ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO FOKKER–PLANCK–KOLMOGOROV EQUATIONS

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Abstract. We study the asymptotic behaviour at infinity of solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation. Our main result is an estimate for the difference of two solutions with different initial conditions in the weighted total variation norm in the case when the coefficients depend on time.

Keywords: Fokker–Planck–Kolmogorov equation, Cauchy problem, asymptotic behaviour.

1. Introduction

We study the behaviour at infinity of solutions $(\mu_t)_{t\ge s}$ to the Cauchy problem for the Fokker–Planck–Kolmogorov equation

$$\partial_t\mu_t = \partial_{x_i}\partial_{x_j}\bigl(a^{ij}(x,t)\mu_t\bigr) - \partial_{x_i}\bigl(b^i(x,t)\mu_t\bigr), \quad \mu_s=\nu,$$

where $s\in\mathbb{R}$, a^{ij} and b^i are Borel functions on $\mathbb{R}^{d+1}$, the matrix

$$A(x,t)=\bigl(a^{ij}(x,t)\bigr)_{1\le i,j\le d}$$

is symmetric and nonnegative definite, ν is a bounded Borel measure on $\mathbb{R}^d$. In addition, throughout we assume summation over repeated indices. Set

$$\begin{aligned} L_tu(x,t) &= \operatorname{trace}\bigl(A(x,t)D^2u(x)\bigr) + \langle b(x,t),\nabla u(x)\rangle \\ &= a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x) + b^i(x,t)\partial_{x_i}u(x). \end{aligned}$$

The Cauchy problem is shortly written in the form

$$\partial_t\mu_t = L_t^*\mu_t, \quad \mu_s=\nu. \tag{1.1}$$

A solution to the Cauchy problem (1.1) is a family $(\mu_t)_{t\ge s}$ of bounded Borel measures on $\mathbb{R}^d$ (possibly, signed) such that for every Borel set B the function $t\mapsto\mu_t(B)$ is Borel measurable, a^{ij}, $b^i\in L^1(U\times[s,T],\mu_t\,dt)$ for every $T>s$ and every ball U, and for every function $\varphi\in C_0^\infty(\mathbb{R}^d)$ for almost all $t\ge s$ the equality

$$\int_{\mathbb{R}^d}\varphi\,d\mu_t = \int_{\mathbb{R}^d}\varphi\,d\nu + \int_s^t\int_{\mathbb{R}^d}L_\tau\varphi\,d\mu_\tau\,d\tau$$

holds.

A solution $(\mu_t)_{t\ge s}$ is called a probability solution if for every $t\ge s$ the measure μ_t is nonnegative and $\mu_t(\mathbb{R}^d)=1$. A survey of modern theory of Fokker–Planck–Kolmogorov equations is given in [4] (see also [7]).

Throughout, unless the otherwise stated, we assume that the coefficients satisfy the following conditions:

$(\mathrm{H}_{\mathrm{A,b}})$ for every ball U, there exist positive numbers $\lambda(U)$, $\Lambda(U)$, and $M(U)$ such that for all $t\in\mathbb{R}$ we have

$$\lambda(U)I\le A(x,t)\le\lambda^{-1}(U)I, \quad |A(x,t)-A(y,t)|\le\Lambda(U)|x-y|,$$

$$|b(x,t)| \le M(U).$$

(H_V) there exists a function $V \in C^2(\mathbb{R}^d)$ along with positive numbers C_1 and C_2 such that $\lim\limits_{|x|\to\infty} V(x) = +\infty$ and

$$L_t V(x,t) \le C_1 - C_2 V(x) \quad \forall x \in \mathbb{R}^d, t \in \mathbb{R}.$$

Such a function V is called a Lyapunov function for the operator L_t.

Let us give a typical example, where conditions ($H_{A,b}$) and (H_V) are fulfilled. Let

$$L_t u(x,t) = \Delta u(x) + \langle b(x,t), \nabla u(x)\rangle,$$

where $\sup_{t\in\mathbb{R},x\in U} |b(x,t)| < \infty$ for every ball $U \subset \mathbb{R}^d$ and there exists a number $C > 0$ such that for all $x \in \mathbb{R}^d$, $t \in \mathbb{R}$ we have

$$\langle b(x,t), x\rangle \le C - C|x|^2.$$

Then condition (H_V) is fulfilled with $V(x) = |x|^2/2$ and $C_1 = d + C$, $C_2 = C$. Condition ($H_{A,b}$) is obviously fulfilled.

According to [4, Corollary 6.6.6], under conditions ($H_{A,b}$) and (H_V), for every $s \ge 0$ and every Borel probability measure ν on $\mathbb{R}^d$ there exists a unique probability solution $(\mu_t)_{t\ge s}$ to the Cauchy problem (1.1). The mapping $t \mapsto \mu_t$ is continuous in the weak topology on the space of probability measures. Recall (see [3]) that convergence in the weak topology is convergence of the integrals of each bounded continuous function. This topology on the space of probability measures is metrizable (for example, one can use the Kantorovich–Rubinshtein metric). In addition, for any $t > s$ the measure μ_t has a density $\varrho(x,t)$ with respect to Lebesgue measure that is continuous in (x,t) and strictly positive.

For every nonnegative Borel function W and a bounded Bore measure σ set

$$\|\sigma\|_W = \|W\sigma\|_{TV},$$

where $\|\mu\|_{TV}$ is the total variation of the measure μ, i.e., $\|\mu\|_{TV} = \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d)$, where $\mu = \mu^+ - \mu^-$ is the Jordan–Hahn decomposition.

The main result of this paper is as follows: under conditions ($H_{A,b}$) and (H_V), for any two Borel probability measures ν^1 and ν^2 on $\mathbb{R}^d$ the estimate

$$\|\mu_t^1 - \mu_t^2\|_{1+\beta V} \le N_1 e^{-N_2(t-s)} \|\nu^1 - \nu^2\|_{1+\beta V}$$

holds, where N_1, N_2, β are positive numbers, μ_t^1 and μ_t^2 are probability solutions to the Cauchy problem (1.1) with initial conditions ν^1 and ν^2, respectively. In addition, if the coefficients are periodic in t, then there exists a unique periodic solution, to which any probability solution tends as $t \to +\infty$. If the coefficients do not depend on t, then this periodic solution does not depend on t and is a unique stationary solution. Moreover, we discuss application of the obtained results to prove convergence to a constant for the solution to the parabolic equation $\partial_t u + Lu = 0$ as $t \to -\infty$.

