THE SUPERPOSITION PRINCIPLE FOR SOLUTIONS TO FOKKER–PLANCK–KOLMOGOROV EQUATIONS WITH POTENTIAL TERMS

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Abstract. We study the superposition principle giving a representation of subprobability solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation with a potential term by means of solutions to martingale problems.

Keywords: Fokker–Planck–Kolmogorov equation, superposition principle, Cauchy problem, martingale problem.

1. Introduction

We consider on $\mathbb{R}^d_T = \mathbb{R}^d \times [0,T]$ the Cauchy problem for the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}\bigl(a^{ij}\mu_t\bigr) - \sum_{i=1}^d \partial_{x_i}\bigl(b^i\mu_t\bigr) + c\mu_t, \quad \mu_0 = \nu. \tag{1.1}$$

We assume throughout that ν is a Borel probability measure, the coefficients a^{ij}, b^i and c are Borel measurable, the matrix $A = (a^{ij})$ is symmetric and nonnegative definite, and for some nonnegative constant c_0 the inequality

$$-c_0 \le c(x,t) \le 0$$

is valid for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

A family of Borel subprobability measures $(\mu_t)_{t\in[0,T]}$ (which means that $\mu_t \ge 0$ and $\mu_t(\mathbb{R}^d) \le 1$) is called a solution if for every Borel set B the function $t \mapsto \mu_t(B)$ is Borel measurable, the coefficients a^{ij} and b^i are integrable with respect to the measure $\mu = \mu_t\,dt$ on $U \times [0,T]$ for every ball $U \subset \mathbb{R}^d$ and for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ and almost all $t \in [0,T]$ we have the equality

$$\int_{\mathbb{R}^d} \varphi\,\mu_t - \int_{\mathbb{R}^d} \varphi\,d\nu = \int_0^t \int_{\mathbb{R}^d} L_{A,b,c}\varphi\,d\mu_s\,ds, \tag{1.2}$$

where

$$L_{A,b,c}\varphi = \mathrm{trace}\bigl(AD^2\varphi\bigr) + \langle b, \nabla\varphi\rangle + c\varphi = \sum_{i,j\le d} a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \sum_{i\le d} b^i\partial_{x_i}\varphi + c\varphi.$$

The measure $\mu = \mu_t\,dt$ is defined by the equality

$$\int_{\mathbb{R}^d\times[0,T]} f\,d\mu = \int_0^T \int_{\mathbb{R}^d} f(x,t)\,\mu_t(dx)\,dt.$$

This measure is also called a solution.

In the papers [13], [24], and [9] the so-called superposition principle for probability solutions was established. In the case where c is zero, this principle states that for every solution $(\mu_t)_{t\in[0,T]}$, defined by a family of Borel probability measures μ_t (i.e.,

$\mu_t \geq 0$ and $\mu_t(\mathbb{R}^d) = 1$) such that the mapping $t \mapsto \mu_t$ is continuous in the weak topology there exists a solution P to the martingale problem with the operator $L_{A,b,0}$ and initial condition ν such that the one-dimensional projections $P \circ e_t^{-1}$ are equal to μ_t. Here and below $P \circ e_t^{-1}$ denotes the image of the measure P under the mapping $e_t \colon \omega \mapsto \omega(t)$. Note that the superposition principle holds for every probability solution given by a continuous curve in the space of probability measures under very weak conditions on the coefficients, for example, it is assumed in [9] that

$$\int_0^T \int_{\mathbb{R}^d} \frac{\|A(x,t)\| + |\langle b(x,t), x\rangle|}{1+|x|^2} \mu_t(dx)\, dt < \infty.$$

In [14] the superposition principle is proved for probability solutions on arbitrary open subsets of $\mathbb{R}^d$ and a justification of the superposition principle with the aid of change of coordinates is suggested. For the continuity equation, where $A = 0$, the superposition principle is established in [1]. In [22], a connection between the superposition principle for the continuity equation and the superposition principle in the theory of flows is found. The papers [24], [16], and [12] discuss the superposition principle for infinite-dimensional spaces. In [21], the superposition principle is obtained for nonlocal Fokker–Planck–Kolmogorov equations satisfied by the distributions of Lévy processes. Applications of the superposition principle to nonlinear Fokker–Planck–Kolmogorov equations are discussed in [17], [2], [18], [3], [5], [9], [4], and [10]. Classical results on existence and uniqueness of solutions to martingale problems can be found in [23] and [15]. The theory of Fokker–Planck–Kolmogorov equations is presented in [8].

This paper continues research initiated in [9] and [14] and is concerned with the superposition principle for solutions to Fokker–Planck–Kolmogorov equation with potential terms. Principal results on existence, uniqueness and regularity of solutions to such equations can be found in [20] and [8]. In place of solutions given by continuous curves in the space of probability measures, in the case of a nonzero potential we consider solutions defined by continuous curves $t \mapsto \mu_t$ in the space of subprobability measures such that

$$\mu_t(\mathbb{R}^d) = 1 + \int_0^t \int_{\mathbb{R}^d} c(x,s)\mu_s(dx)\, ds. \tag{1.3}$$

Our main result states that for every such solution $(\mu_t)_{t\in[0,T]}$ there exists a Borel probability measure P on $C([0,T], \mathbb{R}^d)$ with the following properties: the measure P is a solution to the martingale problem with the operator $L_{A,b,0}$ and initial condition ν and

$$\mu_t = \left(e^{h(\omega,t)} P\right) \circ e_t^{-1},$$

where

$$h(\omega,t) := \int_0^t c(\omega(s), s)\, ds.$$

In the case $A = 0$ for the continuity equation with a potential term the superposition principle was discussed in [19]. However, it should be noted that the justification of this principle in [19] is not satisfactory, because it assumes (see [19, Theorem 4.1, Step 3]) without any explanation that conditional measures corresponding to a weakly convergent sequence of probability measures also converge weakly, while it is well known that in general this is false. The paper [11] deals with the continuity equation with an unbounded potential term, but on a bounded domain. Finally, let us observe that in the case of smooth bounded coefficients the superposition principle for the Fokker–Planck–Kolmogorov equation can be derived from the Feynman–Kac formula

$$u(y,\tau)=\mathbb{E}\biggl[\varphi(\xi_t^{y,\tau})\exp\biggl(\int_\tau^t c(\xi_s^{y,\tau},s)\,ds\biggr)\biggr],$$

where $\varphi\in C_0^\infty(\mathbb{R}^d)$, the function u is the classical solution to the Cauchy problem

$$\partial_s u+L_{A,b,c}u=0,\quad u|_{s=t}=\varphi,$$

bounded along with its derivatives, and $\xi_t^{y,\tau}$ is the solution of the stochastic equation

$$d\xi_t=b(\xi_t,t)\,dt+\sqrt{2A(\xi_t,t)}\,dw_t,\quad \xi_\tau=y.$$

Note that the Feynman–Kac formula can be obtained with the aid of Itô's formula applied to the process $u(\xi_t^{y,\tau},t)$. Let us explain how it can be used to derive the superposition principle. Let $(\mu_t)_{t\in[0,T]}$ be the solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation with initial distribution ν. Then by the definition of a solution we have

