THE STOCHASTIC EVOLUTION OF AN INFINITE POPULATION WITH LOGISTIC-TYPE INTERACTION

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ABSTRACT. An infinite population of point entities dwelling in the habitat $X = \mathbb{R}^d$ is studied. Its members arrive at and depart from X at random. The departure rate has a term corresponding to a logistic-type interaction between the entities. Thereby, the corresponding Kolmogorov operator L has an additive quadratic part, which usually produces essential difficulties in its study. The population's pure states are locally finite counting measures defined on X. The set of such states Γ is equipped with the vague topology and thus with the corresponding Borel σ-field. The population evolution is described at two levels. At the first level, we deal with the Fokker-Planck equation for $(L, \mathcal{F}, \mu_0)$ where $\mathcal{F}$ is an appropriate set of bounded test functions $F : \Gamma \to \mathbb{R}$ (domain of L) and μ_0 is an initial state, which is supposed to belong to the set $\mathcal{P}_{\rm exp}$ of sub-Poissonian probability measures on Γ. We prove that the Fokker-Planck equation has a unique solution $t \mapsto \mu_t$ which also belongs to $\mathcal{P}_{\rm exp}$. Some of the properties of this solution are also obtained. The second level description yields a Markov process such that its one dimensional marginals coincide with the mentioned states μ_t. The process is obtained as the unique solution of the corresponding martingale problem.

CONTENTS

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1. Introduction

More than 200 years ago, T. R. Malthus suggested to describe the evolution of a population by the differential equation $\dot{N}_t = (\lambda - \mu)N_t$. Here N_t is the population size (number of entities) at time t, $\dot{N}_t$ stands for the time derivative, and λ and μ are fertility and mortality rates, respectively. Later on, P. F. Verhulst modified this equation by making the mortality rate state-dependent in the form $\mu = \mu_0 + \mu_1 N$, which takes into account the increase of μ due to the competition between the entities for the resources available at the habitat. This modification would lead to the following evolution equation $\dot{N}_t = (\lambda - \mu_0)N_t - \mu_1 N_t^2$, known now as the logistic growth equation [1]. The appearance of the quadratic term makes the theory more complex. In particular, rough methods which do not take into account the sign of the quadratic term are no longer adequate. At the same time, this change of the model essentially alternates the dependence of N_t on t. In particular, it gets globally bounded in time in contrast to an unbounded growth possible in the Malthus theory.

In the individual-based modeling, the population members are assigned traits – typically, spatial locations $x \in X$. Then counting measures γ defined on the trait space X are naturally employed as the population pure states. That is, for a suitable $\Delta \subset X$, $\gamma(\Delta)$ yields the size of the subpopulation contained in Δ if the state of the whole population is γ. The way back to the aforementioned modeling amounts to restricting the theory to $N = \gamma(X)$. Here, however, one obtains the possibility to study also infinite populations, which corresponds to assuming that $\gamma(\Delta) < \infty$ only for some Δ, e.g., for bounded subsets if $X = \mathbb{R}^d$. The set of all such states Γ is equipped with a suitable topology and hence with the corresponding Borel σ-field $\mathcal{B}(\Gamma)$, which allows one to employ probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ as population states. By $\mathcal{P}(\Gamma)$ we shall denote the set of all such measures. In this setting, pure states γ appear as the corresponding Dirac measures. Another advantage of this approach is that the evolution equation may now appear in its 'dual' form $\dot{F}_t = LF_t$, called the backward Kolmogorov equation. Here $F : \Gamma \to \mathbb{R}$ is a suitable test function, whereas L is the Kolmogorov operator (generator) which contains complete information concerning the elementary acts of the population dynamics.

In this paper, we consider the model in which $X = \mathbb{R}^d$, $d \geq 1$, and the Kolmogorov operator reads

$$
\begin{aligned}
L &= L^+ + L^-, \qquad (1.1)\\
(L^+F)(\gamma) &= \int_X b(x)[F(\gamma \cup x) - F(\gamma)]dx,\\
(L^-F)(\gamma) &= -\int_X \left(m(x) + \int_X a(x-y)(\gamma \setminus x)(dy) \right) [F(\gamma) - F(\gamma \setminus x)]\gamma(dx),
\end{aligned}
$$

where we use notations: $\gamma \cup x = \gamma + \delta_x$, $\gamma \setminus x = \gamma - \delta_x$, δ_x being Dirac's measure centered at x. In this model, point entities at random arrive at and depart from the habitat $X = \mathbb{R}^d$. The arrival rate to a given $\Delta \subset X$ is $\int_\Delta b(x)dx$. It is state-independent and may be infinite for some Δ. The departure part L^- is taken in the logistic form: the departure rate from Δ is

$$
\int_\Delta m(x)\gamma(dx) + \int_\Delta \left(\int_X a(x-y)(\gamma \setminus x)(dy) \right) \gamma(dx),
$$

where the second term corresponds to the departure due to the influence (competition) of the whole population described by

$$\int_X a(x-y)(\gamma \setminus x)(dy).$$

Models of this kind – supported by appropriate simulation methods – find numerous applications in various fields of knowledge, see, e.g., [21] and the publications quoted in that work.

In relatively simple situations, the stochastic evolution of a given model is described by solving the backward Kolmogorov equation $\dot{F}_t = LF$ by constructing a C_0-semigroup acting in suitable Banach spaces of test functions, see, e.g., [10, Chapt. II]. In our case, however, this direct way is rather impossible in view of the complex nature of L given in (1.1) – in particular, due to the fact that it describes an infinite population. Instead, we follow the approach in which the evolution of states is obtained as a map $[0,+\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma)$, which solves the Fokker-Planck equation

$$\mu_t(F) = \mu_0(F) + \int_0^t \mu_u(LF)du, \quad \mu|_{t=0} = \mu_0, \quad \mu(F) := \int F d\mu, \tag{1.2}$$

see [3] for the general theory of such and similar equations. Here μ_0 is an initial state and F is supposed to belong to a sufficiently representative class of functions, $\mathcal{F}$, considered as the domain for L. To stress this, we shall speak of the Fokker-Planck equation for $(L, \mathcal{F}, \mu_0)$.

Let Z be an integer valued random variable and $\varphi_Z(\zeta) = \mathbb{E}\zeta^Z$ its probability generating function. The n-th derivative at $\zeta = 1$ (if it exists) is the corresponding factorial moment of Z, i.e., $\phi_n(Z) = \mathbb{E}Z(Z-1)\cdots(Z-n+1)$, see, e.g., [9, Sect. 5.2, page 112]. If Z is Poissonian with parameter λ, then $\varphi_Z(\zeta) = e^{\lambda(\zeta-1)}$ and hence $\phi_n(Z) = \lambda^n$. For a compact $\Lambda \subset X$, set

$$\Gamma_\Lambda^{(n)} = \{\gamma \in \Gamma : \gamma(\Lambda) = n\}, \qquad n \in \mathbb{N}_0. \tag{1.3}$$

Then each $\mu \in \mathcal{P}(\Gamma)$ by the formula $p_{\Lambda,\mu}(n) = \mu(\Gamma_\Lambda^{(n)})$ determines the distribution of a random variable, $Z_{\mu,\Lambda}$, which is Poissonian with parameter $\kappa(\Lambda)$ if $\mu = \pi_\kappa$, where the latter is the Poisson measure on Γ with intensity measure κ. In dealing with states of infinite 'particle' systems, one often tries to confine the consideration to a suitable subset of $\mathcal{P}(\Gamma)$, which, in particular, may yield additional technical possibilities as well as to shed light on the properties of possible solutions. As in [15, 16], we shall use here the set of *sub-Poissonian* measures $\mathcal{P}_{\rm exp}$. Such a measure μ possesses the property

$$\phi_n(Z_{\mu,\Lambda}) = \int_{\Lambda^n} k_\mu^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \qquad n \in \mathbb{N}, \tag{1.4}$$

with $k_\mu^{(n)}$ being positive symmetric elements of $L^\infty(X^n)$ satisfying Ruelle's bound, see Definition 2.3 below. These $k_\mu^{(n)}$ are called *correlation functions*, cf. [13, 14, 17, 20, 23], which completely characterize the corresponding state.

The aim of this work is constructing a unique Markov process with cadlag paths corresponding to (generated by) L given in (1.1). Here we are going to follow the scheme elaborated in our previous (rather lengthy) works [15, 16] based on solving a restricted martingale problem, see [8, Chapter 5]. Its essential feature is that the one dimensional marginals of the corresponding path measures, which solve this problem, lie in $\mathcal{P}_{\rm exp}$ and solve (1.2). The uniqueness of path measure solutions means that all finite dimensional marginals of two such path measures coincide. The latter is obtained from the fact that their one dimensional marginals coincide, which in turn is obtained by showing that the Fokker-Planck equation for $(L, \mathcal{F}, \mu_0)$ has a unique solution whenever μ_0 is in $\mathcal{P}_{\rm exp}$. It should be pointed out here that L as in (1.1) is a particular case of the generator studied in [12], where the corresponding Markov processes were obtained by solving a stochastic equation involving L. However, uniqueness in [12] was obtained only for the departure part of L which in our notations reads

$$(L^{-}F)(\gamma) = -m\int_X [F(\gamma) - F(\gamma \setminus x)]\gamma(dx).$$

By taking L^{-} as in (1.1) we are going to make the next step in developing this theory.

The present article consists of two parts. First, in Sections 4 and 5 we prove Theorem 3.3 where we state that for each $\mu_0 \in \mathcal{P}_{\rm exp}$, the Fokker-Planck equation (1.2) for $(L, \mathcal{F}, \mu_0)$ has a unique solution $\mu_t \in \mathcal{P}_{\rm exp}$. Certain properties of this solution are also described. Here uniqueness is meant in the class of all measures for which the very solution of this equation can be defined, see Definition 3.2. Among the key ingredients of the proof we mention: (a) the proper choice of the domain $\mathcal{F}$, see (3.10); (b) the proof that every solution of (1.2) for $(L, \mathcal{F}, \mu_0)$ lies in $\mathcal{P}_{\rm exp}$, see Lemma 4.4; (c) the result of [13] where the evolution of states $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ describing the stochastic dynamics governed by (1.1) was obtained with the help of correlation functions. Typically, the domain $\mathcal{F}$ is taken as a subset of the set $C_{\rm b}(\Gamma)$ of all bounded continuous functions, and L is supposed to have the property $L : \mathcal{F} \to C_{\rm b}(\Gamma)$, or at least $L : \mathcal{F} \to B_{\rm b}(\Gamma)$, where the latter is the set of all bounded measurable functions. However, the presence of the quadratic term in (1.1) makes such a property barely possible since in proving that LF is bounded the sign of this quadratic term cannot be taken into account. In our approach, we take $\mathcal{F} \subset C_{\rm b}(\Gamma)$ and define solutions of (1.2) as maps $t \mapsto \mu_t$ for which LF is μ_t-integrable for (Lebesgue) almost all $t > 0$, see Definition 3.2. The construction of $\mathcal{F}$ is made in such a way that $\pm L^{\pm} F \geq 0$ for each $F \in \mathcal{F}$, see (4.18). This allows one to keep track on the sign of the quadratic term in L^{-}.

