Stochastic intrinsic gradient flows on the Wasserstein space*

Panpan Ren[d)], Michael Röckner[b,c)], Feng-Yu Wang[a)] and Simon Wittmann[e)]

a) Center for Applied Mathematics and KL-AAGDM, Tianjin University, 300072 Tianjin, China

b) Faculty of Mathematics, Bielefeld University, 33615 Bielefeld, Germany

c) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100864 Beijing, China

d) Department of Mathematics, City University of Hong Kong, Hong Kong, China

e) Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China

Abstract

We construct stochastic gradient flows on the 2-Wasserstein space $\mathscr{P}_2$ over $\mathbb{R}^d$ for energy functionals of the type $W_F(\rho \mathrm{d}x) = \int_{\mathbb{R}^d} F(x, \rho(x))\mathrm{d}x$. The functions F and $\partial_2 F$ are assumed to be locally Lipschitz on $\mathbb{R}^d \times (0, \infty)$. This includes the relevant examples of W_F as the entropy functional or more generally the Lyapunov function of generalized porous media equations. First, we define a class of Gaussian-based measures Λ on $\mathscr{P}_2$ together with a corresponding class of symmetric Markov processes $(R_t)_{t\geq 0}$. Then, using Dirichlet form techniques we perform stochastic quantization for the perturbations of these objects which result from multiplying such a measure Λ by a density proportional to e^{-W_F}. Then it is proved that the intrinsic gradient $DW_F(\mu)$ is defined for Λ-a.e. μ and that the Gaussian-based reference measure Λ can be chosen in such way that the distorted process $(\mu_t)_{t\geq 0}$ is a martingale solution for the equation $\mathrm{d}\mu_t = -DW_t(\mu_t)\mathrm{d}t + dR_t$, $t \geq 0$.

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1 Introduction

In the pioneering work of [20] the solutions to linear Fokker-Planck-Kolmogorov equations, which are parabolic partial differential equations describing a time-dependent probability density on $\mathbb{R}^d$, have been shown to run along the steepest descent of their corresponding Lyapunov

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function, measured in relation to the Wasserstein distance. Since then, studies about the connection between gradient flows in metric spaces and diffusion equations have been conducted in various settings, see e.g. [34, 24, 25, 17, 14, 22, 26] and the textbooks of [5, 36, 18].

We fix $d \in \mathbb{N}$ and denote the set of all Borel probability measures on $\mathbb{R}^d$ by $\mathscr{P}$. Following the approach in [1, 2, 32, 13, 6, 30, 27], the differential and the gradient of a function $W : \mathscr{P} \to \mathbb{R}$ at a point $\mu \in \mathscr{P}$ may be assigned, by looking at the family of curves $\mu_{\varphi,\varepsilon} := \mu \circ (\mathrm{id} + \varepsilon\varphi)^{-1} \in \mathscr{P}$, $\varphi \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$, parameterized by $\varepsilon \in \mathbb{R}$. Then, W is differentiable at μ if and only if

$$\operatorname{diff} W(\mu)(\varphi) := \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} W(\mu_{\varphi,\varepsilon})\big|_{\varepsilon=0} \tag{1.1}$$

acts as a linear functional in its argument φ which is continuous with respect to the (trace) topology of $L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, as it provides an infinite-dimensional Riemannian-like structure on $\mathscr{P}$ with tangent bundle $(L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu))_{\mu\in\mathscr{P}}$ through the inner product

$$\langle \phi_1, \phi_2 \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)} := \mu(\langle \phi_1, \phi_2 \rangle), \qquad \phi_1, \phi_2 \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu),$$

at $\mu \in \mathscr{P}$. The differential $\operatorname{diff} W(\mu)$ determines a gradient $DW(\mu) \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ via

$$\mu\big(\langle DW(\mu), \varphi \rangle\big) = \operatorname{diff} W(\mu)(\varphi), \qquad \varphi \in C_b^1(\mathbb{R}^d, \mathbb{R}^d), \tag{1.2}$$

which can be taken as definition, regardless of a choice for a metric on $\mathscr{P}$. We refer to (1.1) and (1.2) as the intrinsic derivative, respectively the intrinsic gradient. A discussion about the relation between the intrinsic and the extrinsic derivative can be found in [28]. For $p \in [1, \infty)$ and functions W defined on the p-Wasserstein space

$$\mathscr{P}_p := \big\{ \mu \in \mathscr{P} : \ \mu(|\cdot|^p) < \infty \big\}$$

it is natural to consider the trace topology of $L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ rather than L^2 to assign the differential at μ according to (1.1) (cf. Definition 2.6 below). This leads to a consistent notion of the differential at μ (the class of differentiable functions W will change, but not the value of diff $W(\mu)(\varphi)$ for $\varphi \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ if defined). (1.1) extends to all curves $(\mu_{\varphi,\varepsilon})_\varepsilon$, $\varphi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$.

Sometimes, it is beneficial to consider instead of the standard structure of L^2 an equivalent inner product at μ, leading to a perturbed value for the gradient $DW(\mu)$. For a measurable weight function $\gamma : \mathbb{R}^d \times \mathscr{P} \to [c^{-1}, c]$, $c \in (0, \infty)$, we define $D^\gamma W(\mu) := \gamma(\cdot, \mu) DW(\mu)$. This corresponds to re-defining the gradient w.r.t. the inner product $L^2(\mathbb{R}^d \to \mathbb{R}^d, \gamma(\cdot, \mu)^{-1}\mu)$.

By the findings in [7], under suitable conditions on the coefficients $\beta : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$, the generalized porous media equation

$$\partial_t \rho = \Delta\beta(\rho) + \operatorname{div}\big((\nabla\Phi) b(\rho)\rho\big) \qquad \text{on } (0, \infty) \times \mathbb{R}^d, \tag{1.3}$$

has a mild solution $\rho(t, x)$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$, given by a nonlinear semigroup $(S(t))_{t\geq 0}$ of contractions in $L^1(\mathbb{R}^d, \mathrm{d}x)$, i.e. $\rho(t, \cdot) = S(t)\rho_0$ for an initial value $\rho(0, \cdot) := \rho_0 \in L^1(\mathbb{R}^d, \mathrm{d}x)$. In [8] the uniqueness of this solution is shown in the largest class of solutions, namely the so-called distributional solution. The positivity $\rho_0 \geq 0$ and the mass $\int_{\mathbb{R}^d} \rho_0(x)\mathrm{d}x$ of an initial value are preserved by $S(t)$. (1.3) is an example from the class of nonlinear Fokker–Planck equations, which describe the evolution of the time marginal laws of solutions to distribution dependent stochastic differential equations, known as McKean–Vlasov SDEs. We refer to [9] for a comprehensive study of this topic. By virtue of [27] the curves $\mu_t := \rho(x,t)\mathrm{d}x \in \mathscr{P}$, $t \geq 0$, defined by the probability solutions to (1.3) correspond to the unique gradient flow on $\mathscr{P}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t = -D^\gamma W(\mu_t) \tag{1.4}$$

w.r.t. $\gamma(x, \rho\mathrm{d}x) := b(\rho(x))$ and an explicit energy functional W. The latter has a domain within the set of absolutely continuous measures on which

$$W(\rho\mathrm{d}x) := \int_{\mathbb{R}^d} F(x, \rho(x))\mathrm{d}x$$

with

$$F(x,s) := s\Phi(x) + \int_0^s \int_1^t \frac{\beta'(r)}{rb(r)}\mathrm{d}r\mathrm{d}t, \quad x \in \mathbb{R}^d,\ s \in (0,\infty) \tag{1.5}$$

is well-defined. The functional W coincides with the Lyapunov function determined in [7]. For $\Phi = 0$, $b = \beta' = 1$, (1.3) reduces to the heat equation with $W(\rho\mathrm{d}x) := \int_{\mathbb{R}^d}(\ln(\rho) - 1)\rho\mathrm{d}x$ being the entropy functional. If a suitable space of test functions u is given, (1.4) can be reformulated into

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\mu_t) = -\mu_t\big(\langle\gamma_{\mu_t} DW(\mu_t), Du(\mu_t)\rangle\big)$$

with $\gamma_\mu := \gamma(\cdot,\mu)$.

We are interested in constructing stochastic gradient flows for energy functionals of the type $W_F(\mu) := \int_{\mathbb{R}^d} F(x,\rho_\mu(x))\mathrm{d}x$, $\mu = \rho_\mu\mathrm{d}x$, with $F : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ for example as in (1.5). Adding a stochastic process $(R_t)_{t\geq 0}$ to its right-hand side transforms (1.4) into a stochastic differential equation

$$\mathrm{d}\mu_t = -D^\gamma W_F(\mu_t)\mathrm{d}t + \mathrm{d}R_t, \qquad t \geq 0. \tag{1.6}$$

Let $(R_t)_{t\geq 0}$ be a Λ-symmetric Markov process with state space $\mathscr{P}_p$ for some $p \in [1,2]$ and a probability measure Λ on $\mathscr{P}_p$. Denoting the generator of $(R_t)_{t\geq 0}$ by A we reformulate (1.6) as a martingale problem for the generator of $(\mu_t)_{t\geq 0}$. In this article, solutions to (1.6) are found in the sense that under the laws of a right process $(\Omega, \mathscr{F}, (\mu_t)_{t\geq 0}, (\mathbb{P}^\mu)_{\mu\in\mathscr{P}_p})$,

$$u(\mu_t) - \int_0^t Au(\mu_s)\mathrm{d}s + \mu_s\big(\gamma_{\mu_s}\langle DW_F(\mu_s), Du(\mu_s)\rangle\big)\mathrm{d}s, \qquad t \geq 0, \tag{1.7}$$

is a martingale for a suitable class of test functions $u : \mathscr{P} \to \mathbb{R}$ and starting points μ up to a set of zero capacity. There is no Brownian motion on the set of probability measures. Hence, we do not have a canonical candidate for $(R_t)_{t\geq 0}$. A substitute must be chosen with care, because typical examples of W_F, as e.g. the entropy functional, do not have an intrinsic derivative DW_F defined at all points of $\mathscr{P}_p$, but only within a certain subset of absolute continuous measures. It is a priori not clear how $(R_t)_{t\geq 0}$ can be defined in a way that martingale solutions to (1.6) in the sense of (1.7) exist.

We utilize the method in [29, 31] by which a Markov process $(\phi_t)_{t\geq 0}$ on $L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ for fixed $\lambda \in \mathscr{P}_p$ induces a Markov process on $\mathscr{P}_p$. The main finding of this work is that we can take $(\phi_t)_{t\geq 0}$ as a reflected Ornstein–Uhlenbeck process on $L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ in such a way that (1.7) becomes solvable with A being the generator of the induced process $(R_t)_{t\geq 0}$ on $\mathscr{P}_p$. The test function u must be an element in the L^2-domain $\mathscr{D}(A)$ of the generator, which makes $(A, \mathscr{D}(A))$ a non-positive self-adjoint operator in $L^2(\mathscr{P}_p, \Lambda)$. The Dirichlet form of $(R_t)_{t\geq 0}$ is given by

$$\langle -Au, v\rangle_{L^2(\Lambda)} = \int_{\mathscr{P}_p} \mu\big(\gamma_\mu \langle Du(\mu), Dv(\mu)\rangle\big)\Lambda(\mathrm{d}\mu), \qquad u, v \in \mathscr{D}(A), \tag{1.8}$$

and hence of intrinsic gradient-type.

Let us now briefly point out how to find a solution to (1.7) once a suitable $(R_t)_{t\geq 0}$ has been fixed. We consider a perturbation

$$\Lambda_F(\mathrm{d}\mu) := \frac{1}{Z_F}\mathrm{e}^{-W_F(\mu)}\Lambda(\mathrm{d}\mu) \tag{1.9}$$

of the invariant measure Λ, where for $\mu = \rho_\mu \mathrm{d}x$ we set

$$W_F(\mu) := \begin{cases} \int_{\mathbb{R}^d} F(x, \rho_\mu(x))\mathrm{d}x & \text{if } F(\cdot, \rho_\mu) \in L^1(\mathbb{R}^d, \mathrm{d}x), \\ \infty & \text{otherwise,} \end{cases} \tag{1.10}$$

and

$$Z_F := \int_{\mathscr{P}_p} \mathrm{e}^{-W_F}\mathrm{d}\Lambda \in (0, \infty)$$

is the normalization constant. Defining

$$\Gamma(u, v)(\mu) := \mu\big(\gamma_\mu \langle Du(\mu), Dv(\mu)\rangle\big),$$

(1.8) leads to

$$\big\langle -Au + \mu\big(\gamma_\mu\langle DW(\mu), Du(\mu)\rangle\big), v\big\rangle_{L^2(\Lambda_F)} = \int_{\mathscr{P}_p} \Gamma(u, v)\Lambda_F(\mathrm{d}\mu) \tag{1.11}$$

under suitable assumptions on F, u, v (cf. Lemma 2.1). So, formally a Λ_F-symmetric process $(\mu_t)_{t\geq 0}$ solves (1.7) if its Dirichlet form $\mathscr{E}^F$ reads

$$\mathscr{E}^F(u, v) := \int_{\mathscr{P}_p} \Gamma(u, v)\mathrm{d}\Lambda_F. \tag{1.12}$$

This method is well-known in the theory of Dirichlet forms and has been applied in many different settings, e.g. in [4, 33, 16, 10]. The process $(\mu_t)_{t\geq 0}$ can be obtained by: (1) proving closability of the bilinear form $\mathscr{E}^F$ in (1.12) on a core of differentiable functions and (2) proving quasi-regularity of the minimal closed extension of $\mathscr{E}^F$ in $L^2(\mathscr{P}_p, \Lambda_F)$.

To realize this plan, we appropriately choose $(R_t)_{t\geq 0}$ and Λ as follows. First, a non-degenerate Gaussian measure G on $\{\phi \in C^1(\mathbb{R}^d, \mathbb{R}^d) : \|\nabla\phi\|_\infty < \infty\}$ is fixed, where

$$\|\nabla\phi\|_\infty = \sup_{x\neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|}, \qquad \phi \in C^1(\mathbb{R}^d, \mathbb{R}^d).$$

Then, G assigns a strictly positive value to the set

$$\mathscr{D}_1 := \Big\{\phi \in C^1(\mathbb{R}^d, \mathbb{R}^d) : \phi \text{ is invertible, } \|\nabla\phi\|_\infty + \|\nabla(\phi^{-1})\|_\infty < \infty\Big\}. \tag{1.13}$$

Let $\lambda \in \mathscr{P}_p$ be absolutely continuous. A Gaussian-based measure Λ is defined as the push-forward of the probability measure $\frac{\mathbf{1}_{\mathscr{D}_1}(\phi)G(\mathrm{d}\phi)}{G(\mathscr{D}_1)}$ under the map

$$\Psi_\lambda : \mathscr{D}_1 \ni \phi \mapsto \lambda \circ \phi^{-1} \in \mathscr{P}_p.$$

Since $\mathscr{D}_1 \subset L^p(\mathbb{R}^d \to \mathbb{R}^d)$ for $p \in [1, 2]$ as well as

$$\big(L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)\big)^* \subseteq \big(L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)\big)^* = L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda) \subset L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda),$$

the Gaussian standard gradient-type bilinear form on $\mathscr{D}_1$ is defined in the classical way as

$$\tilde{\mathscr{E}}(f, g) = \int_{\mathscr{D}_1} \lambda\big(\langle \nabla f(\phi), \nabla g(\phi)\rangle\big) \frac{\mathbf{1}_{\mathscr{D}_1}(\phi)G(\mathrm{d}\phi)}{G(\mathscr{D}_1)} \tag{1.14}$$

for f, g in a suitable pre-domain, see [23, Sect. II.3]. It has a closure $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ in $L^2(\mathscr{D}_1, \frac{\mathbf{1}_{\mathscr{D}_1}G}{G(\mathscr{D}_1)})$ and there is a reflected Ornstein–Uhlenbeck process $(\phi_t)_{t\geq 0}$ with associated Dirichlet form $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$. Following the concept of [29, 31], there exists a diffusion process $(R_t)_{t\geq 0}$ on $\mathscr{P}_p$ whose associated Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in $L^2(\mathscr{P}_p, \Lambda)$ has an analogous shape as in (1.14), namely

$$\mathscr{E}(u, v) = \int_{\mathscr{P}_p} \mu\big(\langle Du(\mu), Dv(\mu)\rangle\big)\Lambda(\mathrm{d}\mu).$$

for u, v in a suitable pre-domain. It is referred to as the induced process of $(\phi_t)_{t\geq 0}$ in this text, because $\mathscr{E}$ can be retrieved from the image Dirichlet form of $\tilde{\mathscr{E}}$ under Ψ_λ, where λ is suitably chosen in the further course. By a simple observation, for Λ a.e. μ there is a (reflected) Ornstein–Uhlenbeck process on $\mathscr{D}_1$ which induces $(R_t)_{t\geq 0}$ in the same way via the image structure under $\Psi_\mu(\phi) := \mu \circ \phi^{-1}$ (cf. Remark 2.11).