$$\int_{\mathbb{R}^d}\varphi(y)\mu_t(dy)=\int_{\mathbb{R}^d}u(y,0)\,\nu(dy).$$

The left-hand side can be written as

$$\int_\Omega \varphi(\omega(t))e^{h(\omega,t)}P(d\omega),\quad \Omega=C([0,T],\mathbb{R}^d),$$

where $P=P^y\,\nu(dy)$ and P^y is the distribution of the process $\xi_t^{y,0}$. With the aid of Itô's formula one can verify that P is a solution to the martingale problem with the operator $L_{A,b}$ and initial condition ν and the equality obtained above means that

$$\mu_t=\Bigl(e^{h(\omega,t)}P\Bigr)\circ e_t^{-1}.$$

Of course, the conditions on the coefficients in the reasoning above can be relaxed, but one cannot get rid of the local regularity and restrictions on the growth, because we need the (probabilistically weak) existence of the solution $\xi_t^{x,\tau}$ of the stochastic equation along with the existence of a sufficiently regular solution to the Cauchy problem for the equation $u_t+L_{A,b,c}u=0$ (in order to apply Itô's formula) and to be able to substitute u into the integral identity, defining the solution μ_t. For these reasons, under our very weak assumptions about the coefficients, such a simple method of justification is not applicable. Note also that the situation is complicated by a possible non-uniqueness of solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation and the absence of a priori assumptions of existence of solutions to the corresponding martingale problem and even a probability solution to the Fokker–Planck–Kolmogorov equation with zero potential. Nevertheless, in the case of sufficiently regular coefficients the Feynman–Kac formula enables us to obtain probabilistic representations of solutions to nonlinear Fokker–Planck–Kolmogorov equations of a very general form admitting coefficients that depend not only on the solution density, but also on its gradient (see, e.g., [5], [17], and [18]).

2. Auxiliary results

Recall that the weak topology on the space of bounded Borel measures on $\mathbb{R}^d$ is defined by the seminorms

$$p_f(\sigma)=\biggl|\int_{\mathbb{R}^d}f\,d\sigma\biggr|,$$

where f is a bounded continuous function on $\mathbb{R}^d$. Convergence of a sequence of measures in this topology is convergence of integrals of bounded continuous functions with respect to these measures. This topology is not metrizable, but on the set of nonnegative measures it is metrizable (see [6] or [7]), for example, one can use the Kantorovich–Rubinshtein metric

$$d_{KR}(\mu,\nu)=\sup_{f\in \mathrm{Lip}_1,\,|f|\le 1}\biggl|\int_{\mathbb{R}^d} f\,d\mu-\int_{\mathbb{R}^d} f\,d\nu\biggr|,$$

where Lip_1 is the set of all Lipschitz functions with the Lipschitz constant 1.

Set

$$L_{A,b}\varphi=L_{A,b,0}\varphi=\mathrm{trace}\bigl(AD^2\varphi\bigr)+\langle b,\nabla\varphi\rangle=\sum_{i,j\le d}a^{ij}\partial_{x_i}\partial_{x_j}\varphi+\sum_{i\le d}b^i\partial_{x_i}\varphi.$$

In the following assertion we collect some properties of solutions to the Cauchy problem (1.1) used below.

Proposition 2.1. *Suppose that for every ball U the coefficients a^{ij}, b^i, and c are bounded on $U\times[0,T]$, $c\le 0$ and $A\ge\lambda(U)\cdot I$ for some number $\lambda(U)>0$. Let ν be a Borel probability measure on $\mathbb{R}^d$.*

(i) *There exists a subprobability solution $(\mu_t)_{t\in[0,T]}$ to the Cauchy problem* (1.1) *for which*

$$\mu_t(\mathbb{R}^d)\le 1+\int_0^t\int_{\mathbb{R}^d}c(x,s)\mu_s(dx)\,ds. \tag{2.1}$$

(ii) *If $-c_0\le c(x,t)\le 0$ for all $(x,t)\in\mathbb{R}^d\times[0,T]$, then one can find a solution $(\mu_t)_{t\in[0,T]}$ to the Cauchy problem* (1.1) *and a solution $(\widetilde{\mu}_t)_{t\in[0,T]}$ to the Cauchy problem* (1.1) *with the same coefficients a^{ij}, b^i and zero c such that*

$$\mu_t\le\widetilde{\mu}_t\le e^{c_0t}\mu_t.$$

(iii) *If there exists a function $V\in C^2(\mathbb{R}^d)$ such that $V\ge 0$, $\lim\limits_{|x|\to\infty}V(x)=+\infty$ and*

$$\int_0^T\int_{\mathbb{R}^d}\Bigl(\bigl|\sqrt{A}\nabla V\bigr|^2+\bigl|L_{A,b}V\bigr|\Bigr)\,d\mu_s\,ds<\infty,$$

then

$$\mu_t(\mathbb{R}^d)=1+\int_0^t\int_{\mathbb{R}^d}c(x,s)\mu_s(dx)\,ds.$$

(iv) *If in addition to the conditions in* (i) *and* (iii) *on every ball U we have $|a^{ij}(x,t)-a^{ij}(y,t)|\le\Lambda(U)|x-y|$, then a solution $(\mu_t)_{t\in[0,T]}$ from* (i) *is unique.*

Proof. Items (i), (iii), and (iv) are particular cases of the results in [20] (see Theorem 2.1, Remark 2.9, Theorem 3.5). Let us justify (ii). In [20], solutions are constructed as limits of solutions to equations with smooth bounded coefficients, whose derivatives are also bounded. Hence it suffices to verify that the inequalities are true in the case of such coefficients. Let $\psi\in C_0^\infty(\mathbb{R}^d)$ and $\psi\ge 0$. Consider a bounded solution f to the Cauchy problem

$$\partial_t f+L_{A,b}f=0,\quad f|_{t=s}=\psi.$$

The solution f is infinitely differentiable and bounded along with its first and second order derivatives, moreover, $f\ge 0$ by maximum principle. Substituting f and fe^{c_0t}

into the definition of a solution, we obtain the equalities

$$\int_{\mathbb{R}^d} \psi(x)\,\mu_s(dx) = \int_{\mathbb{R}^d} f(x,0)\,\nu(dx) + \int_0^s \int_{\mathbb{R}^d} c(x,t)f(x,t)\,\mu_t(dx)\,dt,$$

$$\int_{\mathbb{R}^d} \psi(x)\,\widetilde{\mu}_s(dx) = \int_{\mathbb{R}^d} f(x,0)\,\nu(dx),$$

$$\int_{\mathbb{R}^d} \psi(x)e^{c_0 s}\,\mu_s(dx) = \int_{\mathbb{R}^d} f(x,0)\,\nu(dx) + \int_0^s \int_{\mathbb{R}^d} \bigl(c_0 + c(x,t)\bigr)f(x,t)e^{c_0 t}\,\mu_t(dx)\,dt.$$

Taking into account that $f \geq 0$, $c_0 + c \geq 0$ and $c \leq 0$, we arrive at the inequalities

$$\int_{\mathbb{R}^d} \psi(x)\,\mu_s(dx) \leq \int_{\mathbb{R}^d} \psi(x)\,\widetilde{\mu}_s(dx), \qquad \int_{\mathbb{R}^d} \psi(x)e^{c_0 s}\,\mu_s(dx) \geq \int_{\mathbb{R}^d} \psi(x)\,\widetilde{\mu}_s(dx),$$

which yield the inequalities $\mu_s \leq \widetilde{\mu}_s \leq e^{c_0 s}\mu_s$, since ψ was arbitrary. □

The next assertion generalizes Proposition 2.2 and Proposition 2.4 from [9].