A significant property of all $\mu \in \mathcal{P}_{\rm exp}$ is $\mu(\Gamma_*) = 1$, where $\Gamma_* \subset \Gamma$ consists of those γ for which $\psi\gamma$ is a finite measure on X, where $\psi(x) = (1 + |x|^{d+1})^{-1}$. With the help of this property Γ_* can be endowed with the Polish topology induced by the weak topology of the set of all finite measures on X. Then all $\mu \in \mathcal{P}(\Gamma)$ with the property $\mu(\Gamma_*) = 1$ can be redefined as probability measures on Γ_*, see Proposition 2.6 and Remark 2.7. We use this fact in the second part of the article – Sections 6 and 7 – where we construct probability measures on the space $\mathfrak{D}_{[0,+\infty)}(\Gamma_*)$ of all cadlag paths with values in Γ_*. Here we mostly follow the scheme elaborated in our previous works [15, 16]. In particular, the path space measures in question are obtained as unique solutions of the corresponding restricted martingale problems, see Definition 3.8 and Theorem 3.10. In more detail, our approach is presented and commented in Subsect. 3.3 below. In Section 2, we provide technicalities, whereas in Section 3 we formulate the main results as Theorems 3.3 and 3.10.

2. Preliminaries

By $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we mean the set of all nonnegative integers $0, 1, 2, \ldots$, Λ will always denote a compact subset of $X = \mathbb{R}^d$. A Polish space, E in general, is a separable space the topology of which is consistent with a complete metric, see, e.g., [7]. By $\mathcal{B}(E)$, $B_{\rm b}(E)$, $C_{\rm b}(E)$, $C_{\rm cs}(E)$ we denote the corresponding Borel σ-field, the sets of all bounded measurable, bounded continuous, and continuous compactly supported functions, respectively. By $B^+_{\rm b}(E)$, $C^+_{\rm b}(E)$, $C^+_{\rm cs}(E)$ we mean the corresponding cones of positive elements. For a suitable set Δ, by $\mathbb{1}_\Delta$ we denote the indicator function.

2.1. **Configuration spaces and measures.**

As mentioned above, by Γ we denote the standard set of Radon counting measures on $X = \mathbb{R}^d$, which in the sequel are called *configurations.* For $x \in X$ and $\gamma \in \Gamma$, we set $n_\gamma(x) = \gamma(\{x\})$ and $p(\gamma) = \{x \in X : n_\gamma(x) > 0\}$. The set $p(\gamma)$ is called the ground configurations for γ, whereas γ itself is the multiset

$(p(\gamma), n_\gamma)$, see, e.g., [2]. The latter interpretation is consistent with the notations

$$\int_X g(x)\gamma(dx) = \sum_{x\in\gamma} g(x) = \sum_{x\in p(\gamma)} n_\gamma(x) g(x),$$

where g is a suitable numerical function. The weak-hash (vague) topology of Γ is defined as the weakest topology that makes continuous all the maps

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} g(x), \qquad g \in C_{\rm cs}(X).$$

With this topology Γ is a Polish space, see, e.g., [9]. By $\Gamma_{\rm fin}$ we denote the subset of Γ consisting of all finite configurations, i.e., those that satisfy $\gamma(X) < \infty$. Along with the subspace topology induced on $\Gamma_{\rm fin}$ from Γ, one can define the following one, see [19, Sect. 2.1]. For $\xi = \{x_1, \dots, x_n\}$ and $\eta = \{y_1, \dots, y_n\}$, set $\rho_n(\xi, \eta) = \min_{\sigma\in S_n} \sum_{i=1}^n |x_i - y_{\sigma(i)}|$, where S_n is the corresponding symmetric group. Then define

$$\rho_{\rm fin}(\xi,\eta) = \begin{cases} \frac{\rho_{|\xi|}(\xi,\eta)}{1+\rho_{|\xi|}(\xi,\eta)}, & \text{if } \ |\xi| = |\eta|; \\ 1, & \text{otherwise} \end{cases}$$

It turns out that ρ is a complete metric. Moreover, the corresponding Borel σ-field $\mathcal{B}(\Gamma_{\rm fin})$ coincides with the σ-field $\{\Delta \in \mathcal{B}(\Gamma) : \Delta \subset \Gamma_{\rm fin}\}$. Thus, each measurable $G : \Gamma_{\rm fin} \to \mathbb{R}$ is defined by a sequence of symmetric Borel functions $\{G^{(n)}\}_{n\in\mathbb{N}_0}$ such that

$$G(\varnothing) = G^{(0)}, \quad \text{and} \quad G(\gamma) = G^{(n)}(x_1, \dots, x_n), \quad \text{for } \ \gamma = \{x_1, \dots, x_n\}.$$

Here *symmetric* means that

$$\forall \sigma \in S_n \qquad G^{(n)}(x_1, \dots, x_n) = G^{(n)}(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \tag{2.1}$$

with S_n being the corresponding symmetric group.

Definition 2.1. *A measurable function $G : \Gamma_{\rm fin} \to \mathbb{R}$ is said to have bounded support if there exist $n \in \mathbb{N}$ and a compact $\Lambda \subset X$ such that the following holds: (a) $G^{(n)} = 0$ for $n > N$; (b) $G(\gamma) = 0$ whenever $\gamma(\Lambda) < \gamma(X)$. The set of all such bounded functions which are bounded is denoted by $B_{\rm bs}$.*

The Lebesgue-Poisson measure λ on $\Gamma_{\rm fin}$ is defined by the following integrals

$$\int_{\Gamma_{\rm fin}} G(\gamma)\lambda(d\gamma) = G(\varnothing) + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} G^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n. \tag{2.2}$$

It is clear, that each $G \in B_{\rm bs}$ is absolutely λ-integrable. The integral in the left-hand side of (2.2) has the following property, see e.g., [14, Lemma A.1],

$$\int_{\Gamma_{\rm fin}} G(\eta) \sum_{\xi\subset\eta} H(\eta \setminus \xi, \xi)\lambda(d\eta) = \int_{\Gamma^2_{\rm fin}} G(\eta \cup \xi) H(\eta, \xi)\lambda(d\eta)\lambda(d\xi), \tag{2.3}$$

where G and H are suitable functions. In view of the multiset terminology adopted here, for $x \in p(\gamma)$, we write $\gamma\backslash x = \gamma - \delta_x$, i.e., $n_{\gamma\backslash x}(y) = n_\gamma(y)$ for $y \neq x$, and $n_{\gamma\backslash x}(x) = n_\gamma(x) - 1$. Similarly, $\gamma \cup y$, $y \in X$, stands for $\gamma + \delta_y$. We will also use the notations

$$\sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} g(x, y) = \int_{X^2} g(x, y)\gamma(dx)\gamma(dy) - \int_X g(x, x)\gamma(dx),$$

and their extensions

$$\sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_n\in\gamma\setminus\{x_1,\dots,x_{n-1}\}} g(x_1,\dots,x_n) \tag{2.4}$$

$$= \sum_{\mathbb{G}\subset\mathbb{K}_n}(-1)^{l_\mathbb{G}}\int_{X^{n_\mathbb{G}}} g_\mathbb{G}(y_1,\dots,y_{n_\mathbb{G}})\gamma(dy_1)\cdots\gamma(dy_{n_\mathbb{G}}),$$

where each $\mathbb{G}$ is a spanning subgraph of the complete graph $\mathbb{K}_n$ on $\{1,\dots,n\}$, $l_\mathbb{G}$ and $n_\mathbb{G}$ are the number of edges and the connected components of $\mathbb{G}$, respectively; $g_\mathbb{G}(y_1,\dots,y_\mathbb{G})$ is obtained from $g(x_1,\dots,x_n)$ be setting $x_i = y_j$ for all i belonging to j-th connected component of $\mathbb{G}$. For $\gamma\in\Gamma$, by writing $\gamma'\subset\gamma$ we mean a configuration such that $p(\gamma')\subset p(\gamma)$ and $n_{\gamma'}(x)\le n_\gamma(x)$, $x\in p(\gamma')$. Which means that γ' is a multisubset of γ. In this case, we say that γ' is a *sub-configuration* of γ.

Following [14, 17], we introduce now *correlation measures*. For $n\in\mathbb{N}$, a compact $\Delta\subset X^n$ and $\gamma\in\Gamma$, let us consider

$$Q^{(n)}_\gamma(\Delta) = \sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_n\in\gamma\setminus\{x_1,\dots,x_{n-1}\}} \mathbb{1}_\Delta(x_1,\dots x_n). \tag{2.5}$$

Clearly, $Q^{(n)}_\gamma$ is a counting measure: $Q^{(n)}_\gamma(\Delta)$ is the number of tuples $(x_1,\dots,x_n)\in\Delta$ in state γ. In particular, $Q^{(1)}_\gamma=\gamma$. It is known, [17, Theorem 1], that the map $\gamma\mapsto Q^{(n)}_\gamma(\Delta)$ is measurable for each measurable Δ. However, it may be unbounded.

Definition 2.2. *A given $\mu\in\mathcal{P}(\Gamma)$ is said to have all correlations if all $\gamma\mapsto Q^{(n)}_\gamma(\Delta)$, $n\in\mathbb{N}$ are μ-integrable for all compact $\Delta\subset X^n$. By $\mathcal{P}_{\rm cor}(\Gamma)$ we denote the set of all $\mu\in\mathcal{P}(\Gamma)$ that have all correlations.*

For $\mu\in\mathcal{P}_{\rm cor}(\Gamma)$, one can define

$$\chi^{(n)}_\mu(\cdot) = \int_\Gamma Q^{(n)}_\gamma(\cdot)\mu(d\gamma), \tag{2.6}$$

which is called the *correlation measure* of n-th order for μ, cf. [14, 17, 23]. The factorial moment mentioned in (1.4) and the correlation measure are related to each other by

$$\phi_n(Z_{\mu,\Lambda}) = \chi^{(n)}_\mu(\Lambda^n).$$

In view of this, correlation measures are also called *factorial moment measures*, cf. [9, Chapt. 7]. For $G\in B_{\rm bs}$, we write

$$(KG)(\gamma) = \sum_{\eta\Subset\gamma} G(\eta), \tag{2.7}$$

where $\eta\Subset\gamma$ means that the sum is taken over finite sub-configurations of γ, including $\eta=\varnothing$. The advantage of using this K-map can be seen from the following relation

$$\mu(KG) = G(\varnothing) + \sum_{n=1}^\infty \chi^{(n)}_\mu(G^{(n)}), \qquad G\in B_{\rm bs}, \tag{2.8}$$

see [14, Corollary 4.1]. We use this fact to introduce the set of sub-Poissonian measures, which plays the key role in our constructions. Let ϑ be a finite (nonempty) collection of $\theta\in C^+_{\rm cs}(X)$, and $|\vartheta|$ stand for its cardinality. We do not require that all the members of ϑ are distinct, i.e., ϑ is a multiset. For $n=|\vartheta|$, define

$$G^\vartheta(\xi) = \begin{cases} \frac{1}{n!}\sum_{\sigma\in S_n}\prod_{i=1}^n \theta_i(x_{\sigma(i)}), & \text{for } \xi=\{x_1,\dots,x_n\}, \\ 0, & \text{otherwise.}\end{cases} \tag{2.9}$$

For this function, it follows that

$$\left(KG^{\vartheta}\right)(\gamma) = \frac{1}{n!}\sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_n\in\gamma\setminus\{x_1,\dots,x_{n-1}\}}\theta_1(x_1)\cdots\theta_n(x_n). \tag{2.10}$$

Let G_n^{θ} be as in (2.9) with $|\vartheta| = n$ and all the members of ϑ coinciding with a given $\theta \in C_{\rm cs}^{+}(X)$. Then the function

$$F^{\theta}(\gamma) = \prod_{x\in\gamma}(1+\theta(x)) = \exp\left(\sum_{x\in\gamma}\log(1+\theta(x))\right), \tag{2.11}$$

that possibly takes value $+\infty$, can be written in the form

$$F^{\theta}(\gamma) = 1 + \sum_{n=1}^{\infty}(KG_n^{\theta})(\gamma). \tag{2.12}$$

Among all $\mu \in \mathcal{P}(\Gamma)$ we distinguish Poissonian measures. Let κ be a positive Radon measure on X. Then the Poisson measure π_κ, for which κ is the *intensity measure*, is defined as such that its correlation measures are $\chi_{\pi_\kappa}^{(n)} = \kappa^{\otimes n}$. Then by (2.8), (2.9) and (2.12) one gets

$$\pi_\kappa(F^{\theta}) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\kappa^{\otimes n}(G_n^{\theta}) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\kappa(\theta)\right]^n = e^{\kappa(\theta)}. \tag{2.13}$$

If κ is absolutely continuous with respect to Lebesgue's measure on $X = \mathbb{R}^d$, then the Radon-Nikodym derivative $\rho(x) = d\kappa/dx$ may be an element of $L^{\infty}(X)$. A particular case is a constant ρ, i.e., $\rho(x) \equiv \varkappa$ for some $\varkappa > 0$. The corresponding Poisson measure is called *homogeneous*. With certain abuse, we denote it by $\pi_\varkappa$ and call $\varkappa$ the *intensity* of $\pi_\varkappa$. In this case,

$$\pi_\varkappa(F^{\theta}) = \exp\left(\varkappa\int_X \theta(x)dx\right).$$

2.2. **Sup-Poissonian measures.** In this article, the following class of measures on Γ will be employed.