The set-up described in the paragraph above is sufficient to prove the existence of a diffusion on $\mathscr{P}_p$ whose Dirichlet form is as in (1.12). Formally, the process solves (1.6). To ensure the integrability of the drift term in (1.6) which is required for (1.7) we make an amendment regarding the definition of $\mathscr{D}_1$. By $(R_t^{(n)})_{t\geq 0}$ we denote the induced Markov process of a reflected Ornstein–Uhlenbeck process $(\phi_t)_{t\geq 0}$ analogously as above, but regarding a Gaussian measure conditioned to

$$\mathscr{D}^{(n)} := \Big\{\phi \in \mathscr{D}_1 \cap C^2(\mathbb{R}^d, \mathbb{R}^d) : |\phi(0)| + \|\nabla\phi\|_\infty + \|\nabla^2\phi\|_\infty + \|\nabla\phi^{-1}\|_\infty < n\Big\}$$

instead of $\mathscr{D}_1$, for some fixed $n \in \mathbb{N}$. Then, we solve (1.7) for $A = A^{(n)}$, the generator of $(R_t^{(n)})_{t\geq 0}$.

The functional W_F in (1.10) with F as in (1.5) is of major importance, since W_F is a Lyapunov function for the generalized porous media equation (see [7]). In this context, our main results can be summarized as follows.

- We first define a class of Gaussian-based measures Λ on $\mathscr{P}_p$, $p \in [1, 2]$, (see Section 2.2) and then construct a stochastic quantization for the measure Λ_F, i.e. a symmetric diffusion process $(\mu_t)_{t\geq 0}$ on $\mathscr{P}_p$ with invariant measure Λ_F as given in (1.9), by showing:
  - The bilinear form $\mathscr{E}^F$ in (1.12) with the domain of bounded, continuously differentiable functions on $\mathscr{P}_p$ (see Definition 2.6) is well-defined and closable in $L^2(\mathscr{P}_p, \Lambda_F)$.
  - Its minimal closed extension $(\mathscr{E}^F, \mathscr{D}(\mathscr{E}^F))$ is a local, conservative and quasi-regular Dirichlet form in $L^2(\mathscr{P}_p, \Lambda_F)$. Thanks to the one-to-one correspondence between the family of local, quasi-regular Dirichlet forms and the family of diffusion processes on a topological Lusin space (see [23, Chap.'s IV & V]), we obtain a (Λ_F-symmetric) conservative diffusion process $\mathbf{M} = (\Omega, \mathscr{F}, (\mu_t)_{t\geq 0}, (\mathbb{P}^\phi)_{\phi\in\mathscr{P}_p})$ on $\mathscr{P}_p$, properly associated with $(\mathscr{E}^F, \mathscr{D}(\mathscr{E}^F))$.

  The two statements above are proven in Theorem 3.5 and Proposition 2.10. The case with F as in (1.5) is treated in Example 3.6, where we choose $p = 2$.

- We specify a subset of $\lambda \in \mathscr{P}_p$ such that for the corresponding Gaussian-based measure Λ the Radon–Nikodym derivative $\rho_\mu = \frac{\mathrm{d}\mu}{\mathrm{d}x}$ exists and is Lipschitz continuous for Λ-a.e. μ. The gradient of the energy functional for the generalized porous media equation (1.3) is defined for Λ-a.e. μ and computed as

  $$DW_F(\mu) = \nabla\Phi + \frac{\beta'(\rho_\mu)\nabla\rho_\mu}{b(\rho_\mu)\rho_\mu}$$

  in the sense of a local weak intrinsic gradient (see Definition 4.2, Theorem 4.3 and the subsequent discussion).

- For each $n \in \mathbb{N}$ there is a Gaussian-based measure $\Lambda^{(n)}$ (see Section 4, in particular Corollary 4.4) such that:
  - The resulting process $(\mu_t^{(n)})_{t\geq 0}$ solves

    $$\mathrm{d}\mu_t^{(n)} = -DW_F(\mu_t^{(n)})\mathrm{d}t + \mathrm{d}R_t^{(n)}, \qquad t \geq 0, \tag{1.15}$$

    where $(R_t^{(n)})_{t\geq 0}$ is induced by a reflected Ornstein–Uhlenbeck process on $\mathscr{D}^{(n)}$ in the sense explained above.
  - E_n are monotone increasing in n and $\bigcup_{n\in\mathbb{N}} E_n \subset \mathscr{P}_2$ is dense.

The remainder of this article is organized as follows. Section 2.1 contains some preliminaries on strongly local Dirichlet forms and their relation with diffusion processes. The class of suitable reference measures Λ on $\mathscr{P}_p$, including Gaussian-based measures, are introduced in Section 2.2. This framework is applied in Section 3 to prove the existence of a diffusion with associated Dirichlet form as in (1.12) for the energy functional given in (1.10) (Theorem 3.5). Localized versions of the Dirichlet form in (1.12) are introduced in Section 4. As a consequence, we obtain the local weak gradient of the functional (1.10) and show (1.15) (Theorem 4.3 and Corollary 4.4).

2 Diffusion processes on Wasserstein spaces

2.1 Preliminaries

Let E be a metrizable Lusin topological space. We denote its Borel σ-algebra by $\mathscr{B}(E)$ and fix a probability measure Λ on $(E, \mathscr{B}(E))$. Given a measurable function $f : E \to \mathbb{R}$ its Λ-class of measurable functions is again denote by f. All vector spaces in this text are assumed to be real.

Let $\mathscr{D}(\mathscr{E})$ be a dense subspace of $L^2(E, \Lambda)$ and

$$\mathscr{E} : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to \mathbb{R}$$

be a non-negative definite, symmetric bilinear map. $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called closed if $\mathscr{D}(\mathscr{E})$ is complete under $\mathscr{E}_1^{1/2}$-norm induced by the inner product

$$\mathscr{E}_1(u, v) := \mathscr{E}(u, v) + \langle u, v\rangle_{L^2(E,\Lambda)}.$$

It is called a symmetric Dirichlet form if it is closed and

$$u^+ \wedge 1 \in \mathscr{D}(\mathscr{E}) \qquad \text{with} \qquad \mathscr{E}(u^+ \wedge 1, u^+ \wedge 1) \le \mathscr{E}(u, u)$$

for $u \in \mathscr{D}(\mathscr{E})$.

Since E has the strong Lindelöf property, the support, denoted by $\mathrm{supp}[\cdot]$, of a positive measure on $(E, \mathscr{B}(E))$ is defined. For a measurable function $f : E \to \mathbb{R}$ we set $\mathrm{supp}[f] := \mathrm{supp}[|f|\Lambda]$. A symmetric Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is said to possess the local property if

$$\mathscr{E}(u, v) = 0 \qquad \text{for } u, v \in \mathscr{D}(\mathscr{E}) : \mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset.$$

$(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is said to possess the strong local property if

$$\mathscr{E}(u, v) = 0 \qquad \text{for } u, v \in \mathscr{D}(\mathscr{E}) : u \text{ is constant } \Lambda\text{-a.e. on } \mathrm{supp}[v].$$

We call $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ conservative if $\mathbf{1} \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(\mathbf{1}, \mathbf{1}) = 0$. In the conservative case the strong local and the local property are equivalent.

Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Dirichlet form on E. If there exists a bilinear form

$$\Gamma : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to L^1(E, \Lambda)$$

such that

$$\mathscr{E}(uw, v) + \mathscr{E}(vw, u) - \mathscr{E}(w, uv) = 2\int_E w\Gamma(u, v)\mathrm{d}\Lambda, \quad u, v, w \in \mathscr{D}(\mathscr{E}) \cap L^\infty(E, \Lambda), \tag{2.1}$$

then $(\Gamma, \mathscr{D}(\mathscr{E}))$ is called the square-field operator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. In the conservative case (2.1) implies

$$\mathscr{E}(u,v) = \int_E \Gamma(u,v)\mathrm{d}\Lambda, \qquad u,v \in \mathscr{D}(\mathscr{E}).$$

Let it be remarked that (2.1) is satisfied if true for all $u, v, w \in \mathscr{A}$, where $\mathscr{A}$ is some dense subalgebra of $\mathscr{D}(\mathscr{E}) \cap L^\infty(E,\Lambda)$ w.r.t. $\mathscr{E}_1^{1/2}$-norm. Moreover, (2.1) implies continuity of Γ (w.r.t. $\mathscr{E}_1^{1/2} \times \mathscr{E}_1^{1/2}$). We set

$$\mathscr{D}_{b,\mathrm{Lip}}(\mathscr{E}) := \big\{u \in \mathscr{D}(\mathscr{E}) \cap L^\infty(E,\Lambda) : \Gamma(u,u) \in L^\infty(E,\Lambda)\big\}.$$

According to the results in [12, Sect. I.5], a strongly local symmetric Dirichlet form with square field operator Γ satisfies

$$vw \in \mathscr{D}(\mathscr{E}), \qquad \Gamma(u,vw) = v\Gamma(u,w) + w\Gamma(u,v) \in L^1(E,\Lambda)$$

for $u, v \in \mathscr{D}(\mathscr{E})$ and $w \in \mathscr{D}_{b,\mathrm{Lip}}(\mathscr{E})$.

The generator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is the unique non-positive definite, self-adjoint operator $(L, \mathscr{D}(L))$ in $L^2(\Lambda)$ such that $\mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{-L})$ and $\mathscr{E}(u,v) = \langle\sqrt{-L}u, \sqrt{-L}v\rangle_{L^2(\Lambda)}$. It is a Dirichlet operator, i.e. $\langle Lu, (u-1)^+\rangle_{L^2(\Lambda)} \le 0$ for $u \in \mathscr{D}(L)$, and the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\ge0}$ in $L^2(E,\Lambda)$ of Λ-symmetric, sub-Markovian kernel operators,

$$0 \le T_t u \le 1, \ \Lambda\text{a.e.}, \quad \text{if} \quad 0 \le u \le 1, \ \Lambda\text{-a.e.}$$

If $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is conservative, then $T_t\mathbf{1} = \mathbf{1}$.

The Markov semigroup $(T_t)_{t\ge0}$ are contractive operators w.r.t. $\|\cdot\|_{L^p(\Lambda)}$ for $p \in [1,\infty]$, and extend uniquely to a strongly continuous contraction semigroup in $L^p(\Lambda)$ for $p \in [1,\infty)$. All of these coincide with the extension to $L^1(E,\Lambda)$, because of the continuous inclusion $L^p(E,\Lambda) \hookrightarrow L^1(E,\Lambda)$, and are therefore denoted again by $(T_t)_{t\ge0}$. The generator of this semigroup in $L^1(E,\Lambda)$ coincides with the smallest closed extension of $(L, \mathscr{D}(L))$ in $L^1(E,\Lambda)$, see [12, Prop. 2.4.2], and is denoted by $(L, \mathscr{D}_1(L))$. The family of Dirichlet forms in $L^2(E,\Lambda)$ and the family strongly continuous semigroups $(T_t)_{t\ge0}$ which consist of Λ-symmetric, sub-Markovian kernel operators stand in one-to-one correspondence.

Dirichlet forms have been used to analyze symmetric Markov processes. Results on the change of reference measure through the multiplication with a density may for example be found in [4, 35, 16, 19]. We include an elementary and easy-to-prove result on the behaviour of the generator under such a transform here, as it is relevant in the end of Section 3, end of Section 4. We consider another probability measure $\Lambda_\circ$ on $(E, \mathscr{B}(E))$ and a strongly local Dirichlet form $(\mathscr{E}^\circ, \mathscr{D}(\mathscr{E}^\circ))$ in $L^2(E,\Lambda_\circ)$ with generator $(A, \mathscr{D}(A))$ and square-field operator Γ. We assume:

(*a*) Λ is absolutely continuous w.r.t. $\Lambda_\circ$ with Radon–Nikodym density satisfying $\varrho := \frac{\mathrm{d}\Lambda}{\mathrm{d}\Lambda_\circ} > 0$, $\Lambda_\circ$ a.e., and $\varrho \in \mathscr{D}(\mathscr{E}^\circ)$.

(*b*) $(\mathscr{E}^\circ, \mathscr{D}(\mathscr{E}^\circ))$ is a conservative, strongly local Dirichlet form in $L^2(E,\Lambda_\circ)$ with generator $(A, \mathscr{D}(A))$ and square-field operator Γ. $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form in $L^2(E,\Lambda)$ such that there exists a subspace $\mathscr{L} \subseteq \mathscr{D}_{b,\mathrm{Lip}}(\mathscr{E}^\circ) \cap \mathscr{D}(\mathscr{E})$, densely included in $\mathscr{D}(\mathscr{E})$ w.r.t. $\mathscr{E}_1^{1/2}$, with $\mathscr{E}(u,v) = \int_E \Gamma(u,v)\mathrm{d}\Lambda$, $u, v \in \mathscr{L}$.

As above, the generator of the sub-Markovian semigroup $(T_t)_{t\geq 0}$ in $L^1(E,\Lambda)$ corresponding to $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is denoted by $(L,\mathscr{D}_1(L))$.

