NON-UNIQUENESS OF (STOCHASTIC) LAGRANGIAN TRAJECTORIES FOR EULER EQUATIONS

HUAXIANG LÜ, MICHAEL RÖCKNER, AND XIANGCHAN ZHU

ABSTRACT. We are concerned with the (stochastic) Lagrangian trajectories associated with Euler or Navier-Stokes equations. First, we construct solutions to the 3D Euler equations which dissipate kinetic energy with $C_{t,x}^{1/3-}$ regularity, such that the associated Lagrangian trajectories are not unique. The proof is based on the non-uniqueness of positive solutions to the corresponding transport equations, in conjunction with the superposition principle. Second, in dimension $d \geqslant 2$, for any $1 < p < 2$, $\frac{1}{p} + \frac{1}{s} > 1 + \frac{1}{d}$, we construct solutions to the Euler or Navier-Stokes equations in the space $C_t L^p \cap L_t^1 W^{1,s}$, demonstrating that the associated (stochastic) Lagrangian trajectories are not unique. Our result is sharp in 2D in the sense that: (1) in the stochastic case, for any vector field $v \in C_t L^p$ with $p > 2$, the associated stochastic Lagrangian trajectory associated with v is unique (see [KR05]); (2) in the deterministic case, the LPS condition guarantees that for any weak solution $v \in C_t L^p$ with $p > 2$ to the Navier-Stokes equations, the associated (deterministic) Lagrangian trajectory is unique. Our result is also sharp in dimension $d \geqslant 2$ in the sense that for any divergence-free vector field $v \in L_t^1 W^{1,s}$ with $s > d$, the associated (deterministic) Lagrangian trajectory is unique (see [CC21]).

CONTENTS

Date: April 24, 2025.
2010 *Mathematics Subject Classification.* 60H15; 35R60; 35Q30.
Key words and phrases. Euler equations, (stochastic) Lagrangian trajectories, non-uniqueness in law, convex integration.
Research supported by National Key R&D Program of China (No. 2022YFA1006300, 2020YFA0712700) and the NSFC (No. 12426205, 12090014, 12288201). The financial support by the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is greatly acknowledged. Funded by the Deutsche Forschungsgemeinschaft (DFG,German Research Foundation)-Project-ID 317210226-SFB 1283" .

1. Introduction

In this paper, we study the Lagrangian trajectories associated with weak solutions to the incompressible Euler equations on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for $d \geqslant 2$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X_t &= v(t, X_t),\ t \in [0,T], \\ X_0 &= x, \end{aligned} \tag{1.1}$$

where $T > 0, v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a weak solution to the incompressible deterministic Euler equations on $[0,T] \times \mathbb{T}^d$:

$$\begin{aligned} \partial_t v + \mathrm{div}(v \otimes v) + \nabla \pi &= 0, \\ \mathrm{div} v &= 0, \end{aligned} \tag{1.2}$$

where π denotes the pressure field associated with the fluid. In this context, $X : [0,T] \to \mathbb{T}^d$ represents the trajectory of a particle in an incompressible, non-viscous fluid. In this paper, we focus on the Lagrangian trajectory in the following sense.

Definition 1.1. *Let $v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ be a Borel map and $x \in \mathbb{T}^d$. We say that an absolutely continuous function $X^x_\cdot \in AC([0,T]; \mathbb{T}^d)$ is a (deterministic) Lagrangian trajectory of v starting from x if for all $t \in [0,T]$,*

$$X^x_t = x + \int_0^t v(s, X^x_s) \mathrm{d}s.$$

It is now widely believed that both the 3D Euler equations and their corresponding Lagrangian trajectories may develop singularities. The famous Onsager conjecture [Ons49] says that the threshold regularity for energy conservation of weak solutions to the Euler equations (1.2) is $\frac{1}{3}$, which comes from Onsager's attempt to explain the primary mechanism of energy dissipation in turbulence. Consequently, he implicitly suggested that Hölder continuous weak solutions of the Euler equations could provide an appropriate mathematical description of turbulent flows in the inviscid limit. In recent years, this conjecture has been fully proven using the technique of convex integration (see [Ise18, BDLSV19]). For a further discussion, we refer to Section 1.1 below. We also emphasize that $\frac{1}{3}$ regularity is also related to the dissipative length scales in Kolmogorov's 1941 (K41) phenomenological theory of turbulence [Kol41] (see [HPZZ23, Nov23]).

On the other hand, a fundamental insight into the Lagrangian origin of turbulent scalar dissipation is the concept of spontaneous stochasticity, which was predicted by Lorenz [Lor63, Lor69]. This notion suggests that multiscale fluid flows can inherently lose their deterministic nature and become intrinsically random. However, elevating this theory to a rigorous mathematical framework remains a challenge. The pioneering mathematical work of Bernard, Gawedzki, and Kupiainen [BGK97] examined Kraichnan's turbulence model [Kra68], where the advected velocity is represented as a Gaussian random field with white-noise correlation in time. They demonstrated that, due to the spatial roughness of the advected field, Lagrangian trajectories become non-unique and stochastic in the limit of infinite Reynolds number, even for a fixed initial particle position. We refer to [DE17a, DE17b, ED18, TBM20, MR23, JS24] for recent studies on spontaneous stochasticity.

From the above, studying non-uniqueness of Lagrangian trajectories associated with Euler equations is of primary interest for turbulence. In this paper, our first main result provides a rigorous mathematical proof that there exist dissipative solutions to the 3D Euler equation below the Onsager exponent $\frac{1}{3}$ such that deterministic Lagrangian trajectories are not unique.

Theorem 1.2. *Let $T > 0$ and $0 < \beta < \frac{1}{3}$ be fixed. There exists a solution $v \in C^\beta([0,T] \times \mathbb{T}^3)$ to the Euler equation* (1.2) *which dissipates the kinetic energy, such that non-uniqueness of (deterministic) Lagrangian trajectories holds in the sense that:*
There is a measurable set $A(v) \subset \mathbb{T}^3$ with positive Lebesgue measure such that for every $x \in A(v)$ there are at least two trajectories of v starting at x.

To the best of our knowledge, this is the first result on non-uniqueness of Lagrangian trajectories, where the advected vector field itself is the solution to an unforced, deterministic Euler equation which dissipates kinetic energy.

To prove the non-uniqueness of Lagrangian trajectories, we adopt an "Eulerian" perspective, focusing on the evolution of particle densities rather than individual particle paths. This approach involves studying the following transport equation:

$$\begin{aligned} \partial_t \rho + \operatorname{div}(v\rho) &= 0, \\ \rho(0) &= \rho_0. \end{aligned} \tag{1.3}$$

Here, $\rho : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ represents the probability density of the particles. The connection between the Eulerian description (governed by the transport equation) and the Lagrangian description (involving particle trajectories) is established through the superposition principle (see [Amb08, Theorem 3.2] in the case of $\mathbb{R}^d$. For the case of manifolds as our $\mathbb{T}^d$, see [Tre14, Section 7.2]). Specifically, the superposition principle allows us to express the solution ρ as a superposition of time marginal laws of probability measures $\mathbf{Q}^x$ supported on Lagrangian trajectories started at $x \in \mathbb{T}^3$. As a result of the superposition principle, non-uniqueness of the density ρ for the transport equation (1.3) immediately implies non-uniqueness of the Lagrangian trajectories as stated in Theorem 1.2.

Our second main result of the paper is the following sharp non-uniqueness for the transport equations up to the Onsager exponent.

Theorem 1.3. *Let $\beta, \tilde{\beta} \in (0,1)$ and $T > 0$ be fixed.*
(1). Let $\beta + 2\tilde{\beta} > 1$. For any divergence-free vector field $v \in C^{\beta}([0,T] \times \mathbb{T}^3)$, the transport equation (1.3) has a unique solution $\rho \in C^{\tilde{\beta}}([0,T] \times \mathbb{T}^3)$, which conserves the kinetic energy.
(2). Let $0 < \beta + 2\tilde{\beta} < 1, 0 < \beta < \frac{1}{3}$ and $e : [0,T] \to \mathbb{R}$ be a strictly positive smooth function. Then there exists a solution $v \in C^{\beta}([0,T] \times \mathbb{T}^3)$ to the 3D Euler equation (1.2) satisfying

$$\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x = e(t),$$

such that there is a non-constant probability density $\rho \in C^{\tilde{\beta}}([0,T] \times \mathbb{T}^3)$ solving the transport equation (1.3) with initial data $\rho_0 = 1$ the unit volume. Then (1.3) admits at least two probability density solutions in the space $C^{\tilde{\beta}}([0,T] \times \mathbb{T}^3)$ by noticing that $\overline{\rho}(t) \equiv 1$ is always a solution.

The first part of Theorem 1.3 is established using commutator estimates, following the lines of [CET94]. The second part of Theorem 1.3 is our main result and will be established in Section 2 below. The detailed construction using the convex integration method is shown in Section 3. Then by choosing $\beta < \frac{1}{3}$ close to $\frac{1}{3}$, along with a decreasing function $e(t)$, Theorem 1.2 follows by the superposition principle introduced in [Amb08, Theorem 3.2] in the case of $\mathbb{R}^d$. For the case of manifolds as our $\mathbb{T}^d$, see [Tre14, Section 7.2]).

Moreover, we can show some sharp non-uniqueness results of stochastic Lagrangian trajectories associated with the Euler equation, and even with the Navier-Stokes equation in general dimension $d \geqslant 2$, i.e. solutions to

$$\begin{aligned} \mathrm{d}X_t &= v(t, X_t)\mathrm{d}t + \sqrt{2\kappa}\mathrm{d}W_t, \ t \in [0,T], \\ X_0 &= x, \end{aligned} \tag{1.4}$$

where $\kappa \in (0,1]$ be fixed, $X : [0,T] \to \mathbb{T}^d$ is the stochastic process representing the stochastic particle trajectory, and W_t is a standard $\mathbb{T}^d$-valued Brownian motion defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P})$. See below for a precise definition. The drift term $v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a weak solution to the incompressible deterministic Navier-Stokes or Euler equations on $[0,T] \times \mathbb{T}^d$:

$$\begin{aligned} \mathrm{d}v + \mathrm{div}(v \otimes v)\mathrm{d}t - \nu\Delta v\mathrm{d}t + \nabla\pi\mathrm{d}t = 0, \\ \mathrm{div} v = 0, \end{aligned} \tag{1.5}$$

where $\nu \in [0,1]$, π denotes the pressure field associated with the fluid. When $\nu = 0$, (1.5) is the Euler equation (1.2). Now we focus on the stochastic Lagrangian trajectories in the following sense.

Definition 1.4. *Let $v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ be a Borel map and W be a given $\mathbb{T}^d$-valued Brownian motion on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P})$. We say that an $(\mathcal{F}_t)_{t\geqslant 0}$-adapted map $X^x_\cdot \in C([0,T]; \mathbb{T}^d)$, $\mathbf{P}$-a.s. is a stochastic Lagrangian trajectory of v starting from x if*

$$X^x_t = x + \int_0^t v(s, X^x_s)\mathrm{d}s + \sqrt{2\kappa}W_t, \ \ t \in [0,T], \ \ \mathbf{P}\text{−a.s.}$$

Definition 1.5. *We say that uniqueness in law holds for the stochastic Lagrangian trajectories if for any two sets of stochastic Lagrangian trajectories $\{X^x\}_{x\in\mathbb{T}^d}$ and $\{\overline{X}^x\}_{x\in\mathbb{T}^d}$ (which may be defined on different probability spaces), we have $\mathbf{P} \circ (X^x)^{-1} = \overline{\mathbf{P}} \circ (\overline{X}^x)^{-1}$ for a.e. $x \in \mathbb{T}^d$.*

Remark 1.6. The SDE in Definition 1.4 is meant in the sense of solving the corresponding martingale problem. This is sufficient, since we shall only consider the law of the stochastic Lagrangian trajectory.

In the study of SDEs, there is evidence that a suitable stochastic noise may provide a regularizing effect on deterministic ill-posed problems. On the whole space $\mathbb{R}^d$, for vector fields that are only Lebesgue-integrable, specifically $v \in L^r_t L^p := L^r([0,T]; L^p)$ satisfying $\frac{d}{p} + \frac{2}{r} < 1$, Krylov and the second named author [KR05] established strong existence and uniqueness of stochastic Lagrangian trajectories to (1.4), i.e. adapted to the given Brownian motion. By an analogous argument, this result is also valid on the torus. However, beyond the condition $\frac{d}{p} + \frac{2}{r} < 1$, the question whether stochastic Lagrangian trajectories are pathwise unique-or, more weakly, unique in law-is not completely understood. We refer to Section 1.3 for recent results on this problem.

On the other hand, in the study of hydrodynamics, there is a deep connection between SDEs (1.4) and the Navier-Stokes equations (1.5) with $\nu = \kappa > 0$. When v is a smooth solution to the Navier-Stokes equations, Constantin and Iyer [CI08] provided the following stochastic representation:

$$v(t,x) = \mathbb{P}_{\rm H}\mathbf{E}[\nabla^T (X^x_t)^{-1} v_0((X^x_t)^{-1})], \tag{1.6}$$

where $\mathbb{P}_{\rm H}$ is the Leray projection and v_0 is the initial condition. Conversely, if v is smooth and (v, X^x) solves (1.4) and (1.6), then v is also a solution to the Navier-Stokes equations with initial condition v_0. The condition $\frac{d}{p} + \frac{2}{r} < 1$ is special case of the famous Ladyzhenskaya-Prodi-Serrin (LPS) condition $\frac{d}{p} + \frac{2}{r} \leqslant 1$, which provides a sufficient condition for the regularity and uniqueness of weak solutions to the Navier-Stokes equations. Within the LPS condition, both the NS equation and the corresponding Lagrangian trajectories are unique. Beyond this condition, using the convex integration method, sharp non-uniqueness of weak solutions to the Navier-Stokes equations has been shown in [BV19b, BCV21, CL22a, CL23].

In summary, whether from the perspective of SDEs or from the perspective of fluid dynamics, studying the well/ill-posedness of stochastic Lagrangian trajectories with hydrodynamic drifts beyond the LPS condition is of theoretical interest. Our third main result is to show the non-uniqueness in law of the stochastic Lagrangian trajectories of weak solutions to the Navier-Stokes or Euler equations (1.5) in the supercritical regime, i.e. $\frac{d}{p} + \frac{2}{r} > 1$. To state the main result, for $d \geqslant 2$ we define $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 := \left\{ (p,r,s) \in [1,\infty]^3 : \frac{1}{p} + \frac{1}{r} > 1, 1 < s < d \right\},$$
$$\mathcal{A}_2 := \left\{ (p,r,s) \in [1,\infty]^3 : \frac{1}{p} + \frac{1}{r} \leqslant 1, 1 < p < 2, \frac{1}{d} + \frac{1}{2}\frac{1 - \frac{1}{r} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{r}} < \frac{1}{s} < 1 \right\}.$$

Theorem 1.7. *Let $d \geqslant 2, \kappa \in [0,1], \nu \in [0,1]$. For any triple $(p,r,s) \in \mathcal{A}$, there exists a divergence-free vector field $v \in L^r([0,T]; L^p) \cap L^2([0,T] \times \mathbb{T}^d) \cap L^1([0,T]; W^{1,s}) \cap C([0,T]; L^1)$ which is a solution to the Navier-Stokes or Euler equations* (1.5)*, such that the law of stochastic Lagrangian trajectories of v is not unique in the sense that:*
There is a measurable $A(v) \subset \mathbb{T}^d$ with positive Lebesgue measure such that for every $x \in A(v)$ there are at least two stochastic trajectories of v starting at x, admitting distinct laws satisfying $\mathbf{E}[\int_0^T |v(s, X^x_s)|{\rm d}s] < \infty$.

Moreover, when $r = \infty$, the solution v is continuous in L^p-norm, i.e. $v \in C([0,T]; L^p) =: C_t L^p$.

As a corollary, in the stochastic case $\kappa > 0$, by taking $s > 1$ close to 1 in Theorem 1.7, we show that sharp non-uniqueness in law holds for stochastic Lagrangian trajectories of weak solutions to the Navier-Stokes or Euler equations (1.5), within a range of supercritical regimes:

Corollary 1.8. *Let $d \geqslant 2, \kappa \in (0,1], \nu \in [0,1], p, r \in [1,\infty]$ be fixed. There exists a solution*

$$v \in \begin{cases} L^r([0,T]; L^p) \cap L^2([0,T] \times \mathbb{T}^d) \cap L^1([0,T]; W^{1,1}) \cap C([0,T]; L^1), & \text{if } \frac{1}{p} + \frac{1}{r} > 1, \\ C([0,T]; L^p) \cap L^2([0,T] \times \mathbb{T}^d) \cap L^1([0,T]; W^{1,1}), & \text{if } 1 < p < 2, \end{cases}$$

to the Navier-Stokes or Euler equations (1.5)*, such that the law of stochastic Lagrangian trajectories of v is not unique in the sense that:*

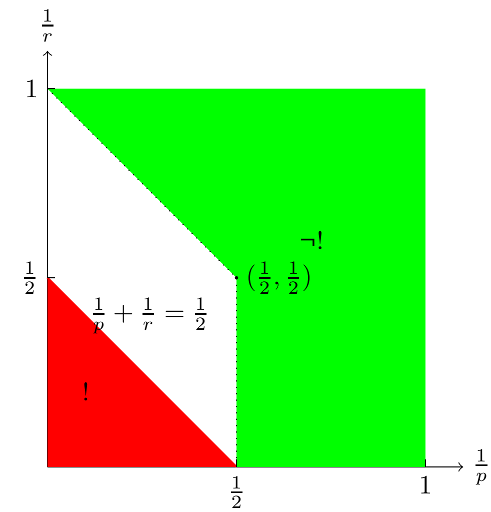

FIGURE 1. State of (non-)uniqueness to SDE (1.4) for vector fields $v \in L_t^r L^p$ in the 2D case.
Red area: in the subcritical case, the SDE admits a unique strong solution.
Red line: in the critical case , the well-posedness of SDE remains open in $d = 2$.
Green area: in the supercritical case, our main result shows the non-uniqueness in law holds in this range.

There is a measurable $A(v) \subset \mathbb{T}^d$ with positive Lebesgue measure such that for every $x \in A(v)$ there are at least two trajectories of v starting at x, admitting distinct laws satisfying $\mathbf{E}[\int_0^T |v(s, X_s^x)| \mathrm{d}s] < \infty$.

In Figure 1, we summarize the (non-)uniqueness in law results on SDE (1.4) for vector fields $v \in L_t^r L^p$ in the case $d = 2$. Our result implies that for any $p < 2$, there exists a solution $v \in C_t L^p$ to the Navier-Stokes or Euler equations such that non-uniqueness in law of stochastic Lagrangian trajectories of v holds, which is sharp in the sense that for any vector field $v \in C_t L^p$, $p > 2$, (1.4) admits a unique strong solution as recalled above. Our result covers the optimal range, except for the endpoint $p = 2$. We also notice that since $\mathrm{div} v = 0$, (1.4) admits an invariant measure $\rho \equiv 1$. We obtain non-uniqueness in law even when starting from this invariant measure.

In contrast to this, in the deterministic case $\kappa = 0$, despite the absence of regularization-by-noise phenomena, the LPS criteria guarantee that any weak solution $v \in C_t L^p$ with $p > d$ to the Navier-Stokes equations is Leray and regular, which still ensures the uniqueness of the associated (deterministic) Lagrangian trajectory. Taking $r = \infty$ in Theorem 1.7, the following corollary presents a sharp counterexample $v \in C_t L^{2-} \cap L_t^1 W^{1,1+}$ for which the Lagrangian trajectories are non-unique, which covers the optimal range, except for the endpoint $p = 2$.

Corollary 1.9. *Let $d \geqslant 2, \nu \in [0, 1]$. For any $1 < p < 2, \frac{1}{p} + \frac{1}{s} > 1 + \frac{1}{d}$, there exists a divergence-free vector field $v \in C([0, T]; L^p) \cap L^2([0, T] \times \mathbb{T}^d) \cap L^1([0, T]; W^{1,s})$ which is a solution to the Navier-Stokes or Euler equations (1.5), such that the non-uniqueness of (deterministic) Lagrangian trajectories holds in the sense that:*
There is a measurable $A(v) \subset \mathbb{T}^d$ with positive Lebesgue measure such that for every $x \in A(v)$ there are at least two trajectories of v starting at x.

Furthermore, for any $s < d$, there is a solution to the Navier-Stokes or Euler equations (1.5) in the space $C_tL^1 \cap L^2_{t,x} \cap L^1_tW^{1,s}$ for which the Lagrangian trajectories are non-unique. Our result is sharp in comparison to the result of Caravenna and Crippa [CC21, Corollary 5.2], which states that for any divergence-free vector field $v \in L^1_tW^{1,s}$ with $s > d$, then for a.e. $x \in \mathbb{T}^d$, there is a unique trajectory of v starting at x. Our result covers the optimal range except for the endpoint $p = d$. We also remark that Brué, Colombo and De Lellis [BCDL21] provided a nice divergence-free counterexample $v \in C_t(W^{1,d-} \cap L^{\infty-})$ using convex integration.

As before, we adopt an "Eulerian" perspective and establish the non-uniqueness of solutions to the corresponding Fokker-Planck equations on $[0,T] \times \mathbb{T}^d$:

$$\begin{aligned} \partial_t\rho - \kappa\Delta\rho + \operatorname{div}(v\rho) &= 0, \\ \rho(0) &= \rho_0. \end{aligned} \tag{1.7}$$

Theorem 1.10. *Let $\kappa, \nu \in [0,1]$. Then for any triple $(p,r,s) \in \mathcal{A}$, there exists a solution $v \in L^r([0,T];L^p) \cap L^2([0,T]\times\mathbb{T}^d) \cap L^1([0,T];W^{1,s}) \cap C([0,T];L^1)$ to (1.5), and a non-constant density $\rho \in L^r([0,T];L^p) \cap L^2([0,T]\times\mathbb{T}^d) \cap C([0,T];L^1)$ satisfying (1.7) with initial data $\rho_0 = 1$.*

Moreover, when $r = \infty$, the solutions v, ρ are also continuous in L^p-norm.

This result is proved in Section 4, while the detailed construction via the convex integration method is given in Section 5. Then Theorem 1.7 follows by the superposition principle proved in [Tre14, Section 7.2].

1.1. Convex integration and Onsager conjecture.

The rigid part of Onsager's conjecture was established using commutator estimates, as demonstrated in [CET94, CCFS08]. The flexible part was proven using the convex integration method by Isett in [Ise18] for the 3D case, with further constructions of strictly dissipative solutions discussed in [BDLSV19]. This convex integration technique was first introduced to fluid dynamics by De Lellis and Székelyhidi Jr. [DLS09, DLS10, DLS13], leading to numerous groundbreaking results. For the L^2-based Sobolev scale, Buckmaster, Masmoudi, Novack, and Vicol [BMNV23] constructed non-conservative weak solutions of the 3D Euler equations in $C_tH^{1/2-}$, and then refined by Novack and Vicol [NV23] to the class $C^0_t(H^{1/2-} \cap L^\infty)$. By interpolation, such solutions belong to $C_tB^{1/3-}_{3,\infty}$, which can be seen as a proof of the L^3-based intermittent Onsager theorem. Notably, the 2D Onsager conjecture was addressed by Giri and Radu [GR24] through convex integration with Newton iteration. We refer to [Cho13, BDLIS15, Buc15, BDLS16, DSJ17, DLK22, Ise22, BHP23, BC23, GKN23, GKN24, BM24a, BM24b] for more results on the Euler equations. We mention that the convex integration method also led to a breakthrough to the non-uniqueness of weak solutions to the Navier-Stokes equations, see for example [BV19b, BCV21, CL22a, CL23, MNY24a, MNY24b, CZZ25]. We refer interested readers to the comprehensive reviews [BV19a, BV21, DLS22] for more details and references.

Very recently, Brué, Colombo and Kumar [BCK24b] introduced a new "asynchronization" idea for the building blocks in the convex integration to derive non-uniqueness of weak solutions in $L^\infty_tL^2$ with vorticity in $L^\infty_tL^{1+}$, which solves a longstanding open problem of non-uniqueness of the 2D Euler equations with integrable vorticity.

We also note that the convex integration method has been successfully applied to the stochastic fluid dynamics (see [BFH20, HZZ22, Yam22a, Yam22b, HZZ23a, HZZ23b, LZ23, HLP24, HZZ24, LRS24, Pap24, HZZ25, LZ25a, LZ25b] and references therein).

1.2. Previous results on the ODE level.

If the advected vector field v is Lipschitz continuous, classical theorems ensure the uniqueness of Lagrangian trajectories starting from any $x \in \mathbb{T}^d$. For less regular vector fields on the whole space $\mathbb{R}^d$, DiPerna and Lions [DL89] proved the uniqueness of trajectories in the class of regular Lagrangian flows under suitable Sobolev regularity conditions. Moreover, for a divergence-free vector field $v \in L_t^1 W^{1,s}$ and every $\rho_0 \in L^p$ with $\frac{1}{p}+\frac{1}{s} \leqslant 1$, there exists a unique weak solution $\rho \in L_t^\infty L^p$ to the transport equation (1.3). An analogous version of this result holds on the torus $\mathbb{T}^d$. Later, the DiPerna-Lions theory was extended by Ambrosio [Amb08] to the bounded variation case $v \in L_t^1(BV)$. However, it is not clear whether for almost every x, there is a unique trajectory of v starting at x. We refer to [DL08, Amb17, CC21, BCDL21] for more related uniqueness results beyond the DiPerna-Lions condition.

Concerning fluid equations on the whole space $\mathbb{R}^3$, if v is a Leray solution to the 3D Navier-Stokes equations as defined in (1.5) and if it is obtained through approximation with $v(0) \in H^{1/2}(\mathbb{R}^3)$, Robinson and Sadowski [RS09a, RS09b] proved the existence and uniqueness of Lagrangian trajectories for a.e. initial point $x \in \mathbb{R}^3$. Then Galeati [Gal24] refined it to the case $v(0) \in L^2(\mathbb{R}^3)$.

Regarding non-uniqueness, to the best of our knowledge, there are currently two distinct methods. The first approach is Lagrangian, which involves using the degeneration of the flow map to show non-uniqueness at the ODE level, and we refer to [DL89, Dep03, YZ17, ACM19, DEIJ22, Kum24, Pap23, BCK24a, MS24]. These constructions are usually quite specific and play an important role in understanding the fluid dynamics. The second approach is Eulerian, by demonstrating non-uniqueness directly at the PDE level. Crippa, Gusev, Spirito, and Wiedemann [CGSW15] were the first to utilize convex integration to derive non-uniqueness results for transport equations (1.3) on torus. Subsequently, numerous significant breakthroughs concerning Sobolev vector fields were achieved through convex integration in [MS18, MS19, MS20, BCDL21, CL21, CL22b, PS23].

1.3. **Previous results on the SDE level.** As mentioned earlier, the regularization-by-noise phenomenon plays a significant role in the well-posedness of SDEs, which comes from the effect of the Laplacian in the Fokker-Planck equations.

On the whole space, when v is a bounded measurable function, Veretennikov [Ver80] proved the uniqueness of probabilistically strong solutions. When $v \in L_t^r L^p$, Krylov and the second named author in [KR05] established the existence and uniqueness of strong solutions to (1.4) in the class $\int_0^T |v(s,X_s)|^2 \mathrm{d}s < \infty$, $\mathbf{P}$-a.s., under the condition $\frac{d}{p}+\frac{2}{r} < 1$. Moreover, the square integrability condition can be removed, see for example [Hao23, Lemma 3.4]. For more well-posedness results in the subcritical case, we refer to [Zha05, Zha11, Zha16, XXZZ20, RZ21a].

In the critical case $\frac{d}{p}+\frac{2}{r} = 1$, on the whole space, Krylov [Kry20a] proved the strong well-posedness of SDEs in the case $v \in L^d(\mathbb{R}^d)$, which is a significant progress on this topic. The second named author and Zhao [RZ21b] showed that for any vector field $v \in C_t L^d$ or $v \in L_t^r L^p, r, p \in (2,\infty)$, (1.4) admits a unique strong solution within a class satisfying a Krylov-type estimate. They [RZ23] also proved weak uniqueness with divergence-free $v \in L_t^\infty L^d$ within a class satisfying a Krylov-type estimate. We refer to [Nam20, Kry20b, Kry20c] for further results.

Beyond the LPS condition $\frac{d}{p}+\frac{2}{r} \leqslant 1$, the known results in the supercritical regime are very limited. When the vector field is not divergence-free, in [BFGM19], there is a counterexample showing that (1.4) may not have weak solutions if v is in the Lorentz space $L^{d,\infty}(\mathbb{R}^d)$. Then Zhao [Zha19, Theorem 5.1] constructed a divergence-free vector field $v \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), p \in (\frac{d}{2}, d), d \geqslant 3$, such that weak uniqueness fails. Given an additional divergence-free property on the drift v, Zhang and Zhao [ZZ21] established weak existence and uniqueness in the sense of approximation for $\frac{d}{p}+\frac{2}{r} \leqslant 2$. Recently, Galeati [Gal24] demonstrated strong existence and pathwise uniqueness for every initial $x \in \mathbb{R}^3$ for the SDEs (1.4), under the assumption that v is a Leray solution to 3D Navier-Stokes equations obtained through approximation with divergence-free $v(0) \in H^{1/2}(\mathbb{R}^3)$. It is also worth mentioning that very recently, for $d \geqslant 1$ and any $\frac{d}{p}+\frac{2}{r} > 1, p > d$, Galeati and Gerencsér in [GG25] constructed an example $v \in L_t^r L^p(\mathbb{R}^d)$ such that non-uniqueness in law holds for (1.4) when starting from $x = 0$. We also refer to [Gal23, BG23, HZ23, GP24, HRZ24] for more results in the supercritical regime.

1.4. **Ideas of the proof.** In this paper, to demonstrate the non-uniqueness of stochastic Lagrangian trajectories, in view of the superposition principle, we work at the PDE level to construct a fluid field such that the transport equation admits two solutions. To achieve this, we concurrently apply the convex integration method to the fluid equation and the transport equation simultaneously. However, when considering two different scales $C^0_{t,x}$ or $L^1_t W^{1,s}$, we face several challenges, which require to address them using distinct strategies tailored to each scale's characteristics.

1.4.1. *Ideas in the $C^0_{t,x}$-scales.* We apply the convex integration method to transport equations and the Euler equations simultaneously. At each step $q \in \mathbb{N}_0$, we construct a pair $(v_q, \rho_q, \mathring{R}_q, M_q)$ satisfying the following system:

$$\begin{aligned} \partial_t \rho_q + \mathrm{div}(v_q \rho_q) &= -\mathrm{div} M_q, \\ \partial_t v_q + \mathrm{div}(v_q \otimes v_q) + \nabla \pi_q &= \mathrm{div} \mathring{R}_q, \ \ \mathrm{div} v_q = 0, \end{aligned} \tag{1.8}$$

where $\mathring{R}_q$ is a trace-free symmetric matrix, and M_q is a vector field. Here, $\mathring{R}_q$ and M_q converge to 0 and (v_q, ρ_q) converge to a weak solution to the transport equation and the Euler equations respectively.

At each iterative step, we need to construct new perturbations to simultaneously cancel two stress terms and obtain smaller residual stress terms. However, a new velocity perturbation constructed to cancel one of the stress terms will also influence the other stress term. To overcome this, inspired by the work of Isett [Ise22], we define the perturbations $(w_{q+1} + \overline{w}_{q+1}, \theta_{q+1})$ as a sum of highly oscillatory Mikado flows such that the support of (w_{q+1}, θ_{q+1}) and that of $\overline{w}_{q+1}$ are disjoint by choosing disjoint building blocks. Here the perturbation (w_{q+1}, θ_{q+1}) is used to cancel the stress term M_q. As we can see, it produces a new term $w_{q+1} \otimes w_{q+1}$ in the Euler equation. Then we construct the perturbation $\overline{w}_{q+1}$ to cancel the stress term $\mathring{R}_q$ and the low frequency part, which comes from $w_{q+1} \otimes w_{q+1}$. More precisely,

$$\begin{aligned} w_{q+1}\theta_{q+1} &\sim M_q + (\text{high frequency error}), \\ \overline{w}_{q+1} \otimes \overline{w}_{q+1} &\sim -\mathring{R}_q - \fint_{\mathbb{T}^3} w_{q+1} \otimes w_{q+1} \mathrm{d}x + (\text{high frequency error}). \end{aligned} \tag{1.9}$$

Once we have the above relations, since the support of (w_{q+1}, θ_{q+1}) and that of $\overline{w}_{q+1}$ are disjoint, it follows that

$$\begin{aligned} (w_{q+1} + \overline{w}_{q+1})\theta_{q+1} &\sim M_q + (\text{high frequency error}), \\ (w_{q+1} + \overline{w}_{q+1}) \otimes (w_{q+1} + \overline{w}_{q+1}) &\sim -\mathring{R}_q + \mathbb{P}_{\neq 0}(w_{q+1} \otimes w_{q+1}) + (\text{high frequency error}). \end{aligned}$$

Here we denote $\mathbb{P}_{\neq 0} f := f - \fint f \mathrm{d}x$. So, the principle part of the oscillation errors has been canceled, while the high frequency errors and the high frequency part of the product $w_{q+1} \otimes w_{q+1}$ are small in C^0-norms.

To achieve the regime $\beta + 2\tilde{\beta} < 1$ and the Onsager regime up to $1/3$ at the same time, we need to choose the parameters carefully. Let us heuristically show the typical estimates of the Nash errors $\mathrm{div}^{-1}((w_{q+1} + \overline{w}_{q+1}) \cdot \nabla v_q)$ and $\mathrm{div}^{-1}((w_{q+1} + \overline{w}_{q+1}) \cdot \nabla \rho_q)$ from the Euler equations and the transport equation respectively. We assume that the frequencies grow hypergeometrically $\lambda_q = \lambda_{q-1}^b$ for some $b > 1$, and the perturbations obey the following Hölder scaling:

$$\|w_{q+1}\|_{C^0} + \|\overline{w}_{q+1}\|_{C^0} \leqslant \lambda_{q+1}^{-\beta}, \ \ \|\theta_{q+1}\|_{C^0} \leqslant \lambda_{q+1}^{-\tilde{\beta}},$$

for $\beta, \tilde{\beta} > 0$. In view of (1.9), we need to ensure

$$\|\mathring{R}_q\|_{C^0} \leqslant \lambda_{q+1}^{-2\beta}, \ \ \|M_q\|_{C^0} \leqslant \lambda_{q+1}^{-\beta-\tilde{\beta}}.$$

Then we need to verify the above bounds at the level $q+1$. As usual in the convex integration method, the iterate (v_q, ρ_q) is a sum of building blocks with frequency not bigger then λ_q. Then by choosing $\lambda_{q+1} \gg \lambda_q$, we notice that the inverse divergence div^{-1} will give us a factor λ_{q+1}^{-1}, while the gradient will give us a factor λ_q. Then two Nash errors can be estimated by

$$\|\mathrm{div}^{-1}((w_{q+1}+\overline{w}_{q+1})\cdot\nabla v_q)\|_{C^0} \lesssim \frac{\|w_{q+1}+\overline{w}_{q+1}\|_{C^0}\|v_q\|_{C^1}}{\lambda_{q+1}} \lesssim \frac{\lambda_{q+1}^{-\beta}\lambda_q^{-\beta}\lambda_q}{\lambda_{q+1}} \leqslant \lambda_{q+2}^{-2\beta},$$
$$\|\mathrm{div}^{-1}((w_{q+1}+\overline{w}_{q+1})\cdot\nabla \rho_q)\|_{C^0} \lesssim \frac{\|w_{q+1}+\overline{w}_{q+1}\|_{C^0}\|\rho_q\|_{C^1}}{\lambda_{q+1}} \lesssim \frac{\lambda_{q+1}^{-\beta}\lambda_q^{-\tilde{\beta}}\lambda_q}{\lambda_{q+1}} \leqslant \lambda_{q+2}^{-\beta-\tilde{\beta}}.$$

To ensure the validity of the iteration, we need

$$1-b-\beta-\beta b < -2\beta b^2, \ 1-b-\tilde{\beta}-\beta b < -\beta b^2-\tilde{\beta}b^2,$$

i.e.

$$(b-1)(\beta(2b+1)-1) < 0, \ (b-1)(\tilde{\beta}(b+1)+\beta b-1) < 0,$$

which is satisfied by choosing $b > 1$ close to 1, $\beta < \frac{1}{3}$ and $\beta + 2\tilde{\beta} < 1$.

Moreover, before the perturbation step, we apply the gluing step introduced by Isett in [Ise18] to achieve the above regularities. Specifically, we combine exact solutions to the Euler equations, following a method similar to that in [BDLSV19]. Correspondingly, we also glue together exact solutions to the transport equation, ensuring that the associated errors $\overline{M}_q$ are supported in disjoint temporal intervals. Here we need to estimate the difference between the glued solution ρ_i and the original solution to the transport equation ρ_q in certain negative order spaces. Since ρ_i and ρ_q are both scalars, we can not apply the Biot-Savart law as done in [BDLSV19]. Instead, we utilize the inverse divergence operator to define $y_i = \mathrm{div}^{-1}\rho_i, y_q = \mathrm{div}^{-1}y_q$, which satisfies an equation of the form:

$$\mathrm{div}[(\partial_t + v_l\cdot\nabla)(y_i - y_q)] = \mathrm{div}[\cdots].$$

Since there is no suitable left-inverse of the div operator, we use the identity $\nabla\mathrm{div} = \Delta + \mathrm{curl}\,\mathrm{curl}$ and the fact that $\mathrm{curl}\,(y_i - y_q) = 0$ to derive that

$$(\partial_t + v_l\cdot\nabla)(y_i - y_q) = \Delta^{-1}\nabla\mathrm{div}[\cdots] - \Delta^{-1}\mathrm{curl}\,\mathrm{curl}\,[v_l\cdot\nabla(y_i - y_q)].$$

By combining various analytic identities (see Lemma 3.4 below), we obtain that $y_i - y_q$ is bounded by estimates on the transport equation together with Gronwall's inequality. We refer to Section 3.2.2 for more details.

1.4.2. *Ideas in the $L_t^1W^{1,s}$-scales.* When we consider the $L_t^1W^{1,s}$-scales or $L_t^rL^p$-scales for SDEs, to prove Theorem 1.10, we once again apply convex integration. At each step $q \in \mathbb{N}_0$, we need to deal with a similar system as (1.8) with an extra Δ in both equations. As in the previous analysis, to eliminate the stress terms, we construct new perturbations of the form $(w_{q+1}+\overline{w}_{q+1}, \theta_{q+1})$, where (w_{q+1}, θ_{q+1}) is to cancel the stress term M_q, while $\overline{w}_{q+1}$ is to cancel the stress term $\mathring{R}_q$. To address the dissipative Laplacian term, we enhance more intermittency in the building blocks by introducing generalized intermittent space-time jets, which are inspired by [LZ23, Section 3], [BCDL21, Section 4] for the spatial direction and by [CL21, Section 4.2] for the temporal direction.

However, because of the existence of extra intermittency, the previous method breaks down. Roughly speaking, we aim to construct a perturbation $w_{q+1} = \sum_\xi a_{(\xi)} W_{(\xi)}, \theta_{q+1} = \sum_\xi a'_{(\xi)} \Theta_{(\xi)}$ with a large oscillation parameter μ. On the one hand, similar to the procedure in [BV19b, Section 4.3], to cancel the additional oscillation errors in the Navier-Stokes equations arising from

$$\operatorname{div}(\mathbb{P}_{\neq 0}(w_{q+1} \otimes w_{q+1})) \sim \frac{1}{\mu}\partial_t(\sum_\xi a^2_{(\xi)} |W_{(\xi)}|^2 \xi) + (\text{high frequency error}),$$

we need to introduce a temporal corrector and choose $\mu > r_\|^{-\frac{1}{2}} r_\perp^{-\frac{d-1}{2}}$ to control this corrector. However, when dealing with the transport equation, the intermittent building blocks should satisfy the equation

$$\partial_t \Theta_{(\xi)} + \mu r_\|^{\frac{1}{2}} r_\perp^{\frac{d-1}{2}} \operatorname{div}(W_{(\xi)} \Theta_{(\xi)}) = 0$$

as seen in (C.3). This condition requires $\mu = r_\|^{-\frac{1}{2}} r_\perp^{-\frac{d-1}{2}}$, which leads to a contradiction.

To address this problem, at the beginning of the iteration, we iterate two equations at different scales. Heuristically speaking, we assume that the frequencies grow hypergeometrically $\lambda_q = \lambda_{q-1}^b$ for some b large enough, and that the perturbations obey the following L^2 scaling:

$$\|\overline{w}_{q+1}\|_{L^2} \leqslant \lambda_{q+1}^{-\beta}, \ \ \|w_{q+1}\|_{L^2} + \|\theta_{q+1}\|_{L^2} \leqslant \lambda_{q+2}^{-\beta},$$

and correspondingly

$$\|\mathring{R}_q\|_{L^1} \leqslant \lambda_{q+1}^{-2\beta}, \ \ \|M_q\|_{L^1} \leqslant \lambda_{q+2}^{-2\beta}.$$

Then the undesired product $w_{q+1} \otimes w_{q+1}$ automatically satisfies the desired estimates for $\mathring{R}_{q+1}$ since

$$\|w_{q+1} \otimes w_{q+1}\|_{L^1} \lesssim \|w_{q+1}\|_{L^2}^2 \lesssim \lambda_{q+2}^{-2\beta}.$$

Finally, we emphasize that although we obtain a spatial range similar to [BCDL21], the building blocks and the corresponding estimates in our work are different. Specifically, in [BCDL21, Section 4], the authors constructed building blocks satisfying a certain L^p-normalization property with some $p > 0$, and derived solutions in the space $C_t(W^{1,d-} \cap L^{\infty -})$. In contrast to that, in the present paper, our construction of building blocks is limited to the L^2-normalization property, which is crucial for solutions to the Navier-Stokes equations. By introducing temporally intermittent jets and a careful choice of parameters, we ultimately achieve a spatial range similar to [BCDL21, Section 4], at the cost that the time integrability of the solutions is only L^1.

1.5. **Organization of the paper.** This paper is organized as follows. First, Section 2 is dedicated to prove our first main result, Theorem 1.2, i.e. to demonstrate the existence of non-unique Lagrangian trajectories for energy-dissipative solutions to the Euler equations with Hölder regularity up to $1/3$. We work at the PDE level by establishing Theorem 1.3 through the convex integration method and then using the superposition principle. The implementation of the main convex integration procedure is presented in Section 3. Then, Section 4 is devoted to the main results in $L^1_t W^{1,s}$-based scales: Theorem 1.7 and the two Corollaries 1.8 and 1.9. We also work at the PDE level by demonstrating non-uniqueness for Fokker-Planck equations advected by solutions to the Navier-Stokes or Euler equations, as stated in Theorem 1.10. The implementation of the main convex integration procedure is presented in Section 5. In Appendix A, we collect some technical tools used in the construction. In Appendix B and Appendix C we provide the building blocks and some auxiliary estimates used in the constructions in Section 3 and Section 5 respectively.

Notations. Let $T > 0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout the manuscript, we write $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for the d-dimensional flat torus. We define the natural projection $\mathrm{Pr} : \mathbb{R}^d \to \mathbb{T}^d$ by $\mathrm{Pr}(x) = x - [x]$, where $[x]$ is the integer part of x, which is continuous. A $\mathbb{T}^d$-valued Brownian motion is seen as the natural projection of

$\mathbb{R}^d$-valued Brownian motion onto $\mathbb{T}^d$. We refer to [Hsu02] for a comprehensive treatment of the Brownian motion on general manifolds. We use the following notations.

- We employ the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$.
- Given a Banach space E with a norm $\|\cdot\|_E$, we write $C_t E = C([0,T];E)$ for the space of continuous functions from $[0,T]$ to E, equipped with the supremum norm. For $p \in [1,\infty]$ we write $L_t^p E = L^p([0,T];E)$ for the space of L^p-integrable functions from $[0,T]$ to E, equipped with the usual L^p-norm.
- For $\alpha \in (0,1)$ we define $C_t^\alpha E$ as the space of α-Hölder continuous functions from $[0,T]$ to E, endowed with the norm $\|f\|_{C_t^\alpha E} = \sup_{s,t\in[0,T],s\neq t} \frac{\|f(s)-f(t)\|_E}{|t-s|^\alpha} + \sup_{t\in[0,T]} \|f(t)\|_E$, and write C_t^α in the case when $E = \mathbb{R}$.
- We use L^p to denote the set of standard L^p-integrable functions on $\mathbb{T}^d$. For $s > 0$, $p > 1$ we set $W^{s,p} := \{f \in L^p; \|(I-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}$ with the norm $\|f\|_{W^{s,p}} = \|(I-\Delta)^{\frac{s}{2}} f\|_{L^p}$.
- For $N \in \mathbb{N}_0$, C^N denotes the space of N-times differentiable functions equipped with the norm
$$\|f\|_{C^N} := \sum_{|\alpha|\leqslant N, \alpha\in\mathbb{N}_0^d} \|D^\alpha f\|_{L_x^\infty}.$$
Similarly, if the norm is taken in space-time, we use $C_{t,x}^N$. For $N \in \mathbb{N}_0$ and $\kappa \in (0,1)$, $C^{N+\kappa}$ denotes the subspace of C^N whose N-th derivatives are κ-Hölder continuous, with the norm
$$\|f\|_{C^{N+\kappa}} := \|f\|_{C^N} + \sum_{|\alpha|=N, \alpha\in\mathbb{N}_0^d} [D^\alpha f]_{C^\kappa},$$
where $[f]_{C_x^\kappa} := \sup_{x\neq y, x,y\in\mathbb{T}^d} \frac{|f(x)-f(y)|}{|x-y|^\kappa}$ is the Hölder seminorm.
- We define the projections $\mathbb{P}_{=0} f := \fint_{\mathbb{T}^d} f \mathrm{d}x$, and $\mathbb{P}_{\neq 0} f := f - \fint_{\mathbb{T}^d} f \mathrm{d}x$.
- For a matrix R, we denote its traceless part by $\mathring{R} := R - \frac{1}{d}\mathrm{tr}(R)\mathrm{Id}$.
- We denote the Lebesgue measures on $\mathbb{T}^d$ by $\mathcal{L}^d$.

2. Construction of non-unique solutions in $C_{t,x}^0$ scales

In this section, we present the proof of Theorem 1.2. First, we work on the PDE level to demonstrate the non-uniqueness of solutions to the transport equations advected by Euler flows, as stated in Theorem 1.3 (2). Specifically, for any fixed $\beta, \tilde{\beta} > 0$ satisfying $0 < \beta + 2\tilde{\beta} < 1, 0 < \beta < \frac{1}{3}$, we apply the convex integration method simultaneously to the Euler equations (1.2) and the transport equations (1.3). We construct a solution to the 3D Euler equations in the space $C^\beta([0,T]\times\mathbb{T}^d)$, which satisfies $e(t) = \|v(t)\|_{L^2}^2$ for a prescribed energy profile, such that the associated transport equation admits a non-constant, positive solution in $C^{\tilde{\beta}}([0,T]\times\mathbb{T}^d)$ and with initial data $\rho_0 = 1$. By selecting β sufficiently close to $1/3$ and choosing a decreasing energy profile $e(t)$, we establish our first main result, Theorem 1.2, with the help of the superposition principle.

Without loss of generality, we assume $T = 1$. When considering the $C_{t,x}^0$ scales, we will primarily focus on spaces equipped with the supremum norm. In this section and the subsequent Section 3, we use the notation $\|\cdot\|_B = \|\cdot\|_{C_t B}$ for any Banach space B.