Proposition 2.2. *Let $\tau \in (0,T]$, $V \in C^2(\mathbb{R}^d)$, $V \geq 0$ and $\lim\limits_{|x|\to+\infty} V(x) = +\infty$.*

(i) *If $(\mu_t)_{t\in[0,T]}$ is a solution to the Cauchy problem* (1.1) *consisting of subprobability measures satisfying* (2.1), *$V \in L^1(\nu)$ and $L_{A,b}V \leq CV+W$, where $W \in L^1(\mu_t\,dt)$, $W \geq 0$ and $C \geq 0$, then equality* (1.3) *is true and for all $t \in [0,\tau]$ we have the estimates*

$$\sup_t \int_{\mathbb{R}^d} V\,d\mu_t \leq e^{Ct}\biggl(\int_{\mathbb{R}^d} V\,d\nu + \int_0^\tau \int_{\mathbb{R}^d} W\,d\mu_s\,ds\biggr),$$

$$\int_0^\tau \int_{\mathbb{R}^d} |L_{A,b}V|\,d\mu_t\,dt \leq 2e^{C\tau}\biggl(\int_{\mathbb{R}^d} V\,d\nu + \int_0^\tau \int_{\mathbb{R}^d} W\,d\mu_s\,ds\biggr).$$

(ii) *There exists a function $\theta \in C^2([0,+\infty))$ with the following properties:*

$$\theta \geq 0, \quad 0 \leq \theta' \leq 1, \quad \theta'' \leq 0, \quad \lim_{u\to+\infty} \theta(u) = +\infty, \quad \theta(V) \in L^1(\nu).$$

(iii) *Let $(\mu_t)_{t\in[0,T]}$ be a solution to the Cauchy problem* (1.1) *consisting of subprobability measures satisfying* (2.1). *If*

$$\int_0^\tau \int_{\mathbb{R}^d} \Bigl(\bigl|\sqrt{A}\nabla V\bigr|^2 + \bigl|L_{A,b}V\bigr|\Bigr)\,d\mu_s\,ds < \infty$$

and θ is the function from (ii), *then*

$$\int_0^\tau \int_{\mathbb{R}^d} \Bigl(\bigl|\sqrt{A}\nabla\theta(V)\bigr|^2 + \bigl|L_{A,b}\theta(V)\bigr|\Bigr)\,d\mu_s\,ds \leq 2e^{C\tau}\int_0^\tau \int_{\mathbb{R}^d} \Bigl(\bigl|\sqrt{A}\nabla V\bigr|^2 + \bigl|L_{A,b}V\bigr|\Bigr)\,d\mu_s.$$

Proof. We justify (i). Taking into account the inequality $c(x,t)V(x,t) \leq 0$ and repeating the reasoning from the proof of Theorem 7.1.1 in [8], we conclude that equality (1.3) is fulfilled and

$$\int_{\mathbb{R}^d} V\,d\mu_t \leq e^{Ct}\biggl(\int_{\mathbb{R}^d} V\,d\nu + \int_0^\tau \int_{\mathbb{R}^d} W\,d\mu_s\,ds\biggr).$$

Write $L_{A,b}V$ as the difference of the functions

$$(L_{A,b}V)^+ = \max\{0, L_{A,b}V\}, \quad (L_{A,b}V)^- = \max\{0, -L_{A,b}V\}.$$

Observe that by nonnegativity of W and CV we have $(L_{A,b}V)^+ \leq W + CV$. Repeating the reasoning from the proof of Proposition 2.1, we obtain the estimate

$$\int_0^\tau \int_{\mathbb{R}^d} |L_{A,b}V|\,d\mu_s\,ds \leq 2e^{C\tau}\biggl(\int_{\mathbb{R}^d} V\,d\nu + \int_0^\tau \int_{\mathbb{R}^d} W\,d\mu_s\,ds\biggr).$$

Item (ii) follows from [8, Proposition 7.1.8]. The justification of (iii) is based on the observation that $L_{A,b}\theta(V) \le |L_{A,b}V|$ and repeats the reasoning from [9, Proposition 2.4]. □

3. Main results

A Borel probability measure P on the path space $\Omega = C([0,T],\mathbb{R}^d)$ is called a solution to the martingale problem with the operator $L_{A,b}$ and initial condition ν if $P(\omega\colon \omega(0) \in B) = \nu(B)$ for every Borel set $B \subset \mathbb{R}^d$ and for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ the process

$$\xi_t(\omega) = \varphi(\omega(t)) - \varphi(\omega(0)) - \int_0^t L_{A,b}\varphi(\omega(s), s)\, ds$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\omega(s), s \le t)$ and the measure P.

Set $e_t(\omega) = \omega(t)$. Here we always assume that for every ball $U \subset \mathbb{R}^d$ one has

$$\int_0^T \int_U \Bigl(\|A(x,s)\| + |b(x,s)|\Bigr) P \circ e_s^{-1}(dx)\, ds < \infty.$$

Proposition 3.1. *Suppose that the measure P is a solution to the martingale problem with the operator $L_{A,b}$ and initial condition ν. Set*

$$\mu_t = \Bigl(e^{h(\omega,t)}P\Bigr) \circ e_t^{-1}, \quad h(\omega,t) = \int_0^t c(\omega(s), s)\, ds.$$

Then the measures μ_t and $P \circ e_t^{-1}$ are equivalent, equality (1.3) *holds, the mapping $t \mapsto \mu_t$ is continuous in the weak topology and the family of measures $(\mu_t)_{t\in[0,T]}$ is a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation* (1.1).

Proof. Let $\varphi \in C_b(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \varphi(x)\, \mu_t(dx) = \int_\Omega \varphi(\omega(t)) e^{h(\omega,t)}\, P(d\omega).$$

Since the right-hand side is continuous in t, the mapping $t \mapsto \mu_t$ is continuous in the weak topology. By the inequality $-c_0 \le c \le 0$ the right-hand side is estimated from above by the integral $\int_\Omega \varphi\, d(P \circ e_t^{-1})$ and is estimated from below by the expression $e^{-c_0 t} \int_\Omega \varphi\, d(P \circ e_t^{-1})$. Therefore, the measures μ_t and $P \circ e_t^{-1}$ are equivalent.