Lemma 2.1. *Assume* (*a*) *and* (*b*). *We have*

$$\mathscr{D}(\mathscr{E})\cap\mathscr{D}(A)\subseteq\mathscr{D}_1(L) \qquad \textit{and} \qquad Lu = Au+\varrho^{-1}\Gamma(u,\varrho), \quad u\in\mathscr{D}(\mathscr{E})\cap\mathscr{D}(A).$$

Proof. We fix $u\in\mathscr{D}(\mathscr{E})\cap\mathscr{D}(A)$. For any $v\in\mathscr{L}$ it holds

$$\int_E \Gamma(u,v)\mathrm{d}\Lambda = \int_E \Gamma(u,v\varrho)-v\Gamma(u,\varrho)\mathrm{d}\Lambda_\circ = \int_E \big(-Au-\varrho^{-1}\Gamma(u,\varrho)\big)v\mathrm{d}\Lambda.$$

Thus, the equality $\mathscr{E}(u,v)=\int_E f_u v\mathrm{d}\Lambda$ with $f_u := -Au-\varrho^{-1}\Gamma(u,\varrho)$ holds for all $v\in\mathscr{D}(\mathscr{E})\cap L^\infty(E,\Lambda)$ by density of $\mathscr{L}$. In particular,

$$\int_E (LT_tu)v\mathrm{d}\Lambda = \int_E LuT_tv\mathrm{d}\Lambda = -\mathscr{E}(u,T_tv) = \int_E (-f_u)T_tv\mathrm{d}\Lambda = \int_E (-T_tf_u)v\mathrm{d}\Lambda.$$

for $v\in\mathscr{D}(\mathscr{E})\cap L^\infty(E,\Lambda)$, $t>0$, and so, $LT_tu=-T_tf_u$ in $L^2(E,\Lambda)$. The claim follows with strong continuity of the semigroup,

$$T_tu \xrightarrow{t\to 0} u \quad \text{and} \quad T_tf_u \xrightarrow{t\to 0} f_u \quad \text{in } L^1(E,\Lambda),$$

and closedness of $(L,\mathscr{D}_1(L))$. □

We recall some basic notions from [23], [21]. We are only interested in the symmetric case. The set of (bounded/non-negative) measurable functions $(E,\mathscr{B}(E))\to\mathbb{R}$ are denoted by $\mathscr{B}(E)$ (respectively $\mathscr{B}_b(E)/\mathscr{B}_+(E)$). Let $\mathbf{M}=(\Omega,\mathscr{F},(\mu_t)_{t\geq 0},(\mathbb{P}^\phi)_{\phi\in E_\Delta})$ be a right process with state space E, life time ζ and shift operator $\theta_t:\Omega\to\Omega$, $t\geq 0$, as defined in [23, Def's. 1.5 & 1.8]. Let $\mathbb{E}^\phi$ denote the expectation w.r.t. $\mathbb{P}^\phi$. A semigroup $(P_t)_{t\geq 0}$ of sub-Markovian kernel operators is given by

$$P_tf(\phi) := \mathbb{E}^\phi[f(\mu_t)], \quad \phi\in E,\ f\in\mathscr{B}_+(E),\ t\geq 0.$$

If $(P_t)_{t\geq 0}$ is Λ-symmetric (in particular it respects Λ-classes), then its action on $\mathscr{B}_b(E)$ uniquely determines a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ in $L^2(E,\Lambda)$. The process $\mathbf{M}$ is called associated with the Dirichlet form $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ which corresponds to $(T_t)_{t\geq 0}$. By virtue of [23, Thm. IV.6.7], for any Dirichlet form $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ in $L^2(E,\Lambda)$, the existence of a right process $\mathbf{M}$ associated with $\mathscr{E}$ is equivalent to quasi-regularity of the form (defined as in [23, Thm. IV.3.1]).

We call an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed sets an $\mathscr{E}$-nest if

$$\bigcup_{k\in\mathbb{N}}\Big\{u\in\mathscr{D}(\mathscr{E}) : u(\phi)=0 \text{ for } \Lambda\text{-a.e. } \phi\in E\setminus F_k\Big\}$$

is dense in $(\mathscr{D}(\mathscr{E}),\mathscr{E}_1^{1/2})$. A subset $N\subseteq\bigcap_{k\in\mathbb{N}}(E\setminus F_k)$ is referred to as $\mathscr{E}$-exceptional. A statement depending on a reference point $\phi\in E$ is said to hold $\mathscr{E}$-quasi-everywhere ($\mathscr{E}$-q.e.) if valid for all $\phi\in E\setminus N$ for some $\mathscr{E}$-exceptional set $N\subset E$. The term $\mathscr{E}$-quasi-continuous applies to a function $f:E\to\mathbb{R}$ which restricts to a continuous function, $f|_{F_k}\in C(F_k)$, on F_K

for all $k \in \mathbb{N}$. If the right process $\mathbf{M}$ is associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, then its transition function $P_t f$ is $\mathscr{E}$-quasi-continuous for all $f \in \mathscr{B}_b(E)$, $t > 0$. A quasi-regular Dirichlet form uniquely determines a Λ-equivalence class of associated right processes on E, see [23, Sect. IV.6]. Within such a class, $\mathbf{M}$ can w.l.o.g. be taken to be Λ-tight and special standard ([23, Def. IV.1.13]). Then, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ has the local property if and only if

$$[0,\zeta) \ni t \mapsto \mu_t \in E \text{ is continuous } \mathbb{P}^{\phi}\text{-almost surely} \tag{2.2}$$

for $\mathscr{E}$-q.e. $\phi \in E$. We call a Λ-tight and special standard process $\mathbf{M} = (\Omega, \mathscr{F}, (\mu_t)_{t\geq 0}, (\mathbb{P}^{\phi})_{\phi\in E})$ a (non-terminating) diffusion, if (2.2) holds with $\zeta = \infty$ for all $\phi \in E$. For a quasi-regular local (conservative) Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ there is a (non-terminating) diffusion $\mathbf{M}$, which is properly associated.

In the following we assume that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is quasi-regular, local, conservative and admits a square-field operator $(\Gamma, \mathscr{D}(\mathscr{E}))$. We recall the definition of a (continuous) additive functional $(A_t)_{t\geq 0}$ of $\mathbf{M}$. The term implies that

- A_t is $\mathscr{F}_t$-measurable with $\{\mathscr{F}_t\}_{t\geq 0}$ being the minimum completed admissible filtration.
- $\exists B \in \mathscr{F}$: $\theta_t(B) \subseteq B\ \forall\, t > 0$ and $\mathbb{P}^{\phi}(B) = 1$ for $\mathscr{E}$-q.e. $\phi \in E$.
- $\forall \omega \in B$: $[0,\infty) \ni t \mapsto A_t(\omega) \in \mathbb{R}$ is càdlàg (resp. continuous), $A_0(\omega) = 0$ and
$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega), \qquad s,t \geq 0.$$

If additionally $A_t(\omega) \geq 0$ for $t \geq 0$ and $\omega \in B$ with B as above, then $(A_t)_{t\geq 0}$ is called positive. An additive functional $(A_t)_{t\geq 0}$ is said to be of finite (resp. zero) energy if the limit

$$\lim_{t\to\infty} \frac{1}{2t} \int_{\Omega} A_t^2(\omega)\mathbb{P}_{\Lambda}(\mathrm{d}\omega) \in [0,\infty)$$

exists (resp. vanishes), where $\mathbb{P}_{\Lambda}$ is the equilibrium measure $\mathbb{P}_{\Lambda}(\mathrm{d}\omega) := \int_E \mathbb{P}^{\phi}(\mathrm{d}\omega)\Lambda(\mathrm{d}\phi)$ on Ω. Two additive functionals $(A_t^{(1)})_{t\geq 0}$ and $(A_t^{(2)})_{t\geq 0}$ are called equivalent if $\mathbb{P}^{\phi}(A_t^{(1)} = A_t^{(2)}) = 1$ for $\mathscr{E}$-q.e. $\phi \in E$, $t \geq 0$. In this case, we write $A_t^{(1)} = A_t^{(2)}$. We identify an additive functional with its equivalence class.

A positive Radon measure m on $(E, \mathscr{B}(E))$ is called smooth if any $\mathscr{E}$-exceptional set is a m-nullset and there is a nest $\{F_k\}_k$ of compact sets such that $m(F_k) < \infty$ for all k. The Revuz correspondence (see [23, Thm. 2.4]) describes a one-to-one assignment between the class all smooth measures m and positive continuous additive functionals $(A_t)_{t\geq 0}$.

Remark 2.2. Let $u \in L^1(E,\Lambda)$. Then, the definition

$$A_t(\omega) := \int_0^t u(\mu_s(\omega))\mathrm{d}s, \qquad \omega \in \Omega,\, t \geq 0,$$

does not depend on the choice of representative of u, up to equivalence. Moreover, $(A_t)_{t\geq 0}$ is a continuous additive functional of $\mathbf{M}$. Writing u^+, u^- for positive and negative part of u, we have the following Revuz correspondences:

$$A_t^+ := \int_0^t u^+(\mu_s(\omega))\mathrm{d}s \qquad \text{and} \qquad u^+(\phi)\Lambda(\mathrm{d}\phi),$$

$$A_t^- := \int_0^t u^-(\mu_s(\omega))\mathrm{d}s \qquad \text{and} \qquad u^-(\phi)\Lambda(\mathrm{d}\phi).$$

Let $u \in \mathscr{D}(\mathscr{E})$ and $\tilde{u}$ denote a $\mathscr{E}$-quasi-continuous representative. The Fukushima decomposition ([23, Thm. 2.5], [21, Thm. 5.2.2]) states that

$$\tilde{u}(\mu_t) - \tilde{u}(\mu_0) = M_t^{[u]} + N_t^{[u]}, \qquad t \geq 0, \tag{2.3}$$

where:

- $(N_t^{[u]})_{t\geq 0}$ is a continuous additive functional of zero energy with $N_t^{[u]} \in L^1(\Omega, \mathbb{P}^\phi)$ for $\mathscr{E}$-q.e. $\phi \in E$, $t \geq 0$.
- $(M_t^{[u]})_{t\geq 0}$ is an additive functional of finite energy with $\int_\Omega M_t^{[u]}(\omega)\mathbb{P}^\phi(\mathrm{d}\omega) = 0$, $M_t^{[u]} \in L^2(\Omega, \mathbb{P}^\phi)$ for $\mathscr{E}$-q.e. $\phi \in E$, $t \geq 0$.

Moreover, $(M_t^{[u]})_{t\geq 0}$, $(N_t^{[u]})_{t\geq 0}$ in (2.3) with the properties above are unique up to equivalence.

Remark 2.3. (i) If $u \in \mathscr{D}_1(L)$, then

$$N_t^{[u]} = \int_0^t Lu(\mu_s)\mathrm{d}s, \quad t \geq 0, \tag{2.4}$$

as follows by [21, Thm. 5.2.4].

(ii) Let $u \in \mathscr{D}(\mathscr{E})$. Then, $((M_t^{[u]})_{t\geq 0}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}^\phi)$ is a square-integrable martingale for $\mathscr{E}$-q.e. $\phi \in E$. The quadratic variation $(\langle M^{[u]}\rangle_t)_{t\geq 0}$ of $(M_t^{[u]})_{t\geq 0}$ is in Revuz correspondence with the energy measure

$$\Gamma(u,u)(\phi)\Lambda(\mathrm{d}\phi),$$

by virtue of [21, Thm. 5.2.3]. Hence, by Remark 2.2 and uniqueness of the Revuz measure,

$$\langle M^{[u]}\rangle_t = \int_0^t \Gamma(u,u)(\mu_s)\mathrm{d}s, \qquad t \geq 0.$$

Moreover,

$$\langle M^{[u]}, M^{[v]}\rangle_t = \int_0^t \Gamma(u,v)(\mu_s)\mathrm{d}s, \qquad t \geq 0,$$

for $u, v \in \mathscr{D}(\mathscr{E})$.

In many cases, gradient-type Dirichlet forms on vector spaces can be constructed using the notion of directional derivatives and related forms. It is done so, for example in [3] and [23, Chap. 2], from which we recall the notion of Λ-admissible elements in E.

Let E and Λ be as above and in addition, E is a locally convex, Hausdorff topological vector space. E^* denotes the space of continuous linear functionals $E \to \mathbb{R}$. A probability measure m on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is said to satisfy the Hamza Condition if there is a probability density $\rho_m : \mathbb{R} \to [0,\infty)$ and moreover an open set $U \subseteq \mathbb{R}$ such that

$$m(\mathrm{d}s) = \rho_m(s)\mathrm{d}s, \quad \rho_m^{-1} \in L_{\mathrm{loc}}(U) \quad \text{and} \quad m(U) = 1. \tag{2.5}$$

In this case, due to continuity of the embedding $L^2(\mathbb{R}, m) \hookrightarrow L^1_{\mathrm{loc}}(U)$, we may define a weighted $(1,2)$-Sobolev space

$$H^{1,2}(m) := \big\{ f \in H^{1,1}_{\mathrm{loc}}(U) \cap L^2(\mathbb{R}, m) : f' \in L^2(\mathbb{R}, m) \big\}$$

with corresponding Sobolev-norm and energy form

$$\|f\|_{H^{1,2}(m)} := \big(\|f\|^2_{L^2(\mathbb{R},m)} + \|f'\|^2_{L^2(\mathbb{R},m)} \big)^{\frac{1}{2}}, \qquad (f,g)_{H^{1,2}(m)} := \langle f', g' \rangle_{L^2(\mathbb{R},m)},$$

for $f, g \in H^{1,2}(m)$. For fixed $\phi \in E \setminus \{0\}$ and $\xi \in E^*$, such that $\xi(\phi) = 1$, we set

$$\pi_\phi : E \ni \eta \mapsto \eta - \xi(\eta)\phi, \quad E_\phi := \pi(E) \quad \text{and} \quad \nu_\phi := \Lambda \circ \pi_\phi^{-1}.$$

As shown in [15, Chap. 10], there exists a Markov kernel $K : E_\phi \times \mathscr{B}(\mathbb{R}) \to [0,1]$ such that

$$\int_E f \,\mathrm{d}\Lambda = \int_{E_\phi} \int_{\mathbb{R}} f(\eta + s\phi) K(\eta, \mathrm{d}s) \nu_\phi(\mathrm{d}\eta), \quad f \in \mathscr{B}_b(E).$$

Hence, we can identify $L^2(E, \Lambda)$ with the measurable field of Hilbert spaces

$$L^2(E, \Lambda) = L^2\big(\mathbb{R} \times E_\phi, K(\eta, \mathrm{d}s)\nu_\phi(\mathrm{d}\eta)\big).$$

A function $\mathbb{R} \times E_\phi \ni (s, \eta) \mapsto u_\eta(s)$ in the latter space determines a unique function $u \in L^2(E, \Lambda)$ such that

$$\int_E (uf) \mathrm{d}\Lambda = \int_{E_\phi} \int_{\mathbb{R}} u_\eta(s) f(\eta + s\phi) K(\eta, \mathrm{d}s) \nu_\phi(\mathrm{d}\eta), \quad f \in L^2(E, \Lambda).$$

Definition 2.4. An element $\phi \in E$ is called Λ-admissible if, either $\phi = 0$, or there exist $\xi \in E^*$ and $K : E_\phi \times \mathscr{B}(\mathbb{R}) \to [0,1]$ as above, such that $K(\eta, \cdot)$ satisfies (2.5) for ν_ϕ-a.e. $\eta \in E_\phi$.

For any Λ-admissible $\phi \in E$,

$$\begin{aligned} \mathscr{E}_\phi(u, v) &:= \int_{E_\phi} (u_\eta, v_\eta)_{H^{1,2}(K(\eta,\cdot))} \nu_\phi(\mathrm{d}\eta), \\ \mathscr{D}(\mathscr{E}_\phi) &:= \Big\{ u = (u_\eta)_{\eta \in E_\phi} \in L^2(E, \Lambda) : \ u_\eta \in H^{1,2}(K(\eta, \cdot)) \text{ for } \nu_\phi\text{-a.e. } \eta, \ \mathscr{E}_\phi(u, u) < \infty \Big\} \end{aligned}$$

defines a Dirichlet form in $L^2(E, \Lambda)$.

