{-1} \|\nabla \theta_{q+1}^{(loc,i)}\|_{C_{t,x}^1} \lesssim \lambda_{q+1}^{(12d+40)\alpha+d+\frac{7}{2}}. \qquad (4.22)$$

To end this section, we estimate $\theta_{q+1}^{(p,i)}$ in $W^{1,1+\epsilon}$-norm for some $\epsilon > 0$ small enough, which will be used below in the estimate of stress term. By the bounds for the building blocks in (2.8), (2.10) and the bounds for the amplitude functions in Lemma 4.1 we have

$$\|\theta_{q+1}^{(p,i)}\|_{L_t^1 W^{1,1+\epsilon}} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\| \chi(\zeta|M_l^i| - n) \left(\frac{n}{\zeta}\right)^{1/d_0'} \Gamma_\xi \right\|_{C_{t,x}^1} \|\Theta_{(\xi,n,i)}\|_{C_t W^{1,1+\epsilon}(\mathbb{T}^d)} \|g_{(\xi,i,d_0')}\|_{L_t^1}$$

$$\lesssim l^{-4d-12} \lambda_{q+1} r_\parallel^{\frac{1}{1+\epsilon} - \frac{1}{d_0'}} r_\perp^{\frac{d-1}{1+\epsilon} - \frac{d-1}{d_0'}} \eta^{1-\frac{1}{d_0'}} \lesssim \lambda_{q+1}^{(8d+24)\alpha - \frac{1}{N} + d\epsilon} \lesssim \lambda_{q+1}^{-\alpha}, \qquad (4.23)$$

where we used the choice of parameters in (4.4) and chose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used conditions on the parameters to have $(8d + 26)\alpha < \frac{1}{N}$.

4.7. **Estimates of F_{q+1}^i.** We recall the functions F_{q+1}^i defined in (4.13). In this section, we provide estimates for F_{q+1}^i, which will be used to derive the corresponding estimates for $\theta_{q+1}^{(g,i)}$.

Proposition 4.3. *The functions F_{q+1}^i defined in* (4.13) *satisfy the following properties: F_{q+1}^i are mean-zero, $\operatorname{supp} F_{q+1}^i \subset \Omega_{q+1}$ and $F_{q+1}^i = 0$ on $[0, T_{q+1}]$. Moreover, it holds that for $k, j \in \mathbb{N}$*

$$\|\partial_t^k \nabla^j F_{q+1}^i\|_{C_{t,x}^0} \lesssim \lambda_{q+1}^{2k+j-1}.$$

Proof. The first three properties follow directly from the definition. To establish the Hölder continuity, we begin by considering the case $k = j = 0$ and analyze each term separately. We obverse that $\Theta_{(\xi,n,i)}$ and $W_{(\xi,n,i)}\Theta_{(\xi,n,i)}$ are both $(\mathbb{T}/r_\perp \lambda_{q+1})^d$-periodic, and $\nabla^m \Delta^{-m}$ are $(-m)$-homogeneous operators. By the estimates for the amplitude functions and for the building blocks in Lemma 4.1, (2.8), (2.10) and (4.4) respectively, we have

$$\|F_{osc,t}^i\|_{C_{t,x}^0} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|F_{(N)}^{1,(\xi,n,i)}\|_{C_{t,x}^0}$$

$$\lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|\nabla^{N^2} \partial_t [\chi(\zeta|M_l^i| - n)(\frac{n}{\zeta})^{1/d_0'} \Gamma_\xi g_{(\xi,i,d_0')}]\|_{C_{t,x}^0} \|\nabla^{N^2} \Delta^{-N^2} \mathbb{P}_{\neq 0} \Theta_{(\xi,n,i)}\|_{C_{t,x}^0}$$

$$\lesssim l^{-4d-12-(2d+8)N^2} (r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \sigma \eta^{-\frac{1}{d_0'}}) r_\perp^{-d+1} r_\parallel^{-1} \eta^{-1} (r_\perp \lambda_{q+1})^{-N^2}$$

$$\lesssim \lambda_{q+1}^{(8d+24+(4d+16)N^2)\alpha+d+1-N} \lesssim \lambda_{q+1}^{d+2-3N/4} \lesssim \lambda_{q+1}^{-1},$$