The convex integration iteration is indexed by a parameter $q \in \mathbb{N}_0$. We define the frequency parameter $\{\lambda_q\}_{q\in\mathbb{N}_0} \subset \mathbb{N}$ which diverges to ∞, and the amplitude parameters $\{\delta_q, \tilde{\delta}_q\}_{q\in\mathbb{N}_0} \subset (0,1]$ which are decreasing to 0 by
$$\lambda_q = a^{(b^q)}, \ \ \delta_q = \lambda_q^{-2\beta}, \ \ \tilde{\delta}_q = \lambda_q^{-2\tilde{\beta}}, \ q \geqslant 0,$$

where $a > 1$ is a large parameter and $b > 1$ is close to 1. Here we recall that $\beta, \tilde{\beta} > 0$ are given in Theorem 1.3 satisfying $0 < \beta < 1/3$ and $0 < \beta + 2\tilde{\beta} < 1$. In the following, without loss of generality, we additionally assume $\beta \leqslant \tilde{\beta}$. In addition, we have

$$\sum_{q \geqslant 1} \tilde{\delta}_q^{1/2} \lesssim \frac{1}{a^{b\tilde{\beta}} - 1} < \frac{1}{3} \tag{2.1}$$

by choosing a large enough in terms of b and $\tilde{\beta}$.

At each step $q \in \mathbb{N}_0$, a pair $(v_q, \rho_q, \mathring{R}_q, M_q)$ is constructed solving the following systems on $\mathbb{T}^d$:

$$\partial_t \rho_q + \mathrm{div}(v_q \rho_q) = -\mathrm{div} M_q, \tag{2.2}$$

$$\partial_t v_q + \mathrm{div}(v_q \otimes v_q) + \nabla \pi_q = \mathrm{div} \mathring{R}_q, \ \ \mathrm{div} v_q = 0, \tag{2.3}$$

where $\mathring{R}_q$ is assumed as a trace-free symmetric matrix, and M_q is a vector field.

To take into account the initial condition, we require that $\rho_q = 1$ near the origin, more precisely, that $\rho_q = 1$ on $[0, T_q]$, where

$$T_q := \frac{1}{3} - \sum_{1 \leqslant r \leqslant q} \tilde{\delta}_r^{1/2}. \tag{2.4}$$

Applying (2.1) we obtain $0 < T_q \leqslant \frac{1}{3}$. Here we define $\sum_{1 \leqslant r \leqslant 0} := 0$.

Our main iteration on the approximate solution $(v_q, \rho_q, \mathring{R}_q, M_q)$ reads as follows:

Proposition 2.1. *Let $\beta \in (0, \frac{1}{3}), \tilde{\beta} \in [\beta, \frac{1-\beta}{2})$ and $1 < b < \frac{1-\tilde{\beta}}{\beta+\tilde{\beta}}$. Let $e : [0, 1] \to \mathbb{R}$ be a strictly positive function satisfying $|e'(t)| \leqslant 1$. Then there exists a choice of parameters $\alpha \in (0, 1)$ and $a > 1$ such that the following holds true: let $(v_q, \rho_q, \mathring{R}_q, M_q)$ be a solution to (2.2)-(2.3) satisfying $\int \rho_q \mathrm{d}x = 1$,*

$$\|v_q\|_{C^0} \leqslant M \sum_{i=0}^{q} \delta_i^{1/2}, \ \ \|\rho_q\|_{C^0} \leqslant 2 + \sum_{i=0}^{q} \tilde{\delta}_i^{1/2}, \tag{2.5}$$

$$\|v_q\|_{C^1} \leqslant M \delta_q^{1/2} \lambda_q, \ \ \|\rho_q\|_{C^1} \leqslant \tilde{\delta}_q^{1/2} \lambda_q, \tag{2.6}$$

$$\|\mathring{R}_q\|_{C^0} \leqslant \delta_{q+1} \lambda_q^{-3\alpha}, \ \ \|M_q\|_{C^0} + \frac{1}{\lambda_q} \|M_q\|_{C^1} \leqslant \delta_{q+1}^{1/2} \tilde{\delta}_{q+1}^{1/2} \lambda_q^{-3\alpha}, \tag{2.7}$$

$$\rho_q - 1 = M_q = 0 \text{ on } [0, T_q], \tag{2.8}$$

where $M > 1$ is a universal geometric constant. Moreover, for any $t \in [0, 1]$

$$\delta_{q+1} \lambda_q^{-\alpha/3} \leqslant e(t) - \|v_q(t)\|_{L^2}^2 \leqslant \delta_{q+1}. \tag{2.9}$$

Then there exists $(v_{q+1}, \rho_{q+1}, \mathring{R}_{q+1}, M_{q+1})$ which solves (2.2)-(2.3), satisfies (2.5)-(2.9) at the level $q + 1$ and

$$\|v_{q+1} - v_q\|_{C^0} \leqslant M \delta_{q+1}^{1/2}, \ \ \|\rho_{q+1} - \rho_q\|_{C^0} \leqslant \tilde{\delta}_{q+1}^{1/2}. \tag{2.10}$$

As noted in [BDLSV19], the wiggle room provided by the factor $\lambda_q^{-3\alpha}$ is beneficial during the gluing step. Additionally, the extra bound on M_q in the C^1 norm is utilized to establish the time regularity of ρ, see the proof of Theorem 1.3 below.

Proof of Theorem 1.3. We start with the proof of the first part of the theorem by in a similar way as [CET94]. Specifically, we prove that for $\beta + 2\tilde{\beta} > 1$ and vector field $v \in C^{\beta}_{t,x}$, any weak solution $\rho \in C^{\tilde{\beta}}_{t,x}$ to the transport equation (1.3) conserves energy. Then the uniqueness follows from the linearity.

Let $\rho_l := \rho * \varphi_l$ be the spatial mollification of ρ with a length scale l. Similarly, we define ρ_l and $(v\rho)_l$. Then ρ_l satisfies for any $t \in [0, 1]$

$$\int_{\mathbb{T}^3} |\rho_l(x,t)|^2 \mathrm{d}x - \int_{\mathbb{T}^3} |\rho_l(x,0)|^2 \mathrm{d}x = 2\int_0^t \langle (v\rho)_l, \nabla \rho_l \rangle_{L^2} \mathrm{d}s.$$

Since

$$\langle v_l \rho_l, \nabla \rho_l \rangle_{L^2} \equiv 0,$$

we have

$$\int_{\mathbb{T}^3} |\rho_l(x,t)|^2 \mathrm{d}x - \int_{\mathbb{T}^3} |\rho_l(x,0)|^2 \mathrm{d}x = 2\int_0^t \langle (v\rho)_l - v_l\rho_l, \nabla \rho_l \rangle_{L^2} \mathrm{d}s.$$

Using the commutator estimate Lemma A.6 we have

$$\big| \|\rho_l(t)\|_{L^2}^2 - \|\rho_l(0)\|_{L^2}^2 \big| \lesssim \|(v\rho)_l - v_l\rho_l\|_{C^0_{t,x}} \|\rho_l\|_{C^1_{t,x}} \lesssim l^{\beta+2\tilde\beta-1} \|v\|_{C^\beta_{t,x}} \|\rho\|^2_{C^{\tilde\beta}_{t,x}}.$$

Thus, as $\beta + 2\tilde\beta > 1$, the right hand side converges to zero as $l \to 0$. We conclude the proof with the observation that ρ_l converges to ρ in $C_t L^2$.

For the second part, since $0 < \beta < \frac{1}{3}$, it suffices to prove the statement in the case where $\beta \leqslant \tilde\beta < 1 - 2\beta$ and $T = 1$. Without loss of generality, by the same argument as in [BDLSV19, Proof of Theorem 1.1], we may assume that

$$\inf_{t\in[0,1]} e(t) \geqslant \delta_1 \lambda_0^{-\alpha/3}, \quad \sup_{t\in[0,1]} e(t) \leqslant \delta_1, \quad \sup_{t\in[0,1]} e'(t) \leqslant 1.$$

By choosing $\alpha > 0$ small enough, there exists $\gamma \in \mathbb{Q}$ such that $3\alpha + (\tilde\beta + \beta)b < \gamma < 1 - \tilde\beta$. Then we define $\lambda = \lambda_0^\gamma \in 2\mathbb{N}$ by choosing a suitable a large enough, and start the iteration from

$$\rho_0(t,x) = 1 + \frac{\sin \lambda\pi x_1}{2} \chi_0(t),\ v_0 = 0,\ \mathring{R}_0 = 0,\ M_0(t,x) = \partial_t \chi_0(t) \frac{\cos \lambda \pi x_1}{2\lambda\pi} (1,0,0)^T,$$

where $x = (x_1, x_2, x_3)$ and χ_0 is a smooth function with $\chi_0(t) = 0$ on $[0, \frac{1}{3}]$, $\chi_0(t) = 1$ on $[\frac{2}{3}, 1]$.

Then $\int \rho_0 \mathrm{d}x = 1$, and by choosing a large enough to absorb the constant, we have

$$\|\rho_0\|_{C^0} \leqslant 2,\ \ \|\rho_0\|_{C^1} \lesssim \lambda \leqslant \tilde\delta_0^{1/2} \lambda_0,\ \ \|M_0\|_{C^0} + \frac{1}{\lambda} \|M_0\|_{C^1} \lesssim \frac{1}{\lambda} \leqslant \delta_1^{1/2} \tilde\delta_1^{1/2} \lambda_0^{-3\alpha},$$

which implies that (2.5)-(2.8) hold at the level $q = 0$. (2.9) is automatically satisfied by the assumption on $e(t)$ and the fact that $v_0 = 0$.

Applying Proposition 2.1 iteratively, we obtain a sequence of fields $(v_q, \rho_q, \mathring{R}_q, M_q)$ satisfying (2.5)-(2.9) and converging to a weak solution (v, ρ) to (1.2), (1.3) by (2.7). Using the estimate (2.10) yields for any $\beta' < \beta$:

$$\sum_{q=0}^\infty \|v_{q+1} - v_q\|_{C^{\beta'}} \lesssim \sum_{q=0}^\infty \|v_{q+1} - v_q\|_{C^0}^{1-\beta'} \|v_{q+1} - v_q\|_{C^1}^{\beta'} \lesssim \sum_{q=0}^\infty \delta_{q+1}^{\frac{1-\beta'}{2}} \left(\delta_{q+1}^{1/2} \lambda_{q+1}\right)^{\beta'} \lesssim \sum_{q=0}^\infty \lambda_{q+1}^{\beta'-\beta}.$$

Hence we obtain that $v \in C^0_t C^{\beta'}_x$. By the same argument and (2.10) we obtain that $\rho \in C^0_t C^{\beta''}_x$ for any $\beta'' < \tilde\beta$. The time regularity of v is obtained by the same argument as in [Ise18] or [BDLSV19]. We thus obtain $v \in C_x C^{\beta'}_t$ and then $v \in C^{\beta'}([0,1] \times \mathbb{T}^3)$ for arbitrary $\beta' < \beta$.

For the time regularity of ρ, using (2.6), (2.7) above, we immediately have for $q \in \mathbb{N}_0$

$$\|\partial_t \rho_q\|_{C^0} \lesssim \|v_q\|_{C^0} \|\nabla \rho_q\|_{C^0} + \|\mathrm{div} M_q\|_{C^0} \lesssim \tilde\delta_q^{1/2} \lambda_q + \lambda_q \delta_{q+1}^{1/2} \tilde\delta_{q+1}^{1/2} \lesssim \tilde\delta_q^{1/2} \lambda_q,$$

which together with interpolation and (2.10) implies that

$$\sum_{q=0}^{\infty} \|\rho_{q+1} - \rho_q\|_{C^0_x C^{\beta''}_t} \lesssim \sum_{q=0}^{\infty} \|\rho_q - \rho_{q+1}\|_{C^0}^{1-\beta''} \|\partial_t \rho_q - \partial_t \rho_{q+1}\|_{C^0}^{\beta''}$$
$$\lesssim \sum_{q=0}^{\infty} \tilde{\delta}_{q+1}^{\frac{1-\beta''}{2}} \left(\tilde{\delta}_{q+1}^{1/2} \lambda_{q+1}\right)^{\beta''} \lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{\beta''-\tilde{\beta}}.$$

As a consequence, $\rho_q \to \rho \in C^0_x C^{\beta''}_t$. Thus we obtain $\rho \in C^{\beta''}([0,1] \times \mathbb{T}^3)$ for any $\beta'' < \tilde{\beta}$.

Moreover, (2.8) ensures that $\rho(t) \equiv 1$ for every t sufficiently close to 0, and (2.9) ensures that $e(t) = \|v(t)\|_{L^2}^2$ for any $t \in [0,1]$.

Clearly by (2.1) and (2.10) we have

$$\|\rho - \rho_0\|_{C^0} \leqslant \sum_{q=0}^{\infty} \|\rho_{q+1} - \rho_q\|_{C^0} \leqslant \sum_{q=0}^{\infty} \tilde{\delta}_{q+1}^{1/2} \leqslant \frac{1}{3},$$

which implies that ρ is nonnegative on $\mathbb{T}^3$:

$$\inf_{t\in[0,1]} \rho \geqslant \inf_{t\in[0,1]} \rho_0 - \|\rho - \rho_0\|_{C^0} \geqslant \frac{1}{2} - \frac{1}{3} > 0,$$

and ρ does not coincide with the solution which is constantly 1, as

$$\|\rho - 1\|_{C^0} \geqslant \|1 - \rho_0\|_{C^0} - \|\rho - \rho_0\|_{C^0} \geqslant \frac{1}{2} - \frac{1}{3} > 0.$$

□

Remark 2.2. If one applies the same argument as in [BDLSV19, (2.16)] to derive the time regularity for ρ, one can only achieve the time regularity up to $\tilde{\beta}/(1+\tilde{\beta}-\beta) < \tilde{\beta}$, due to the low regularity of v.

In Theorem 1.3 we choose $\beta < \frac{1}{3}$ close enough to $\frac{1}{3}$ and choose $e(t)$ to be decreasing. Then the proof of Theorem 1.2 can be established using an argument analogous to [BCDL21, Theorem 1.3] (see also the proof of Theorem 1.7 detailed in Section 4).

3. Proof of Proposition 2.1

The proof follows a series of main steps. First, we fix some necessary parameters and proceed with a mollification step in Section 3.1. Next, in Section 3.2, we apply the gluing procedure for both (v_q, ρ_q). The gluing step for the Euler equation is similar to the approach in [BDLSV19], but with a distinct gluing parameter τ_q. Additionally, we introduce the gluing step for the transport equation to ensure that the associated error term is supported on disjoint temporal intervals. In Section 3.3, we define the new iteration (v_{q+1}, ρ_{q+1}) and provide inductive estimates. Specifically, we first construct the perturbation (w_{q+1}, θ_{q+1}) for the transport equation to cancel the glued stress term $\overline{M}_q$. Subsequently, we construct $\overline{w}_{q+1}$ for the Euler equation to cancel the glued stress term $\overline{\mathring{R}}_q$ as well as the low frequency part term of $w_{q+1} \otimes w_{q+1}$. In Section 3.4, we define the new stress terms $(\mathring{R}_{q+1}, M_{q+1})$ and establish the required estimates. Finally, in Section 3.5, we derive the energy estimates, completing the proof.

3.1. **Choice of parameters and mollification.** In the sequel, additional parameters will be indispensable and their value has to be carefully chosen in order to respect all the compatibility conditions appearing in the estimates below. First, for any fixed $\beta \in (0, \frac{1}{3})$, $\tilde{\beta} \in [\beta, \frac{1-\beta}{2})$, $\tilde{\beta}(b+1) + \beta b < 1$, and $\alpha > 0$ small enough we have

$$\frac{\lambda_q \tilde{\delta}_q^{1/2} \delta_{q+1}}{\tilde{\delta}_{q+1}^{1/2} \lambda_{q+1}} \leqslant \frac{\delta_{q+2}}{\lambda_{q+1}^{8\alpha}}, \quad \frac{\lambda_q \tilde{\delta}_q^{1/2} \delta_{q+1}^{1/2}}{\lambda_{q+1}} \leqslant \frac{\delta_{q+2}^{1/2} \tilde{\delta}_{q+2}^{1/2}}{\lambda_{q+1}^{8\alpha}}. \tag{3.1}$$

To see this, we take logarithms and obtain

$$1 - b - \tilde{\beta} - 2\beta b + \tilde{\beta} b < -2\beta b^2 - 8\alpha b, \quad 1 - b - \tilde{\beta} - \beta b < -\beta b^2 - \tilde{\beta} b^2 - 8\alpha b,$$

i.e.

$$8\alpha b + (b-1)(2\beta b + \tilde{\beta} - 1) < 0, \quad 8\alpha b + (b-1)(\tilde{\beta}(b+1) + \beta b - 1) < 0,$$

which are satisfied by the choice of b and by choosing $\alpha > 0$ small enough.

Finally, we increase a such that (2.1) holds. In the sequel, we increase a to absorb the various implicit and universal constants in the following estimates.

We take a sufficiently small $\alpha \in (0, 1)$ such that (3.1) holds, and define $l > 0$ as

$$l = \frac{\tilde{\delta}_{q+1}^{1/2}}{\tilde{\delta}_q^{1/2} \lambda_q^{1+\frac{3\alpha}{2}}} \leqslant \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2} \lambda_q^{1+\frac{3\alpha}{2}}}, \tag{3.2}$$

where we used $\beta \leqslant \tilde{\beta}$ to deduce $\tilde{\delta}_{q+1}^{1/2} \tilde{\delta}_q^{-1/2} \leqslant \delta_{q+1}^{1/2} \delta_q^{-1/2}$.

Then we replace the pair $(v_q, \rho_q, \mathring{R}_q, M_q)$ with a mollified pair $(v_l, \rho_l, \mathring{R}_l, M_l)$. We define

$$\begin{aligned} v_l :=& v_q *_x \phi_l, \quad \mathring{R}_l := \mathring{R}_q *_x \phi_l - (v_q \mathring{\otimes} v_q) *_x \phi_l + v_l \mathring{\otimes} v_l, \\ \rho_l :=& \rho_q *_x \phi_l, \quad M_l := M_q *_x \phi_l + (v_q \rho_q) *_x \phi_l - v_l \rho_l, \end{aligned}$$

which obey (2.2) and (2.3) for a suitable π_l. Here $\phi_l := \frac{1}{l^3} \phi(\frac{\cdot}{l})$ is a family of standard radial mollifiers on $\mathbb{R}^3$. Since the mollification does not depend on time, we still have $\rho_l = 1$ on $[0, T_q]$. Since $\rho_l = \rho_q = 1$ on the interval $[0, T_q]$, we have $M_l = v_q *_x \phi_l - v_l = 0$ on the interval $[0, T_q]$.

In order to bound $\mathring{R}_l$ and M_l, we use the basic mollification estimate, Lemma A.6 and the fact that $\lambda_q^{-3/2} \leqslant l \leqslant \lambda_q^{-1}$ (by choosing $\tilde{\beta}(b-1) + \frac{3}{2}\alpha < \frac{1}{2}$) to obtain for $N \in \mathbb{N}_0$

$$\|v_l - v_q\|_{C^0} \lesssim l \|v_q\|_{C^1} \lesssim l \lambda_q \delta_q^{1/2} \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\alpha}, \quad \|v_l\|_{C^{N+1}} \lesssim l^{-N} \|v_q\|_{C^1} \lesssim \delta_q^{1/2} \lambda_q l^{-N}, \tag{3.3}$$

$$\|\rho_l - \rho_q\|_{C^0} \lesssim l \|\rho_q\|_{C^1} \lesssim l \lambda_q \tilde{\delta}_q^{1/2} \lesssim \tilde{\delta}_{q+1}^{1/2} \lambda_q^{-\alpha}, \quad \|\rho_l\|_{C^{N+1}} \lesssim l^{-N} \|\rho_q\|_{C^1} \lesssim \tilde{\delta}_q^{1/2} \lambda_q l^{-N}, \tag{3.4}$$

$$\|\mathring{R}_l\|_{C^{N+\alpha}} \lesssim l^{-N-\alpha} \|\mathring{R}_q\|_{C^0} + l^{2-N-\alpha} \|v_q\|_{C^1}^2 \lesssim l^{-N-\alpha} \lambda_q^{-3\alpha} \delta_{q+1} \lesssim \delta_{q+1} l^{-N+\alpha}, \tag{3.5}$$

$$\|M_l\|_{C^{N+\alpha}} \lesssim l^{-N-\alpha} \|M_q\|_{C^0} + l^{2-N-\alpha} \|v_q\|_{C^1} \|\rho_q\|_{C^1} \lesssim l^{-N-\alpha} \lambda_q^{-3\alpha} \delta_{q+1}^{1/2} \tilde{\delta}_{q+1}^{1/2} \lesssim \delta_{q+1}^{1/2} \tilde{\delta}_{q+1}^{1/2} l^{-N+\alpha}. \tag{3.6}$$

Similarly, we use the same computation as for (3.3) to obtain

$$\left| \int_{\mathbb{T}^3} |v_q|^2 - |v_l|^2 \mathrm{d}x \right| = \left| \int_{\mathbb{T}^3} |v_q|^2 * \phi_l - |v_l|^2 \mathrm{d}x \right| \lesssim \left\| |v_q|^2 * \phi_l - |v_l|^2 \right\|_{C^0} \lesssim l^2 \|v_q\|_{C^1}^2 \lesssim \delta_{q+1} l^{\alpha}. \tag{3.7}$$

3.2. **The gluing procedure.** To achieve the regime $\beta+2\tilde{\beta}<1$ and the Onsager regime up to $1/3$, we employ the gluing technique introduced by Isett in [Ise18]. In Section 3.2.1, we glue together the exact solutions to the Euler equations similarly to [BDLSV19]. This ensures that the associated error term $\overline{\mathring{R}}_q$ is supported on disjoint temporal intervals. Subsequently, in Section 3.2.2, we glue together the exact solutions of the transport equations, ensuring that the corresponding error term $\overline{M}_q$ is also localized on pairwise disjoint temporal intervals. To this end, we define the length of the time intervals as

$$\tau_q := \frac{l^{2\alpha}\tilde{\delta}_{q+1}^{1/2}}{\tilde{\delta}_q^{1/2}\lambda_q\delta_{q+1}^{1/2}}.$$

Then we define

$$t_i = i\tau_q, \qquad I_i = [t_i + \frac{\tau_q}{3}, t_i + \frac{2\tau_q}{3}] \cap [0,1], \qquad J_i = (t_i - \frac{\tau_q}{3}, t_i + \frac{\tau_q}{3}) \cap [0,1]. \tag{3.8}$$

3.2.1. *Gluing step for* $(v_l, \mathring{R}_l)$. The gluing step for the Euler equations in (1.2) is analogous to that in [BDLSV19], with the exception of the choice of the parameters τ_q and l. In this section, we will only utilize the facts that

$$\tau_q \leqslant \frac{l^{2\alpha}}{\delta_q^{1/2}\lambda_q}, \quad \tau_q\delta_{q+1}^{1/2}l^{-1-\alpha/2} = (l\lambda_q)^{\frac{3\alpha}{2}} \leqslant 1, \tag{3.9}$$

which hold by using again that $\tilde{\delta}_{q+1}^{1/2}\tilde{\delta}_q^{-1/2} \leqslant \delta_{q+1}^{1/2}\delta_q^{-1/2}$ because $\beta \leqslant \tilde{\beta}$. Here we note that the right-hand side of the first term corresponds to the definition of τ_q in [BDLSV19]. We will repeat the procedures as in [BDLSV19] and provide some details of the proof in Appendix B.2.

First, we recall that by (3.3) and (3.9), τ_q obeys the CFL-like condition:

$$\tau_q \|v_l\|_{C^{1+\alpha}} \lesssim \tau_q\delta_q^{1/2}\lambda_q l^{-\alpha} \lesssim l^\alpha \ll 1. \tag{3.10}$$

Therefore, for each i, we can uniquely solve the Euler equations locally on time interval $[t_i - \tau_q, t_i + \tau_q]$ with initial datum $v_l(\cdot, t_i) \in C^{1+\alpha}$:

$$\begin{aligned} \partial_t v_i + v_i \cdot \nabla v_i + \nabla \pi_i &= 0, \\ \mathrm{div} v_i &= 0, \\ v_i(\cdot, t_i) &= v_l(\cdot, t_i). \end{aligned} \tag{3.11}$$

Then we aim to establish the estimates on v_i, the differences between v_i and v_l in Hölder space and in the some negative order spaces. For this purpose we apply the Biot-Savart operator to define

$$z_i = (-\Delta)^{-1}\mathrm{curl}\, v_i, \quad z_l = (-\Delta)^{-1}\mathrm{curl}\, v_l.$$

It follows that $\mathrm{div} z_i = \mathrm{div} z_l = 0$ and $\mathrm{curl}\, z_i = v_i, \mathrm{curl}\, z_l = v_l$.

Then by the same argument as in [BDLSV19, Corollary 3.2. Proposition 3.3, Proposition 3.4] we have

Proposition 3.1. *For all* $t \in [t_i - \tau_q, t_i + \tau_q]$ *and all* $N \in \mathbb{N}_0$

$$\|v_i(t)\|_{C^{N+1+\alpha}} \lesssim \tau_q^{-1} l^{-N+\alpha}, \tag{3.12}$$

$$\|(v_i - v_l)(t)\|_{C^{N+\alpha}} + \tau_q \|(\partial_t + v_l \cdot \nabla)(v_i - v_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1} l^{-N-1+\alpha}, \tag{3.13}$$

$$\|(z_i - z_l)(t)\|_{C^{N+\alpha}} + \tau_q \|(\partial_t + v_l \cdot \nabla)(z_i - z_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1} l^{-N+\alpha}. \tag{3.14}$$

Here we remark that although the choice of parameters are different from [BDLSV19], all the estimates can be established analogously by using (3.9). We put the proof in Appendix B.2 for completeness.

Then like the usual gluing step, we introduce the time-dependent cut-off functions $\{\chi_i\}_i$, which is a partition of unity in time for $[0,1]$ with the property that $\operatorname{supp}\chi_i\cap\operatorname{supp}\chi_{i+2}=\emptyset$ and moreover for $N\in\mathbb{N}_0$

$$\operatorname{supp}\chi_i\subset[t_i-\frac{2\tau_q}{3},t_i+\frac{2\tau_q}{3}],\quad \chi_i=1 \text{ on } (t_i-\frac{\tau_q}{3},t_i+\frac{\tau_q}{3}),\quad \left\|\partial_t^N\chi_i\right\|_{C^0}\lesssim\tau_q^{-N}.$$

The glued stress term can be defined following the same argument as in [BDLSV19, Section 4.2]. We glue the constructed exact solutions v_i together and define

$$\overline{v}_q(x,t):=\sum_i\chi_i(t)v_i(x,t)\,. \tag{3.15}$$

We note that the glued vector field $\overline{v}_q$ is also divergence-free, as the cutoffs χ_i only depend on time. Furthermore, for $t\in J_i$, we have $\mathring{\overline{R}}_q=0$, and for $t\in I_i$ we define

$$\mathring{\overline{R}}_q=\partial_t\chi_i\mathcal{R}(v_i-v_{i+1})-\chi_i(1-\chi_i)(v_i-v_{i+1})\mathring{\otimes}(v_i-v_{i+1})\,, \tag{3.16}$$

where we used the inverse divergence operator $\mathcal{R}$ introduced in Section A.1. By construction, $\mathring{\overline{R}}_q$ is traceless and symmetric, and we know that $(\overline{v}_q,\mathring{\overline{R}}_q)$ solves (2.3) for a suitable $\overline{\pi}_q$.

To finish this section, it remains to estimate the glued velocity field and Reynolds stress defined in (3.15) and (3.16). We define $\overline{z}_q:=(-\Delta)^{-1}\operatorname{curl}\overline{v}_q$. Then the desired estimates are obtained by the same argument as in [BDLSV19, Proposition 4.2, Proposition 4.3, Proposition 4.4]. We put the details in Appendix B.2.

Proposition 3.2. *For* $N\in\mathbb{N}_0$,

$$\left\|\overline{v}_q-v_l\right\|_{C^{N+\alpha}}+l^{-1}\left\|\overline{z}_q-z_l\right\|_{C^{N+\alpha}}\lesssim\tau_q\delta_{q+1}l^{-1-N+\alpha}\lesssim\delta_{q+1}^{1/2}l^{-N+\alpha}, \tag{3.17}$$

$$\left|\int_{\mathbb{T}^3}|\overline{v}_q|^2-|v_l|^2\mathrm{d}x\right|\lesssim\delta_{q+1}l^{\alpha}\,, \tag{3.18}$$

$$\left\|\mathring{\overline{R}}_q\right\|_{C^{N+\alpha}}+\tau_q\left\|(\partial_t+\overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\right\|_{C^{N+\alpha}}\lesssim\delta_{q+1}l^{-N+\alpha}. \tag{3.19}$$

In particular, from the bounds (3.3), (3.17) and the choice of parameters in (3.9) we obtain for all $N\in\mathbb{N}_0$

$$\|\overline{v}_q\|_{C^{N+1}}\lesssim\|\overline{v}_q-v_l\|_{C^{N+1}}+\|v_l\|_{C^{N+1}}\lesssim\tau_q\delta_{q+1}l^{-2-N+\alpha}+\delta_q^{1/2}\lambda_q l^{-N}\lesssim\tau_q^{-1}l^{2\alpha-N}. \tag{3.20}$$

Here the estimate is slightly different from [BDLSV19, (4.7)] due to the the different choice of parameters. However, the new bound is still suitable for our proof.

3.2.2. *Gluing step for* (ρ_l,M_l). Now we glue the exact solutions ρ_i of the transport equations at the same times t_i defined above, to make sure the glued stress $\overline{M}_q$ is located in disjoint time intervals I_i.

For each i, we solve the following transport equations on $[t_i-\tau_q,t_i+\tau_q]$:

$$\partial_t\rho_i+\overline{v}_q\cdot\nabla\rho_i=0,$$
$$\rho_i(\cdot,t_i)=\rho_l(\cdot,t_i).$$

By (3.20) we have

$$\tau_q\left\|\overline{v}_q\right\|_{C^{1+\alpha}}\lesssim l^{\alpha}\ll 1.$$

Then using the estimates for the transport equations in Proposition A.7, the bounds in (3.4), (3.20) and interpolation we have that for all $t\in[t_i-\tau_q,t_i+\tau_q]$ and $N\geqslant 1$

$$\|\rho_i(t)\|_{C^{N+\alpha}}\lesssim\|\rho_l(t_i)\|_{C^{N+\alpha}}+\tau_q\|\overline{v}_q\|_{C^{N+\alpha}}\|\rho_l\|_{C^1}$$

$$\lesssim \tilde{\delta}_q^{1/2}\lambda_q l^{1-N-\alpha} + \tau_q\tau_q^{-1}\tilde{\delta}_q^{1/2}\lambda_q l^{1-N} \lesssim \tilde{\delta}_q^{1/2}\lambda_q l^{1-N-\alpha}. \tag{3.21}$$

Moreover, we have the following result.

Proposition 3.3. *For $t \in [t_i - \tau_q, t_i + \tau_q]$ and all $N \geqslant 0$,*

$$\|(\rho_i - \rho_l)(t)\|_{C^{N+\alpha}} + \tau_q \|(\partial_t + v_l \cdot \nabla)(\rho_i - \rho_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N-1+\alpha}. \tag{3.22}$$

Proof. It is easy to see that $\rho_l - \rho_i$ obeys

$$\partial_t(\rho_l - \rho_i) + v_l \cdot \nabla(\rho_l - \rho_i) = -(v_l - \overline{v}_q) \cdot \nabla\rho_i - \mathrm{div} M_l \tag{3.23}$$

with initial condition 0 at $t = t_i$. Then by the estimates (3.6), (3.17), (3.21) and the definition of τ_q we obtain for all $N \geqslant 0$ and $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}\|(\partial_t + v_l \cdot \nabla)(\rho_i - \rho_l)(t)\|_{C^{N+\alpha}} &\lesssim \|v_l - \overline{v}_q\|_{C^{N+\alpha}}\|\nabla\rho_i\|_{C^\alpha} + \|v_l - \overline{v}_q\|_{C^\alpha}\|\nabla\rho_i\|_{C^{N+\alpha}} + \|M_l\|_{C^{N+1+\alpha}} \\ &\lesssim \tilde{\delta}_q^{1/2}\lambda_q\tau_q\delta_{q+1} l^{-N-1} + \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N-1+\alpha} \lesssim \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N-1+\alpha}.\end{aligned}$$

Then by standard estimates for the transport equations in Proposition A.7, it follows immediately that for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\|(\rho_i - \rho_l)(t)\|_{C^\alpha} \lesssim \left|\int_{t_0}^t \|(\partial_t + v_l \cdot \nabla)(\rho_i - \rho_l)(s)\|_{C^\alpha}\,\mathrm{d}s\right| \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-1+\alpha},$$

and for $N \geqslant 1$

$$\begin{aligned}\|(\rho_i - \rho_l)(t)\|_{C^{N+\alpha}} &\lesssim \left|\int_{t_0}^t \|(\partial_t + v_l \cdot \nabla)(\rho_i - \rho_l)\|_{C^{N+\alpha}} + \tau_q\|v_l\|_{C^{N+\alpha}}\|(\partial_t + v_l \cdot \nabla)(\rho_i - \rho_l)\|_{C^1}\,\mathrm{d}s\right| \\ &\lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N-1+\alpha} + \tau_q^2\lambda_q\delta_q^{1/2} l^{1-N-\alpha}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-2+\alpha} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N-1+\alpha}.\end{aligned}$$

□

We also introduce vector potentials using the inverse divergence operator $\mathcal{R}_1 = \nabla\Delta^{-1}$. More precisely, as $\int_{\mathbb{T}^3}\rho_i \mathrm{d}x = \int_{\mathbb{T}^3}\rho_l \mathrm{d}x = 1$, defining

$$y_i := \nabla\Delta^{-1}(\rho_i - 1), \;\; y_l := \nabla\Delta^{-1}(\rho_l - 1),$$

we have

$$\rho_i = \mathrm{div} y_i + 1, \;\; \rho_l = \mathrm{div} y_l + 1, \;\; \mathrm{curl}\, y_i = \mathrm{curl}\, y_l = 0.$$

With this notation, now we need to estimate $y_l - y_i$ in some Hölder spaces. To achieve this, first we establish the following analytic identities, which can be used to derive the equation for $y_l - y_i$ by applying $\mathcal{R}_1$ on both sides of (3.23). This also allows us to utilize the basic estimates for transport equations in Appendix A.7 in this context.

Lemma 3.4. *For any smooth vector fields $z, v : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $\mathrm{div} v = 0$, and smooth function $\rho : \mathbb{R}^3 \to \mathbb{R}$, we have*

$$\begin{aligned}(\mathrm{curl}\, z) \cdot \nabla\rho &= \mathrm{div}(z \times \nabla\rho), \\ v \cdot \nabla(\mathrm{div} z) &= \mathrm{div}(v \cdot \nabla z - z \cdot \nabla v), \\ \mathrm{curl}\,(v \cdot \nabla z) &= -\mathrm{div}((z \times \nabla)v) + v \cdot \nabla(\mathrm{curl}\, z).\end{aligned}$$

The proof of this Lemma is provided in Appendix B.2. Then we have the following result.

Proposition 3.5. *For $t \in [t_i - \tau_q, t_i + \tau_q]$, and all $N \geqslant 0$*

$$\|(y_i - y_l)(t)\|_{C^{N+\alpha}} + \tau_q \|(\partial_t + v_l \cdot \nabla)(y_i - y_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{-N+\alpha}. \tag{3.24}$$

Proof. By lemma 3.4, we use the fact that $\mathrm{div} v_l = 0$ to deduce

$$(v_l - \overline{v}_q)\cdot\nabla\rho_i = \mathrm{div}\Big((z_l - \overline{z}_q)\times\nabla\rho_i\Big),$$
$$v_l\cdot\nabla(\rho_l-\rho_i) = \mathrm{div}\Big(v_l\cdot\nabla(y_l-y_i) - (y_l-y_i)\cdot\nabla v_l\Big).$$

Consequently, from (3.23) one deduces that

$$\mathrm{div}\Big(\partial_t(y_l-y_i)+v_l\cdot\nabla(y_l-y_i)\Big) = \mathrm{div}\Big((y_l-y_i)\cdot\nabla v_l - (z_l-\overline{z}_q)\times\nabla\rho_i - M_l\Big).$$

Taking gradient on both side of the above equation, and using the identity $\nabla\mathrm{div} = \Delta + \mathrm{curl}\,\mathrm{curl}$ we obtain

$$\begin{aligned}\Delta\Big(\partial_t(y_l-y_i)+v_l\cdot\nabla(y_l-y_i)\Big) &= \nabla\mathrm{div}\Big((y_l-y_i)\cdot\nabla v_l - (z_l-\overline{z}_q)\times\nabla\rho_i - M_l\Big)\\ &\quad - \mathrm{curl}\,\mathrm{curl}\,\Big(\partial_t(y_l-y_i)+v_l\cdot\nabla(y_l-y_i)\Big).\end{aligned}$$

Since $\mathrm{curl}\, y_l = \mathrm{curl}\, y_i = 0, \mathrm{div} v_l = 0$, by Lemma 3.4 we have

$$\mathrm{curl}\,\Big(v_l\cdot\nabla(y_l-y_i)\Big) = -\mathrm{div}\Big(((y_l-y_i)\times\nabla)v_l\Big),$$

which implies that

$$\begin{aligned}\partial_t(y_l-y_i)+v_l\cdot\nabla(y_l-y_i) &= \Delta^{-1}\nabla\mathrm{div}\Big((y_l-y_i)\cdot\nabla v_l - (z_l-\overline{z}_q)\times\nabla\rho_i - M_l\Big)\\ &\quad + \Delta^{-1}\mathrm{curl}\,\mathrm{div}\Big(((y_l-y_i)\times\nabla)v_l\Big).\end{aligned}$$

As a result, for $N\in\mathbb{N}_0$

$$\begin{aligned}\|(\partial_t+v_l\cdot\nabla)(y_i-y_l)\|_{C^{N+\alpha}} &\lesssim \|y_l-y_i\|_{C^\alpha}\|\nabla v_l\|_{C^{N+\alpha}} + \|y_l-y_i\|_{C^{N+\alpha}}\|\nabla v_l\|_{C^\alpha}\\ &\quad + \|z_l-\overline{z}_q\|_{C^{N+\alpha}}\|\nabla\rho_i\|_{C^\alpha} + \|z_l-\overline{z}_q\|_{C^\alpha}\|\nabla\rho_i\|_{C^{N+\alpha}} + \|M_l\|_{C^{N+\alpha}}.\end{aligned}$$

Then together with the estimates of $v_l, M_l, z_l-\overline{z}_q$ and ρ_i in (3.3), (3.6), (3.17), and (3.21) respectively we deduce that

$$\begin{aligned}\|(\partial_t+v_l\cdot\nabla)(y_i-y_l)\|_{C^\alpha} &\lesssim \delta_q^{1/2}\lambda_q l^{-\alpha}\|y_i-y_l\|_{C^\alpha} + \tilde{\delta}_q^{1/2}\lambda_q\tau_q\delta_{q+1} + \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^\alpha\\ &\lesssim \delta_q^{1/2}\lambda_q l^{-\alpha}\|y_i-y_l\|_{C^\alpha} + \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^\alpha.\end{aligned}$$

Using (3.10) and Proposition A.7 we obtain for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\|(y_i-y_l)(t)\|_{C^\alpha} \lesssim |t-t_i|\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^\alpha + \left|\int_{t_i}^t \delta_q^{1/2}\lambda_q l^{-\alpha}\|(y_i-y_l)(s)\|_{C^\alpha}ds\right|,$$

which by Gronwall's inequality implies that for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\|(y_i-y_l)(t)\|_{C^\alpha} + \tau_q\,\|(\partial_t+v_l\cdot\nabla)(y_i-y_l)(t)\|_{C^\alpha} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^\alpha.$$

Finally, commuting the derivatives in $N+\alpha, N\geqslant 0$ with $\partial_t+v_l\cdot\nabla$ as in the proof of [BDLSV19, Proposition 3.4] we obtain for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\|(y_i-y_l)(t)\|_{C^{N+\alpha}} + \tau_q\,\|(\partial_t+v_l\cdot\nabla)(y_i-y_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}.$$

□

As before, we glue the constructed ρ_i together using the same time-dependent cut-off functions χ_i:

$$\overline{\rho}_q(x,t) := \sum_i \chi_i(t)\rho_i(x,t), \tag{3.25}$$

which inherits the identity $\int_{\mathbb{T}^3} \overline{\rho}_q \mathrm{d}x = 1$ from ρ_i. Moreover, we denote

$$i_q := \max\{i : t_i < T_q\}, \tag{3.26}$$

where we recall that $t_i = i\tau_q$ and T_q is defined in (2.4). For any $i \leqslant i_q, \rho_l(t_i) = 1$, which implies that $\rho_i(t) = 1$ on the interval $[t_i - \tau_q, t_i + \tau_q]$ by solving the transport equation and using the fact that $\mathrm{div}\overline{v}_q = 0$. By the definition of the gluing solution, we know that $\overline{\rho}_q(t) = 1$ on $[0, T_q - \tau_q] \subset [0, t_{i_q}]$.

By the definition of the cutoff functions, on every J_i interval we have $\overline{\rho}_q = \rho_i$, so $(\overline{v}_q, \overline{\rho}_q)$ is an exact solution of the transport equations (2.2). On the other hand, on every interval I_i we have

$$\overline{\rho}_q = \chi_i\rho_i + (1-\chi_i)\rho_{i+1},$$

which leads to

$$\partial_t\overline{\rho}_q + \mathrm{div}(\overline{v}_q\overline{\rho}_q) = \partial_t\chi_i(\rho_i - \rho_{i+1}).$$

So for all $t \in I_i$ we define

$$\overline{M}_q = -\partial_t\chi_i(y_i - y_{i+1}), \tag{3.27}$$

which together with (3.16) implies that $(\overline{v}_q, \overline{\rho}_q, \mathring{\overline{R}}_q, \overline{M}_q)$ solves (2.2)-(2.3) on $\mathbb{T}^3 \times [0,1]$. Since $\rho_i = 1$ on $[t_i - \tau_q, t_i + \tau_q]$ for any $i \leqslant i_q$, we obtain $y_i = 0$ on these intervals. Then we have $\mathrm{supp}(\overline{M}_q) \subset \mathbb{T}^3 \times \cup_{i \geqslant i_q} I_i$.

Now we have the following estimates on the glued solution and stress term.

Proposition 3.6. *For $N \in \mathbb{N}_0$, we have*

$$\|\overline{\rho}_q - \rho_l\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-1-N+\alpha} \lesssim \tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}, \tag{3.28}$$

$$\|\overline{\rho}_q\|_{C^{N+1}} \lesssim \tilde{\delta}_q^{1/2}\lambda_q l^{-N}, \tag{3.29}$$

$$\|\overline{M}_q\|_{C^{N+\alpha}} + \tau_q\left\|(\partial_t + \overline{v}_q\cdot\nabla)\overline{M}_q\right\|_{C^{N+\alpha}} \lesssim \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}. \tag{3.30}$$

Proof. By (3.4), (3.22) and the fact that $\tau_q\delta_{q+1}^{1/2}l^{-1} \leqslant 1$ in (3.9), we obtain for all $N \geqslant 0$

$$\|\overline{\rho}_q - \rho_l\|_{C^{N+\alpha}} \lesssim \sum_i \chi_i\|\rho_i - \rho_l\|_{C^{N+\alpha}} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-1-N+\alpha} \lesssim \tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha},$$

$$\begin{aligned}\|\overline{\rho}_q\|_{C^{N+1}} &\lesssim \|\overline{\rho}_q - \rho_l\|_{C^{N+1}} + \|\rho_l\|_{C^{N+1}} \lesssim \tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-2-N+\alpha} + \tilde{\delta}_q^{1/2}\lambda_q l^{-N}\\ &\lesssim ((l\lambda_q)^{3\alpha} + 1)\tilde{\delta}_q^{1/2}\lambda_q l^{-N} \lesssim \tilde{\delta}_q^{1/2}\lambda_q l^{-N}.\end{aligned}$$

On the other hand, for the glued stress $\overline{M}_q$, by (3.27), the property of the cut-off functions and the estimate in (3.24) we have for $t \in I_i$,

$$\|\overline{M}_q(t)\|_{C^{N+\alpha}} \lesssim \|\partial_t\chi_i\|_{C_t^0}\|(y_i - y_{i+1})(t)\|_{C^{N+\alpha}} \lesssim \tau_q^{-1}\tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha} \lesssim \delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}.$$

Next we have for $t \in I_i$,

$$(\partial_t + v_l\cdot\nabla)\overline{M}_q = -\partial_t^2\chi_i(y_i - y_{i+1}) - \partial_t\chi_i(\partial_t + v_l\cdot\nabla)(y_i - y_{i+1}),$$

which by (3.24) and the property of the cut-off functions implies that

$$\|(\partial_t + v_l\cdot\nabla)\overline{M}_q(t)\|_{C^{N+\alpha}} \lesssim \tau_q^{-2}\tau_q\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha} + \tau_q^{-1}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha} \lesssim \tau_q^{-1}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}.$$

Then using the fact that $\delta_{q+1}^{1/2}\tau_q l^{-1}\leqslant 1$ in (3.9) and (3.17) we deduce that

$$\begin{aligned}\left\|(\partial_t+\overline{v}_q\cdot\nabla)\overline{M}_q\right\|_{C^{N+\alpha}} &\lesssim \left\|(\partial_t+v_l\cdot\nabla)\overline{M}_q\right\|_{C^{N+\alpha}}+\left\|(\overline{v}_q-v_l)\cdot\nabla\overline{M}_q\right\|_{C^{N+\alpha}}\\ &\lesssim \left\|(\partial_t+v_l\cdot\nabla)\overline{M}_q\right\|_{C^{N+\alpha}}+\|\overline{v}_q-v_l\|_{C^{N+\alpha}}\left\|\nabla\overline{M}_q\right\|_{C^\alpha}+\|\overline{v}_q-v_l\|_{C^\alpha}\left\|\nabla\overline{M}_q\right\|_{C^{N+\alpha}}\\ &\lesssim \tau_q^{-1}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}+\tau_q\delta_{q+1}^{3/2}\tilde{\delta}_{q+1}^{1/2}l^{-N-2+2\alpha}\lesssim\tau_q^{-1}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{-N+\alpha}.\end{aligned}$$

□

3.3. **The perturbation procedure.** To proceed the procedure, inspired by Isett's work [Ise18], we proceed with the construction of the perturbation $(w_{q+1}+\overline{w}_{q+1},\theta_{q+1})$, where the support of (w_{q+1},θ_{q+1}) and that of $\overline{w}_{q+1}$ are disjoint. This can be achieved by choosing building blocks being disjoint, as detailed in Appendix B.1. Here the perturbation (w_{q+1},θ_{q+1}) is used to cancel the low frequency part of the stress term $\overline{M}_q$. However, it brings a new term $w_{q+1}\otimes w_{q+1}$ into the Euler equation. On the way, the perturbation $\overline{w}_{q+1}$ is used to cancel the stress term $\mathring{\overline{R}}_q$ and the low frequency part from $w_{q+1}\otimes w_{q+1}$. To achieve this, we need the the Mikado flows recalled in Appendix B.1. Then we define the perturbation (w_{q+1},θ_{q+1}) in Section 3.3.1 and define the perturbation $\overline{w}_{q+1}$ in Section 3.3.2. The corresponding estimates are shown in Section 3.3.3.