Since for all $t \in [0,T]$ we have

$$e^{h(\omega,t)} = 1 + \int_0^t c(\omega(s), s) e^{h(\omega,s)}\, ds,$$

by Fubini's theorem

$$\int_\Omega \varphi(\omega(t)) e^{h(\omega,t)}\, P(d\omega)$$
$$= \int_\Omega \varphi(\omega(t))\, P(d\omega) + \int_0^t \int_\Omega \varphi(\omega(t)) c(\omega(s), s) e^{h(\omega,s)}\, P(d\omega)\, ds.$$

For $\varphi = 1$ we obtain

$$\mu_t(\mathbb{R}^d) = 1 + \int_0^t \int_\Omega c(\omega(s), s) e^{h(\omega,s)}\, P(d\omega)\, ds = 1 + \int_0^t \int_{\mathbb{R}^d} c(x,s)\, \mu_s(dx)\, ds.$$

We now verify that the family of measures $(\mu_t)_{t\in[0,T]}$ satisfies the Fokker–Planck–Kolmogorov equation. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Consider the partition of $[0,T]$ by the points $0 = t_0 < t_1 < \ldots, t_N = t$, where $t_k = kt/N$. We have

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t - \int_{\mathbb{R}^d} \varphi\, d\nu = \sum_{k=1}^N \biggl(\int_{\mathbb{R}^d} \varphi\, d\mu_{t_k} - \int_{\mathbb{R}^d} \varphi\, d\mu_{t_{k-1}}\biggr).$$

By the definition of the measures μ_{t_k} and $\mu_{t_{k-1}}$ we have the equality

$$\int_{\mathbb{R}^d} \varphi\, d\mu_{t_k} - \int_{\mathbb{R}^d} \varphi\, d\mu_{t_{k-1}} = \int_\Omega \Bigl(\varphi(\omega(t_k))e^{h(\omega,t_k)} - \varphi(\omega(t_{k-1}))e^{h(\omega,t_{k-1})}\Bigr)\, P(d\omega).$$

Observe that

$$\begin{aligned}
&\varphi(\omega(t_k))e^{h(\omega,t_k)} - \varphi(\omega(t_{k-1}))e^{h(\omega,t_{k-1})}\\
&\qquad = \biggl(\varphi(\omega(t_k)) - \varphi(\omega(t_{k-1})) - \int_{t_{k-1}}^{t_k} L_{A,b}\varphi(\omega(s),s)\, ds\biggr)e^{h(\omega,t_{k-1})}\\
&\quad + \biggl(\int_{t_{k-1}}^{t_k} L_{A,b}\varphi(\omega(s),s)\, ds\biggr)e^{h(\omega,t_{k-1})} + \varphi(\omega(t_k))\biggl(\int_{t_{k-1}}^{t_k} c(\omega(s),s)e^{h(\omega,s)}\, ds\biggr),
\end{aligned}$$

where we used the equality

$$e^{h(\omega,t_k)} - e^{h(\omega,t_{k-1})} = \int_{t_{k-1}}^{t_k} c(\omega(s),s)e^{h(\omega,s)}\, ds.$$

Since the measure P is a solution to the martingale problem and the function $e^{h(\omega,t_{k-1})}$ is measurable with respect to $\mathcal{F}_{t_{k-1}}$, we obtain

$$\int_\Omega \biggl(\varphi(\omega(t_k)) - \varphi(\omega(t_{k-1})) - \int_{t_{k-1}}^{t_k} L_{A,b}\varphi(\omega(s),s)\, ds\biggr)e^{h(\omega,t_{k-1})}\, P(d\omega) = 0.$$

We also have the equality

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} L_{A,b}\varphi(\omega(s),s)\, ds\, e^{h(\omega,t_{k-1})}\\
&\qquad = \int_{t_{k-1}}^{t_k} e^{h(\omega,s)}L_{A,b}\varphi(\omega(s),s)\, ds + \int_{t_{k-1}}^{t_k} \Bigl(e^{h(\omega,t_{k-1})} - e^{h(\omega,s)}\Bigr)L_{A,b}\varphi(\omega(s),s)\, ds.
\end{aligned}$$

Since

$$\Bigl|e^{h(\omega,t_{k-1})} - e^{h(\omega,s)}\Bigr| = e^{h(\omega,s)}\biggl|\exp\biggl(-\int_{t_{k-1}}^{s} c(\omega(\tau),\tau)\, d\tau\biggr) - 1\biggr| \le |t_{k-1} - s|e^{h(\omega,s)}e^{c_0 t},$$

we have

$$\biggl|\int_{t_{k-1}}^{t_k} \Bigl(e^{h(\omega,t_{k-1})} - e^{h(\omega,s)}\Bigr)L_{A,b}\varphi(\omega(s),s)\, ds\biggr| \le e^{c_0 t}tN^{-1}\int_{t_{k-1}}^{t_k} e^{h(\omega,s)}\bigl|L_{A,b}\varphi(\omega(s),s)\bigr|\, ds.$$

Observe also that

$$\begin{aligned}
&\varphi(\omega(t_k))\biggl(\int_{t_{k-1}}^{t_k} c(\omega(s),s)e^{h(\omega,s)}\, ds\biggr)\\
&= \int_{t_{k-1}}^{t_k} \varphi(\omega(s))c(\omega(s),s)e^{h(\omega,s)}\, ds + \int_{t_{k-1}}^{t_k} \bigl(\varphi(\omega(t_k)) - \varphi(\omega(s))\bigr)c(\omega(s),s)e^{h(\omega,s)}\, ds,
\end{aligned}$$

in addition,

$$\Bigl|\int_{t_{k-1}}^{t_k} \bigl(\varphi(\omega(t_k))-\varphi(\omega(s))\bigr)c(\omega(s),s)e^{h(\omega,s)}\,ds\Bigr| \le \frac{c_0 t}{N} \sup_{|\tau_1-\tau_2|\le tN^{-1}} \bigl|\varphi(\omega(\tau_1))-\varphi(\omega(\tau_2))\bigr|.$$

Thus, we arrive at the equality

$$\int_{\mathbb{R}^d}\varphi\,d\mu_t-\int_{\mathbb{R}^d}\varphi\,d\nu = \int_\Omega\biggl(\int_0^t \bigl(e^{h(\omega,s)}L_{A,b}\varphi(\omega(s),s)+e^{h(\omega,s)}c(\omega(s),s)\varphi(\omega(s))\,ds\biggr)\,P(d\omega)+\delta_N,$$

where

$$\bigl|\delta_N\bigr|\le\frac{e^{c_0t}t}{N}\int_0^t\int_{\mathbb{R}^d}|L_{A,b}\varphi|\,d\mu_s\,ds+c_0t\int_\Omega\sup_{|\tau_1-\tau_2|\le tN^{-1}}\bigl|\varphi(\omega(\tau_1))-\varphi(\omega(\tau_2))\bigr|\,P(d\omega).$$

Letting $N\to\infty$, we obtain the equality

$$\int_{\mathbb{R}^d}\varphi\,d(\mu_t-\nu)=\int_\Omega\int_0^t\Bigl(e^{h(\omega,s)}L_{A,b}\varphi(\omega(s),s)+e^{h(\omega,s)}c(\omega(s),s)\varphi(\omega(s)\Bigr)\,ds\,P(d\omega).$$

Applying Fubini's theorem and the change of variable formula we arrive at the equality

$$\int_{\mathbb{R}^d}\varphi\,d(\mu_t-\nu)=\int_0^t\int_{\mathbb{R}^d}\bigl(L_{A,b}\varphi+c\varphi\bigr)\,d\mu_s\,ds$$

from the definition of a solution. □

Remarkably, under very general conditions, the converse assertion is true.