Example 2.5. *Let Λ be a probability measure on $(E, \mathscr{B}(E))$ such that, for each $\xi \in E^*$ the image measure $\Lambda \circ \xi^{-1}$ on $\mathbb{R}$ is Gaussian (including the case of a Dirac measure). Then, Λ is called Gaussian measure on E and there is a criterion for Λ-admissibility via quasi-shift-invariance.*

Let $\overline{E^*}^\Lambda$ denote the closure of the set $\{\xi - \Lambda(\xi) : \xi \in E^*\}$ in $L^2(E, \Lambda)$. If $\phi \in E$ such that

$$\exists\, \xi_\phi \in \overline{E^*}^\Lambda : \qquad \xi(\phi) = \int_E \xi_\phi\big(\xi - \Lambda(\xi)\big) \mathrm{d}\Lambda \qquad \forall\, \xi \in E^*, \tag{2.6}$$

then Λ and the image measure under the shift, $\Lambda(\,\cdot\, - \phi)$, are absolutely continuous w.r.t. each other. The linear space of all elements ϕ with the property in (2.6) *is called Cameron–Martin space $\mathcal{H}_\Lambda$ of Λ. The corresponding Radon–Nikodym density is given by*

$$\frac{\mathrm{d}\Lambda(\,\cdot\, - \phi)}{\mathrm{d}\Lambda} = \exp\big(\xi_\phi - \tfrac{1}{2}\|\xi_\phi\|_{L^2(E,\Lambda)}\big)$$

(see [11, Chap. 2]). Hence, the Λ-admissibility of every element in the Cameron–Martin space follows with [3, Prop. 4.2].

To conclude the preliminaries, we give the definition of an image form under a transform of the state space. Let E, Λ be as in the beginning of this section, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a closed symmetric form in $L^2(E, \Lambda)$, and $\Psi : E \to S$ a measurable map into a measurable space (S, σ). The closed symmetric form $(\mathscr{E}^{\mathrm{im}}, \mathscr{D}(\mathscr{E}^{\mathrm{im}}))$ in $L^2(S, \Lambda \circ \Psi^{-1})$ which is defined

$$\begin{aligned} \mathscr{E}^{\mathrm{im}}(u, v) &:= \mathscr{E}(u \circ \Psi, v \circ \Psi) \qquad \text{for } u, v \text{ of its domain,} \\ \mathscr{D}(\mathscr{E}^{\mathrm{im}}) &:= \big\{w \in L^2(S, \Lambda \circ \Psi^{-1}) : w \circ \Psi \in \mathscr{D}(\mathscr{E})\big\}, \end{aligned}$$

is called the image form of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ under Ψ.

2.2 Induced diffusion processes on $\mathscr{P}_p$

In this section, we specify the basic framework on which Sections 3 and 4 build. The p-Wasserstein space $\mathscr{P}_p$ is a separable Polish topological space w.r.t. the p-Wasserstein distance

$$\mathbb{W}_p(\mu, \nu) := \inf_\pi \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y)\Big)^{\frac{1}{p}}, \qquad \mu, \nu \in \mathscr{P}_p(\mathbb{R}^d),$$

where the infimum rus over all couplings and $p \in [1, \infty)$. In [6], the intrinsic derivative on $\mathscr{P}_p$ and the class $C_b^1(\mathscr{P}_p)$ is introduced, in the spirit of [1, 2, 32], as follows.

Definition 2.6. Let $p \in [1, \infty)$ and $\mathrm{id}(x) = x$ for $x \in \mathbb{R}^d$.

(1) A continuous function $f : \mathscr{P}_p \to \mathbb{R}$ is called intrinsically differentiable, if

$$L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto D_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (\mathrm{id} + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{R}$$

is a well-defined bounded linear functional for any $\mu \in \mathscr{P}_p$. In this case, the intrinsic derivative $Df(\mu)$ of f at μ is the unique element in $L^{\frac{p}{p-1}}(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$D_\phi f(\mu) = \mu\big(\langle \phi, Df(\mu)\rangle\big), \quad \phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu).$$

(2) An intrisically differentiable function f is called L-differentiable if

$$\lim_{\|\phi\|_{L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)} \downarrow 0} \frac{|f(\mu \circ (\mathrm{id} + \phi)^{-1}) - f(\mu) - D_\phi f(\mu)|}{\|\phi\|_{L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)}} = 0$$

for each $\mu \in \mathscr{P}_p$. The class $C_b^1(\mathscr{P}_p)$ contains all L-differentiable functions f for which μ-versions of $Df(\mu)$, $\mu \in \mathscr{P}_p$, can be chosen such that

$$\mathbb{R}^d \times \mathscr{P}_p \ni (x,\mu) \mapsto Df(x,\mu) := Df(\mu)(x) \in \mathbb{R}^d$$

is bounded and continuous.

Below, we define a Dirichlet form $\mathscr{E}$ with state space $\mathscr{P}_p$, $p \in [1,2]$, and square-field operator of the type

$$\Gamma(u,v)(\mu) = \mu\big(\gamma_\mu \langle Du(\mu), Dv(\mu)\rangle\big), \qquad \mu \in \mathscr{P}_p,\ u,v \in C_b^1(\mathscr{P}_p),$$

where $\gamma_\mu(x) := \gamma(x,\mu)$ and $\gamma : \mathbb{R}^d \times \mathscr{P}_p \to (0,\infty)$ is measurable with $c^{-1} \le \gamma(\cdot,\cdot) \le c$ for some constant $c \in (0,\infty)$. This construction is analogous to the classical situation in which the state space $\mathbb{B}$ is a Banach space and $\mathbb{H} \subseteq \mathbb{B}$ a densely embedded Hilbert space. Then $\Gamma(f,g)(x) = \langle \nabla f(x), \nabla g(x)\rangle_{\mathbb{H}}$, $x \in \mathbb{B}$, for suitable differentiable functions $f,g : \mathbb{B} \to \mathbb{R}$, is the square-field operator of standard gradient-type Dirichlet forms on $\mathbb{B}$ (e.g. see [3], [23, Sect. II.3]).

Let $\mathscr{P}_p^{\mathrm{ac}} := \{\mu \in \mathscr{P}_p : \text{a probability density } \rho_\mu(x) = \frac{\mathrm{d}\mu}{\mathrm{d}x},\ x \in \mathbb{R}^d, \text{ exists}\}$. First, we formulate a condition on a reference probability measure on $L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$, $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$. Subsequently, with the map in (2.8) below, we consider its push-forward onto $(\mathscr{P}_p, \mathscr{B}(\mathscr{P}_p))$. In this manner, we can obtain a gradient-type Dirichlet form and diffusion process on $\mathscr{P}_p$.

(C_1) Let $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ for some $p \in [1,2]$ and $\mathscr{D} \subseteq L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ be a linear subspace, equipped with a locally convex Hausdorff topology which makes it a Lusin space, such that $\mathscr{D}$ is densely and continuously embedded into $L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$. Moreover, let $G_{\mathscr{D}}$ be a probability measure on $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$ such that $L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ has an orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ consisting of $G_{\mathscr{D}}$-admissible elements.

Remark 2.7. If $G_{\mathscr{D}}$ satisfies condition (C_1), then so does the measure $\frac{\mathbf{1}_U(\phi) G_{\mathscr{D}}(\mathrm{d}\phi)}{G_{\mathscr{D}}(U)}$ for any open set $U \subseteq \mathscr{D}$ with $G_{\mathscr{D}}(U) > 0$. This is an immediate consequence of the fact that $\frac{\mathbf{1}_D(s) m(\mathrm{d}s)}{m(D)}$ satisfies the Hamza Condition (2.5) if a probability m on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ does and $D \subseteq \mathbb{R}$ is open with $m(D) > 0$.

In Sections 3, 4 we focus on a particular choice for the space $\mathscr{D}$ introduced in the next example. Gaussian measures on $\mathscr{D}$, as defined Example 2.5, provide suitable choices for $G_{\mathscr{D}}$ if the Cameron–Martin space $\mathcal{H}_{G_{\mathscr{D}}}$ is dense in $L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$.

Example 2.8. *Let $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ for some $p \in [1,2]$. The space*

$$\mathscr{D} := \big\{\phi \in C^1(\mathbb{R}^d, \mathbb{R}^d) :\ \|\nabla\phi\|_\infty < \infty\big\} \tag{2.7}$$

$$\text{with metric} \qquad d_{\mathscr{D}}(\phi_1,\phi_2) := |\phi_1 - \phi_2|(0) + \big\|\nabla(\phi_1 - \phi_2)\big\|_\infty, \qquad \phi_1,\phi_2 \in \mathscr{D},$$

is complete and separable. Given any Gaussian measure $G_{\mathscr{D}}$ on $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$, ($C1$) is satisfied if the inclusion

$$\mathcal{H}_{G_{\mathscr{D}}} \cap L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda) \subseteq L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$$

is dense.

For $\mathscr{D}$ in (2.7) a Gaussian measure satisfying $(C1)$ can be constructed in a natural way from any sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ which is an orthonormal basis of $L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$, as demonstrated in the next example.

Example 2.9. *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be as stated and $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$, $p \in [1,2]$. We set*

$$a_k := \big(2^k \|\nabla \phi_k\|_\infty^2\big) \vee 1.$$

Defining $T_{\lambda,2} := L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ and

$$\mathbb{H} := \Big\{ \phi \in T_{\lambda,2} : \sum_{k=1}^\infty a_k \langle \phi, \phi_k \rangle_{T_{\lambda,2}}^2 < \infty \Big\},$$

$$\langle \phi_1, \phi_2 \rangle_{\mathbb{H}} := \sum_{k=1}^\infty a_k \langle \phi_1, \phi_k \rangle_{T_{\lambda,2}} \langle \phi_2, \phi_k \rangle_{T_{\lambda,2}}, \quad \phi_1, \phi_2 \in \mathbb{H},$$

yields a Hilbert space with $\mathbb{H} \subset L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda) \cap \mathscr{D}$, because of the estimate

$$\|\nabla \phi\|_\infty \leq \sum_{k=1}^\infty \langle \phi, \phi_k \rangle_{T_{\lambda,2}} \|\nabla \phi_k\|_\infty \leq \|\phi\|_{\mathbb{H}} \Big(\sum_{k=1}^\infty \tfrac{\|\nabla \phi_k\|_\infty^2}{a_k} \Big)^{1/2} \leq \|\phi\|_{\mathbb{H}}$$

for $\phi \in \mathbb{H}$. If we choose a sequence $(b_k)_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} \frac{a_k}{b_k} < \infty$, then the measure

$$G_{\mathscr{D}}(\mathrm{d}\phi) := \prod_{k=1}^\infty m_k(\mathrm{d}\langle \phi_k, \phi \rangle_{T_{\lambda,2}}) \quad \textit{with} \quad m_k(\mathrm{d}r) := \Big(\frac{b_k}{2\pi}\Big)^{\frac{1}{2}} \exp\Big[-\frac{b_k r^2}{2}\Big] \mathrm{d}r$$

guarantees that $(C1)$ holds true, as each ϕ_k is admissible and $G_{\mathscr{D}}$ is a Gaussian measure on $\mathbb{H}$, in particular on $\mathscr{D}$. The latter is true, since $G_{\mathscr{D}}$ can be rewritten as $G_{\mathscr{D}}(\mathrm{d}\phi) = \prod_{k=1}^\infty \tilde{m}_k(\mathrm{d}\langle \tilde{\phi}_k, \phi \rangle_{T_{\lambda,2}})$ with $\tilde{m}_k(\mathrm{d}r) := (\frac{b_k}{2a_k\pi})^{\frac{1}{2}} \exp[-\frac{b_k r^2}{2a_k}]\mathrm{d}r$ in terms of the basis $\tilde{\phi}_k := (\alpha_k)^{-1/2}\phi_k$, $k \in \mathbb{N}$, normalized in $\mathbb{H}$.

Assuming $(C1)$, the push-forward of $G_{\mathscr{D}}$ under the map

$$\Psi_\lambda : \mathscr{D} \ni \phi \mapsto \lambda \circ \phi^{-1} \in \mathscr{P}_p \tag{2.8}$$

yields a probability measure

$$\Lambda := G_{\mathscr{D}} \circ \Psi_\lambda^{-1} \tag{2.9}$$

on $(\mathscr{P}_p, \mathscr{B}(\mathscr{P}_p))$ suitable for the purpose of defining a gradient-type Dirichlet form.

Dirichlet forms and diffusion processes on $\mathscr{P}_p$ related to the intrinsic derivative have been studied in [29, 31] for such type of reference measures Λ. The next proposition sums up the relevant result in our context.

Proposition 2.10. *Assume* (C_1). *Let* Λ *be as in* (2.9) *and* $\gamma : \mathbb{R}^d \times \mathscr{P} \to (0,\infty)$ *be measurable such that* $c^{-1} \le \gamma(\cdot,\cdot) \le c$ *for some constant* $c \in (0,\infty)$. *We set* $\gamma_\mu(\cdot) := \gamma(\cdot,\mu)$.

(i) The bilinear form $(\mathscr{E}^{\gamma,\Lambda}, C_b^1(\mathscr{P}_p))$, *defined*

$$\mathscr{E}^{\gamma,\Lambda}(u,v) := \int_{\mathscr{P}_p} \mu\big(\gamma_\mu\langle Du(\mu), Dv(\mu)\rangle\big)\Lambda(\mathrm{d}\mu), \qquad u,v \in C_b^1(\mathscr{P}_p),$$

is closable in $L^2(\mathscr{P}_p,\Lambda)$. *Its smallest closed extension yields a quasi-regular, strongly local Dirichlet form* $(\mathscr{E}^{\gamma,\Lambda}, \mathscr{D}(\mathscr{E}^{\gamma,\Lambda}))$. *In particular, there is a non-terminating diffusion* $(\Omega, \mathscr{F}, (\mu_t)_{t\ge 0}, (\mathbb{P}^\mu)_{\mu\in\mathscr{P}_p})$ *which is properly associated with* $\mathscr{E}^{\gamma,\Lambda}$.

(ii) There exist $\phi_{\mu,k} \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, $k \in \mathbb{N}$, $\mu \in \mathscr{P}_p$, *such that* $D_{\phi_{\mu,k}}u : \mathscr{P}_p \to \mathbb{R}$ *is measurable for* $u \in C_b^1(\mathscr{P}_p)$ *and* $(\mathscr{E}^{\gamma,\Lambda}, \mathscr{D}(\mathscr{E}^{\gamma,\Lambda}))$ *admits a square-field operator* Γ *with*

$$\Gamma(u,v)(\mu) = \mu\big(\gamma_\mu\langle Du(\mu), Dv(\mu)\rangle\big) = \sum_{k=1}^{\infty} D_{\gamma_\mu\phi_{\mu,k}}u(\mu)D_{\phi_{\mu,k}}v(\mu)$$

for Λ*-a.e.* $\mu \in \mathscr{P}_p$, $u,v \in C_b^1(\mathscr{P}_p)$.

Proof. In the proof, we write $\mathscr{E} := \mathscr{E}^{\gamma,\Lambda}$ for short.