Definition 2.3. *$\mu \in \mathcal{P}(\Gamma)$ is said to be sub-Poissonian if it has all correlations, i.e., $\mu \in \mathcal{P}_{\rm cor}(\Gamma)$, see Definition 2.2, and for each $n \in \mathbb{N}$ and $\vartheta = \{\theta_1,\dots,\theta_n\}$, $\theta_i \in C_{\rm cs}^{+}(X)$, and thus for G^{ϑ} as in (2.9), (2.10), the following holds*

$$n!\mu(KG^{\vartheta}) = \chi_\mu^{(n)}(G^{\vartheta}) \le \varkappa^n\langle\theta_1\rangle\cdots\langle\theta_n\rangle, \qquad \langle\theta_i\rangle := \int_X \theta_i(x)dx, \tag{2.14}$$

for one and the same $\varkappa > 0$. The least such $\varkappa$ will be called the type of μ; $\mathcal{P}_{\rm exp}$ will denote the set of all sub-Poissonian measures whereas $\mathcal{P}_{\rm exp}^{\alpha}$, $\alpha \in \mathbb{R}$, is to denote the set of all those $\mu \in \mathcal{P}_{\rm exp}$ the type of which does not exceed e^{α}.

Remark 2.4. By this definition and (2.10), each $\mu \in \mathcal{P}_{\rm exp}$ has the following properties:

(a) For each $n \in \mathbb{N}$, the map $(\theta_1,\dots,\theta_n) \mapsto \mu(KG^{\vartheta})$, cf. (2.9), can be continued to a continuous n-linear functional on the real Banach space $L^1(X^n)$.
(b) For $\theta \in C_{\rm cs}^{+}(X)$, the map $\theta \mapsto \mu(F^{\theta})$, see (2.11), can be continued to a real exponential entire function of normal type defined on $L^1(X)$.
(c) For each $H \in B_{\rm b}^{+}(\Gamma)$ such that $\mu(H) =: C_H > 0$, the measure $\mu_H := C_H^{-1}H\mu$, lies in $\mathcal{P}_{\rm exp}$ and the types of μ and μ_H satisfy

$$\varkappa_{\mu_H} \le \varkappa_\mu \max\{1; C_H^{-1}\sup H\}.$$

Proof. Claim (a) follows by the fact that the linear span of vectors $(\theta_1, \ldots, \theta_n)$ is dense in $L^1(X^n)$. First, one proves that it is dense in $C_{\rm cs}(X^n)$ – by the Stone-Weierstrass theorem [4], and then one applies [5, Theorem 4.3, page 90]. Thereafter, one employs the estimate in (2.14). To prove claim (b), we first note that the function

$$F_N^\theta(\gamma) := 1 + \sum_{n=1}^N (KG_n^\theta)(\gamma)$$

is certainly μ-integrable, and then by (2.12) and (2.14), we get

$$\mu(F_N^\theta) \leq \exp\left(\varkappa_\mu \langle\theta\rangle\right),$$

where $\varkappa_\mu$ is the type of μ. By the Beppo Levi (monotone convergence) theorem this yields the proof of claim (b). The proof of (c) is immediate. □

Let μ have all correlations. Similarly as in (2.2), by employing its correlation measures χ_μ, $\chi_\mu^{(n)}$ defined in (2.6) one may write

$$\int_{\Gamma_{\rm fin}} G(\eta)\chi_\mu(d\eta) = G(\varnothing) + \sum_{n=1}^\infty \frac{1}{n!}\int_{X_n} G^{(n)}(x_1, \ldots, x_n)\chi_\mu^{(n)}(dx_1, \ldots, dx_n). \qquad (2.15)$$

Such integrals do exist for $G \in B_{\rm bs}$.

Definition 2.5. *Let $q : X \to [0, 1]$ be measurable. For a given $\mu \in \mathcal{P}_{\rm cor}(\Gamma)$, its q-thinning is the measure $\mu^q \in \mathcal{P}_{\rm cor}(\Gamma)$ defined by the correlation measures χ_{μ^q} that have the following form*

$$\chi_{\mu^q}(d\eta) = e(\eta; q)\chi_\mu(d\eta), \qquad e(\eta; q) := \prod_{x\in\eta} q(x). \qquad (2.16)$$

By standard arguments, see e.g., [5, Theorem 4.14, page 99], it follows that, for each $\mu \in \mathcal{P}_{\rm exp}$, its correlation measure χ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure λ. Its Radon-Nikodym derivative

$$k_\mu := \frac{d\chi_\mu}{d\lambda} \qquad (2.17)$$

is such that $k_\mu(\varnothing) = 1$ and, for $\xi = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, the following holds, cf. (1.4),

$$k_\mu(\xi) = k_\mu^{(n)}(x_1, \ldots, x_n) := \frac{d\chi^{(n)}}{dx_1 \cdots dx_n}(x_1, \ldots, x_n), \qquad (2.18)$$

where $k_\mu^{(n)}$ is a symmetric element of the corresponding $L^\infty(X^n)$, see (2.1). Then for $\mu \in \mathcal{P}_{\rm exp}$ and $G \in B_{\rm bs}$, we get

$$\mu(KG) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)\lambda(d\eta) =: \langle\!\langle k_\mu, G\rangle\!\rangle. \qquad (2.19)$$

Since $\chi_\mu^{(n)}$ is positive, see (2.5), (2.6), and in view of (2.14), we have that

$$0 \leq k_\mu^{(n)}(x_1, \ldots, x_n) \leq \varkappa_\mu^n, \qquad (2.20)$$

holding for all n and almost all $(x_1, \ldots, x_n)$. For $G \geq 0$, by (2.19) and (2.20) one readily gets that

$$\mu(KG) \leq \langle\!\langle k_{\pi_{\varkappa_\mu}}, G\rangle\!\rangle = \pi_{\varkappa_\mu}(KG), \qquad (2.21)$$

which may be interpreted as the 'sub-Poissonocity' of μ. The upper estimate in (2.20) is known as Ruelle's bound. It turns out that the set of measures possessing correlation functions satisfying (2.20) contains states of thermal equilibrium of physical particles interacting via super-stable potentials, see [20]. Further iformation concerning sub-Poissonian measures can be found in [15, sect. 2.2].

2.3. **Tempered configurations and cadlag paths.** States from $\mathcal{P}_{\rm exp}$ have one more significant property which allows one to confine the theory to so called *tempered* confgurations, the set of which Γ_* is defined by the condition $\mu(\Gamma_*) = 1$ that ought to hold for all $\mu \in \mathcal{P}_{\rm exp}$. In the present work, this set is defined by means of the function

$$\psi(x) = \frac{1}{1 + |x|^{d+1}}, \qquad x \in X, \tag{2.22}$$

for which we have

$$0 < \psi(x) \leq 1, \qquad \text{and} \qquad \int_X \psi(x)dx =: \langle \psi \rangle < \infty. \tag{2.23}$$

Define

$$\Gamma_* = \{\gamma \in \Gamma : \varPhi(\gamma) := \gamma(\psi) = \sum_{x\in\gamma} \psi(x) < \infty\}. \tag{2.24}$$

By (2.21) we then have

$$\mu(\varPhi) = \int_X k_\mu^{(1)}(x)\psi(x)dx \leq \varkappa_\mu \langle \psi \rangle < \infty, \tag{2.25}$$

which yields that $\mu(\Gamma_*) = 1$. Next we define

$$\mathcal{A}_* = \{\mathbb{A} \in \mathcal{B}(\Gamma) : \mathbb{A} \subset \Gamma_*\}. \tag{2.26}$$

By (2.24) it follows that Γ_* is the set of all those configurations γ for which $\psi\gamma$ is a finite Borel measure on X. This fact allows one to define a kind of weak topology on Γ_*, which we do as follows. Set

$$\rho(\gamma, \gamma') = \max \left\{ 1; \sup_{g\in C^1_{\rm L}} |\gamma(\psi g) - \gamma'(\psi g)| \right\}, \qquad \gamma, \gamma' \in \Gamma_*, \tag{2.27}$$

where

$$C^1_{\rm L} := \left\{ g \in C_{\rm b}(X) : \sup_{x\in X} |g(x)| + \sup_{x,y\in X,\ x\neq y} \frac{|g(x) - g(y)|}{|x - y|} \leq 1 \right\}.$$

It is clear that ρ defined in (2.27) is a metric on Γ_*.

Proposition 2.6. [15, Lemma 2.7 and Corollary 2.8] *The metric space* (Γ_*, ρ) *is complete and separable. Its Borel σ-field and the collection of sets defined in* (2.26) *satisfy* $\mathcal{B}(\Gamma_*) = \mathcal{A}_*$.

Remark 2.7. The space of tempered configurations defined in (2.22) and (2.24) is exactly the same as that in [15], where one can find more information on the properties of this space. Here we only mention that, in view of the equality $\mathcal{B}(\Gamma_*) = \mathcal{A}_*$, each $\mu \in \mathcal{P}(\Gamma)$ with the property $\mu(\Gamma_*) = 1$ can be redefined as an element of $\mathcal{P}(\Gamma_*)$.

Similarly as in [15], see also [8, Sect. V] and [11, Chapter 4], we introduce the spaces of cadlag paths with values in Γ_*. For $s \geq 0$, we let $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ stand for the set of all cadlag maps $[s, +\infty) \ni t \mapsto \gamma_t \in \Gamma_*$, where we mean the metric topology of Γ_*, see Proposition 2.6. For $s = 0$, we write $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$. The elemets of $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ will be denoted $\bar{\gamma}$. The restriction of a given $\bar{\gamma} \in \mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ is usually considered as an element of $\mathfrak{D}_{[s',+\infty)}(\Gamma_*)$ for every $s' > s$. For $t \geq 0$, by ϖ_t we denote the evaluation map, that is, $\varpi_t(\bar{\gamma})$ is the corresponding value γ_t of $\bar{\gamma}$. For $t, t' \geq 0$, $t' > t$, by $\mathfrak{F}^0_{t,t'}$ we mean the σ-field of subsets of $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$ generated by the collection of maps $\{\varpi_u : u \in [t, t']\}$. Next, set

$$\mathfrak{F}_{t,t'} = \bigcap_{\epsilon>0} \mathfrak{F}^0_{t,t'+\epsilon}, \qquad \mathfrak{F}_{t,+\infty} = \bigcup_{n\in\mathbb{N}} \mathfrak{F}_{t,t+n}. \tag{2.28}$$

By [11, Theorem 5.6, page 121] the Skorohod topology turns each $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ into a Polish space, measurably isomorphic to the measurable space $(\mathfrak{D}_{[s,+\infty)}(\Gamma_*), \sigma(\mathfrak{F}_{s,+\infty}))$.