Theorem 3.2. *Suppose that a mapping $t\mapsto\mu_t$ from $[0,T]$ to the space of subprobability measures is continuous in the weak topology, the family of measures $(\mu_t)_{t\in[0,T]}$ is a solution to the Cauchy problem* (1.1), *equality* (1.3) *holds, and*

$$(1+|x|)^{-2}\Bigl(\|A(x,t)\|+\bigl|\langle b(x,t),x\rangle\bigr|\Bigr)\in L^1(\mathbb{R}^d\times[0,T],\mu),\quad -c_0\le c(x,t)\le 0.$$

Then there exists a solution P to the martingale problem with the operator $L_{A,b}$ and initial condition ν such that

$$\mu_t=\Bigl(e^{h(\omega,t)}P\Bigr)\circ e_t^{-1}.$$

Proof. Since the proof repeats conceptually and partly technically the proof of a similar theorem for the case $c=0$ in [9], we omit part of the reasoning and give references to the corresponding places in [9]. The scheme of our proof consists of the following steps: 1) smoothing the coefficients of the operator $L_{A,b,c}$, 2) applying the superposition principle known in the smooth case along with the theorem of uniqueness of a subprobability solution, 3) justifying the limit procedure. The way of approximating the coefficients by smooth functions in this proof differs slightly from approximations in [9]. Namely, in place of convolutions with a smooth kernel with respect to the variable t we use expressions of the form $\varepsilon^{-1}\int_t^{t+\varepsilon} f(s)\,ds$, which is simpler and enables us to shorten the proof a bit.

According to Proposition 2.2 there exists a function θ for which

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}\theta(\ln(1+|x|^2))\,\mu_t(dx)<\infty.$$

Let $0 < T' < T$ and $T - T' < 1$. We first construct an auxiliary measure P on $C[0,T']$. Then the construction of the desired measure P on $C[0,T]$ repeats verbatim to reasoning from [9], namely, the measure P on $C[0,T]$ is obtained as the weak limit of some subsequence of measures P_n on $C[0,T-n^{-1}]$.

Let $\zeta \in C^\infty([0,\infty))$, $\zeta(0)=1$, $0 \le \zeta \le 1$, $\zeta' \le 0$ and $\zeta(v)=0$ if $v>1$. Set

$$\eta(v) = \int_v^{+\infty} \zeta(u)\,du, \quad v \in [0,+\infty).$$

Observe that $\eta(v)=0$ if $v>1$ and $\eta'(v) = -\zeta(v)$.

If $0 < \varepsilon < T - T'$, we denote the function $c_d\varepsilon^{-d}\zeta(\varepsilon^{-2}|x|^2)$ by h_ε, where the constant c_d is picked such that

$$\int_{\mathbb{R}^d} h_\varepsilon(x)\,dx = 1.$$

Let $\gamma(x) = (2\pi)^{-d/2}e^{-|x|^2/2}$ and

$$\kappa_\varepsilon = \varepsilon^{-1}\int_0^\varepsilon \mu_s(\mathbb{R}^d)\,ds.$$

Then $\lim\limits_{\varepsilon\to 0}\kappa_\varepsilon = 1$, hence we can assume that $\kappa_\varepsilon > 1/2$, decreasing ε. Set

$$\sigma_t^\varepsilon(x,t) = \varepsilon\gamma(x) + \frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} h_\varepsilon(x-y)\,\mu_s(dy)\,ds,$$

$$a_\varepsilon^{ij}(x,t) = \frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon\sigma_t^\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} a^{ij}(y,s)h_\varepsilon(x-y)\,\mu_s(dy)\,ds,$$

$$b_\varepsilon^{i}(x,t) = \frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon\sigma_t^\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} b^{i}(y,s)h_\varepsilon(x-y)\,\mu_s(dy)\,ds,$$

$$c_\varepsilon(x,t) = \frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon\sigma_t^\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} c(y,s)h_\varepsilon(x-y)\,\mu_s(dy)\,ds.$$

Below we use also the notation

$$\alpha_\varepsilon(x,t) = A_\varepsilon(x,t) + \frac{\varepsilon\gamma(x)}{\sigma_t^\varepsilon(x)}I, \quad \beta_\varepsilon(x,t) = b_\varepsilon(x,t) - \frac{\varepsilon\gamma(x)}{\sigma_t^\varepsilon(x)}x.$$

The coefficients α_ε^{ij}, β_ε^i, c_ε are infinitely differentiable in x and continuous in t, moreover, $-2c_0 \le c_\varepsilon(x,t) \le 0$.

The family of measures $(\sigma_t^\varepsilon)_{t\in[0,T']}$ is a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation with the operator

$$\mathcal{L}^\varepsilon_{\alpha,\beta,c}\varphi = L^\varepsilon_{A,b,c}\varphi + \frac{\varepsilon\gamma}{\sigma_t^\varepsilon}\big(\Delta\varphi - \langle x, \nabla\varphi\rangle\big),$$

where

$$L^\varepsilon_{A,b,c}\varphi = \mathrm{trace}\big(A_\varepsilon D^2\varphi\big) + \langle b_\varepsilon, \nabla\varphi\rangle + c_\varepsilon\varphi,$$

and initial condition

$$\nu_\varepsilon(x) = \varepsilon^{-1}\kappa_\varepsilon^{-1}(1-\varepsilon)\int_0^\varepsilon\int_{\mathbb{R}^d} h_\varepsilon(x-y)\mu_s(dy)\,ds + \varepsilon\gamma(x).$$

Observe that $\nu_\varepsilon(\mathbb{R}^d)=1$. Moreover, since

$$\int_0^t\int_{\mathbb{R}^d} c_\varepsilon(y,s)\sigma_s^\varepsilon(dy)\,ds=\kappa_\varepsilon^{-1}\varepsilon^{-1}(1-\varepsilon)\int_0^t\int_s^{s+\varepsilon}\int_{\mathbb{R}^d} c(y,\tau)\mu_\tau(dy)\,d\tau\,ds$$
$$=\kappa_\varepsilon^{-1}\varepsilon^{-1}(1-\varepsilon)\int_0^t\Bigl(\mu_{s+\varepsilon}(\mathbb{R}^d)-\mu_s(\mathbb{R}^d)\Bigr)\,ds=\sigma_t^\varepsilon(\mathbb{R}^d)-\nu_\varepsilon(\mathbb{R}^d),$$

the family $(\sigma_t^\varepsilon)_{t\in[0,T']}$ satisfies equality (1.3).