A great number of papers is devoted to the study of solutions to parabolic equations as $t \to \infty$. A survey of the state-of-the-art in this area is given in [10]. One of pioneering works was the paper [16], where the asymptotic behaviour of solutions to parabolic equations with time dependent coefficients is studied, in particular, for solutions to Fokker–Planck–Kolmogorov equations. The paper [16] deals with classical solutions, the coefficients are assumed to be sufficiently smooth, and the following restriction on the growth of coefficients is imposed: $\|A(x,t)\| \le C + C|x|^2$ and $|b(x,t)| \le C + C|x|$. In this paper, we generalize some results from [16] to the case of unbounded and non-smooth coefficients. Since we establish exponential convergence in the total variation with a weight, we employ some stronger conditions with Lyapunov functions than in [16]. In the case of time periodic coefficients (i.e., periodic in t) the existence and uniqueness of a time periodic probability solution to the Fokker–Planck–Kolmogorov equation and also convergence to it of other probability solutions as $t \to +\infty$ are justified in [17]. Below we give a short proof of these results under more restrictive assumptions about the coefficients, but we obtain a stronger assertion about the mode of convergence to the periodic solution. Below we also recall one of the main results of [17]. Merging of solutions to Fokker–Planck–Kolmogorov equations as $t \to +\infty$ is established in [20] under the condition of monotonicity and dissipativity of the drift coefficient b. The latter condition implies the existence of a radial Lyapunov function V for the operator L_t. In [20], a new definition of a solution to the Fokker–Planck–Kolmogorov equation is employed and a new approach to the study of the behaviour of solutions as $t \to +\infty$ is suggested, based on the theory of viscosity solutions. In addition, the Ishii–Jensen lemma is substantially used along with the method of doubling variables. Our results presented below are analogues of some assertions from [20], but without the assumption of monotonicity of b and with a general Lyapunov function V. One of the main results of [20] is also formulated below.

When the coefficients are independent of time, the classical problem consists in constructing a stationary solution, which is an invariant measure for the corresponding diffusion process. Existence of an invariant measure and convergence to it of transition probabilities was first investigated by Hasminskii [19] and [15]. There are several approaches to justification of convergence of transition probabilities to invariant measures, among which we mention the following three ones: 1) application of Harris's theorem and the method of Lyapunov functions (see, for example, [19], [14], [11], and [18]), 2) application of entropy estimates and the Sobolev and Poincaré inequalities (see, for example, [1], and [9]), 3) a probabilistic approach based on the method of coupling (see, for example, [12] and [13]). The results presented below are based on Harris's theorem, i.e., belong to the first group. Finally, note that active research is conducted on convergence to stationary solutions for probability solutions to nonlinear Fokker–Planck–Kolmogorov equations (see [8], [5], [6], and [2]).

2. Main results

Recall that throughout we assume conditions $(\mathrm{H}_{\mathrm{A,b}})$ and $(\mathrm{H}_{\mathrm{V}})$. Let $s \in \mathbb{R}$. A probability solution to the Cauchy problem (1.1) with initial condition $\nu = \delta_y$ at $t = s$ possesses a positive density $\varrho(s, y, t, x)$ that is continuous in (t, x) and jointly Borel measurable (which can be derived from its construction as a limit of classical solutions to initial boundary value problems, see [4, Section 6.6]).

Proposition 2.1. *Under the stated assumptions the Chapman–Kolmogorov equations hold:*

$$\varrho(s, y, t, x) = \int_{\mathbb{R}^d} \varrho(v, z, t, x)\varrho(s, y, v, z)\, dz \quad \textit{for all } t \geq s, x, y \in \mathbb{R}^d.$$

Proof. Observe that the right-hand side is a probability density in x and the corresponding family of probability measures satisfies the Fokker–Planck–Kolmogorov equation. Indeed, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have the equality

$$\int_{\mathbb{R}^d} \varphi(x)\biggl(\int_{\mathbb{R}^d} \varrho(v,z,t,x)\varrho(s,y,v,z)\,dz\biggr)\,dx$$
$$= \varphi(v) + \int_s^t \int_{\mathbb{R}^d} L_t\varphi(x,t)\biggl(\int_{\mathbb{R}^d} \varrho(v,z,t,x)\varrho(s,y,v,z)\,dz\biggr)\,dx\,d\tau.$$

Now our assertion follows from the uniqueness of a probability solution to the Cauchy problem 1.1 with initial condition $\nu = \delta_y$ at $t = s$. □

Proposition 2.2. *For every bounded Borel measure ν on $\mathbb{R}^d$ (possibly, signed) the formula*

$$\mu_t(B) = \int_B \int_{\mathbb{R}^d} \varrho(s,y,t,x)\,\nu(dy)\,dx$$

defines a family of bounded Borel measures $(\mu_t)_{t\ge s}$ that is a solution to the Cauchy problem (1.1) *with initial condition ν at $t = s$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \varphi\,d\mu_t - \int_{\mathbb{R}^d} \varphi\,d\sigma = \int_{\mathbb{R}^d} \biggl(\int_{\mathbb{R}^d} \varphi(x)\varrho(s,y,t,x)\,dx - \varphi(y)\biggr)\,\sigma(dy)$$
$$= \int_{\mathbb{R}^d} \biggl(\int_s^t \int_{\mathbb{R}^d} L_\tau\varphi(x,\tau)\varrho(s,y,\tau,x)\,dx\,d\tau\biggr)\,\sigma(dy) = \int_s^t \int_{\mathbb{R}^d} L_\tau\varphi(x,\tau)\,\mu_\tau(dx)\,d\tau,$$

which completes the proof. □

For every $s \in \mathbb{R}$ we consider the mapping Q_s associating to a bounded Borel measure σ on $\mathbb{R}^d$ the measure $Q_s\sigma$ defined by the formula

$$Q_s\sigma(B) = \int_B \biggl(\int_{\mathbb{R}^d} \varrho(s,y,s+1,x)\,\sigma(dy)\biggr)\,dx.$$

Note that Q_s depends on s, since the coefficients of the operator depend on time.