3.3.1. *The construction of the perturbation* (w_{q+1},θ_{q+1}). In this section, we aim to construct the perturbation (w_{q+1},θ_{q+1}), which is used to cancel the stress term $\overline{M}_q$. Recall that $\overline{M}_q$ has support in $\mathbb{T}^3\times\cup_{i\geqslant i_q}I_i$, where I_i is defined in (3.8) and i_q is defined in (3.26). Then we define a family of smooth temporal cut-off functions $\{\eta_i\}_{i\geqslant i_q}$ with the following properties:

(1) $0\leqslant\eta_i\leqslant 1$, $\eta_i\equiv 1$ on I_i , $\mathrm{supp}(\eta_i)\subset(t_i+\frac{1}{6}\tau_q,t_i+\frac{5}{6}\tau_q)$,
(2) $\|\partial_t^n\eta_i\|_{C_t}\lesssim\tau_q^{-n}$, for all $n\geqslant 0$.

Since $\eta_i\equiv 1$ on I_i, $\eta_i\eta_j\equiv 0$ for $i\neq j$, and $\mathrm{supp}(\overline{M}_q)\subset\mathbb{T}^3\times\cup_{i\geqslant i_q}I_i$, we have that

$$\sum_i\eta_i^2\overline{M}_q=\overline{M}_q\,. \tag{3.31}$$

Then we define the flow maps Φ_i for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{aligned}(\partial_t+\overline{v}_q\cdot\nabla)\Phi_i&=0\,,\\ \Phi_i\,(x,t_i)&=x,\end{aligned} \tag{3.32}$$

for all $t\in(t_i-\frac{\tau_q}{3},t_i+\frac{4\tau_q}{3})$. In the following we use the notation $D_{t,q}=\partial_t+\overline{v}_q\cdot\nabla_x$.

We have the following estimates for the flow maps.

Proposition 3.7. *For all* $t\in(t_i-\frac{\tau_q}{3},t_i+\frac{4\tau_q}{3})$ *and* $N\geqslant 0$, *we have*

$$\|\nabla\Phi_i(t)-\mathrm{Id}\|_{C^0}\lesssim l^\alpha, \tag{3.33}$$

$$\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^N}+\|\nabla\Phi_i(t)\|_{C^N}\lesssim l^{-N}, \tag{3.34}$$

$$\left\|D_{t,q}(\nabla\Phi_i)^{-1}(t)\right\|_{C^N}+\|D_{t,q}\nabla\Phi_i(t)\|_{C^N}\lesssim\tau_q^{-1}l^{-N}. \tag{3.35}$$

The proof of this Proposition is given in Appendix B.3.

Next for $t\in\mathrm{supp}(\eta_i)$ we define the vector field

$$M_{q,i}(t)=\nabla\Phi_i(t)\left((\frac{3}{4},0,0)^T+\frac{\overline{M}_q(t)}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\right). \tag{3.36}$$

Remark 3.8. Here we introduce additional translation in the definition of $M_{q,i}$, which on an one hand ensures that $M_{q,i}$ remains in the annulus $\overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$ such that Lemma B.1 is applicable. On the other hand, this translation also guarantees that the supplementary term in (3.44) below maintains divergence-free as the cut-offs η_i are only functions of t.

Furthermore, using the estimates on $\overline{M}_q, \nabla\Phi_i$ in (3.30) and (3.33) respectively, we deduce that for $t \in \mathrm{supp}(\eta_i)$

$$\left\| M_{q,i}(t) - (\frac{3}{4}, 0, 0)^T \right\|_{C^0} \lesssim \|\nabla\Phi_i - \mathrm{Id}\|_{C^0} + \frac{\left\|\nabla\Phi_i \overline{M}_q\right\|_{C^0}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2} l^{\alpha/2}} \lesssim l^{\alpha/2} \leqslant \frac{1}{4}, \tag{3.37}$$

where we choose a large enough to absorb the universal constant.

Thus, due to (3.37), for $\xi \in \Lambda^1 \cup \Lambda^2$, we have $\frac{1}{2} \leqslant |M_{q,i}| \leqslant 1$, then we apply Lemma B.1 and define the amplitude functions

$$A_{(\xi,i)} = l^{\alpha/4}\delta_{q+1}^{1/2}\eta_i\,\Gamma_\xi^{1/2}(M_{q,i}), \;\; \tilde{A}_{(\xi,i)} = l^{\alpha/4}\tilde{\delta}_{q+1}^{1/2}\eta_i\,\Gamma_\xi^{1/2}(M_{q,i}), \tag{3.38}$$

where Γ_ξ are the functions from Lemma B.1. In particular, by the definition of the cut-off functions and i_q in (3.26), we have $\eta_i = 0$ on $[0, T_q - \tau_q]$. Hence it follows that $\tilde{A}_{(\xi,i)} = 0$ on $[0, T_q - \tau_q]$.

Then we have the following estimates on the amplitude functions.

Proposition 3.9. *For $\xi \in \Lambda, i = 1, 2$ and $N \in \mathbb{N}_0$, we have*

$$\left\|A_{(\xi,i)}\right\|_{C^N} + \tau_q \left\|D_{t,q}A_{(\xi,i)}\right\|_{C^N} \lesssim \delta_{q+1}^{1/2} l^{\alpha/4-N}, \tag{3.39}$$

$$\left\|\tilde{A}_{(\xi,i)}\right\|_{C^N} + \tau_q \left\|D_{t,q}\tilde{A}_{(\xi,i)}\right\|_{C^N} \lesssim \tilde{\delta}_{q+1}^{1/2} l^{\alpha/4-N}. \tag{3.40}$$

The proof of this Proposition is given in Appendix B.3.

Now we proceed with the construction of the principle part of (w_{q+1}, θ_{q+1}). To this end, we employ the Mikado flows which are defined as in Appendix B.1 with $\lambda = \lambda_{q+1}$, i.e.

$$W_{(\xi)}(x) = W_{\xi,\lambda_{q+1}}(x), \;\; \Theta_{(\xi)}(x) = \Theta_{\xi,\lambda_{q+1}}(x)\,.$$

Here for two index sets Λ^1, Λ^2 as in Lemma B.1, we use the notation $\Lambda^i = \Lambda^1$ for i odd, and $\Lambda^i = \Lambda^2$ for i even. With this notation, we now use the amplitudes defined in (3.38) to define the principal part as

$$w_{q+1}^{(p)}(x,t) = \sum_i \sum_{\xi\in\Lambda^i} A_{(\xi,i)}(x,t)(\nabla\Phi_i(x,t))^{-1} W_{(\xi)}(\Phi_i(x,t))\,, \tag{3.41}$$

$$\theta_{q+1}^{(p)}(x,t) = \sum_i \sum_{\xi\in\Lambda^i} \tilde{A}_{(\xi,i)}(x,t)\Theta_{(\xi)}(\Phi_i(x,t))\,. \tag{3.42}$$

We then remark that the vector field $U_{i,\xi} = (\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)$ satisfies the following Lie-advection identity:

$$D_{t,q}U_{i,\xi} = (U_{i,\xi}\cdot\nabla)\overline{v}_q = (\nabla\overline{v}_q)^T U_{i,\xi}\,. \tag{3.43}$$

Using (B.2), (B.4), the identity (3.31), and the fact that the η_i have mutually disjoint supports, we obtain

$$\begin{aligned}
w_{q+1}^{(p)}\theta_{q+1}^{(p)} &= \sum_i \sum_{\xi\in\Lambda^i} A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}(W_{(\xi)}(\Phi_i))(\Theta_{(\xi)}(\Phi_i)) \\
&= \sum_i l^{\alpha/2}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}\eta_i^2(\nabla\Phi_i)^{-1}\sum_{\xi\in\Lambda^i}\Gamma_\xi(M_{q,i})\left((W_{(\xi)}\Theta_{(\xi)})(\Phi_i)\right) \\
&= \sum_i l^{\alpha/2}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}\eta_i^2(\nabla\Phi_i)^{-1}M_{q,i} + \sum_i\sum_{\xi\in\Lambda^i} A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\left(\left(\mathbb{P}_{\neq 0}(W_{(\xi)}\Theta_{(\xi)})\right)(\Phi_i)\right)
\end{aligned}$$

$$= \sum_i l^{\alpha/2}\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}\eta_i^2(\frac{3}{4},0,0)^T + \overline{M}_q + \sum_i\sum_{\xi\in\Lambda^i} A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\Big((\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(W_{(\xi)}\Theta_{(\xi)}))(\Phi_i)\Big), \tag{3.44}$$

where we recall the notation $\mathbb{P}_{\neq 0}f(x) = f(x) - \fint_{\mathbb{T}^3} f(y)\mathrm{d}y$. For the last term we used the fact that $W_{(\xi)}\Theta_{(\xi)}$ is $(\mathbb{T}/\lambda_{q+1})^3$-periodic. Here we remark that the divergence of the first term in the last identity is zero since the cut-off functions η_i only depend on t.

To define the incompressibility corrector, we note that

$$(\nabla\Phi_i)^{-1}(W_{(\xi)}(\Phi_i)) = \mathrm{curl}\,\left((\nabla\Phi_i)^T V_{(\xi)}(\Phi_i)\right).$$

Then we define

$$w_{q+1}^{(c)}(x,t) = \sum_i\sum_{\xi\in\Lambda^i}\nabla A_{(\xi,i)}(x,t)\times\left((\nabla\Phi_i(x,t))^T V_{(\xi)}(\Phi_i(x,t))\right), \tag{3.45}$$

and the total velocity increment w_{q+1} as

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} = \mathrm{curl}\Big(\sum_i\sum_{\xi\in\Lambda^i} A_{(\xi,i)}\,(\nabla\Phi_i)^T(V_{(\xi)}(\Phi_i))\Big), \tag{3.46}$$

which is automatically incompressible. In particular, by definition we have $\mathrm{supp}\, w_{q+1} \subset \cup_i \mathrm{supp}(\eta_i) \subset \cup_i(t_i + \frac{1}{6}\tau_q, t_i + \frac{5}{6}\tau_q)$.

We recall that the perturbation θ_{q+1} needs to be mean-zero, so we define the corresponding corrected perturbation

$$\theta_{q+1}^{(c)} : = -\mathbb{P}_0\theta_{q+1}^{(p)}, \quad \text{and } \theta_{q+1} := \theta_{q+1}^{(p)} + \theta_{q+1}^{(c)}.$$

Then since $\tilde{A}_{(\xi,i)} = 0$ on $[0, T_q - \tau_q]$, by definition we know $\theta_{q+1} = 0$ on $[0, T_q - \tau_q]$.

3.3.2. *The construction of the perturbation* $\overline{w}_{q+1}$. In this section, we construct the new perturbation $\overline{w}_{q+1}$ which is used to cancel the stress term $\mathring{\overline{R}}_q$ and the low frequency part from $w_{q+1}\otimes w_{q+1}$. We first note that the support of $\mathring{\overline{R}}_q$ is located in $\mathbb{T}^3\times\cup_i I_i$, where I_i is defined in (3.8) and the support of w_{q+1} is located in $\cup_i(t_i + \frac{1}{6}\tau_q, t_i + \frac{5}{6}\tau_q)$. Then we define a family of smooth cutoff functions $\{\overline{\eta}_i\}_{i\geqslant 0}$ with the following properties:

(1) $0\leqslant\overline{\eta}_i\leqslant 1, \overline{\eta}_i\equiv 1$, on $\mathbb{T}^3\times(t_i+\frac{1}{6}\tau_q, t_i+\frac{5}{6}\tau_q)$, $\mathrm{supp}(\overline{\eta}_i)\subset\mathbb{T}^3\times(t_i - \frac{\tau_q}{3}, t_i + \frac{4\tau_q}{3})$,
(2) $\overline{\eta}_i\overline{\eta}_j\equiv 0$ for every $i\neq j$,
(3) for all $t\in[0,1]$,

$$c_{\overline{\eta}}\leqslant\sum_i\int_{\mathbb{T}^3}\overline{\eta}_i^2(x,t)\mathrm{d}x, \tag{3.47}$$

where $c_{\overline{\eta}} > 0$ is a universal constant,
(4) $\|\partial_t^n\overline{\eta}_i\|_{C^m}\lesssim\tau_q^{-n}$, for all $n, m\geqslant 0$.

In contrast to [BDLSV19, Lemma 5.3], our construction requires the cutoff function to satisfy $\overline{\eta}_i\equiv 1$ over an extended domain to ensure that the support covers not only the support of $\mathring{\overline{R}}_q$, but also that of w_{q+1}. The existence of such cut-off functions follows by using the following domains:

$$\overline{O}_i = \{(x,t) : t_i + \frac{\tau_q}{12}(\sin(2\pi x_1) + \frac{1}{2}) \leqslant t \leqslant t_{i+1} + \frac{\tau_q}{12}(\sin(2\pi x_1) - \frac{1}{2})\}.$$

Then the desired cut-off functions are defined as

$$\overline{\eta}_i := \mathbf{1}_{\overline{O}_i} *_t \varphi_{\epsilon_0\tau_q} *_x \phi_{\epsilon_0},$$

where $\phi_{\epsilon_0} := \frac{1}{\epsilon_0^d}\phi(\frac{\cdot}{\epsilon_0})$ is a family of standard mollifiers on $\mathbb{R}^3$, and $\varphi_{\epsilon_0\tau_q} := \frac{1}{\epsilon_0\tau_q}\varphi(\frac{\cdot}{\epsilon_0\tau_q})$ is a family of standard mollifiers with support in $(0,1)$. Then by choosing $\epsilon_0 > 0$ small enough, the desired conclusions hold by a similar argument as in [BDLSV19, Lemma 5.3].

Before defining the second amplitude function, we define the low frequency of the product $w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}$ by

$$R_q^{(1)} := \sum_i \sum_{\xi\in\Lambda^i} A_{(\xi,i)}^2 (\nabla\Phi_i)^{-1}\,\xi\otimes\xi\,(\nabla\Phi_i)^{-T}.$$

In fact, by the fact that the building blocks have mutually disjoint support (B.2), the fact that the η_i have mutually disjoint supports, and (B.3) we have

$$w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - R_q^{(1)} = \sum_i \sum_{\xi\in\Lambda^i} A_{(\xi,i)}^2 (\nabla\Phi_i)^{-1}\Big(\left(\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)})\right)(\Phi_i)\Big)(\nabla\Phi_i)^{-T}. \tag{3.48}$$

By the Leibniz rule, the fact that η_i have disjoint supports, the estimate for $A_{(\xi,i)}$ in (3.39) and the estimates for $\nabla\Phi_i$ in (3.34), (3.35), we have for $N \geqslant 0$

$$\left\|R_q^{(1)}\right\|_{C^N} + \tau_q\left\|D_{t,q}R_q^{(1)}\right\|_{C^N} \lesssim \delta_{q+1}l^{\alpha/2-N}. \tag{3.49}$$

To prescribe the energy profile, we define the energy gap

$$\Upsilon_q(t) := \frac{1}{3}\left(e(t) - \frac{\delta_{q+2}}{2} - \|\overline{v}_q(t)\|_{L^2}^2 - \int_{\mathbb{T}^3}\mathrm{tr}R_q^{(1)}(t,x)\mathrm{d}x\right), \tag{3.50}$$

and then by (3.47) we decompose Υ_q by

$$\Upsilon_{q,i}(t,x) := \frac{\overline{\eta}_i^2(t,x)}{\sum_j\int_{\mathbb{T}^3}\overline{\eta}_j^2(t,y)\mathrm{d}y}\Upsilon_q(t). \tag{3.51}$$

From the construction, it follows that $\sum_i\int_{\mathbb{T}^3}\Upsilon_{q,i}\mathrm{d}x = \Upsilon_q$ for all $t\in[0,1]$.

Proposition 3.10. *For any $t\in[0,1]$ and $N\geqslant 0$ we have*

$$\frac{\delta_{q+1}}{6\lambda_q^{\alpha/3}} \leqslant \Upsilon_q(t) \leqslant \delta_{q+1},\ \ \Upsilon_{q,i}(t) \leqslant \frac{\delta_{q+1}}{c_{\overline{\eta}}}, \tag{3.52}$$

$$\|\Upsilon_{q,i}\|_{C^N} + \tau_q\|\partial_t\Upsilon_{q,i}\|_{C^N} \lesssim \delta_{q+1}. \tag{3.53}$$

Proof. First, by the estimate for the energy gaps in (2.9), (3.7), (3.18) and the bound in (3.49) we have for some $C>0$

$$\begin{aligned}\frac{\delta_{q+1}}{2\lambda_q^{\alpha/3}} &\leqslant \frac{\delta_{q+1}}{\lambda_q^{\alpha/3}} - \frac{\delta_{q+2}}{2} - C\delta_{q+1}l^{\alpha/2} \leqslant 3\Upsilon_q(t)\\ &= e(t) - \|v_q(t)\|_{L^2}^2 - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3}\mathrm{tr}R_q^{(1)}(t,x)\mathrm{d}x\\ &\quad + \left[\|v_q(t)\|_{L^2}^2 - \|v_l(t)\|_{L^2}^2\right] + \left[\|v_l(t)\|_{L^2}^2 - \|\overline{v}_q(t)\|_{L^2}^2\right]\\ &\leqslant \delta_{q+1} + C\delta_{q+1}l^{\alpha/2} \leqslant 2\delta_{q+1},\end{aligned}$$

where we choose $0<\alpha<6\beta b(b-1)$ small enough to deduce that $\delta_{q+2}\leqslant\frac{1}{2}\delta_{q+1}\lambda_q^{-\alpha/3}$. We also used $l\lambda_q \ll 1$ and choose a large enough to absorb the universal constant. Then together with the definition of the cut-off functions, we obtain the second inequality in (3.52).

By the properties of the cutoff functions and (3.52), the bound for the first term in (3.53) follows. Now we derive the time regularity of $\Upsilon_{q,i}$. Since $(\overline{v}_q, \mathring{\overline{R}}_q)$ obeys (2.3), by the basic energy estimate, the bounds in (3.19) and (3.20) we have for $t \in [0,1]$

$$\left|\partial_t \|\overline{v}_q(t)\|_{L^2}^2\right| \lesssim \left|\int_{\mathbb{T}^d} \mathring{\overline{R}}_q(t,x) \cdot \nabla \overline{v}_q(t,x)\mathrm{d}x\right| \lesssim \delta_{q+1}\tau_q^{-1} l^\alpha.$$

Since $\int_{\mathbb{T}^3} \overline{v}_q \cdot \nabla R_q^{(1)}(t,x)\mathrm{d}x = 0$, by (3.19) we have for $t \in [0,1]$

$$\left|\int_{\mathbb{T}^3} \mathrm{tr}\partial_t R_q^{(1)}(t,x)\mathrm{d}x\right| = \left|\int_{\mathbb{T}^3} \mathrm{tr} D_{t,q} R_q^{(1)}(t,x)\mathrm{d}x\right| \lesssim \|D_{t,q}R_q^{(1)}\|_{C^0} \lesssim \tau_q^{-1}\delta_{q+1} l^\alpha,$$

which implies that

$$\|\partial_t \Upsilon_q\|_{C_t^0} \lesssim |e'(t)| + \left|\partial_t \|\overline{v}_q(t)\|_{L^2}^2\right| + \left|\int_{\mathbb{T}^3} \mathrm{tr}\partial_t R_q^{(1)}(t,x)\mathrm{d}x\right| \lesssim \tau_q^{-1}\delta_{q+1} l^\alpha, \tag{3.54}$$

where we used that $|e'(t)| \leqslant 1$ and choose a large enough. Then by the Leibniz rule and the properties of the cutoff functions it is easy to see that $\|\partial_t[\frac{\overline{\eta}_i^2(t,x)}{\sum_j \int_{\mathbb{T}^3} \overline{\eta}_j^2(t,y)\mathrm{d}y}]\|_{C^N} \lesssim \tau_q^{-1}$. Thus, the bound for the second term in (3.53) follows by applying the Leibniz rule again. □

Since $\overline{\eta}_i \equiv 1$ on $\mathbb{T}^3 \times (t_i + \frac{1}{6}\tau_q, t_i + \frac{5}{6}\tau_q)$, $\overline{\eta}_i\overline{\eta}_j \equiv 0$ for $i \neq j$, and since $\mathrm{supp}(\mathring{\overline{R}}_q), \mathrm{supp}\, R_q^{(1)} \subset \mathbb{T}^3 \times \cup_i (t_i + \frac{1}{6}\tau_q, t_i + \frac{5}{6}\tau_q)$, we have that

$$\sum_i \overline{\eta}_i^2 \mathring{\overline{R}}_q = \mathring{\overline{R}}_q\,, \quad \sum_i \overline{\eta}_i^2 R_q^{(1)} = R_q^{(1)}\,. \tag{3.55}$$

Now we define the symmetric tensor

$$R_{q,i} = \nabla\Phi_i\Big(\mathrm{Id} - \frac{\overline{\eta}_i^2(\mathring{\overline{R}}_q + \mathring{R}_q^{(1)})}{\Upsilon_{q,i}}\Big)\nabla\Phi_i^T = \nabla\Phi_i\Big(\mathrm{Id} - \frac{\sum_j \int_{\mathbb{T}^3} \overline{\eta}_j^2(t,y)\mathrm{d}y}{\Upsilon_q(t)}(\mathring{\overline{R}}_q + \mathring{R}_q^{(1)})\Big)\nabla\Phi_i^T \tag{3.56}$$

for all $(x,t) \in \mathrm{supp}(\overline{\eta}_i)$. Here $\mathring{R}_q^{(1)}$ means the trace-free part of $R_q^{(1)}$. By the bounds in (3.19), (3.33), (3.49) and (3.52), we have that on $\mathrm{supp}(\overline{\eta}_i)$

$$\begin{aligned}\|R_{q,i}(t) - \mathrm{Id}\|_{C^0} &\lesssim \|\nabla\Phi_i\nabla\Phi_i^T - \mathrm{Id}\|_{C^0} + \|\nabla\Phi_i \frac{\sum_j \int_{\mathbb{T}^3} \overline{\eta}_j^2(t,y)\mathrm{d}y}{\Upsilon_q(t)}(\mathring{\overline{R}}_q + \mathring{R}_q^{(1)})\nabla\Phi_i^T\|_{C^0} \\ &\lesssim l^{\alpha/2}\lambda_q^{\alpha/3} \leqslant 1/2\,,\end{aligned}$$

where we used the fact that $l\lambda_q \ll 1$, and choose a large enough to absorb the universal constant.

One last important property of the stress $R_{q,i}$ is obtained by recalling (3.55)

$$\sum_i \Upsilon_{q,i}(\nabla\Phi_i)^{-1}R_{q,i}(\nabla\Phi_i)^{-T} = \sum_i \Upsilon_{q,i}\mathrm{Id} - \mathring{\overline{R}}_q - \mathring{R}_q^{(1)}. \tag{3.57}$$

Thus, since $R_{q,i}$ obeys the conditions of Lemma B.2 on $\mathrm{supp}(\overline{\eta}_i)$, for $\xi \in \overline{\Lambda}^1 \cup \overline{\Lambda}^2$, we can define the amplitude functions as

$$a_{(\xi,i)} = \Upsilon_{q,i}^{1/2}\gamma_\xi(R_{q,i}), \tag{3.58}$$

where the γ_ξ are the functions from Lemma B.2.

Then we have the following estimates for the amplitude functions $a_{(\xi,i)}$.

Proposition 3.11. *For $\xi \in \overline{\Lambda}^1 \cup \overline{\Lambda}^2$, $i = 1, 2$ and $N \in \mathbb{N}_0$, we have*

$$\left\| a_{(\xi,i)} \right\|_{C^N} + \tau_q \left\| D_{t,q} a_{(\xi,i)} \right\|_{C^N} \lesssim \delta_{q+1}^{1/2} l^{-N}. \tag{3.59}$$

We put the proof of this proposition in Appendix B.3.

Now we define the principle part of the perturbation $\overline{w}_{q+1}$. We shall use the index sets $\overline{\Lambda}^1, \overline{\Lambda}^2$ from Lemma B.2. We use the notation $\overline{\Lambda}^i = \overline{\Lambda}^1$ for i odd, and $\overline{\Lambda}^i = \overline{\Lambda}^2$ for i even. We use the amplitude functions $a_{(\xi,i)}$ in (3.58) to define

$$\overline{w}_{q+1}^{(p)}(x,t) = \sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)}(x,t)(\nabla \Phi_i(x,t))^{-1} W_{(\xi)}(\Phi_i(x,t)). \tag{3.60}$$

Using (B.2), (B.4), (3.57), (3.48) above and the fact that the $\overline{\eta}_i$ have mutually disjoint supports, we obtain

$$\begin{aligned}
(w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}) \otimes (w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}) + \mathring{\overline{R}}_q &= \overline{w}_{q+1}^{(p)} \otimes \overline{w}_{q+1}^{(p)} + w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{\overline{R}}_q \\
&= \sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)}^2 (\nabla \Phi_i)^{-1} \Big((W_{(\xi)}(\Phi_i)) \otimes (W_{(\xi)}(\Phi_i)) \Big) (\nabla \Phi_i)^{-T} + w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{\overline{R}}_q \\
&= \sum_i \Upsilon_{q,i} (\nabla \Phi_i)^{-1} \sum_{\xi \in \overline{\Lambda}^i} \gamma_\xi^2 (R_{q,i}) \Big((W_{(\xi)} \otimes W_{(\xi)})(\Phi_i) \Big) (\nabla \Phi_i)^{-T} + w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{\overline{R}}_q \\
&= \sum_i \Upsilon_{q,i} (\nabla \Phi_i)^{-1} R_{q,i} (\nabla \Phi_i)^{-T} + w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{\overline{R}}_q \\
&\quad + \sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)}^2 (\nabla \Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0} (W_{(\xi)} \otimes W_{(\xi)}) \right) (\Phi_i) \Big) (\nabla \Phi_i)^{-T} \\
&= \sum_i \Upsilon_{q,i} \mathrm{Id} + \frac{1}{3} \mathrm{tr} R_q^{(1)} \mathrm{Id} + \sum_i \sum_{\xi \in \Lambda^i} A_{(\xi,i)}^2 (\nabla \Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0} (W_{(\xi)} \otimes W_{(\xi)}) \right) (\Phi_i) \Big) (\nabla \Phi_i)^{-T} \\
&\quad + \sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)}^2 (\nabla \Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0} (W_{(\xi)} \otimes W_{(\xi)}) \right) (\Phi_i) \Big) (\nabla \Phi_i)^{-T}.
\end{aligned} \tag{3.61}$$

Similar to (3.45), we define the incompressibility corrector

$$\overline{w}_{q+1}^{(c)}(x,t) = \sum_i \sum_{\xi \in \overline{\Lambda}^i} \nabla a_{(\xi,i)}(x,t) \times \Big((\nabla \Phi_i(x,t))^T (V_{(\xi)}(\Phi_i(x,t)) \Big). \tag{3.62}$$

Then the total velocity increment $\overline{w}_{q+1}$ is defined as

$$\overline{w}_{q+1} = \overline{w}_{q+1}^{(p)} + \overline{w}_{q+1}^{(c)} = \mathrm{curl} \Big(\sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)} \, (\nabla \Phi_i)^T (V_{(\xi)}(\Phi_i)) \Big), \tag{3.63}$$

which is automatically incompressible. Finally we define

$$v_{q+1} = \overline{v}_q + w_{q+1} + \overline{w}_{q+1}, \quad \rho_{q+1} = \overline{\rho}_q + \theta_{q+1}. \tag{3.64}$$

Moreover, recalling the fact that $\tau_q \leqslant \lambda_q^{-1} \delta_q^{-1/2} \leqslant \tilde{\delta}_{q+1}^{1/2}$ due to $\tilde{\beta} b + \beta < 1$, we have $\theta_{q+1} = \overline{\rho}_q - 1 = 0$ on $[0, T_{q+1}] \subset [0, T_q - \tau_q]$, hence $\rho_{q+1} = 1$ on $[0, T_{q+1}] \subset [0, T_q - \tau_q]$. Moreover, it is easy to see that θ_{q+1} is mean-zero. Consequently, we have $\int \rho_{q+1} \mathrm{d}x = \int \overline{\rho}_q \mathrm{d}x = 1$.

To conclude this section, we note that

$$\overline{w}_{q+1} \theta_{q+1}^{(p)} = 0, \tag{3.65}$$

since the building blocks have disjoint supports.

3.3.3. *Estimate of perturbations.* In this section we establish the desired estimates on the perturbations and derive the estimates (2.5), (2.6) and (2.10) at the level of $q+1$.

First we estimate the principle part of the perturbations $w_{q+1}^{(p)}, \theta_{q+1}^{(p)}$ and $\overline{w}_{q+1}^{(p)}$ as defined in (3.41), (3.42) and (3.60) respectively. From the estimates for $\nabla\Phi_i$, $A_{(\xi,i)}$ and $\tilde{A}_{(\xi,i)}$ given in (3.33), (3.39), the estimates for the building blocks in (B.5), and (3.40) respectively, and the fact that the $\overline{\eta}_i$ have disjoint supports we have for some $M>1$

$$\left\|w_{q+1}^{(p)}\right\|_{C^0} \lesssim \sum_i \sum_{\xi\in\Lambda^i} \|A_{(\xi,i)}(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\|_{C^0} \leqslant \frac{M}{8}\delta_{q+1}^{1/2}, \tag{3.66}$$

$$\|\theta_{q+1}\|_{C^0} \lesssim \|\theta_{q+1}^{(p)}\|_{C^0} \lesssim \sum_i \sum_{\xi\in\Lambda^i} \|\tilde{A}_{(\xi,i)}\Theta_{(\xi)}(\Phi_i)\|_{C^0} \lesssim \tilde{\delta}_{q+1}^{1/2} l^{\alpha/4} \leqslant \frac{1}{2}\delta_{q+1}^{1/2}, \tag{3.67}$$

where in the last inequality we choose a large enough to absorb the universal constant. Here and in the following the sum over$i$ is finite, since by definition the amplitude functions have disjoint supports for different i.

When considering the C^1-norm, from the bounds for $\nabla\Phi_i$, $A_{(\xi,i)}$, and $\tilde{A}_{(\xi,i)}$ in (3.34), (3.39), (3.40) respectively, from (B.5) we lose a factor l^{-1} from the gradient, and lose a factor λ_{q+1} from the gradient of $W_{(\xi)}$, i.e.

$$\left\|w_{q+1}^{(p)}\right\|_{C^1} \leqslant \sum_i \sum_{\xi\in\Lambda^i} \|A_{(\xi,i)}(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\|_{C^1} \lesssim \delta_{q+1}^{1/2}(\lambda_{q+1}+l^{-1}) \leqslant \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1},$$

$$\|\theta_{q+1}\|_{C^1} \lesssim \|\theta_{q+1}^{(p)}\|_{C^1} \lesssim \sum_i \sum_{\xi\in\Lambda^i} \|\tilde{A}_{(\xi,i)}\Theta_{(\xi)}(\Phi_i)\|_{C^1} \lesssim \tilde{\delta}_{q+1}^{1/2} l^{\alpha/4}(\lambda_{q+1}+l^{-1}) \leqslant \frac{1}{2}\delta_{q+1}^{1/2}\lambda_{q+1}, \tag{3.68}$$

where we used that by choosing $\alpha>0$ small enough

$$\frac{l^{-1}}{\lambda_{q+1}} = \frac{\tilde{\delta}_q^{1/2}\lambda_q^{1+\frac{3\alpha}{2}}}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}} = \frac{\lambda_q^{1-\tilde{\beta}+\frac{3\alpha}{2}}}{\lambda_{q+1}^{1-\tilde{\beta}}} \leqslant \lambda_q^{\frac{3\alpha}{2}-(b-1)(1-\tilde{\beta})} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}} \ll 1\,. \tag{3.69}$$

We also choose a large enough to absorb the universal constant in the last inequality.

Then we note that the definition of the perturbation $\overline{w}_{q+1}^{(p)}$ is similar to that of $w_{q+1}^{(p)}$ by replacing Λ^i, $A_{(\xi,i)}$ by $\overline{\Lambda}^i$, $a_{(\xi,i)}$ respectively. Then by the estimate of $a_{(\xi,i)}$ in (3.59) and a similar calculation we have for $j=0,1$

$$\left\|\overline{w}_{q+1}^{(p)}\right\|_{C^j} \leqslant \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1}^j. \tag{3.70}$$

For the incompressibility correctors $w_{q+1}^{(c)}$ and $\overline{w}_{q+1}^{(c)}$ defined in (3.45) and (3.62) respectively, we see that from the estimates for $\nabla\Phi_i$, $A_{(\xi,i)}$ and $V_{(\xi)}$ in (3.34), (3.39) and (B.5) respectively, we obtain that for $j=0,1$ and $M>1$

$$\left\|w_{q+1}^{(c)}\right\|_{C^j} \lesssim \sum_i \sum_{\xi\in\Lambda^i} \left\|\nabla A_{(\xi,i)} \times \left((\nabla\Phi_i)^T(V_{(\xi)}(\Phi_i)\right)\right\|_{C^j} \lesssim \delta_{q+1}^{1/2}\frac{l^{-1}}{\lambda_{q+1}}(\lambda_{q+1}+l^{-1})^j \leqslant \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1}^j,$$

where we used (3.69). Similarly,

$$\left\|\overline{w}_{q+1}^{(c)}\right\|_{C^j} \lesssim \delta_{q+1}^{1/2}\frac{l^{-1}}{\lambda_{q+1}}(\lambda_{q+1}+l^{-1})^j \leqslant \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1}^j. \tag{3.71}$$

Combining all the bounds above, together with (3.3), (3.17), (3.20) we have that

$$\|v_{q+1}-v_q\|_{C^0}\leqslant\|w_{q+1}+\overline{w}_{q+1}\|_{C^0}+\|\overline{v}_q-v_l\|_{C^0}+\|v_l-v_q\|_{C^0}\leqslant\frac{M}{2}\delta_{q+1}^{1/2}+M'\delta_{q+1}^{1/2}\lambda_q^{-\alpha}\leqslant M\delta_{q+1}^{1/2},$$

$$\|v_{q+1}\|_{C^1}\leqslant\|w_{q+1}+\overline{w}_{q+1}\|_{C^1}+\|\overline{v}_q\|_{C^1}\leqslant\frac{M}{2}\delta_{q+1}^{1/2}\lambda_{q+1}+M'\tau_q^{-1}l^{2\alpha}\leqslant M\delta_{q+1}^{1/2}\lambda_{q+1},$$

where M' is a universal constant and we note that by choosing a large enough, we have

$$\tau_q l^{-2\alpha}\delta_{q+1}^{1/2}\lambda_{q+1}=\lambda_q^{(b-1)(1-\tilde{\beta})}\gg 1.$$

We also choose a large enough to absorb the constant. Then (2.6) and (2.10) are satisfied for v_{q+1}, and, moreover, the bound (2.5) holds for v_{q+1}.

By combining the above estimates with (3.4), (3.28) and (3.29), we deduce that

$$\|\rho_{q+1}-\rho_q\|_{C^0}\leqslant\|\theta_{q+1}\|_{C^0}+\|\overline{\rho}_q-\rho_l\|_{C^0}+\|\rho_l-\rho_q\|_{C^0}\leqslant\frac{1}{2}\tilde{\delta}_{q+1}^{1/2}+M'\tilde{\delta}_{q+1}^{1/2}\lambda_q^{-\alpha}\leqslant\tilde{\delta}_{q+1}^{1/2},$$

$$\|\rho_{q+1}\|_{C^1}\leqslant\|\theta_{q+1}\|_{C^1}+\|\overline{\rho}_q\|_{C^1}\leqslant\frac{1}{2}\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}+M'\tilde{\delta}_q^{1/2}\lambda_q\leqslant\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1},$$

where M' is a universal constant and we choose a large enough to absorb the constant. Then (2.6) and (2.10) are satisfied for ρ_{q+1}, and, moreover, the bound (2.5) holds for ρ_{q+1}.

3.4. **The estimate of the stress terms** $(M_{q+1},\mathring{R}_{q+1})$**.** In this section we aim to establish the desired estimates on the stress terms. We recall that the stress term $\overline{M}_q$ in the transport equation is canceled by the perturbation (w_{q+1},θ_{q+1}), while it brings a new term $w_{q+1}\otimes w_{q+1}$ into the Euler equations. Then the perturbation $\overline{w}_{q+1}$ is used to cancel the stress term $\overline{\mathring{R}}_q$ and the low frequency part from $w_{q+1}\otimes w_{q+1}$. This procedure together with the definition of new stress terms M_{q+1} and $\mathring{R}_{q+1}$is stated in Section 3.4.1. Then we establish the desired estimates in Section 3.4.2 and 3.4.3 for M_{q+1} and $\mathring{R}_{q+1}$ respectively.

3.4.1. *The definition of stress terms M_{q+1} and $\mathring{R}_{q+1}$.* In order to define M_{q+1}, we notice that $(\overline{v}_q,\overline{\rho}_q,\overline{M}_q)$ obeys the transport equation (2.2). Using the definition of the perturbation (v_{q+1},ρ_{q+1}), along with the fact that

$$\mathrm{div}((w_{q+1}+\overline{w}_{q+1})\theta_{q+1}^{(c)})=\theta_{q+1}^{(c)}\mathrm{div}(w_{q+1}+\overline{w}_{q+1})=0,$$

and (3.65), we have that

$$\begin{aligned}-\mathrm{div}M_{q+1}&=D_{t,q}\theta_{q+1}+\mathrm{div}((w_{q+1}+\overline{w}_{q+1})\theta_{q+1}-\overline{M}_q)+(w_{q+1}+\overline{w}_{q+1})\cdot\nabla\overline{\rho}_q\\&=D_{t,q}\theta_{q+1}+\mathrm{div}(w_{q+1}\theta_{q+1}^{(p)}-\overline{M}_q)+(w_{q+1}+\overline{w}_{q+1})\cdot\nabla\overline{\rho}_q.\end{aligned}$$

Using the inverse divergence operator $\mathcal{R}_1$ introduced in Section A.1, and (3.44) we define the transport error, Nash error and oscillation error respectively as

$$\begin{aligned}&M_{tr}:=\mathcal{R}_1(D_{t,q}\theta_{q+1}),\quad M_{Nash}:=\mathcal{R}_1((w_{q+1}+\overline{w}_{q+1})\cdot\nabla\overline{\rho}_q),\\&M_{osc}:=\sum_i\sum_{\xi\in\Lambda^i}\mathcal{R}_1\mathrm{div}\left(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\left((\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(W_{(\xi)}\Theta_{(\xi)}))(\Phi_i)\right)\right)+w_{q+1}^{(c)}\theta_{q+1}^{(p)}.\end{aligned}$$

Then

$$-M_{q+1}:=M_{tr}+M_{osc}+M_{Nash}\,. \tag{3.72}$$

Since $\overline{\rho}_q-1=\theta_{q+1}=\theta_{q+1}^{(p)}=\tilde{A}_{(\xi,i)}=0$ on $[0,T_{q+1}]$, it follows that $M_{q+1}=0$ on $[0,T_{q+1}]$.

In order to define $\mathring{R}_{q+1}$, recalling that $(\overline{v}_q, \mathring{\overline{R}}_q)$ solves the system (2.3), we write

$$\begin{aligned}\operatorname{div}\mathring{R}_{q+1} - \nabla p_{q+1} + \nabla \overline{p}_q = D_{t,q}(w_{q+1} + \overline{w}_{q+1}) + \operatorname{div}((w_{q+1} + \overline{w}_{q+1}) \otimes (w_{q+1} + \overline{w}_{q+1}) + \mathring{\overline{R}}_q) \\ + (w_{q+1} + \overline{w}_{q+1}) \cdot \nabla \overline{v}_q.\end{aligned}$$

Then using the inverse divergence operator $\mathcal{R}$ introduced in Section A.1, we define the transport error and Nash error, respectively, as

$$R_{tr} := \mathcal{R}(D_{t,q}(w_{q+1} + \overline{w}_{q+1})), \quad R_{Nash} := \mathcal{R}((w_{q+1} + \overline{w}_{q+1}) \cdot \nabla \overline{v}_q).$$

For the oscillation error, we use (3.61) to define

$$\begin{aligned}R_{osc} := &\mathcal{R}\operatorname{div}\left(\sum_i \sum_{\xi \in \Lambda^i} A^2_{(\xi,i)} (\nabla \Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right)(\Phi_i)\Big) (\nabla \Phi_i)^{-T}\right) (:= R_{osc,1}) \\ &+ \mathcal{R}\operatorname{div}\left(\sum_i \sum_{\xi \in \overline{\Lambda}^i} a^2_{(\xi,i)} (\nabla \Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right)(\Phi_i)\Big) (\nabla \Phi_i)^{-T}\right) (:= R_{osc,2}) \\ &+ (w^{(c)}_{q+1} + \overline{w}^{(c)}_{q+1}) \mathring{\otimes} (w^{(c)}_{q+1} + \overline{w}^{(c)}_{q+1}) + 2(w^{(p)}_{q+1} + \overline{w}^{(p)}_{q+1}) \mathring{\otimes}_s (w^{(c)}_{q+1} + \overline{w}^{(c)}_{q+1}) (:= R_{osc,3}),\end{aligned}$$

where we use the notation $a \otimes_s b = \frac{1}{2}(a \otimes b + b \otimes a)$, and $\mathring{\otimes}_s$ is defined as the trace-free part of the symmetric tensor. Here the divergence of first two terms in the last inequality in (3.61) can be written as pressure terms, so we put them into ∇p_{q+1}.

Then the Reynolds stress at the level of $q+1$ is defined by

$$\mathring{R}_{q+1} := R_{tr} + R_{osc} + R_{Nash}\,. \tag{3.73}$$

To estimate the above stress terms, we will apply the stationary phase bounds in Proposition A.5 to the building blocks defined in Appendix B.1. More precisely, by (3.34), (B.5) and a similar argument as in [BV19a, Section 6.6.1], we have for $a \in C^\infty(\mathbb{T}^d; \mathbb{R}), b \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$

$$\begin{aligned}\left\|\mathcal{R}(b(W_{(\xi)} \circ \Phi_i))\right\|_{C^\alpha} + \lambda_{q+1} \left\|\mathcal{R}(b(V_{(\xi)} \circ \Phi_i))\right\|_{C^\alpha} + \left\|\mathcal{R}\left(b\ \left((\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)})) \circ \Phi_i\right)\right)\right\|_{C^\alpha} \\ \lesssim \frac{\|b\|_{C^0}}{\lambda^{1-\alpha}_{q+1}} + \frac{\|b\|_{C^{m+\alpha}} + \|b\|_{C^0}\, l^{-m-\alpha}}{\lambda^{m-\alpha}_{q+1}},\end{aligned} \tag{3.74}$$

$$\left\|\mathcal{R}_1(a(W_{(\xi)} \circ \Phi_i))\right\|_{C^\alpha} + \left\|\mathcal{R}_1\left(a\ \left((\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)})) \circ \Phi_i\right)\right)\right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda^{1-\alpha}_{q+1}} + \frac{\|a\|_{C^{m+\alpha}} + \|a\|_{C^0}\, l^{-m-\alpha}}{\lambda^{m-\alpha}_{q+1}}. \tag{3.75}$$

3.4.2. *The estimate for M_{q+1}.* In this section, we aim to show the estimate of M_{q+1}, which is defined in (3.72) by estimating the three errors above separately.

First, for the transport error M_{tr}, recalling that Φ_i satisfies the equation (3.32), the definition of the perturbations θ_{q+1}, and the fact the $\theta^{(c)}_{q+1}$ is a function of time, we write

$$\mathcal{R}_1(D_{t,q}\theta_{q+1}) = \mathcal{R}_1(D_{t,q}\theta^{(p)}_{q+1}) = \sum_i \sum_{\xi \in \Lambda^i} \mathcal{R}_1\left((D_{t,q}\tilde{A}_{(\xi,i)})\Theta_{(\xi)}(\Phi_i)\right).$$

By the estimate of the derivatives of $\tilde{A}_{(\xi,i)}$ in (3.40) we have

$$\left\|D_{t,q}\tilde{A}_{(\xi,i)}\right\|_{C^{m+\alpha}} \lesssim \tilde{\delta}^{1/2}_{q+1}\tau_q^{-1} l^{-m-\alpha},$$

which together with (3.75) implies that

$$\|M_{tr}\|_{C^\alpha} \lesssim \left\|\mathcal{R}_1(D_{t,q}\theta^{(p)}_{q+1})\right\|_{C^\alpha} \lesssim \frac{\tilde{\delta}^{1/2}_{q+1}\tau_q^{-1}}{\lambda^{1-\alpha}_{q+1}}\left(1+\frac{l^{-m-\alpha}}{\lambda^{m-1}_{q+1}}\right) \lesssim \frac{\delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q}\lambda_q}{\lambda^{1-3\alpha}_{q+1}},$$

where we used (3.69) to deduce $(l\lambda_{q+1})^{-1} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$, and then choose m large enough such that $(l\lambda_{q+1})^{-m}l^{-\alpha}\lambda_{q+1} \lesssim 1$ in the last inequality. Here and in the following the sum over i is finite, since by the definition the amplitude functions have disjoint supports for different i.

For the oscillation error M_{osc}, we recall the identity

$$\nabla(\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i) = (\nabla\Phi_i)^T\Big((\nabla\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i)\Big).$$

Therefore, we have

$$\begin{aligned}
&\operatorname{div}\Big(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\Big((\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(W_{(\xi)}\Theta_{(\xi)}))(\Phi_i)\Big)\Big)\\
&= \operatorname{div}\Big(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\xi\Big((\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i)\Big)\Big)\\
&= (\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i)\operatorname{div}\Big(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\xi\Big)\\
&\quad + A_{(\xi,i)}\tilde{A}_{(\xi,i)}\Big((\nabla\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i)\Big)^T(\nabla\Phi_i)(\nabla\Phi_i)^{-1}\xi,
\end{aligned}$$

which by the fact that $(\xi\cdot\nabla)\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}) = 0$ implies

$$M_{osc} = \sum_i\sum_{\xi\in\Lambda^i}\mathcal{R}_1\Big((\mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}(\phi^2_{(\xi)}))(\Phi_i)\operatorname{div}\Big(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\xi\Big)\Big) + w^{(c)}_{q+1}\theta^{(p)}_{q+1}.$$

By the estimate for the derivatives of $A_{(\xi,i)}, \tilde{A}_{(\xi,i)}$ and $(\nabla\Phi_i)^{-1}$ in (3.39), (3.40) and (3.34) respectively, we have

$$\left\|\operatorname{div}(A_{(\xi,i)}\tilde{A}_{(\xi,i)}(\nabla\Phi_i)^{-1}\xi)\right\|_{C^{m+\alpha}} \lesssim \delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q+1}l^{-1-m-\alpha},$$

which together with (3.75), the estimate of $w^{(c)}_{q+1}$ in (3.71), the estimate of $\theta^{(p)}_{q+1}$ in (3.67), (3.68) and interpolation implies that

$$\|M_{osc}\|_{C^\alpha} \lesssim \frac{\delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q+1}l^{-1}}{\lambda^{1-\alpha}_{q+1}}\left(1+\frac{l^{-m-\alpha}}{\lambda^{m-1}_{q+1}}\right) + \left\|w^{(c)}_{q+1}\right\|_{C^\alpha}\left\|\theta^{(p)}_{q+1}\right\|_{C^\alpha} \lesssim \frac{\delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q+1}l^{-1}}{\lambda^{1-2\alpha}_{q+1}} \lesssim \frac{\delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q}\lambda_q}{\lambda^{1-4\alpha}_{q+1}}.$$

Here we used again the bound $(l\lambda_{q+1})^{-1} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and then choose m large enough.