$$\|F_{osc,x}^i\|_{C_{t,x}^0} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|F_{(N)}^{2,(\xi,n,i)}\|_{C_{t,x}^0}$$

$$\lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|\nabla^{N^2+1} [\chi(\zeta|M_l^i| - n)\frac{n}{\zeta} \Gamma_\xi] g_{(\xi,i,d_0)} g_{(\xi,i,d_0')}\|_{C_{t,x}^0}$$

$$\times \|\nabla^{N^2} \Delta^{-N^2} \mathbb{P}_{\neq 0} (W_{(\xi,n,i)}\Theta_{(\xi,n,i)})\|_{C_{t,x}^0}$$

$$\lesssim l^{-4d-12-(2d+8)N^2} \eta^{-1} r_\perp^{-d+1} r_\parallel^{-1} (r_\perp \lambda_{q+1})^{-N^2}$$

$$\lesssim \lambda_{q+1}^{(8d+24+(4d+16)N^2)\alpha+d+1-N} \lesssim \lambda_{q+1}^{d+2-3N/4} \lesssim \lambda_{q+1}^{-1},$$

where we used to conditions on the parameters to deduce that $(8d + 32)\alpha < \frac{1}{2N}$.

When taking a space derivative on F_{q+1}^i, by Lemma 4.1, (2.8) and (2.10), we will obtain an extra power of $l^{-2d-8} \lesssim \lambda_{q+1}$ from the amplitude functions and a extra power of λ_{q+1} from the building blocks. Similarly, when taking a time derivative, we will obtain an extra power of $l^{-2d-8} + \sigma\eta^{-1} + \frac{r_\perp \lambda \mu}{r_\parallel} \lesssim \lambda_{q+1}^2$. Analogous to the previous analysis, we obtain the estimate on the derivatives. $\qquad\square$

4.8. **Estimates of $\theta_{q+1}^{(g,i)}$.** Since the functions $\theta_{q+1}^{(g,i)}$ vanish identically on the interval $[0, T_{q+1}]$, our subsequent analysis can be restricted to the regime $t \geqslant T_{q+1} \geqslant \frac{1}{6}$. We begin by establishing a key estimate for heat kernel convolutions:

Proposition 4.4. *Let $d \geqslant 2$. For any $t \in [\frac{1}{12}, 1]$ and $x \in \mathbb{R}^d$, we have*

$$\int_0^t \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(s, x - y) \mathrm{d}s \mathrm{d}y \lesssim t \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(t, x - y) \mathrm{d}y.$$

Proof. If $|x| \geqslant 9$, we have $|x - y|^2/4t \geqslant 1$, then it suffices to prove that $\int_0^t p(s, x) \mathrm{d}s \lesssim tp(t, x)$ for $a := |x|^2/4t \geqslant 1$. By change of variable we have that

$$\int_0^t (4\pi s)^{-d/2} e^{-|x|^2/4s} \mathrm{d}s = \int_{|x|^2/4t}^\infty (\pi |x|^2/u)^{-d/2} e^{-u} \frac{|x|^2}{4u^2} \mathrm{d}u$$

$$= t(4\pi t)^{-d/2} a^{-d/2+1} \int_a^\infty u^{d/2-2} e^{-u} \mathrm{d}u \leqslant t(4\pi t)^{-d/2} \int_a^\infty (u/a)^{d/2-1} e^{-u} \mathrm{d}u$$

$$\lesssim t(4\pi t)^{-d/2} \int_a^\infty (1 + u - a)^{d/2-1} e^{-u} \mathrm{d}u \leqslant t(4\pi t)^{-d/2} e^{-a} \int_1^\infty u^{d/2-1} e^{-u} \mathrm{d}u \lesssim tp(t, x).$$