Recall Harris's ergodic theorem (see, for example, [14]). Suppose that on a measurable space $(X,\mathcal{B})$ we are given a Markov transition kernel $P(\,\cdot\,,\,\cdot\,)$, i.e., for every $x \in X$ the function $B \mapsto P(x,B)$ is a probability measure on $\mathcal{B}$ and for every $B \in \mathcal{B}$ the function $x \mapsto P(x,B)$ is measurable with respect to the σ-algebra $\mathcal{B}$. Set

$$\mathcal{P}f(y) = \int_X f(x)\,P(y,\,dx), \quad \mathcal{P}\sigma(B) = \int_X P(y,B)\,\sigma(dy).$$

Suppose that

(i) there exists a measurable function $\Phi\colon X \to [0,+\infty)$ along with numbers $\delta \in (0,1)$ and $K > 0$ such that

$$\mathcal{P}\Phi(y) \le \delta\Phi(y) + K \quad \forall y \in X;$$

(ii) there exists a probability measure σ along with a number $q \in (0,1)$ such that

$$\inf_{y\colon\ \Phi(y)\le R} \mathcal{P}(y,\,\cdot\,) \ge q\sigma(\,\cdot\,)$$

for some $R > 2K/(1-\delta)$.

Then according to [14, Theorem 3.1] there exist two numbers $\beta_0 \in (0,1)$ and $\beta > 1$ such that

$$\|\mathcal{P}\sigma_1 - \mathcal{P}\sigma_2\|_{1+\beta\Phi} \le \beta_0\|\sigma_1 - \sigma_2\|_{1+\beta\Phi},$$

moreover, β_0 and β can be explicitly expressed through δ, K, R and q in the following way:

$$\beta = \frac{q_0}{K}, \quad \beta_0 = \frac{\max\{1-(q-q_0), 2+R\beta r_0\}}{2+R\beta},$$

where q_0 is an arbitrary number from the interval $(0,q)$ and r_0 is an arbitrary number from the interval $(\delta+2K/R, 1)$.

The next result plays the key role in our subsequent reasoning.

Theorem 2.3. *There exist two numbers $0<\beta_0<1$ and $\beta>0$ such that for any two Borel probability measures σ_1 and σ_2 on $\mathbb{R}^d$ the estimate*

$$\|Q_s\sigma_1 - Q_s\sigma_2\|_{1+\beta V} \le \beta_0\|\sigma_1-\sigma_2\|_{1+\beta V}$$

holds, moreover, β_0 and β depend only on the dimension d, the constants C_1 and C_2 from condition $(\mathrm{H_V})$ *and the constants $\lambda(U)$, $\Lambda(U)$ and $M(U)$ from condition* $(\mathrm{H_{A,b}})$*, but do not depend on s.*

Proof. Let us verify conditions (i) and (ii) from Harris's ergodic theorem for

$$P(y,\,dx) = \varrho(s,y,s+1,x)\,dx, \quad \Phi(x) = V(x).$$

Let us verify condition (i). Since $L_tV(x,t)\le C_1 - C_2V(x)$, we have

$$\partial_t(e^{-C_2(t-s)}V) + L_t(e^{-C_2(t-s)}V) \le C_1e^{C_2(t-s)}$$

and according to [4, Theorem 7.1.1] the estimate

$$\int_{\mathbb{R}^d} V(x)\varrho(s,y,s+1,x)\,dx \le e^{-C_2}V(y) + \frac{C_1}{C_2}\big(1-e^{-C_2}\big)$$

holds. Setting

$$\delta = e^{-C_2}, \quad K = \frac{C_1}{C_2}\big(1-e^{-C_2}\big),$$

we rewrite this estimate in the form

$$\int_{\mathbb{R}^d} V(x)\varrho(s,y,s+1,x)\,dx \le \delta V(x) + K.$$

For the verification of condition (ii) we apply Harnack's inequality from [4, Theorem 8.2.1]. Let α, η and γ be positive continuous functions on $[0,+\infty)$ such that

$$\sup_{|x|\le 2r}\Big|b^i(x,t) - \sum_j \partial_{x_j}a^{ij}(x,t)\Big| \le \eta(r),$$

$$\sup_{|x|\le 2r}\|A(x,t)^{-1}\| \le \alpha(r), \quad \sup_{|x|\le 2r}\|A(x,t)\| \le \gamma(r).$$

Then there exists a positive number $K(d)$, depending only on the dimension d, such that for every $\tau\in(0,1)$ we have

$$\varrho(s,y,s+1,x) \ge \varrho(s,y,s+\tau,0)\exp\Big(-K(d)\big|1+\alpha(|x|) + \big(\alpha(|x|)^{1/2}+\alpha(|x|)\big)\gamma(|x|)\big|^2\Big(1+\frac{1-\tau}{s+\tau}\eta(|x|)^2 + \frac{1}{\tau}|x|^2\Big)\Big).$$

Observe that $\frac{1-\tau}{s+\tau}\le\frac{1-\tau}{\tau}$. Set

$$W(x,\tau) = K(d)\big|1+\alpha(|x|) + \big(\alpha(|x|)^{1/2}+\alpha(|x|)\big)\gamma(|x|)\big|^2\Big(1+\frac{1-\tau}{\tau}V(|x|)^2 + \frac{1}{\tau}|x|^2\Big).$$

Then

$$\varrho(s,y,s+1,x) \geq \varrho(s,y,s+\tau,0)e^{-K(d)W(x,\tau)}.$$

Let us fix $R > 2K/(1-\delta)$ and estimate the function $\varrho(s,y,s+\tau,0)$ from below. Set $U(0,r) = \{x\colon\ |x| < r\}$. Let us fix $R' > 0$ such that $\{x\colon\ V(x) \leq R\} \subset U(0,R')$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $x \in U(0,R')$ and $\psi(x) = 0$ if $x \notin U(0,3R')$. We have

$$\int_{\mathbb{R}^d} \psi(x)\varrho(s,y,s+\tau/2,x)\,dx = \psi(y) + \int_s^{s+\tau/2}\int_{\mathbb{R}^d} L_v\psi(x,v)\varrho(s,y,v,x)\,dx\,dv.$$