The Nash error is written as $M_{Nash} = \mathcal{R}_1(w_{q+1}\cdot\nabla\overline{\rho}_q) + \mathcal{R}_1(\overline{w}_{q+1}\cdot\nabla\overline{\rho}_q)$, where

$$\begin{aligned}
\mathcal{R}_1(w_{q+1}\cdot\nabla\overline{\rho}_q) = \sum_i\sum_{\xi\in\Lambda^i}&\mathcal{R}_1((\nabla\overline{\rho}_q)^TA_{(\xi,i)}(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i))\\
&+\mathcal{R}_1\left((\nabla\overline{\rho}_q)^T\nabla A_{(\xi,i)}\times((\nabla\Phi_i)^T(V_{(\xi)}(\Phi_i))\right),
\end{aligned}$$

and the term $\mathcal{R}_1(\overline{w}_{q+1}\cdot\nabla\overline{\rho}_q)$ can be written in a similar form with $A_{(\xi,i)}, \Lambda^i$ replaced by $a_{(\xi,i)}, \overline{\Lambda}^i$ respectively.

By the estimate of the derivatives of $A_{(\xi,i)}$, $\nabla\overline{\rho}_q$, and $(\nabla\Phi_i)^{-1}$ in (3.39), (3.29), and (3.34) respectively, we have

$$\|(\nabla\overline{\rho}_q)^TA_{(\xi,i)}(\nabla\Phi_i)^{-1}\|_{C^{m+\alpha}} + l\|(\nabla\overline{\rho}_q)^T\nabla A_{(\xi,i)}\times((\nabla\Phi_i)^T\cdot)\|_{C^{m+\alpha}} \lesssim \delta^{1/2}_{q+1}\tilde{\delta}^{1/2}_{q}\lambda_ql^{-m-\alpha},$$

at which point we apply (3.75) to deduce that

$$\left\|\mathcal{R}_1(w_{q+1}\cdot\nabla\overline{\rho}_q)\right\|_{C^\alpha}\lesssim\frac{\delta_{q+1}^{1/2}\tilde{\delta}_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{l^{-1}}{\lambda_{q+1}}+\frac{l^{-1-m-\alpha}}{\lambda_{q+1}^m}\right)\lesssim\frac{\delta_{q+1}^{1/2}\tilde{\delta}_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}}.$$

Here we used again the bound $(l\lambda_{q+1})^{-1}\leqslant\lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and then choose m large enough.

The second term $\mathcal{R}_1(\overline{w}_{q+1}\cdot\nabla\overline{\rho}_q)$ is bounded by the same argument, but we omit the details. Furthermore, we have

$$\|M_{Nash}\|_{C^\alpha}\lesssim\left\|\mathcal{R}_1(w_{q+1}\cdot\nabla\overline{\rho}_q)\right\|_{C^\alpha}+\left\|\mathcal{R}_1(\overline{w}_{q+1}\cdot\nabla\overline{\rho}_q)\right\|_{C^\alpha}\lesssim\frac{\delta_{q+1}^{1/2}\tilde{\delta}_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}}.$$

The above bounds together with (3.1) immediately imply the desired estimate (2.7) for M_{q+1}:

$$\|M_{q+1}\|_{C^\alpha}\lesssim\frac{\delta_{q+1}^{1/2}\tilde{\delta}_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}\leqslant\frac{1}{2}\frac{\delta_{q+2}^{1/2}\tilde{\delta}_{q+2}^{1/2}}{\lambda_{q+1}^{3\alpha}},$$

where the extra power of $\lambda_{q+1}^{-\alpha}$ is used to absorb the implicit constant by choosing a sufficiently large.

Additionally, when considering the C^1-norm, we lose a factor l^{-1} in the estimates of the derivatives of $A_{(\xi,i)}, \tilde{A}_{(\xi,i)}$, $\nabla\overline{\rho}_q$, and $(\nabla\Phi_i)^{-1}$ as presented in (3.39), (3.40), (3.29), and (3.34) respectively. Moreover, we lose a factor λ_{q+1} from the estimates of $W_{(\xi)}, V_{(\xi)}$ in (B.5). Consequently, we have

$$\|M_{q+1}\|_{C^1}\lesssim\frac{\delta_{q+1}^{1/2}\tilde{\delta}_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}\lambda_{q+1}\leqslant\frac{1}{2}\frac{\delta_{q+2}^{1/2}\tilde{\delta}_{q+2}^{1/2}}{\lambda_{q+1}^{3\alpha-1}}.$$

3.4.3. *The estimate for* $\mathring{R}_{q+1}$. With the new Reynolds stress $\mathring{R}_{q+1}$ established, we shall see that it satisfies the estimate (2.7) at level $q+1$ by individually estimating the three errors mentioned above.

First, we consider the transport error R_{tr}. Using the definition of $w_{q+1}^{(p)}$ in (3.41), and the Lie-advection identity (3.43) we rewrite the transport stress as

$$\mathcal{R}\Big(D_{t,q}w_{q+1}^{(p)}\Big)=\sum_i\sum_{\xi\in\Lambda^i}\mathcal{R}\Big(A_{(\xi,i)}(\nabla\overline{v}_q)^T(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\Big)+\mathcal{R}\Big((D_{t,q}A_{(\xi,i)})(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)\Big).$$

By the estimate of the derivatives of $A_{(\xi,i)}$, $\overline{v}_q$, and $(\nabla\Phi_i)^{-1}$ in (3.39), (3.20) and (3.34) respectively we have

$$\|A_{(\xi,i)}(\nabla\overline{v}_q)^T(\nabla\Phi_i)^{-1}\|_{C^{m+\alpha}}\lesssim\delta_{q+1}^{1/2}\tau_q^{-1}l^{-m-\alpha},\ \ \|(D_{t,q}A_{(\xi,i)})(\nabla\Phi_i)^{-1}\|_{C^{m+\alpha}}\lesssim\delta_{q+1}^{1/2}\tau_q^{-1}l^{-m-\alpha}.$$

Then by (3.74) we obtain

$$\left\|\mathcal{R}(D_{t,q}w_{q+1}^{(p)})\right\|_{C^\alpha}\lesssim\frac{\delta_{q+1}^{1/2}\tau_q^{-1}}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{l^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right)\lesssim\frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}},$$

where we used the fact that $(l\lambda_{q+1})^{-1}\leqslant\lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and the last inequality was obtained by taking the parameter m sufficiently large (in terms of $\tilde{\beta}$ and b). Here and in the following the sum over i is finite, since by the definition the amplitude functions have disjoint supports for distinct i

For the incompressibility corrector defined in (3.45), since Φ_i obeys (3.32), we have

$$D_{t,q}w_{q+1}^{(c)}=\sum_i\sum_{\xi\in\Lambda^i}D_{t,q}\nabla A_{(\xi,i)}\times\left((\nabla\Phi_i)^TV_{(\xi)}(\Phi_i)\right)+\sum_i\sum_{\xi\in\Lambda^i}\nabla A_{(\xi,i)}\times\left((D_{t,q}\nabla\Phi_i)^TV_{(\xi)}(\Phi_i)\right).$$

By the estimates for $A_{(\xi,i)}$, $\overline{v}_q$ in (3.39), (3.20) respectively, we have

$$\begin{aligned}\|D_{t,q}\nabla A_{(\xi,i)}\|_{C^{m+\alpha}} &\lesssim \|D_{t,q}A_{(\xi,i)}\|_{C^{m+1+\alpha}} + \|\overline{v}_q\|_{C^{1+\alpha}}\|A_{(\xi,i)}\|_{C^{m+1+\alpha}} + \|\overline{v}_q\|_{C^{m+1+\alpha}}\|A_{(\xi,i)}\|_{C^{1+\alpha}} \\ &\lesssim \delta_{q+1}^{1/2}\tau_q^{-1}l^{-1-m-\alpha},\end{aligned}$$

which together with the estimate for $(\nabla\Phi_i)^T$ in (3.34) implies that

$$\|D_{t,q}\nabla A_{(\xi,i)} \times \left((\nabla\Phi_i)^T\cdot\right)\|_{C^{m+\alpha}} + \|\nabla A_{(\xi,i)} \times \left((D_{t,q}\nabla\Phi_i)^T\cdot\right)\|_{C^{m+\alpha}} \lesssim \delta_{q+1}^{1/2}\tau_q^{-1}l^{-1-m-\alpha},$$

while we gain a factor λ_{q+1}^{-1} from $V_{(\xi)}$ compared with $W_{(\xi)}$. Then by applying (3.74) we have

$$\left\|\mathcal{R}(D_{t,q}w_{q+1}^{(c)})\right\|_{C^\alpha} \lesssim \frac{\delta_{q+1}^{1/2}\tau_q^{-1}}{\lambda_{q+1}^{1-\alpha}}\frac{l^{-1}}{\lambda_{q+1}}\left(1+\frac{l^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right) \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}},$$

where we used the fact that $(l\lambda_{q+1})^{-1} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and the last inequality was obtained by taking the parameter m sufficiently large.

Then compared to $\mathcal{R}(D_{t,q}w_{q+1}^{(p)})$ and $\mathcal{R}(D_{t,q}w_{q+1}^{(c)})$, we obverse that the definition of $\mathcal{R}(D_{t,q}\overline{w}_{q+1}^{(p)})$ and $\mathcal{R}(D_{t,q}\overline{w}_{q+1}^{(c)})$ can be obtained by replacing $\Lambda^i, A_{(\xi,i)}$ by $\overline{\Lambda}^i, a_{(\xi,i)}$ respectively. Then by the estimate of $a_{(\xi,i)}$ in (3.59) we have the same bound as above:

$$\left\|\mathcal{R}(D_{t,q}\overline{w}_{q+1}^{(p)})\right\|_{C^\alpha} + \left\|\mathcal{R}(D_{t,q}\overline{w}_{q+1}^{(c)})\right\|_{C^\alpha} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}}.$$

To address the oscillation error R_{osc}, we begin by bounding $R_{osc,1}$, and the second term $R_{osc,2}$ can be bounded by a similar argument. We recall the identity

$$\nabla(\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2))(\Phi_i) = (\nabla\Phi_i)^T\Big((\nabla\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2))(\Phi_i)\Big).$$

Therefore, we have

$$\begin{aligned}&\operatorname{div}\left(A_{(\xi,i)}^2(\nabla\Phi_i)^{-1}\left((\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(W_{(\xi)}\otimes W_{(\xi)}))(\Phi_i)\right)(\nabla\Phi_i)^{-T}\right) \\ &= \operatorname{div}\left(A_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\left((\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2))(\Phi_i)\right)\right) \\ &= \left((\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2))(\Phi_i)\right)\operatorname{div}\left(A_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}\right) \\ &\quad + A_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T}(\nabla\Phi_i)^T\left((\nabla\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2))(\Phi_i)\right),\end{aligned}$$

where the last term equals to 0 since $(\xi\cdot\nabla)\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{2}}(\phi_{(\xi)}^2) = 0$. By the estimate of the derivatives of $A_{(\xi,i)}$ in (3.39) and $(\nabla\Phi_i)^{-1}$ in (3.34) we have

$$\|\operatorname{div}(A_{(\xi,i)}^2(\nabla\Phi_i)^{-1}(\xi\otimes\xi)(\nabla\Phi_i)^{-T})\|_{C^{m+\alpha}} \lesssim \delta_{q+1}l^{-1-m-\alpha}.$$

Then we apply (3.74) to obtain

$$\|R_{osc,1}\|_{C^\alpha} \lesssim \frac{\delta_{q+1}l^{-1}}{\lambda_{q+1}^{1-\alpha}}\left(1+\frac{l^{-m-\alpha}}{\lambda_{q+1}^{m-1}}\right) \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}},$$

where we used the fact that $(l\lambda_{q+1})^{-1} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and then choose m large enough.

By replacing Λ^i, $A_{(\xi,i)}$ with $\overline{\Lambda}^i$, $a_{(\xi,i)}$ and applying a similar argument as before, we obtain

$$\|R_{osc,2}\|_{C^\alpha} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}}.$$

The last term $R_{osc,3}$ is estimated by using the bounds for the perturbations in (3.66)-(3.71) and the definition of l in (3.2) as

$$\|R_{osc,3}\|_{C^\alpha} \lesssim \|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}\|_{C^\alpha}\left(\|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}\|_{C^\alpha} + \|w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}\|_{C^\alpha}\right) \lesssim \frac{\delta_{q+1}l^{-1}}{\lambda_{q+1}^{1-2\alpha}} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-4\alpha}}.$$

In the end, we only need to estimate the Nash error $R_{Nash} = \mathcal{R}(w_{q+1}\cdot\nabla\overline{v}_q) + \mathcal{R}(\overline{w}_{q+1}\cdot\nabla\overline{v}_q)$. For the first term, due to the definition of the perturbation w_{q+1}, we have

$$\begin{aligned}\mathcal{R}(w_{q+1}\cdot\nabla\overline{v}_q) = &\sum_i\sum_{\xi\in\Lambda^i}\mathcal{R}((\nabla\overline{v}_q)^T A_{(\xi,i)}(\nabla\Phi_i)^{-1}W_{(\xi)}(\Phi_i)) \\ &+ \mathcal{R}\left((\nabla\overline{v}_q)^T\nabla A_{(\xi,i)}\times((\nabla\Phi_i)^T V_{(\xi)}(\Phi_i)\right).\end{aligned}$$

By the estimate of the derivatives of $A_{(\xi,i)}$, $\nabla\overline{v}_q$, and $(\nabla\Phi_i)^{-1}$ in (3.39), (3.20) and (3.34) respectively, we have

$$\|(\nabla\overline{v}_q)^T A_{(\xi,i)}(\nabla\Phi_i)^{-1}\|_{C^{m+\alpha}} + l\|(\nabla\overline{v}_q)^T\nabla A_{(\xi,i)}\times((\nabla\Phi_i)^T\cdot)\|_{C^{m+\alpha}} \lesssim \delta_{q+1}^{1/2}\tau_q^{-1}l^{-m-\alpha}.$$

Then we use (3.74) to show that

$$\|\mathcal{R}(w_{q+1}\cdot\nabla\overline{v}_q)\|_{C^0} \lesssim \frac{\delta_{q+1}^{1/2}\tau_q^{-1}}{\lambda_{q+1}^{1-\alpha}}\left(1 + \frac{l^{-1}}{\lambda_{q+1}} + \frac{l^{-1-m-\alpha}}{\lambda_{q+1}^m}\right) \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}},$$

where we used the fact that $(l\lambda_{q+1})^{-1} \leqslant \lambda_q^{-\frac{(b-1)(1-\tilde{\beta})}{2}}$ by (3.69), and the last inequality was obtained by taking the parameter m sufficiently large.

Then the expression of $\mathcal{R}(\overline{w}_{q+1}\cdot\nabla\overline{v}_q)$ is similar to $\mathcal{R}(w_{q+1}\cdot\nabla\overline{v}_q)$ with Λ^i, $A_{(\xi,i)}$ replaced by $\overline{\Lambda}^i$, $a_{(\xi,i)}$ respectively. Then by the estimate of $a_{(\xi,i)}$ in (3.59) we deduce

$$\|\mathcal{R}(\overline{w}_{q+1}\cdot\nabla\overline{v}_q)\|_{C^\alpha} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-3\alpha}}.$$

By all the bounds above together with (3.1) we have

$$\left\|\mathring{R}_{q+1}\right\|_{C^\alpha} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-4\alpha}} \leqslant \frac{\delta_{q+2}}{\lambda_{q+1}^{3\alpha}},$$

where the extra power of $\lambda_{q+1}^{-\alpha}$ is used to absorb the implicit constant by choosing a sufficiently large.

3.5. **Estimates on the energy.** To conclude the proof of Proposition 2.1, we have to check that the iterative condition on the energy (2.9) holds at the level $q+1$. To this end, by the definition of v_{q+1} in (3.64), we first have

$$\begin{aligned}e(t) - \frac{\delta_{q+2}}{2} - \|v_{q+1}(t)\|_{L^2}^2 = e(t) - \frac{\delta_{q+2}}{2} - \|\overline{v}_q(t)\|_{L^2}^2 - \int_{\mathbb{T}^3}|w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}|^2(t)\mathrm{d}x \\ - \int_{\mathbb{T}^3}\left[|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}|^2 + 2(w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)})\cdot(w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)})\right](t)\mathrm{d}x\end{aligned}$$

$$- \int_{\mathbb{T}^3} 2\overline{v}_q \cdot (w_{q+1} + \overline{w}_{q+1})(t)\mathrm{d}x. \tag{3.76}$$

For the integrand in the first line on the right hand side, by taking the trace on both sides of (3.61) and using the fact that $\mathring{\overline{R}}_q$ is traceless, we deduce that

$$\begin{aligned}
&|w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}|^2 \\
&\quad = 3\sum_i \Upsilon_{q,i} + \mathrm{tr} R_q^{(1)} + \sum_i \sum_{\xi \in \Lambda^i} \mathrm{tr}\left[A_{(\xi,i)}^2 (\nabla\Phi_i)^{-1} \Big(\left(\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right)(\Phi_i)\Big)(\nabla\Phi_i)^{-T}\right] \\
&\qquad + \sum_i \sum_{\xi \in \overline{\Lambda}^i} \mathrm{tr}\left[a_{(\xi,i)}^2 (\nabla\Phi_i)^{-1} \Big(\Big(\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\Big)(\Phi_i)\Big)(\nabla\Phi_i)^{-T}\right].
\end{aligned}$$

By integrating on both sides, together with the fact that $\sum_i \int_{\mathbb{T}^3} \Upsilon_{q,i}\mathrm{d}x = \Upsilon_q$, the definitions in (3.50), (3.51), and the definition of the building blocks we obtain for $t \in [0,1]$

$$\begin{aligned}
\int_{\mathbb{T}^3} |(w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)})(t)|^2 \mathrm{d}x &= e(t) - \frac{\delta_{q+2}}{2} - \|\overline{v}_q(t)\|_{L^2}^2 \\
&\quad + \sum_i \sum_{\xi \in \Lambda^i} \int_{\mathbb{T}^3} \mathrm{tr}\left[A_{(\xi,i)}^2 (\nabla\Phi_i)^{-1} \xi \otimes \xi (\nabla\Phi_i)^{-T}\right] \Big((\mathbb{P}_{\neq 0}(\phi_{(\xi)}^2))(\Phi_i)\Big)(t)\mathrm{d}x \\
&\quad + \sum_i \sum_{\xi \in \overline{\Lambda}^i} \int_{\mathbb{T}^3} \mathrm{tr}\left[a_{(\xi,i)}^2 (\nabla\Phi_i)^{-1} \xi \otimes \xi (\nabla\Phi_i)^{-T}\right] \Big((\mathbb{P}_{\neq 0}(\phi_{(\xi)}^2))(\Phi_i)\Big)(t)\mathrm{d}x.
\end{aligned}$$

Then we use the estimates for $a_{(\xi,i)}$, $A_{(\xi,i)}$, and $(\nabla\Phi_i)^{-1}$ in (3.59), (3.39) and (3.34) respectively, to obtain for any $t \in [0,1]$

$$\|\mathrm{tr}[a_{(\xi,i)}^2 (\nabla\Phi_i)^{-1}(\xi \otimes \xi)(\nabla\Phi_i)^{-T}]\|_{C^m} + \|\mathrm{tr}[A_{(\xi,i)}^2 (\nabla\Phi_i)^{-1}(\xi \otimes \xi)(\nabla\Phi_i)^{-T}]\|_{C^m} \lesssim \delta_{q+1} l^{-m},$$

which together with (A.1) with $N = 1$ implies that

$$\left| e(t) - \frac{\delta_{q+2}}{2} - \|\overline{v}_q(t)\|_{L^2}^2 - \int_{\mathbb{T}^3} |(w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)})(t)|^2 \mathrm{d}x \right| \lesssim \frac{\delta_{q+1} l^{-1}}{\lambda_{q+1}} \lesssim \frac{\delta_{q+1} \tilde{\delta}_q^{1/2} \lambda_q}{\tilde{\delta}_{q+1}^{1/2} \lambda_{q+1}^{1-2\alpha}},$$

where we recall that the sum over i is finite, since by the definition the amplitude functions have disjoint supports for different i.

Then the second term on the right hand side of (3.76) is estimated by the bounds for the perturbations in (3.66)-(3.71). We have for $t \in [0,1]$

$$\begin{aligned}
&\left| \int_{\mathbb{T}^3} \left[|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}|^2 + 2(w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}) \cdot (w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}) \right](t)\mathrm{d}x \right| \\
&\quad \lesssim \|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}\|_{C^0}^2 + \|w_{q+1}^{(p)} + \overline{w}_{q+1}^{(p)}\|_{C^0} \|w_{q+1}^{(c)} + \overline{w}_{q+1}^{(c)}\|_{C^0} \lesssim \frac{\delta_{q+1} l^{-1}}{\lambda_{q+1}} \lesssim \frac{\delta_{q+1} \tilde{\delta}_q^{1/2} \lambda_q}{\tilde{\delta}_{q+1}^{1/2} \lambda_{q+1}^{1-2\alpha}}.
\end{aligned}$$

For the last term on the right hand side of (3.76), we recall that $w_{q+1} + \overline{w}_{q+1}$ can be written as

$$w_{q+1} + \overline{w}_{q+1} = \mathrm{curl}\Big(\sum_i \sum_{\xi \in \Lambda^i} A_{(\xi,i)} (\nabla\Phi_i)^T (V_{(\xi)}(\Phi_i)) + \sum_i \sum_{\xi \in \overline{\Lambda}^i} a_{(\xi,i)} (\nabla\Phi_i)^T (V_{(\xi)}(\Phi_i)) \Big).$$

Then we use integration by parts, the estimates for $\overline{v}_q$ in (3.20), the estimates for $a_{(\xi,i)}$, $A_{(\xi,i)}$, and $(\nabla\Phi_i)^{-1}$ in (3.59), (3.39) and (3.34) respectively, and the bound on the building block in (B.5) to obtain for any

$t \in [0,1]$

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^3} \overline{v}_q \cdot (w_{q+1} + \overline{w}_{q+1})(t)\mathrm{d}x\Big| \\
&\lesssim \Big(\sum_i \sum_{\xi\in\Lambda^i} \|A_{(\xi,i)}\, (\nabla \Phi_i)^T (V_{(\xi)}(\Phi_i))\|_{C^0} + \sum_i \sum_{\xi\in\overline{\Lambda}^i} \|a_{(\xi,i)}\, (\nabla \Phi_i)^T (V_{(\xi)}(\Phi_i))\|_{C^0}\Big) \|\overline{v}_q\|_{C^1} \\
&\lesssim \frac{\delta_{q+1}^{1/2}\tau_q^{-1}}{\lambda_{q+1}} \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-2\alpha}}.
\end{aligned}$$

Combining the above estimates we obtain by (3.1) that

$$\Big| e(t) - \frac{\delta_{q+2}}{2} - \|v_{q+1}(t)\|_{L^2}^2 \Big| \lesssim \frac{\delta_{q+1}\tilde{\delta}_q^{1/2}\lambda_q}{\tilde{\delta}_{q+1}^{1/2}\lambda_{q+1}^{1-2\alpha}} \leqslant \delta_{q+2}\lambda_{q+1}^{-\alpha},$$

where we the extra power of $\lambda_{q+1}^{-\alpha}$ is used to absorb the constant by choosing a large enough. Then we obtain (2.9) at the level $q+1$ by choosing a large enough again:

$$\delta_{q+2}\lambda_{q+1}^{-\alpha/3} \leqslant \frac{\delta_{q+2}}{2} - \delta_{q+2}\lambda_{q+1}^{-\alpha} \leqslant e(t) - \|v_{q+1}(t)\|_{L^2}^2 \leqslant \frac{\delta_{q+2}}{2} + \delta_{q+2}\lambda_{q+1}^{-\alpha} \leqslant \delta_{q+2}.$$

This completes the proof of Proposition 2.1.

4. Construction of non-unique solutions in $L_t^1 W^{1,s}$ scales

In this section, our primary objective is to establish the non-uniqueness of stochastic Lagrangian trajectories for solutions to the Navier-Stokes or Euler equations that possess a specific level of Sobolev regularity, as stated in Theorem 1.7. By applying the superposition principle, it suffices to demonstrate the non-uniqueness of the corresponding Fokker-Planck equations, which is exactly the claim of Theorem 1.10. More precisely, for any triple $(p,r,s)\in\mathcal{A}$, we construct a solution $v\in L_t^rL^p\cap L_t^2L^2\cap L_t^1W^{1,s}\cap C_tL^1$ to the Navier-Stokes equations (1.5) such that the related advection-diffusion equations (1.7) admit two positive solutions in the space $L_t^rL^p\cap L_t^2L^2\cap C_tL^1$ with initial data $\rho_0=1$. To achieve this, we apply the convex integration method again. However, in this section, in contrast to Section 2, we will add more intermittency in the building blocks in both the spatial and temporal directions, and handle the advection-diffusion equations (1.7) and the Navier-Stokes equations (1.5) at different frequency scales.

Without loss of generality, we set $T=1$. The convex integration iteration is indexed by a parameter $q\in\mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}_0}\subset\mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q\in\mathbb{N}_0}\subset(0,1]$ which is decreasing to 0. Let

$$\lambda_q = a^{(b^q)}, q\geqslant 0, \quad \delta_q = \frac{1}{48^2}\lambda_2^{2\beta}\lambda_q^{-2\beta}, q\geqslant 2,\ \ \delta_0=1,\ \ \delta_1=\frac{1}{48^2}.$$

where $\beta>0$ will be chosen sufficiently small and a,b will be chosen sufficiently large. In addition, we use that $\sum_{q\geqslant 1}\delta_q^{1/2}\leqslant \frac{1}{48}(1+\sum_{q\geqslant 2}a^{(2-q)b\beta})\leqslant \frac{1}{48}(1+\frac{1}{1-a^{-b\beta}})<\frac{1}{16}$ which boils down to

$$a^{b\beta} > 2, \tag{4.1}$$

assumed from now on.

At each step q, a pair $(v_q,\rho_q,\mathring{R}_q,M_q)$ is constructed solving the following system on $[0,1]$:

$$\begin{aligned}
\partial_t\rho_q - \kappa\Delta\rho_q + \mathrm{div}(v_q\rho_q) &= -\mathrm{div}M_q, \\
\partial_t v_q + \mathrm{div}(v_q\otimes v_q) - \nu\Delta v_q + \nabla\pi_q &= \mathrm{div}\mathring{R}_q,\ \ \mathrm{div}v_q=0.
\end{aligned}\tag{4.2}$$

where $\kappa, \nu \in [0,1]$, $\mathring{R}_q$ is a trace-free symmetric matrix and M_q is some vector field.

As before, to handle the initial condition, we let $T_q := \frac{1}{3} - \sum_{1\leqslant r\leqslant q} \delta_r^{1/2} \in (0, \frac{1}{3}]$. Here we define $\sum_{1\leqslant r\leqslant 0} := 0$.

Under the above assumptions, our main iteration reads as follows:

Proposition 4.1. *Let $d \geqslant 2$. For any triple $(p,r,s) \in \mathcal{A}$, there exists a choice of parameters a, b, β such that the following holds: Let $(v_q, \rho_q, \mathring{R}_q, M_q)$ be a solution to (4.2) satisfying $\int \rho_q \mathrm{d}x = 1$,*

$$\|v_q\|_{L_t^2 L^2} \leqslant C_v C_0 \sum_{m=1}^{q} \delta_m^{1/2}, \quad \|\rho_q\|_{L_t^2 L^2} \leqslant C_\rho C_0 \sum_{m=0}^{q+1} \delta_m^{1/2} \tag{4.3}$$

for some universal constants $C_0, C_v, C_\rho \geqslant 1$, and

$$\|v_q\|_{C^2_{t,x}} \leqslant C_0 \lambda_q^{4d+3}, \quad \|\rho_q\|_{C^1_{t,x}} \leqslant C_0 \lambda_q^{3d+2}, \tag{4.4}$$

$$\|\mathring{R}_q\|_{L_t^1 L^1} \leqslant C_0^2 \delta_{q+1}, \quad \|M_q\|_{L_t^1 L^1} \leqslant C_0^2 \delta_{q+2}^2, \tag{4.5}$$

$$\rho_q - 1 = v_q = \mathring{R}_q = M_q = 0 \text{ on } [0, T_q]. \tag{4.6}$$

Then there exists $(v_{q+1}, \rho_{q+1}, \mathring{R}_{q+1}, M_{q+1})$ which solves (4.2) and satisfies (4.3)-(4.6) at the level $q+1$ and

$$\|v_{q+1} - v_q\|_{L_t^2 L^2} \leqslant C_v C_0 \delta_{q+1}^{1/2}, \quad \|\rho_{q+1} - \rho_q\|_{L_t^2 L^2} \leqslant C_\rho C_0 \delta_{q+2}^{1/2}. \tag{4.7}$$

Moreover,

$$\|v_{q+1} - v_q\|_{L_t^r L^p} \leqslant \delta_{q+1}^{1/2}, \quad \|v_{q+1} - v_q\|_{C_t L^1} \leqslant \delta_{q+1}^{1/2}, \quad \|v_{q+1} - v_q\|_{L_t^1 W^{1,s}} \leqslant \delta_{q+1}^{1/2}, \tag{4.8}$$

$$\|\rho_{q+1} - \rho_q\|_{L_t^r L^p} \leqslant \delta_{q+2}^{1/2}, \quad \|\rho_{q+1} - \rho_q\|_{C_t L^1} \leqslant \delta_{q+2}^{1/2}, \quad \inf_{t\in[0,1]} (\rho_{q+1} - \rho_q) \geqslant -\delta_{q+2}^{1/2}. \tag{4.9}$$

Here C_0 is determined by the choice of the starting iterations, and C_v, C_ρ are two constants determined by the generalized Holder inequality for v_q, ρ_q respectively, and other implicit geometrical constants in the proof.

Here we remark that all the parameters are independent of the choices of κ and ν, the extra power in the bound on M_q in (4.5) is used to absorb the universal constant to avoid exponential explosion during the iterative process, see (5.20) for the details.

The proof of Proposition 4.1 is presented in Section 5 below. With Proposition 4.1 in hand, the proof of Theorem 1.10 follows by a similar argument as for Theorem 1.3.

Proof of Theorem 1.10. For any triple $(p,r,s) \in \mathcal{A}$, without loss of generality we assume that $p > 1$ and $T = 1$. As before, we intend to start the iteration from

$$\rho_0(t,x) = 1 + \frac{\sin \pi x_1}{4} \chi_0(t), \ v_0 = 0, \ \mathring{R}_0 = 0, \ M_0(t,x) = (\partial_t \chi_0(t) + \kappa \chi_0(t) \pi^2) \frac{\cos \pi x_1}{4\pi} (1, 0, ..., 0).$$

where $x = (x_1, ..., x_d)$, χ_0 is a smooth function with $\chi_0(t) = 0$ on $[0, \frac{1}{3}]$, $\chi_0(t) = 1$ on $[\frac{2}{3}, 1]$. By choosing C_0 large enough, we have for $\kappa \in [0,1]$

$$\|\rho_0\|_{L_t^2 L^2} + \|\rho_0\|_{C^1_{t,x}} \lesssim 1 \leqslant C_0, \quad \|M_0\|_{L_t^1 L^1} \lesssim 1 \leqslant \frac{1}{48^4} C_0^2.$$

Then (4.3)-(4.6) are satisfied.

Next, we use Proposition 4.1 to build inductively $(v_q, \rho_q, \mathring{R}_q, M_q)$ for every $q \geqslant 1$. By (4.7)-(4.9), the sequence $\{v_q\}_{q\in\mathbb{N}}$ is Cauchy in

$$C([0,1]; L^1) \cap L^r([0,1]; L^p) \cap L^2([0,1] \times \mathbb{T}^d) \cap L^1([0,1]; W^{1,s})$$

and the sequence $\{\rho_q\}_{q\in\mathbb{N}}$ is Cauchy in

$$L^r([0,1];L^p)\cap L^2([0,1]\times\mathbb{T}^d)\cap C([0,1];L^1).$$

We then denote by (v,ρ) the limit, where v is also divergence-free. For the case $r=\infty$, we have $v,\rho\in C([0,1];L^p)$. Clearly by (4.5), (ρ,v) solves (1.7) and (1.5). Then from the fact $\int\rho_q\mathrm{d}x=1$ we deduce that $\int\rho\mathrm{d}x=1$. (4.6) ensures that $\rho(t)\equiv 1$ for every t sufficiently close to 0.

Moreover, ρ is non-negative on $\mathbb{T}^d$ by (4.1) and (4.9):

$$\inf_{t\in[0,1]}\rho\geqslant\inf_{t\in[0,1]}\rho_0+\sum_{q=0}^{\infty}\inf_{t\in[0,1]}(\rho_{q+1}-\rho_q)\geqslant\frac{3}{4}-\sum_{q=0}^{\infty}\delta_{q+1}^{1/2}\geqslant\frac{1}{2},$$

and ρ does not coincide with the solution which is constantly equal to 1, since by (4.1) and (4.9)

$$\|\rho-1\|_{C_tL^1}\geqslant\|1-\rho_0\|_{C_tL^1}-\sum_{q=0}^{\infty}\|\rho_{q+1}-\rho_q\|_{C_tL^1}\geqslant\frac{1}{16}-\sum_{q=0}^{\infty}\delta_{q+1}^{1/2}>0.$$

□

Then the non-uniqueness of stochastic Lagrangian trajectories follows from Theorem 1.10 and the superposition principle.

Proof of Theorem 1.7. We only prove the case where $\kappa>0$, while the case $\kappa=0$ follows similarly and more easily. For any triple $(p,r,s)\in\mathcal{A}$, without loss of generality we assume that $p>1$ and $T=1$. By Theorem 1.10, there exists $v\in L^r([0,1];L^p)\cap L^2([0,1]\times\mathbb{T}^d)\cap L^1([0,1];W^{1,s})\cap C([0,1];L^1)$ and a non-constant positive density $\rho\in L^r([0,1];L^p)\cap L^2([0,1]\times\mathbb{T}^d)\cap C([0,1];L^1)$ satisfying (1.7) and (1.5). If $r=\infty$, we have additionally $v\in C([0,1];L^p)$.

Moreover, by (4.4), (4.7) and interpolation we conclude that $v\in L^{2(1+\epsilon)}([0,1]\times\mathbb{T}^d)$ for some $\epsilon>0$ small enough. Then, it follows that

$$\int_0^1\int_{\mathbb{T}^d}|v(s,x)|^{1+\epsilon}\rho(s,x)\mathrm{d}x\mathrm{d}s\leqslant\|v\|_{L^{2+2\epsilon}_{t,x}}^{1+\epsilon}\|\rho\|_{L^2_{t,x}}<\infty,$$

and that $t\to\rho(t,x)\mathrm{d}x$ is weakly continuous on $[0,1]$ since $\rho\in C([0,1];L^1)$. Using the superposition principle (see [Tre14, Section 7.2]) for (1.7), there exists a probability measure $\mathbf{Q}$ on $C([0,1];\mathbb{T}^d)$ equipped with its Borel σ-algebra and its natural filtration generated by the canonical process $\Pi_t, t\in[0,1]$, defined by

$$\Pi_t(\omega):=\omega(t),\ \ \omega\in C([0,1];\mathbb{T}^d),$$

which is a martingale solution associated to diffusion operator

$$L:=\kappa\Delta+v\cdot\nabla.$$

More precisely, for every smooth function f on $\mathbb{T}^d$, the process

$$f(\Pi_t)-f(\Pi_0)-\int_0^t Lf(\Pi_s)\mathrm{d}s$$

is a $\mathbf{Q}$-martingale with respect to the natural filtration with the initial law $\mathbf{Q}\circ\Pi_0^{-1}=\mathcal{L}^d$. Here $\mathcal{L}^d$ denotes the Lebesgue measure on the torus. Moreover, for every $t\in[0,1]$, it holds that $\rho(t)\mathcal{L}^d=\mathbf{Q}\circ\Pi_t^{-1}$. Since $\overline{\rho}\equiv 1$ is also a solution satisfying all the above conditions, we have another martingale solution $\overline{\mathbf{Q}}$ satisfying $\mathcal{L}^d=\overline{\mathbf{Q}}\circ\Pi_t^{-1}$.

Now we define $\{\mathbf{Q}^x\}_{x\in\mathbb{T}^d}, \{\overline{\mathbf{Q}}^x\}_{x\in\mathbb{T}^d}$ as the regular conditional probabilities with respect to Π_0. Then for a.e. $x \in \mathbb{T}^d$, $\mathbf{Q}^x, \overline{\mathbf{Q}}^x$ are both martingale solutions to (1.4) with initial condition x. We define

$$A(v) := \{x \in \mathbb{T}^d : \mathbf{Q}^x, \overline{\mathbf{Q}}^x \text{ are two distinct martingale solutions associated to } L\}.$$

Then we prove that the Lebesgue measure of $A(v)$ is positive. We assume by contradiction that the Lebesgue measure is zero. In this case, we have $\mathbf{Q}^x = \overline{\mathbf{Q}}^x$ for a.e. $x \in \mathbb{T}^d$. Consequently, we have for any smooth function f and $t \in [0, 1]$

$$\int_{\mathbb{T}^d} f\rho(t)\mathrm{d}x = \int f(\Pi_t)\mathrm{d}\mathbf{Q} = \int_{\mathbb{T}^d}\int f(\Pi_t)\mathrm{d}\mathbf{Q}^x\mathrm{d}x = \int_{\mathbb{T}^d}\int f(\Pi_t)\mathrm{d}\overline{\mathbf{Q}}^x\mathrm{d}x = \int_{\mathbb{T}^d} f\mathrm{d}x,$$

which leads to a contradiction as ρ is non-constant.

Since $|v|, \rho \in L^2([0, 1] \times \mathbb{T}^d)$, we have

$$\int_{\mathbb{T}^d} \mathbf{E}^x \int_0^1 |v(s,\Pi_s)|\mathrm{d}s\mathrm{d}x = \int\int_0^1 |v(s,\Pi_s)|\mathrm{d}s\mathrm{d}\mathbf{Q} = \int_0^1\int_{\mathbb{T}^d} |v(s,x)|\rho(s,x)\mathrm{d}x\mathrm{d}s < \infty,$$

which implies that $\mathbf{E}^x[\int_0^1 |v(s,\Pi_s)|\mathrm{d}s] < \infty$ for a.e. $x \in \mathbb{T}^d$. Similarly, we have $\overline{\mathbf{E}}^x[\int_0^1 |v(s,\Pi_s)|\mathrm{d}s] < \infty$ for a.e. $x \in \mathbb{T}^d$.

□

5. Proof of Proposition 4.1

The proof is also based on convex integration schemes. Compared with Section 3, here we use two distinct scales for two equations during the iteration. We begin by fixing the parameters and then proceed with a mollification step in Section 5.1. Section 5.2 presents the perturbation construction of $(w_{q+1}+\overline{w}_{q+1}, \theta_{q+1})$ and the new iteration (v_{q+1}, ρ_{q+1}). Here, the perturbation (w_{q+1}, θ_{q+1}) is designed to cancel the stress term M_q in the advection-diffusion equations, while the perturbation w_{q+1} is to cancel the stress term $\mathring{R}_q$ in the fluid equations. Here, unlike the previous case, the term $w_{q+1} \otimes w_{q+1}$ is automatically small in L^1 space due to the choice of smaller scales for the advection-diffusion equation. We emphasize that the method employed in Section 3 can not be applied directly in this context, since the additional intermittency in the building blocks introduces extra oscillation errors (see Section 1.4.2 for more explanation). Then we establish the inductive estimates. Finally, in Section 5.3, we define the new stress components $(\mathring{R}_{q+1}, M_{q+1})$ and establish the inductive estimates respectively.

5.1. **Choice of parameters and mollification.** In the sequel, additional parameters will be indispensable and their value have to be carefully chosen to respect all the compatibility conditions appearing in the estimates below. First, for a sufficiently small $\alpha \in (0, 1)$ to be chosen, we take $l := \lambda_{q+1}^{-\frac{3\alpha}{2}}\lambda_q^{-2d-\frac{3}{2}}$ and have

$$l^{-1} \leqslant \lambda_{q+1}^{2\alpha}, \quad l\lambda_q^{4d+3} \ll \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+3}^2, \quad \lambda_q^{4d+3} \leqslant \lambda_{q+1}^{\alpha} \tag{5.1}$$

provided $\alpha b \geqslant 4d + 3, \alpha > 4\beta b^2$.

For fixed $d \geqslant 2$ and $(p, r, s) \in \mathcal{A}$, without loss of generality, we assume $p > 1$. Then we introduce a large constant $N := N(p, r, s, d)$ to be chosen in Lemma 5.1 below. In the sequel, we also need

$$\alpha b \geqslant 4d + 3, \quad \alpha > 4\beta b^2, \quad (12d + 43)\alpha < \frac{1}{2N}.$$

The above can be obtained by choosing $\alpha > 0$ small such that $(12d+43)\alpha < \frac{1}{2N}$, and choosing $b \in \mathbb{N}$ large enough such that $b > \frac{4d+3}{\alpha}$, and finally choosing $\beta > 0$ small such that $\alpha > 4\beta b^2$.

Then we increase a such that (4.1) holds. In the sequel, we also increase a to absorb various implicit and universal constants in the subsequent estimates.

Now we replace (v_q,ρ_q) by a mollified field (v_l,ρ_l), and define

$$v_l=(v_q*_x\phi_l)*_t\varphi_l,\ \ \rho_l=(\rho_q*_x\phi_l)*_t\varphi_l,$$

where $\phi_l:=\frac{1}{l^d}\phi(\frac{\cdot}{l})$ is a family of standard radial mollifiers on $\mathbb{R}^d$, and $\varphi_l:=\frac{1}{l}\varphi(\frac{\cdot}{l})$ is a family of standard radial mollifiers with support in $(0,1)$. For the mollification around $t=0$, since $v_q,\rho_q,\mathring{R}_q$ and M_q are constants around $t=0$, see (4.6), we can directly extend these definitions to $t\leqslant 0$ by their values at $t=0$.

By straightforward calculations and (4.2) we obtain

$$\begin{aligned}\partial_t\rho_l-\kappa\Delta\rho_l+\mathrm{div}(v_l\rho_l)&=-\mathrm{div}M_l,\\ \partial_t v_l-\nu\Delta v_l+\mathrm{div}(v_l\otimes v_l)+\nabla\pi_l&=\mathrm{div}\mathring{R}_l,\ \ \mathrm{div}v_l=0\end{aligned}\tag{5.2}$$

for some suitable π_l, where

$$\begin{aligned}M_l&:=(M_q*_x\phi_l)*_t\varphi_l-v_l\rho_l+(v_q\rho_q)*_x\phi_l*_t\varphi_l,\\ \mathring{R}_l&:=(\mathring{R}_q*_x\phi_l)*_t\varphi_l+v_l\mathring{\otimes}v_l-(v_q\mathring{\otimes}v_q)*_x\phi_l*_t\varphi_l.\end{aligned}\tag{5.3}$$

Moreover, since $l\leqslant\frac12\delta_{q+1}^{1/2}=\frac{T_q-T_{q+1}}{2}$, by (4.6) we know that $\rho_l-1=v_l=\mathring{R}_l=M_l=0$ on $[0,\frac{T_q+T_{q+1}}{2}]$.

Then by the mollification estimates in Lemma A.6, the space-time embedding $W^{d+1+\epsilon,1}\subset L^\infty$ and the bounds (4.4), (4.5), (5.1) we obtain

$$\|\mathring{R}_l\|_{L^1_tL^1}\leqslant\|\mathring{R}_q\|_{L^1_tL^1}+Cl^2\|v_q\|^2_{C^1_{t,x}}\leqslant C_0^2\delta_{q+1}+CC_0^2l^2\lambda_q^{8d+6}\leqslant 2C_0^2\delta_{q+1},\tag{5.4}$$

$$\|M_l\|_{L^1_tL^1}\leqslant\|M_q\|_{L^1_tL^1}+Cl^2\|v_q\|_{C^1_{t,x}}\|\rho_q\|_{C^1_{t,x}}\leqslant C_0^2\delta_{q+2}^2+CC_0^2l^2\lambda_q^{7d+5}\leqslant 2C_0^2\delta_{q+2}^2,\tag{5.5}$$

and for $N\geqslant 0$,

$$\|\mathring{R}_l\|_{C^N_{t,x}}\lesssim l^{-d-1-\epsilon-N}\|\mathring{R}_q\|_{L^1_tL^1}+l^{2-N}\|v_q\|^2_{C^1_{t,x}}\lesssim l^{-d-1-\epsilon-N}+l^{2-N}\lambda_q^{8d+6}\lesssim l^{-d-2-N},\tag{5.6}$$

$$\|M_l\|_{C^N_{t,x}}\lesssim l^{-d-1-\epsilon-N}\|M_q\|_{L^1_tL^1}+l^{2-N}\|v_q\|_{C^1_{t,x}}\|\rho_q\|_{C^1_{t,x}}\lesssim l^{-d-1-\epsilon-N}+l^{2-N}\lambda_q^{7d+5}\lesssim l^{-d-2-N},\tag{5.7}$$

where we use l^{-1} to absorb the implicit constants by choosing a large enough.

5.2. **The construction of perturbations and inductive estimates.** As outlined in Section 1.4, we proceed with the construction of the perturbation $(w_{q+1}+\overline{w}_{q+1},\theta_{q+1})$, ensuring that the supports of (w_{q+1},θ_{q+1}) and $\overline{w}_{q+1}$ are disjoint. The perturbation (w_{q+1},θ_{q+1}) is designed to cancel the stress terms M_q. At the same time, the perturbations w_{q+1} are exclusively utilized to cancel the stress terms $\mathring{R}_q$. In contrast to the previous case, this extra product $w_{q+1}\otimes w_{q+1}$ is already sufficiently small since we use two distinct scales for two equations during the iteration (see the inductive condition (4.5)).

5.2.1. *Construction of w_{q+1} and θ_{q+1}.* Let us now proceed with the construction of the perturbation (w_{q+1},θ_{q+1}) by employing the building blocks and temporal jets introduced in Section C.1.