Note that for every ball $U(0,R)\subset\mathbb{R}^d$ we have

$$\int_0^{T'}\int_{U(0,R)}\Bigl(\|\alpha_\varepsilon\|+|\beta_\varepsilon|\Bigr)\sigma_t^\varepsilon\,dx\,dt\le 2\int_0^T\int_{U(0,R+1)}\Bigl(\|A\|+|b|\Bigr)\,d\mu_t\,dt+2T.$$

We show that

$$\int_0^{T'}\int_{\mathbb{R}^d}\biggl(\frac{\|\alpha_\varepsilon(x,t)\|+|\langle\beta_\varepsilon(x,t),x\rangle|}{1+|x|^2}\biggr)\sigma_t^\varepsilon(x)\,dx\,dt\le C,$$

and moreover there exists a function $V\in C^2(\mathbb{R}^d)$ such that $\lim\limits_{|x|\to\infty}V(x)=+\infty$ and

$$\int_{\mathbb{R}^d}V\,d\nu_\varepsilon+\int_0^{T'}\int_{\mathbb{R}^d}\Bigl(|\sqrt{\alpha_\varepsilon}\nabla V|^2+|L_{\alpha,\beta}^\varepsilon V|\Bigr)\,d\sigma_t^\varepsilon\,dt\le\widetilde{C},$$

where the numbers C and $\widetilde{C}$ do not depend on ε. We now observe that

$$|\langle\beta_\varepsilon(x,t),x\rangle|\le|\langle b_\varepsilon(x,t),x\rangle|+\frac{\varepsilon\gamma(x)|x|^2}{\sigma_t^\varepsilon},$$

where

$$\langle b_\varepsilon(x,t),x\rangle=\frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon\sigma_t^\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),y\rangle h_\varepsilon(x-y)\,\mu_s(dy)\,ds$$
$$+\frac{(1-\varepsilon)}{\varepsilon\kappa_\varepsilon\sigma_t^\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),x-y\rangle h_\varepsilon(x-y)\,\mu_s(dy)\,ds.$$

Note that

$$\varepsilon^{-1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),x-y\rangle h_\varepsilon(x-y)\,\mu_s(dy)\,ds$$
$$=2\varepsilon^{-d+1}c_d\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),\nabla_y\eta(\varepsilon^{-2}|x-y|^2)\rangle\,\mu_s(dy)\,ds.$$

Since $(\mu_t)_{t\in[0,T]}$ satisfies the Fokker–Planck–Kolmogorov equation, we have the equality

$$\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),\nabla_y\eta(\varepsilon^{-2}|x-y|^2)\rangle\,\mu_s(dy)\,ds$$
$$=\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_{t+\varepsilon}(dy)-\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_t(dy)$$
$$-\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\Bigl(\mathrm{trace}\bigl(A(y,s)D_y^2\eta(\varepsilon^{-2}|x-y|^2)\bigr)+c(y,s)\eta(\varepsilon^{-2}|x-y|^2)\Bigr)\,\mu_s(dy)\,ds.$$

Note that

$$\begin{aligned}&\operatorname{trace}\big(A(y,s)D_y^2\eta(\varepsilon^{-2}|x-y|^2)\big)\\&\qquad=4\varepsilon^{-4}\eta''(\varepsilon^{-2}|x-y|^2)\langle A(y,s)(x-y),x-y\rangle-2\varepsilon^{-2}\eta'(\varepsilon^{-2}|x-y|^2)\mathrm{tr}A(y,s).\end{aligned}$$

Since $\eta'=-\zeta$ and $\eta''=-\zeta'$, we have

$$\begin{aligned}&\Big|\operatorname{trace}\big(A(y,s)D_y^2\eta(\varepsilon^{-2}|x-y|^2)\big)\Big|\\&\qquad\le 4\varepsilon^{-4}|x-y|^2|\zeta'(\varepsilon^{-2}|x-y|^2)|\|A(y,s)\|+2\varepsilon^{-2}\zeta(\varepsilon^{-2}|x-y|^2)\operatorname{trace}A(y,s).\end{aligned}$$

Thus, we obtain the estimate

$$\begin{aligned}&\varepsilon^{-1}\Big|\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\langle b(y,s),x-y\rangle h_\varepsilon(x-y)\,\mu_s(dy)\,ds\Big|\\&\quad\le 2\varepsilon^{-d+1}c_d\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_{t+\varepsilon}(dy)+2\varepsilon^{-d+1}c_d\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_t(dy)\\&\qquad+8\varepsilon^{-d-3}c_d\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}|x-y|^2|\zeta'(\varepsilon^{-2}|x-y|^2)|\|A(y,s)\|\,\mu_s(dy)\,ds\\&\qquad+4\varepsilon^{-d-1}c_d\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\zeta(\varepsilon^{-2}|x-y|^2)\operatorname{trace}A(y,s)\,\mu_s(dy)\,ds\\&\qquad+2c_0c_d\varepsilon^{-d+1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_s(dy)\,ds.\end{aligned}$$

It follows that there is a number $N_1>0$ such that

$$\frac{|\langle\beta_\varepsilon(x,t),x\rangle|}{1+|x|^2}\le\frac{N_1}{\sigma_t^\varepsilon}\Big(\gamma(x)+B_\varepsilon(x,t)+\varepsilon^{-1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\frac{|\langle b(y,s),y\rangle|}{1+|y|^2}h_\varepsilon(x-y)\,\mu_s(dy)\,ds\Big),$$

where

$$\begin{aligned}B_\varepsilon(x,t)&=2\varepsilon^{-d+1}c_d\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_{t+\varepsilon}(dy)+2\varepsilon^{-d+1}c_d\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_t(dy)\\&\quad+8\varepsilon^{-d-3}c_d\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}|x-y|^2|\zeta'(\varepsilon^{-2}|x-y|^2)|\|A(y,s)\|(1+|y|^2)^{-1}\,\mu_s(dy)\,ds\\&\quad+4\varepsilon^{-d-1}c_d\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\zeta(\varepsilon^{-2}|x-y|^2)\|A(y,s)\|(1+|y|^2)^{-1}\,\mu_s(dy)\,ds\\&\quad+2c_0c_d\varepsilon^{-d+1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\eta(\varepsilon^{-2}|x-y|^2)\,\mu_s(dy)\,ds.\end{aligned}$$