(i) We may w.l.o.g. assume $\gamma(\cdot,\cdot) = 1$ regarding the claim of (i), as it does not affect the properties of closability, locality and quasi-regularity. Let $X := L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ and $\mathrm{id}_{\mathscr{D}\to X}$ be the identification of an element in $\mathscr{D}$ with its λ-class. For $k \in \mathbb{N}$ and ϕ_k as in Condition $(C1)$ there exists a Dirichlet form $(\mathscr{E}_{\phi_k}, \mathscr{D}(\mathscr{E}_{\phi_k}))$ on $\mathscr{D}$ corresponding to the directional derivative w.r.t. ϕ_k, as introduced at the end of Section 2.1. For simplicity, the image structure under $\mathrm{id}_{\mathscr{D}\to X}$, i.e. image form and image measure on X, are again denoted by $\mathscr{E}_{\phi_k}$ and $G_{\mathscr{D}}$. Let $C_b^1(X)$ denote the space of functions $f : X \to \mathbb{R}$ with continuous, bounded Fréchet derivative $\nabla f : X \to L^\infty(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$. Clearly, $C_b^1(X)$ on X is contained in

$$\mathscr{D}(\tilde{\mathscr{E}}) := \Big\{ f \in \bigcap_{k\in\mathbb{N}} \mathscr{D}(\mathscr{E}_{\phi_k}) : \sum_{k\in\mathbb{N}} \mathscr{E}_{\phi_k}(f,f) < \infty \Big\}.$$

Moreover,

$$\mathscr{D}(\tilde{\mathscr{E}}) \times \mathscr{D}(\tilde{\mathscr{E}}) \ni (f,g) \mapsto \sum_{k\in\mathbb{N}} \mathscr{E}_{\phi_k}(f,g)$$

is a closed symmetric form in $L^2(X, G_{\mathscr{D}})$.

Now, the proof can be completed by using the arguments presented in the proofs of [29, Thm. 3.2] and [31, Thms. 2.1 & 3.1]. First, Ψ_λ can be understood as a continuous map on X, because the assignment in (2.8) respects λ-classes. From [6, Thm. 2.1], it follows $u \circ \Psi_\lambda \in C_b^1(X)$ for $u \in C_b^1(\mathscr{P}_p)$ with

$$\nabla(u \circ \Psi_\lambda)(\phi) = (Du \circ \Psi_\lambda) \circ \phi, \qquad \phi \in X \tag{2.10}$$

(see also [31, Lem. 3.2]). This, in turn, implies that the image form of $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ under Ψ_λ is an extension of $(\mathscr{E}, C_b^1(\mathscr{P}_p))$. Since such an extension, say $\mathscr{E}$ with domain $\mathscr{D}(\mathscr{E})_{\mathrm{large}} \subset L^2(\mathscr{P}_p,\Lambda)$, can be restricted to the topological closure $\overline{C_b^1(\mathscr{P}_p)}^{\mathscr{E}_1}$ of $C_b^1(\mathscr{P}_p)$ in $\mathscr{D}(\mathscr{E})_{\mathrm{large}}$ w.r.t. $\mathscr{E}_1^{1/2}$-norm, there exists a minimal closed symmetric form $(\mathscr{E},\mathscr{D}(\mathscr{E}))=\overline{C_b^1(\mathscr{P}_p)}^{\mathscr{E}_1}$ in $L^2(\mathscr{P}_p,\Lambda)$ extending $(\mathscr{E},C_b^1(\mathscr{P}_p))$. The Markovian property of $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is inherited via the relation in (2.10) and the fact that $\{\kappa\circ u : u\in C_b^1(\mathscr{P}_p)\}\subseteq C_b^1(\mathscr{P}_p)$ for $\kappa\in C_b^1(\mathbb{R})$. The strong local property and the existence of a square-field operator (i.e. the fact that $\mathscr{E}$ satisfies (2.1) with $\Gamma(u,v)(\mu)=\langle Du(\mu),Dv(\mu)\rangle_{T_{\mu,2}}$) is a consequence of the strong local property, respectively the product rule, of the gradient operator on $C_b^1(X)$ and again (2.10). Finally, $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is a quasi-regular Dirichlet form on $\mathscr{P}_p$ via the criterion in [31, Thms. 2.1], since $\mathscr{D}(\mathscr{E})$ contains all differentiable cylinder functions on $\mathscr{P}_p$, has a dense subset of continuous functions and

$$\mathscr{E}(u,u)\le\sup_{\mu\in\mathscr{P}_p}\|Du(\mu)\|^2_{L^{\frac{p}{p-1}}(\mathbb{R}^d\to\mathbb{R}^d,\mu)},\qquad u\in C_b^1(\mathscr{P}_p).$$

Hence, a diffusion on $\mathscr{P}_p$ as claimed exists and the proof of (i) is complete.

(ii) Let $u,v\in C_b^1(\mathscr{P}_p)$ and $k\in\mathbb{N}$. We refer to [31, Sect. 3.2, in part. Thm. 3.4 & Exa. 3.7] for the existence of $\phi_{\mu,k}\in L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)$, $\mu\in\mathscr{P}_p$, as claimed such that

$$\begin{aligned}(2.11)\quad &\int_{\mathscr{P}_p}\mu\big(\gamma_\mu\langle Dv(\mu),\phi_{\mu,k}\rangle\big)\mu\big(\langle Dv(\mu),\phi_{\mu,k}\rangle\big)\Lambda(\mathrm{d}\mu)\\ &\qquad=\int_{\mathscr{D}}\lambda\big(\gamma(\Psi_\lambda(\phi),\phi(\cdot))\langle\nabla(u\circ\Psi_\lambda)(\phi),\phi_k\rangle\big)\lambda\big(\langle\nabla(u\circ\Psi_\lambda)(\phi),\phi_k\rangle\big)G_{\mathscr{D}}(\mathrm{d}\phi).\end{aligned}$$

Summing up over $k\in\mathbb{N}$, the right-hand side of (2.11) equals

$$\int_{\mathscr{D}}\lambda\big(\gamma(\Psi_\lambda(\phi),\phi(\cdot))\langle\nabla(u\circ\Psi_\lambda)(\phi),\nabla(v\circ\Psi_\lambda)(\phi)\rangle\big)G_{\mathscr{D}}(\mathrm{d}\phi)=\mathscr{E}(u,v)$$

because of (2.10) and the fact that $\{\phi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d\to\mathbb{R}^d,\lambda)$. The claim now follows easily by (2.11). □

We consider $\mathscr{D}_1$ as in (1.13). The composition makes $(\mathscr{D}_1,\circ)$ a group. Hence, the sets

$$(2.12)\qquad [\lambda]^\sim:=\Psi_\lambda(\mathscr{D}_1),\qquad \lambda\in\mathscr{P}_p^{\mathrm{ac}},$$

are equivalence classes on $\mathscr{P}_p^{\mathrm{ac}}$. The definition of Λ depends on a fixed choice for $\lambda\in\mathscr{P}_p^{\mathrm{ac}}$. In the situation of Example 2.8, if instead of λ and (2.8) we consider an element $\mu\in[\lambda]^\sim$, then Λ may as well be represented as a push-forward measure under Ψ_μ, after a linear transform of the Gaussian $G_{\mathscr{D}}$.

Remark 2.11. Let $(\mathscr{D},d_{\mathscr{D}})$ be as in (2.7) and $\mathscr{N}_{\mathscr{D}}$ be the set of all Gaussian measures on $(\mathscr{D},\mathscr{B}(\mathscr{D}))$ with full topological support.

(i) Given $\phi\in\mathscr{D}_1$ the vector space isomorphism

$$K_\phi:\mathscr{D}\ni\tilde\phi\mapsto\tilde\phi\circ\phi\in\mathscr{D}$$

defines permutation on $\mathscr{N}_{\mathscr{D}}$ through the assignment

$$G_{\mathscr{D}}\mapsto G_{\mathscr{D}}\circ K_\phi^{-1}.$$

By the elementary relation

$$(G_{\mathscr{D}} \circ K_{\phi}^{-1}) \circ \Psi_{\lambda}^{-1} = G_{\mathscr{D}} \circ \Psi_{\Psi_{\lambda}(\phi)}^{-1}$$

the family $\{G_{\mathscr{D}} \circ \Psi_{\lambda}^{-1} : G_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}}\}$ coincides with $\{G_{\mathscr{D}} \circ \Psi_{\mu}^{-1} : G_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}}\}$ if $\mu, \lambda \in \mathscr{P}_p^{\mathrm{ac}}$ are equivalent and is henceforth denoted by $\mathscr{G}_{[\lambda]}$.

(ii) Since $K_{\phi}(\mathscr{D}_1) = \mathscr{D}_1$ for $\phi \in \mathscr{D}_1$, the same reasoning applies to the family

$$\left\{ G_{\mathscr{D}} \circ \Psi_{\lambda}^{-1} : G_{\mathscr{D}}(\mathrm{d}\phi) = \frac{\mathbf{1}_{\mathscr{D}_1}(\phi) G'_{\mathscr{D}}(\mathrm{d}\phi)}{G'_{\mathscr{D}}(\mathscr{D}_1)} \text{ for some } G'_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}} \right\} =: \mathscr{G}_{[\lambda]}^1$$

for $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$. We note that $\Lambda(\mathscr{P}_p^{\mathrm{ac}}) = 1$ for $\Lambda \in \mathscr{G}_{[\lambda]}^1$, because every function in $\mathscr{D}_1$ is a diffeomorphism.

(iii) Let $p = 2$. Since $\mathscr{D}$ is densely contained in $L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ for all $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$, so is the Cameron–Martin space $\mathcal{H}_{G_{\mathscr{D}}}$ for every $G_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}}$. Hence, Condition (C_1) is always satisfied if $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ and $G_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}}$. The same is true for

$$G_{\mathscr{D}} := \frac{\mathbf{1}_{\mathscr{D}_1} G'_{\mathscr{D}}}{G'_{\mathscr{D}}(\mathscr{D}_1)}, \qquad G'_{\mathscr{D}} \in \mathscr{N}_{\mathscr{D}},$$

by Remark 2.7.

Lemma 2.12. *Let $\lambda \in \mathscr{P}_2^{\mathrm{ac}}$. Any measure $\Lambda \in \mathscr{G}_{[\lambda]} \cup \mathscr{G}_{[\lambda]}^1$ has full topological support on $\mathscr{P}_2$.*

Proof. It suffices to show that $\Psi_{\lambda}(\mathscr{D})$ and $\Psi_{\lambda}(\mathscr{D}_1)$ are dense in $\mathscr{P}_2$. If so, for any open neighborhood U of an element $\mu \in \mathscr{P}_2$, the sets $\Psi_{\lambda}^{-1}(U)$ and $\Psi_{\lambda}^{-1}(U) \cap \mathscr{D}_1$ are non-empty open in $\mathscr{D}$ and hence they are assigned a positive probability w.r.t. any non-degenerate Gaussian on $\mathscr{D}$. The density of $\Psi_{\lambda}(\mathscr{D})$ is trivial, since $\mathscr{D}$ is dense in $L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$ and for each $\mu \in \mathscr{P}_2$ there is a measurable transport map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\lambda \circ \phi^{-1} = \mu$. In the case of $\Psi_{\lambda}(\mathscr{D}_1)$, we argue as follows. For an arbitrary element $\mu \in \mathscr{P}_2$ we can choose an optimal pair (φ, φ^c) such that

$$\lambda(\varphi) + \mu(\varphi^c) = \sup_{(f,g)} (\lambda(f) + \mu(g)) = \mathbb{W}_2(\mu, \lambda),$$

where the supremum runs over all $(f, g) \in C_b$ with $f(x) + g(y) \leq |x - y|^2$ for $x, y \in \mathbb{R}^d$ and $\varphi^c(y) := \inf_{x \in \mathbb{R}^d} |x - y|^2 - \varphi(x)$. Then, we know from the theory of optimal transport that $|\cdot|^2 - \varphi$ is convex, φ has an approximate differential $\nabla\varphi$ and

$$\phi : \mathbb{R}^d \ni x \mapsto x - \tfrac{1}{2}\nabla\varphi(x) \in \mathbb{R}^d$$

is an optimal transport map with $\lambda \circ \phi^{-1} = \mu$, see [5, Thm. 6.2.4]. We can approximate ϕ by elements in $\mathscr{D}_1$. For this purpose, let $\{k_n\}_{n \in \mathbb{N}}$ be a smooth approximate identity on $\mathbb{R}^d$ and

$$\phi_n : \mathbb{R}^d \ni x \mapsto (1 + \tfrac{1}{n})x - \tfrac{1}{2}\nabla(k_n * \varphi)(x) \in \mathbb{R}^d, \quad n \geq 1.$$

Since ϕ_n is the gradient field of a strictly convex function, the Jacobi matrix $\nabla\phi_n$ of ϕ_n is symmetric and strictly positive definite, uniformly on $\mathbb{R}^d$. This implies that ϕ_n is injective and the image set $\mathrm{Im}(\phi_n)$ is open. Moreover, by Lipschitz continuity of the inverse ϕ_n^{-1} :

$\mathrm{Im}(\phi_n) \to \mathbb{R}^d$, every Cauchy-sequence in $\mathrm{Im}(\phi_n)$ has a limit inside $\mathrm{Im}(\phi_n)$. Thus, $\mathrm{Im}(\phi_n) = \mathbb{R}^d$. The map $\mathbb{R}^d \to x \mapsto \phi_n^{-1}(x) + \frac{x}{m} \in \mathbb{R}^d$ is bi-Lipschitz by construction and hence a diffeomorphism on $\mathbb{R}^d$. So, the claimed density of $\Psi_\lambda(\mathscr{D}_1)$ follows since the family

$$\phi_{n,m} : \mathbb{R}^d \ni x \mapsto \Big(\phi_n^{-1} + \frac{\mathrm{id}}{m}\Big)^{-1}(x) \in \mathbb{R}^d, \quad m \geq 1,$$

belong to $\mathscr{D}_1$ and

$$\Psi_\lambda(\phi_{n,m}) = \lambda \circ (\phi_n^{-1} + \tfrac{1}{m}\mathrm{id}) \overset{m\to\infty}{\longrightarrow} \lambda \circ \phi_n^{-1} \overset{n\to\infty}{\longrightarrow} \lambda \circ \phi =: \mu$$

w.r.t. $\mathbb{W}_2$. $\square$

3 Perturbation by energy functionals

From here on, we always consider $(\mathscr{D}, d_{\mathscr{D}})$ as in (2.7) and $\mathscr{D}_1$ as in (1.13). For $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ the map $\Psi_\lambda : \mathscr{D} \to \mathscr{P}_p$ is defined in (2.8). In this section, we treat perturbations for a diffusion process $(R_t)_{t\geq 0}$ with Dirichlet form $\mathscr{E}^{\gamma,\Lambda}$ given in Proposition 2.10. The invariant measure Λ is multiplied by a factor proportional to e^{-W_F} while the square-field operator Γ remains the same. The energy functional W_F is of the type (1.10) and thus takes values smaller infinity only on a certain domain within $\mathscr{P}^{\mathrm{ac}}$. Conditions $(C2)$ and $(C3)$ below make sure the measure Λ_F in (1.9) is well-defined and a stochastic quantization exists, i.e. Λ_F is the stationary distribution of a gradient diffusion process $(\mu_t)_{t\geq 0}$ (see Theorem 3.5 below). Consequently, as stated in Corollary 3.8 below, $(\mu_t)_{t\geq 0}$ solves (1.6) in case $W \in \mathscr{D}(\mathscr{E}^{\gamma,\Lambda})$ such that $e^{-W_F} \in \mathscr{D}(\mathscr{E}^{\gamma,\Lambda}) \cap L^\infty(\Lambda)$.