Otherwise if $|x| < 9$, since $t \geqslant 1/12$, we have

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(t, x - y) \mathrm{d}y \geqslant \int_{[-\frac{1}{2}, \frac{1}{2}]^d} (4\pi t)^{-d/2} e^{-25/t} \mathrm{d}y \geqslant (4\pi)^{-d/2} e^{-300},$$

which implies that

$$\int_0^t \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(s, x - y) \mathrm{d}s \mathrm{d}y \leqslant t \lesssim t \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(t, x - y) \mathrm{d}y.$$

$$\square$$

Then, we are in position to bound $\theta_{q+1}^{(g,i)}$. By Proposition 4.3 and Proposition 4.4, we have that for $t - \frac{1}{12} \geqslant \frac{1}{12}$

$$|\theta_{q+1}^{(g,i)}(t, x)| \leqslant \int_0^t \int_{\mathbb{R}^d} p(t - s, x - y) |F_{q+1}^i(s, y)| \mathrm{d}s \mathrm{d}y \lesssim \lambda_{q+1}^{-1} \int_0^{t-\frac{1}{12}} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(s, x - y) \mathrm{d}s \mathrm{d}y$$

$$\lesssim \lambda_{q+1}^{-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} p(t - \frac{1}{12}, x - y) \mathrm{d}y \lesssim \lambda_{q+1}^{-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} e^{-c|x-y|^2} \mathrm{d}y \lesssim \lambda_{q+1}^{-1} e^{-c|x|^2}, \quad (4.24)$$

which implies that

$$\|\theta_{q+1}^{(g,i)}\|_{L_t^{d_0'} L^{d_0'}} + \|\theta_{q+1}^{(g,i)}\|_{C_t L^1} + \|\theta_{q+1}^{(g,i)}\|_{C_{t,x}^0} \lesssim \lambda_{q+1}^{-1}. \quad (4.25)$$

Then together with the estimates on $\theta_{q+1}^{(loc,i)}$ in Section 4.6 and the choice of parameters in (4.1), (4.4) we imply

$$\|\rho_{q+1}^i - \rho_q^i\|_{L_t^{d_0'} L^{d_0'}} \leqslant \|\theta_{q+1}^i\|_{L_t^{d_0'} L^{d_0'}} \leqslant \frac{C_\rho}{2} C_0^{1/d_0'} \delta_{q+1}^{1/d_0'} + C\lambda_{q+1}^{-\alpha} \leqslant C_\rho C_0^{1/d_0'} \delta_{q+1}^{1/d_0'},$$

$$\|\rho_{q+1}^i - \rho_q^i\|_{C_t L^1} \leqslant \|\theta_{q+1}^i\|_{C_t L^1} \lesssim \lambda_{q+1}^{(4d+8)\alpha - \frac{1}{N}} + \lambda_{q+1}^{-\alpha} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/d_0'}, \quad (4.26)$$

which implies (3.11), (3.13) and then (3.6) for ρ_{q+1}^i. Here we used $(4d+9)\alpha < \frac{1}{N}$, and chose a large enough to absorb the universal constant.

Next, by the fact that supp $\theta_{q+1}^{(c,i)}$, supp $\theta_{q+1}^{(o,i)} \subset [\frac{1}{6}, 1] \times [-\frac{1}{2}, \frac{1}{2}]^d$, together with (4.20), (4.21) we know that

$$\theta_{q+1}^{(c,i)}(t, x) + \theta_{q+1}^{(o,i)}(t, x) \geqslant -\lambda_{q+1}^{-\alpha} 1_{\{[\frac{1}{6}, 1] \times [-\frac{1}{2}, \frac{1}{2}]^d\}}(t, x) \geqslant -\frac{1}{c_d} \lambda_{q+1}^{-\alpha} \overline{p}(t - \frac{1}{12}, x).$$

By the fact that $\theta_{q+1}^{(p,i)}$ is non-negative, and the choice of parameters in (4.1), (4.24) we have

$$(\rho_{q+1}^i - \rho_q^i)(t, x) \geqslant -C\lambda_{q+1}^{-\alpha} \overline{p}(t - \frac{1}{12}, x) \geqslant -\delta_{q+1}^{1/d_0'} \overline{p}(t - \frac{1}{12}, x),$$

which yields (3.14). Here we chose a large enough to absorb the universal constant.