For given p,r,s and d, we choose the parameters $\lambda,r_\perp,r_\parallel,\eta$ as follows:

Lemma 5.1. *Let $d\geqslant 2$, $(p,r,s)\in\mathcal{A}$ and $p>1$. There exists a choice of parameters $\lambda,r_\perp,r_\parallel,\eta$ and $N=N(p,r,s,d)\in\mathbb{N}$, such that*

$$\lambda^{-1}\ll r_\perp\ll r_\parallel\ll 1,\ \ \eta^{-1}\leqslant\lambda^d,\tag{5.8}$$

and

$$\max\{\frac{r_\perp}{r_\parallel},\ \ r_\perp^{-1}\lambda^{-1},\ \ \lambda r_\perp^{\frac{d-1}{s}-\frac{d-1}{2}}r_\parallel^{\frac1s-\frac12}\eta^{\frac12},\ \ r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\eta^{-\frac12},\ \ r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}}r_\parallel^{\frac1p-\frac12}\eta^{\frac1r-\frac12}\}\leqslant\lambda^{-\frac1N}.\tag{5.9}$$

Proof. For any $(p,r,s)\in\mathcal{A}$ and $p>1$, it is easy to see that there exists $k\in\mathbb{Q}, k>1$ such that

$$\frac{1}{r}-\frac{1}{2}+k(\frac{1}{p}-\frac{1}{2})>0,\ \ \frac{1}{d}<\frac{1}{2k}-\frac{1}{2}+\frac{1}{s}.$$

Then there exists $M\in\mathbb{N}$ such that $\frac{1}{r}-\frac{1}{2}+k(\frac{1}{p}-\frac{1}{2})>\frac{1}{M}, \frac{1}{d}<(1-\frac{1}{M})(\frac{1}{2k}-\frac{1}{2}+\frac{1}{s})$. We choose $N\in\mathbb{N}$ large enough such that $N\geqslant\max\{M,\frac{2kM^2}{(M-1)d},\frac{4Mk}{d(M-1)(k-1)}\}$ and $\frac{1}{d}-(1-\frac{1}{M})(\frac{1}{2k}-\frac{1}{2}+\frac{1}{s})\leqslant-\frac{2}{dN}$. Then we define $\eta:=\lambda^{-\frac{d}{k}(1-\frac{1}{M})}, r_\perp:=\lambda^{-1+\frac{1}{M}}\gg\lambda^{-1}, r_\parallel:=\lambda^{-1+\frac{1}{M}+\frac{1}{N}}$ and have

$$\frac{r_\perp}{r_\parallel}=\lambda^{-\frac{1}{N}},\ r_\perp^{-1}\lambda^{-1}=\lambda^{-\frac{1}{M}}\leqslant\lambda^{-\frac{1}{N}},\ r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac{1}{2}}\eta^{-\frac{1}{2}}\leqslant\lambda^{-(1-\frac{1}{M})\frac{d}{2}(1-\frac{1}{k})+\frac{1}{N}}\leqslant\lambda^{-\frac{1}{N}},$$

$$\lambda r_\perp^{\frac{d-1}{s}-\frac{d-1}{2}}r_\parallel^{\frac{1}{s}-\frac{1}{2}}\eta^{\frac{1}{2}}\leqslant\lambda^{1-(1-\frac{1}{M})d(\frac{1}{2k}-\frac{1}{2}+\frac{1}{s})+\frac{1}{N}}\leqslant\lambda^{-\frac{1}{N}},$$

$$r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}}r_\parallel^{\frac{1}{p}-\frac{1}{2}}\eta^{\frac{1}{r}-\frac{1}{2}}\leqslant\lambda^{-(1-\frac{1}{M})\frac{d}{k}(\frac{1}{r}-\frac{1}{2}+k(\frac{1}{p}-\frac{1}{2}))+\frac{1}{N}}\leqslant\lambda^{-(1-\frac{1}{M})\frac{d}{k}\frac{1}{M}+\frac{1}{N}}\leqslant\lambda^{-\frac{1}{N}}.$$

□

Then we choose $\lambda:=\lambda_{q+1}$, and $r_\perp, r_\parallel, \eta$ in terms of λ_{q+1} according to Lemma 5.1. Moreover, we define

$$\lambda:=\lambda_{q+1},\ \ \overline{\mu}:=r_\parallel^{-\frac{1}{2}}r_\perp^{-\frac{d-1}{2}}\lambda_{q+1}^{\frac{1}{2N}}\leqslant\lambda_{q+1}^{\frac{d}{2}},\ \ \mu:=r_\parallel^{-\frac{1}{2}}r_\perp^{-\frac{d-1}{2}}\leqslant\lambda_{q+1}^{\frac{d}{2}},\ \ \sigma:=\lambda_{q+1}^{\frac{1}{2N}}. \tag{5.10}$$

It is required that b is a multiple of M to ensure that $\lambda r_\perp=a^{(b^{q+1})/M}\in\mathbb{N}$, where M is given in Lemma 5.1.

Next, using the building blocks introduced in Section C.1, we define the perturbation (w_{q+1},θ_{q+1}) similarly to that in [BCDL21, Section 5.3]. Let $\chi\in C_c^\infty(-\frac{3}{4},\frac{3}{4})$ be a non-negative function such that $\sum_{n\in\mathbb{Z}}\chi(t-n)=1$ for every $t\in\mathbb{R}$. Let $\tilde{\chi}\in C_c^\infty(-\frac{4}{5},\frac{4}{5})$ be a non-negative function satisfying $\tilde{\chi}=1$ in $[-\frac{3}{4},\frac{3}{4}]$ and $\sum_{n\in\mathbb{Z}}\tilde{\chi}(t-n)\leqslant 2$.

We fix a parameter $\zeta=20/\delta_{q+3}^2$ and consider two disjoint sets Λ^1,Λ^2 introduced in Lemma B.1 with $d\geqslant 2$. Next, we use the notation $\Lambda^i=\Lambda^1$ for i odd, and $\Lambda^i=\Lambda^2$ for i even. In the following we abuse the notation and define for $n\in\mathbb{N}$

$$W_{(\xi,g)}(x,t):=W_{(\xi)}(x,(\frac{n}{\zeta})^{1/2}H_{(\xi)}(t)).$$

Similarly, we could define $\Theta_{(\xi,g)}, V_{(\xi,g)}$ and all other terms appearing in Section C.1.1. Now by the identity (C.3), the definition of $H_{(\xi)}(t)$ in (C.15) and the choice of μ in (5.10) we have

$$\partial_t\Theta_{(\xi,g)}+(\frac{n}{\zeta})^{1/2}g_{(\xi)}\mathrm{div}(W_{(\xi,g)}\Theta_{(\xi,g)})=0. \tag{5.11}$$

As the next step, we define the principle part of the perturbation w_{q+1} by

$$w_{q+1}^{(p)}:=\sum_{n\geqslant 3}\tilde{\chi}(\zeta|M_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/2}\sum_{\xi\in\Lambda^n}W_{(\xi,g)}g_{(\xi)}.$$

We remark that here and in the following the first sum runs for n in the range

$$3\leqslant n\leqslant 1+\zeta|M_l|\leqslant 1+l^{-2/3-d-1-\epsilon}\|M_l\|_{L_t^1L^1}\leqslant 1+Cl^{-d-2} \tag{5.12}$$

because of the bounds (5.1), (5.4) and the space-time Sobolev embedding $W^{d+1+\epsilon,1}\subset L^\infty$. Here we choose a large enough to absorb the embedding constant.

Moreover, we define the incompressibility corrector by

$$w_{q+1}^{(c)}:=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big(-\tilde{\chi}(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\frac{1}{(n_*\lambda_{q+1})^2}\nabla\Phi_{(\xi,g)}\xi\cdot\nabla\psi_{(\xi,g)}$$
$$+\nabla(\tilde{\chi}(\zeta|M_l|-n))(\frac{n}{\zeta})^{1/2}:V_{(\xi,g)}\Big)g_{(\xi)}.$$

Here we denote $(\nabla(\tilde{\chi}(\zeta|M_l|-n)):V_{(\xi,g)})^i:=\sum_{j=1}^d\partial_j(\tilde{\chi}(\zeta|M_l|-n))V_{(\xi,g)}^{ij},\ i=1,2,...,d.$

By the identity (C.4) we have

$$w_{q+1}^{(p)}+w_{q+1}^{(c)}=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\mathrm{div}\Big(\tilde{\chi}(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}V_{(\xi,g)}\Big)g_{(\xi)}.\tag{5.13}$$

Since $V_{(\xi,g)}$ is skew-symmetric, we obtain

$$\mathrm{div}(w_{q+1}^{(p)}+w_{q+1}^{(c)})=0.$$

Then we define the principle part of the perturbation θ_{q+1} by

$$\theta_{q+1}^{(p)}:=\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\sum_{\xi\in\Lambda^n}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Theta_{(\xi,g)}g_{(\xi)},\tag{5.14}$$

where Γ_ξ is introduced in Lemma B.1. $\theta_{q+1}^{(p)}$ is non-negative since all the components are non-negative.

The mean corrector is defined by

$$\theta_{q+1}^{(c)}:=-\mathbb{P}_0\theta_{q+1}^{(p)},$$

where we recall that $\mathbb{P}_0 f=\fint_{\mathbb{T}^d}f\mathrm{d}x$.

Using the fact that $g_{(\xi)}$ have disjoint support, the identity (C.2) above, the geometry Lemma B.1 and the fact that $\chi\tilde{\chi}=\chi$ we obtain

$$\begin{aligned}w_{q+1}^{(p)}\theta_{q+1}^{(p)}&=\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\sum_{\xi\in\Lambda^n}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)W_{(\xi,g)}\Theta_{(\xi,g)}g_{(\xi)}^2\\&=\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\mathbb{P}_{\neq 0}(W_{(\xi,g)}\Theta_{(\xi,g)})g_{(\xi)}^2\\&\quad+\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi(g_{(\xi)}^2-1)+\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\frac{M_l}{|M_l|}.\end{aligned}\tag{5.15}$$

We observe that in (5.15), there is an undesirable term of the form $(\cdot)(g_{(\xi)}^2-1)$ arising from the temporal intermittency. To deal with this term, we define the temporal corrector as follows:

$$\theta_{q+1}^{(o)}:=-\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}h_{(\xi)}\mathrm{div}\Big(\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big).$$

Recalling the definition of $h_{(\xi)}(t)$ in (C.15) we have

$$\begin{aligned}&\partial_t\theta_{q+1}^{(o)}+\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}(g_{(\xi)}^2-1)\mathrm{div}\Big(\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big)\\&\quad=-\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}h_{(\xi)}\partial_t\mathrm{div}\Big(\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big).\end{aligned}\tag{5.16}$$

Finally, the perturbations are defined as

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)}, \quad \theta_{q+1} := \theta_{q+1}^{(p)} + \theta_{q+1}^{(c)} + \theta_{q+1}^{(o)},$$

where w_{q+1} is mean-zero and divergence-free, and θ_{q+1} is mean-zero. Since $M_l(t) = 0$ for $t \in [0, \frac{T_q+T_{q+1}}{2}]$, by definition we know that $w_{q+1}(t) = \theta_{q+1}(t) = 0$ for $t \in [0, \frac{T_q+T_{q+1}}{2}]$.

The new scalar ρ_{q+1} is defined by

$$\rho_{q+1} := \theta_{q+1} + \rho_l,$$

which satisfies $\int_{\mathbb{T}^d} \rho_{q+1} \mathrm{d}x = 1$. Consequently, since $\rho_l = 1$ for $t \in [0, \frac{T_q+T_{q+1}}{2}]$, we have $\rho_{q+1}(t) = 1$ for $t \in [0, \frac{T_q+T_{q+1}}{2}]$.

5.2.2. *Estimates of* w_{q+1}. First we establish the estimate of the amplitude functions defined in Section 5.2.1.

Proposition 5.2. *For* $N \in \mathbb{N}_0$ *we have*

$$\sum_{n\geqslant 3} \|\chi(\zeta|M_l| - n)\|_{C^N_{t,x}} + \sum_{n\geqslant 3} \|\tilde{\chi}(\zeta|M_l| - n)\|_{C^N_{t,x}} \lesssim l^{-(d+4)N-(d+2)},$$

$$\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} \left\| \chi(\zeta|M_l| - n)\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\right\|_{C^N_{t,x}} \lesssim l^{-(2d+8)N-(d+2)},$$

$$\Big(\frac{n}{\zeta}\Big)^N 1_{\{\chi(\zeta|M_l|-n)>0\}} + \Big(\frac{n}{\zeta}\Big)^N 1_{\{\tilde{\chi}(\zeta|M_l|-n)>0\}} \lesssim l^{-N(d+2)}.$$

We give the proof of this lemma in Appendix B.3.

Recalling that $w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)}$ is defined in Section 5.2.1, we first estimate $w_{q+1}^{(p)}$ in the $L^2_t L^2$-norm. Similar as in (5.15) we have

$$|w_{q+1}^{(p)}|^2 \lesssim \sum_{n\geqslant 3} \tilde{\chi}(\zeta|M_l| - n)\frac{n}{\zeta} \sum_{\xi\in\Lambda^n} |W_{(\xi,g)}|^2 g^2_{(\xi)}.$$

Then by the improved Hölder inequality in Lemma A.8, the estimate of $\tilde{\chi}$ in Proposition 5.2, the bounds (C.7) and (5.9) we have

$$\begin{aligned}
\|w_{q+1}^{(p)}(t)\|^2_{L^2} &\lesssim \sum_{n\geqslant 3} \left\|\tilde{\chi}(\zeta|M_l(t)| - n)\frac{n}{\zeta}\right\|_{L^1} \sum_{\xi\in\Lambda^n} \|W_{(\xi,g)}\|^2_{C_tL^2} g^2_{(\xi)}(t) \\
&\quad + (r_\perp\lambda_{q+1})^{-1} \left\|\tilde{\chi}(\zeta|M_l| - n)\frac{n}{\zeta}\right\|_{C^1_{t,x}} \sum_{\xi\in\Lambda^n} \|W_{(\xi,g)}\|^2_{C_tL^2} g^2_{(\xi)}(t) \\
&\lesssim \Big(\|\sum_{n\geqslant 3} \tilde{\chi}(\zeta|M_l(t)| - n)(|M_l(t)| + \zeta^{-1})\|_{L^1} + l^{-3d-8}\lambda_{q+1}^{-\frac{1}{N}}\Big) \sum_{\xi\in\Lambda^1\cup\Lambda^2} g^2_{(\xi)}(t) \\
&\lesssim (\|M_l(t)\|_{L^1} + \delta^2_{q+2}) \sum_{\xi\in\Lambda^1\cup\Lambda^2} g^2_{(\xi)}(t),
\end{aligned}$$

where we used the facts that $\tilde{\chi}$ is non-negative, $\sum_{n\in\mathbb{Z}} \tilde{\chi}(t-n) \leqslant 2$, and used conditions on the parameters to have $(6d+16)\alpha - \frac{1}{N} < -\alpha < -4\beta b$. Then we apply the improved Hölder inequality of Lemma A.8 again in time. Together with the bounds on $g_{(\xi)}$ in (C.16), M_l in (5.5), (5.7) and the choice of parameters in (5.1) we obtain

$$\|w_{q+1}^{(p)}\|^2_{L^2_tL^2} \lesssim (\|M_l\|_{L^1_tL^1} + \delta^2_{q+2} + \sigma^{-1}\|M_l\|_{C^1_{t,x}})\| \sum_{\xi\in\Lambda^1\cup\Lambda^2} g^2_{(\xi)}\|_{L^1_t}$$

$$\lesssim C_0^2(\delta_{q+2}^2 + \lambda_{q+1}^{(2d+6)\alpha - \frac{1}{2N}}) \lesssim C_0^2(\delta_{q+2}^2 + \lambda_{q+1}^{-\alpha}) \lesssim \frac{C_0^2}{16}\delta_{q+2}^2, \tag{5.17}$$

where we used conditions on the parameters to have $(2d+8)\alpha < \frac{1}{2N}$.

For the general $L_t^u L^m$-norm with $u, m \in [1, \infty]$, by the estimates for the building blocks in (C.5)-(C.7) and the estimates for $\tilde{\chi}$ in Proposition 5.2 we obtain

$$\begin{aligned}\|w_{q+1}^{(p)}\|_{L_t^u L^m} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\tilde{\chi}(\zeta|M_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/2}\right\|_{C^0_{t,x}}\|W_{(\xi,g)}\|_{C_tL^m}\|g_{(\xi)}\|_{L_t^u}\\ &\lesssim l^{-2d-4} r_\perp^{\frac{d-1}{m}-\frac{d-1}{2}} r_\parallel^{\frac{1}{m}-\frac{1}{2}}\eta^{\frac{1}{u}-\frac{1}{2}},\end{aligned} \tag{5.18}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{L_t^u L^m} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\left\|\tilde{\chi}(\zeta|M_l|-n)\Big(\frac{n}{\zeta}\Big)^{1/2}\right\|_{C^1_{t,x}}\\ &\qquad\times\Big(\frac{1}{\lambda_{q+1}^2}\|\nabla\Phi_{(\xi,g)}\xi\cdot\nabla\psi_{(\xi,g)}\|_{L^m}+\|V_{(\xi,g)}\|_{L^m}\Big)\|g_{(\xi)}\|_{L_t^u}\\ &\lesssim l^{-3d-8} r_\perp^{\frac{d-1}{m}-\frac{d-1}{2}} r_\parallel^{\frac{1}{m}-\frac{1}{2}}\Big(\frac{r_\perp}{r_\parallel}+\lambda_{q+1}^{-1}\Big)\eta^{\frac{1}{u}-\frac{1}{2}} \lesssim l^{-3d-8} r_\perp^{\frac{d-1}{m}-\frac{d-1}{2}} r_\parallel^{\frac{1}{m}-\frac{1}{2}}\eta^{\frac{1}{u}-\frac{1}{2}}\lambda_{q+1}^{-\frac{1}{N}}.\end{aligned} \tag{5.19}$$

Combining these with the choice of parameters in (5.1), (5.9), and the bound (5.17) we obtain

$$\|w_{q+1}\|_{L_t^2L^2} \lesssim \frac{C_0}{4}\delta_{q+2} + l^{-3d-8}\lambda_{q+1}^{-\frac{1}{N}} \leqslant \frac{3C_0}{8}\delta_{q+2}^{1/2}, \tag{5.20}$$

where we used conditions on the parameters to have $(6d+16)\alpha - \frac{1}{N} < -\alpha < -\beta b$. We also selected $\delta_{q+2}^{1/2}$ to be small enough by choosing a large enough to absorb the universal constant.

By the above bounds (5.18), (5.19) and the choice of parameters in (5.1), (5.9) we have

$$\|w_{q+1}\|_{L_t^r L^p} \lesssim l^{-3d-8} r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}} r_\parallel^{\frac{1}{p}-\frac{1}{2}}\eta^{\frac{1}{r}-\frac{1}{2}} \lesssim \lambda_{q+1}^{(6d+16)\alpha-\frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha}, \tag{5.21}$$

$$\|w_{q+1}\|_{C_tL^1} \lesssim l^{-3d-8} r_\perp^{\frac{d-1}{2}} r_\parallel^{\frac{1}{2}}\eta^{-\frac{1}{2}} \lesssim \lambda_{q+1}^{(6d+16)\alpha-\frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha}, \tag{5.22}$$

where we used conditions on the parameters to have $(6d+17)\alpha < \frac{1}{N}$.

Next, we estimate the $C^2_{t,x}$-norm. By the fact that

$$\partial_t(V_{(\xi,g)}(t)) = \Big(\frac{n}{\zeta}\Big)^{1/2} g_{(\xi)}\Big(\partial_t V_{(\xi)}\Big)\Big(\Big(\frac{n}{\zeta}\Big)^{1/2}H_{(\xi)}(t)\Big),$$

and the estimates for the building blocks in (C.7), (C.16), the identity (5.13) and the estimates for the amplitude functions in Proposition 5.2 we have

$$\begin{aligned}\|w_{q+1}\|_{C^2_{t,x}} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|\tilde{\chi}(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\|_{C^3_{t,x}}\Big(\|g_{(\xi)}\nabla V_{(\xi,g)}\|_{C^2_{t,x}}+\|g_{(\xi)}V_{(\xi,g)}\|_{C^2_{t,x}}\Big)\\ &\lesssim \sum_{n\geqslant 3}\|\tilde{\chi}(\zeta|M_l|-n)(\frac{n}{\zeta})^{3/2}\|_{C^3_{t,x}}\lambda_{q+1}^2\mu^2 r_\parallel^{-\frac{1}{2}} r_\perp^{-\frac{d-1}{2}}\sigma^2\eta^{-\frac{5}{2}} \lesssim \lambda_{q+1}^{(12d+36)\alpha+4d+\frac{5}{2}} \leqslant \lambda_{q+1}^{4d+3},\end{aligned} \tag{5.23}$$

where we used (5.8) (5.10), and conditions on the parameters to have $(12d+36)\alpha < \frac{1}{2}$.

We conclude this part with estimates in $W^{1,s}$-norms. By the estimates for the building blocks in (C.7), (C.16), and the estimate for the amplitude functions in Proposition 5.2 we obtain

$$\begin{aligned}\|w_{q+1}\|_{L^1_tW^{1,s}} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|\tilde{\chi}(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\|_{C^2_{t,x}}\|V_{(\xi,g)}\|_{C_tW^{2,s}}\|g_{(\xi)}\|_{L^1_t}\\ &\lesssim l^{-4d-12}\lambda_{q+1}r_\perp^{\frac{d-1}{s}-\frac{d-1}{2}}r_\parallel^{\frac{1}{s}-\frac{1}{2}}\eta^{\frac{1}{2}}\lesssim\lambda_{q+1}^{(8d+24)\alpha-\frac{1}{N}}\lesssim\lambda_{q+1}^{-\alpha},\end{aligned}\tag{5.24}$$

where we used (5.1), (5.9) and conditions on the parameters to have $(8d+25)\alpha<\frac{1}{N}$.

5.2.3. *Estimates of θ_{q+1}.* Recall that θ_{q+1} is defined in Section 5.2.1. We first estimate $\theta^{(p)}_{q+1}$ in $L^2_tL^2$-norm by a similar argument as in (5.17). Noting the fact that Γ_ξ are uniformly bounded, we have

$$|\theta^{(p)}_{q+1}|^2\lesssim\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\sum_{\xi\in\Lambda^n}\left|\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Theta_{(\xi,g)}\right|^2g^2_{(\xi)}\lesssim\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)\frac{n}{\zeta}\sum_{\xi\in\Lambda^n}\left|\Theta_{(\xi,g)}\right|^2g^2_{(\xi)}.$$

Then by the same argument as in (5.17), we have for some $C_\rho\geqslant 1$

$$\begin{aligned}\|\theta^{(p)}_{q+1}\|^2_{L^2_tL^2}&\lesssim(\|M_l\|_{L^1_tL^1}+\delta_{q+2}+\sigma^{-1}\|M_l\|_{C^1_{t,x}})\left\|\sum_{\xi\in\Lambda^1\cup\Lambda^2}g^2_{(\xi)}\right\|_{L^1_t}\\ &\lesssim C_0^2(\delta_{q+2}+\lambda_{q+1}^{(2d+6)\alpha-\frac{1}{2N}})\lesssim C_0^2(\delta_{q+2}+\lambda_{q+1}^{-\alpha})\leqslant\frac{C_\rho^2C_0^2}{4}\delta_{q+2},\end{aligned}\tag{5.25}$$

where we used (5.1) and conditions on the parameters to have $(2d+7)\alpha<\frac{1}{2N}$.

For the general $L^u_tL^m$-norm with $m,u\in[1,\infty]$, by the same argument as (5.18), we have

$$\|\theta^{(p)}_{q+1}\|_{L^u_tL^m}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{m}-\frac{d-1}{2}}r_\parallel^{\frac{1}{m}-\frac{1}{2}}\eta^{\frac{1}{u}-\frac{1}{2}}.\tag{5.26}$$

Moreover, by the estimate for $h_{(\xi)}$ in (C.16) and Proposition 5.2 we have

$$\|\theta^{(c)}_{q+1}\|_{C_t}\lesssim\|\theta^{(p)}_{q+1}\|_{C_tL^1}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac{1}{2}}\eta^{-\frac{1}{2}}\lesssim\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{N}}\lesssim\lambda_{q+1}^{-\alpha},\tag{5.27}$$

$$\begin{aligned}\|\theta^{(o)}_{q+1}\|_{C_tW^{1,\infty}}&\lesssim\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|h_{(\xi)}\|_{L^\infty_t}\left\|\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\right\|_{W^{2,\infty}}\\ &\lesssim\sigma^{-1}l^{-6d-20}\lesssim\lambda_{q+1}^{(12d+40)\alpha-\frac{1}{2N}}\lesssim\lambda_{q+1}^{-\alpha}.\end{aligned}\tag{5.28}$$

where we used (5.9), (5.10) and the conditions on the parameters to deduce $(12d+41)\alpha<\frac{1}{2N}$ and choose a large enough to absorb the universal constant. Then together with the bound (4.4) and (5.1) we show that

$$\|\rho_{q+1}-\rho_q\|_{L^2_tL^2}\leqslant\|\theta_{q+1}\|_{L^2_tL^2}+l\|\rho_q\|_{C^1_{t,x}}\leqslant\frac{C_\rho C_0}{2}\delta^{1/2}_{q+2}+C\lambda_{q+1}^{-\alpha}+C_0l\lambda_q^{3d+2}\leqslant C_\rho C_0\delta^{1/2}_{q+2},$$

once we choose a large enough in order to absorb the universal constant. Then we have (4.7) and then (4.3) for ρ_{q+1}.

Furthermore, together with the bounds (4.4) and (5.26)-(5.28) we have

$$\begin{aligned}\|\rho_{q+1}-\rho_q\|_{L^r_tL^p}&\leqslant\|\theta_{q+1}\|_{L^r_tL^p}+l\|\rho_q\|_{C^1_{t,x}}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}}r_\parallel^{\frac{1}{p}-\frac{1}{2}}\eta^{\frac{1}{r}-\frac{1}{2}}+\lambda_{q+1}^{-\alpha}+C_0l\lambda_q^{3d+2}\\ &\lesssim\lambda_{q+1}^{(4d+8)\alpha-\frac{1}{N}}+\lambda_{q+1}^{-\alpha}\lesssim\lambda_{q+1}^{-\alpha}\leqslant\delta^{1/2}_{q+2},\end{aligned}$$

where we used (5.1), (5.9) and conditions on the parameters to have $(4d+9)\alpha<\frac{1}{N}$ and choose a large enough to absorb the universal constant. Consequently, we obtain the first bound in (4.9).

The second bound in (4.9) can be derived by using the above bounds again:

$$\|\rho_{q+1}-\rho_q\|_{C_tL^1}\leqslant \|\theta_{q+1}\|_{C_tL^1}+l\|\rho_q\|_{C^1_{t,x}}\lesssim l^{-2d-4}r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\eta^{-\frac12}+\lambda_{q+1}^{-\alpha}+C_0l\lambda_q^{3d+2}$$
$$\lesssim \lambda_{q+1}^{(4d+8)\alpha-\frac1N}+\lambda_{q+1}^{-\alpha}\lesssim\lambda_{q+1}^{-\alpha}\leqslant\delta_{q+2}^{1/2},$$

where we use conditions on the parameters to have $(4d+9)\alpha<\frac1N$ choose a large enough to absorb the universal constant.

Since $\theta^{(p)}_{q+1}$ is non-negative, by the estimates for $\rho_q,\theta^{(c)}_{q+1}$ and $\theta^{(o)}_{q+1}$ in (4.4), (5.27) and (5.28) respectively we obtain

$$\inf_{t\in[0,1]}(\rho_{q+1}-\rho_q)\geqslant -\|\theta^{(c)}_{q+1}\|_{C_t}-\|\theta^{(o)}_{q+1}\|_{C^0_{t,x}}-\|\rho_l-\rho_q\|_{C^0_{t,x}}\geqslant -C\lambda_{q+1}^{-\alpha}-lC_0\lambda_q^{3d+2}\geqslant-\delta_{q+2}^{1/2},$$

which yields the last bound in (4.9). Here we used (5.1) and choose a large enough to absorb the universal constant.

Now we estimate θ_{q+1} in $C^1_{t,x}$-norm. By the estimates for the building blocks and the amplitude functions in (C.8), (C.16) and Proposition 5.2 respectively, we have

$$\|\theta^{(c)}_{q+1}\|_{C^1_{t,x}}\lesssim\|\theta^{(p)}_{q+1}\|_{C^1_{t,x}}\lesssim\sum_{n\geqslant3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,g)}\|_{C^1_{t,x}}\|g_{(\xi)}\|_{C^1_t}$$
$$\lesssim l^{-4d-12}\lambda_{q+1}\mu r_\parallel^{-\frac12}r_\perp^{-\frac{d-1}{2}}\sigma\eta^{-2}\lesssim\lambda_{q+1}^{(8d+24)\alpha+3d+\frac32},$$
$$\|\theta^{(o)}_{q+1}\|_{C^1_{t,x}}\lesssim\sigma^{-1}\sum_{n\geqslant3}\sum_{\xi\in\Lambda^n}\|h_{(\xi)}\|_{C^1_t}\Big\|\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big\|_{W^{2,\infty}}\lesssim\lambda_{q+1}^{(12d+40)\alpha+d}.$$

where we used (5.8) and (5.10). By choosing $(8d+24)\alpha<1/2$ we deduce

$$\|\rho_{q+1}\|_{C^1_{t,x}}\leqslant\|\rho_l\|_{C^1_{t,x}}+\|\theta_{q+1}\|_{C^1_{t,x}}\leqslant C_0\lambda_q^{3d+2}+\lambda_{q+1}^{3d+2}\leqslant C_0\lambda_{q+1}^{3d+2}.$$

which implies (4.4) for ρ_{q+1}.

Finally, we consider the bound of $\theta^{(p)}_{q+1}$ in $W^{1,s}$-norm which will be used below. By the estimates on the building blocks in (C.8), (C.16) and on the amplitude functions in Proposition 5.2 we obtain

$$\|\theta^{(p)}_{q+1}\|_{L^1_tW^{1,s}}\lesssim\sum_{n\geqslant3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,g)}\|_{C_tW^{1,s}}\|g_{(\xi)}\|_{L^1_t}$$
$$\lesssim l^{-4d-12}\lambda_{q+1}r_\parallel^{\frac1s-\frac12}r_\perp^{\frac{d-1}{s}-\frac{d-1}{2}}\eta^{\frac12}\lesssim\lambda_{q+1}^{(8d+24)\alpha-\frac1N}\lesssim\lambda_{q+1}^{-\alpha},\qquad(5.29)$$

where we used (5.9) and conditions on the parameters to have $(8d+25)\alpha<\frac1N$.

5.2.4. *Construction of the perturbation* $\overline{w}_{q+1}$. Let us now proceed with the construction of the perturbation $\overline{w}_{q+1}$ by employing the generalized intermittent jets and temporal jets introduced in Section C.1.

As the next step, we shall define certain amplitude functions used in the definition of the perturbations $\overline{w}_{q+1}$. For any $\xi\in\overline{\Lambda}$, we define

$$A:=2\sqrt{l^2+|\mathring{R}_l|^2},\ \ a_{(\xi)}:=A^{1/2}\gamma_\xi(\mathrm{Id}-\frac{\mathring{R}_l}{A}),$$

where γ_ξ is introduced in Lemma B.2. Since we have

$$|\mathrm{Id}-\frac{\mathring{R}_l}{A}-\mathrm{Id}|\leqslant1/2,$$

by Lemma B.2 it follows that

$$A\,\mathrm{Id}-\mathring{R}_l=\sum_{\xi\in\overline{\Lambda}}A\gamma_\xi^2(\mathrm{Id}-\frac{\mathring{R}_l}{A})\xi\otimes\xi=\sum_{\xi\in\overline{\Lambda}}a_{(\xi)}^2\xi\otimes\xi. \tag{5.30}$$

We have the following estimate for the amplitude function.

Proposition 5.3. *For $\xi\in\overline{\Lambda}$ and $N\in\mathbb{N}_0$ we have*

$$\|a_{(\xi)}\|_{C^N_{t,x}}\lesssim l^{-2d-3-(d+4)N}. \tag{5.31}$$

The proof of this proposition is given in Appendix B.3.

Now recalling the temporal jets $H_{(\xi)}(t)$ in Section C.1.3 we define

$$\overline{W}_{(\xi,g)}(x,t)=\overline{W}_{(\xi)}(x,H_{(\xi)}(t)),$$

and similarly define $\overline{V}_{(\xi,g)},\overline{\varphi}_{(\xi,g)}$ and other terms appearing in Section C.1.2.

With these preparations in hand, we define the principal part

$$\overline{w}^{(p)}_{q+1}:=\sum_{\xi\in\overline{\Lambda}}a_{(\xi)}\overline{W}_{(\xi,g)}g_{(\xi)},$$

and by the identity (5.30) and the fact that $g_{(\xi)}$ have disjoint support we have

$$\begin{aligned}(\overline{w}^{(p)}_{q+1}+w^{(p)}_{q+1})\otimes(\overline{w}^{(p)}_{q+1}+w^{(p)}_{q+1})+\mathring{R}_l&=\overline{w}^{(p)}_{q+1}\otimes\overline{w}^{(p)}_{q+1}+w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}+\mathring{R}_l\\&=\sum_{\xi\in\overline{\Lambda}}a_{(\xi)}^2\overline{W}_{(\xi,g)}\otimes\overline{W}_{(\xi,g)}g_{(\xi)}^2-\sum_{\xi\in\overline{\Lambda}}a_{(\xi)}^2\xi\otimes\xi+w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}+A\mathrm{Id}\\&=\sum_{\xi\in\overline{\Lambda}}a_{(\xi)}^2\mathbb{P}_{\neq0}(\overline{W}_{(\xi,g)}\otimes\overline{W}_{(\xi,g)})g_{(\xi)}^2+w^{(p)}_{q+1}\otimes w^{(p)}_{q+1},\end{aligned} \tag{5.32}$$

where we use the notation $\mathbb{P}_{\neq0}f:=f-\fint_{\mathbb{T}^d}f\,\mathrm{d}x$.

We define the incompressibility corrector by

$$\overline{w}^{(c)}_{q+1}:=\sum_{\xi\in\overline{\Lambda}}-\frac{1}{(\overline{n}_*\lambda_{q+1})^2}a_{(\xi)}\nabla\overline{\Phi}_{(\xi,g)}\xi\cdot\nabla\overline{\psi}_{(\xi,g)}g_{(\xi)}+\nabla a_{(\xi)}:\overline{V}_{(\xi,g)}g_{(\xi)}. \tag{5.33}$$

Here $(\nabla a_{(\xi)}:\overline{V}_{(\xi,g)})^i=\sum_{j=1}^d\partial_ja_{(\xi)}\overline{V}^{ij}_{(\xi,g)},\ i=1,2,...,d.$

By identity (C.11) we have

$$\overline{w}^{(p)}_{q+1}+\overline{w}^{(c)}_{q+1}=\sum_{\xi\in\overline{\Lambda}}(a_{(\xi)}\mathrm{div}\overline{V}_{(\xi,g)}+\nabla a_{(\xi)}:\overline{V}_{(\xi,g)})g_{(\xi)}=\sum_{\xi\in\overline{\Lambda}}\mathrm{div}(a_{(\xi)}\overline{V}_{(\xi,g)})g_{(\xi)}. \tag{5.34}$$

Since $a_{(\xi)}\overline{V}_{(\xi,g)}$ is skew-symmetric, we obtain

$$\mathrm{div}(\overline{w}^{(p)}_{q+1}+\overline{w}^{(c)}_{q+1})=0.$$

Next we looking back at (5.32), we see that there are still two bad terms that we need to deal with (in fact, we need to deal with the divergence of the these terms). To this end, we introduce the temporal corrector

$$\overline{w}^{(t)}_{q+1}:=\frac{1}{\mu}\sum_{\xi\in\overline{\Lambda}}\mathbb{P}_{\neq0}\mathbb{P}_{\mathrm{H}}(a_{(\xi)}^2\overline{\psi}^2_{(\xi,g)}\overline{\phi}^2_{(\xi,g)}\xi)g_{(\xi)}, \tag{5.35}$$

where $\mathbb{P}_{\rm H}$ is the Helmholtz projection. By a direct computation and (C.9) we obtain

$$\begin{aligned}
&\partial_t \overline{w}_{q+1}^{(t)} + \sum_{\xi\in\overline{\Lambda}} \mathbb{P}_{\neq 0}(a_{(\xi)}^2 g_{(\xi)}^2 \mathrm{div}(\overline{W}_{(\xi,g)} \otimes \overline{W}_{(\xi,g)})) \\
&\quad = \frac{1}{\mu}\sum_{\xi\in\overline{\Lambda}} \mathbb{P}_H \mathbb{P}_{\neq 0}\partial_t (a_{(\xi)}^2 g_{(\xi)} \overline{\psi}_{(\xi,g)}^2 \overline{\phi}_{(\xi,g)}^2 \xi) - \frac{1}{\mu}\sum_{\xi\in\overline{\Lambda}} \mathbb{P}_{\neq 0}(a_{(\xi)}^2 g_{(\xi)} \partial_t (\overline{\psi}_{(\xi,g)}^2 \overline{\phi}_{(\xi,g)}^2 \xi)) \\
&\quad = (\mathbb{P}_H - \mathrm{Id})\frac{1}{\mu}\sum_{\xi\in\overline{\Lambda}} \mathbb{P}_{\neq 0}\partial_t (a_{(\xi)}^2 g_{(\xi)} \overline{\psi}_{(\xi,g)}^2 \overline{\phi}_{(\xi,g)}^2 \xi) + \frac{1}{\mu}\sum_{\xi\in\overline{\Lambda}} \mathbb{P}_{\neq 0}(\partial_t (a_{(\xi)}^2 g_{(\xi)}) \overline{\psi}_{(\xi,g)}^2 \overline{\phi}_{(\xi,g)}^2 \xi). \qquad (5.36)
\end{aligned}$$

Note that the first term on the right hand side can be viewed as a pressure term.

Similarly as before, to handle the undesirable term of the form $(\cdot)(g_{(\xi)}^2 - 1)$ in (5.32), we define a new perturbation term

$$\overline{w}_{q+1}^{(o)} := -\sigma^{-1}\mathbb{P}_{\rm H}\mathbb{P}_{\neq 0}\sum_{\xi\in\overline{\Lambda}} h_{(\xi)} \mathrm{div}(a_{(\xi)}^2 \xi\otimes\xi),$$

which implies that

$$\partial_t \overline{w}_{q+1}^{(o)} + \sum_{\xi\in\overline{\Lambda}} (g_{(\xi)}^2 - 1)\mathrm{div}(a_{(\xi)}^2 \xi\otimes\xi) = \nabla p - \sigma^{-1}\mathbb{P}_{\rm H}\mathbb{P}_{\neq 0}\sum_{\xi\in\overline{\Lambda}} h_{(\xi)}\partial_t \mathrm{div}(a_{(\xi)}^2 \xi\otimes\xi). \qquad (5.37)$$

Here p denotes some pressure term.

Finally, the total perturbation $\overline{w}_{q+1}$ and the new velocity v_{q+1} are defined by

$$\overline{w}_{q+1} := \chi_q \overline{w}_{q+1}^{(p)} + \chi_q \overline{w}_{q+1}^{(c)} + \chi_q^2 \overline{w}_{q+1}^{(t)} + \chi_q^2 \overline{w}_{q+1}^{(o)}, \quad v_{q+1} := v_l + w_{q+1} + \overline{w}_{q+1},$$

which are mean-zero and divergence-free. Here χ_q is a smooth time cut off satisfying $\chi_q = 0$ for $t \leqslant T_{q+1}$, $\chi_q = 1$ for $t \geqslant \frac{T_q + T_{q+1}}{2}$ and for $n \geqslant 0$

$$\|\chi_q\|_{C_t^n} \lesssim (T_q - T_{q+1})^{-n} \lesssim l^{-n}. \qquad (5.38)$$

Then we obtain $v_{q+1} = 0$ on $[0, T_{q+1}]$.

5.2.5. *Estimates of* $\overline{w}_{q+1}$. Recall that $\overline{w}_{q+1}$ is defined in Section 5.2.4. First we estimate $\overline{w}_{q+1}^{(p)}$ in $L_t^2 L^2$-norm by applying the improved Hölder inequality from Lemma A.8 in the spatial variable. By the definition of $a_{(\xi)}$, the bounds on $\overline{W}_{(\xi)}$ and $a_{(\xi)}$ in (C.14) and (5.31) we obtain

$$\begin{aligned}
\|\overline{w}_{q+1}^{(p)}(t)\|_{L^2}^2 &\lesssim \sum_{\xi\in\overline{\Lambda}} \Big(\|a_{(\xi)}(t)\|_{L^2}^2 + (r_\perp \lambda_{q+1})^{-1}\|a_{(\xi)}\|_{C_{t,x}^1}^2\Big)\|\overline{W}_{(\xi,g)}\|_{C_t L^2}^2 g_{(\xi)}^2(t) \\
&\lesssim \sum_{\xi\in\overline{\Lambda}} (\|\mathring{R}_l(t)\|_{L^1} + l + \lambda_{q+1}^{(12d+28)\alpha - \frac{1}{N}}) g_{(\xi)}^2(t) \lesssim (\|\mathring{R}_l(t)\|_{L^1} + \delta_{q+1})\sum_{\xi\in\overline{\Lambda}} g_{(\xi)}^2(t),
\end{aligned}$$

where we used the choice of parameters in (5.9) and conditions on the parameters to have $(12d+28)\alpha - \frac{1}{N} < -\alpha < -2\beta$. Then we again apply the improved Hölder inequality in Lemma A.8 in time. By the bounds on $g_{(\xi)}$ and $\mathring{R}_l$ in (C.16), (5.4) and (5.6) we obtain for some universal constant $C_v \geqslant 1$

$$\begin{aligned}
\|\overline{w}_{q+1}^{(p)}\|_{L_t^2 L^2}^2 &\lesssim (\|\mathring{R}_q\|_{L_t^1 L^1} + \delta_{q+1} + \sigma^{-1}\|\mathring{R}_l\|_{C_{t,x}^1})\|\sum_{\xi\in\overline{\Lambda}} g_{(\xi)}^2\|_{L_t^1} \lesssim C_0^2(\delta_{q+1} + \sigma^{-1} l^{-d-3}) \\
&\lesssim C_0^2(\delta_{q+1} + \lambda_{q+1}^{(2d+6)\alpha - \frac{1}{2N}}) \leqslant \frac{C_v^2 C_0^2}{16}\delta_{q+1}, \qquad (5.39)
\end{aligned}$$

where we used conditions on the parameters to have $(2d+6)\alpha - \frac{1}{2N} < -\alpha < -2\beta$.

Then we turn to bound the perturbations $\overline{w}_{q+1}$ in general $L^m_t L^n$-norm with $m \in [1,\infty], n \in (1,\infty)$. Recalling the choice of parameters in (5.9) and (5.10), the estimates on the building blocks and the amplitude functions $a_{(\xi)}$ in (C.12)-(C.14), (C.16) and (5.31) respectively, we have that

$$\|\overline{w}^{(p)}_{q+1}\|_{L^m_t L^n} \lesssim \sum_{\xi\in\overline{\Lambda}} \|a_{(\xi)}\|_{C^0_{t,x}} \|\overline{W}_{(\xi,g)}\|_{C_tL^n} \|g_{(\xi)}\|_{L^m_t} \lesssim l^{-2d-3} r_\perp^{\frac{d-1}{n}-\frac{d-1}{2}} r_\|^{\frac{1}{n}-\frac{1}{2}} \eta^{\frac{1}{m}-\frac{1}{2}}, \tag{5.40}$$

$$\begin{aligned}\|\overline{w}^{(c)}_{q+1}\|_{L^m_t L^n} &\lesssim \sum_{\xi\in\overline{\Lambda}} \frac{1}{\lambda_{q+1}^2} \|a_{(\xi)} \nabla\overline{\Phi}_{(\xi,g)}\xi\cdot\nabla\overline{\psi}_{(\xi,g)}\|_{C_tL^n} \|g_{(\xi)}\|_{L^m_t} + \sum_{\xi\in\overline{\Lambda}} \|\nabla a_{(\xi)} : \overline{V}_{(\xi,g)}\|_{C_tL^n} \|g_{(\xi)}\|_{L^m_t} \\ &\lesssim l^{-3d-7} r_\perp^{\frac{d-1}{n}-\frac{d-1}{2}} r_\|^{\frac{1}{n}-\frac{1}{2}} \left(\frac{r_\perp}{r_\|} + \frac{1}{\lambda_{q+1}}\right) \eta^{\frac{1}{m}-\frac{1}{2}} \lesssim l^{-3d-7} r_\perp^{\frac{d-1}{n}-\frac{d-1}{2}} r_\|^{\frac{1}{n}-\frac{1}{2}} \eta^{\frac{1}{m}-\frac{1}{2}} \lambda_{q+1}^{-\frac{1}{N}},\end{aligned} \tag{5.41}$$

$$\begin{aligned}\|\overline{w}^{(t)}_{q+1}\|_{L^m_t L^n} &\lesssim \overline{\mu}^{-1} \sum_{\xi\in\overline{\Lambda}} \|a_{(\xi)}\|^2_{C^0_{t,x}} \|\overline{\psi}_{(\xi,g)}\|^2_{C_tL^{2n}} \|\overline{\phi}_{(\xi,g)}\|^2_{C_tL^{2n}} \|g_{(\xi)}\|_{L^m_t} \\ &\lesssim l^{-4d-6} \overline{\mu}^{-1} r_\perp^{\frac{d-1}{n}-d+1} r_\|^{\frac{1}{n}-1} \eta^{\frac{1}{m}-\frac{1}{2}} \lesssim l^{-4d-6} r_\perp^{\frac{d-1}{n}-\frac{d-1}{2}} r_\|^{\frac{1}{n}-\frac{1}{2}} \eta^{\frac{1}{m}-\frac{1}{2}} \lambda_{q+1}^{-\frac{1}{2N}},\end{aligned} \tag{5.42}$$

and together with the boundedness of $h_{(\xi)}$ in (C.16), we obtain

$$\|\overline{w}^{(o)}_{q+1}\|_{C_tW^{1,\infty}} \lesssim \sigma^{-1} \|h_{(\xi)}\|_{L^\infty_t} \|a^2_{(\xi)}\|_{C_tW^{2+\alpha,\infty}} \lesssim \sigma^{-1} l^{-6d-15} \lesssim \lambda_{q+1}^{(12d+30)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha}, \tag{5.43}$$

where we used conditions on the parameters to have $(12d+31)\alpha < \frac{1}{2N}$. Combining the choice of parameters in (5.9), the bound (5.39) and the fact that $\|\chi_q\|_{L^\infty_t} \leqslant 1$ we obtain

$$\|\overline{w}_{q+1}\|_{L^2_tL^2} \leqslant \frac{C_vC_0}{4}\delta_{q+1}^{1/2} + Cl^{-4d-6}\lambda_{q+1}^{-\frac{1}{2N}} + C\lambda_{q+1}^{-\alpha} \leqslant \frac{3C_vC_0}{8}\delta_{q+1}^{1/2}, \tag{5.44}$$

where we used the conditions on the parameters to have $(8d+12)\alpha - \frac{1}{2N} < -\alpha < -\beta$. In the last inequality we choose a large enough to absorb the universal constant.

The above inequality together with the bounds in (4.4), (5.1) and (5.20) yields:

$$\|v_{q+1}-v_q\|_{L^2_tL^2} \leqslant \|w_{q+1}\|_{L^2_tL^2} + \|\overline{w}_{q+1}\|_{L^2_tL^2} + l\|v_q\|_{C^1_{t,x}} \leqslant \frac{3}{4}C_vC_0\delta_{q+1}^{1/2} + lC_0\lambda_q^{4d+3} \leqslant C_vC_0\delta_{q+1}^{1/2}.$$

Then (4.7) and (4.3) holds for v_{q+1}.