2.4. **Banach spaces of functions.** For a given $\mu \in \mathcal{P}_{\rm exp}$, its correlation functions k_μ, $k_\mu^{(n)}$, $n \in \mathbb{N}_0$, are defined in (2.17), (2.18). Recall that the latter is a symmetric element of $L^\infty(X^n)$ satisfying the Ruelle estimate (2.20). Having this in mind we introduce Banach spaces which contain such k_μ. As each $k : \Gamma_{\rm fin} \to \mathbb{R}$ is defined by its restrictions $k^{(n)}$ to $\xi = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}_0$, cf. (2.18), we set

$$\|k\|_\alpha = \sup_{n\in\mathbb{N}_0} \|k^{(n)}\|_{L^\infty} e^{-\alpha n}, \qquad \alpha \in \mathbb{R}, \tag{2.29}$$

where

$$\|k^{(n)}\|_{L^\infty} = {\rm esssup}_{(x_1,\dots,x_n)\in X^n} |k^{(n)} x_1, \dots, x_n)|.$$

Let $\mathcal{K}_\alpha$ be the real Banach space of $k : \Gamma_{\rm fin} \to \mathbb{R}$ for which $\|k\|_\alpha < \infty$. It is obvious that

$$\mathcal{K}_{\alpha'} \hookrightarrow \mathcal{K}_\alpha, \qquad \text{for} \ \ \alpha' < \alpha, \tag{2.30}$$

where we mean continuous embedding. Let k be in $\mathcal{K}_\alpha$, $\alpha \in \mathbb{R}$, and G be in $B_{\rm bs}$, see Definition 2.1. By (2.15) and (2.4) one readily gets that

$$\langle\!\langle |k|, |G| \rangle\!\rangle = \int_{\Gamma_{\rm fin}} |k(\eta)||G(\eta)|\lambda(d\eta) < \infty.$$

Recall that KG is defined for all $G \in B_{\rm bs}$, see (2.7). Keeping this in mind we set

$$B^\star_{\rm bs} = \{G \in B_{\rm bs} : (KG)(\gamma) \geq 0 \quad \text{for all } \gamma \in \Gamma\}.$$

Note that the cone of poinwise positive $G \in B_{\rm bs}$ is a proper subset of $B^\star_{\rm bs}$. By [14, Theorems 6.1 and 6.2 and Remark 6.3] one has the following fact.

Proposition 2.8. *For each $\alpha \in \mathbb{R}$, the following is true. If $k \in \mathcal{K}_\alpha$ is such that: (i) $k(\varnothing) = 1$; (ii) $\langle\!\langle k, G \rangle\!\rangle \geq 0$ for all $G \in B^\star_{\rm bs}$, then k is the correlation function for a unique $\mu \in \mathcal{P}_{\rm exp}$ the type of which does not exceed e^α.*

Let now $G : \Gamma_{\rm fin} \to \mathbb{R}$ be such that each $G^{(n)}$, $n \in \mathbb{N}$, cf. (2.15), is a symmetric element of $L^1(X^n)$. We denote its corresponding norm $\|G^{(n)}\|_{L^1}$ and set

$$|G|_\alpha = |G(\varnothing)| + \sum_{n=1}^\infty \frac{1}{n!} e^{n\alpha} \|G^{(n)}\|_{L^1} = \int_{\Gamma_{\rm fin}} e^{\alpha|\eta|} |G(\eta)|\lambda(d\eta), \tag{2.31}$$

and also

$$\mathcal{G}_\alpha = \{G : |G|_\alpha < \infty\}, \qquad \alpha \in \mathbb{R}. \tag{2.32}$$

Thus, each $\mathcal{G}_\alpha$ is a weighted L^1-type real Banach space. Similarly as in (2.30), we have

$$\mathcal{G}_\alpha \hookrightarrow \mathcal{G}_{\alpha'}, \qquad \text{for} \ \ \alpha' < \alpha. \tag{2.33}$$

However, here the embedding is also dense. Recall that the set of functions $B_{\rm bs}$ is defined in Definition 2.1.

Remark 2.9. Regarding the spaces $\mathcal{G}_\alpha$, $\alpha \in \mathbb{R}$, the following is true:

(i) For each $\alpha \in \mathbb{R}$, $B_{\rm bs}$ is a dense subset of $\mathcal{G}_\alpha$.
(ii) By (2.13), (2.21) and (2.31) one may write

$$|G|_\alpha = \pi_{e^\alpha}(K|G|), \tag{2.34}$$

by which the K-map defined in (2.7) can be extended to $\mathcal{G}_\alpha$ with an arbitrary $\alpha \in \mathbb{R}$. In this case, $K : \mathcal{G}_\alpha \to L^1(\Gamma, \pi_{e^\alpha})$.

3. The Results

3.1. **Solving the Fokker-Planck equation.** The Kolmogorov operator L introduced in (1.1) is subject to the following

Assumption 3.1. *The parameters of L satisfy: (a) $a(0) > 0$; m, a and b are nonnegative and continuous; (b) the following quantities are finite*

$$\sup_{x\in X} \frac{a(x)}{\psi(x)} =: \|a\|, \qquad \sup_{x\in X} b(x) =: \|b\|, \qquad \sup_{x\in X} m(x) =: \|m\|, \tag{3.1}$$

where $\psi(x)$ is as in (2.22).

According to (3.1) a is integrable, cf. (2.23); hence,

$$\int_X a(x)dx =: \langle a\rangle < \infty. \tag{3.2}$$

Moreover, by the triangle inequality and (2.22) we have

$$a(x-y) \leq \|a\|\psi(y)\left(1 + \sum_{l=0}^{d+1} \binom{d+1}{l} |x|^l\right) =: \|a\|\psi(y)\ell_a(x). \tag{3.3}$$

By this estimate it follows that, for each $\theta \in C^+_{\rm cs}(X)$, the following holds, see (2.24) and (3.3),

$$\forall \gamma \in \Gamma_* \quad \sum_{x\in\gamma} \theta(x) \sum_{y\in\gamma\setminus x} a(x-y) \leq \gamma(\psi)\|a\| \sum_{x\in\gamma} \theta(x)\ell_a(x) < \infty, \tag{3.4}$$

since the support of θ is compact.

Below we use the following functions of $t \geq 0$ and $x \in X$

$$q_t(x) = e^{-m(x)t}, \qquad \varrho_t(x) = \begin{cases} \left(1 - e^{-m(x)t}\right)\frac{b(x)}{m(x)}, & \text{if } \ m(x) > 0; \\ b(x)t, & \text{if } \ m(x) = 0. \end{cases} \tag{3.5}$$

Let us turn now to defining solutions of (1.2), which we precede by the following reminder, see e.g., [11, pages 112, 113]. A subset $\mathcal{C} \subset C_{\rm b}(\Gamma)$ is said to be separating if, for each $\mu_1, \mu_2 \in \mathcal{P}$, the equality $\mu_1(F) = \mu_2(F)$ holding for all $F \in \mathcal{C}$ implies $\mu_1 = \mu_2$. If $\mathcal{C}$ is closed under multiplication and separates points of Γ, it is separating. The latter property means that, for each $\gamma_1 \neq \gamma_2$, one finds $F \in \mathcal{C}$ such that $F(\gamma_1) \neq F(\gamma_2)$.

Definition 3.2. *Fix separating $\mathcal{F} \subset C_{\rm b}(\Gamma)$ and $\mu_0 \in \mathcal{P}(\Gamma)$. A map $\mathbb{R}_+ \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma)$ is said to be a solution of the Fokker-Planck equation (1.2) for $(L, \mathcal{F}, \mu_0)$ if, for each $F \in \mathcal{F}$, the following holds:*

(i) *LF is absolutely μ_t-integrable for Lebesgue-almost all t, and the map $t \mapsto \mu_t(LF)$ is measurable and Lebesgue-integrable on each $[0, T]$, $T > 0$.*
(ii) *The equality in (1.2) holds true.*

Obviously, the domain $\mathcal{F}$ should be separating if one strives for uniqueness of the solutions of (1.2). Typically, see, e.g., [8, page 78], generators $(L, \mathcal{F})$ are chosen in such a way that $L : \mathcal{F} \to B_{\rm b}(\Gamma)$, where the latter is the set of all bounded measurable functions $F : \Gamma \to \mathbb{R}$. In our case, however, it is barely possible in view of the quadratic term in L^-. It might also be clear from this definition that the proper choice of $\mathcal{F}$ is one of the main technical aspects of the current research.

For $\theta \in C^+_{\rm cs}(X)$, we set

$$\varPhi^\theta(\gamma) = \gamma(\theta) = \sum_{x\in\gamma} \theta(x), \tag{3.6}$$

and then

$$\varPhi^\theta_\tau(\gamma) = \frac{\varPhi^\theta(\gamma)}{1+\tau\varPhi^\theta(\gamma)} = \int_0^{+\infty} \varPhi^\theta(\gamma) \exp\Big(-\alpha\Big[1+\tau\varPhi^\theta(\gamma)\Big]\Big) d\alpha, \quad \tau\in(0,1/2]. \tag{3.7}$$

It is obvious that both $\varPhi^\theta$ and $\varPhi^\theta_\tau$ are vaguely continuous, and also

$$0 \le \varPhi^\theta_\tau(\gamma) \le 1/\tau \qquad \gamma\in\Gamma.$$

By ϑ we denote a finite multiset consisting of the elements of $C^+_{\rm cs}(X)$, see (2.9), (2.14). Define

$$\varPsi^\vartheta_\tau(\gamma) = \prod_{\theta\in\vartheta} \varPhi^\theta_\tau(\gamma), \qquad \varPsi^\varnothing_\tau(\gamma)\equiv 1. \tag{3.8}$$

Clearly, all $\varPsi^\vartheta_\tau$ are bounded and continuous. Thereafter, we set

$$\Theta = \{\theta\in C^+_{\rm cs}(X) : \langle\theta\rangle \le \langle\psi\rangle, \ \ \theta(x)\le 1, \ x\in X\}, \tag{3.9}$$

see (2.14), (2.22) and (2.23), and

$$\mathcal{F}_\tau = \{\varPsi^\vartheta_\tau : \text{all possible finite } \vartheta\subset\Theta\}, \qquad \mathcal{F} = \bigcup_{\tau\in(0,1/2]} \mathcal{F}_\tau. \tag{3.10}$$

Obviously, each $\mathcal{F}_\tau$ separates points of Γ. For one can take $\theta\in\Theta$ the support of which contains some $x\in p(\gamma_1)$ and such that $\theta(y)=0$ for all $y\in p(\gamma_2)$. Also, by the very definition (3.8), each $\mathcal{F}_\tau$ is closed under multiplication and hence separating, see e.g., [11, Theorem 4.5, page 113]. Thus, so is $\mathcal{F}$. The choice of the upper bounds in (3.9) will be explained later. Here we just recall that $\psi(x)\le 1$ and hence Θ is closed under multiplication.

To proceed further, we introduce the following notions. First we recall that the K-map and the spaces $\mathcal{G}_\alpha$ are defined in (2.7) and (2.32), respectively. Then we set

$$\mathcal{F}_{\rm max} = \{F = KG : G\in\mathcal{G}_\alpha \text{ for all } \alpha\in\mathbb{R}\}, \tag{3.11}$$

see Remark 2.9. Note that $\mathcal{F}_{\rm max}$ contains also unbounded functions. For a compact $\Lambda\subset X$ and $\gamma\in\Gamma$, define

$$N_\Lambda(\gamma) = \sum_{x\in\gamma} \mathbb{1}_\Lambda(x) = \gamma(\Lambda), \tag{3.12}$$

which is the total number of the elements of γ contained in Λ, cf. (1.3). Now we can formulate our first statement. Recall that F^θ is defined in (2.12).