Therefore, for any $y \in U(0,R')$ the following estimate holds:

$$\sup_{x\in U(0,3R')} \varrho(s,y,s+\tau/2,x) \geq 1 - \frac{\tau}{2}\sup_{x\in U(0,3R'),t\geq 0}|L\psi(x,t)|.$$

There is $\tau \in (0,1)$, depending only on $\lambda(U(0,3R'))$, $M(U(0,3R'))$ and ψ, such that

$$\sup_{x\in U(0,3R')} \varrho(s,y,s+\tau/2,x) \geq \frac{1}{2}.$$

According to [4, Theorem 8.1.3], there exists a number C_R, depending only on d, $\lambda(U(0,4R'))$, $\Lambda(U(0,4R'))$, $M(U(0,4R'))$ and τ, such that for all $y \in U(0,R')$ we have

$$\varrho(s,y,s+\tau,0) \geq C_R \sup_{x\in U(0,3R')} \varrho(s,y,s+\tau/2,x) \geq \frac{C_R}{2}.$$

Hence we obtain the inequality

$$\inf_{y\colon\ V(y)\leq R} \varrho(s,y,s+1,x) \geq \frac{C_R}{2}e^{-K(d)W(x,\tau)},$$

which yields the estimate

$$\inf_{y\colon\ V(y)\leq R} P(y,dx) \geq q\sigma(dx),$$

where

$$\sigma(dx) = C^{-1}e^{-K(d)W(x,\tau)}\,dx, \quad C = \int e^{-K(d)W(x,\tau)}\,dx, \quad q = \frac{C_R C}{2}.$$

Thus, all conditions in Harris's theorem are fulfilled, moreover, the constants q, δ, K do not depend on s. □

Corollary 2.4. *There exist positive numbers N_1, N_2 and β such that for every s and for any two probability solutions μ_t^1 and μ_t^2 to the Cauchy problem* (1.1) *with initial conditions ν^1 and ν^2 at $t = s$ the estimate*

$$\|\mu_t^1 - \mu_t^2\|_{1+\beta V} \leq N_1 e^{-N_2(t-s)}\|\nu^1 - \nu^2\|_{1+\beta V}$$

holds.

Proof. Let $t - s = m + \eta$, where $m \in \mathbb{N}\cup\{0\}$ and $\eta \in (0,1)$. Then due to the Chapman–Kolmogorov equations (see Proposition 2.1) the equalities

$$\mu_t^1 = Q_{m-1}Q_{m-2}\ldots Q_0\mu_{s+\eta}^1, \quad \mu_t^2 = Q_{m-1}Q_{m-2}\ldots Q_0\mu_{s+\eta}^2$$

are valid. According to Theorem 2.3 we have

$$\|\mu_t^1 - \mu_t^2\|_{1+\beta V} \leq \beta_0^m\|\mu_{s+\eta}^1 - \mu_{s+\eta}^2\|_{1+\beta V}, \tag{2.1}$$

where $\beta_0 \in (0,1)$ and $\beta > 0$ are the constants from Theorem 2.3. Observe that for every function $\psi \in C_0^\infty(\mathbb{R}^d)$ satisfying the inequality $|\psi| \le 1 + \beta V$ we have the estimate

$$\int_{\mathbb{R}^d} \psi \, d(\mu^1_{s+\eta} - \mu^2_{s+\eta}) \le \int_{\mathbb{R}^d} \biggl(\int_{\mathbb{R}^d} \bigl(1 + \beta V(x)\bigr) \varrho(s, y, s+\eta, x)\, dx \biggr) |\nu^1 - \nu^2|(dy).$$

The assumption $L_t V(x,t) \le C_1 - C_2 V(x)$ yields the inequality

$$\partial_t(e^{-C_2(t-s)}V) + L(e^{-C_2(t-s)}V) \le C_1 e^{C_2(t-s)},$$

which according to [4, Theorem 7.1.1] ensures the inequality

$$\int_{\mathbb{R}^d} V(x) \varrho(s, y, s+\eta, x)\, dx \le e^{-C_2\eta} V(y) + \frac{C_1}{C_2}\bigl(1 - e^{-C_2\eta}\bigr).$$

Taking into account that $\eta \in (0,1)$, we arrive at the inequality

$$\|\mu^1_{s+\eta} - \mu^2_{s+\eta}\|_{1+\beta V} \le C_3 \|\nu^1 - \nu^2\|_{1+\beta V}, \tag{2.2}$$

where C_3 depends on C_1, C_2 and β. Combining (2.1) and (2.2), we obtain our assertion. $\square$

It is shown in Proposition 4.1 below that there exists a family of probability measures $(\pi_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^d$ such that the mapping $t \mapsto \pi_t$ is continuous and satisfies the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = L_t^* \mu_t$$

on $\mathbb{R}^{d+1}$, i.e., whenever $t_1 < t_2$, for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi \, d\pi_{t_2} - \int \varphi \, d\pi_{t_1} = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} L_t \varphi \, d\pi_t \, dt.$$

Using $(\pi_t)_{t\in\mathbb{R}}$, Corollary 2.4 can be reformulated as follows. For every $s \in \mathbb{R}$ and every probability measure ν on $\mathbb{R}^d$, the probability solution μ_t to the Cauchy (1.1) problem with initial condition ν at $t = s$ tends to π_t as $t \to +\infty$ and

$$\|\mu_t - \pi_t\|_{1+\beta V} \le N_1 e^{-N_1(t-s)} \|\pi_s - \nu\|_{1+\beta V}.$$

Note that the probability solution π_t does not depend on s and ν. In this form Corollary 2.4 generalizes [16, Theorem 5] to the case of non-smooth and unbounded coefficients.

3. Applications to periodic solutions and non probability solutions

It should be noted that in the assertions above the equation's coefficients depend on time in an arbitrary way. We now consider the case of periodic coefficients.

Let $\mathcal{P}_V(\mathbb{R}^d)$ denote the set of all Borel probability measures μ on $\mathbb{R}^d$ satisfying the condition $V \in L^1(\mu)$.