Similarly, recalling the choice of parameters in (5.9) we have the following bounds:

$$\|\overline{w}_{q+1}\|_{L^r_tL^p} \lesssim l^{-4d-6} r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}} r_\|^{\frac{1}{p}-\frac{1}{2}} \eta^{\frac{1}{r}-\frac{1}{2}} + \lambda_{q+1}^{-\alpha} \lesssim \lambda_{q+1}^{(8d+12)\alpha-\frac{1}{N}} + \lambda_{q+1}^{-\alpha} \lesssim \lambda_{q+1}^{-\alpha}, \tag{5.45}$$

$$\begin{aligned}\|\overline{w}_{q+1}\|_{C_tL^1} &\lesssim \|\overline{w}_{q+1}\|_{C_tL^{1+\epsilon}} \lesssim l^{-4d-6} r_\perp^{\frac{d-1}{2}} r_\|^{\frac{1}{2}} \eta^{-\frac{1}{2}} \lambda_{q+1}^{d\epsilon} + \sigma^{-1} l^{-6d-15} \\ &\lesssim \lambda_{q+1}^{(8d+13)\alpha-\frac{1}{N}} + \lambda_{q+1}^{(12d+30)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},\end{aligned} \tag{5.46}$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$ and choose the parameters such that $(12d+31)\alpha < \frac{1}{2N}$. Together with the bounds on v_q, w_{q+1} in (4.4), (5.21) and (5.22) respectively we obtain

$$\|v_{q+1}-v_q\|_{L^r_tL^p} \lesssim \|w_{q+1}\|_{L^r_tL^p} + \|\overline{w}_{q+1}\|_{L^r_tL^p} + l\|v_q\|_{C^1_{t,x}} \lesssim \lambda_{q+1}^{-\alpha} + lC_0\lambda_q^{4d+3} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/2},$$

$$\|v_{q+1}-v_q\|_{C_tL^1} \lesssim \|w_{q+1}\|_{C_tL^1} + \|\overline{w}_{q+1}\|_{C_tL^1} + l\|v_q\|_{C^1_{t,x}} \lesssim \lambda_{q+1}^{-\alpha} + lC_0\lambda_q^{4d+3} \lesssim \lambda_{q+1}^{-\alpha} \leqslant \delta_{q+1}^{1/2},$$

where in the last inequality we choose a large enough to absorb the universal constant. Then we obtain the first two bounds in (4.8).

Next we estimate the $C^2_{t,x}$-norm. Taking into account the fact that

$$\partial_t\Big(\overline{V}_{(\xi,g)}(t)\Big) = g_{(\xi)}(t)(\partial_t\overline{V}_{(\xi)})(H_{(\xi)}(t)),$$

and using the estimates on the building blocks $\overline{\psi}_{(\xi)}, \overline{\phi}_{(\xi)}, \overline{V}_{(\xi)}, g_{(\xi)}$ in (C.12)-(C.14), (C.16) respectively, the estimates on $a_{(\xi)}$ in (5.31) and the identity (5.34) we have

$$\begin{aligned}\|\overline{w}^{(p)}_{q+1}+\overline{w}^{(c)}_{q+1}\|_{C^2_{t,x}} &\lesssim \sum_{\xi\in\overline{\Lambda}}\|a_{(\xi)}\|_{C^3_{t,x}}(\|g_{(\xi)}\nabla\overline{V}_{(\xi,g)}\|_{C^2_{t,x}}+\|g_{(\xi)}\overline{V}_{(\xi,g)}\|_{C^2_{t,x}})\\ &\lesssim l^{-5d-15}\lambda^2_{q+1}\overline{\mu}^2 r_{\|}^{-\frac12} r_{\perp}^{-\frac{d-1}{2}}\sigma^2\eta^{-\frac52}\lesssim \lambda_{q+1}^{(10d+30)\alpha+4d+\frac52},\end{aligned}$$

$$\begin{aligned}\|\overline{w}^{(t)}_{q+1}\|_{C^2_{t,x}} &\lesssim \frac{1}{\overline{\mu}}\sum_{\xi\in\overline{\Lambda}}\sum_{i=0}^{2}\|a^2_{(\xi)}\overline{\psi}^2_{(\xi,g)}\overline{\phi}^2_{(\xi,g)}g_{(\xi)}\|_{C^i_tW^{2-i+\alpha,\infty}}\\ &\lesssim l^{-6d-14}\lambda^{2+\alpha}_{q+1}\overline{\mu}r_{\|}^{-3}r_{\perp}^{-d+3}\sigma^2\eta^{-\frac52}\lesssim \lambda_{q+1}^{(12d+29)\alpha+4d+\frac52},\end{aligned}$$

and by the estimates on $h_{(\xi)}, a_{(\xi)}$ in (C.16), (5.31), we have

$$\|\overline{w}^{(o)}_{q+1}\|_{C^2_{t,x}}\lesssim \sigma^{-1}\sum_{\xi\in\overline{\Lambda}}\|h_{(\xi)}\|_{C^2}\|\mathrm{div}(a^2_{(\xi)}\xi\otimes\xi)\|_{C^{2+\alpha}_{t,x}}\lesssim \eta^{-2}l^{-7d-19}\lesssim\lambda_{q+1}^{(14d+38)\alpha+2d}.$$

Thus by using (5.38) and $(12d+33)\alpha<\frac12$ we obtain

$$\|\overline{w}_{q+1}\|_{C^2_{t,x}}\leqslant(\|\chi_q\|^2_{C^1_t}+\|\chi_q\|_{C^2_t})\lambda_{q+1}^{(12d+29)\alpha+4d+\frac52}\leqslant\lambda^{4d+3}_{q+1}.$$

Combining the bounds on w_{q+1} in (5.23) we get (4.4) for v_{q+1}.

$$\|v_{q+1}\|_{C^2_{t,x}}\leqslant\|v_q\|_{C^2_{t,x}}+\|w_{q+1}\|_{C^2_{t,x}}+\|\overline{w}_{q+1}\|_{C^2_{t,x}}\leqslant C_0\lambda_q^{4d+3}+2\lambda^{4d+3}_{q+1}\leqslant C_0\lambda^{4d+3}_{q+1}.$$

We conclude this part with the bound in $W^{1,s}$-norm. By the estimates for the building blocks $\overline{\psi}_{(\xi)}, \overline{\phi}_{(\xi)}$, $\overline{V}_{(\xi)}, g_{(\xi)}$ in (C.12)-(C.14), (C.16) respectively and the estimates for $a_{(\xi)}$ in (5.31) it follows that

$$\begin{aligned}\|\overline{w}^{(p)}_{q+1}+\overline{w}^{(c)}_{q+1}\|_{L^1_tW^{1,s}} &\lesssim\sum_{\xi\in\overline{\Lambda}}\|a_{(\xi)}\|_{C^2_{t,x}}\|\overline{V}_{(\xi,g)}\|_{C_tW^{2,s}}\|g_{(\xi)}\|_{L^1_t}\\ &\lesssim l^{-4d-11}r_{\perp}^{\frac{d-1}{s}-\frac{d-1}{2}}r_{\|}^{\frac1s-\frac12}\lambda_{q+1}\eta^{\frac12}\lesssim\lambda_{q+1}^{(8d+22)\alpha-\frac1N}\lesssim\lambda^{-\alpha}_{q+1},\end{aligned}$$

$$\begin{aligned}\|\overline{w}^{(t)}_{q+1}\|_{L^1_tW^{1,s}}&\lesssim\frac{1}{\overline{\mu}}\sum_{\xi\in\overline{\Lambda}}\|a^2_{(\xi)}\|_{C^1_{t,x}}\|\overline{\psi}^2_{(\xi,g)}\overline{\phi}^2_{(\xi,g)}\|_{C_tW^{1,s}}\|g_{(\xi)}\|_{L^1_t}\lesssim l^{-5d-10}\overline{\mu}^{-1}r_{\perp}^{\frac{d-1}{s}-d+1}r_{\|}^{\frac1s-1}\lambda_{q+1}\eta^{\frac12}\\ &\lesssim l^{-5d-10}r_{\perp}^{\frac{d-1}{s}-\frac{d-1}{2}}r_{\|}^{\frac1s-\frac12}\lambda_{q+1}\eta^{\frac12}\lesssim\lambda_{q+1}^{(10d+20)\alpha-\frac1N}\lesssim\lambda^{-\alpha}_{q+1},\end{aligned}$$

where we used (5.9) and conditions on the parameters to have $(10d+21)\alpha<\frac1N$. Then since $\|\chi_q\|_{L^\infty_t}\leqslant 1$, together with the the estimate on $\overline{w}^{(o)}_{q+1}$ in (5.43) we obtain

$$\|\overline{w}_{q+1}\|_{L^1_tW^{1,s}}\lesssim\lambda^{-\alpha}_{q+1}. \tag{5.47}$$

Hence together with (5.1) and the bound on w_{q+1} in (5.24) we deduce

$$\|v_{q+1}-v_q\|_{L^1_tW^{1,s}}\leqslant\|w_{q+1}\|_{L^1_tW^{1,s}}+\|\overline{w}_{q+1}\|_{L^1_tW^{1,s}}+l\|v_q\|_{C^2_{t,x}}\leqslant\lambda^{-\alpha}_{q+1}+lC_0\lambda_q^{4d+3}\leqslant\delta^{1/2}_{q+1},$$

which implies the last estimate in (4.8). Here in the last inequality we choose a large enough to absorb the universal constant.

5.3. **The estimates of the stress terms.** In this section we complete the proof of Proposition 4.1 by proving the remaining estimates on the stress terms in (4.5). The stress term M_l will be canceled by the perturbation (w_{q+1}, θ_{q+1}), as will be showed in Section 5.3.1. The estimate of the new stress term M_{q+1} will be estimated in Section 5.3.2. The stress term $\mathring{R}_l$ will be canceled by the perturbation $\overline{w}_{q+1}$ will be shown in Section 5.3.3. The estimate of the new stress term $\mathring{R}_{q+1}$ is contained in Section 5.3.4.

5.3.1. *Construction of the stress term* M_{q+1}. First we recall that the supports of $g_{(\xi)}$ are disjoint for different $\xi \in \Lambda^1 \cup \Lambda^2 \cup \overline{\Lambda}$. As a result, based on the definitions of the perturbations, we have that

$$(\chi_q \overline{w}_{q+1}^{(p)} + \chi_q \overline{w}_{q+1}^{(c)} + \chi_q^2 \overline{w}_{q+1}^{(t)})\theta_{q+1}^{(p)} = 0.$$

Then together with the identity $\mathrm{div}(v_{q+1}\theta_{q+1}^{(c)}) = \theta_{q+1}^{(c)}\mathrm{div} v_{q+1} = 0$ we obtain that

$$\begin{aligned}
&-\mathrm{div} M_{q+1}\\
&\quad = \partial_t \theta_{q+1} + \mathrm{div}(w_{q+1}^{(p)}\theta_{q+1}^{(p)} - M_l)(:= \mathrm{div} M_{osc})\\
&\quad - \kappa\Delta\theta_{q+1} + \mathrm{div}\Big(v_l\theta_{q+1} + (w_{q+1} + \overline{w}_{q+1})(\rho_l + \theta_{q+1}^{(o)}) + (w_{q+1}^{(c)} + \chi_q^2\overline{w}_{q+1}^{(o)})\theta_{q+1}^{(p)}\Big), (:= \mathrm{div} M_{lin})
\end{aligned}$$

where we use the inverse divergence operator $\mathcal{R}_1$ defined in Section A.1 to define the linear error by

$$M_{lin} := -\kappa\mathcal{R}_1\Delta\theta_{q+1} + v_l\theta_{q+1} + (w_{q+1} + \overline{w}_{q+1})(\rho_l + \theta_{q+1}^{(o)}) + (w_{q+1}^{(c)} + \chi_q^2\overline{w}_{q+1}^{(o)})\theta_{q+1}^{(p)}.$$

To define the oscillation error M_{osc}, by the identities (5.15) and (5.16) we have

$$\begin{aligned}
\mathrm{div} M_{osc} &= \partial_t\theta_{q+1} + \mathrm{div}(\theta_{q+1}^{(p)}w_{q+1}^{(p)} - M_l)\\
&= \mathbb{P}_{\neq 0}\Big(\partial_t\theta_{q+1}^{(p)} + \mathrm{div}(\theta_{q+1}^{(p)}w_{q+1}^{(p)} - M_l) + \partial_t\theta_{q+1}^{(o)}\Big)\\
&= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\mathbb{P}_{\neq 0}\Big(\partial_t\Big[\chi(\zeta|M_l| - n)(\frac{n}{\zeta})^{\frac12}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)g_{(\xi)}\Big]\Theta_{(\xi,g)}\Big)(:= \mathrm{div} M_{osc,t})\\
&+ \mathbb{P}_{\neq 0}\Big[\nabla[\chi(\zeta|M_l| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)]g_{(\xi)}^2\mathbb{P}_{\neq 0}(W_{(\xi,g)}\Theta_{(\xi,g)})\Big](:= \mathrm{div} M_{osc,x})\\
&+ \mathbb{P}_{\neq 0}\Big[\chi(\zeta|M_l| - n)(\frac{n}{\zeta})^{\frac12}g_{(\xi)}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)(\partial_t\Theta_{(\xi,g)} + (\frac{n}{\zeta})^{\frac12}g_{(\xi)}\mathrm{div}(W_{(\xi,g)}\Theta_{(\xi,g)}))\Big]\\
&+ \mathrm{div}\Big(\sum_{n\geqslant 3}\chi(\zeta|M_l| - n)\frac{n}{\zeta}\frac{M_l}{|M_l|} - M_l\Big)(:= \mathrm{div} M_{osc,c})\\
&- \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}h_{(\xi)}\partial_t\mathrm{div}\Big[\chi(\zeta|M_l| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big](:= \mathrm{div} M_{osc,o}),
\end{aligned}$$

where the last third term equals to 0 by (5.11). Now by using the inverse divergence operators $\mathcal{R}_1, \mathcal{B}_1$ introduced in Section A.1 we define $M_{osc} := M_{osc,t} + M_{osc,x} + M_{osc,c} + M_{osc.o}$, where

$$\begin{aligned}
M_{osc,t} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\mathcal{R}_1\Big(\partial_t[\chi(\zeta|M_l| - n)(\frac{n}{\zeta})^{1/2}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)g_{(\xi)}]\Theta_{(\xi,g)}\Big),\\
M_{osc,x} &:= \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\mathcal{B}_1\Big(\nabla[\chi(\zeta|M_l| - n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)], \mathbb{P}_{\neq 0}(W_{(\xi,g)}\Theta_{(\xi,g)})\Big)g_{(\xi)}^2,\\
M_{osc,c} &:= \sum_{n\geqslant 3}\chi(\zeta|M_l| - n)\frac{n}{\zeta}\frac{M_l}{|M_l|} - M_l,
\end{aligned}$$

$$M_{osc,o} := -\sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n} h_{(\xi)}\partial_t\Big(\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\xi\Big).$$

Then we have $-M_{q+1} := M_{osc} + M_{lin}$. It is easy to see that $M_{q+1}(t) = 0$ on $[0, T_{q+1}]$, since $M_l(t) = w_{q+1}(t) = \overline{w}_{q+1}(t) = 0$ on $[0, T_{q+1}]$.

5.3.2. *Estimates of the* M_{q+1}. To conclude the proof of Proposition 4.1 we verify the last bound in (4.5) for M_{q+1} by estimating each term in the definition of M_{q+1} separately, as done previously.

For the oscillation error, we consider $M_{osc,t}$ first. By the improved Hölder inequality in Lemma A.8, the estimates for amplitude functions in Proposition 5.2, and the estimates for the building blocks in (C.8), (C.16) we obtain

$$\begin{aligned}\|M_{osc,t}\|_{L^1_tL^1} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta|M_l|-n)(\frac{n}{\zeta})^{1/2}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Big\|_{C^1_{t,x}}\|\Theta_{(\xi,g)}\|_{C_tL^1}\|g_{(\xi)}\|_{W^{1,1}_t}\\ &\lesssim l^{-4d-12}r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\eta^{-\frac12}\sigma \lesssim \lambda_{q+1}^{(8d+24)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},\end{aligned}$$

where we used the choice of parameters in (5.1), (5.9), (5.10), and conditions on the parameters to have $(8d+25)\alpha < \frac{1}{2N}$.

For the second term $M_{osc,x}$, we obverse that $W_{(\xi)}\Theta_{(\xi)}$ is $(\mathbb{T}/r_\perp\lambda_{q+1})^d$-periodic. So, together with the bounds for the amplitude functions in Proposition 5.2, and the estimates for the building blocks in (C.7), (C.8), (C.16), we apply Theorem A.4 to obtain

$$\begin{aligned}\|M_{osc,x}\|_{L^1_tL^1} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big\|\mathcal{B}_1\Big(\nabla[\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)], \mathbb{P}_{\neq 0}(W_{(\xi,g)}\Theta_{(\xi,g)})\Big)\Big\|_{C_tL^1}\Big\|g^2_{(\xi)}\Big\|_{L^1_t}\\ &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\Big\|\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\Big\|_{C^2_{t,x}}\|W_{(\xi,g)}\Theta_{(\xi,g)}\|_{C_tL^1}(r_\perp\lambda_{q+1})^{-1}\\ &\lesssim l^{-6d-20}(r_\perp\lambda_{q+1})^{-1}\|\Theta_{(\xi,g)}\|_{C_tL^2}\|W_{(\xi,g)}\|_{C_tL^2} \lesssim \lambda_{q+1}^{(12d+40)\alpha-\frac{1}{N}} \lesssim \lambda_{q+1}^{-\alpha},\end{aligned}$$

where we used (5.1), (5.9) and conditions on the parameters to have $(12d+41)\alpha < \frac{1}{N}$.

For the third term $M_{osc,c}$, by a similar argument as in [BCDL21, (31)] we have

$$\begin{aligned}|M_{osc,c}| &\leqslant \left|\sum_{n=-1}^{2}\chi(\zeta|M_l|-n)M_l\right| + \left|\sum_{n\geqslant 3}\chi(\zeta|M_l|-n)(\frac{n}{\zeta}\frac{M_l}{|M_l|}-M_l)\right|\\ &\leqslant \frac{3}{\zeta} + \sum_{n\geqslant 3}\chi(\zeta|M_l|-n)\left|\frac{n}{\zeta}-|M_l|\right| \leqslant \frac{3}{20}\delta^2_{q+3} + \frac{1}{20}\delta^2_{q+3} \leqslant \frac15 C_0^2\delta^2_{q+3}.\end{aligned}$$

By the bounds on $h_{(\xi)}$ in (C.16) and the bounds on the amplitude functions in Proposition 5.2 we have

$$\begin{aligned}\|M_{osc,o}\|_{L^1_tL^1} &\lesssim \sigma^{-1}\sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|h_{(\xi)}\|_{L^\infty_t}\|\chi(\zeta|M_l|-n)\frac{n}{\zeta}\Gamma_\xi\xi\|_{C^1_{t,x}}\\ &\lesssim \sigma^{-1}l^{-4d-12} \lesssim \lambda_{q+1}^{(8d+24)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},\end{aligned}$$

where we used conditions on the parameters to have $(8d+25)\alpha < \frac{1}{2N}$.

Now we turn to the linear error M_{lin}, where we note that all terms have already been estimated in the previous sections. More precisely, by the estimates on $\theta^{(p)}_{q+1}$ and $\theta^{(o)}_{q+1}$ in (5.29) and (5.28) respectively, we have for $0 \leqslant \kappa \leqslant 1$

$$\|\kappa \mathcal{R}_1 \Delta \theta_{q+1}\|_{L^1_t L^1} \lesssim \|\theta^{(p)}_{q+1}\|_{L^1_t W^{1,s}} + \|\theta^{(o)}_{q+1}\|_{L^1_t W^{1,\infty}} \lesssim \lambda_{q+1}^{-\alpha}.$$

Moreover, according to the estimates of v_l, ρ_l in (4.4), the estimates of θ_{q+1} in (5.26)-(5.28), the estimates of w_{q+1} in (5.22), and the estimates of $\overline{w}_{q+1}$ in (5.46) we obtain

$$\begin{aligned}
&\|v_l \theta_{q+1} + (w_{q+1} + \overline{w}_{q+1})(\rho_l + \theta^{(o)}_{q+1}) + (w^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1})\theta^{(p)}_{q+1}\|_{L^1_t L^1} \\
&\quad \leqslant \|v_l\|_{C^0_{t,x}} \|\theta_{q+1}\|_{C_t L^1} + (\|w_{q+1}\|_{C_t L^1} + \|\overline{w}_{q+1}\|_{C_t L^1})(\|\rho_l\|_{C^0_{t,x}} + \|\theta^{(o)}_{q+1}\|_{C^0_{t,x}}) \\
&\qquad + (\|w^{(c)}_{q+1}\|_{L^2_t L^2} + \|\overline{w}^{(o)}_{q+1}\|_{C^0_{t,x}}) \|\theta^{(p)}_{q+1}\|_{L^2_t L^2} \\
&\quad \lesssim C_\rho C_0 (\lambda_q^{4d+3} + 1) l^{-6d-20} (r_\perp^{\frac{d-1}{2}} r_\parallel^{\frac{1}{2}} \eta^{-\frac{1}{2}} \lambda_{q+1}^{d\epsilon} + \sigma^{-1}) \lesssim C_\rho C_0 \lambda_{q+1}^{(12d+42)\alpha - \frac{1}{2N}} \lesssim C_0 \lambda_{q+1}^{-\alpha},
\end{aligned}$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used the bounds in (5.1), (5.9), and conditions on the parameters to have $(12d+43)\alpha < \frac{1}{2N}$. In the last inequality we choose a large enough to absorb the universal constant.

Summarizing all the bounds above we obtain (4.5) for M_{q+1}:

$$\|M_{q+1}\|_{L^1_t L^1} \leqslant \frac{1}{5} C_0^2 \delta_{q+3}^2 + C C_0^2 \lambda_{q+1}^{-\alpha} \leqslant C_0^2 \delta_{q+3}^2,$$

where we choose a large to absorb the universal constant.

5.3.3. *Construction of the Reynolds stress* $\mathring{R}_{q+1}$. From (4.2), (5.2) and the definition of the perturbations $w_{q+1}, \overline{w}_{q+1}$ we obtain

$$\begin{aligned}
&\operatorname{div} \mathring{R}_{q+1} - \nabla \pi_{q+1} + \nabla \pi_l \\
&= \partial_t (\chi_q \overline{w}^{(p)}_{q+1} + \chi_q \overline{w}^{(c)}_{q+1} + w_{q+1}) - \nu \Delta (w_{q+1} + \overline{w}_{q+1}) + \operatorname{div}\Big(v_l \otimes (w_{q+1} + \overline{w}_{q+1}) + (w_{q+1} + \overline{w}_{q+1}) \otimes v_l\Big) \\
&\quad + \operatorname{div}\Big((\chi_q \overline{w}^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1} + w^{(c)}_{q+1}) \otimes (w_{q+1} + \overline{w}_{q+1})\Big) \\
&\quad + \operatorname{div}\Big((\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) \otimes (\chi_q \overline{w}^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1} + w^{(c)}_{q+1})\Big) \\
&\quad + \partial_t (\chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1}) + \operatorname{div}\Big((\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) \otimes (\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) + \mathring{R}_l\Big),
\end{aligned}$$

where by using the inverse divergence operator $\mathcal{R}$ introduced in Section A.1 we define

$$\begin{aligned}
R_{lin} &:= \mathcal{R} \partial_t (\chi_q \overline{w}^{(p)}_{q+1} + \chi_q \overline{w}^{(c)}_{q+1} + w_{q+1}) - \nu \mathcal{R} \Delta (w_{q+1} + \overline{w}_{q+1}) \\
&\quad + v_l \mathring{\otimes} (w_{q+1} + \overline{w}_{q+1}) + (w_{q+1} + \overline{w}_{q+1}) \mathring{\otimes} v_l. \\
R_{cor} &:= (\chi_q \overline{w}^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1} + w^{(c)}_{q+1}) \mathring{\otimes} (w_{q+1} + \overline{w}_{q+1}) \\
&\quad + (\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) \mathring{\otimes} (\chi_q \overline{w}^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1} + w^{(c)}_{q+1}).
\end{aligned}$$

In order to define the remaining oscillation error in the last line, first by the definition of χ_q and the fact that $\mathring{R}_l = w_{q+1} = w^{(p)}_{q+1} = 0$ for $t \in [0, \frac{T_q + T_{q+1}}{2}]$, we know that $\mathring{R}_l = \chi_q \mathring{R}_l, w^{(p)}_{q+1} = \chi_q w^{(p)}_{q+1}$ and $w_{q+1} = \chi_q w_{q+1}$. Then we apply the identities (5.32), (5.36)-(5.37) to obtain

$$\begin{aligned}
&\partial_t (\chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1}) + \operatorname{div}\Big((\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) \otimes (\chi_q \overline{w}^{(p)}_{q+1} + w^{(p)}_{q+1}) + \mathring{R}_l\Big) \\
&\quad = \chi_q^2 \partial_t \overline{w}^{(t)}_{q+1} + \chi_q^2 \sum_{\xi \in \overline{\Lambda}} \operatorname{div}\Big(a^2_{(\xi)} \mathbb{P}_{\neq 0} (\overline{W}_{(\xi,g)} \otimes \overline{W}_{(\xi,g)}) g^2_{(\xi)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \chi_q^2\partial_t\overline{w}_{q+1}^{(o)} + \chi_q^2\sum_{\xi\in\overline{\Lambda}}\mathrm{div}\Big(a_{(\xi)}^2(g_{(\xi)}^2-1)\xi\otimes\xi\Big)\\
&+ \chi_q^2\nabla A + \mathrm{div}(w_{q+1}\otimes w_{q+1}) + (\chi_q^2)'(\overline{w}_{q+1}^{(t)}+\overline{w}_{q+1}^{(o)})\\
=& \sum_{\xi\in\overline{\Lambda}}\chi_q^2\mathbb{P}_{\neq 0}\left(\nabla a_{(\xi)}^2 g_{(\xi)}^2\mathbb{P}_{\neq 0}(\overline{W}_{(\xi,g)}\otimes\overline{W}_{(\xi,g)})\right) + \frac{1}{\overline{\mu}}\chi_q^2\sum_{\xi\in\overline{\Lambda}}\mathbb{P}_{\neq 0}\left(\partial_t(a_{(\xi)}^2 g_{(\xi)})\overline{\phi}_{(\xi,g)}^2\overline{\psi}_{(\xi,g)}^2\xi\right)\\
&- \sigma^{-1}\chi_q^2\mathbb{P}_{\mathrm{H}}\mathbb{P}_{\neq 0}\sum_{\xi\in\overline{\Lambda}}h_{(\xi)}\partial_t\mathrm{div}(a_{(\xi)}^2\xi\otimes\xi) + \partial_t(\chi_q^2)(\overline{w}_{q+1}^{(t)}+\overline{w}_{q+1}^{(o)}) + \nabla p_1 + \mathrm{div}(w_{q+1}\otimes w_{q+1}).
\end{aligned}
$$

Here p_1 denotes the pressure term.

Therefore using the inverse divergence operators $\mathcal{R}$, $\mathcal{B}$ introduced in Section A.1 we have

$$
\begin{aligned}
R_{osc} :=& \sum_{\xi\in\overline{\Lambda}} g_{(\xi)}^2\mathcal{B}\left(\nabla a_{(\xi)}^2\chi_q^2, \mathbb{P}_{\neq 0}(\overline{W}_{(\xi,g)}\otimes\overline{W}_{(\xi,g)})\right)(:= R_{osc,x})\\
&+ \frac{1}{\overline{\mu}}\sum_{\xi\in\overline{\Lambda}}\mathcal{R}\left(\partial_t(a_{(\xi)}^2 g_{(\xi)})\chi_q^2\overline{\phi}_{(\xi,g)}^2\overline{\psi}_{(\xi,g)}^2\xi\right)(:= R_{osc,t})\\
&+ (\chi_q^2)'\mathcal{R}(\overline{w}_{q+1}^{(t)}+\overline{w}_{q+1}^{(o)}) - \sigma^{-1}\chi_q^2\mathbb{P}_{\neq 0}\sum_{\xi\in\overline{\Lambda}}h_{(\xi)}\partial_t(a_{(\xi)}^2\xi\mathring{\otimes}\xi)(:= R_{osc,o})\\
&+ w_{q+1}\mathring{\otimes}w_{q+1}.
\end{aligned}
$$

Then the Reynolds stress at the level $q+1$ is given by $\mathring{R}_{q+1} = R_{lin} + R_{cor} + R_{osc}$. It is easy to see that $\mathring{R}_{q+1}$ is a trace-free and symmetric matrix satisfying $\mathring{R}_{q+1} = 0$ on $[0, T_{q+1}]$.

5.3.4. *Estimate of* $\mathring{R}_{q+1}$. We estimate each term in the definition of $\mathring{R}_{q+1}$ separately.

For the linear error R_{lin}, by the estimates for the building blocks in (C.14), (C.16), the estimate for $a_{(\xi)}$ in (5.31) and the identity (5.34) we obtain for $\epsilon > 0$ small enough

$$
\begin{aligned}
\|\mathcal{R}\partial_t(\chi_q\overline{w}_{q+1}^{(p)} + \chi_q\overline{w}_{q+1}^{(c)})\|_{L_t^1L^1} &\lesssim \sum_{\xi\in\overline{\Lambda}}\|\chi_q\|_{C_t^1}\|a_{(\xi)}\|_{C_{t,x}^1}(\|\partial_t(g_{(\xi)}V_{(\xi,g)})\|_{L_t^1L^{1+\epsilon}} + \|g_{(\xi)}V_{(\xi,g)}\|_{L_t^1L^{1+\epsilon}})\\
&\lesssim l^{-3d-8}r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\Big(\frac{r_\perp\overline{\mu}}{r_\parallel} + \frac{\sigma\eta^{-\frac12}}{\lambda_{q+1}}\Big)r_\perp^{-d\epsilon} \lesssim l^{-3d-8}\Big(\frac{r_\perp}{r_\parallel}\lambda_{q+1}^{\frac{1}{2N}} + r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\eta^{-\frac12}\Big)\lambda_{q+1}^{d\epsilon}\\
&\lesssim \lambda_{q+1}^{(6d+17)\alpha-\frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},
\end{aligned}
\tag{5.48}
$$

where we note that $\mathcal{R}\mathrm{div}$ is not L^p bounded for $p > 1$, and choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used the choice of parameters in (5.1), (5.9), (5.10) and used the conditions on the parameters to have $(6d+18)\alpha < \frac{1}{2N}$.

The definition of w_{q+1} is analogous to that of $\overline{w}_{q+1}$, allowing us to bound it in a similar manner. By the estimates on the building blocks in (C.7), (C.16), and the estimate on $\tilde{\chi}$ in Proposition 5.2 we obtain for $\epsilon > 0$ small enough

$$
\begin{aligned}
\|\mathcal{R}\partial_t w_{q+1}\|_{L_t^1L^1} &\lesssim \sum_{n\geqslant 3}\sum_{\xi\in\Lambda^n}\|\tilde{\chi}(\zeta|M_l| - n)(\frac{n}{\zeta})^{\frac12}\|_{C_{t,x}^1}\Big(\|\partial_t(g_{(\xi)}V_{(\xi,g)})\|_{L_t^1L^{1+\epsilon}} + \|g_{(\xi)}V_{(\xi,g)}\|_{L_t^1L^{1+\epsilon}}\Big)\\
&\lesssim l^{-3d-8}\Big(\frac{r_\perp}{r_\parallel} + r_\perp^{\frac{d-1}{2}}r_\parallel^{\frac12}\eta^{-\frac12}\Big)\lambda_{q+1}^{d\epsilon} \lesssim \lambda_{q+1}^{(6d+17)\alpha-\frac1N} \lesssim \lambda_{q+1}^{-\alpha},
\end{aligned}
$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$ and used conditions on the parameters to have $(6d+18)\alpha < \frac{1}{N}$.

Using (5.1), the bounds in (5.47), (5.24) and the boundedness property of the inverse divergence operator in Theorem A.1 we have for $0 \leqslant \nu \leqslant 1$

$$\nu \|\mathcal{R}\Delta(w_{q+1}+\overline{w}_{q+1})\|_{L^1_tL^1} \lesssim \|w_{q+1}\|_{L^1_tW^{1,s}} + \|\overline{w}_{q+1}\|_{L^1_tW^{1,s}} \lesssim \lambda_{q+1}^{-\alpha}.$$

By the estimates of v_q in (4.4), and the estimates of the components of $w_{q+1}, \overline{w}_{q+1}$ in (5.40)-(5.43), (5.18) and (5.19) we deduce

$$\begin{aligned}
&\|v_l \otimes (w_{q+1}+\overline{w}_{q+1}) + (w_{q+1}+\overline{w}_{q+1}) \otimes v_l\|_{L^1_tL^1} \lesssim \|v_l\|_{C^0_{t,x}} (\|\overline{w}_{q+1}\|_{C_tL^{1+\epsilon}} + \|w_{q+1}\|_{C_tL^{1+\epsilon}}) \\
&\quad \lesssim C_0 \lambda_q^{4d+3} l^{-6d-15} (r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{2}} r_\parallel^{\frac{1}{1+\epsilon}-\frac{1}{2}} \eta^{-\frac{1}{2}} + \sigma^{-1}) \lesssim C_0 \lambda_{q+1}^{(12d+30)\alpha - \frac{1}{2N}+\epsilon d} \lesssim C_0 \lambda_{q+1}^{-\alpha},
\end{aligned}$$

where we used (5.1), (5.9) and choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used the conditions on the parameters to have $(12d+32)\alpha < \frac{1}{2N}$.

The corrector error is estimated using the choice of parameters (5.1), (5.9) and the bounds on perturbations in (5.39)-(5.43), (5.17)-(5.19) as

$$\begin{aligned}
\|R_{cor}\|_{L^1_tL^1} &\leqslant \|\chi_q \overline{w}^{(c)}_{q+1} + \chi_q^2 \overline{w}^{(t)}_{q+1} + \chi_q^2 \overline{w}^{(o)}_{q+1} + w^{(c)}_{q+1}\|_{L^2_tL^2} \\
&\qquad \times (\|w_{q+1}\|_{L^2_tL^2} + \|\overline{w}_{q+1}\|_{L^2_tL^2} + \|w^{(p)}_{q+1}\|_{L^2_tL^2} + \|\overline{w}^{(p)}_{q+1}\|_{L^2_tL^2}) \\
&\lesssim (l^{-4d-6} \lambda_{q+1}^{-\frac{1}{2N}} + \lambda_{q+1}^{-\alpha}) C_v C_0 \lesssim C_0 \lambda_{q+1}^{-\alpha},
\end{aligned}$$

where we used conditions on the parameters to have $(8d+13)\alpha < \frac{1}{2N}$. We choose a large enough to absorb the constant C_v and the implicit constant.

Now we consider the oscillation term. In order to bound the first term $R_{osc,x}$, by the boundedness property of inverse divergence operator in Theorems A.1 and A.2, the estimates for the building blocks in (C.14), (C.16) and the estimates for $a_{(\xi)}$ in (5.31) we have

$$\begin{aligned}
\|R_{osc,x}\|_{L^1_tL^1} &\lesssim \sum_{\xi \in \overline{\Lambda}} \|a^2_{(\xi)}\|_{C^2_{t,x}} \|\mathcal{R}(\overline{W}_{(\xi,g)} \otimes \overline{W}_{(\xi,g)})\|_{C_tL^{1+\epsilon}} \|g^2_{(\xi)}\|_{L^1_t} \\
&\lesssim l^{-6d-14} (r_\perp \lambda_{q+1})^{-1} r_\perp^{(d-1)(\frac{1}{1+\epsilon}-1)} r_\parallel^{\frac{1}{1+\epsilon}-1} \lesssim \lambda_{q+1}^{(12d+28)\alpha - \frac{1}{N}+d\epsilon} \lesssim \lambda_{q+1}^{-\alpha},
\end{aligned}$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We also used (5.1), (5.9), (5.10) and conditions on the parameters to have $(12d+30)\alpha < \frac{1}{N}$.

For the second term $R_{osc,t}$ we use the estimates for the building blocks $g_{(\xi)}, \overline{\phi}_{(\xi)}, \overline{\psi}_{(\xi)}$ and for $a_{(\xi)}$ in (C.16), (C.12), (C.13), and (5.31) respectively to deduce

$$\begin{aligned}
\|R_{osc,t}\|_{L^1_tL^1} &\lesssim \frac{1}{\mu} \sum_{\xi \in \overline{\Lambda}} \|a^2_{(\xi)}\|_{C^1_{t,x}} \|g_{(\xi)}\|_{W^{1,1}} \|\overline{\phi}^2_{(\xi,g)} \overline{\psi}^2_{(\xi,g)} \xi\|_{C_tL^{1+\epsilon}} \\
&\lesssim l^{-5d-10} \overline{\mu}^{-1} r_\perp^{(d-1)(\frac{1}{1+\epsilon}-1)} r_\parallel^{\frac{1}{1+\epsilon}-1} \sigma \eta^{-\frac{1}{2}} \\
&\lesssim l^{-5d-10} \lambda_{q+1}^{d\epsilon} r_\perp^{\frac{d-1}{2}} r_\parallel^{\frac{1}{2}} \eta^{-\frac{1}{2}} \sigma \lesssim \lambda_{q+1}^{(10d+21)\alpha - \frac{1}{2N}} \lesssim \lambda_{q+1}^{-\alpha},
\end{aligned}$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$ and used (5.1), (5.9) and condition on the parameters to have $(10d+22)\alpha < \frac{1}{2N}$.

We continue with $R_{osc,o}$. By the bound on the cut-off function in (5.38), the bounds on temporal jets $g_{(\xi)}, h_{(\xi)}$ in (C.16), the bounds on perturbations $\overline{w}^{(t)}_{q+1}, \overline{w}^{(o)}_{q+1}$ in (5.42), (5.43) and the bounds on $a_{(\xi)}$ in

(5.31) we have

$$\|R_{osc,o}\|_{L_t^1L^1} \lesssim \|\chi_q^2\|_{C_t^1}(\|\overline{w}_{q+1}^{(t)}\|_{C_tL^{1+\epsilon}} + \|\overline{w}_{q+1}^{(o)}\|_{C_tL^{1+\epsilon}}) + \sigma^{-1}\sum_{\xi\in\overline{\Lambda}}\|h_{(\xi)}\|_{L_t^\infty}\|a_{(\xi)}^2\|_{C_{t,x}^1}$$

$$\lesssim l^{-4d-7}r_\perp^{\frac{d-1}{1+\epsilon}-\frac{d-1}{2}}r_\parallel^{\frac{1}{1+\epsilon}-\frac{1}{2}}\eta^{-\frac{1}{2}} + l^{-6d-16}\sigma^{-1} \lesssim C_0^2\lambda_{q+1}^{(12d+32)\alpha-\frac{1}{2N}} \lesssim C_0^2\lambda_{q+1}^{-\alpha},$$

where we choose $\epsilon > 0$ small enough such that $d\epsilon < \alpha$. We used (5.1), (5.9), (5.10) and conditions on the parameters to have $(12d+33)\alpha < \frac{1}{2N}$. We choose a large enough to absorb the universal constant.

By (5.20) we have that

$$\|w_{q+1}\mathring{\otimes}w_{q+1}\|_{L_t^1L^1} \leqslant \frac{C_0^2}{2}\delta_{q+2}.$$

Summarizing all the estimates above we obtain the first term in (4.5):

$$\|\mathring{R}_{q+1}\|_{L_t^1L^1} \leqslant \frac{C_0^2}{2}\delta_{q+2} + CC_0^2\lambda_{q+1}^{-\alpha} \leqslant C_0^2\delta_{q+2}.$$

Here we used the condition (5.1) to deduce the last inequality.

Thus we finish the proof of Proposition 4.1.

Appendix A. Some technical tools

We collect some technical tools used in the construction of convex integration schemes.

A.1. **Inverse divergence operators.**

We first recall the following inverse divergence operator $\mathcal{R}$ as in [CL22a, Appendix B.2], which acts on vector fields v with $\int_{\mathbb{T}^d} v\mathrm{d}x = 0$ as

$$(\mathcal{R}v)_{ij} = \mathcal{R}_{ijk}v_k,$$

where

$$\mathcal{R}_{ijk} = \frac{2-d}{d-1}\Delta^{-2}\partial_i\partial_j\partial_k - \frac{1}{d-1}\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.$$

Then $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^d$, and $\mathcal{R}$ is a right inverse of the div operator, i.e. $\mathrm{div}(\mathcal{R}v) = v$. Here we use the notation $\mathcal{R}v := \mathcal{R}(v - \int v\mathrm{d}x)$ for a general vector field. In the following we use $C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ to denote the space of smooth functions from $\mathbb{T}^d$ to $\mathbb{R}^d$, and we use $C_0^\infty(\mathbb{T}^d;\mathbb{R}^d)$ to denote the subspace of functions with zero spatial mean. Similarly, we define $C^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})$ and $C_0^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})$. By $\mathcal{S}_0^{d\times d}$ we denote the space of symmetric trace-free matrices.

Theorem A.1. ([CL22a, Theorem B.3]) *Let $1 \leqslant p \leqslant \infty$. For any vector field $f \in C_0^\infty(\mathbb{T}^d;\mathbb{R}^d)$, $\sigma \in \mathbb{N}$,*

$$\|\mathcal{R}f(\sigma\cdot)\|_{L^p} \lesssim \sigma^{-1}\|f\|_{L^p}.$$

We also introduce the bilinear version $\mathcal{B} : C^\infty(\mathbb{T}^d;\mathbb{R}^d) \times C_0^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d}) \to C^\infty(\mathbb{T}^d;\mathcal{S}_0^{d\times d})$ by

$$(\mathcal{B}(v,A))_{ij} = v_m\mathcal{R}_{ijk}A_{mk} - \mathcal{R}(\partial_i v_m\mathcal{R}_{ijk}A_{mk}).$$

Theorem A.2. ([CL22a, Theorem B.4]) *Let $1 \leqslant p \leqslant \infty$. For any $v \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ and $A \in C_0^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})$, we have $\mathrm{div}(\mathcal{B}(v,A)) = vA - \int_{\mathbb{T}^d} vA\mathrm{d}x$, and*

$$\|\mathcal{B}(v,A)\|_{L^p} \lesssim \|v\|_{C^1}\|\mathcal{R}A\|_{L^p}.$$

We also need to define the inverse divergence operator acting on scalars. We define $\mathcal{R}_1 := \nabla\Delta^{-1}$ as a right inverse of the div operator, i.e. $\operatorname{div}(\mathcal{R}_1 v) = v$ for scalars v with $\int_{\mathbb{T}^d} v\mathrm{d}x = 0$. Here we use the notation $\mathcal{R}_1 v := \mathcal{R}_1(v - \int v\mathrm{d}x)$ for a general scalar function v. Then since $\mathcal{R}_1$ is a Calderón-Zygmund operator, we have

Theorem A.3. *Let $1 \leqslant p \leqslant \infty$. For any vector field $f \in C_0^\infty(\mathbb{T}^d;\mathbb{R})$, $\sigma \in \mathbb{N}$,*

$$\|\mathcal{R}_1 f(\sigma\cdot)\|_{L^p} \lesssim \sigma^{-1}\|f\|_{L^p}.$$

We introduce the the bilinear version $\mathcal{B}_1 : C^\infty(\mathbb{T}^d;\mathbb{R}) \times C_0^\infty(\mathbb{T}^d;\mathbb{R}) \to C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ by

$$\mathcal{B}_1(v,f) = v\mathcal{R}_1 f - \mathcal{R}_1\Big(\nabla v\cdot\mathcal{R}_1 f + \int vf\mathrm{d}x\Big).$$

Theorem A.4. ([BCDL21, Lemma 3.3]) *Let $1 \leqslant p \leqslant \infty$. For any $v \in C^\infty(\mathbb{T}^d;\mathbb{R})$ and $f \in C_0^\infty(\mathbb{T}^d;\mathbb{R})$, we have $\operatorname{div}(\mathcal{B}_1(v,f)) = vf - \int_{\mathbb{T}^d} vf\mathrm{d}x$, and for $\sigma \in \mathbb{N}$,*

$$\|\mathcal{B}_1(v,f(\sigma\cdot))\|_{W^{k,p}} \lesssim \sigma^{k-1}\|v\|_{C^{k+1}}\|f\|_{W^{k,p}}.$$

We recall the following estimates stationary phase bounds from [DSJ17, Lemma 2.2], which are useful for estimating the oscillation errors.

Proposition A.5. *Let $\lambda\xi \in \mathbb{Z}^d, N \geqslant 1$, $a \in C^\infty(\mathbb{T}^d;\mathbb{R}), b \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, $\Phi \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ be smooth functions and assume that there exists a constant $\hat{C} \geqslant 1$ such that $\hat{C}^{-1} \leqslant |\nabla\Phi| \leqslant \hat{C}$ holds on $\mathbb{T}^d$. Then*

$$\left|\int_{\mathbb{T}^d} a(x)e^{i\lambda\xi\cdot\Phi(x)}\mathrm{d}x\right| \lesssim \frac{\|a\|_{C^N} + \|a\|_{C^0}\|\nabla\Phi\|_{C^N}}{\lambda^N}, \tag{A.1}$$

and for the operators $\mathcal{R}$ and $\mathcal{R}_1$ defined above, we have for $\alpha \in (0,1)$

$$\left\|\mathcal{R}\left(b(x)e^{i\lambda\xi\cdot\Phi(x)}\right)\right\|_{C^\alpha} \lesssim \frac{\|b\|_{C^0}}{\lambda^{1-\alpha}} + \frac{\|b\|_{C^{N+\alpha}} + \|b\|_{C^0}\|\nabla\Phi\|_{C^{N+\alpha}}}{\lambda^{N-\alpha}},$$
$$\left\|\mathcal{R}_1\left(a(x)e^{i\lambda\xi\cdot\Phi(x)}\right)\right\|_{C^\alpha} \lesssim \frac{\|a\|_{C^0}}{\lambda^{1-\alpha}} + \frac{\|a\|_{C^{N+\alpha}} + \|a\|_{C^0}\|\nabla\Phi\|_{C^{N+\alpha}}}{\lambda^{N-\alpha}},$$

where the implicit constants depend on $\hat{C}$, α and N, but not on the frequency λ.

A.2. **Commutator estimate.** We recall the following commutator estimate which can be seen as a generalization of [BDLSV19, Proposition A.2]:

Lemma A.6. *Let $f, g \in C^\infty(\mathbb{T}^3 \times [0,1])$ and ψ a standard radial smooth and compactly supported kernel. For any $r \geqslant 0$ and $\theta_1, \theta_2 \in (0,1]$ we have the estimate*

$$\|(f*\psi_l)(g*\psi_l) - (fg)*\psi_l\|_{C^r} \lesssim l^{\theta_1+\theta_2-r}\|f\|_{C^{\theta_1}}\|g\|_{C^{\theta_2}},$$

where the implicit constant depends only on r and ψ.

A.3. **Estimates for transport equations.** Now we recall some standard estimates for solutions to the transport equation:

$$\begin{aligned} \partial_t f + v\cdot\nabla f &= g, \\ f(0) &= f_0, \end{aligned} \tag{A.2}$$

where v is a given smooth vector field. We have the following proposition.

Proposition A.7. [BDLSV19, Proposition B.1] *Assume* $t\|v\|_{C^1} \leqslant 1$. *Then, any solution f of* (A.2) *satisfies*

$$\|f(t)\|_{C^0} \leqslant \|f_0\|_{C^0} + \int_0^t \|g(\tau)\|_{C^0} \mathrm{d}\tau,$$

$$\|f(t)\|_{C^\alpha} \leqslant e^\alpha \left(\|f_0\|_{C^\alpha} + \int_0^t \|g(\tau)\|_{C^\alpha} \mathrm{d}\tau \right),$$

for all $\alpha \in [0,1)$. *More generally, for any* $N \geqslant 1$ *and* $\alpha \in [0,1)$

$$\|f(t)\|_{C^{N+\alpha}} \lesssim \|f_0\|_{C^{N+\alpha}} + |t|\|v\|_{C^{N+\alpha}}\|f_0\|_{C^1} + \int_0^t \left(\|g(\tau)\|_{C^{N+\alpha}} + (t-\tau)\|v\|_{C^{N+\alpha}}\|g(\tau)\|_{C^1} \right) \mathrm{d}\tau,$$

where the implicit constant depends on N *and* α.