There is also a number $N_2>0$ independent of ε such that

$$\int_0^{T'}\int_{\mathbb{R}^d}B_\varepsilon(x,t)\,dx\,dt\le N_2\Big(1+\int_0^T\int_{\mathbb{R}^d}\frac{\|A(y,s)\|}{1+|y|^2}\,\mu_s(dy)\,ds\Big).$$

Therefore,

$$\begin{aligned}&\int_0^{T'}\int_{\mathbb{R}^d}\frac{|\langle\beta_\varepsilon(x,t),x\rangle|}{1+|x|^2}\,\sigma_t^\varepsilon(x)\,dx\,dt\\&\qquad\le N_1+N_2+(N_1+N_2)\int_0^T\int_{\mathbb{R}^d}\frac{\|A(y,s)\|+|\langle b(y,s),y\rangle|}{1+|y|^2}\,\mu_s(dy)\,ds.\end{aligned}$$

Since

$$\frac{\|\alpha_\varepsilon(x,t)\|}{1+|x|^2} \le \frac{N_3}{\sigma_t^\varepsilon}\biggl(\gamma(x) + \varepsilon^{-1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} h_\varepsilon(x-y)\frac{\|A(y,s)\|}{1+|y|^2}\,\mu_s(dy)\,ds\biggr),$$

we have

$$\int_0^{T'}\int_{\mathbb{R}^d}\frac{\|\alpha_\varepsilon(x,t)\|}{1+|x|^2}\,\sigma_t^\varepsilon(x)\,dx\,dt \le N_3 + N_3\int_0^T\int_{\mathbb{R}^d}\frac{\|A(y,s)\|}{1+|y|^2}\,\mu_s(dy)\,ds.$$

Thus we obtain

$$\int_0^{T'}\int_{\mathbb{R}^d}\biggl(\frac{\|\alpha_\varepsilon(x,t)\| + |\langle\beta_\varepsilon(x,t),x\rangle|}{1+|x|^2}\biggr)\sigma_t^\varepsilon(x)\,dx\,dt \le N_4,$$

where the number N_4 does not depend on ε. Therefore, for the function $W(x) = \ln(1+|x|^2)$ the estimate

$$\int_0^{T'}\int_{\mathbb{R}^d}\Bigl(|\sqrt{\alpha_\varepsilon}\nabla W|^2 + |\mathcal{L}^\varepsilon_{\alpha,\beta}W|\Bigr)\,\sigma_t^\varepsilon\,dx\,dt \le N_5$$

holds, where N_5 does not depend on ε. According to assertion (iii) of Proposition 2.2 for the function $V = \theta(W)$ we have

$$\int_{\mathbb{R}^d} V(x)\,d\nu_\varepsilon + \int_0^{T'}\int_{\mathbb{R}^d}\Bigl(|\sqrt{\alpha_\varepsilon}\nabla V|^2 + |\mathcal{L}^\varepsilon_{\alpha,\beta}V|\Bigr)\sigma_t^\varepsilon\,dx\,dt \le N_6,$$

where N_6 does not depend on ε. By assertion (iv) of Proposition 2.2 the family of subprobability measures $\sigma_t^\varepsilon\,dx$ is a unique subprobability solution to the Cauchy problem

$$\partial_t\sigma = (\mathcal{L}^\varepsilon_{\alpha,\beta})^*\sigma + c_\varepsilon\sigma, \quad \sigma|_{t=0} = \nu_\varepsilon.$$

Moreover, by assertion (ii) in Proposition 2.2 there exists a subprobability solution $\varrho_t^\varepsilon\,dx$ to the Cauchy problem

$$\partial_t\varrho = (\mathcal{L}^\varepsilon_{\alpha,\beta})^*\varrho, \quad \varrho|_{t-0} = \nu_\varepsilon$$

such that

$$\sigma_t^\varepsilon \le \varrho_t^\varepsilon \le e^{2c_0t}\sigma_t^\varepsilon.$$

Therefore,

$$\int_0^{T'}\int_{\mathbb{R}^d}\biggl(\frac{\|\alpha_\varepsilon(x,t)\| + |\langle\beta_\varepsilon(x,t),x\rangle|}{1+|x|^2}\biggr)\sigma_t^\varepsilon(x)\,dx\,dt \le N_4e^{2c_0T'},$$

and

$$\int_0^{T'}\int_{\mathbb{R}^d}\Bigl(|\sqrt{\alpha_\varepsilon}\nabla V|^2 + |\mathcal{L}^\varepsilon_{\alpha,\beta}V|\Bigr)\varrho_t^\varepsilon\,dx\,dt \le N_6e^{2c_0T'}.$$

Hence $\varrho_t^\varepsilon\,dx$ is a unique probability solution and by the superposition principle from [9] there exists a solution P^ε to the martingale problem with the operator $\mathcal{L}^\varepsilon_{\alpha,\beta}$ such that $\varrho_t^\varepsilon = P^\varepsilon\circ e_t^{-1}$. Moreover, according to Proposition 3.1 we have

$$\sigma_t^\varepsilon = \Bigl(\exp\int_0^t c_\varepsilon\,ds\cdot P^\varepsilon\Bigr)\circ e_t^{-1}.$$

Applying [9, Proposition 2.5] and [24, Corollary A5]), we conclude that P^ε contains a subsequence P^{ε_n} with $\varepsilon_n\to 0$ weakly convergent to some probability measure P. Let $\varrho_t = P\circ e_t^{-1}$. Since the measures $\varrho_t^\varepsilon = P^\varepsilon\circ e_t^{-1}$ satisfy the inequality $\varrho_t^\varepsilon \le e^{2c_0t}\sigma_t^\varepsilon$, their limiting probability measure ϱ_t satisfies an analogous estimate with respect to μ_t. In particular, the coefficients a^{ij} and b^i are locally integrable with respect to $\varrho_t\,dt$. In addition, the measure P is a solution to the martingale problem with the operator $L_{A,b}$ and for the measure μ_t we have the representation

$$\mu_t = \Bigl(e^{h(\omega,t)}P\Bigr)\circ e_t^{-1}.$$

These assertions are obtained by means of passing to the limit as $\varepsilon\to 0$, the justification of which practically repeats verbatim the reasoning from [9]. Let us show, for example, how we can pass to the limit in the expression

$$\int_\Omega \varphi(\omega(t))\Bigl(\exp\int_0^t c_\varepsilon(\omega(s),s)\,ds\Bigr)\,P^\varepsilon(d\omega).$$

Let $q\in C_0^\infty(\mathbb{R}^d\times(0,T))$ and $-2c_0\le q\le 0$. Set

$$q_\varepsilon(x,t)=\frac{(1-\varepsilon)\bigl(q\mu_t\bigr)_\varepsilon}{\kappa_\varepsilon\sigma_t^\varepsilon},$$

where

$$\bigl(q\mu_t\bigr)_\varepsilon=\varepsilon^{-1}\int_t^{t+\varepsilon}\int_{\mathbb{R}^d} c(y,s)h_\varepsilon(x-y)\,\mu_s(dy)\,ds.$$