Corollary 3.1. *Suppose that in addition to conditions* $(\mathrm{H}_{\mathrm{A,b}})$ *and* $(\mathrm{H}_{\mathrm{V}})$ *the coefficients A and b are periodic in t with a period $T > 0$, i.e., for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^d$ we have*

$$A(x, t+T) = A(x,t), \quad b(x, t+T) = b(x,t).$$

Then there exists a unique family of probability measures $(\pi_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^d$ such that the equality $\pi_{t+T} = \pi_t$ holds for all $t \in \mathbb{R}$ and $(\pi_t)_{t\in\mathbb{R}}$ satisfies the Fokker–Planck–Kolmogorov equation on $\mathbb{R}^{d+1}$. Moreover, for every s and for every probability solution μ_t to the Cauchy problem (1.1) *with initial condition ν at $t = s$ one has*

$$\|\mu_t - \pi_t\|_{1+\beta V} \le N_1 e^{-N_2(t-s)} \|\nu - \pi_s\|_{1+\beta V},$$

where N_1 and N_2 are the constants from Corollary 2.4. *If the coefficients do not depend on t, then $\pi_t = \pi_0$ for all t and π_0 is a stationary solution.*

Proof. Let $T = 1$. The general case reduces to this by scaling. The space $\mathcal{P}_V(\mathbb{R}^d)$ with the metric $\|\mu^1 - \mu^2\|_{1+\beta V}$ is complete. According to Theorem 2.3 the mapping $\mu \mapsto Q_0\mu$ is contracting with the coefficient β_0. By the Banach contracting mapping theorem there exists a unique fixed point π_0 for which $\pi_0 = Q_0\pi_0$. Let π_t be a probability solution to the Cauchy problem (1.1) with initial condition $\nu = \pi_0$ at $t = 0$. Then $\pi_1 = \pi_0$. By the uniqueness of a probability solution and periodicity of the coefficients the equality $\pi_{t+1} = \pi_t$ holds for all $t \ge 0$. Defining π_t for $t < 0$ by the equality $\pi_t = \pi_{-t}$, we obtain the desired periodic solution. Convergence of every probability solution to π_t as $t \to +\infty$ and the estimate for the rate of convergence follow from Corollary 2.4.

If the coefficients do not depend on t, then they are periodic with any period $T > 0$. In addition, the mapping $t \mapsto \pi_t$ is continuous in the weak topology. Therefore, $\pi_t = \pi_0$ for all $t \ge 0$. $\square$

Note that in [17] an analogous assertion about existence and uniqueness of a periodic probability solution is obtained under weaker assumptions about the coefficients and a weaker condition on the Lyapunov function. However, our justification of the existence of a periodic solution is shorter and convergence to it is established in a stronger mode, namely, in place of the total variation we use the total variation with a growing weight.

Let us also mention a result on the existence of a periodic solution from [17], which we formulate in the case of $\mathbb{R}^d$. In [17], an arbitrary open subset of $\mathbb{R}^d$ is considered and a more general time-dependent Lyapunov function is used.

Theorem 3.2. *Let $p > d+2$. Suppose that the coefficients are periodic with a period $T > 0$ and for every ball $U \subset \mathbb{R}^d$ we have*

$$a^{ij} \in L^\infty(\mathbb{R}, W^{1,p}(U)), \quad b^i \in L^p(U \times \mathbb{R}),$$
$$\lambda(U)^{-1} I \le A(x,t) \le \lambda(U) I, \quad \forall x \in U, t \in \mathbb{R}.$$

Suppose also that there is a nonnegative function $V \in C^2(\mathbb{R}^d)$ such that $\lim\limits_{|x|\to+\infty} V(x) = +\infty$ and for some $R > 0$ and $C > 0$ one has

$$L_t V(x,t) \le -C \quad \text{if} \quad |x| > R, t \in \mathbb{R}.$$

Then there exists a unique time periodic probability solution to the Fokker–Planck–Kolmogorov equation $\partial_t \mu_t = L_t^* \mu_t$.

Recall that it is proved in Proposition 2.2 above that the formula

$$\mu_t(B) = \int_B \left(\int_{\mathbb{R}^d} \varrho(0, y, t, x)\, \nu(dy) \right) dx \tag{3.1}$$

defines a solution (possibly, signed) to the Cauchy (1.1) problem with initial condition ν at $t = 0$.

Corollary 3.3. *Suppose that conditions* $(\mathrm{H}_{\mathrm{A,b}})$ *and* $(\mathrm{H}_{\mathrm{V}})$ *are fulfilled and the family of bounded Borel measures* $(\mu_t)_{t\geq 0}$ *is defined by formula* (3.1)*, where* ν *is a bounded Borel measure on* $\mathbb{R}^d$ *and* $\nu(\mathbb{R}^d)=0$*. Then*

$$\|\mu_t\|_{1+\beta V}\leq N_1 e^{-tN_2}\|\nu\|_{1+\beta V}.$$

Proof. First we consider the case where $\nu\neq 0$. Let $\nu=\nu^+-\nu^-$ be the Jordan–Hahn decomposition, where ν^+ and ν^- are nonnegative bounded measures. Then

$$\nu^+(\mathbb{R}^d)=\nu^-(\mathbb{R}^d)>0.$$

After normalization we can assume that ν^+ and ν^{-1} are probability measures (this does not affect the constants in our estimate, because it is homogeneous). According to Corollary 2.4 we have

$$\|\mu_t^+-\mu_t^-\|_{1+\beta V}\leq N_1 e^{-N_2 t}\|\nu^+-\nu^{-1}\|_{1+\beta V},$$

where

$$\mu_t^+(B)=\int_B\biggl(\int_{\mathbb{R}^d}\varrho(0,y,t,x)\,\nu^+(dy)\biggr)\,dx,\ \ \mu_t^-(B)=\int_B\biggl(\int_{\mathbb{R}^d}\varrho(0,y,t,x)\,\nu^-(dy)\biggr)\,dx.$$

It remains to use the equality $\mu_t=\mu_t^+-\mu_t^-$. □

Note that for an arbitrary solution to the Cauchy problem (1.1) given by a family of bounded Borel measures μ_t this assertion is false. Let us consider the following example (see [4, Problem 9.8.47]). Let $d=1$, $A=1$ and

$$b(x)=-\frac{2x}{1+x^2}-(1+x^2)\mathrm{arctg}\,x,\quad \nu(dx)=\frac{1}{\pi(1+x^2)}\,dx.$$

Let $V(x)=1+x^2$. Observe that

$$L_tV(x,t)=2+2b(x)x=2-\frac{4x^2}{1+x^2}-2x(1+x^2)\mathrm{arctg}\,x.$$

Since $2x\mathrm{arctg}\,x\geq\pi/2$ if $|x|>1$, we have $L_tV(x,t)\leq C_1-C_2V(x)$ for some positive numbers C_1 and C_2. Therefore, in this situation Corollary 2.4 is applicable. However, the Cauchy problem (1.1) with the same initial condition has also the solution

$$\mu_t(dx)=\frac{e^t}{\pi(1+x^2)}\,dx,$$

which is not a probability solution and which does not satisfy the estimate from Corollary 2.4 (valid for probability solutions).