Theorem 3.3. *Let the parameters of the Kolmogorov operator L introduced in (1.1) satisfy Assumption 3.1 and $\mathcal{F}$ be as in (3.10). Then, for each $\mu_0\in\mathcal{P}_{\rm exp}$, the Fokker-Planck equation for $(L,\mathcal{F},\mu_0)$ has a unique solution such that $\mu_t\in\mathcal{P}_{\rm exp}$ for all $t\ge 0$. This solution has the following properties:*

(a) *For each $\theta\in C^+_{\rm cs}(X)$,*

$$\mu_t(F^\theta) \le \pi_{\varrho_t}(F^\theta)\mu_0^{q_t}(F^\theta), \qquad t>0, \tag{3.13}$$

ϱ_t and q_t are as in (3.5) and $\mu_0^{q_t}$ is the q_t-thinning of μ_0, see Definition 2.5.

(b) *For each compact $\Lambda\subset X$ and $n\in\mathbb{N}$, there exists $C_{n,\Lambda}>0$ such that*

$$\forall t>0 \qquad \mu_t(N^n_\Lambda)\le C_{n,\Lambda}, \tag{3.14}$$

i.e., the moments of the observable (3.12) are globally bounded in time.

(c) *The sets defined in* (3.10) *and* (3.11) *satisfy $\mathcal{F}\subset\mathcal{F}_{\rm max}$, and μ_t solves the Fokker-Planck equation* (1.2) *with each $F\in\mathcal{F}_{\rm max}$.*

Let us make some comments to this statement. A priori we look for solutions among all $\mu \in \mathcal{P}(\Gamma)$ satisfying item (i) of Definition 3.2 – restricting only the choice of the initial state μ_0. The result is that the solution is unique and lies in $\mathcal{P}_{\rm exp}$; i.e., the evolution leaves invariant the set of sub-Poissonian measures. Note that item (i) of Definition 3.2 means

$$\forall F \in \mathcal{F} \quad \forall T > 0 \qquad \int_0^T \mu_t(|LF|)dt < \infty. \tag{3.15}$$

The measure $\tilde{\mu}_t$ such that $\tilde{\mu}_t(F^\theta) = {\rm RHS}(3.13)$ is the convolution, see [18, page 15], of two states: the Poissonian state π_{ϱ_t} describing the distribution of the newcomers; the thinned initial state. The inequality in (3.13) indicates that the sign of the quadratic term in L^- was taken into account properly. Finally, the boundedness as in (3.14), which holds also if $m(x) \equiv 0$, is of the same nature as the boundedness of N_t in the original Verhulst model. As already mentioned, the set $\mathcal{F}_{\rm max}$ contains also unbounded functions, which are μ_t-integrable for all $t > 0$, see (2.19).

3.2. **The Markov process.** Theorem 3.3 yields the evolution of states $\mu_0 \to \mu_t$ of the model corresponding to the Kolmogorov operator (1.1). A more comprehensive description of the evolution of this model can be obtained by constructing a Markov process. In this work, we follow the way elaborated in [15, 16] in which the process in question is obtained by solving a restricted martingale problem, which yields probability measures on the corresponding space of cadlag paths. The central role here is played by the Fokker-Planck equation, especially stated in Theorem 3.3 uniqueness, the facts that its solutions lie in the class of sup-Poissonian measures and that the unique solution satisfies (1.2) with all $F \in \mathcal{F}_{\rm max}$.

As just mentioned, the evolution $\mu_0 \to \mu_t$ related to (1.2) leaves invariant the set of measures $\mathcal{P}_{\rm exp}$ the elements of which have the property $\mu(\Gamma_*) = 1$, see Remark 2.7. Therefore, it might be natural to construct a process with values in Γ_*, cf. (2.28), such that its one dimensional marginals solve the Fokker-Planck equation. Such a process will be obtained as a solution of the restricted martingale problem involving L, the domain of which should be consistent with the domain used in solving the Fokker-Planck equation. Thus, we start by setting

$$\widetilde{F}_v(\gamma) = \exp\left(- \sum_{x\in\gamma} v(x)\psi(x) \right), \qquad v \in C_{\rm b}^+(X). \tag{3.16}$$

Concerning such functions it is known the following, see [8, Lemma 3.2.5 and Theorem 3.2.6, page 43].

Proposition 3.4. *There exists a countable set $\mathcal{V} \subset C_{\rm b}^+(X)$, that contains constants and is closed under addition, such that the set $\widetilde{\mathcal{F}} = \{\widetilde{F}_v : v \in \mathcal{V}\}$ has the following properties:*

(i) *The space of $\mathcal{B}(\Gamma_*)$-measurable functions is the bounded pointwise closure of $\widetilde{\mathcal{F}}$, see* [8, page 41]*, and $\mathcal{B}(\Gamma_*)$ is generated by $\widetilde{\mathcal{F}}$, i.e., $\mathcal{B}(\Gamma_*) = \sigma(\widetilde{\mathcal{F}})$.*
(ii) *$\widetilde{\mathcal{F}}$ is strongly separating and hence separating. The former implies that this set is weak convergence determining. That is, if $\mu_n(\widetilde{F}_v) \to \mu(\widetilde{F}_v)$ for all $\widetilde{F}_v \in \widetilde{\mathcal{F}}$, then $\mu_n \Rightarrow \mu$, holding for $\{\mu_n\} \subset \mathcal{P}(\Gamma_*)$ and $\mu \in \mathcal{P}(\Gamma_*)$.*

For $\widetilde{F}_v$ as in (3.16), one can write, cf. (2.7) and (2.16),

$$\widetilde{F}_v(\gamma) = \prod_{x\in\gamma}(1 + h_v(x)) = \sum_{\xi \Subset \gamma} e(\xi; h_v) = (Ke(\cdot; h_v))(\gamma), \tag{3.17}$$

$$h_v(x) = e^{-v(x)\psi(x)} - 1.$$

It is clear that $C^{-1}_{v,\mu} := \mu(\widetilde{F}_v) > 0$ for each $v \in \mathcal{V}$ and $\mu \in \mathcal{P}(\Gamma_*)$. In the sequel, we use the following measures, cf. item (c) of Remark 2.4,

$$\mu_v = C_{v,\mu} \widetilde{F}_v \mu. \tag{3.18}$$

Proposition 3.5. *Along with the properties mentined in Proposition 3.4, the set $\widetilde{\mathcal{F}}$ has the following ones:*

(a) *for each $\mu \in \mathcal{P}_{\rm exp}$ and $v \in \mathcal{V}$, the measure μ_v introduced in* (3.18) *is in $\mathcal{P}_{\rm exp}$, and the types of the two measures, see Definition 2.3, varify $\varkappa_{\mu_v} = \max\{C_{v,\mu}\varkappa_\mu; \varkappa_\mu\}$. Moreover, for all appropriate $G : \Gamma_{\rm fin} :\to \mathbb{R}$, the correlation functions of μ_v and μ satisfy*

$$\langle\!\langle k_{\mu_v}, G \rangle\!\rangle = C_{v,\mu} \langle\!\langle k_\mu, G_v \rangle\!\rangle, \tag{3.19}$$

$$G_v(\eta) = \exp\left(- \sum_{x\in\eta} v(x)\psi(x) \right) \sum_{\xi \subset \eta} e(\xi; h_v) G(\eta \setminus \xi).$$

where $e(\cdot; h_v))$ is as in (3.17).

(b) $\widetilde{\mathcal{F}} \subset \mathcal{F}_{\max}$, *where the latter set is defined in* (3.11).

(c) *For each $\mu \in \mathcal{P}_{\rm exp}$ and $F \in \widetilde{\mathcal{F}}$, it follows that $\mu(|LF|) < \infty$.*

Proof. The validity of the bound for $\varkappa_{\mu_v}$ readily follows from (2.14) and (3.16). To prove the validity of (3.19) we use the following formula, see [14, Definition 3.2],

$$(KG_1)(\gamma)(KG_2)(\gamma) = K(G_1 \star G_2)(\gamma), \tag{3.20}$$

$$(G_1 \star G_2)(\eta) = \sum_{\xi_1 \subset \eta} \sum_{\xi_2 \subset \eta \setminus \xi_1} G_1(\xi_1 \cup \xi_2) G_2(\eta \setminus \xi_2),$$

where both G_1, G_1 are suitable functions on $\Gamma_{\rm fin}$. For G as in (3.19), by (2.19) and (3.20), and then by (2.3), we get

$$\begin{aligned}
\langle\!\langle k_{\mu_v}, G \rangle\!\rangle &= C_{v,\mu} \langle\!\langle k_\mu, e(\cdot, h_v) \star G \rangle\!\rangle \qquad (3.21)\\
&= C_{v,\mu} \int_{\Gamma_{\rm fin}} k_\mu(\eta) \sum_{\xi_1 \subset \eta} \sum_{\xi_2 \subset \eta \setminus \xi_1} e(\xi_1 \cup \xi_2; h_v) G(\eta \setminus \xi_2) \lambda(d\eta)\\
&= C_{v,\mu} \int_{\Gamma^3_{\rm fin}} k_\mu(\eta \cup \xi_1 \cup \xi_2) e(\xi_1; h_v) e(\xi_2; h_v) G(\eta \cup \xi_1) \lambda(d\eta) \lambda(d\xi_1) \lambda(d\xi_2)\\
&= C_{v,\mu} \int_{\Gamma^2_{\rm fin}} k_\mu(\eta \cup \xi_2) e(\xi_2; h_v) G(\eta) \left(\sum_{\xi_1 \subset \eta} e(\xi_1; h_v) \right) \lambda(d\eta) \lambda(d\xi_2)\\
&= C_{v,\mu} \int_{\Gamma_{\rm fin}} k_\mu(\eta) e(\eta; 1 + h_v) \left(\sum_{\xi \subset \eta} e(\xi; h_v) G(\eta \setminus \xi) \right) \lambda(d\eta),
\end{aligned}$$

which yields (3.19) if one takes into account the following evident equality, see (3.17),

$$\sum_{\xi \subset \eta} \prod_{x \in \xi} h_v(x) = \prod_{x\in\eta} (1 + h_v(x)) = \exp\left(- \sum_{x\in\eta} v(x)\psi(x) \right).$$

To prove the validity of (b), we have to show that $e(\cdot; h_v) \in \mathcal{G}_\alpha$, $\alpha \in \mathbb{R}$. By (2.34) it follows that

$$
\begin{aligned}
|e(\cdot; h_v)|_\alpha &= \int_{\Gamma_{\rm fin}} e^{\alpha|\xi|} \prod_{x\in\xi}\left(1 - e^{-v(x)\psi(x)}\right)\lambda(d\xi) \qquad (3.22)\\
&\leq \int_{\Gamma_{\rm fin}} e^{\alpha|\xi|}\left(\prod_{x\in\xi} v(x)\psi(x)\right)\lambda(d\xi)\\
&\leq \exp\left(e^\alpha \|v\|\langle\psi\rangle\right), \qquad \|v\| = \sup_{x\in X} v(x),
\end{aligned}
$$

which completes the proof of item (b). By the same calculations as in (3.21) and (3.22) one shows that the functions that appear (3.19) satisfy

$$
|G_v|_\alpha \leq \exp\left(e^\alpha \|v\|\langle\psi\rangle\right)|G|_\alpha \qquad (3.23)
$$

holding for all $\alpha \in \mathbb{R}$.