We assume that λ and F satisfy:

(C_2) $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$, $F : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is measurable such that $\int_{\mathbb{R}^d} \bar{F}_\alpha(x)\mathrm{d}x < \infty$ for any $\alpha \in (1,\infty)$, where

$$\bar{F}_\alpha(x) := \sup\Big\{\big|F\big(y, t\rho_\lambda(x)\big)\big| : \ (y,t) \in \mathbb{R}^d \times \mathbb{R}, \ |y| \leq \alpha(1+|x|), \ t \in [\alpha^{-1}, \alpha]\Big\}.$$

Remark 3.1. (i) Every $\phi \in \mathscr{D}_1$ is a bi-Lipschitz C^1-diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ with

$$\inf_{x\in\mathbb{R}^d} \big|\det[\nabla\phi(x)]\big| > 0$$

and for $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ the transformation rule yields

$$\int_{\mathbb{R}^d} (f\circ\phi)\lambda(\mathrm{d}x) = \int_{\mathbb{R}^d} \frac{f(x)\rho_\lambda(\phi^{-1}(x))\mathrm{d}x}{|\det[\nabla\phi(\phi^{-1}(x))]|}$$

for bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$. Hence, if $\mu = \Psi_\lambda(\phi)$, then

$$\rho_\mu(x) := \frac{\mu(\mathrm{d}x)}{\mathrm{d}x} = \frac{\rho_\lambda(\phi^{-1}(x))}{|\det[\nabla\phi(\phi^{-1}(x))]|}.$$

(ii) With Condition $(C2)$ the density e^{-W_F} is strictly positive on $\Psi_\lambda(\mathscr{D}_1)$. Indeed, for $\phi \in \mathscr{D}_1$,

$$\int_{\mathbb{R}^d} |F\big(\phi(x), |\det[\nabla\phi(x)]|^{-1}\rho_\lambda(x)\big)\det[\nabla\phi(x)]|\mathrm{d}x < \infty.$$

Thus, if $\mu = \lambda \circ \phi^{-1}$, then $e^{-W_F(\mu)} > 0$ due to

$$\begin{aligned} W_F(\mu) &= \int_{\mathbb{R}^d} F(x, \rho_\mu(x))\mathrm{d}x = \int_{\mathbb{R}^d} F(x, \rho_\mu(x))\rho_\mu(x)^{-1}\mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} F\big(\phi(x), |\det[\nabla\phi(x)]|^{-1}\rho_\lambda(x)\big)\det[\nabla\phi(x)]\mathrm{d}x. \end{aligned} \tag{3.1}$$

We present a relevant example for a function F, which is further discussed in Example 3.6 and Corollary 4.4 below.

Example 3.2. *If $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ with $\lambda(\ln|\rho_\lambda|) < \infty$, then (C_2) is satisfied for*

$$F(x,s) := sV(x) + \int_0^s \int_1^t \frac{q(r)}{r}\mathrm{d}r\mathrm{d}t, \quad x \in \mathbb{R}^d,\ s \in [0,\infty),$$

where $V \in C(\mathbb{R}^d)$ and $q : (0,\infty) \to (0,\infty)$ is measurable such that

$$V(\cdot) \le c(1 + |\cdot|), \qquad c^{-1} \le q(\cdot) \le c,$$

for some constant $c \in (0,\infty)$. Indeed, we can find a constant $\tilde{c} \in (0,\infty)$ such that

$$|F(x,s)| \le \tilde{c}s(1 + |x| + |\ln(s)|),$$

by which $\int_{\mathbb{R}^d} \bar{F}_\alpha(x)\mathrm{d}x < \infty$ for $\alpha \in (1,\infty)$ is obvious.

Remark 3.3. Let $\beta : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to (0,\infty)$, $\Phi \in \mathbb{R}^d \to \mathbb{R}$ be as in [27, Hypothesis 1], in particular

$$\begin{aligned} &\beta \in C^1(\mathbb{R}), \quad \beta(0) = 0, \quad c^{-1} \le \beta' \le c, \\ &b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R}), \quad c^{-1} \le b, \\ &\Phi \in C^1(\mathbb{R}^d), \quad \nabla\phi \in C_b(\mathbb{R}^d, \mathbb{R}^d), \end{aligned}$$

for some constant $c \in (0,\infty)$ and

$$F(x,s) := s\Phi(x) + \int_0^s \int_1^t \frac{\beta'(r)}{rb(r)}\mathrm{d}r\mathrm{d}t, \quad x \in \mathbb{R}^d,\ s \in [0,\infty). \tag{3.2}$$

By [27, Thm.'s 2, 3.7 & 3.8] there is a one-to-one correspondence between the probability solutions to the generalized porous media equation (1.3) and the gradient flow in (1.4). If $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ with $\lambda(\ln|\rho_\lambda|) < \infty$, then $(C2)$ holds for F in (3.2), as this is a special case of Example 3.2. In Example 3.6 below, we choose Λ from the family $\mathscr{G}^1_{[\lambda]}$ of Gaussian-based measures (see Remark 2.11 (ii)) and give an existence statement for a diffusion $(\mu_t)_{t\ge0}$ on $\mathscr{P}_2$ which has invariant measure Λ_F as defined in (1.9). The representation of $(\mu_t)_{t\ge0}$ as a stochastic gradient flow is discussed in Section 4.

The next condition makes sure the pre-Dirichlet form in (1.12) is well-defined on $C_b^1(\mathscr{P}_p)$, closable in $L^2(\mathscr{P}_p, \Lambda_F)$ and yields a diffusion process on $\mathscr{P}_p$.

(C_3) For given $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ together with a probability measure $G_{\mathscr{D}}$ on $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$ and a function $F : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ we assume:

- (C_1), (C_2) hold true.
- $G_{\mathscr{D}}(\mathscr{D}_1) = 1$.
- $Z_F := \displaystyle\int_{\mathscr{D}_1} \mathrm{e}^{-W_F(\lambda\circ\phi^{-1})} G_{\mathscr{D}}(\mathrm{d}\phi) < \infty$.

Remark 3.4. With Condition ($C3$) and $\Lambda := G_{\mathscr{D}} \circ \Psi_\lambda^{-1}$, it follows $\Lambda(\mathscr{P}_p^{\mathrm{ac}}) = 1$ and the function e^{-W_F} is strictly positive, Λ-almost surely, because of (3.1). Moreover, $\int_{\mathscr{P}_p} \mathrm{e}^{-W_F(\mu)}\Lambda(\mathrm{d}\mu) = \int_{\mathscr{D}_1} \mathrm{e}^{-W_F(\lambda\circ\phi^{-1})} G_{\mathscr{D}}(\mathrm{d}\phi)$ and so, (1.9) defines a probability measure on $\mathscr{P}_p^{\mathrm{ac}}$.

In Theorem 3.5 below, we assume (C_3) and consider the bilinear form

$$\mathscr{E}^F(u,v) := \int_{\mathscr{P}_p} \Gamma(u,v)\mathrm{d}\Lambda_F, \quad u,v \in C_b^1(\mathscr{P}_p), \tag{3.3}$$

where Γ is the square-field operator of $\mathscr{E}^{\gamma,\Lambda}$ as in Proposition 2.10.

Theorem 3.5. *$(\mathscr{E}^F, C_b^1(\mathscr{P}_p))$ is closable in $L^2(\mathscr{P}_p, \Lambda_F)$ and its closure $(\mathscr{E}^F, \mathscr{D}(\mathscr{E}^F))$ is a quasi-regular, strongly local Dirichlet form. There is a non-terminating diffusion $(\Omega, \mathscr{F}, (\mu_t)_{t\geq 0}, (\mathbb{P}^\mu)_{\mu\in\mathscr{P}_p})$ on $\mathscr{P}_p$ which is properly associated with $(\mathscr{E}^F, \mathscr{D}(\mathscr{E}^F))$.*

Proof. By Proposition 2.10, it suffices to verify condition (C_1) for the measure $(\mathrm{e}^{-W_F} \circ \Psi_\lambda)\mathrm{d}G_{\mathscr{D}}$ replacing $G_{\mathscr{D}}$.

We observe, that Condition (2.5) for a probability measure m on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is inherited to ϱm, if $\varrho : \mathbb{R} \to [0,\infty)$ is measurable, $\int_{\mathbb{R}} \varrho \mathrm{d}m = 1$ and

$$m\big(\{t \in \mathbb{R} : \varrho \text{ is continuous on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}\big) = 1.$$

So, inferring that $\mathscr{D}_1$ is an open set in $\mathscr{D}$, the claim follows once we show that $\mathrm{e}^{-W_F} \circ \Psi_\lambda$ is continuous on $\mathscr{D}_1$. Let $\xi \in \mathscr{D}_1$ and

$$B(\xi) := \big\{\phi \in \mathscr{D} : 2d_{\mathscr{D}}(\phi,\xi)^d < 2 \wedge \inf_{x\in\mathbb{R}^d} |\det[\nabla\xi(x)]|\big\}.$$

Then, $B(\xi) \subset \mathscr{D}_1$, since $\phi \in B(\xi)$ implies

$$\begin{aligned}\inf_{x\in\mathbb{R}^d} \det[\nabla\phi(x)] &\geq \inf_{x\in\mathbb{R}^d} |\det[\nabla\xi(x)]| - \|\nabla(\phi-\xi)\|_\infty^d \\ &\geq \inf_{x\in\mathbb{R}^d} |\det[\nabla\xi(x)]| - d_{\mathscr{D}}(\phi,\xi)^d \geq \frac{1}{2}\inf_{x\in\mathbb{R}^d} |\det[\nabla\xi(x)]| > 0.\end{aligned} \tag{3.4}$$

Analogously to (3.1),

$$\int_{\mathbb{R}^d} F(x, \rho_{\lambda\circ\phi^{-1}}(x))\mathrm{d}x = \int_{\mathbb{R}^d} F\big(\phi(x), |\det[\nabla\phi(x)]|^{-1}\rho_\lambda(x)\big)|\det[\nabla\phi(x)]|\mathrm{d}x \tag{3.5}$$

for $\phi \in B(\xi)$. Due to (C_2) and (3.4), we can find a dx-integrable function which dominates the integrand of the right-hand side of (3.5), uniformly in $\phi \in B(\xi)$. Hence, using Lebesgue's dominated convergence and (3.5), if $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathscr{D}_1$, $\phi \in B(\xi)$, such that $d_{\mathscr{D}}(\phi_n, \phi) = 0$, then

$$\begin{aligned}\lim_{n\to\infty} (W_F \circ \Psi_\lambda)(\phi_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} F\big(\phi_n(x), |\det[\nabla\phi_n(x)]|^{-1}\rho_\lambda(x)\big)|\det[\nabla\phi_n(x)]|\mathrm{d}x \\ &= \int_{\mathbb{R}^d} F\big(\phi(x), |\det[\nabla\phi(x)]|^{-1}\rho_\lambda(x)\big)|\det[\nabla\phi(x)]|\mathrm{d}x \\ &= \int_{\mathbb{R}^d} F(x, \rho_{\lambda\circ\phi^{-1}}(x))\mathrm{d}x = (W_F \circ \Psi_\lambda)(\phi).\end{aligned}$$

This implies continuity of $\mathrm{e}^{-W_F} \circ \Psi_\lambda$ on $\mathscr{D}_1$ and concludes the proof. □

The next example shows that Theorem 3.5 is applicable for F as in Example 3.2 and $\Lambda \in \mathscr{G}^1_{[\lambda]}$ as defined in Remark 2.11 (ii). We choose $p = 2$ to ensure Condition $(C1)$ through the argument of Remark 2.11 (iii).

Example 3.6. *Let $\lambda \in \mathscr{P}_2^{\mathrm{ac}}$, $\lambda(\ln|\rho_\lambda|) < \infty$, $F : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be as in Example 3.2 and $\Lambda \in \mathscr{G}^1_{[\lambda]}$. We find constants $c_1, c_2 \in (0, \infty)$ such that for all $\phi \in \mathscr{D}_1$ it holds*

$$\begin{aligned}(3.6)\qquad -W_F(\lambda\circ\phi^{-1}) &= -\int_{\mathbb{R}^d} F\big(\phi(x), \tfrac{\rho_\lambda}{|\det[\nabla\phi]|}(x)\big)|\det[\nabla\phi(x)]|\mathrm{d}x \\ &\le c_1 \int_{\mathbb{R}^d} \big(1 + |\phi(x)| - \ln\big(\tfrac{\rho_\lambda}{|\det[\nabla\phi]|}(x)\big)\big)\rho_\lambda(x)\mathrm{d}x \le c_2(1 + \|\phi\|_{\mathscr{D},2}),\end{aligned}$$

where

$$\|\phi\|_{\mathscr{D},2} := \Big(\int_{\mathbb{R}^d} (|\phi|^2 + |\nabla\phi|^2)\mathrm{d}\lambda\Big)^{\frac{1}{2}}.$$

Since $\|\cdot\|_{\mathscr{D},2}$ is a measurable norm on $(\mathscr{D}, d_{\mathscr{D}})$, for any Gaussian measure $G_{\mathscr{D}}$ there exists $\alpha > 0$ such that $G_{\mathscr{D}}(\mathrm{e}^{\alpha\|\cdot\|^2_{\mathscr{D},2}}) < \infty$, see [11, Thm. 2.8.5]. This implies

$$Z_F = \int_{\mathscr{D}_1} \mathrm{e}^{-W_F(\lambda\circ\phi^{-1})} G_{\mathscr{D}}(\mathrm{d}\phi) < \infty$$

in view of (3.6) for any Gaussian measure $G_{\mathscr{D}}$ on $\mathscr{D}$. In particular, the measure $\Lambda_F(\mathrm{d}\mu) := Z_F^{-1}\mathrm{e}^{-W_F(\mu)}\Lambda(\mathrm{d}\mu)$ is well-defined, since $\Lambda \in \mathscr{G}^1_{[\lambda]}$. By Theorem 3.5 there exists a non-terminating diffusion on $\mathscr{P}_2$ which is properly associated with the closure of $(\mathscr{E}^F, C_b^1(\mathscr{P}_p))$ as in (3.3).

The diffusion obtained under $(C3)$ by Theorem 3.5 is denoted by $(\mu_t)_{t\ge0}$ in the following. It has invariant measure Λ_F as in (1.9). We can regard $(\mu_t)_{t\ge0}$ as a perturbation of the process $(R_t)_{t\ge0}$ with invariant measure Λ, which arises from the choice $F = 0$ and has Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E})) := (\mathscr{E}^{\gamma,\Lambda}, \mathscr{D}(\mathscr{E}^{\gamma,\Lambda}))$ as in Proposition 2.10.

To make sense of the drift term involving $D^\gamma W_F$ in (1.6), (1.7), we introduce the "weak intrinsic derivative", which is the intrinsic gradient D from Definition 2.6 extended naturally to $\mathscr{D}(\mathscr{E})$. According to the proof of Proposition 2.10, for any $u \in \mathscr{D}(\mathscr{E})$, we find a sequence $\{f_m\}_{m\in\mathbb{N}} \subset C_b^1(L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda))$ such that:

(a) $f_m \to u \circ \Psi_\lambda$ in $L^2(\mathscr{D}, G_{\mathscr{D}})$.