In [20], the assertion of Corollary 3.3 is obtained under other conditions. Let us formulate the corresponding result.

Suppose that $A=I$, b is continuous and there exist positive numbers α, γ, R and c_0 such that

$$\langle b(x,t),x\rangle\leq-\alpha|x|^\gamma\quad\text{ if }\quad |x|>R,\, t\geq 0,$$

$$\langle b(x,t)-b(y,t),x-y\rangle\leq c_0|x-y|\quad\forall\, x,y\in\mathbb{R}^d, t\geq 0.$$

The next result is a particular case of [20, Theorem 4.1].

Theorem 3.4. *Let* $k>0$*,* $W(x)=(1+|x|^2)^{k/2}$*, and let* ν *be a bounded measure given by a density* ϱ *with respect to Lebesgue measure such that* $W|\varrho|\in L^1(\mathbb{R}^d)$ *and* $\nu(\mathbb{R}^d)=0$*. Assume that the two conditions stated above hold. Let* μ_t *be a solution* *to the Cauchy problem* (1.1) *with initial condition* ν *given by formula* (3.1)*. Then, whenever* $\gamma \geq 2$*, we have*

$$\|\mu_t\|_W \leq M_1 e^{-M_1 t}\|\nu\|_W,$$

and if $\gamma \in (0,2)$ *and* $r > k$ *is such that* $W^{r/k}|\varrho| \in L^1(\mathbb{R}^d)$ *we have*

$$\|\mu_t\|_W \leq M_1(1+t)^{-q}\|\nu\|_{W^{r/k}}, \quad q = \frac{r-k}{2-\gamma},$$

where the numbers M_1 *and* M_2 *depend only on* α*,* γ*,* R*,* c_0*,* k*,* r*, and* d*.*

The condition

$$\langle b(x,t), x\rangle \leq -\alpha|x|^\gamma \quad \text{if} \quad |x| > R,\, t \geq 0,$$

implies the estimate $L_t(|x|^2) \leq 2d - 2\alpha|x|^\gamma$. If $\gamma \geq 2$, then for suitable positive numbers C_1 and C_2 we have $L_t(|x|^2) \leq C_1 - C_2|x|^2$. In Theorem 3.4 we also assume the monotonicity condition

$$\langle b(x,t) - b(y,t), x - y\rangle \leq c_0|x-y|,$$

and in Corollary 2.4 this condition is absent, but it is assumed instead that for every ball U one has $\sup_{x\in U, t\geq 0}|b(x,t)| < \infty$. If $\gamma \in (0,2)$, then the inequality $L_t(|x|^2) \leq C_1 - C_2|x|^2$ is not fulfilled, so Corollary 2.4 does not enable us to estimate $\|\mu_t\|_{1+|x|^2}$. In [20], another definition of a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation is used. Namely, a continuous mapping $t \mapsto m_t$ on $[0,T)$ with values in the space of bounded Borel measures on $\mathbb{R}^d$ equipped with the weak topology is called a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation with initial condition m_0, which is a bounded Borel measure on $\mathbb{R}^d$, provided that

$$\int_{\mathbb{R}^d} \xi\, dm_t + \int_0^t \int_{\mathbb{R}^d} f\, dm_s\, ds = \int_{\mathbb{R}^d} \varphi(x,0)\, m_0(dx)$$

for all $t \in (0,T)$, $\xi \in C_b(\mathbb{R}^d)$, $f \in C_b(\mathbb{R}^d \times [0,T])$ and $\varphi \in C_b(\mathbb{R}^d \times [0,T])$ such that φ is a viscosity solution to the equation $-\partial_t\varphi + L_{I,b}\varphi = f$ in $\mathbb{R}^d \times (0,t)$ with $\varphi(x,t) = \xi(x)$. However, according to [20, Theorem 2.5], one has the uniqueness of a solution and the property that the mapping $\nu \mapsto \mu_t$ preserves the nonnegativity and the total mass of $\mathbb{R}^d$. Since a solution in the sense of [20] is a solution in our sense and a probability solution is unique under the considered conditions, the solution in the sense of [20] is given by formula (3.1). Note also that in [20] the Fokker–Planck–Kolmogorov equation with a nonconstant diffusion matrix and a nonlocal Lévy operator is considered.

4. Stabilization of solutions to parabolic equations

In this section we discuss connections between Corollary 2.4 and stabilization as $t \to -\infty$ of the solution u to the Cauchy problem

$$\partial_t u + L_t u = 0, \quad u(x,0) = \psi(x). \tag{4.1}$$

Here $\psi \in C(\mathbb{R}^d)$ and a solution is a function $u \in C(\mathbb{R}^d \times (-\infty, 0])$ with Sobolev derivatives $\partial_t u$, $\partial_{x_i} u$, $\partial_{x_i}\partial_{x_j} u$ belonging to $L^p_{loc}(\mathbb{R}^d \times (-\infty,0])$, where $p > 1$, and the equation is understood in the sense of equality almost everywhere.

We need an auxiliary assertion of independent interest. Recall that we assume conditions $(\mathrm{H_{A,b}})$ and $(\mathrm{H_V})$.