Consequently, the flow Φ *of* v *starting at time* 0 *(i.e.* $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = v(\Phi(t),t)$ *and* $\Phi(0) = \mathrm{Id}$*) satisfies*

$$\|\nabla\Phi(t) - \mathrm{Id}\|_{C^0} \lesssim |t|\|v\|_{C^1},$$

$$\|\Phi(t)\|_{C^N} \lesssim |t|\|v\|_{C^N}, \; N \geqslant 2.$$

A.4. **Improved Hölder inequality on** $\mathbb{T}^d$**.** This lemma improves the usual Hölder inequality by using the decorrelation between frequencies, which establish the desired estimates.

Lemma A.8. ([CL22a, Theorem B.1]) *Let* $d \geqslant 2, p \in [1,\infty]$ *and* $a, f : \mathbb{T}^d \to \mathbb{R}$ *be smooth functions. Then for any* $\sigma \in \mathbb{N}$,

$$|\|af(\sigma\cdot)\|_{L^p} - \|a\|_{L^p}\|f\|_{L^p}| \lesssim \sigma^{-1/p}\|a\|_{C^1}\|f\|_{L^p}.$$

Appendix B. Building blocks and auxiliary estimates in Section 3

In this section, we first introduce the building blocks used in the convex integration method. Then we provide some auxiliary estimates in the gluing and perturbation steps in Section 3.

B.1. **Mikado flows.** In this section we revisit the Mikado flows introduced in [DSJ17] and extend their applicability to transport equations.

First we introduce the following geometric lemmas in $\mathbb{R}^d$, where $d \geqslant 2$. The first lemma is to show that every vector in the annulus $\overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$ can be written as a positive combination of vectors in some subset of $\mathbb{S}^{d-1} \cap \mathbb{Q}^d$, which generalizes [BCDL21, Lemma 3.1] about the sphere $\partial B_1(0)$ to the annulus. The proof follows from a similar argument as in [BCDL21, Lemma 3.1].

Lemma B.1. *Let* $B_r(0)$ *denote the ball of radius* r *around 0 in* $\mathbb{R}^d$. *There exists a finite set* $\Lambda \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ *and a non-negative* C^∞*-function* $\Gamma_\xi : \overline{B}_1(0)\backslash B_{\frac{1}{2}}(0) \to \mathbb{R}$ *such that for every* $\frac{1}{2} \leqslant |R| \leqslant 1$

$$R = \sum_{\xi\in\Lambda} \Gamma_\xi(R)\xi.$$

Proof. For each vector $v \in \overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$, we consider a collection $\Lambda(v) = \{\xi_1(v), ..., \xi_d(v)\} \subset \partial B_1(0)$ of linearly independent unit vectors in $\mathbb{Q}^d$ with the property that the d-dimensional open symplex $\Sigma(v)$ with vertices $0, 2\xi_1(v), ..., 2\xi_d(v)$ contains v. Since $\{\Sigma(v) : v \in \overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)\}$ is an open cover of $\overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$, we consider a finite sub-cover and the corresponding collections $\Lambda_i = \Lambda(v_i), \; i = 1, ..., N$. We set

$$\Lambda := \cup_{j=1}^N \Lambda_j.$$

For each fixed j, each vector $R \in \overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$ can be written in a unique way as linear combination of the vectors in $\Lambda_j = \{\xi_{j,i}\}_{i=1,...,d}$. If denotes $b_{j,i}(R)$ the corresponding coefficients which depend linearly on

R, then the latter are all strictly positive if R belongs to $\Sigma(v_j)$. Then we consider a partition of unity χ_j on $\overline{B}_1(0)\backslash B_{\frac{1}{2}}(0)$ associated to this cover and for every $\xi_{j,i} = \xi \in \Lambda$ we set

$$\Gamma_\xi(R) := \chi_j(R) b_{j,i}(R).$$

The coefficients Γ_ξ are then smooth nonnegative functions of R. □

Then the next lemma shows that symmetric matrices in $\overline{B}_{\frac{1}{2}}(\mathrm{Id})$ can be written as a combination of some symmetric tensors.

Lemma B.2. [CL22a, Lemma 4.2] *Let $\overline{B}_{\frac{1}{2}}(\mathrm{Id})$ denote the closed ball of radius $\frac{1}{2}$ around the identity matrix* Id*, in the space of $d\times d$ symmetric matrices. There exists a finite set $\overline{\Lambda} \in \mathbb{S}^{d-1}\cap\mathbb{Q}^d$ such that for each $\xi \in \overline{\Lambda}$ there exists a C^∞-function γ_ξ: $\overline{B}_{\frac{1}{2}}(\mathrm{Id}) \to \mathbb{R}$ such that for every symmetric matrix satisfying $|R-\mathrm{Id}| \leqslant 1/2$*

$$R = \sum_{\xi\in\overline{\Lambda}} \gamma_\xi^2(R)(\xi\otimes\xi).$$

Then by taking suitable rational rotations, there are four disjoint sets $\Lambda^1, \Lambda^2, \overline{\Lambda}^1, \overline{\Lambda}^2$ such that both Λ^1 and Λ^2 enjoy the property of Lemma B.1, and similarly both $\overline{\Lambda}^1$ and $\overline{\Lambda}^2$ enjoy the property of Lemma B.2. For convenience, in the following we set $\Lambda := \Lambda^1 \cup \Lambda^2 \cup \overline{\Lambda}^1 \cup \overline{\Lambda}^2$.

Now we apply the two lemmas in dimension 3 and obtain the set Λ. For each $\xi \in \Lambda$ we define $A_\xi^i \in \mathbb{S}^2 \cap \mathbb{Q}^3,\ i = 1, 2$ such that $\{\xi, A_\xi^i, i = 1, 2\}$ form an orthonormal basis in $\mathbb{R}^3$. We label by n_* the smallest natural number such that

$$\{n_*\xi, n_*A_\xi^i, i = 1, 2\} \subset \mathbb{Z}^3.$$

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with support in a ball of radius ϵ_Λ, where $\epsilon_\Lambda > 0$ will be chosen later in terms of Λ. We normalize Φ such that $\phi = -\Delta\Phi$ obeys

$$\int_{\mathbb{R}^2} \phi^2(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 = 1.$$

By definition we know $\int_{\mathbb{R}^2} \phi \mathrm{d}x = 0$.

We periodize them so that they are viewed as periodic functions on $\mathbb{T}^2$. Consider a large parameter $\lambda \in \mathbb{N}$. For every $\xi \in \Lambda$ we introduce

$$\Phi_{(\xi)}(x) := \Phi(n_*\lambda(x-\alpha_\xi)\cdot A_\xi^1, n_*\lambda(x-\alpha_\xi)\cdot A_\xi^2),$$
$$\phi_{(\xi)}(x) := \phi(n_*\lambda(x-\alpha_\xi)\cdot A_\xi^1, n_*\lambda(x-\alpha_\xi)\cdot A_\xi^2),$$

where $\alpha_\xi \in \mathbb{Q}^2$ are shifts to ensure that $\{\phi_{(\xi)}\}_{\xi\in\Lambda}$ have mutually disjoint supports by choosing $\epsilon_\Lambda > 0$ correspondingly. For the existence of such shifts α_ξ, we refer to [BV19a, Section 6.4].

By construction, we know that $\xi\cdot\nabla\Phi_{(\xi)} = \xi\cdot\nabla\phi_{(\xi)} = 0$ and $(n_*\lambda)^2\phi_{(\xi)} = -\Delta\Phi_{(\xi)}$.

With this notation, the Mikado flows $W_{(\xi)}\colon \mathbb{T}^3 \to \mathbb{R}^3, \Theta_{(\xi)}\colon \mathbb{T}^3 \to \mathbb{R}$ are defined as

$$W_{(\xi)}(x) := W_{\xi,\lambda}(x) := \xi\,\phi_{(\xi)}(x),\ \ \Theta_{(\xi)}(x) := \Theta_{\xi,\lambda}(x) := \phi_{(\xi)}(x). \tag{B.1}$$

Since $\xi\cdot\nabla\phi_{(\xi)} = 0$, we immediately deduce that

$$\mathrm{div} W_{(\xi)} = 0,\ \ \mathrm{div}\left(W_{(\xi)}\otimes W_{(\xi)}\right) = 0 \ \text{ and } \ \mathrm{div}\left(W_{(\xi)}\Theta_{(\xi)}\right) = 0.$$

By construction, the functions $W_{(\xi)}$ have zero mean on $\mathbb{T}^3$ and are in fact $(\mathbb{T}/\lambda)^3$-periodic. Moreover, by our choice of α_ξ we have that

$$W_{(\xi)}\otimes W_{(\xi')} \equiv 0,\ \ W_{(\xi)}\Theta_{(\xi')} \equiv 0 \qquad \text{whenever} \qquad \xi \neq \xi' \in \Lambda\,, \tag{B.2}$$

and the normalization of $\phi_{(\xi)}$ ensures that

$$\fint_{\mathbb{T}^3} W_{(\xi)}(x)\otimes W_{(\xi)}(x)\,\mathrm{d}x = \xi\otimes\xi, \quad \fint_{\mathbb{T}^3} W_{(\xi)}(x)\Theta_{(\xi)}(x)\,\mathrm{d}x = \xi\,. \tag{B.3}$$

Lastly, using (B.3), the definition of the functions γ_ξ, Γ_ξ in Lemma B.1, Lemma B.2 and the L^2 normalization of the functions $\phi_{(\xi)}$ we have that

$$\sum_{\xi\in\overline{\Lambda}^i}\gamma_\xi^2(R)\fint_{\mathbb{T}^3} W_{(\xi)}(x)\otimes W_{(\xi)}(x)\mathrm{d}x = R\,, \quad \sum_{\xi\in\Lambda^i}\Gamma_\xi(M)\fint_{\mathbb{T}^3} W_{(\xi)}(x)\Theta_{(\xi)}(x)\mathrm{d}x = M, \tag{B.4}$$

for every $i=1,2$, any symmetric matrix $R\in\overline{B}_{\frac12}(\mathrm{Id})$, and $M\in\overline{B}_1(0)\backslash B_{\frac12}(0)$.

To define the incompressibility corrector in Section 3.3.1-Section 3.3.2, we note that $W_{(\xi)}$ can be written as curl $V_{(\xi)}$ similar as [BV19a, (6.31)], where we define

$$V_{(\xi)} := \frac{1}{(n_*\lambda)^2}\nabla\Phi_{(\xi)}\times\xi.$$

With this notation we have the bounds for $N\geqslant 0$

$$\left\|W_{(\xi)}\right\|_{C^N} + \left\|\Theta_{(\xi)}\right\|_{C^N} + \lambda\left\|V_{(\xi)}\right\|_{C^N} \lesssim \lambda^N. \tag{B.5}$$

B.2. **The estimates in gluing steps.** In this section, we provide some estimates on the glued solutions for the Euler equations, which follow by a similar argument as in [BDLSV19, Section3, Section4]. Here we give the detailed calculations taking into account the different definitions of the parameters.

Proof of Proposition 3.1. First by [BDLSV19, Proposition 3.1], (3.3) and (3.9) we have for $t\in[t_i-\tau_q, t_i+\tau_q], N\geqslant 0$

$$\|v_i(t)\|_{C^{N+1+\alpha}} \lesssim \|v_l\|_{C^{N+1+\alpha}} \lesssim \delta_q^{1/2}\lambda_q l^{-N-\alpha} \lesssim \tau_q^{-1}l^{-N+\alpha}. \tag{B.6}$$

We note that $v_l - v_i$ obeys

$$(\partial_t + v_l\cdot\nabla)(v_l-v_i) = -(v_l-v_i)\cdot\nabla v_i - \nabla(\pi_l-\pi_i) + \mathrm{div}\mathring{R}_l,$$

and

$$\nabla(\pi_l-\pi_i) = \nabla\Delta^{-1}\mathrm{div}\Big((v_l-v_i)\cdot\nabla v_l + (v_l-v_i)\cdot\nabla v_i\Big) + \nabla\Delta^{-1}\mathrm{div}\mathrm{div}\mathring{R}_l.$$

Then by the transport estimate in Proposition A.7, (3.5) and (B.6) we have for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\begin{aligned}\|(v_l-v_i)(t)\|_{C^\alpha} &\lesssim \left|\int_{t_i}^t \|(v_l-v_i)\cdot\nabla v_i(s)\|_{C^\alpha} + \|\nabla(\pi_l-\pi_i)(s)\|_{C^\alpha} + \|\mathring{R}_l\|_{C^{1+\alpha}}\mathrm{d}s\right|\\ &\lesssim \tau_q\|\mathring{R}_l\|_{C^{1+\alpha}} + \left|\int_{t_i}^t \|\left(v_l-v_i\right)(s)\|_{C^\alpha}\left(\|v_i(s)\|_{C^{1+\alpha}} + \|v_l\|_{C^{1+\alpha}}\right)\mathrm{d}s\right|\\ &\lesssim \tau_q\delta_{q+1}l^{-1+\alpha} + \left|\int_{t_i}^t \|\left(v_l-v_i\right)(s)\|_{C^\alpha}\tau_q^{-1}l^\alpha\mathrm{d}s\right|.\end{aligned}$$

By Gronwall's inequality we conclude that for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\|(v_l-v_i)(t)\|_{C^\alpha} \lesssim \tau_q\delta_{q+1}l^{-1+\alpha}.$$

Moreover, by (3.5) and (B.6) we have for $t\in[t_i-\tau_q, t_i+\tau_q]$

$$\begin{aligned}\|\nabla(\pi_l-\pi_i)(t)\|_{C^\alpha} &\lesssim \|(v_l-v_i)\cdot\nabla v_l(t)\|_{C^\alpha} + \|(v_l-v_i)\cdot\nabla v_i(t)\|_{C^\alpha} + \|\mathring{R}_l\|_{C^{1+\alpha}}\\ &\lesssim \tau_q\delta_{q+1}l^{-1+\alpha}\tau_q^{-1}l^\alpha + \delta_{q+1}l^{-1+\alpha} \lesssim \delta_{q+1}l^{-1+\alpha},\end{aligned}$$

and then we obtain for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}\|(\partial_t + v_l \cdot \nabla)(v_i - v_l)(t)\|_{C^\alpha} &\lesssim \|\nabla v_i (v_l - v_i)(t)\|_{C^\alpha} + \|\mathring{R}_l\|_{C^{1+\alpha}} + \delta_{q+1} l^{-1+\alpha} \\ &\lesssim \tau_q \delta_{q+1} l^{-1+\alpha} \tau_q^{-1} l^\alpha + \delta_{q+1} l^{-1+\alpha} \lesssim \delta_{q+1} l^{-1+\alpha}.\end{aligned}$$

Let θ be a multi-index with $|\theta| = N$, then by (3.5) and (B.6) we have for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}\|\partial^\theta \nabla(\pi_l - \pi_i)(t)\|_{C^\alpha} &\lesssim \|\mathring{R}_l\|_{C^{1+N+\alpha}} + \|(v_l - v_i)(t)\|_{C^\alpha}(\|v_i(s)\|_{C^{1+N+\alpha}} + \|v_l\|_{C^{1+N+\alpha}}) \\ &\quad + \|(v_l - v_i)(t)\|_{C^{N+\alpha}}(\|v_i(t)\|_{C^{1+\alpha}} + \|v_l\|_{C^{1+\alpha}}) \\ &\lesssim \delta_{q+1} l^{-N-1+\alpha} + \tau_q \delta_{q+1} l^{-1-N+\alpha} \tau_q^{-1} l^\alpha + \|v_l - v_i\|_{C^{N+\alpha}} \tau_q^{-1} l^\alpha \\ &\lesssim \delta_{q+1} l^{-N-1+\alpha} + \|v_l - v_i\|_{C^{N+\alpha}} \tau_q^{-1} l^\alpha.\end{aligned}$$

Moreover, by a similar calculation we have for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\|\partial^\theta (\partial_t + v_l \cdot \nabla)(v_l - v_i)(t)\|_{C^\alpha} \lesssim \delta_{q+1} l^{-N-1+\alpha} + \|(v_l - v_i)(t)\|_{C^{N+\alpha}} \tau_q^{-1} l^\alpha.$$

Using the transport estimate in Proposition A.7 once again, we have for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}\|\partial^\theta (v_l - v_i)(t)\|_{C^\alpha} &\lesssim \left| \int_{t_i}^t \|(\partial_t + v_l \cdot \nabla)\partial^\theta\,(v_l - v_i)\,(s)\|_{C^\alpha} \mathrm{d}s \right| \\ &\lesssim \left| \int_{t_i}^t \|[(\partial_t + v_l \cdot \nabla), \partial^\theta](v_l - v_i)(s)\|_{C^\alpha} + \delta_{q+1} l^{-N-1+\alpha} + \|(v_l - v_i)(s)\|_{C^{N+\alpha}} \tau_q^{-1} l^\alpha \mathrm{d}s \right|.\end{aligned}$$

By interpolation and (B.6) we have for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}\|[(\partial_t + v_l \cdot \nabla), \partial^\theta](v_l - v_i)(t)\|_{C^\alpha} &\lesssim \|v_l\|_{C^{1+\alpha}} \|(v_l - v_i)(t)\|_{C^{N+\alpha}} + \|v_l\|_{C^{1+N+\alpha}} \|(v_l - v_i)(t)\|_{C^\alpha} \\ &\lesssim \tau_q^{-1} l^\alpha \|(v_l - v_i)(t)\|_{C^{N+\alpha}} + \delta_{q+1} l^{-N-1+\alpha},\end{aligned}$$

which implies that for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\|\partial^\theta (v_l - v_i)(t)\|_{C^\alpha} \lesssim \left| \int_{t_i}^t l^{-N-1+\alpha} \delta_{q+1} + \|(v_l - v_i)(s)\|_{C^{N+\alpha}} \tau_q^{-1} l^\alpha \mathrm{d}s \right|.$$

Using Gronwall's inequality, we obtain (3.13): for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\|(v_l - v_i)(t)\|_{C^{N+\alpha}} + \tau_q \|(\partial_t + v_l \cdot \nabla)(v_i - v_l)(t)\|_{C^{N+\alpha}} \lesssim \tau_q \delta_{q+1} l^{-N-1+\alpha}.$$

By a similar calculation as in [BDLSV19, (3.20)], $z_i - z_l$ obeys

$$\begin{aligned}(\partial_t + v_l \cdot \nabla)(z_i - z_l) &= \Delta^{-1} \mathrm{curl}\,\mathrm{div} \mathring{R}_l + \Delta^{-1} \nabla \mathrm{div}\Big(((z_i - z_l) \cdot \nabla) v_l\Big) \\ &\quad + \Delta^{-1} \mathrm{curl}\,\mathrm{div}\Big(((z_i - z_l) \times \nabla) v_l + ((z_i - z_l) \times \nabla) v_i^T\Big).\end{aligned}$$

Consequently, by interpolation, (3.5) and (B.6) we have for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\begin{aligned}&\|(\partial_t + v_l \cdot \nabla)(z_i - z_l)(t)\|_{C^{N+\alpha}} \\ &\lesssim (\|v_i(t)\|_{C^{N+1+\alpha}} + \|v_l\|_{C^{N+1+\alpha}}) \|(z_i - z_l)(t)\|_{C^\alpha} \\ &\quad + (\|v_i(t)\|_{C^{1+\alpha}} + \|v_l\|_{C^{1+\alpha}}) \|(z_i - z_l)(t)\|_{C^{N+\alpha}} + \|\mathring{R}_l\|_{C^{N+\alpha}} \\ &\lesssim \tau_q^{-1} l^{-N+\alpha} \|(z_i - z_l)(t)\|_{C^\alpha} + \tau_q^{-1} l^\alpha \|(z_i - z_l)(t)\|_{C^{N+\alpha}} + \delta_{q+1} l^{-N+\alpha}.\end{aligned}$$

When $N = 0$, we have by Gronwall's inequality that for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\|(z_i - z_l)(t)\|_{C^\alpha} + \tau_q \|(\partial_t + v_l \cdot \nabla)(z_i - z_l)(t)\|_{C^\alpha} \lesssim \tau_q \delta_{q+1} l^\alpha.$$

By commuting the derivatives in $N + \alpha, N \geqslant 0$ with $\partial_t + v_l \cdot \nabla$ as before, we finish the proof of (3.14). $\square$

Proof of Proposition 3.2. By (3.9), (3.13), (3.14) and the definition of the cut-off function we have

$$\|\overline{v}_q - v_l\|_{C^{N+\alpha}} \lesssim \sum_i \chi_i \|v_i - v_l\|_{C^{N+\alpha}} \lesssim \tau_q \delta_{q+1} l^{-1-N+\alpha} \lesssim \delta_{q+1}^{1/2} l^{-N+\alpha},$$
$$\|\overline{z}_q - z_l\|_{C^{N+\alpha}} \lesssim \sum_i \chi_i \|z_i - z_l\|_{C^{N+\alpha}} \lesssim \tau_q \delta_{q+1} l^{-N+\alpha},$$

which implies (3.17).

To estimate the energy gap, by the same calculation as in [BDLSV19, Proposition 4.4], we obtain on the time interval I_i

$$|\overline{v}_q|^2 - |v_l|^2 = \chi_i(|v_i|^2 - |v_l|^2) + (1-\chi_i)(|v_{i+1}|^2 - |v_l|^2) - \chi_i(1-\chi_i)|v_i - v_{i+1}|^2.$$

Then since v_i solves the Euler equation, $(v_l, \mathring{R}_l)$ solves (2.3), by the basic energy estimate, (3.5) and (B.6), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{T}^3} |v_i|^2 - |v_l|^2 \mathrm{d}x\Big)\right| = \left|2\int_{\mathbb{T}^3} \nabla v_l : \mathring{R}_l \mathrm{d}x\right| \lesssim \|\nabla v_l\|_{C^0} \|\mathring{R}_l\|_{C^0} \lesssim \tau_q^{-1}\delta_{q+1} l^{\alpha},$$

Moreover, after integrating in time we deduce for $t \in [t_i - \tau_q, t_i + \tau_q]$

$$\left|\int_{\mathbb{T}^3} (|v_i|^2 - |v_l|^2)(t)\mathrm{d}x\right| \lesssim \delta_{q+1} l^{\alpha}.$$

Furthermore, together with (3.17) we obtain

$$\left|\int_{\mathbb{T}^3} (|\overline{v}_q|^2 - |v_l|^2)(t)\mathrm{d}x\right| \lesssim \delta_{q+1} l^{\alpha} + \|(v_i - v_{i+1})(t)\|_{C^\alpha}^2 \lesssim \delta_{q+1} l^{\alpha},$$

where we rewrite $v_i - v_{i+1} = (v_i - v_l) - (v_{i+1} - v_l)$. This concludes the proof of (3.18).

For the Reynolds stress, by the definition of the cut-off functions, the definition in (3.16), the choice of parameters in (3.9), the bounds (3.13) and (3.14) we have for $t \in I_i$

$$\begin{aligned}\left\|\mathring{\overline{R}}_q(t)\right\|_{C^{N+\alpha}} &\lesssim \|\partial_t \chi_i\|_{C_t^0} \|\mathcal{R}(v_i - v_{i+1})(t)\|_{C^{N+\alpha}} + \|(v_i - v_{i+1})(t)\|_{C^{N+\alpha}} \|(v_i - v_{i+1})(t)\|_{C^\alpha} \\ &\lesssim \tau_q^{-1}\tau_q \delta_{q+1} l^{-N+\alpha} + \tau_q^2 \delta_{q+1}^2 l^{-2-N+\alpha} \lesssim \delta_{q+1} l^{-N+\alpha},\end{aligned}$$

where we rewrite $v_i - v_{i+1} = (v_i - v_l) - (v_{i+1} - v_l)$. By direct calculation we have for $t \in I_i$

$$\begin{aligned}(\partial_t + v_l \cdot \nabla)\mathring{\overline{R}}_q &= \partial_t^2 \chi_i \mathcal{R}\mathrm{curl}\,(z_i - z_{i+1}) + \partial_t \chi_i \mathcal{R}\mathrm{curl}\,(\partial_t + v_l \cdot \nabla)(z_i - z_{i+1}) \\ &\quad + \partial_t \chi_i [v_l \cdot \nabla, \mathcal{R}\mathrm{curl}\,](z_i - z_{i+1}) + \partial_t(\chi_i^2 - \chi_i)(v_i - v_{i+1}) \mathring{\otimes} (v_i - v_{i+1}) \\ &\quad + (\chi_i^2 - \chi_i)\left((\partial_t + v_l \cdot \nabla)(v_i - v_{i+1})\right) \mathring{\otimes} (v_i - v_{i+1}), \\ &\quad + (\chi_i^2 - \chi_i)(v_i - v_{i+1}) \mathring{\otimes} \left((\partial_t + v_l \cdot \nabla)(v_i - v_{i+1})\right),\end{aligned}$$

which together with the definition of the cut-off functions, the choice of parameters in (3.9) and the bounds in (3.3), (3.12)-(3.14) implies that for $t \in I_i$

$$\begin{aligned}\|(\partial_t + v_l \cdot \nabla)\mathring{\overline{R}}_q(t)\|_{C^{N+\alpha}} &\lesssim \tau_q^{-2} \|(z_i - z_{i+1})(t)\|_{C^{N+\alpha}} + \tau_q^{-1} \|(\partial_t + v_l \cdot \nabla)(z_i - z_{i+1})(t)\|_{C^{N+\alpha}} \\ &\quad + \tau_q^{-1} \|v_l\|_{C^{1+\alpha}} \|(z_i - z_{i+1})(t)\|_{C^{N+\alpha}} + \tau_q^{-1} \|v_l\|_{C^{N+1+\alpha}} \|(z_i - z_{i+1})(t)\|_{C^\alpha} \\ &\quad + \tau_q^{-1} \|(v_i - v_{i+1})(t)\|_{C^{N+\alpha}} \|(v_i - v_{i+1})(t)\|_{C^\alpha} \\ &\quad + \|(\partial_t + v_l \cdot \nabla)(v_i - v_{i+1})(t)\|_{C^{N+\alpha}} \|(v_i - v_{i+1})(t)\|_{C^\alpha} \\ &\quad + \|(\partial_t + v_l \cdot \nabla)(v_i - v_{i+1})(t)\|_{C^\alpha} \|(v_i - v_{i+1})(t)\|_{C^{N+\alpha}} \\ &\lesssim \tau_q^{-1} \delta_{q+1} l^{-N+\alpha} + \tau_q \delta_{q+1}^2 l^{-N-2+\alpha} \lesssim \tau_q^{-1} \delta_{q+1} l^{-N+\alpha},\end{aligned}$$

where the commutator is bounded using [BDLSV19, Proposition D.1] as $\mathcal{R}\mathrm{curl}$ is a Calderón-Zygmund operator.

Then we use interpolation, (3.9) and the estimate (3.17) to deduce that for $t \in I_i$

$$\begin{aligned}\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q(t)\|_{C^{N+\alpha}} &\lesssim \|(\partial_t + v_l\cdot\nabla)\mathring{\overline{R}}_q(t)\|_{C^{N+\alpha}}\\ &\quad + \|(v_l-\overline{v}_q)(t)\|_{C^\alpha}\|\mathring{\overline{R}}_q(t)\|_{C^{1+N+\alpha}} + \|(v_l-\overline{v}_q)(t)\|_{C^{N+\alpha}}\|\mathring{\overline{R}}_q(t)\|_{C^{1+\alpha}}\\ &\lesssim \tau_q^{-1}\delta_{q+1}l^{-N+\alpha} + \tau_q\delta_{q+1}^2 l^{-N-2+\alpha} \lesssim \tau_q^{-1}\delta_{q+1}l^{-N+\alpha}.\end{aligned}$$

□

To conclude this section, we provide the derivation of the analytic identities in Lemma 3.4.

Proof of Lemma 3.4. The proof follows by direct calculation:

$$\begin{aligned}(\mathrm{curl}\, z)\cdot\nabla\rho &= \sum_{i,j,k}\epsilon_{ijk}\partial_j z^k\partial_i\rho = \sum_{i,j,k}\Big(\partial_j(\epsilon_{ijk}z^k\partial_i\rho) - \epsilon_{ijk}z^k\partial_i\partial_j\rho\Big)\\ &= \sum_{i,j,k}\partial_j(\epsilon_{ijk}z^k\partial_i\rho) = \mathrm{div}(z\times\nabla\rho),\end{aligned}$$

where the second term equals to 0 since $\epsilon_{ijk}\partial_i\partial_j\rho = -\epsilon_{jik}\partial_j\partial_i\rho$. Here ϵ_{ijk} is the Levi-Civita symbol in 3D which is defined as follows: $\epsilon_{ijk} = 1/-1$ if $\{i,j,k\}$ are an even/odd permutation of the indices $\{1,2,3\}$. Otherwise, $\epsilon_{ijk} = 0$. For the second term, since $\mathrm{div} v = 0$, we have

$$\begin{aligned}v\cdot\nabla(\mathrm{div} z) &= \sum_{i,j} v^i\partial_i\partial_j z^j = \sum_{i,j}\Big(\partial_j(v^i\partial_i z^j) - \partial_j v^i\partial_i z^j\Big) = \mathrm{div}(v\cdot\nabla z) - \sum_{i,j}\Big(\partial_i(\partial_j v^i z^j) - \partial_i\partial_j v^i z^j\Big)\\ &= \mathrm{div}(v\cdot\nabla z - z\cdot\nabla v).\end{aligned}$$

For the last term, since $\mathrm{div} v = 0$, we have for $i = 1,2,3$

$$\begin{aligned}[\mathrm{curl}\,(v\cdot\nabla z)]^i &= \sum_{j,k,l}\epsilon_{ijk}\partial_j(v^l\partial_l z^k) = \sum_{j,k,l}\Big(\epsilon_{ijk}\partial_j v^l\partial_l z^k + \epsilon_{ijk}v^l\partial_j\partial_l z^k\Big)\\ &= \sum_{j,k,l}\Big(\partial_l(\epsilon_{ijk}\partial_j v^l z^k) - \epsilon_{ijk}\partial_l\partial_j v^l z^k + v^l\partial_l(\epsilon_{ijk}\partial_j z^k)\Big)\\ &= -[\mathrm{div}((z\times\nabla)v)]^i + [v\cdot\nabla(\mathrm{curl}\, z)]^i,\end{aligned}$$

where we use the notation $[(z\times\nabla)v]^{il} = \sum_{j,k}\epsilon_{ikj}z^k\partial_j v_l$. □

B.3. **The estimates in perturbation steps.** In this section, we show the estimates of the amplitude functions appearing in Section 3. Before that, we give the estimates of the flow maps.

Proof of Proposition 3.7. Since for every $t \in (t_i - \frac{\tau_q}{3}, t_i + \frac{4\tau_q}{3})$ we have $|t - t_i| \leqslant 2\tau_q$, using the estimates (3.20) and Proposition A.7 we obtain

$$\|\nabla\Phi_i(t) - \mathrm{Id}\|_{C^0} \lesssim \tau_q\|\nabla\overline{v}_q\|_{C^\alpha} \lesssim l^\alpha,$$

which implies the estimate in (3.33). Then it is easy to see that $(\nabla\Phi_i)^{-1}(t)$ is well-defined for $t \in (t_i - \frac{\tau_q}{3}, t_i + \frac{4\tau_q}{3})$ and $\big\|(\nabla\Phi_i)^{-1}(t)\big\|_{C^0} \lesssim 1$.

By applying Proposition A.7 again we have for $N \geqslant 1$

$$\|\nabla\Phi_i(t)\|_{C^N} \lesssim \tau_q\|\nabla\overline{v}_q\|_{C^N} \lesssim l^{-N},$$

which together with the Leibniz rule implies that for $N \geqslant 1$

$$\begin{aligned}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^N} &\lesssim \left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^0}\sum_{m=0}^{N-1}\|\nabla\Phi_i(t)\|_{C^{N-m}}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^m}\\&\lesssim \sum_{m=0}^{N-1} l^{-N+m}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^m}.\end{aligned}$$

By induction we obtain (3.34). To derive (3.35), since $D_{t,q}\nabla\Phi_i = -\nabla\Phi_i D\overline{v}_q$, we have for $N \geqslant 0$

$$\|D_{t,q}\nabla\Phi_i(t)\|_{C^N} \lesssim \|\nabla\Phi_i(t)\|_{C^0}\|\overline{v}_q\|_{C^{N+1}} + \|\nabla\Phi_i(t)\|_{C^N}\|\overline{v}_q\|_{C^1} \lesssim \tau_q^{-1}l^{-N}, \tag{B.7}$$

and

$$\begin{aligned}&\left\|D_{t,q}(\nabla\Phi_i)^{-1}(t)\right\|_{C^N}\\&\lesssim \|D_{t,q}\nabla\Phi_i(t)\|_{C^N}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^0}^2 + \|D_{t,q}\nabla\Phi_i(t)\|_{C^0}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^N}\left\|(\nabla\Phi_i)^{-1}(t)\right\|_{C^0}\\&\lesssim \tau_q^{-1}l^{-N}.\end{aligned}$$

□

Proof of Proposition 3.9. We recall the definition of $M_{q,i}$ in (3.36). Using the Leibniz rule, the estimates on $\overline{M}_q, \nabla\Phi_i$ in (3.30), and (3.33)-(3.35) we have for all $t \in (t_i - \frac{\tau_q}{3}, t_i + \frac{4\tau_q}{3})$ and $N \geqslant 0$

$$\|M_{q,i}(t)\|_{C^N} \lesssim \|\nabla\Phi_i(t)\|_{C^0}\Big(1 + \frac{\|\overline{M}_q\|_{C^N}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big) + \|\nabla\Phi_i(t)\|_{C^N}\Big(1 + \frac{\|\overline{M}_q\|_{C^0}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big) \lesssim l^{-N},$$

$$\begin{aligned}\|D_{t,q}M_{q,i}(t)\|_{C^N} &\lesssim \|D_{t,q}\nabla\Phi_i(t)\|_{C^0}\Big(1 + \frac{\|\overline{M}_q\|_{C^N}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big) + \|D_{t,q}\nabla\Phi_i(t)\|_{C^N}\Big(1 + \frac{\|\overline{M}_q\|_{C^0}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big)\\&\quad + \|\nabla\Phi_i(t)\|_{C^0}\Big(1 + \frac{\|D_{t,q}\overline{M}_q\|_{C^N}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big) + \|\nabla\Phi_i(t)\|_{C^N}\Big(1 + \frac{\|D_{t,q}\overline{M}_q\|_{C^0}}{\delta_{q+1}^{1/2}\tilde{\delta}_{q+1}^{1/2}l^{\alpha/2}}\Big) \lesssim \tau_q^{-1}l^{-N}.\end{aligned}$$

By applying [BDLIS15, Proposition C.1] we have for $N \geqslant 1$

$$\|\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^N} \lesssim \|\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^0} + \|\Gamma_\xi^{1/2}\|_{C^1}\|M_{q,i}(t)\|_{C^N} + \|\Gamma_\xi^{1/2}\|_{C^N}\|M_{q,i}(t)\|_{C^1}^N \lesssim l^{-N}.$$

It is easy to see that the above is still valid for $N = 0$.

Then we calculate $D_{t,q}\Gamma_\xi^{1/2}(M_{q,i}) = \nabla\Gamma_\xi^{1/2}(M_{q,i})\cdot D_{t,q}M_{q,i}$, by the same argument we have for $N \geqslant 0$

$$\|\nabla\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^N} \lesssim l^{-N},$$

which together with the Leibniz rule implies that

$$\begin{aligned}&\|D_{t,q}\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^N}\\&\quad\lesssim \|\nabla\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^0}\|D_{t,q}M_{q,i}(t)\|_{C^N} + \|\nabla\Gamma_\xi^{1/2}(M_{q,i})(t)\|_{C^N}\|D_{t,q}M_{q,i}(t)\|_{C^0} \lesssim \tau_q^{-1}l^{-N}.\end{aligned}$$

By applying the Leibniz rule again, and by the properties of the cut-off functions we have

$$\begin{aligned}\|A_{(\xi,i)}\|_{C^N} &\lesssim l^{\alpha/4}\delta_{q+1}^{1/2}\|\eta_i\|_{C^0}\|\Gamma_\xi^{1/2}(M_{q,i})\|_{C^N} + l^{\alpha/4}\delta_{q+1}^{1/2}\|\eta_i\|_{C^N}\|\Gamma_\xi^{1/2}(M_{q,i})\|_{C^0} \lesssim \delta_{q+1}^{1/2}l^{\alpha/4-N},\\ \|D_{t,q}A_{(\xi,i)}\|_{C^N} &\lesssim l^{\alpha/4}\delta_{q+1}^{1/2}\Big(\|D_{t,q}\eta_i\Gamma_\xi^{1/2}(M_{q,i})\|_{C^N} + \|\eta_i D_{t,q}\Gamma_\xi^{1/2}(M_{q,i})\|_{C^N}\Big) \lesssim \tau_q^{-1}\delta_{q+1}^{1/2}l^{\alpha/4-N},\end{aligned}$$

where we used $\tau_q^{-1} \leqslant l^{-1}$, since $\delta_{q+1}^{1/2} \leqslant l^{2\alpha}\lambda_q^{3\alpha/2}$ by choosing $\alpha > 0$ small enough. The other term $\tilde{A}_{(\xi,i)}$ can be estimated by the same calculation. So we omit the proof. □

Proof of Proposition 3.11. First we estimate the derivatives of $R_{q,i}$ defined in (3.56). More precisely, we show that for $N \geqslant 0$ and $t \in \mathrm{supp}(\overline{\eta}_i)$

$$\|R_{q,i}(t)\|_{C^N} + \tau_q \|D_{t,q} R_{q,i}(t)\|_{C^N} \lesssim l^{-N}. \tag{B.8}$$

In fact, by the properties of the cutoff functions, the C^N-bounds in (3.19), (3.49) and (3.52), we have for $N \geqslant 0$

$$\left\| \frac{\sum_j \int \overline{\eta}_j^2(t,y)\mathrm{d}y}{\Upsilon_q(t)} (\mathring{\overline{R}}_q + \mathring{R}_q^{(1)}) \right\|_{C^N} \lesssim \frac{\lambda_q^{\alpha/3}}{\delta_{q+1}} \left\| \mathring{\overline{R}}_q + \mathring{R}_q^{(1)} \right\|_{C^N} \lesssim \lambda_q^{\alpha/3} l^{\alpha/2-N} \lesssim l^{-N},$$

where we used the fact that $(l\lambda_q)^\alpha \ll 1$. Then by the Leibniz rule and (3.34), the bound for the first term follows.

Then, to estimate the material derivative, we have

$$D_{t,q}\Big(\frac{\sum_j \int \overline{\eta}_j^2 \mathrm{d}x}{\Upsilon_q} (\mathring{\overline{R}}_q + \mathring{R}_q^{(1)}) \Big) = \partial_t \left(\frac{\sum_j \int \overline{\eta}_j^2 \mathrm{d}x}{\Upsilon_q} \right) (\mathring{\overline{R}}_q + \mathring{R}_q^{(1)}) + \frac{\sum_j \int \overline{\eta}_j^2 \mathrm{d}x}{\Upsilon_q} D_{t,q}(\mathring{\overline{R}}_q + \mathring{R}_q^{(1)}).$$

By the Leibniz rule for the derivative of the product, (3.52) and (3.54) we derive

$$\left| \partial_t \left(\frac{\sum_j \int \overline{\eta}_j^2 \mathrm{d}x}{\Upsilon_q} \right) \right| \lesssim \frac{\sup_j \|\partial_t \overline{\eta}_j\|_{C^0}}{\Upsilon_q} + \left| \frac{\partial_t \Upsilon_q}{\Upsilon_q^2} \right| \lesssim \delta_{q+1}^{-1} \lambda_q^{\alpha/3} \tau_q^{-1} + \delta_{q+1}^{-2} \lambda_q^{2\alpha/3} \tau_q^{-1} \delta_{q+1} l^\alpha \leqslant \delta_{q+1}^{-1} \lambda_q^{\alpha/3} \tau_q^{-1},$$

where we used the fact that $(l\lambda_q)^\alpha \ll 1$. Then together with (3.52), (3.19) and (3.49) we obtain that

$$\left\| D_{t,q} \Big(\frac{\overline{\eta}_i^2 (\mathring{\overline{R}}_q + \mathring{R}_q^{(1)})}{\Upsilon_{q,i}} \Big) \right\|_{C^N} \lesssim \delta_{q+1}^{-1} \lambda_q^{\alpha/3} \tau_q^{-1} \delta_{q+1} l^{-N+\alpha/2} + \delta_{q+1}^{-1} \lambda_q^{\alpha/3} \tau_q^{-1} \delta_{q+1} l^{-N+\alpha/2} \lesssim \tau_q^{-1} l^{-N},$$

where we used the fact that $(l\lambda_q)^\alpha \ll 1$. By applying the Leibniz rule again, together with the estimates on Φ_i in (3.34), (3.35) we obtain the bound for the second term in (B.8).

By applying [BDLIS15, Proposition C.1] and (B.8) we have for $N \geqslant 1$

$$\|\gamma_\xi(R_{q,i})\|_{C^N} \lesssim \|\gamma_\xi\|_{C^1} \|R_{q,i}\|_{C^N} + \|\gamma_\xi\|_{C^N} \|R_{q,i}\|_{C^1}^N \lesssim l^{-N}. \tag{B.9}$$

By the definition of $\Upsilon_{q,i}$ in (3.51), (3.52) and the properties of the cut-off functions $\overline{\eta}_i$, we deduce that

$$\|\Upsilon_{q,i}^{1/2}\|_{C^N} \lesssim \|\Upsilon_q^{1/2}\|_{C_t^0} \|\overline{\eta}_i\|_{C^N} \lesssim \delta_{q+1}^{1/2}.$$

Then together with (B.9), we obtain for $N \geqslant 1$

$$\|a_{(\xi,i)}\|_{C^N} \lesssim \|\Upsilon_{q,i}^{1/2}\|_{C^0} \|\gamma_\xi(R_{q,i})\|_{C^N} + \|\Upsilon_{q,i}^{1/2}\|_{C^N} \|\gamma_\xi(R_{q,i})\|_{C^0} \lesssim \delta_{q+1}^{1/2} l^{-N}.$$

It is easy to see that the bound is also valid for $N = 0$.

Then we calculate that

$$D_{t,q}(\Upsilon_{q,i}^{1/2}) = \left[\partial_t \left(\frac{\overline{\eta}_i}{(\sum_j \int \overline{\eta}_j^2 \mathrm{d}x)^{1/2}} \right) + \frac{\overline{v}_q \cdot \nabla \overline{\eta}_i}{(\sum_j \int \overline{\eta}_j^2 \mathrm{d}x)^{1/2}} \right] \Upsilon_q^{1/2} + \frac{\overline{\eta}_i \partial_t (\Upsilon_q^{1/2})}{(\sum_j \int \overline{\eta}_j^2 \mathrm{d}x)^{1/2}}.$$

By the properties of the cut-off functions and (3.20) we have for $N \geqslant 1$

$$\left\| \partial_t \left(\frac{\overline{\eta}_i}{(\sum_j \int \overline{\eta}_j^2 \mathrm{d}x)^{1/2}} \right) \right\|_{C^N} \lesssim \tau_q^{-1},$$

$$\|\overline{v}_q \cdot \nabla \overline{\eta}_i\|_{C^N} \lesssim \|\overline{v}_q\|_{C^1} \|\nabla \overline{\eta}_i\|_{C^N} + \|\overline{v}_q\|_{C^N} \|\nabla \overline{\eta}_i\|_{C^0} \lesssim \tau_q^{-1} l^{-N+1} \lesssim \tau_q^{-1} l^{-N}.$$

It is easy to see that the bounds are also valid for $N = 0$, since

$$\|\overline{v}_q \cdot \nabla \overline{\eta}_i\|_{C^0} \lesssim \|\overline{v}_q\|_{C^1} \|\nabla \overline{\eta}_i\|_{C^0} \lesssim \tau_q^{-1}.$$

By (3.52) and (3.54) we have

$$|\partial_t(\Upsilon_q^{1/2})| \lesssim \left| \frac{\partial_t \Upsilon_q}{\Upsilon_q^{1/2}} \right| \lesssim \tau_q^{-1} \delta_{q+1}^{1/2} l^{\alpha} \lambda_q^{\alpha/6} \lesssim \tau_q^{-1} \delta_{q+1}^{1/2},$$

where we used the fact that $l\lambda_q \leqslant 1$ in the last inequality. Then it follows that for $N \geqslant 0$

$$\|D_{t,q}(\Upsilon_{q,i}^{1/2})\|_{C^N} \lesssim \tau_q^{-1} l^{-N} \|\Upsilon_q^{1/2}\|_{C_t^0} + \tau_q^{-1} \delta_{q+1}^{1/2} \lesssim \tau_q^{-1} \delta_{q+1}^{1/2} l^{-N}.$$

Moreover,

$$D_{t,q}(\gamma_\xi(R_{q,i})) = \nabla \gamma_\xi(R_{q,i}) \cdot D_{t,q} R_{q,i}.$$

By applying [BDLIS15, Proposition C.1] and (B.8) we have for $N \geqslant 1$

$$\|\nabla \gamma_\xi(R_{q,i})\|_{C^N} \lesssim \|\nabla \gamma_\xi\|_{C^1} \|R_{q,i}\|_{C^N} + \|\nabla \gamma_\xi\|_{C^N} \|R_{q,i}\|_{C^1}^N \lesssim l^{-N},$$

which together with (B.8) implies that for $N \geqslant 1$

$$\|D_{t,q}(\gamma_\xi(R_{q,i}))\|_{C^N} \lesssim \|\nabla \gamma_\xi(R_{q,i})\|_{C^N} \|D_{t,q} R_{q,i}\|_{C^0} + \|\nabla \gamma_\xi(R_{q,i})\|_{C^0} \|D_{t,q} R_{q,i}\|_{C^N} \lesssim \tau_q^{-1} l^{-N}.$$

It is easy to see that the bound is also valid for $N = 0$.

In the end, by applying the chain rule again, we summarize all the bounds above and obtain $N \geqslant 0$

$$\|D_{t,q} a_{(\xi,i)}\|_{C^N} \lesssim \|D_{t,q}(\Upsilon_{q,i}^{1/2}) \gamma_\xi(\tilde{R}_{q,i})\|_{C^N} + \|\Upsilon_{q,i}^{1/2} D_{t,q} \gamma_\xi(\tilde{R}_{q,i})\|_{C^N} \lesssim \tau_q^{-1} \delta_{q+1}^{1/2} l^{-N}.$$

□

APPENDIX C. BUILDING BLOCKS AND AUXILIARY ESTIMATES IN SECTION 5

In this section, we first introduce the building blocks used in the convex integration method. Then we provide some estimates on the amplitude functions appearing in Section 5.

C.1. Generalized intermittent spatial-time jets.

C.1.1. *Building blocks for the transport equations.* In this section, we present the building blocks for advection-diffusion equations, which can be viewed as a generalization of those in [BCDL21, Section 4], incorporating more intermittency in the spatial domain.

For parameters $\lambda, r_\perp, r_\| > 0$, we assume

$$\lambda^{-1} \ll r_\perp \ll r_\| \ll 1, \quad \lambda r_\perp \in \mathbb{N}.$$

We recall the geometric Lemma B.1 and the two disjoint families Λ^1, Λ^2 discussed in that lemma for $d \geqslant 2$. For each $\xi \in \Lambda^1 \cup \Lambda^2$ let us define $A_\xi^i \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$, $i = 1, 2, ..., d-1$ such that $\{\xi, A_\xi^i, i = 1, ..., d-1\}$ form an orthonormal basis in $\mathbb{R}^d$. We label by n_* the smallest natural number such that

$$\{n_* \xi, n_* A_\xi^i, i = 1, ..., d-1\} \subset \mathbb{Z}^d,$$

for every $\xi \in \Lambda^1 \cup \Lambda^2$.