By (3.16) one gets

$$
\begin{aligned}
\widetilde{F}_v(\gamma\cup x) - \widetilde{F}_v(\gamma) &= -\widetilde{F}_v(\gamma)\left[1 - e^{-v(x)\psi(x)}\right], \qquad (3.24)\\
|\widetilde{F}_v(\gamma\cup x) - \widetilde{F}_v(\gamma)| &\leq v(x)\psi(x),
\end{aligned}
$$

cf. (3.22). Then by (1.1) it follows that

$$
|L^+\widetilde{F}_v(\gamma)| \leq \int_X b(x)v(x)\psi(x)dx \leq \|b\|\|v\|\langle\psi\rangle, \qquad (3.25)
$$

see (2.23). In dealing with $L^-\widetilde{F}_v$, we first take $\widetilde{F}_{v_n}$ with $v_n = v\mathbb{1}_{\Delta_n}$, $\Delta_n := \{x \in X : |x| \leq n\}$. In this case, by (3.24) and (3.4) we get

$$
\begin{aligned}
|L^-\widetilde{F}_{v_n}(\gamma)| &\leq \|m\|\|v\|\sum_{x\in\gamma}\psi(x) + \sum_{x\in\gamma} v_n(x)\psi(x)\sum_{y\in\gamma\setminus x} a(x-y) \qquad (3.26)\\
&\leq \|m\|\|v\|\gamma(\psi) + \gamma(\psi)\|a\|\sum_{x\in\gamma} v_n(x)\ell_a(x) < \infty.
\end{aligned}
$$

At the same time, by (2.19), (2.20), the first line in (3.26) and (3.2) it follows that

$$
\begin{aligned}
\mu(|L^-\widetilde{F}_{v_n}|) &\leq \|m\|\|v\|\int_X k_\mu^{(1)}(x)dx + \int_{X^2} k_\mu^{(2)}(x,y)v_n(x)\psi(x)a(x-y)dxdy \qquad (3.27)\\
&\leq \|v\|\varkappa_\mu\langle\psi\rangle(\|m\| + \varkappa_\mu\langle a\rangle).
\end{aligned}
$$

By the monitione convergence theorem one then gets that

$$
\mu(|L^-\widetilde{F}_v|) \leq \mathrm{RHS}(3.27),
$$

which together with the estimate in (3.25) yields the proof of claim (c). □

Remark 3.6. By (3.27) one readily gets the following extension of claim (c) of Proposition 3.5. For a subset, $\mathcal{P} \subset \mathcal{P}_{\rm exp}$, assume that $\sup_{\mu\in\mathcal{P}} \varkappa_\mu =: \varkappa < \infty$. Then

$$
\sup_{\mu\in\mathcal{P}} \mu(|L\widetilde{F}_v|) \leq \|v\|\langle\psi\rangle(\|b\| + \|m\|\varkappa + \langle a\rangle\varkappa^2).
$$

Fix now some $t_2 > t_1 \geq 0$, $\widetilde{F}_v$ as in (3.16) and consider

$$
\mathsf{Q}^\pm(\bar\gamma) = \int_{t_1}^{t_2} (L^\pm\widetilde{F}_v)(\varpi_u(\bar\gamma))du, \qquad \mathsf{Q} = \mathsf{Q}^+ + \mathsf{Q}^-.
$$

Clearly, Q is $\mathfrak{F}_{[t_1,+\infty)}$-measurable. By the first line of (3.24) and (3.25) Q^+ is bounded, but Q^- may take value $-\infty$.

Proposition 3.7. *Let $P \in \mathcal{P}(\mathfrak{D}_{[0,+\infty)}(\Gamma_*))$ be such that $P \circ \varpi_u^{-1} \in \mathcal{P}_{\rm exp}$ for all $u \geq 0$. Moreover, for each $t > 0$, assume that $\sup_{u\in[0,t]} \varkappa_u =: \varkappa < \infty$, where $\varkappa_u$ is the type of $P \circ \varpi_u^{-1}$, see Definition 2.3. Then $P(|\mathsf{Q}|) < \infty$.*

Proof. As mentioned above, Q^+ is bounded; hence, it remains to prove that $P(|\mathsf{Q}^-|) < \infty$. First, as in (3.26) we take

$$\mathsf{Q}_n^-(\bar{\gamma}) = \int_{t_1}^{t_2} (L^- \widetilde{F}_{v_n})(\varpi_u(\bar{\gamma}))du, \qquad n \in \mathbb{N}.$$

Note that $|\mathsf{Q}_n^-(\bar{\gamma})| = -\mathsf{Q}_n^-(\bar{\gamma})$, see (3.24), and $\mathsf{Q}_n^- : \mathfrak{D}_{[0,+\infty)}(\Gamma_*) \to \mathbb{R}$, see (3.26). By the assumption of this statement it follows that $P \circ \varpi_u^{-1} =: \mu_u \in \mathcal{P}_{\rm exp}$; hence,

$$\int_{t_1}^{t_2} P(|(L^- \widetilde{F}_{v_n}) \circ \varpi_u|)du = \int_{t_1}^{t_2} (P \circ \varpi_u^{-1})(|(L^- \widetilde{F}_{v_n})|)du \tag{3.28}$$

$$= \int_{t_1}^{t_2} \mu_u(|(L^- \widetilde{F}_{v_n})|)du \leq (t_2 - t_1)\|v\|\varkappa\langle\psi\rangle(\|m\| + \varkappa\langle a\rangle),$$

see (3.27) and Remark 3.6. By the Tonelli and Fubini theorems, see [5, Theorems 4.4 and 4.5, page 91], it then follows

$$P(|\mathsf{Q}_n^-|) \leq \mathrm{RHS}(3.28),$$

which by the monotone convergence theorem yields the proof. □

For some $s \geq 0$ and $0 \leq t_1 < t_2$, let $\mathsf{J} : \mathfrak{D}_{[s,+\infty)} \to \mathbb{R}$ be $\mathfrak{F}_{s,t_1}$-measurable. Then for $F \in \widetilde{\mathcal{F}}$, define

$$\mathsf{H}(\bar{\gamma}) = \left[F(\varpi_{t_2}(\bar{\gamma})) - F(\varpi_{t_1}(\bar{\gamma})) - \int_{t_1}^{t_2} (LF)(\varpi_u(\bar{\gamma}))du\right] \mathsf{J}(\bar{\gamma}). \tag{3.29}$$

The next definition is an adaptation of the corresponding definition in [8, Sect. 5.1, pages 78, 79], see also [15, Definition 3.3]. Herein, for $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, we deal with probability measures on $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$, cf. Proposition 3.7.

Definition 3.8. *A family of probability measures $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ is said to be a solution of the restricted martingale problem if for all $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, the following holds: (a) $P_{s,\mu} \circ \varpi_s^{-1} = \mu$; (b) $P_{s,\mu} \circ \varpi_u^{-1} \in \mathcal{P}_{\rm exp}$ for all $u > s$; (c) for all $t > s$, the types $\varkappa_u$ of $P_{s,\mu} \circ \varpi_u^{-1}$ satisfy $\sup_{u\in[s,t]} \varkappa_u < \infty$; (d) $P_{s,\mu}(\mathsf{H}) = 0$, holding for H as in* (3.29) *with each $F \in \widetilde{\mathcal{F}}$ and every bounded function $\mathsf{J} : \mathfrak{D}_{[s,+\infty)}(\Gamma_*) \to \mathbb{R}$ which is $\mathfrak{F}_{s,t_1}$-measurable, see* (2.28)*. The restricted martingale problem is said to be well-posed if it has a unique solution in the following sense: if $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ and $\{P'_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ solve the problem, then all finite dimensional marginals of $P_{s,\mu}$ and $P'_{s,\mu}$ coincide for all s and μ, see* [11, page 182]*.*

Remark 3.9. Concerning the notions introduced in Definition 3.8 one should remark the following:

(a) By Proposition 3.7 H as given in (3.29) is absolutely $P_{s,\mu}$-integrable for each s and μ.

(b) The map $[0, +\infty) \ni t \mapsto P_{0,\mu_0} \circ \varpi_t$, $t > 0$ solves the Fokker-Planck equation for $(L, \mathcal{F}, \mu_0)$. Indeed, by claim (c) of Theorem 3.3 and claim (b) of Proposition 3.5, the solution of the Fokker-Planck equation (1.2) for $(L, \mathcal{F}, \mu)$ solves this equation also with each $F \in \widetilde{\mathcal{F}}$. At the same time, the map in question solves (1.2) with each $F \in \widetilde{\mathcal{F}}$, which can be obtained by taking $\mathsf{J} \equiv 1$ and interchanging integrations as in the proof of Proposition 3.7. Since the solution of (1.2) is unique the two discussed solutions coincide, which yields the mentioned property.

(c) The adjective "restricted" points to condition (b) of Definition 3.8 wich forces the one dimensional marginals to be in $\mathcal{P}_{\rm exp}$.

Another important remark is that the function J in (3.29) can be taken in the form

$$\mathsf{J}(\bar{\gamma}) = J_1(\varpi_{s_1}(\bar{\gamma})) \cdots J_m(\varpi_{s_m}(\bar{\gamma})), \tag{3.30}$$

with all possible choices of $m \in \mathbb{N}$, $J_1, \dots, J_m \in C_{\rm b}^{+}(\Gamma_*)$ such that $J_l(\gamma) > 0$ for all $\gamma \in \Gamma_*$, $l = 1, \dots, m$, and $s \leq s_1 < s_2 < \cdots < s_m \leq t_1$, see eq. (3.4) on page 174 of [11]. We are going to use this in the proof of Theorem 3.10 which we formulate now.

Theorem 3.10. *Let L and $\widetilde{\mathcal{F}}$ be as* (1.1) *and Proposition 3.4, respectively. Then*

(a) *the restricted martingale problem has a unique solution in the sense of Definition 3.8;*

(b) *the stochastic process related to the family*

$$\{\mathfrak{D}_{[s,+\infty)}(\Gamma_*), \mathfrak{F}_{s,+\infty}, \{\mathfrak{F}_{s,t} : t \geq s\}, \{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\} : s \geq 0\}$$

is Markov.

3.3. **The scheme of the proof of both theorems and comments.**

3.3.1. *Concerning Theorem 3.3.* As mentioned above, in [13] there was proved the existence of a map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$, which describes the evolution of states of the model with the generator (1.1). It was done by constructing the evolution of the corresponding correlation functions, see Proposition 5.3 below. However, it has remained unclear whether this map describes the evolution in question in a unique way – an effect of the lack of a 'canonical' way of constructing solutions, e.g., by means of a C_0-semigroup. Our approach to this problem is based on the expectation that the Fokker-Planck equation (1.2) – being a weaker version of the evolution equation – would be more accessible for solving it by existing methods. In this case, however, the weakness just mentioned imposes the necessity of *specifying* its solutions, which we do by means of the triple $(L, \mathcal{F}, \mu_0)$, see Definition 3.2. In view of this, the choice of $\mathcal{F}$ and of the class of initial states becomes a crucial aspect of the theory. As a benefit, one can raise and solve the problem of uniqueness of solutions understood as the coincidence of any two of them specified by the same triple. In Theorem 3.3 we state that the solution constructed in [13] is the unique solution of the the Fokker-Planck equation (1.2) corresponding to the triple $(L, \mathcal{F}, \mu_0)$ with $\mu_0 \in \mathcal{P}_{\rm exp}$. This is done in the following steps.