Proposition 4.1. *There exists a family of probability measures $(\pi_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^d$ such that the mapping $t \mapsto \pi_t$ is continuous in the weak topology, the measure π_t has a density $\varrho(x,t)$ with respect to Lebesgue measure that is jointly continuous in (x,t), and $(\pi_t)_{t\in\mathbb{R}}$ is a solution to the Fokker–Planck–Kolmogorov equation on $\mathbb{R}^{d+1}$. Moreover,*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^d} V(x)\,\pi_t(dx) \le V(0) + \frac{C_1}{C_2},$$

where C_1 and C_2 are the constants from condition (H$_\mathrm{V}$).

Proof. Let $(\pi_t^n)_{t\ge -n}$ be the probability solution to the Cauchy problem (1.1) with initial condition $\mu_{-n} = \delta_0$, i.e., $\pi_t^n(dx) = \varrho(-n,0,t,x)\,dx$. Then

$$\int_{\mathbb{R}^d} V(x)\varrho(-n,0,t,x)\,dx \le e^{-C_2(t+n)}V(0) + \frac{C_1}{C_2}\bigl(1 - e^{-C_2(t+n)}\bigr) \le V(0) + \frac{C_1}{C_2}.$$

According to [4, Corollary 6.4.3], for every ball $U \subset \mathbb{R}^d$ and every bounded interval $J \subset \mathbb{R}$ we can find a number $C(U,J) > 0$ and a number n_0 such that for all $n > n_0$ the Hölder norm of the density $\varrho(-n,0,x,t)$ is bounded by $C(U,J)$. Therefore, there exists a subsequence $\varrho(-n_k,0,x,t)$ that converges locally uniformly on $\mathbb{R}^{d+1}$ to a nonnegative continuous function $\varrho(x,t)$). Since

$$\int_{\mathbb{R}^d} V(x)\varrho(-n_k,0,t,x)\,dx \le V(0) + \frac{C_1}{C_2},$$

for every t the function $x \mapsto \varrho(x,t)$ is a probability density and

$$\int_{\mathbb{R}^d} V(x)\varrho(x,t)\,dx \le V(0) + \frac{C_1}{C_2}.$$

Whenever $-n_k < t_1 < t_2$, for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\varrho(-n_k,0,t_2,x)\,dx &- \int_{\mathbb{R}^d} \varphi(x)\varrho(-n_k,0,t_1,x)\,dx \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d} L_t\varphi(x,t)\varrho(-n_k,0,t,x)\,dx\,dt.\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\int_{\mathbb{R}^d} \varphi(x)\varrho(x,t)\,dx - \int_{\mathbb{R}^d} \varphi(x)\varrho(x,t)\,dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^d} L_t\varphi(x,t)\varrho(x,t)\,dx\,dt.$$

Thus, $\pi_t = \varrho(x,t)\,dx$ is the desired probability solution. □

Proposition 4.2. *Let $\pi_t = \varrho(x,t)\,dx$ be the probability solution on $\mathbb{R}^{d+1}$ constructed in Proposition* 4.1*. For all s, y and $t > s$ we have*

$$\int_{\mathbb{R}^d} |\varrho(s,y,t,x) - \varrho(x,t)|(1+\beta V(x))\,dx \le N_1 e^{-N_2(t-s)}\Bigl(2 + \beta(V(0)+V(y)) + \frac{\beta C_1}{C_2}\Bigr),$$

where β, N_1 and N_2 are the constants from Corollary 2.4 *and C_1 and C_2 are the constants from condition* (H$_\mathrm{V}$)*. Hence, as $s \to -\infty$, the density $\varrho(s,y,t,\,\cdot\,)$ converges to $\varrho(\,\cdot\,,t)$ in the weighted L^1 space, and the probability solution constructed in Proposition* 4.1 *is unique.*

Proof. Since $(\pi_t)_{t\geq s}$ is a probability solution to the Cauchy problem (1.1) with initial condition $\nu=\pi_s$ at $t=s$, by Corollary 2.4 we have the estimate

$$\int_{\mathbb{R}^d} |\varrho(s,y,t,x)-\varrho(x,t)|(1+\beta V(x))\,dx \le N_1 e^{-N_2(t-s)}\|\delta_y-\pi_s\|_{1+\beta V}.$$

It remains to observe that

$$\|\delta_y-\pi_s\|_{1+\beta V}\le 2+\beta(V(0)+V(y))+\frac{\beta C_1}{C_2},$$

which completes the proof. □

Our next theorem generalizes [16, Theorem 3'] and is an analogue of [20, Theorem 3.5].

Theorem 4.3. *Suppose that u is a solution to the Cauchy problem* (4.1) *such that for every $T>0$ the functions*

$$|a^{ij}||u_{x_ix_j}|,\ (1+|x|)^{-1}|a^{ij}||u_{x_i}|,\ (1+|x|)^{-2}|a^{ij}||u|,\ |b^i||u_{x_i}|,\quad (1+|x|)^{-1}|b^i||u|,\ |u|$$

are majorized on $\mathbb{R}^d\times[-T,0]$ from above by a function of the form C_T+C_TV, where C_T is a positive number. Then the equality

$$u(y,s)=\int_{\mathbb{R}^d}\psi(x)\varrho(s,y,0,x)\,dx$$

holds. In addition, for all y,s the estimate

$$\left|u(y,s)-\int_{\mathbb{R}^d}\psi(x)\pi_0(x)\,dx\right|\le N_1'(1+V(y))e^{N_2 s}$$

holds, where π_t is the probability solution constructed in Proposition 4.1*, the constant N_1' is expressed through N_1, β, C_T, $V(0)$, C_1, C_2, the numbers N_1 and N_2 are the constants from Corollary* 2.4*, and the numbers C_1 and C_2 are the constants from condition* ($\mathrm{H_V}$)*. In particular, for all y the equality*

$$\lim_{s\to-\infty}u(y,s)=\int_{\mathbb{R}^d}\psi(x)\pi_0(x)\,dx$$

holds.

Proof. It suffices to justify the equality

$$u(y,s)=\int_{\mathbb{R}^d}\psi(x)\varrho(s,y,0,x)\,dx,$$

because the remaining assertions will then follow from Proposition 4.2.