By a similar calculation as (4.24) and Proposition 4.3, we obtain that

$$|\partial_t \theta_{q+1}^{(g,i)}| + |\nabla \theta_{q+1}^{(g,i)}| \leqslant \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)(|\partial_t F_{q+1}^i(s,y)| + |\nabla F_{q+1}^i(s,y)|)\mathrm{d}s\mathrm{d}y \lesssim \lambda_{q+1}^2.$$

Then together with (4.22) we imply

$$\|\rho_{q+1}^i\|_{C_{t,x}^1} \leqslant \|\rho_q^i\|_{C_{t,x}^1} + \|\theta_{q+1}^i\|_{C_{t,x}^1} \leqslant C_0^{1/d_0'}\lambda_q^{d+4} + \frac{1}{2}\lambda_{q+1}^{d+4} \leqslant C_0^{1/d_0'}\lambda_{q+1}^{d+4},$$

which implies (3.7) for ρ_{q+1}^i. Here we chose a large enough to absorb the universal constant.

4.9. Estimates of M_{q+1}^i.

We estimate each terms in the definition of M_{q+1}^i separately.

4.9.1. *Estimate of oscillation error M_{osc}^i.* Since $\Theta_{(\xi,n,i)}$ and $W_{(\xi,n,i)}\Theta_{(\xi,n,i)}$ are both $(\mathbb{T}/r_\perp\lambda_{q+1})^d$-periodic, by the estimates for the amplitude functions and for the building blocks in Lemma 4.1, (2.8) and (2.10) respectively we obtain

$$\|M_{osc,x}^i\|_{L_t^1 L^1} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \sum_{m=0}^{N^2-1} \|\nabla^m \partial_t[\chi(\zeta|M_l^i| - n)(\frac{n}{\zeta})^{1/d_0'}\Gamma_\xi g_{(\xi,i,d_0')}]\|_{L_t^1 C^0}$$
$$\times (r_\perp\lambda_{q+1})^{-m-1}\|\Theta_{(\xi,n,i)}\|_{C_t L^1(\mathbb{T}^d)},$$
$$\lesssim l^{-4d-12} r_\perp^{d-1-\frac{d-1}{d_0'}} r_\parallel^{1-\frac{1}{d_0'}} \sigma \eta^{-\frac{1}{d_0'}} \sum_{m=0}^{N^2-1} \lambda_{q+1}^{((4d+16)\alpha - \frac{1}{N})m - \frac{1}{N}} \lesssim \lambda_{q+1}^{(8d+24)\alpha - \frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha},$$

$$\|M_{osc,t}^i\|_{L_t^1 L^1} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \sum_{m=0}^{N^2-1} \|\nabla^{1+m}[\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi]\|_{C_{t,x}^0}\|g_{(\xi,i,d_0)} g_{(\xi,i,d_0')}\|_{L_t^1}$$
$$\times (r_\perp\lambda_{q+1})^{-m-1}\|W_{(\xi,n,i)}\Theta_{(\xi,n,i)}\|_{C_t L^1(\mathbb{T}^d)},$$
$$\lesssim l^{-4d-12}\lambda_{q+1}^{-\frac{1}{N}} \sum_{m=0}^{N^2-1} \lambda_{q+1}^{((4d+16)\alpha - \frac{1}{N})m} \lesssim \lambda_{q+1}^{(8d+24)\alpha - \frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha},$$

where we used conditions on the parameters to have $(8d+25)\alpha < \frac{1}{N}$.

By Lemma 4.1, (2.8) and (2.10) again we obtain

$$\|M_{osc,a}^i\|_{L_t^1 L^1} \lesssim \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\|\chi(\zeta|M_l^i| - n)\frac{n}{\zeta}\Gamma_\xi\right\|_{C_{t,x}^0} \|W_{(\xi,n,i)}\|_{C_t L^1(\mathbb{T}^d)}\|\Theta_{(\xi,n,i)}\|_{C_t L^1(\mathbb{T}^d)}$$
$$\lesssim l^{-2d-4} r_\perp^{d-1} r_\parallel \lesssim \lambda_{q+1}^{(4d+8)\alpha - \frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha},$$

where we used conditions on the parameters to have $(4d+9)\alpha < \frac{1}{N}$.