- In order to be able to control the sign of the quadratic term in L^-, we introduce functions $F : \Gamma \to \mathbb{R}$, see (3.10), in such a way that $\pm L^{\pm} F \geq 0$, which is done in subsect. 4.2, see (4.18). This allows one to control $\mu_t(F)$ for possible solutions μ_t, see (4.24), (4.25), and thereby to extend the domain of L to a special class of unbounded functions, see Lemma 4.2. By means of this extension we then prove, see Lemma 4.4, that each solution lies in $\mathcal{P}_{\rm exp}$, which is a key point of the whole theory.
- In Lemma 4.1, we prove the inclusion $\mathcal{F} \subset \mathcal{F}_{\rm max}$, see claim (c) of Theorem 3.3. By (2.19) this result allows one to pass to the correlation functions and thus to prove (see Section 5) that the evolution $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ constructed in [13] solves the Fokker-Planck equation for any $F \in \mathcal{F}_{\rm max}$ whenever $\mu_0 \in \mathcal{P}_{\rm exp}$. The proof of uniqueness is mainly based on the fact proved in Lemma 4.4. Note, however, that the presence of the quadratic term in L^- produces essential difficulties also in this part.

3.3.2. *Concerning Theorem 3.10.* In constructing the Markov process describing the stochastic evolution of the model corresponding to L we mostly follow the way elaborated in our works [15, 16]. It is based on solving the restricted martingale problem, which in an intrinsic way is related with the Fokker-Planck equation. To see this, it is enough to compare (1.2) and (3.29). In particular, uniqueness in this case is a consequence of the same property proved in Theorem 3.3. The proof consists in the following steps:

- We modify the model described by L given in (1.1) by multiplying the model parameters by a certain function ψ_σ, see (6.1), (6.2). As a result, we obtain the family of generators $\{L_\sigma : \sigma \in (0,1]\}$ corresponding to so called 'auxiliary models'. For these models, all the results of [13] and the first part of this work hold true. At the same time, the mentioned modification allows one to directly construct Markov transition functions p_t^σ, see subsect. 6.2, which is impossible for the initial model. By standard methods one then gets, see (7.4), the finite dimensional laws of the Markov processes corresponding to the auxiliary models.
- As mentioned above, by means of methods used in the first part we construct the evolution $t \mapsto \mu_t^\sigma \in \mathcal{P}_{\rm exp}$. At the same time, for $\sigma \in (0,1]$, the transition functions p_t^σ define the evolution $t \mapsto \hat{\mu}_t^\sigma$ by the formula $\hat{\mu}_t^\sigma(\cdot) = \int p_t^\sigma(\gamma, \cdot)\mu_0(d\gamma)$, cf. (6.34). In Lemma 6.6, we show that $\hat{\mu}_t^\sigma = \mu_t^\sigma$, which is one of the crucial points of the construction. Then in Lemma 7.5 we prove that $\mu_t^\sigma \Rightarrow \mu_t$ as $\sigma \to 0$, where the latter is the evolution constructed in Theorem 3.3. This fact and the Chentsov-like estimates obtained in Lemma 7.3 yield the following: (a) the Markov processes corresponding to L_σ, $\sigma \in (0,1]$, have cadlag paths; (b) these processes weakly converge as $\sigma \to 0$ to a unique process with cadlag paths, which is the process in question.

4. Properties of Possible Solutions of the Fokker-Planck Equation

In this section, we obtain a number of a priori properties of the solutions of (1.2) which then allow us to develop the tools for proving Theorem 3.3.

4.1. **A property of the domain.** The principal technical aspect of our approach consists in dealing with correlation functions of the states in question rather than with the states themselves. In view of this, we use their relationship expressed in (2.19), which is based on the possibility to present the elements of $\mathcal{F}$ in the form $F = KG$, see (2.7). Recall that the K-map can be extended to $\mathcal{G}_\alpha$, see Remark 2.9. Below by K we understand this extension.

Lemma 4.1. *For each $\alpha \in \mathbb{R}$, finite $\vartheta \subset \Theta$, see* (3.9)*, and $\tau \in (0, 1/2]$, there exists a unique $G_\tau^\vartheta \in \mathcal{G}_\alpha$ such that $\Psi_\tau^\vartheta = KG_\tau^\vartheta$. In other words, $\mathcal{F} \subset \mathcal{F}_{\max}$.*

Proof. Fix some $\vartheta \subset \Theta$, and write $\vartheta = \{\theta_1, \dots, \theta_n\}$, $n \in \mathbb{N}$. By (3.7) and (3.8) one writes

$$\Psi_\tau^\vartheta(\gamma) = \int_0^{+\infty} \cdots \int_0^{+\infty} e^{-(\beta_1 + \cdots + \beta_n)} \Psi^\vartheta(\gamma) \exp\left(-\tau \sum_{x\in\gamma} \sum_{j=1}^n \beta_j \theta_j(x) \right) d\beta_1 \cdots d\beta_n, \quad (4.1)$$

where

$$\Psi^\vartheta(\gamma) = \prod_{\theta\in\vartheta} \Phi^\theta(\gamma) = \left(\sum_{x\in\gamma} \theta_1(x) \right) \cdots \left(\sum_{x\in\gamma} \theta_n(x) \right). \quad (4.2)$$

For a positive integer $l \leq n$, let $d = (\delta_1, \dots, \delta_l)$ be a division of $\{1, \dots, n\}$ into nonempty subsets satisfying $|\delta_j| \geq |\delta_k|$ for $j < k$. Let also $\mathfrak{d}_l$ be the set of all such divisions. Noteworthy, its cardinality is $S(n, l)$ – Stirling's number of second kind. For $d \in \mathfrak{d}_l$, we set

$$\widehat{\theta}_k(x) = \prod_{i\in\delta_k} \theta_i(x), \qquad k = 1, \dots, l. \quad (4.3)$$

Then by (2.9), (2.10) one gets the following relation, inverse to that in (2.4),

$$\begin{aligned} \varPsi^{\vartheta}(\gamma) &= \sum_{l=1}^{n}\sum_{d\in\mathfrak{d}_l}\left(\sum_{x_1\in\gamma}\widehat{\theta}_1(x_1)\sum_{x_2\in\gamma\setminus x_1}\widehat{\theta}_2(x_2)\cdots\sum_{x_l\in\gamma\setminus\{x_1,\dots,x_{l-1}\}}\widehat{\theta}_l(x_l)\right) \\ &= \sum_{l=1}^{n} l!\sum_{d\in\mathfrak{d}_l}(KG^{\widehat{\vartheta}})(\gamma), \qquad \widehat{\vartheta}:=\{\widehat{\theta}_1,\dots,\widehat{\theta}_l\}, \end{aligned} \tag{4.4}$$

where $G^{\widehat{\vartheta}}$ is as in (2.9). Note that each $\widehat{\theta}_k$ is in Θ – since $\psi(x)\leq 1$, see (2.22). Now we set, cf. (3.17),

$$H^{\vartheta}_{\beta}(\eta)=\prod_{x\in\eta}\left(e^{-\tau\tilde{\theta}_\beta(x)}-1\right), \qquad \tilde{\theta}_\beta(x)=\sum_{j=1}^{n}\beta_j\theta_j(x), \qquad \eta\in\Gamma_{\rm fin}. \tag{4.5}$$

Then

$$\begin{aligned} &\exp\left(-\tau\sum_{x\in\gamma}\sum_{j=1}^{n}\beta_j\theta_j(x)\right)=\exp\left(-\tau\sum_{x\in\gamma}\tilde{\theta}_\beta(x)\right) \\ &=\prod_{x\in\gamma}\left(1+[e^{-\tau\tilde{\theta}_\beta(x)}-1]\right)=\sum_{\eta\Subset\gamma}H^{\vartheta}_{\beta}(\eta)=(KH^{\vartheta}_{\beta})(\gamma), \end{aligned}$$

and further, see (4.1),

$$\varPsi^{\vartheta}_{\tau}(\gamma)=\sum_{l=1}^{n}\sum_{d\in\mathfrak{d}_l}\int_0^{+\infty}\cdots\int_0^{+\infty}e^{-(\beta_1+\cdots+\beta_n)}l!(KG^{\widehat{\vartheta}})(\gamma)(KH^{\vartheta}_{\beta})(\gamma)d\beta_1\cdots d\beta_n. \tag{4.6}$$

In view of (3.20), we have

$$\begin{aligned} (KG^{\widehat{\vartheta}})(\gamma)(KH^{\vartheta}_{\beta})(\gamma) &= (K(G^{\widehat{\vartheta}}\star H^{\vartheta}_{\beta}))(\gamma), \\ (G^{\widehat{\vartheta}}\star H^{\vartheta}_{\beta})(\eta) &= \sum_{\xi_1\subset\eta}\sum_{\xi_2\subset\eta\setminus\xi_1}G^{\widehat{\vartheta}}(\xi_1\cup\xi_2)H^{\vartheta}_{\beta}(\eta\setminus\xi_2). \end{aligned} \tag{4.7}$$

Note that the sums in the right-hand side of (4.7) are finite since η is a finite multiset. This allows for interchanging K and the integration in (4.6). Then (4.6) implies

$$\varPsi^{\vartheta}_{\tau}=KG^{\vartheta}_{\tau},$$

with

$$G^{\vartheta}_{\tau}(\eta)=\sum_{l=1}^{n}\sum_{d\in\mathfrak{d}_l}\int_0^{+\infty}\cdots\int_0^{+\infty}e^{-(\beta_1+\cdots+\beta_n)}l!(G^{\widehat{\vartheta}}\star H^{\vartheta}_{\beta})(\eta)d\beta_1\cdots d\beta_n. \tag{4.8}$$

To prove the lemma we have to estimate the norms of G^{ϑ}_{τ}, see (2.31) and (2.32). To interchange the β-integration with that over $\Gamma_{\rm fin}$ we employ the Tonelli-Fubini theorems [5, Theorems 4.4 and 4.5, page 91]. By the evident inequality $1-e^{-u}\leq\sqrt{2u}$, $u\geq 0$, we have, cf. (4.5),

$$|H^{\vartheta}_{\beta}(\eta)|\leq\prod_{x\in\eta}(1-e^{-\tau\tilde{\theta}_\beta(x)})\leq(2\tau)^{|\eta|/2}e(\eta;\theta_\beta)\leq e(\eta;\theta_\beta), \quad \theta_\beta(x):=\sqrt{\tilde{\theta}_\beta(x)}, \tag{4.9}$$

see (2.16). According to (2.31) and by means of (2.3), (4.9) we then get

$$|G^{\widehat{\vartheta}} \star H^{\vartheta}_{\beta}|_\alpha \leq \int_{\Gamma_{\rm fin}} e^{\alpha|\eta|} \left(\sum_{\xi_1 \subset \eta} \sum_{\xi_2 \subset \eta \setminus \xi_1} G^{\widehat{\vartheta}}(\xi_1 \cup \xi_2) e(\eta \setminus \xi_2; \theta_\beta) \right) \lambda(d\eta) \tag{4.10}$$

$$= \int_{\Gamma^2_{\rm fin}} e^{\alpha|\eta| + \alpha|\xi_1|} \left(\sum_{\xi_2 \subset \eta} G^{\widehat{\vartheta}}(\xi_1 \cup \xi_2) e(\eta \setminus \xi_2 \cup \xi_1; \theta_\beta) \right) \lambda(d\eta)\lambda(d\xi_1)$$

$$= \int_{\Gamma^3_{\rm fin}} e^{\alpha|\eta| + \alpha|\xi_1| + \alpha|\xi_2|} G^{\widehat{\vartheta}}(\xi_1 \cup \xi_2) e(\xi_1; \theta_\beta) e(\eta; \theta_\beta) \lambda(d\eta)\lambda(d\xi_1)\lambda(d\xi_2)$$

$$= \exp\left(e^\alpha \int_X \theta_\beta(x) dx \right) \int_{\Gamma^2_{\rm fin}} e^{\alpha|\xi_1| + \alpha|\xi_2|} G^{\widehat{\vartheta}}(\xi_1 \cup \xi_2) e(\xi_1; \theta_\beta) \lambda(d\xi_1)\lambda(d\xi_2).$$