Now we introduce the cut-off functions used in the construction. First, there is a smooth mean-zero function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ with support in $B(0,1)$ satisfying $\phi \equiv 1$ on $B(0, \frac{1}{3})$. Moreover, the function Φ, defined by $\phi = -\Delta \Phi$, is a smooth function with support in $B(0,1)$. In fact, let $\phi_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ be a smooth function with support in $B(0,1)$ satisfying $\phi_0 \equiv 1$ on $B(0, \frac{1}{3})$. By [MB03, Lemma 1.12] we can define

$\Phi_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ by solving $\phi_0 = -\Delta\Phi_0$. We let $\Phi := \Phi_0\phi_0$ which is a smooth function with support in a ball of radius 1. We define $\phi = -\Delta\Phi$, which is a smooth function with support in a ball of radius 1 and mean-zero. It is easy to check that on $B(0, \frac{1}{3})$, $\phi = -\Delta(\Phi_0\phi_0) = -\Delta\Phi_0 = \phi_0 = 1$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, mean-zero function with support in $B(0, 1)$ satisfying $\psi \equiv 1$ on $B(0, \frac{1}{3})$. Define $\phi' : \mathbb{R}^{d-1} \to \mathbb{R}$ to be a smooth non-negative function with support in $B(0, \frac{1}{3})$ satisfying

$$\int_{\mathbb{R}^{d-1}} \phi'(x_1, x_2, ..., x_{d-1})\mathrm{d}x_1\mathrm{d}x_2..\mathrm{d}x_{d-1} = 1,$$

and let $\psi' : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with support in $B(0, \frac{1}{3})$ such that

$$\int_{\mathbb{R}} \psi'(x_d)\mathrm{d}x_d = 1.$$

Then it is easy to see that

$$\phi\phi' = \phi', \quad \psi\psi' = \psi'. \tag{C.1}$$

We define the rescaled cut-off functions by

$$\begin{aligned}
\phi_{r_\perp}(x_1, x_2, ..., x_{d-1}) &= \frac{1}{r_\perp^{(d-1)/2}}\phi(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}),\\
\Phi_{r_\perp}(x_1, x_2, ..., x_{d-1}) &= \frac{1}{r_\perp^{(d-1)/2}}\Phi(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}),\\
\psi_{r_\parallel}(x_d) &= \frac{1}{r_\parallel^{1/2}}\psi(\frac{x_d}{r_\parallel}).
\end{aligned}$$

Similarly, we define the rescaled cut-off $\phi'_{r_\perp}, \psi'_{r_\parallel}$ with respect to ϕ', ψ'. We periodize $\phi_{r_\perp}, \Phi_{r_\perp}, \psi_{r_\parallel}, \phi'_{r_\perp}, \psi'_{r_\parallel}$ so that they can be viewed as functions on $\mathbb{T}^{d-1}$ and $\mathbb{T}$ respectively. Consider a large time oscillation parameter $\mu > 0$. For every $\xi \in \Lambda^1 \cup \Lambda^2$ we introduce

$$\begin{aligned}
\psi_{(\xi)}(t, x) : &= \psi_{r_\parallel}(n_* r_\perp \lambda(x \cdot \xi - \mu t)),\\
\Phi_{(\xi)}(x) : &= \Phi_{r_\perp}(n_* r_\perp \lambda x \cdot A^1_\xi, ..., n_* r_\perp \lambda x \cdot A^{d-1}_\xi),\\
\phi_{(\xi)}(x) : &= \phi_{r_\perp}(n_* r_\perp \lambda x \cdot A^1_\xi, ..., n_* r_\perp \lambda x \cdot A^{d-1}_\xi).
\end{aligned}$$

Here we remark that we do not need to translate the building blocks such that the supports are disjoint, since the disjoint support property will be achieved by choosing suitable time jets in the following section. Similarly, we define the building blocks $\phi'_{(\xi)}, \psi'_{(\xi)}$.

The building blocks $W_{(\xi)} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d, \Theta_{(\xi)} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ are defined as

$$W_{(\xi)}(t, x) := \xi\psi_{(\xi)}(t, x)\phi_{(\xi)}(x), \quad \Theta_{(\xi)}(t, x) := \psi'_{(\xi)}(t, x)\phi'_{(\xi)}(x),$$

at which point, together with identities in (C.1) we have that

$$\int_{\mathbb{T}^d} W_{(\xi)}\Theta_{(\xi)}\mathrm{d}x = \xi, \tag{C.2}$$

$$\partial_t\Theta_{(\xi)} + \mu r_\perp^{\frac{d-1}{2}} r_\parallel^{\frac{1}{2}} \mathrm{div}(W_{(\xi)}\Theta_{(\xi)}) = 0. \tag{C.3}$$

Since $W_{(\xi)}$ is not divergence-free, inspired by [CL22a, Section 4.1] we introduce the skew-symmetric corrector term

$$V_{(\xi)} := \frac{1}{(n_*\lambda)^2}(\xi \otimes \nabla\Phi_{(\xi)} - \nabla\Phi_{(\xi)} \otimes \xi)\psi_{(\xi)}.$$

Then by a direct computation

$$\mathrm{div} V_{(\xi)} = \psi_{(\xi)}\phi_{(\xi)}\xi - \frac{1}{(n_*\lambda)^2}\nabla\Phi_{(\xi)}\xi\cdot\nabla\psi_{(\xi)} = W_{(\xi)} - \frac{1}{(n_*\lambda)^2}\nabla\Phi_{(\xi)}\xi\cdot\nabla\psi_{(\xi)}. \tag{C.4}$$

Finally, we obtain that for $N, M \geqslant 0$ and $p \in [1,\infty]$ the following holds

$$\|\nabla^N\partial_t^M\psi_{(\xi)}\|_{C_tL^p} \lesssim r_\|^{\frac{1}{p}-\frac{1}{2}}\Big(\frac{r_\perp\lambda}{r_\|}\Big)^N\Big(\frac{r_\perp\lambda\mu}{r_\|}\Big)^M, \tag{C.5}$$

$$\|\nabla^N\phi_{(\xi)}\|_{L^p} + \|\nabla^N\Phi_{(\xi)}\|_{L^p} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}}\lambda^N, \tag{C.6}$$

$$\|\nabla^N\partial_t^M W_{(\xi)}\|_{C_tL^p} + \lambda\|\nabla^N\partial_t^M V_{(\xi)}\|_{C_tL^p} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}} r_\|^{\frac{1}{p}-\frac{1}{2}}\lambda^N\Big(\frac{r_\perp\lambda\mu}{r_\|}\Big)^M, \tag{C.7}$$

$$\|\nabla^N\partial_t^M\Theta_{(\xi)}\|_{C_tL^p} \lesssim r_\perp^{\frac{d-1}{p}-\frac{d-1}{2}} r_\|^{\frac{1}{p}-\frac{1}{2}}\lambda^N\Big(\frac{r_\perp\lambda\mu}{r_\|}\Big)^M, \tag{C.8}$$

where the implicit constants may depend on p, N and M, but are independent of $\lambda, r_\perp, r_\|, \mu$. These estimates can be easily deduced from the definitions.

C.1.2. *Building blocks for the Navier-Stokes or Euler equations.* In this section we recall the generalized intermittent jets introduced in [LZ23, Section3].

We recall the geometric Lemma B.2 and the set $\overline{\Lambda} := \overline{\Lambda}^1$ discussed in that lemma for $d \geqslant 2$. Additionally, we use the parameters $\lambda, r_\perp, r_\| > 0$ defined in the previous section. For each $\xi \in \overline{\Lambda}$ let us define $\overline{A}_\xi^i \in \mathbb{S}^{d-1}\cap\mathbb{Q}^d,\ i = 1, 2, ..., d-1$, such that $\{\xi, A_\xi^i, i = 1, ..., d-1\}$ form an orthonormal basis in $\mathbb{R}^d$. We label by $\overline{n}_*$ the smallest natural number such that for every $\xi \in \overline{\Lambda}$

$$\{\overline{n}_*\xi, \overline{n}_*\overline{A}_\xi^i, i = 1, ..., d-1\} \subset \mathbb{Z}^d.$$

Let $\overline{\Phi} : \mathbb{R}^{d-1} \to \mathbb{R}$ be a smooth function with support in a ball of radius 1. We normalize $\overline{\Phi}$ such that $\overline{\phi} = -\Delta\overline{\Phi}$ obeys

$$\int_{\mathbb{R}^{d-1}} \overline{\phi}^2(x_1, x_2, ..., x_{d-1})\mathrm{d}x_1\mathrm{d}x_2..\mathrm{d}x_{d-1} = 1.$$

By definition we know that $\int_{\mathbb{R}^{d-1}}\overline{\phi}\mathrm{d}x_1\mathrm{d}x_2..\mathrm{d}x_{d-1} = 0$.

Define $\overline{\psi} : \mathbb{R} \to \mathbb{R}$ to be a smooth, mean-zero function with support in a ball of radius 1 satisfying

$$\int_{\mathbb{R}} \overline{\psi}^2(x_d)\mathrm{d}x_d = 1.$$

We define the rescaled cut-off functions

$$\overline{\phi}_{r_\perp}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/2}}\overline{\phi}\Big(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}\Big),$$

$$\overline{\Phi}_{r_\perp}(x_1, x_2, ..., x_{d-1}) = \frac{1}{r_\perp^{(d-1)/2}}\overline{\Phi}\Big(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}, ..., \frac{x_{d-1}}{r_\perp}\Big),$$

$$\overline{\psi}_{r_\|}(x_d) = \frac{1}{r_\|^{1/2}}\overline{\psi}\Big(\frac{x_d}{r_\|}\Big).$$

We periodize them so that they can be viewed as functions on $\mathbb{T}^{d-1}$ and $\mathbb{T}$ respectively. Consider a new large time oscillation parameter $\overline{\mu} > 0$. For every $\xi \in \overline{\Lambda}$ we introduce

$$\overline{\psi}_{(\xi)}(t, x) := \overline{\psi}_{r_\|}(\overline{n}_* r_\perp\lambda(x\cdot\xi - \overline{\mu}t)),$$

$$\overline{\Phi}_{(\xi)}(x) := \overline{\Phi}_{r_\perp}(\overline{n}_* r_\perp \lambda x \cdot \overline{A}_\xi^1, ..., \overline{n}_* r_\perp \lambda x \cdot \overline{A}_\xi^{d-1}),$$
$$\overline{\phi}_{(\xi)}(x) := \overline{\phi}_{r_\perp}(\overline{n}_* r_\perp \lambda x \cdot \overline{A}_\xi^1, ..., \overline{n}_* r_\perp \lambda x \cdot \overline{A}_\xi^{d-1}).$$

Here, similarly as before, we do not need to translate the building blocks since the the disjoint support property will be achieved by selecting suitable time jets in the following section.

The intermittent jets $\overline{W}_{(\xi)} : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^d$ are defined as

$$\overline{W}_{(\xi)}(t,x) = \xi \overline{\psi}_{(\xi)}(t,x) \overline{\phi}_{(\xi)}(x).$$

By definition and a basic calculation we have that

$$\partial_t(\overline{\psi}^2_{(\xi)} \overline{\phi}^2_{(\xi)} \xi) + \overline{\mu} \mathrm{div}(\overline{W}_{(\xi)} \otimes \overline{W}_{(\xi)}) = 0, \tag{C.9}$$

$$\int_{\mathbb{T}^d} \overline{W}_{(\xi)} \otimes \overline{W}_{(\xi)} \mathrm{d}x = \xi \otimes \xi. \tag{C.10}$$

Since $\overline{W}_{(\xi)}$ is not divergence-free, we introduce the skew-symmetric corrector term

$$\overline{V}_{(\xi)} := \frac{1}{(\overline{n}_* \lambda)^2} (\xi \otimes \nabla \overline{\Phi}_{(\xi)} - \nabla \overline{\Phi}_{(\xi)} \otimes \xi) \overline{\psi}_{(\xi)}.$$

Then by a direct computation

$$\mathrm{div} \overline{V}_{(\xi)} = \overline{W}_{(\xi)} - \frac{1}{(\overline{n}_* \lambda)^2} \nabla \overline{\Phi}_{(\xi)} \xi \cdot \nabla \overline{\psi}_{(\xi)}. \tag{C.11}$$

Finally, we obtain that for $N, M \geqslant 0$ and $p \in [1, \infty]$ the following holds

$$\|\nabla^N \partial_t^M \overline{\psi}_{(\xi)}\|_{C_t L^p} \lesssim r_\|^{\frac{1}{p} - \frac{1}{2}} (\frac{r_\perp \lambda}{r_\|})^N (\frac{r_\perp \lambda \overline{\mu}}{r_\|})^M, \tag{C.12}$$

$$\|\nabla^N \overline{\Phi}_{(\xi)}\|_{L^p} + \|\nabla^N \overline{\phi}_{(\xi)}\|_{L^p} \lesssim r_\perp^{\frac{d-1}{p} - \frac{d-1}{2}} \lambda^N, \tag{C.13}$$

$$\|\nabla^N \partial_t^M \overline{W}_{(\xi)}\|_{C_t L^p} + \lambda \|\nabla^N \partial_t^M \overline{V}_{(\xi)}\|_{C_t L^p} \lesssim r_\perp^{\frac{d-1}{p} - \frac{d-1}{2}} r_\|^{\frac{1}{p} - \frac{1}{2}} \lambda^N (\frac{r_\perp \lambda \overline{\mu}}{r_\|})^M, \tag{C.14}$$

where the implicit constants may depend on p, N and M, but are independent of $\lambda, r_\perp, r_\|, \overline{\mu}$.

C.1.3. *Temporal jets.* In this section, we introduce additional intermittency in the time direction similarly as in [CL21, Section 4.2]. For $\xi \in \Lambda^1 \cup \Lambda^2 \cup \overline{\Lambda}$, let us choose temporal functions $g_{(\xi)}(t)$ and $h_{(\xi)}(t)$ to oscillate the building blocks intermittently in time. Let $G \in C_c^\infty(0,1)$ be non-negative and

$$\int_0^1 G^2(t) \mathrm{d}t = 1.$$

Letting $\eta > 0$ be a small constant satisfying $\eta \cdot \mathrm{card}(\Lambda^1 \cup \Lambda^2 \cup \overline{\Lambda}) \ll 1$, we define $\tilde{g}_{(\xi)} : \mathbb{T} \to \mathbb{R}$ as the 1-periodic extension of $\eta^{-\frac{1}{2}} G(\frac{t - t_\xi}{\eta})$, where t_ξ are chosen so that $\tilde{g}_{(\xi)}$ have disjoint supports for different $\xi \in \Lambda^1 \cup \Lambda^2 \cup \overline{\Lambda}$. We will also oscillate the perturbations at a large frequency $\sigma \in \mathbb{N}$. So, we define

$$g_{(\xi)}(t) = \tilde{g}_{(\xi)}(\sigma t).$$

For the corrector term we define $H_{(\xi)}, h_{(\xi)} : \mathbb{T} \to \mathbb{R}$ by

$$H_{(\xi)}(t) = \int_0^t g_{(\xi)}(s) \mathrm{d}s, \ \ h_{(\xi)}(t) = \int_0^{\sigma t} (\tilde{g}^2_{(\xi)}(s) - 1) \mathrm{d}s. \tag{C.15}$$

In view of the zero-mean condition for $\tilde{g}^2_{(\xi)}(t)-1$, we obtain that $h_{(\xi)}$ is $\mathbb{T}/\sigma$-periodic, and for any $n\geqslant 0, p\geqslant 1$

$$\|g_{(\xi)}\|_{W^{n,p}_t}\lesssim (\frac{\sigma}{\eta})^n\eta^{\frac{1}{p}-\frac{1}{2}},\ \ \|h_{(\xi)}\|_{L^\infty_t}\leqslant 1. \tag{C.16}$$

C.2. **The estimates on the amplitude functions.** In this section, we show the estimates of the amplitude functions appearing in Section 5.

Proof of Proposition 5.2. First, we estimate $\chi(\zeta|M_l|-n)$ in $C^N_{t,x}$-norm for $N\in\mathbb{N}$. We recall that from (5.7)

$$\|M_l\|_{C^N_{t,x}}\lesssim l^{-d-2-N}.$$

On $\operatorname{supp}\chi(\zeta|M_l|-n), n\geqslant 3$, we have $|M_l|\geqslant\zeta^{-1}$. Then we apply [BDLIS15, Proposition C.1] to a smooth function $f(z)$ satisfying $f(z)=|z|$ on $|z|\geqslant\zeta^{-1}$. Since $|D^Nf(z)|\lesssim\zeta^{N-1}$ on $|z|\geqslant\zeta^{-1}$, we have for $N\geqslant 1$

$$\begin{aligned}\||M_l|\|_{C^N_{t,x}}&\lesssim\||M_l|\|_{C^0_{t,x}}+\|Df\|_{C^0}\|M_l\|_{C^N_{t,x}}+\|Df\|_{C^{N-1}}\|M_l\|^N_{C^1_{t,x}}\\&\lesssim l^{-d-2-N}+\zeta^{N-1}l^{-(d+3)N}\lesssim\zeta^{N-1}l^{-(d+3)N}.\end{aligned}$$

Then we apply the chain rule from [BDLIS15, Proposition 4.1] to $f(z)=\chi(\zeta z-n), |D^mf|\lesssim\zeta^m$ to obtain

$$\begin{aligned}\|\chi(\zeta|M_l|-n)\|_{C^N_{t,x}}&\lesssim\|\chi(\zeta|M_l|-n)\|_{C^0_{t,x}}+\|Df\|_{C^0}\,\||M_l|\|_{C^N_{t,x}}+\|Df\|_{C^{N-1}}\,\||M_l|\|^N_{C^1_{t,x}}\\&\lesssim\zeta^Nl^{-N(d+3)}\lesssim l^{-N(d+4)},\end{aligned}$$

where we used the condition $\zeta\lesssim l^{-1}$. Then by (5.12) we have for $N\in\mathbb{N}$

$$\sum_{n\geqslant 3}\|\chi(\zeta|M_l|-n)\|_{C^N_{t,x}}\lesssim\sum_{n=3}^{1+Cl^{-d-2}}l^{-N(d+4)}\lesssim l^{-N(d+4)-(d+2)}.$$

This bound is also valid for $N=0$. The bound for $\tilde{\chi}(\zeta|M_l|-n)$ is similar to the one described above.

Next we estimate $\chi(\zeta|M_l|-n)\Gamma_\xi(\frac{M_l}{|M_l|})$ in $C^N_{t,x}$-norm. By the Leibniz rule we get

$$\left\|\frac{M_l}{|M_l|}\right\|_{C^N_{t,x}}\lesssim\sum_{m=0}^N\|M_l\|_{C^{N-m}_{t,x}}\left\|\frac{1}{|M_l|}\right\|_{C^m_{t,x}}.$$

On $\operatorname{supp}\chi(\zeta|M_l|-n), n\geqslant 3$, we have $|M_l|\geqslant\zeta^{-1}$. Then we apply [BDLIS15, Proposition C.1] to a smooth function $f(z)$ satisfying $f(z)=\frac{1}{|z|}$ on $|z|\geqslant\zeta^{-1}$. Since $|D^Nf(z)|\lesssim\zeta^{N+1}$ on $|z|\geqslant\zeta^{-1}$, we have for $N\geqslant 1$

$$\left\|\frac{1}{|M_l|}\right\|_{C^m_{t,x}}\lesssim\zeta+\zeta^2l^{-d-2-m}+\zeta^{m+1}l^{-m(d+3)}\lesssim l^{-m(d+4)-1},$$

which implies that

$$\left\|\frac{M_l}{|M_l|}\right\|_{C^N_{t,x}}\lesssim\sum_{m=1}^N l^{-d-2-N+m}l^{-m(d+4)-1}+l^{-d-2-N}l^{-1}\lesssim l^{-(d+5)N-(d+3)}.$$

We apply the chain rule from [BDLIS15, Proposition 4.1] to $f(z)=\Gamma_\xi(z), |D^mf|\lesssim 1$ to obtain for $N\geqslant 1$

$$\left\|\Gamma_\xi\Big(\frac{M_l}{|M_l|}\Big)\right\|_{C^N_{t,x}}\lesssim\left\|\frac{M_l}{|M_l|}\right\|_{C^N_{t,x}}+\left\|\frac{M_l}{|M_l|}\right\|^N_{C^1_{t,x}}\lesssim l^{-(2d+8)N}.$$

By the chain rule we get for $N \in \mathbb{N}$

$$\sum_{n \geqslant 3} \sum_{\xi \in \Lambda^n} \left\| \chi(\zeta |M_l| - n) \Gamma_\xi \Big(\frac{M_l}{|M_l|} \Big) \right\|_{C^N_{t,x}} \lesssim l^{-(2d+8)N-(d+2)}.$$

This bound is also valid for $N = 0$.

For the last term, we have for $N \in \mathbb{N}_0$

$$\Big(\frac{n}{\zeta}\Big)^N 1_{\{\chi(\zeta|M_l|-n)>0\}} + \Big(\frac{n}{\zeta}\Big)^N 1_{\{\tilde\chi(\zeta|M_l|-n)>0\}} \lesssim (\|M_l\|_{C^0_{t,x}} + \zeta^{-1})^N \lesssim l^{-N(d+2)}.$$

□

Proof of Proposition 5.3. First we recall the definition of A:

$$A := 2\sqrt{l^2 + |\mathring{R}_l|^2},$$

which together with (5.6) implies that $\|A\|_{C^0_{t,x}} \lesssim l^{-d-2}$.

Next we estimate the $C^N_{t,x}$-norm for $N \in \mathbb{N}$. We apply the chain rule in [BDLIS15, Proposition C.1] to $f(z) = \sqrt{l^2 + z^2}, |D^m f(z)| \lesssim l^{-m+1}$ to obtain

$$\begin{aligned}\left\| \sqrt{l^2 + |\mathring{R}_l|^2} \right\|_{C^N_{t,x}} &\lesssim \left\| \sqrt{l^2 + |\mathring{R}_l|^2} \right\|_{C^0_{t,x}} + \|Df\|_{C^0} \|\mathring{R}_l\|_{C^N_{t,x}} + \|Df\|_{C^{N-1}} \|\mathring{R}_l\|^N_{C^1_{t,x}} \\ &\lesssim l^{-d-2-N} + l^{-N+1} l^{-(d+3)N},\end{aligned}$$

which implies that for $N \geqslant 1$

$$\|A\|_{C^N_{t,x}} \lesssim \left\| \sqrt{l^2 + |\mathring{R}_l|^2} \right\|_{C^N_{t,x}} \lesssim l^{1-(d+4)N}. \tag{C.17}$$

Let us now estimate the $C^N_{t,x}$-norm. By the Leibniz rule we get

$$\|a_{(\xi)}\|_{C^N_{t,x}} \lesssim \sum_{m=0}^N \left\| A^{1/2} \right\|_{C^m_{t,x}} \left\| \gamma_\xi (\mathrm{Id} - \frac{\mathring{R}_l}{A}) \right\|_{C^{N-m}_{t,x}}.$$

Applying [BDLIS15, Proposition C.1] to $f(z) = z^{1/2}$, $|D^m f(z)| \lesssim |z|^{1/2-m}$, for $m = 1, ..., N$, and using (C.17) we obtain for $m \geqslant 1$

$$\|A^{1/2}\|_{C^m_{t,x}} \lesssim \|A^{1/2}\|_{C^0_{t,x}} + l^{-1/2} \|A\|_{C^m_{t,x}} + l^{1/2-m} \|A\|^m_{C^1_{t,x}} \lesssim l^{1/2-(d+4)m}.$$

Next we estimate $\gamma_\xi(\mathrm{Id} - \frac{\mathring{R}_l}{A})$. By [BDLIS15, Proposition C.1] we need to estimate

$$\left\| \frac{\mathring{R}_l}{A} \right\|_{C^{N-m}_{t,x}} + \left\| \frac{\nabla_{t,x} \mathring{R}_l}{A} \right\|^{N-m}_{C^0_{t,x}} + \left\| \frac{\mathring{R}_l}{A^2} \right\|^{N-m}_{C^0_{t,x}} \|A\|^{N-m}_{C^1_{t,x}}.$$

We use $A \geqslant l$ to have that

$$\left\| \frac{\nabla_{t,x} \mathring{R}_l}{A} \right\|^{N-m}_{C^0_{t,x}} \lesssim l^{-N+m} l^{-(d+3)(N-m)} \lesssim l^{-(d+4)(N-m)},$$

and in view of $|\frac{\mathring{R}_l}{A}| \leqslant 1$ that

$$\Big\|\frac{\mathring{R}_l}{A^2}\Big\|_{C^0_{t,x}}^{N-m} \lesssim \Big\|\frac{1}{A}\Big\|_{C^0_{t,x}}^{N-m} \lesssim l^{-N+m},$$

and by (C.17) that

$$\|A\|_{C^1_{t,x}}^{N-m} \lesssim l^{-(d+3)(N-m)}.$$

Moreover, we write

$$\Big\|\frac{\mathring{R}_l}{A}\Big\|_{C^{N-m}_{t,x}} \lesssim \sum_{k=0}^{N-m} \|\mathring{R}_l\|_{C^k_{t,x}} \Big\|\frac{1}{A}\Big\|_{C^{N-m-k}_{t,x}}.$$

Using (C.17) and [BDLIS15, Proposition C.1] we obtain

$$\begin{aligned}\Big\|\frac{1}{A}\Big\|_{C^{N-m-k}_{t,x}} &\lesssim \Big\|\frac{1}{A}\Big\|_{C^0_{t,x}} + l^{-2}\|A\|_{C^{N-m-k}_{t,x}} + l^{-N+m+k-1}\|A\|_{C^1_{t,x}}^{N-m-k} \\ &\lesssim l^{-2}l^{1-(d+4)(N-m-k)} + l^{-(N-m-k)-1}l^{-(d+3)(N-m-k)} \lesssim l^{-(d+4)(N-m-k)-1}.\end{aligned}$$

Thus, we obtain

$$\|\frac{\mathring{R}_l}{A}\|_{C^{N-m}_{t,x}} \lesssim \sum_{k=0}^{N-m-1} l^{-d-2-k}l^{-(d+4)(N-m-k)-1} + l^{-d-2-(N-m)}l^{-1} \lesssim l^{-(d+3)-(d+4)(N-m)}.$$

Finally, the above bounds lead to

$$\Big\|\gamma_\xi\Big(\mathrm{Id}-\frac{\mathring{R}_l}{A}\Big)\Big\|_{C^{N-m}_{t,x}} \lesssim l^{-(d+3)-(d+4)(N-m)}.$$

Combining this with the bounds for $A^{1/2}$ above yields for $N\in\mathbb{N}$

$$\begin{aligned}\|a_{(\xi)}\|_{C^N_{t,x}} &\lesssim l^{-(d+2)/2}l^{-(d+3)-(d+4)N} + \sum_{m=1}^{N-1} l^{1/2-(d+4)m}l^{-(d+3)-(d+4)(N-m)} + l^{1/2-(d+4)N} \\ &\lesssim l^{-2d-3-(d+4)N},\end{aligned}$$

where the final bound is also valid for $N=0$.

□

References

[ACM19] G. Alberti, G. Crippa, and A. L. Mazzucato. Exponential self-similar mixing by incompressible flows. J. Amer. Math. Soc., 32(2):445–490, 2019.

[Amb08] L. Ambrosio. Transport equation and Cauchy problem for non-smooth vector fields. In Calculus of variations and nonlinear partial differential equations, volume 1927 of Lecture Notes in Math., pp. 1–41. Springer, Berlin 2008.

[Amb17] L. Ambrosio. Well posedness of ODE's and continuity equations with nonsmooth vector fields, and applications. Rev. Mat. Complut., 30(3):427–450, 2017.

[Alb12] G. Alberti. Generalized N-property and Sard theorem for Sobolev maps. Rendiconti Lincei, 2012, 23(4): 477-491.

[BC23] E. Brué and M. Colombo. Nonuniqueness of solutions to the Euler equations with vorticity in a Lorentz space. Commun. Math. Phys., 403(2):1171–1192, 2023.

[BCDL21] E. Brué, M. Colombo, C. De Lellis. Positive solutions of transport equations and classical non-uniqueness of characteristic curves. Archive for Rational Mechanics and Analysis, 2021, 240: 1055-1090.

[BCK24a] E. Brué, M. Colombo, and A. Kumar. Sharp non-uniqueness in the Transport Equation with Sobolev Velocity Field. arXiv:2405.01670. 2024.

[BCK24b] E. Brué, M. Colombo, and A. Kumar. Flexibility of Two-Dimensional Euler Flows with Integrable Vorticity[J]. arXiv preprint arXiv:2408.07934, 2024.

[BCV21] T. Buckmaster, M. Colombo, and V. Vicol. Wild solutions of the navier-stokes equations whose singular sets in time have hausdorff dimension strictly less than 1. Journal of the European Mathematical Society, 2021, 24(9): 3333-3378.

[BDLIS15] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi Jr. Anomalous dissipation for 1/5-Hölder Euler flows. Annals of Mathematics 182(1):127–172, 2015.

[BDLS16] T. Buckmaster, C. De Lellis, and L. Székelyhidi, Jr. Dissipative Euler flows with Onsager-critical spatial regularity. Comm. Pure Appl. Math., 69(9):1613–1670, 2016. doi:10.1002/cpa.21586.

[BDLSV19] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. Comm. Pure Appl. Math., 72(2):229–274, 2019.

[BFH20] D. Breit, E. Feireisl, and M. Hofmanová. On solvability and ill-posedness of the compressible Euler system subject to stochastic forces. Anal. PDE, 13(2):371–402, 2020.

[BFGM19] L. Beck, F. Flandoli, M. Gubinelli, and M. Maurelli. Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. Electron. J. Probab. 24: 1-72 (2019)

[BHP23] A. Bulut, M.K. Huynh, S. Palasek. Convex integration above the Onsager exponent for the forced Euler equations. arXiv:2301.00804, 2023.

[BG23] O. Butkovsky, S. Gallay. Weak existence for SDEs with singular drifts and fractional Brownian or Levy noise beyond the subcritical regime[J]. arXiv:2311.12013, 2023.

[BGK97] D. Bernard, K. Gawedzki, A. Kupiainen. Slow modes in passive advection. 1997.

[BM24a] M. Buck and S. Modena. Compactly supported anomalous weak solutions for 2d euler equations with vorticity in hardy spaces. 2024, 2405.19214.

[BM24b] M. Buck and S. Modena. Non-uniqueness and energy dissipation for 2d euler equations with vorticity in hardy spaces. Journal of Mathematical Fluid Mechanics, 26(26), 2024. doi:10.1007/s00021-024-00860-9

[BMNV23] t. Buckmaster, N. Masmoudi, M. Novack, V. Vicol. Intermittent Convex Integration for the 3D Euler equations. Ann. of Math. Studies, Vol. 217, (2023)

[Buc15] T. Buckmaster. Onsager's conjecture almost everywhere in time[J]. Communications in Mathematical Physics, 2015, 333: 1175-1198.

[BV19a] T. Buckmaster and V. Vicol. Convex integration and phenomenologies in turbulence. EMS Surv. Math. Sci., 6(1-2):173–263, 2019.

[BV19b] T. Buckmaster and V. Vicol. non-uniqueness of weak solutions to the Navier-Stokes equation. Ann. of Math. (2), 189(1):101–144, 2019.

[BV21] T. Buckmaster and V. Vicol. Convex integration constructions in hydrodynamics. Bull. Amer. Math. Soc. (N.S.), 58(1):1–44, 2021.

[CC21] L. Caravenna, G. Crippa. A directional Lipschitz extension lemma, with applications to uniqueness and Lagrangianity for the continuity equation. Communications in Partial Differential Equations, 2021, 46(8): 1488-1520.

[CCFS08] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity, 21(6):1233–1252, 2008.

[CET94] P. Constantin, W. E, and E. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm. Math. Phys., 165(1):207–209, 1994.

[CGSW15] G. Crippa, N. Gusev, S. Spirito, and E. Wiedemann. Non-uniqueness and prescribed energy for the continuity equation. Commun. Math. Sci., 13(7):1937–1947, 2015.

[Cho13] A. Choffrut. h-principles for the incompressible Euler equations. Arch. Ration. Mech. Anal., 210(1):133–163, 2013.

[CI08] P. Constantin and G. Iyer. A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 61(3):330–345, 2008.

[CL21] A. Cheskidov, and X. Luo. non-uniqueness of weak solutions for the transport equation at critical space regularity. Annals of PDE 7: 1-45. 2021.

[CL22a] A. Cheskidov, and X. Luo. Sharp non-uniqueness for the Navier–Stokes equations. Inventiones mathematicae: 1-68, 2022.

[CL22b] A. Cheskidov, and X. Luo. Extreme temporal intermittency in the linear Sobolev transport: almost smooth nonunique solutions. arXiv:2204.08950, 2022.

[CL23] A. Cheskidov, and X. Luo. L^2-Critical Nonuniqueness for the 2D Navier-Stokes Equations. Annals of PDE, 2023, 9(2): 13.

[CZZ25] A. Cheskidov, Z. Zeng, D. Zhang. Global dissipative solutions of the 3D Naiver-Stokes and MHD equations. arXiv:2503.05692. 2025.

[DE17a] T. D. Drivas, G. L. Eyink. A Lagrangian fluctuation–dissipation relation for scalar turbulence. Part I. Flows with no bounding walls. Journal of Fluid Mechanics, 2017, 829: 153-189.

[DE17b] T. D. Drivas, G. L. Eyink. A Lagrangian fluctuation-dissipation relation for scalar turbulence. Part II. Wall-bounded flows, J. Fluid Mech. 829 (2017), 236–279. MR3698054

[DEIJ22] T. D. Drivas, T. M. Elgindi, G. Iyer, and I. Jeong. Anomalous Dissipation in Passive Scalar Transport. Archive for Rational Mechanics and Analysis, 2022: 1-30.

[Dep03] N. Depauw. Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan. C. R. Math. Acad. Sci. Paris, 337(4):249–252, 2003.

[DL89] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math., 98(3):511–547, 1989.

[DL08] C. De Lellis. ODEs with Sobolev coefficients: the Eulerian and the Lagrangian approach. Discrete Contin. Dyn. Syst. Ser. S, 1(3):405–426, 200.

[DLK22] C. De Lellis, H. Kwon. On non-uniqueness of Hölder continuous globally dissipative Euler flows. Anal. and PDE. 15(8), 2003–2059 (2022)

[DLS09] C. De Lellis and L. Székelyhidi Jr. The Euler equations as a differential inclusion. Ann. of Math. (2), 170(3):1417–1436, 2009.

[DLS10] C. De Lellis and L. Székelyhidi Jr. On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal., 195(1):225–260, 2010.

[DLS13] C. De Lellis and L. Székelyhidi Jr. Dissipative continuous Euler flows. Invent. Math., 193(2):377–407, 2013.

[DLS22] C. De Lellis and L. Székelyhidi Jr. Weak stability and closure in turbulence Phil. Trans. R. Soc. A. 380, 20210091 (2022)

[DSJ17] S. Daneri and L. Székelyhidi. Jr. Non-uniqueness and h-principle for H ölder-continuous weak solutions of the Euler equations. Arch. Ration. Mech. Anal., 224(2):471–514, 2017.

[ED18] G. L. Eyink and T. D. Drivas, A Lagrangian fluctuation-dissipation relation for scalar turbulence. Part III. Turbulent Rayleigh-Bénard convection, J. Fluid Mech. 836 (2018), 560–598.

[Gal23] L. Galeati. A note on weak existence for singular SDEs[J]. arXiv preprint arXiv:2309.06295, 2023.

[Gal24] L. Galeati. Almost-everywhere uniqueness of Lagrangian trajectories for 3D Navier–Stokes revisited. arXiv:2406.12788. 2024.

[GG25] L. Galeati, M. Gerencsér. Solution theory of fractional SDEs in complete subcritical regimes[C]//Forum of Mathematics, Sigma. Cambridge University Press, 2025, 13: e12.

[GHV14] N. Glatt-Holtz and V. Vicol, Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. Ann. Probab. 42(1), 80-145, 2014.

[GKN23] V. Giri, H. Kwon, M. Novack. The L^3-based strong Onsager theorem[J]. arXiv preprint arXiv:2305.18509, 2023.

[GKN24] V. Giri, H. Kwon, M. Novack. A wavelet-inspired L^3-based convex integration framework for the euler equations. Annals of PDE, 2024, 10(2): 19.

[GP24] L. Gräfner, N. Perkowski. Weak well-posedness of energy solutions to singular SDEs with supercritical distributional drift. arXiv:2407.09046. 2024.

[GR24] V. Giri, R. O. Radu. The 2d onsager conjecture: a newton-nash iteration. Inventiones mathematicae, 2024, 238(2): 691-768.

[Hao23] Z. Hao. McKean-Vlasov SDEs with singular drifts. Thesis (Ph.D.)- Bielefeld: Universität Bielefeld; 2023.

[HLP24] M. Hofmanová, T. Lange, U. Pappalettera. Global existence and non-uniqueness of 3D Euler equations perturbed by transport noise[J]. Probability Theory and Related Fields, 2024, 188(3): 1183-1255.

[HPZZ23] M. Hofmanová, U. Pappalettera, R. Zhu, and X. Zhu. Anomalous and total dissipation due to advection by solutions of randomly forced Navier-Stokes equations, arXiv:2305.08090, 2023.

[HRZ24] Z. Hao, M. Röckner, X. Zhang. Flow-distribution dependent SDEs and Navier-Stokes equations with $\mathbf{f}$ B $\mathbf{m}$[J]. arXiv:2405.19034, 2024.

[Hsu02] EP. Hsu. Stochastic analysis on manifolds[M]. American Mathematical Soc., 2002.

[HZ23] Z. Hao and X. Zhang. SDEs with supercritical distributional drifts. arXiv:2312.11145. 2023.

[HZZ22] M. Hofmanová, R. Zhu, and X. Zhu. On ill- and well-posedness of dissipative martingale solutions to stochastic 3D Euler equations. Commun. Pure Appl. Math Vol. LXXV, 2446–2510, 2022.

[HZZ23a] M. Hofmanová, R. Zhu, and X. Zhu. Global-in-time probabilistically strong and Markov solutions to stochastic 3D Navier-Stokes equations: existence and non-uniqueness. The Annals of probability, Vol. 51, No. 2, 524-579, 2023.

[HZZ23b] M. Hofmanová, R. Zhu, and X. Zhu. Global existence and non-uniqueness for 3D Navier-Stokes equations with space-time white noise. Archive for Rational Mechanics and Analysis, 247 (2023), no.3, paper No. 46.

[HZZ24] M. Hofmanová, R. Zhu, and X. Zhu. Non-uniqueness in law of stochastic 3D Navier–Stokes equations. J. Eur. Math. Soc. 26, 163–260 (2024).

[HZZ25] M. Hofmanová, R. Zhu, and X. Zhu. Non-unique ergodicity for deterministic and stochastic 3D Navier–Stokes and Euler equations. To appear in Archive for Rational Mechanics and Analysis.

[Ise18] P. Isett. A proof of Onsager's conjecture. Ann. of Math. (2), 188(3):871–963, 2018.

[Ise22] P. Isett. non-uniqueness and existence of continuous, globally dissipative Euler flows. Archive for Rational Mechanics and Analysis, 2022, 244(3): 1223-1309.

[JS24] C. J. P. Johansson and M. Sorella. Anomalous dissipation via spontaneous stochasticity with a two-dimensional autonomous velocity field. arXiv:2409.03599. 2024.

[Kol41] A. N. Kolmogorov, Dissipation of energy in the locally isotropic turbulence, C. R. (Doklady) Acad. Sci. URSS (N.S.) 32 (1941), 16-18.

[KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. Probability Theory and Related Fields, 131(2):154–196, 2005.

[Kra68] R. H. Kraichnan. Small-scale structure of a scalar field convected by turbulence. The Physics of Fluids, 1968, 11(5): 945-953.

[Kry20a] N. V. Krylov. On strong solutions of Itô's equations with $A \in W_d^1$ and $b \in L_d$. arXiv:2007.06040v1, 2020.

[Kry20b] N. V. Krylov. On stochastic Itô processes with drift in L_d. arXiv:2001.03660, 2020.

[Kry20c] N. V. Krylov. On strong solutions of Itô's equations with $A \in W_d^1$ and $b \in L_d$. arXiv:2007.06040v1, 2020.

[Kum24] A. Kumar. non-uniqueness of trajectories on a set of full measure for sobolev vector fields. Archive for Rational Mechanics and Analysis, 2024, 248(6): 114.

[Lor63] E. N. Lorenz. Deterministic nonperiodic flow. J. Atmos. Sci. 20, 130–141 (1963).

[Lor69] E. N. Lorenz. The predictability of a flow which possesses many scales of motion. Tellus 21, 289–307 (1969).

[LRS24] T. Lange, M. Rehmeier, A. Schenke. Non-uniqueness of Leray–Hopf solutions for the $3D$ fractional Navier–Stokes equations perturbed by transport noise. arXiv:2412.16532, 2024.

[LZ23] H. Lü, X. Zhu. Global-in-time probabilistically strong solutions to stochastic power-law equations: existence and non-uniqueness. Stochastic Processes and their Applications, 2023, 164: 62-98.

[LZ25a] H. Lü, X. Zhu. Sharp Non-uniqueness of Solutions to 2D Navier-Stokes Equations with Space-Time White Noise. To appear in The Annals of applied probability.

[LZ25b] H. Lü, X. Zhu. Non-unique Ergodicity for the 2D Stochastic Navier-Stokes Equations with Derivative of Space-Time White Noise. Journal of Differential Equations, 2025, 425 383-433.

[MB03] A. Majda, A Bertozzi. Vorticity and incompressible flow. World Publishing Corporation, 2003.

[MR23] A. A. Mailybaev, A. Raibekas. Spontaneous stochasticity and renormalization group in discrete multi-scale dynamics[J]. Communications in Mathematical Physics, 2023, 401(3): 2643-2671.

[MS18] S. Modena, L. Székelyhidi. Non-uniqueness for the transport equation with Sobolev vector fields. Ann. PDE 4(2), 38 (2018)

[MS19] S. Modena, L. Székelyhidi. Non-renormalized solutions to the continuity equation. Calc. Var. Partial Differ. Equ. 58(6), 30 (2019)

[MS20] S. Modena, and G. Sattig. Convex integration solutions to the transport equation with full dimensional concentration. Ann. Inst. H. Poincaré Anal. Non Linéaire 37, 5, 1075–1108, 2020.

[MS24] T. Moerschell, M. Sorella. Non-uniqueness of parabolic solutions for advection-diffusion equation. arXiv:2410.03308, 2024.

[MNY24a] C. Miao, Y. Nie, W. Ye. Non-uniqueness of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. arXiv:2412.10404, 2024.

[MNY24b] C. Miao, Y. Nie, W. Ye. Sharp non-uniqueness for the Navier-Stokes equations in $\mathbb{R}^3$. arXiv preprint arXiv:2412.09637, 2024.

[Nam20] Kyeongsik Nam. Stochastic differential equations with critical drifts. Stochastic Processes and their Applications, 130(9):5366–5393, 2020.

[Nov23] M. Novack. Scaling laws and exact results in turbulence[J]. Nonlinearity, 2024, 37(9): 095002.

[NV23] M. Novack, V. Vicol. An intermittent Onsager theorem. Inventiones mathematicae, 2023, 233(1): 223-323.

[Ons49] L. Onsager. Statistical hydrodynamics. Il Nuovo Cimento (1943-1954), 6(2):279–287, 1949.

[Pap23] U. Pappalettera. On measure-preserving selection of solutions of ODEs. arXiv:2311.15661.

[Pap24] U. Pappalettera. Global existence and non-uniqueness for the Cauchy problem associated to 3D Navier–Stokes equations perturbed by transport noise[J]. Stochastics and Partial Differential Equations: Analysis and Computations, 2024, 12(3): 1769-1804.

[PS23] J. Pitcho, M Sorella. Almost everywhere non-uniqueness of integral curves for divergence-free sobolev vector fields. SIAM Journal on Mathematical Analysis, 2023, 55(5): 4640-4663.

[RS09a] J. C. Robinson and W. Sadowski. Almost-everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier-Stokes equations. Nonlinearity, 22(9):2093–2099, 2009.

[RS09b] J. C. Robinson and W. Sadowski. A criterion for uniqueness of Lagrangian trajectories for weak solutions of the 3D Navier-Stokes equations. Comm. Math. Phys., 290(1):15–22, 2009.

[RZ21a] M. Röckner, X. Zhang. Well-posedness of distribution dependent SDEs with singular drifts. Bernoulli, 2021, 27(2).

[RZ21b] M. Röckner, G. Zhao. SDEs with critical time dependent drifts: strong solutions. arXiv:2103.05803. 2021.

[RZ23] M. Röckner, G. Zhao. SDEs with critical time dependent drifts: weak solutions. Bernoulli, 2023.

[Str87] D.W. Stroock, Lectures on stochastic analysis: difusion theory, LMS Student Texts, 1987

[TBM20] S. Thalabard, J. Bec, A. A. Mailybaev. From the butterfly effect to spontaneous stochasticity in singular shear flows. Communications Physics, 2020, 3(1): 122.

[Tre14] D. Trevisan. Well-posedness of diffusion processes in metric measure spaces. PhD thesis, 2014.

[Ver80] A. Y. Veretennikov. On strong solutions and explicit formulas for solutions of stochastic integral equations. Matematicheskii Sbornik, 153(3):434–452, 1980.

[XXZZ20] P. Xia, L. Xie, X. Zhang, G. Zhao, $L^q(L^p)$-theory of stochastic differential equations, Stochastic Processes and their Applications, Volume 130, Issue 8, 2020.

[Yam22a] K. Yamazaki. NonUniqueness in Law for Two-Dimensional Navier–Stokes Equations with Diffusion Weaker than a Full Laplacian. SIAM journal on mathematical analysis, 2022, 54(4): 3997-4042.

[Yam22b] K. Yamazaki. Remarks on the non-uniqueness in law of the Navier–Stokes equations up to the J.-L. Lions' exponent. Stochastic Processes and their Applications, 2022, 147: 226-269.

[YZ17] Y. Yao and A. Zlatoš. Mixing and un-mixing by incompressible flows. J. Eur. Math. Soc. (JEMS), 19(7):1911–1948, 2017.

[Zha05] X. Zhang. Strong solutions of SDES with singular drift and Sobolev diffusion coefficients, Stochastic Processes and their Applications, Volume 115, Issue 11, 2005.

[Zha11] X. Zhang. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. Electronic Journal of Probability, 16:1096–1116, 2011.

[Zha16] X. Zhang. Stochastic differential equations with Sobolev diffusion and singular drift and applications. The Annals of Applied Probability, 26(5):2697–2732, 2016.

[Zha19] G. Zhao. Stochastic Lagrangian flows for SDEs with rough coefficients. arXiv:1911.05562. 2019.

[ZZ21] X. Zhang and G. Zhao. Stochastic Lagrangian path for Leray's solutions of 3D Navier-Stokes equations. Comm. Math. Phys., 381(2):491–525, 2021.

(H. Lü) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: lvhuaxiang22@mails.ucas.ac.cn

(M. Röckner) Fakultät für Mathematik, Bielefeld Universität, D 33615 Bielefeld, Germany

Email address: roeckner@math.uni-bielefeld.de

(X. Zhu) State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: zhuxiangchan@126.com