The stress term $M_{osc,c}^i$ is bounded similarly as in [LRZ25, Section 5.4.2].

$$|M_{osc,c}^i| \leqslant \left|\sum_{n=-1}^{2} \chi(\zeta|M_l^i| - n)M_l^i\right| + \left|\sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)(\frac{n}{\zeta}\frac{M_l^i}{|M_l^i|} - M_l^i)\right|$$
$$\leqslant \frac{3}{\zeta} + \sum_{n \geqslant 3} \chi(\zeta|M_l^i| - n)\left|\frac{n}{\zeta} - |M_l^i|\right| \leqslant \frac{3}{20}\delta_{q+2} + \frac{1}{20}\delta_{q+2} \leqslant \frac{1}{5}C_0\delta_{q+2}.$$

By the bounds (2.10), (3.8) and (4.1) we have

$$\|M^i_{osc,o}\|_{L^1_t L^1} \lesssim \sigma^{-1} \sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \|h_{(\xi,d_0)}\|_{L^\infty_t} \left\| \partial_t [\chi(\zeta|M^i_l| - n) \frac{n}{\zeta} \Gamma_\xi \xi] \right\|_{C^0_{t,x}}$$
$$\lesssim \sigma^{-1} l^{-4d-12} \lesssim \lambda_{q+1}^{(8d+24)\alpha - \frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},$$

where we used conditions on the parameters to have $(8d+25)\alpha < \frac{1}{2N}$.

In summary, we have

$$\|M^i_{osc}\|_{L^1_t L^1} \leqslant C C_0 \lambda_{q+1}^{-\alpha} + \frac{1}{5} C_0 \delta_{q+2} \leqslant \frac{1}{3} C_0 \delta_{q+2}, \tag{4.27}$$

where we choose a large enough to absorb the constant.

Moreover, similar to the case of F^i_{q+1}, by Lemma 4.1, (2.8) and (2.10), if we take a space/time derivative on $M^i_{osc} - M^i_{osc,c}$, we will obtain an extra power of λ_{q+1}^2 at most, so we have

$$\|\partial_t (M^i_{osc} - M^i_{osc,c})\|_{L^1_t L^1} + \|\nabla (M^i_{osc} - M^i_{osc,c})\|_{L^1_t L^1} \lesssim \lambda_{q+1}^{2-\alpha}.$$

As for $M^i_{osc,c}$, by Lemma 4.1 we have

$$\|M^i_{osc,c}\|_{C^1_{t,x}} \lesssim \|M^i_q\|_{C^1_{t,x}} + \sum_{n \geqslant 3} \left\| \chi(\zeta|M^i_l| - n) \frac{n}{\zeta} \frac{M^i_l}{|M^i_l|} \right\|_{C^1_{t,x}} \lesssim \lambda_q^{2d+8} + l^{-(4d+12)} \lesssim \lambda_{q+1}^{2d+7},$$

which implies that

$$\|\partial_t M^i_{osc}\|_{L^1_t L^1} + \|\nabla M^i_{osc}\|_{L^1_t L^1} \lesssim \lambda_{q+1}^{2d+7} \leqslant \frac{1}{3} \lambda_{q+1}^{2d+8}, \tag{4.28}$$

where we chose a large enough to absorb the universal constant.

4.9.2. *Estimate of linear error M^i_{lin}.* By the bounds in (4.21)-(4.23) we have

$$\|\nabla \theta^{(loc,i)}_{q+1}\|_{L^1_t L^1} \lesssim \|\theta^{(p,i)}_{q+1}\|_{L^1_t W^{1+\epsilon}} + \|\theta^{(o,i)}_{q+1}\|_{L^1_t W^{1+\epsilon}} \lesssim \lambda_{q+1}^{-\alpha}.$$

By the estimates in (3.7), (4.17), (4.20), (4.21), (4.25) and (4.26) respectively we obtain

$$\|v_q \theta^i_{q+1} + w_{q+1}(\rho^i_q + \theta^{(c,i)}_{q+1} + \theta^{(o,i)}_{q+1} + \theta^{(g,i)}_{q+1})\|_{L^1_t L^1}$$
$$\leqslant \|v_q\|_{C^0_{t,x}} \|\theta^i_{q+1}\|_{C_t L^1} + (\|\rho^i_q\|_{C^0_{t,x}} + \|\theta^{(c,i)}_{q+1}\|_{C^0_{t,x}} + \|\theta^{(o,i)}_{q+1}\|_{C^0_{t,x}} + \|\theta^{(g,i)}_{q+1}\|_{C^0_{t,x}}) \|w_{q+1}\|_{L^r_t L^p}$$
$$\lesssim (C_0 \lambda_q^{d+4} + 1) \lambda_{q+1}^{(12d+40)\alpha - \frac{1}{2N}} \lesssim C_0 \lambda_{q+1}^{(12d+41)\alpha - \frac{1}{2N}} \lesssim C_0 \lambda_{q+1}^{-\alpha},$$

where we used (4.1) and conditions on the parameters to have $(12d+42)\alpha < \frac{1}{2N}$.