(b) f_m is measurable w.r.t $\sigma(\Psi_\lambda)$, the σ-algebra $\sigma(\Psi_\lambda)$ generated by Ψ_λ.

(c) $\{\nabla f_m\}_m$ is a Cauchy sequence in $L^2(\mathscr{D} \to L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda), G_{\mathscr{D}})$.

Due to $(C1)$ the limit

$$\mathscr{D} \ni \phi \mapsto \tilde{D}^u(\phi) := \lim_{m\to\infty} \nabla f_m(\phi) \qquad \text{in } L^2(\mathscr{D} \to L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda), G_{\mathscr{D}})$$

only depends on u (but not on the choice of an approximating sequence $\{f_m\}_m$) and is measurable w.r.t. $\sigma(\Psi_\lambda)$. In view of $\Lambda := G_{\mathscr{D}} \circ \Psi_\lambda^{-1}$ and $G_{\mathscr{D}}(\mathscr{D}_1) = 1$, we obtain the following notion of a weak intrinsic gradient.

Definition 3.7 (Weak intrinsic gradient)**.** For $u \in \mathscr{D}(\mathscr{E})$ the unique element

$$Du \in \bigcap_{n=1}^{\infty} L^2\big(\mathbb{R}^d \times \mathscr{P}_p \to \mathbb{R}^d, \mu(\mathrm{d}x)\Lambda(\mathrm{d}\mu)\big)$$

such that

$$Du(\Psi_\lambda(\phi)) \circ \phi = \tilde{D}^u(\phi), \qquad G_{\mathscr{D}}\text{-a.e. } \phi \in \mathscr{D}_1,$$

is called *weak intrinsic gradient* of u.

It is clear that for $u \in C_b^1(\mathscr{P}_p)$, the weak intrinsic gradient coincides with the intrinsic derivative Du. Moreover, the square-field operator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is given by

$$\Gamma(u,v)(\mu) = \int_{\mathbb{R}^d} \gamma_\mu(x)\langle Du(x,\mu), Dv(x,\mu)\rangle \mu(\mathrm{d}x), \quad \Lambda\text{-a.e. } \mu \in \mathscr{P}_p,\ u, v \in \mathscr{D}(\mathscr{E}). \tag{3.7}$$

We formulate a corollary of Theorem 3.5.

Corollary 3.8. *In addition to $(C3)$ we assume $W_F \in \mathscr{D}(\mathscr{E})$ and $\mathrm{e}^{-W_F} \in \mathscr{D}(\mathscr{E}) \cap L^\infty(\mathscr{P}_p, \Lambda)$. Let $(A, \mathscr{D}(A))$ be the generator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in $L^2(\mathscr{P}_p, \Lambda)$. The diffusion $(\Omega, \mathscr{F}, (\mu_t)_{t\geq 0}, (\mathbb{P}^\mu)_{\mu\in\mathscr{P}_p})$ on $\mathscr{P}_p$ which is properly associated with $(\mathscr{E}^F, \mathscr{D}(\mathscr{E}^F))$ yields a solution to* (1.6), (1.7)*. More precisely:*

(i) Let $u \in \mathscr{D}(A)$ and $\tilde{u}$ denotes an $\mathscr{E}^F$-quasi-continuous version. Then,

$$\tilde{u}(\mu_t) - \int_0^t \Big[Au(\mu_s) - \mu_s\big(\langle \gamma_{\mu_s} DW_F(\cdot, \mu_s), Du(\cdot, \mu_s)\rangle\big)\Big]\mathrm{d}s, \qquad t \geq 0,$$

is a $\mathbb{P}^\mu$-martingale for $\mathscr{E}^F$-q.e. $\mu \in \mathscr{P}_p$.

(ii) If additionally $\mathrm{e}^{-\frac{1}{2}W_F} \in \mathscr{D}(\mathscr{E})$, the set $\mathscr{P}_p \setminus \mathscr{P}_p^{\mathrm{ac}}$ is $\mathscr{E}^F$-exceptional.

Proof. (i) We note that $e^{-W_F} \in L^\infty(\mathscr{P}_p, \Lambda)$ implies $\mathscr{D}(A) \subset \mathscr{D}(\mathscr{E}) \subseteq \mathscr{D}(\mathscr{E}^F)$. Due to integrability of $\mu \mapsto \mu\big(\gamma_\mu\langle DW_F(\cdot,\mu), Du(\cdot,\mu)\rangle\big)$ w.r.t. $\Lambda_F(\mathrm{d}\mu)$ for $u \in \mathscr{D}(\mathscr{E}^F)$, the term $\int_0^t \mu_s(\gamma_{\mu_s}\langle DW_F(\cdot,\mu_s), Du(\cdot,\mu_s)\rangle)ds$ is well-defined as an additive functional of the process $(\mu_t)_{t\geq 0}$. The statement follows by Lemma 2.1, (3.7) and Remark 2.3.

(ii) Under the assumption $e^{-\frac{1}{2}W_F} \in \mathscr{D}(\mathscr{E})$, an $\mathscr{E}^F$-nest which is contained in $\{\mu \in \mathscr{P}_p : W_F(\mu) \in (0,\infty)\}$ can be constructed analogously as in the proof of [16, Lem. 3.2]. The statement follows since $\{\mu \in \mathscr{P}_p : W_F(\mu) \in (0,\infty)\} \subseteq \mathscr{P}_p^{\mathrm{ac}}$. $\square$

4 Functionals with local weak gradient

This section is motivated by the following observation. In case of Example 3.6 we cannot show that W_F has a weak intrinsic gradient in the sense of Definition 3.7, because the candidate for its gradient $DW_F(\mu) = \nabla V + \frac{q(\rho_\mu)\nabla\rho_\mu}{\rho_\mu}$ doesn't have the desired integrability properties w.r.t. $\Lambda(\mathrm{d}\mu)$. We introduce the notion of a "local weak intrinsic gradient". Under suitable Lipschitz conditions on F and $\partial_2 F$, by Theorem 4.3 below, W_F has a local gradient in this sense. The application to Example 3.6 is addressed subsequently.

Let λ, $G_{\mathscr{D}}$ satisfy $(C1)$ with $(\mathscr{D}, d_{\mathscr{D}})$ as in (2.7). Defining

$$\mathscr{D}_2 := \left\{\phi \in \mathscr{D} : \nabla\phi \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})\right\}$$

and $\mathscr{D}_1$ as in (1.13), we additionally assume

$$G_{\mathscr{D}}(\mathscr{D}_1 \cap \mathscr{D}_2) = 1.$$

The linear space $\mathscr{D}_2$ is complete regarding the metric

$$d_{\mathscr{D}_2}(\phi_1, \phi_2) := d_{\mathscr{D}}(\phi_1, \phi_2) + \left\|\nabla^2\phi_1 - \nabla^2\phi_2\right\|_\infty, \quad \phi_1, \phi_2 \in \mathscr{D}_2, \tag{4.1}$$

where

$$\|\nabla^2\phi\|_\infty = \sup_{x\neq y} \frac{|\nabla\phi(x) - \nabla\phi(y)|_{\mathrm{op}}}{|x-y|}$$

with $|\cdot|_{\mathrm{op}}$ being the operator norm for matrices.

Remark 4.1. A choice for $G_{\mathscr{D}}$ which fulfills these requirements can always be found in the class of Gaussian-based measures. First, we define a non-degenerate Gaussian measure on $\mathscr{D}_2$ analogously as done in Example 2.9 for the space $\mathscr{D}$, by regarding

$$a_k := \left(2^k(\|\nabla\phi_k\|_\infty^2 \vee \|\nabla^2\phi_k\|_\infty^2)\right) \vee 1, \qquad k \in \mathbb{N},$$

instead and proceeding the exact same way. Then, property $(C1)$ is preserved if we condition to the set $\mathscr{D}_1$, which is open in $(\mathscr{D}, d_{\mathscr{D}})$, as pointed out in Remark 2.7. So, we end up with a measure $G_{\mathscr{D}}$ on $\mathscr{D}_2$ such that $G_{\mathscr{D}}(\mathscr{D}_1 \cap \mathscr{D}_2) = 1$ and Condition $(C1)$ holds true.

To define the local weak intrinsic gradient we introduce the sets

$$\mathscr{D}^{(n)} := \Big\{\phi \in \mathscr{D}_2 \cap \mathscr{D}_1 : \|\nabla\phi^{-1}\|_\infty + d_{\mathscr{D}_2}(\phi, 0) < n\Big\}, \qquad n \in \mathbb{N},$$

which are increasing to $\mathscr{D}_1 \cap \mathscr{D}_2$ as $n \uparrow \infty$. For $n \in \mathbb{N}$ let

$$\Lambda^{(n)} := \Big(\frac{\mathbf{1}_{\mathscr{D}^{(n)}} G_{\mathscr{D}}}{G_{\mathscr{D}}(\mathscr{D}^{(n)})}\Big) \circ \Psi_\lambda^{-1}. \tag{4.2}$$

By Proposition 2.10, we obtain Dirichlet forms $(\mathscr{E}^{(n)}, \mathscr{D}(\mathscr{E}^{(n)})) := (\mathscr{E}^{\gamma,\Lambda^{(n)}}, \mathscr{D}(\mathscr{E}^{\gamma,\Lambda^{(n)}}))$ in $L^2(\mathscr{P}_p, \Lambda^{(n)})$ respectively for $n \in \mathbb{N}$, as well as $(\mathscr{E}, \mathscr{D}(\mathscr{E})) := (\mathscr{E}^{\gamma,\Lambda}, \mathscr{D}(\mathscr{E}^{\gamma,\Lambda}))$ with $\Lambda := G_{\mathscr{D}} \circ \Psi_\lambda^{-1}$. The domains $\mathscr{D}(\mathscr{E}^{(n)})$ are decreasing in n and

$$\mathscr{D}(\mathscr{E}) \subseteq \mathscr{D}(\mathscr{E}^{(\infty)}) := \bigcap_{n=1}^{\infty} \mathscr{D}(\mathscr{E}^{(n)}).$$

It should be mentioned that $\mathrm{supp}[\Lambda^{(n)}]$ is compact w.r.t. the p-Wasserstein topology, due to continuity of the map

$$L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda) \ni \phi \mapsto \Psi_\lambda(\phi) = \lambda \circ \phi^{-1} \in \mathscr{P}_p$$

and the fact that bounded sets in $(\mathscr{D}_2, d_{\mathscr{D}_2})$ are precompact in $L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda)$.

Definition 4.2 below is analogue to Definition 3.7, but using the notion of the weak gradient coming from the family of Dirichlet forms $\{\mathscr{E}^{(n)}\}_n$ instead of $\mathscr{E}$. According to the proof of Proposition 2.10 with $\mathbf{1}_{\mathscr{D}^{(n)}} G_{\mathscr{D}}$ replacing $G_{\mathscr{D}}$, for any $u \in \mathscr{D}(\mathscr{E}^{(n)})$, we find a sequence $\{f_m\}_{m\in\mathbb{N}} \subset C_b^1(L^p(\mathbb{R}^d \to \mathbb{R}^d, \lambda))$ such that the analogues of (a), (b), (c) at the end of Section 3 hold true regarding $\mathbf{1}_{\mathscr{D}^{(n)}} G_{\mathscr{D}}$. The limit

$$\mathscr{D} \ni \phi \mapsto \tilde{D}^u(\phi) := \lim_{n\to\infty} \nabla f_m(\phi) \qquad \text{in } L^2(\mathscr{D} \to L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda), \mathbf{1}_{\mathscr{D}^{(n)}} G_{\mathscr{D}}) \tag{4.3}$$

only depends on u (but not on the choice of an approximating sequence $\{f_m\}_m$) and is measurable w.r.t. $\sigma(\Psi_\lambda)$. Moreover, by the hierarchy of open sets,

$$\mathscr{D}^{(n_1)} \subset \mathscr{D}^{(n_2)} \quad \text{for} \quad n_1 \leq n_2,$$

we conclude that for $G_{\mathscr{D}}$-a.e. ϕ, $\tilde{D}^u(\phi)$ does not depend on a particular choice of n with $\phi \in \mathscr{D}^{(n)}$, given that $u \in \mathscr{D}(\mathscr{E}^{(\infty)})$. So,

$$\tilde{D}^u \in \bigcap_{n=1}^{\infty} L^2\big(\mathscr{D} \to L^2(\mathbb{R}^d \to \mathbb{R}^d, \lambda), \mathbf{1}_{\mathscr{D}^{(n)}} G_{\mathscr{D}}\big)$$

is well-defined. In view of (4.2) and $\Lambda^{(n)} \uparrow \Lambda$, up to normalization constants, we have the following notion of a local weak intrinsic derivative.

Definition 4.2 (Local weak intrinsic gradient)**.** For $u \in \mathscr{D}(\mathscr{E}^{(\infty)})$ the unique element

$$Du \in \bigcap_{n=1}^{\infty} L^2\big(\mathbb{R}^d \times \mathscr{P}_p \to \mathbb{R}^d, \mu(\mathrm{d}x)\Lambda^{(n)}(\mathrm{d}\mu)\big)$$

such that

$$Du(\Psi_\lambda(\phi)) \circ \phi = \tilde{D}^u(\phi), \qquad G_{\mathscr{D}}\text{-a.e. } \phi,$$

is called *local weak intrinsic gradient* of u.

It is clear that for $u \in C_b^1(\mathscr{P}_p)$, respectively $u \in \mathscr{D}(\mathscr{E})$, the local weak intrinsic gradient coincides with the (weak) intrinsic gradient.

Let ∇_1 and ∂_2 denote the gradient in x and derivative in s for $(x, s) \in \mathbb{R}^d \times (0, \infty)$.

Theorem 4.3. *We assume (C_1), $(C2)$ with $G_{\mathscr{D}}(\mathscr{D}_2 \cap \mathscr{D}_1) = 1$ and $\lambda \in \mathscr{P}_p^{\mathrm{ac}}$ such that ρ_λ is strictly positive, Lipschitz continuous. Then:*

(i) ρ_μ is Lipschitz continuous for Λ-a.e. μ.

(ii) If F and $\partial_2 F$ are locally Lipschitz continuous on $\mathbb{R}^d \times (0, \infty)$ and

$$\Lambda^{(n)}\Big(\big\||(\nabla_1\partial_2 F)(\cdot, \rho_\mu)| + \big|(\partial_2\partial_2 F)(\cdot, \rho_\mu)\nabla\rho_\mu\big| + |\partial_2 F(\cdot, \rho_\mu)|\big\|^2_{L^2(\mathbb{R}^d,\mu)}\Big) < \infty$$

for $n \in \mathbb{N}$, then the functional W_F defined in (1.10) *has local weak intrinsic derivative satisfying*

$$DW_F(\mu)(x) = H_F(x,\mu) := (\nabla_1\partial_2 F)(x,\rho_\mu(x)) + (\partial_2\partial_2 F)(x,\rho_\mu(x))\nabla\rho_\mu(x) \tag{4.4}$$

for $\mu(\mathrm{d}x)\Lambda(\mathrm{d}\mu)$-a.e. $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_p$.