Let $\zeta\in C_0^\infty(\mathbb{R}^d)$, $0\le\zeta\le1$, $\zeta(x)=1$ if $|x|<1$ and $\zeta(x)=0$ if $|x|>2$. Set $\zeta_N(x)=\zeta(x/N)$. For $s<0$ we have

$$\int_{\mathbb{R}^d}\zeta_N(x)u(x,0)\varrho(s,y,0,x)\,dx-\zeta_N(y)u(y,s)$$
$$=\int_s^0\int_{\mathbb{R}^d}\Big(\zeta_N(x)\partial_t u(x,t)+L_t\big(\zeta_N u\big)(x,t)\Big)\varrho(s,y,t,x)\,dx\,dt.$$

Due to the equality

$$L_t\big(\zeta_N u\big)(x,t)=\zeta_N(x)L_tu(x,t)+2\langle A(x,t)\nabla\zeta_N(x),\nabla u(x,t)\rangle+u(x,t)L_t\zeta_N(x,t)$$

we have

$$\int_{\mathbb{R}^d} \zeta_N(x)u(x,0)\varrho(s,y,0,x)\,dx - \zeta_N(y)u(y,s)$$
$$= \int_s^0 \int_{\mathbb{R}^d} \Big(2\langle A(x,t)\nabla\zeta_N(x), \nabla u(x,t)\rangle + u(x,t)L_t\zeta_N(x,t)\Big)\varrho(s,y,t,x)\Big)\,dx\,dt.$$

Since

$$\nabla\zeta_N(x) = N^{-1}I_{N<|x|<2N}\nabla\zeta(x/N), \quad D^2\zeta_N(x) = N^{-2}I_{N<|x|<2N}D^2\zeta(x/N),$$

taking into account our conditions on u and the estimate

$$\int_{\mathbb{R}^d} V(x)\varrho(s,y,t,x)\,dx \le e^{-C_2(t-s)}V(y) + \frac{C_1}{C_2}\big(1 - e^{-C_2(t-s)}\big)$$

we obtain the equality

$$\lim_{N\to+\infty}\int_s^0 \int_{\mathbb{R}^d} \Big(2\langle A(x,t)\nabla\zeta_N(x), \nabla u(x,t)\rangle + u(x,t)L_t\zeta_N(x,t)\Big)\varrho(s,y,t,x)\Big)\,dx\,dt = 0.$$

Using that $\zeta_N(x) \to 1$ as $M \to \infty$ and letting $N \to \infty$, we obtain

$$\int_{\mathbb{R}^d} u(x,0)\varrho(s,y,0,x)\,dx - u(y,s) = 0,$$

which completes the proof. □

It would be interesting to obtain analogues of Corollaries 2.4 and 3.1 and also Propositions 4.1 and 4.2 for solutions to nonlinear Fokker–Planck–Kolmogorov equations.

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References

[1] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differ. Equ. 26:1-2 (2001), 43–100.

[2] V. Barbu, M. Röckner, The evolution to equilibrium of solutions to nonlinear Fokker–Planck equation, Indiana Univ. Math. J. 72:1 (2023), 89–131.

[3] V.I. Bogachev, Weak convergence of measures, Amer. Math. Soc., Rhode Island, Providence, 2018.

[4] V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov, Fokker–Planck–Kolmogorov equations, Amer. Math. Soc., Providence, Rhode Island, 2015.

[5] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, Convergence in variation of solutions of nonlinear Fokker–Planck–Kolmogorov equations to stationary measures, J. Funct. Anal. 276:12 (2019), 3681–3713.

[6] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, On convergence to stationary distributions for solutions to nonlinear Fokker–Planck–Kolmogorov equations, J. Math. Sci. 242:1 (2019), 69–84.

[7] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, Kolmogorov problems on equations for stationary and transition probabilities of diffusion processes, Teor. Veroyatn. Primen. 68 (2023), 420–455 (in Russian); English transl.: Theory Probab. Appl. 68:3 (2023), 342–369.

[8] V.I. Bogachev, S.V. Shaposhnikov, Nonlinear Fokker–Planck–Kolmogorov equations, Uspehi Mat. Nauk 79:5 (2024), 3–60 (in Russian); English transl.: Russian Math. Surveys 79:5 (2024).

[9] P. Cattiaux, Long time behavior of Markov processes, ESAIM Proc. 44 (2014), 110–128.

[10] V.N. Denisov, On the behaviour of solutions of parabolic equations for large values of time, Russian Math. Surveys 60:4 (2005), 721–790.

[11] D. Down, S.P. Meyn, R.L. Tweedie, Exponential and uniform ergodicity of Markov processes, Ann. Probab. 23 (1996), 1671–1691.

[12] A. Eberle, Reflection couplings and contraction rates for diffusions, Probab. Theory Related Fields 166 (2016), 851–886.

[13] A. Eberle, A. Guillin, R. Zimmer, Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. Trans. Amer. Math. Soc. 371:10 (2019), 7135–7173.

[14] M. Hairer, J.C. Mattingly, Yet another look at Harris' ergodic theorem for Markov chains, In: Seminar on Stochastic Analysis, Random Fields and Applications VI, Progress in Probab., vol. 63, 2011, pp. 109–117.

[15] R.Z. Has'minskii, Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations, Teor. Veroyatnost. i Primenen. 5:2 (1960), 196–214 (in Russian); English transl.: Theory Probab. Appl. 5:2 (1960), 179–196.

[16] A.M. Il'in, R.Z. Has'minskii, Asymptotic behavior of solutions of parabolic equations and an ergodic property of non-homogeneous diffusion processes, Mat. Sbornik 102:3 (1963), 366–392 (in Russian); English transl.: in: Ten Papers on Functional Analysis and Measure Theory, Amer. Math. Soc. Transl. Ser. 2 49, Amer. Math. Soc., Providence, RI, 1966, pp. 241–268.

[17] M. Ji, W. Qi, Z. Shen, Y. Yi, Convergence to periodic probability solutions in Fokker–Planck equations, SIAM J. Math. Anal. 53:2 (2021), 1958–1992.

[18] M. Ji, Z. Shen, Y. Yi, Convergence to equilibrium in Fokker–Planck equations, J. Dyn. Differ. Equ. 31 (2019), 1591–1615.

[19] R.Z. Khasminskii, Stochastic stability of differential equations. 2nd ed. Springer, Heidelberg, 2012.

[20] A. Porretta, Decay rates of convergence for Fokker–Planck equations with confining drift, Adv. Math. 436 (2024), Paper No. 109393, 57 pp.

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