Since $w_{q+1}(t) = 0$ on $[0, T_{q+1}]$, by (3.3) and (4.17) we obtain

$$\|w_{q+1} \rho^{in,i}\|_{L^1_t L^1} \lesssim \|\rho^{in,i}\|_{C^0_{[\frac{1}{6},1],x}} \|w_{q+1}\|_{L^r_t L^p} \lesssim C_{in} \lambda_{q+1}^{-\alpha}.$$

By the estimates in (4.16) and (4.18) we have

$$\|w^{(c,i)}_{q+1} \theta^{(p,i)}_{q+1}\|_{L^1_t L^1} \leqslant \|\theta^{(p,i)}_{q+1}\|_{L^{d'_0}_t L^{d'_0}} \|w^{(c,i)}_{q+1}\|_{L^{d_0}_t L^{d_0}} \lesssim C_\rho C_0^{1/d'_0} l^{-3d-8} \frac{r_\perp}{r_\parallel} \lesssim C_0 \lambda_{q+1}^{(6d+16)\alpha - \frac{1}{N}} \lesssim C_0 \lambda_{q+1}^{-\alpha},$$

where we used (4.1) and conditions on the parameters to have $(6d+17)\alpha < \frac{1}{N}$. We choose a large enough to absorb the universal constant.

In summary, we have

$$\|M^i_{lin}\|_{L^1_t L^1} \lesssim C_0 \lambda_{q+1}^{-\alpha} \leqslant \frac{1}{3} C_0 \delta_{q+2}, \tag{4.29}$$

where we choose a large enough to absorb the constant.

Moreover, by (3.7), (4.22) and all the Hölder's estimate in Section 4.5-4.8, we have

$$\|\partial_t M_{lin}^i\|_{L_t^1 L^1} + \|\nabla M_{lin}^i\|_{L_t^1 L^1}$$
$$\lesssim \|\nabla\theta_{q+1}^{(loc,i)}\|_{C_{t,x}^1} + \|v_q\theta_{q+1}^i + w_{q+1}(\rho_q^i + \theta_{q+1}^{(c,i)} + \theta_{q+1}^{(o,i)} + \theta_{q+1}^{(g,i)} + \rho^{in,i}) + w_{q+1}^{(c,i)}\theta_{q+1}^{(p,i)}\|_{C_{t,x}^1}$$
$$\lesssim \lambda_{q+1}^{d+5} + \lambda_{q+1}^{(24d+80)\alpha+2d+7} \leqslant \frac{1}{3}\lambda_{q+1}^{2d+8}, \tag{4.30}$$

where we used $(12d+40)\alpha < \frac{1}{3}$, and chose a large to abosrb the universal constant.

4.9.3. *Estimate of commutator error M_{com}^i.* By (3.8) and (4.1) we obtain

$$\|M_{com}^i\|_{L_t^1 L^1} \lesssim l(\|\partial_t M_q^i\|_{L_t^1 L^1} + \|\nabla M_q^i\|_{L_t^1 L^1}) \lesssim C_0 l\lambda_q^{2d+8} \lesssim C_0\lambda_{q+1}^{-\alpha} \leqslant \frac{1}{3}C_0\delta_{q+2},$$

$$\|\partial_t M_{com}^i\|_{L_t^1 L^1} + \|\nabla M_{com}^i\|_{L_t^1 L^1} \lesssim \|\partial_t M_q^i\|_{L_t^1 L^1} + \|\nabla M_q^i\|_{L_t^1 L^1} \lesssim \lambda_q^{2d+8} \leqslant \frac{1}{3}\lambda_{q+1}^{2d+8},$$

which together with (4.27)-(4.30) implies that

$$\|M_{q+1}^i\|_{L_t^1 L^1} \leqslant C_0\delta_{q+2}, \quad \|\partial_t M_{q+1}^i\|_{L_t^1 L^1} + \|\nabla M_{q+1}^i\|_{L_t^1 L^1} \leqslant \lambda_{q+1}^{2d+8}.$$

We finish the proof of Proposition 3.1.