Proof. We complete the proof by four steps. Claim (i) follows from Step (1). Claim (ii) is verified in Steps (2)-(4).

(1) First, we show that ρ_μ is Lipschitz continuous for $\Lambda^{(n)}$-a.e. μ and any $n \in \mathbb{N}$, with a Lipschitz constant only depending on $n \in \mathbb{N}$. We recall

$$\rho_\mu = \Big(\frac{\rho_\lambda}{|\mathrm{det}\nabla\phi|}\Big)\circ\phi^{-1}, \qquad \phi \in \mathscr{D}_1,\ \mu = \Psi_\lambda(\phi) = \lambda\circ\phi^{-1}. \tag{4.5}$$

For any $\phi \in \mathscr{D}^{(n)}$, we have

$$\|\nabla\phi\|_\infty \vee \|\nabla^2\phi\|_\infty \vee \|\nabla\phi^{-1}\|_\infty \vee |\phi(0)| \le n. \tag{4.6}$$

Then

$$|\phi^{-1}(0)| \le n + |\phi^{-1}(0) - \phi(0)| \le n + n|\phi(\phi(0))| \le n + n|\phi(0)| + n^2|\nabla\phi\|_\infty \le 3n^3$$

and hence

$$|\phi^{-1}(x)| \le 3n^3 + \|\nabla\phi^{-1}\|_\infty|x| \le 3n^3 + n|x|.$$

This together with (4.6) yields

$$\begin{aligned}\inf_{|x|\le r}\Big(\frac{\rho_\lambda}{|\mathrm{det}\nabla\phi|}\Big)(\phi^{-1}(x)) &\ge n^{-d}\inf_{|x|\le 3n^3+nr}\rho_\lambda(x) > 0,\\ \sup_{|x|\le r}\Big(\frac{\rho_\lambda}{|\mathrm{det}\nabla\phi|}\Big)(\phi^{-1}(x)) &\le n^{d}\sup_{|x|\le 3n^3+nr}\rho_\lambda(x) < \infty, \qquad r\in(0,\infty).\end{aligned} \tag{4.7}$$

Moreover, for $\mu := \Psi_\lambda(\phi)$, $\phi \in \mathscr{D}^{(n)}$, (4.5) implies

$$\nabla\rho_\mu = \left(\frac{(\nabla\phi)^{-1}\nabla\rho_\lambda}{|\mathrm{det}\nabla\phi|} - \mathrm{sgn}(\mathrm{det}[\nabla\phi])\frac{\rho_\lambda(\nabla\phi)^{-1}(\nabla\mathrm{det}\nabla\phi)}{|\mathrm{det}\nabla\phi|^2}\right)\circ\phi^{-1} \tag{4.8}$$

and thus by (4.6) we find a constant $c_n \in (0,\infty)$ such that

$$\|\nabla\rho_\mu\|_\infty \le c_n, \quad \Lambda^{(n)}\text{-a.e. } \mu. \tag{4.9}$$

(2) In this step we calculate the local weak intrinsic gradient DW_F for F satisfying

$$F \in C^2(\mathbb{R}^d\times(0,\infty)), \quad \bigcup_{s>0}\mathrm{supp}[F(\cdot,s)] \subset [-l,l]^d \ \text{ for some } \ l>0. \tag{4.10}$$

Let $\tau \in C_0^\infty(\mathbb{R}^d)$ such that for some constant $\varepsilon > 0$

$$0\le\tau\le 1, \quad \tau(x) = 1 \text{ if } |x|\le\varepsilon, \quad \int_{\mathbb{R}^d}\tau(x)\mathrm{d}x = 1.$$

For any $m \in \mathbb{N}$ and $x \in \mathbb{R}^d$, let $\tau_m(x) := m^d \tau(mx)$. Define

$$(\tau_m * \mu)(x) := \int_{\mathbb{R}^d} \tau_m(x-y)\mu(\mathrm{d}y),\ \ \mu \in \mathscr{P},\ x \in \mathbb{R}^d.$$

By (4.5)-(4.9), we have

$$\begin{aligned} &\tau_m * \mu \in C_b^\infty(\mathbb{R}^d), \quad \inf_{[-l,l]^d} \tau_m * \mu > 0, \quad m \in \mathbb{N},\ \mu \in \Psi_\lambda(\mathscr{D}^{(n)}), \\ &\lim_{m\to\infty} \sup_{\mu \in \Psi_\lambda(\mathscr{D}^{(n)})} \big(\|\rho_\mu - \tau_m * \mu\|_\infty + \|\nabla \rho_\mu - \nabla(\tau_m * \mu)\|_{L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)} \big) = 0, \quad n \in \mathbb{N}. \end{aligned} \tag{4.11}$$

Moreover, in view of (4.10), there exists $\tilde{F} \in C_c^2(\mathbb{R}^{d+1})$ such that

$$F(x, (\tau_m * \mu)(x)) = \tilde{F}(x, (\tau_m * \mu)(x))$$

for $x \in \mathbb{R}^d$, $m \in \mathbb{N}$, $\mu \in \Psi_\lambda(\mathscr{D}^{(n)})$, and hence defining $u_m(\mu) := \int_{\mathbb{R}^d} \tilde{F}\big(x, (\tau_m * \mu)(x)\big)\mathrm{d}x$, $\mu \in \mathscr{P}_p$, we have $u_m \in C_b^1(\mathscr{P}_p)$ and

$$\begin{aligned} u_m(\mu) &= \int_{\mathbb{R}^d} F\big(x, (\tau_m * \mu)(x)\big)\mathrm{d}x, \\ Du_m(\mu)(x) &= \int_{\mathbb{R}^d} \big[\partial_2 F(y, (\tau_m * \mu)(y))\big](-\nabla \tau_m)(y-x)\mathrm{d}y \end{aligned}$$

for $\mu \in \Psi_\lambda(\mathscr{D}^{(n)})$. With integration by parts we get

$$Du_m(\mu)(x) = \int_{\mathbb{R}^d} \big[(\nabla_1 \partial_2 F)(\cdot, \tau_m * \mu) + (\partial_2\partial_2 F)(\cdot, \tau_m * \mu)\nabla(\tau_m * \mu)\big](y)\tau_m(y-x)\mathrm{d}y.$$

Using (4.10) and (4.11), we obtain

$$\lim_{m\to\infty} \sup_{\mu \in \Psi_\lambda(\mathscr{D}^{(n)})} \big(|u_m(\mu) - W_F(\mu)| + \big\|Du_m(\mu) - H_F(\cdot, \mu)\big\|_{L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)} \big) = 0,\ \ n \in \mathbb{N}.$$

Hence, W_F has local weak intrinsic derivative $DW_F = H_F$ as claimed.

(3) We calculate the local weak intrinsic derivative DW_F for F as in the assumptions, but keep the support condition

$$\bigcup_{s>0} \mathrm{supp}[F(\cdot, s)] \subset [-l, l]^d \ \text{ for some } \ l > 0 \tag{4.12}$$

of the previous step. For $m \in \mathbb{N}$ and $(x, s) \in \mathbb{R}^d \times (0, \infty)$ let

$$F_m(x, s) := \frac{1}{m} \int_s^{s+m^{-1}} \mathrm{d}r \int_{\mathbb{R}^d} F(y, r)\tau_m(x-y)\mathrm{d}x.$$

Then F_m satisfies (4.10). Hence, by Step (2), W_{F_m} has local weak intrinsic derivative $DW_{F_m} = H_{F_m}$ and from (4.5)-(4.9), (4.12) we conclude

$$\sup_{m \in \mathbb{N}} \sup_{\mu \in \Psi_\lambda(\mathscr{D}^{(n)})} \big\| |W_{F_m}(\mu)| + |H_{F_m}(\cdot, \mu)| \big\|_{L^\infty(\mathbb{R}^d, \mu)} < \infty, \qquad n \in \mathbb{N}.$$

Consequently, for each n,

$$\text{(4.13)} \qquad \lim_{m\to\infty}\Big(\big\|W_{F_m}-W_F\big\|_{L^2(\mathscr{P}_2,\Lambda^{(n)})}+\big\|DW_{F_m}-H_F\big\|_{L^2(\mathbb{R}^d\times\mathscr{P}_2\to\mathbb{R}^d,\mu(\mathrm{d}x)\Lambda^{(n)}(\mathrm{d}\mu))}\Big)=0.$$

So, W_F has local weak intrinsic gradient $DW_F = H_F$.

(4) Let F be as in the assumptions of the theorem. For any $m\in\mathbb{N}$, let

$$F_m(x,s) := \tau(mx)F(x,s).$$

Then F_m satisfies Condition (4.12), so that Step (3) implies that W_{F_m} has local weak intrinsic derivative $DW_{F_m} = H_{F_m}$. Since

$$|H_{F_m}(\cdot,\mu)| \le c_\tau\Big(\big|(\nabla_1\partial_2 F)(\cdot,\rho_\mu)\big|+\big|(\partial_2\partial_2 F)(\cdot,\rho_\mu)\nabla\rho_\mu\big|+\big|(\partial_2 F)(\cdot,\rho_\mu)\big|\Big)$$

for $m\in\mathbb{N}$ and some constant c_τ depending only on τ, we obtain (4.13) for each n by Lebesgue's dominated convergence. So, W_F has local weak intrinsic derivative $DW_F = H_F$. □

From here on, we fix $\lambda\in\mathscr{P}_2^{\mathrm{ac}}$ with ρ_λ being strictly positive, Lipschitz continuous and

$$\int_{\mathbb{R}^d}\Big(|\ln(\rho_\lambda)|+\tfrac{|\nabla\rho_\lambda|}{\rho_\lambda}\Big)^2\mathrm{d}\lambda<\infty.$$

Let F, V, q be as in Examples 3.2 & 3.6 with Lipschitz continuous V and $G_{\mathscr{D}}$ be as in Remark 4.1. We define $\Lambda, \Lambda^{(n)}$ as in (2.9) and (4.2) for $n\in\mathbb{N}$. Up to normalization constants $\Lambda^{(n)}\uparrow\Lambda$. Let

$$\Lambda_F^{(n)} := \frac{e^{-W_F}\mathrm{d}\Lambda^{(n)}}{\Lambda^{(n)}(e^{-W_F})}$$

analogous to (1.9), (1.10) and $(\mu_t^{(n)})_{t\ge0}$ denote the diffusion on $\mathscr{P}_2$ with Dirichlet form $(\mathscr{E}^{F,n},\mathscr{D}(\mathscr{E}^{F,n})) := (\mathscr{E}^{\gamma,\Lambda_F^{(n)}},\mathscr{D}(\mathscr{E}^{\gamma,\Lambda_F^{(n)}}))$ as defined by Proposition 2.10 in combination with Theorem 3.5. The corresponding objects for the undisturbed case $F=0$ are denoted by $(R_t^{(n)})_{t\ge0}$ and $(\mathscr{E}^{(n)},\mathscr{D}(\mathscr{E}^{(n)})) := (\mathscr{E}^{\gamma,\Lambda^{(n)}},\mathscr{D}(\mathscr{E}^{\gamma,\Lambda^{(n)}}))$.

Combining the results of Sections 3 and Theorem 4.3, $(\mu_t^{(n)})_{t\ge0}$ can be interpreted as intrinsic stochastic gradient flow on $\mathscr{P}_2$ satisfying

$$\text{(4.14)} \qquad \mathrm{d}\mu_t^{(n)} = -D^\gamma W_F(\mu_t^{(n)})\mathrm{d}t+\mathrm{d}R_t^{(n)}, \qquad t\ge0,$$

where

$$\text{(4.15)} \qquad DW_F(\mu) = \nabla V+\frac{q(\rho_\mu)\nabla\rho_\mu}{\rho_\mu}$$

is the local weak intrinsic gradient of W_F for F as in Examples 3.2 & 3.6. The process $(\mu_t^{(n)})_{t\ge0}$ lives on the compact set $E_n := \mathrm{supp}[\Lambda^{(n)}]\subset\mathscr{P}_2^{\mathrm{ac}}$. The sets E_n are increasing in n and

$$\bigcup_{n\in\mathbb{N}} E_n \subseteq \mathscr{P}_2 \quad \text{densely w.r.t. } \mathbb{W}_2,$$

as follows by the proof of Lemma 2.12. The precise formulation of (4.14) involves the generator $A^{(n)}$ of $\mathscr{E}^{(n)}$, analogous to (1.7) and Corollary 3.8.

Corollary 4.4. *Let $n \in \mathbb{N}$. The diffusion $((\mu_t^{(n)})_{t\geq 0}, (\mathbb{P}^{\mu,n})_{\mu\in E_n})$ on E_n which is properly associated with $(\mathscr{E}^{F,n}, \mathscr{D}(\mathscr{E}^{F,n}))$ yields a solution to (4.14), (4.15) in the following sense:*

For $u \in \mathscr{D}(A^{(n)})$,

$$\tilde{u}(\mu_t^{(n)}) - \int_0^t \Big[A^{(n)}u(\mu_s^{(n)}) - \mu_s^{(n)}\big(\gamma_{\mu_s^{(n)}}\langle DW_F(\mu_s^{(n)}), Du(\mu_s^{(n)})\rangle\big)\Big]\mathrm{d}s, \qquad t \geq 0,$$

is a $\mathbb{P}^{\mu,n}$-martingale for $\mathscr{E}^{F,n}$-q.e. $\mu \in E_n$, where $\tilde{u}$ denotes an $\mathscr{E}^{F,n}$-quasi-continuous version.

Proof. The statement follows applying Theorem 4.3 and then Corollary 3.8 regarding the localized objects. To verify the assumptions we use (4.5)-(4.8) and

$$\partial_2 F(x,s) = V(x) + \int_1^s \frac{q(r)}{r}\mathrm{d}r, \quad \nabla_1\partial_2 F(x,s) = \nabla V(x), \quad \partial_2\partial_2 F(x,s) = \frac{q(s)}{s}.$$

For $n \in \mathbb{N}$ and $\mu \in \Psi_\lambda(\mathscr{D}^{(n)})$ there exists a constant $c_n \in (0,\infty)$ such that

$$\begin{aligned} \|\partial_2 F(\cdot,\rho_\mu(\cdot))\|_{L^2(\mathbb{R}^d,\mu)} &\leq c_n \big\|1 + |\cdot| + |\ln(\rho_\lambda)|\big\|_{L^2(\mathbb{R}^d,\lambda)}, \\ \|\nabla_1\partial_2 F(\cdot,\rho_\mu(\cdot))\|_{L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)} &\leq c_n, \\ \|\partial_2\partial_2 F(\cdot,\rho_\mu(\cdot))\nabla\rho_\mu\|_{L^2(\mathbb{R}^d,\mu)} &\leq c_n \big\|\tfrac{\nabla\rho_\lambda}{\rho_\lambda}\big\|_{L^2(\mathbb{R}^d\to\mathbb{R}^d,\lambda)}. \end{aligned}$$

Hence, we have $e^{-W_F} \in D(\mathscr{E}^{(n)}) \cap L^\infty(\Lambda^{(n)})$ and $De^{-W_F} = -e^{-W_F}DW_F$. □

Remark 4.5. Corollary 4.4 incorporates the choice of F as in Remark 3.3 setting $V := \Phi$ and $q := \frac{\beta'}{b}$. With $\gamma(x,\mu) := b(\rho_\mu)(x)$ the deterministic counterpart of (4.14) yields a solution to the generalized porous media equation (1.3).

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