As $\varepsilon\to 0$, the functions q_ε converge uniformly to q, and the function

$$\exp\int_0^t q(\omega(s),s)\,ds$$

is bounded and continuous in ω. Therefore,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(\omega(t))\Bigl(\exp\int_0^t q_\varepsilon(\omega(s),s)\,ds\Bigr)\,P^\varepsilon(d\omega)$$
$$=\int_\Omega \varphi(\omega(t))\Bigl(\exp\int_0^t q(\omega(s),s)\,ds\Bigr)\,P(d\omega).$$

It remains to estimate the expressions

$$\Bigl|\int_\Omega \varphi(\omega(t))\Bigl(\exp\int_0^t c(\omega(s),s)\,ds-\exp\int_0^t q(\omega(s),s)\,ds\Bigr)\,P(d\omega)\Bigr|,$$
$$\Bigl|\int_\Omega \varphi(\omega(t))\Bigl(\exp\int_0^t c_\varepsilon(\omega(s),s)\,ds-\exp\int_0^t q_\varepsilon(\omega(s),s)\,ds\Bigr)\,P^\varepsilon(d\omega)\Bigr|.$$

To this end it suffices to estimate the integrals

$$\int_0^T\int_{\mathbb{R}^d}|c_\varepsilon(x,t)-q_\varepsilon(x,t)|\varrho_t^\varepsilon(x)\,dx\,dt,$$
$$\int_0^T\int_{\mathbb{R}^d}|c(x,t)-q(x,t)|\,\varrho_t(dx)\,dt.$$

For estimating the first integral we use the inequality $\varrho_t^\varepsilon\le e^{2c_0t}\sigma_t^\varepsilon$ and observe that

$$|c_\varepsilon(x,t)-q_\varepsilon(x,t)|\sigma_t^\varepsilon(x)=\frac{(1-\varepsilon)}{\kappa_\varepsilon}\bigl((c-q)\mu_t\bigr)_\varepsilon(x,t).$$

Hence

$$\int_0^T\int_{\mathbb{R}^d}|c_\varepsilon(x,t)-q_\varepsilon(x,t)|\varrho_t^\varepsilon(x)\,dx\,dt\le 2e^{2c_0T}\int_0^T\int_{\mathbb{R}^d}|c(x,t)-q(x,t)|\,\mu_t(dx)\,dt.$$

Since $\varrho_t(dx) \le e^{2c_0 t}\mu_t$, also for estimating the expression

$$\int_0^T \int_{\mathbb{R}^d} |c(x,t) - q(x,t)|\, \varrho_t(dx)\, dt$$

it remains to estimate the integral

$$\int_0^T \int_{\mathbb{R}^d} |c(x,t) - q(x,t)|\, \mu_t(dx)\, dt,$$

which can be made arbitrarily small by approximating the coefficient c by smooth functions q in $L^1(\mu_t\, dt)$. □

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References

[1] L. Ambrosio, Transport equation and Cauchy problem for non-smooth vector fields. Lect. Notes Math. 1927 (2008), 2–41.

[2] V. Barbu, M. Röckner, Probabilistic representation for solutions to nonlinear Fokker–Planck equations. SIAM J. Math. Anal. 50:4 (2018), 4246–4260.

[3] V. Barbu, M. Röckner, From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE. Ann. Probab. 48:4 (2020), 1902–1920.

[4] V. Barbu, M. Röckner, Nonlinear Fokker–Planck flows and their probabilistic counterparts. Lecture Notes in Mathematics, Springer Cham, 2024.

[5] Ya.I. Belopol'skaya, Systems of nonlinear backward and forward Kolmogorov equations: generalized solutions, Theory Probab. Appl. 66:1 (2021), 15–43.

[6] V.I. Bogachev, Measure theory. V. 2. Springer, New Your, 2007.

[7] V.I. Bogachev, Weak convergence of measures. Amer. Math. Soc., Rhode Island, Providence, 2018.

[8] V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov, Fokker–Planck–Kolmogorov equations. Amer. Math. Soc., Providence, Rhode Island, 2015.

[9] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, On the Ambrosio–Figalli–Trevisan superposition principle for probability solutions to Fokker–Planck–Kolmogorov equations. J. Dyn. Differ. Equ. 33:2 (2021), 715–739.

[10] V.I. Bogachev, S.V. Shaposhnikov, Noninear Fokker–Planck–Kolmogorov equations. Russian Math. Surveys 79:5 (2024)

[11] K. Bredies, M. Carioni, S. Fanzon, A superposition principle for the inhomogeneous continuity equation with Hellinger–Kantorovich-regular coefficients. Comm. Partial Differ. Equ. 47:10 (2022), 2023–2069.

[12] M. Dieckmann, A restricted superposition principle for (non-)linear Fokker–Planck–Kolmogorov equations on Hilbert spaces. J. Evol. Equ. 22:2 (2022), Paper No. 55, 28 pp.

[13] A. Figalli, Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. J. Funct. Anal. 254:1 (2008), 109–153.

[14] T.I. Krasovitskii, S.V. Shaposhnikov, Superposition principle for the Fokker–Planck–Kolmogorov equations with unbounded coefficients. Funct. Anal. Appl. 56 (2022), 282–298.

[15] T.G. Kurtz, Equivalence of stochastic equations and martingale problems. In: Crisan, D. (ed.) Stochastic Analysis 2010, pp. 113–130. Springer, Heidelberg, 2011.

[16] D. Lacker, M. Shkolnikov, J. Zhang, Superposition and mimicking theorems for conditional McKean–Vlasov equations, arXiv: 2004.00099.

[17] A. Le Cavil, N. Oudjane, F. Russo, Probabilistic representation of a class of non-conservative nonlinear partial differential equations, ALEA Lat. Am. J. Probab. Math. Stat. 13:2 (2016), 1189–1233.

[18] A. Le Cavil, N. Oudjane, F. Russo, Forward Feynman–Kac type representation for semilinear non-conservative partial differential equations, Stochastics 91:8 (2019), 1206–1248.

[19] S. Maniglia, Probabilistic representation and uniqueness results for measure-valued solutions of transport equations. J. Math. Pures Appl. 87:6 (2007), 601–626.
[20] O.A. Manita, S.V. Shaposhnikov, On the Cauchy problem for Fokker–Planck–Kolmogorov equations with potential terms on arbitrary domains. J. Dyn. Differ. Equ. 28 (2016), 493–518.
[21] M. Röckner, L. Xie, X. Zhang, Superposition principle for non-local Fokker–Planck–Kolmogorov operators. Probab. Theory Related Fields 178:3-4 (2020), 699–733.
[22] E. Stepanov, D. Trevisan, Three superposition principles: currents, continuity equations and curves of measures. J. Funct. Anal. 272:3 (2017), 1044–1103.
[23] D.W. Stroock, S.R.S. Varadhan, Multidimensional diffusion processes. Springer, Berlin, 2006.
[24] D. Trevisan, Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. Electron. J. Probab. 21 (2016), Paper No. 22, 41 pp.

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