Appendix A. An overview of convex integration on the torus

In this section, we present an overview of how the convex integration method applies to the continuity equation on the torus. For detailed derivations and estimates, we refer the reader to [BCDL21], and to [LRZ25] for the advection-diffusion case on the torus. Our goal here is to outline the main ideas and the structure of the construction rather than to reproduce the full technical details. We also add remarks on the additional difficulties that arise when extending the argument to the whole space $\mathbb{R}^d$. The new ideas required to overcome these issues are presented in detail in Section 3.3.

We consider the following continuity equation on $\mathbb{T}^d$, with $d \geqslant 2$:

$$\partial_t\rho + \text{div}(v\rho) = 0, \ \text{div}v = 0,$$
$$\rho(0) = 1. \tag{A.1}$$

By the divergence-free condition on the drift v, equation (A.1) admits the constant solution $\overline{\rho}(t) = 1$ which is a probability density, w.r.t. the normed volume measures on $\mathbb{T}^d$. Our goal is therefore to construct a divergence-free drift v such that (A.1) admits another positive, non-constant solution.

The construction proceeds via induction indexed by $q \in \mathbb{N}$. At each step $q \in \mathbb{N}_0$, we construct a pair (v_q, ρ_q, M_q) satisfying the following system:

$$\partial_t\rho_q + \text{div}(v_q\rho_q) = -\text{div}M_q, \ \text{div}v_q = 0, \tag{A.2}$$

Here M_q denotes a vector field. As $q \to \infty$, we aim to prove that in some topology, $M_q \to 0$ and $(v_q, \rho_q) \to (v, \rho)$, which is a weak solution of the transport equation. At the same time, we require that $\int_{\mathbb{T}^d} \rho_q = 1$ and $\rho_q \geqslant 0$, so that the solution can be interpreted as a probability density.

At each iterative step, we construct perturbations $w_{q+1} = v_{q+1} - v_q, \theta_{q+1} = \rho_{q+1} - \rho_q$ such that (v_{q+1}, ρ_{q+1}) is a solution of (A.1) at level $q+1$ with a smaller residual stress term M_{q+1}, which can be written as

$$-\text{div} M_{q+1} = \partial_t \theta_{q+1} + \text{div}(v_q \theta_{q+1} + w_{q+1} \rho_q) + \text{div}(w_{q+1} \theta_{q+1} - M_q).$$

The stress term M_{q+1} is then defined by solving the above divergence equation with the aid of a linear differential operator of order -1, denoted by div^{-1}. We remark that the inverse divergence of a periodic function remains periodic. However, on the whole space, handling the inverse operator div^{-1} becomes substantially more delicate.

The construction of the perturbations forms the core of the convex integration scheme. We define the perturbations (w_{q+1}, θ_{q+1}) as a sum of highly oscillatory building blocks in order to achieve a cancellation between the low-frequency part of the quadratic term $w_{q+1} \theta_{q+1}$ and the stress M_q. Roughly speaking, we introduce the principal part of the perturbation $(w_{q+1}^{(p)}, \theta_{q+1}^{(p)})$, which takes the form

$$w_{q+1}^{(p)} = \sum_\xi a_\xi(M_q) W_\xi(\lambda_{q+1} x), \quad \theta_{q+1}^{(p)} = \sum_\xi a_\xi(M_q) \Theta_\xi(\lambda_{q+1} x),$$

where Θ_ξ is a carefully chosen positive solution to the continuity equation transported by W_ξ. These building blocks are also oscillating at a high frequency $\lambda_{q+1} \in \mathbb{N}$. The amplitude coefficients a_ξ are chosen such that

$$\sum_\xi a_\xi^2(M_q) \cdot \int_{\mathbb{T}^d} (W_\xi \Theta_\xi)(\lambda_{q+1} x) \mathrm{d}x = \sum_\xi a_\xi^2(M_q) \xi = M_q.$$

Here, the existence of such coefficients a_ξ is ensured by the Geometric Lemma 2.1. Roughly speaking, the low-frequency part of the product $w_{q+1}^{(p)} \theta_{q+1}^{(p)}$ cancels the stress term M_q. Additionally, to ensure that w_{q+1} is divergence-free, we introduce a divergence-free corrector $w_{q+1}^{(c)}$ such that

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)}$$

is divergence-free. Similarly, to guarantee that $\int_{\mathbb{T}^d} \rho_{q+1} \, dx - \int_{\mathbb{T}^d} \rho_q \, dx = 0$, the actual perturbation for ρ_{q+1} is defined as

$$\theta_{q+1} := \theta_{q+1}^{(p)} - \int_{\mathbb{T}^d} \theta_{q+1}^{(p)} \mathrm{d}x.$$

Moreover, by taking advantage of the fact that the constructed $\theta_{q+1}^{(p)}$ is a positive function, we have

$$\rho_{q+1} - \rho_q \geqslant - \int_{\mathbb{T}^d} \theta_{q+1}^{(p)} \mathrm{d}x,$$

which implies that the limited function ρ satisfies $\rho \geqslant \rho_0 + \sum_q (\rho_{q+1} - \rho_q) \geqslant \inf_x \rho_0 - \sum_q \|\theta_{q+1}\|_{L^1} > 0$ by choosing suitable parameters. Here, we use the compactness of the domain to ensure that ρ_0 is bounded below, so that the solution remains positive by keeping the perturbations θ_{q+1} sufficiently small. In contrast, on the whole space $\mathbb{R}^d$, this argument no longer applies, since the density ρ_0 decays to zero at infinity. Therefore, in the whole-space setting, the perturbations must be estimated carefully to ensure compatibility with the decay of ρ_0.

It then suffices to verify that M_{q+1} is small in the Lebesgue space L^1. We choose the frequencies super-geometrically, with $\lambda_{q+1} \gg \lambda_q$ for $q \in \mathbb{N}$. The function $v_q = \sum_{i \leqslant q} (v_i - v_{i-1}) + v_0 = \sum_{i \leqslant q} w_i + v_0$ is thus a sum of perturbations of frequencies higher than λ_q, which are considered

as low-frequencies compared with the building blocks $W_\xi(\lambda_{q+1}x)$ and $\Theta_\xi(\lambda_{q+1}x)$. Similarly, the functions ρ_q and M_q are also sums of perturbations with frequencies exceeding λ_q. Then, we have

$$\text{div}(w_{q+1}^{(p)}\theta_{q+1}^{(p)} - M_q) \sim \text{div}\Big(\sum_\xi a_\xi^2(M_q)\mathbb{P}_{\geqslant \lambda_{q+1}/2}[(W_\xi\Theta_\xi)(\lambda_{q+1}x)]\Big)$$

$$\sim \sum_\xi (\nabla a_\xi^2(M_q)) \cdot \mathbb{P}_{\geqslant \lambda_{q+1}/2}[(W_\xi\Theta_\xi)(\lambda_{q+1}x)],$$

and as a consequence,

$$\|\text{div}^{-1}\text{div}(w_{q+1}\theta_{q+1} - M_q)\|_{L^1} \lesssim \frac{1}{\lambda_{q+1}} \sum_\xi \|a_\xi^2(M_q)\|_{C^1}\|\mathbb{P}_{\geqslant \lambda_{q+1}/2}(W_\xi\Theta_\xi)\|_{L^1} \lesssim \frac{\sum_\xi \|a_\xi^2(M_q)\|_{C^1}}{\lambda_{q+1}}.$$

Here, the factor λ_{q+1}^{-1} arises from the -1 order inverse divergence operator div^{-1}. In the last term, the amplitude a_ξ is of low frequency, of order at most λ_q, and is therefore very small due to the super-geometric growth $\lambda_{q+1} \gg \lambda_q$.

Here, we note that on the whole space, the stress term must be assumed to have compact support in order to fit within the convex integration framework. However, in this setting, one cannot directly apply the inverse divergence operator, since even for a function with compact support, the inverse of its divergence may still have support extending over the entire space. In this paper, we introduce a new decomposition of the stress term that separates the stress term into two distinct components.

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