Here we have taken into account that

$$e(\xi_1 \cup \xi_2; \theta_\beta) = e(\xi_1; \theta_\beta) e(\xi_2; \theta_\beta),$$

see (2.16), and also (2.21). By (2.9) and the latter we have

$$\Upsilon_l := l! \int_{\Gamma^2_{\rm fin}} e^{\alpha|\xi_1| + \alpha|\xi_2|} G^{\widehat{\vartheta}}(\xi_1 \cup \xi_2) e(\xi_1; \theta_\beta) \lambda(d\xi_1)\lambda(d\xi_2) \tag{4.11}$$

$$= l! \int_{\Gamma_{\rm fin}} e^{\alpha|\eta|} G^{\widehat{\vartheta}}(\eta) \left(\sum_{\xi \subset \eta} e(\xi; \theta_\beta) \right) \lambda(d\eta).$$

Now we recall that all $\theta_i \in \vartheta$ are in Θ, see (3.9); hence, $\theta_i(x) \leq 1$. By (4.5) and (4.9) we then have

$$0 \leq \theta_\beta(x) \leq \omega_\beta := \sqrt{\beta_1 + \cdots + \beta_n}, \tag{4.12}$$

holding for all $x \in X$. Hence, $e(\xi; \theta_\beta) \leq \omega_\beta^{|\xi|}$, see (2.16). By means of this estimate and (2.9), (4.3) we get in (4.11) the following one

$$\Upsilon_l \leq l! \int_{\Gamma_{\rm fin}} [e^\alpha (1 + \omega_\beta)]^{|\eta|} G^{\widehat{\vartheta}}(\eta) \lambda(d\eta) \tag{4.13}$$

$$= [e^\alpha (1 + \omega_\beta)]^l \prod_{k=1}^{l} \langle \widehat{\theta}_k \rangle \leq [e^\alpha \langle \psi \rangle (1 + \omega_\beta)]^l,$$

see (2.14) and (3.9). Here we used the upper bound $\langle \theta \rangle \leq \langle \psi \rangle$, see (3.9), which yields an estimate uniform in ϑ. Now we employ (4.13) and (4.12) in (4.10) and finally get

$$l! |G^{\widehat{\vartheta}} \star H^{\vartheta}_{\beta}|_\alpha \leq \exp\left(e^\alpha \langle \psi \rangle \omega_\beta \right) [e^\alpha \langle \psi \rangle (1 + \omega_\beta)]^l,$$

see also (2.23). This yields in (4.8) the following

$$|G^{\vartheta}_{\tau}|_\alpha \leq \int_0^{+\infty} \cdots \int_0^{+\infty} \exp\left(-\beta_1 \cdots - \beta_n + e^\alpha \langle \psi \rangle \omega_\beta \right) \tag{4.14}$$

$$\times \sum_{l=1}^{n} S(n,l) [e^\alpha \langle \psi \rangle (1 + \omega_\beta)]^l d\beta_1 \cdots d\beta_n$$

$$= \int_0^{+\infty} \cdots \int_0^{+\infty} \exp\left(-\beta_1 \cdots - \beta_n \right) W_\alpha(\beta_1, \ldots, \beta_n) d\beta_1 \cdots d\beta_n$$

where

$$W_\alpha(\beta_1, \ldots, \beta_n) := \exp\left(e^\alpha \langle \psi \rangle \omega_\beta \right) T_n \left(e^\alpha \langle \psi \rangle (1 + \omega_\beta) \right),$$

Here $T_n(u) = \sum_{l=0}^n S(n,l)u^l$ is Touchard's polynomial, $\deg T_n = n$. Thus, the integral in the last line of (4.14) is obviously convergent for all $\alpha \in \mathbb{R}$, see (4.12). This completes the proof. □

4.2. **Useful estimates and their consequences.** Here we derive a number of estimates by means of which we then obtain properties of possible solution of the Fokker-Planck equation (1.2) basing on Definition 3.2 with the domain $\mathcal{F}$ defined in (3.10). As a result, we extend the domain of L to some unbounded functions, which in particular will allow us to prove that each solution lies in $\mathcal{P}_{\rm exp}$.

For $F \in \mathcal{F}$, write

$$\nabla_x F(\gamma) = F(\gamma \cup x) - F(\gamma), \qquad x \in X. \tag{4.15}$$

By taking $F = \Phi^\theta_\tau$, see (3.7), we then get

$$\nabla_x \Phi^\theta_\tau(\gamma) = \frac{\theta(x)}{[1+\tau\Phi^\theta_\tau(\gamma\cup x)][1+\tau\Phi^\theta_\tau(\gamma)]},$$

which immediately yields

$$0 \le \nabla_x \Phi^\theta_\tau(\gamma) \le \theta(x), \qquad \frac{\partial}{\partial\tau}\nabla_x \Phi^\theta_\tau(\gamma) \le 0, \tag{4.16}$$

Next, see (3.8),

$$\begin{aligned} 0 \le \nabla_x \Psi^\vartheta_\tau(\gamma) &= \prod_{\theta\in\vartheta}\Big[\nabla_x\Phi^\theta_\tau(\gamma) + \Phi^\theta_\tau(\gamma)\Big] - \prod_{\theta\in\vartheta}\Phi^\theta_\tau(\gamma) \\ &\le \sum_{\varnothing\neq\vartheta'\subset\vartheta}\left(\prod_{\theta\in\vartheta'}\theta(x)\right)\Psi^{\vartheta\setminus\vartheta'}_\tau(\gamma). \end{aligned} \tag{4.17}$$

By (1.1) we then conclude that

$$\pm L^{\pm}\Psi^\vartheta_\tau(\gamma) \ge 0, \tag{4.18}$$

and also

$$L^+\Psi^\vartheta_\tau(\gamma) \le \sum_{\varnothing\neq\vartheta'\subset\vartheta}\left(\int_X b(x)\prod_{\theta\in\vartheta'}\theta(x)dx\right)\Psi^{\vartheta\setminus\vartheta'}_\tau(\gamma) \le \|b\|\langle\psi\rangle\sum_{\varnothing\neq\vartheta'\subset\vartheta}\Psi^{\vartheta\setminus\vartheta'}_\tau(\gamma), \tag{4.19}$$

where $\langle\psi\rangle = \int\psi(x)dx$, see (2.23) and (3.9). By (4.18) it follows that

$$\mu_t(\Psi^\vartheta_\tau) \le \mu_0(\Psi^\vartheta_\tau) + \int_0^t \mu_s(L^+\Psi^\vartheta_\tau)ds, \tag{4.20}$$

which should hold for any solution μ_t. By (4.1) we have

$$\Psi^\vartheta_\tau(\gamma) \le \Psi^\vartheta(\gamma), \qquad \lim_{\tau\to 0}\Psi^\vartheta_\tau(\gamma) = \Psi^\vartheta(\gamma). \tag{4.21}$$

For $\mu_0 \in \mathcal{P}_{\rm exp}$, similarly as in (4.13) by means of (4.21), (4.4) and (2.14) we obtain

$$\begin{aligned} \mu_0(\Psi^\vartheta_\tau) \le \mu_0(\Psi^\vartheta) &= \sum_{l=1}^{|\vartheta|}\sum_{d\in\mathfrak{d}_l}\int_{\Gamma_{\rm fin}} k^{(l)}_{\mu_0}(x_1,\dots,x_l)\widehat{\theta}_1(x_1)\cdots\widehat{\theta}_l(x_l)dx_1\cdots dx_l \\ &\le \sum_{l=1}^{|\vartheta|}\sum_{d\in\mathfrak{d}_l}\varkappa^l_{\mu_0}\langle\widehat{\theta}_1\rangle\cdots\langle\widehat{\theta}_l\rangle \le \sum_{l=1}^{|\vartheta|}S(|\vartheta|,l)\,[\varkappa_{\mu_0}\langle\psi\rangle]^l = T_{|\vartheta|}(\varkappa_{\mu_0}\langle\psi\rangle), \end{aligned} \tag{4.22}$$

where $\varkappa_{\mu_0}$ is the type of μ_0, see Definition 2.3. Recall that T_n stands for Touchard's polynomial. By (4.16), (4.20) and (4.22) we then get

$$\mu_t(\Phi^\theta_\tau) \le T_1(\varkappa_{\mu_0}\langle\psi\rangle) + \|b\|\langle\psi\rangle t, \qquad \theta\in\Theta,$$

see (3.1) and (3.9). Now by (4.17) this can be generalized to the following recursion

$$\mu_t(\Psi^{\vartheta}_\tau) \leq T_{|\vartheta|}(\varkappa_{\mu_0}\langle\psi\rangle) + \int_0^t \left(\sum_{\varnothing\neq\vartheta'\subset\vartheta} \mu_s(\Psi^{\vartheta\setminus\vartheta'}_\tau) \right) ds, \tag{4.23}$$

by which we obtain

$$\mu_t(\Psi^{\vartheta}_\tau) \leq Q_{|\vartheta|}(t) := \sum_{l=0}^{|\vartheta|} \frac{1}{l!} T_{|\vartheta|-l}(\varkappa_{\mu_0}\langle\psi\rangle) \left[\|b\|\langle\psi\rangle\right]^l t^l. \tag{4.24}$$

By this estimate and (4.19) we also get

$$\mu_t(L^+\Psi^{\vartheta}_\tau) \leq \|b\|\langle\psi\rangle \sum_{k=1}^{|\vartheta|} \binom{|\vartheta|}{k} Q_k(t) =: Q^+_{|\vartheta|}(t). \tag{4.25}$$

In particular, this yields that $\mu_t(L^+\Psi^{\vartheta}_\tau)$ is dominated by a polynomial in t, which yields in turn the local integrability of the map $t \mapsto \mu_t(L^+\Psi^{\vartheta}_\tau)$. Then the $\mu_t(d\gamma)dt$-integrability of $|L^-\Psi^{\vartheta}_\tau(\gamma)|$ follows by the triangle inequality from the assumed corresponding integrability of $|L\Psi^{\vartheta}_\tau(\gamma)|$, see Definition 3.2, cf. (3.15), and (4.25).

Now we are able to prove the following important statement.

Lemma 4.2. *Let μ_t be a solution of the Fokker-Planch equation for $(L, \mathcal{F}, \mu_0)$ for some $\mu_0 \in \mathcal{P}_{\rm exp}$. Then it solves* (1.2) *also with $F = \Psi^{\vartheta}$, see* (4.2).

Proof. In view of (4.21), to prove the lemma one should show that, for all $t \geq 0$, the following holds

$$\lim_{\tau\to 0} \mu_t(\Psi^{\vartheta}_\tau) = \mu_t(\Psi^{\vartheta}), \qquad \lim_{\tau\to 0} \int_0^t \mu_s(L^{\pm}\Psi^{\vartheta}_\tau)ds = \int_0^t \mu_s(L^{\pm}\Psi^{\vartheta})ds. \tag{4.26}$$

By (3.7), (3.6) and (4.16) the maps $\tau \mapsto \mu_t(\Psi^{\vartheta}_\tau)$ and $\tau \mapsto \int_0^t \mu_s(L^{\pm}\Psi^{\vartheta}_\tau)ds$ are monotone. Then the convergence as in (4.26) follows by the monotone convergence theorem, the bounds in (4.23), (4.25) and the following one, see also (4.18),

$$\begin{aligned} -\int_0^t \mu_s(L^-\Psi^{\vartheta}_\tau)ds &= \int_0^t \mu_s(|L^-\Psi^{\vartheta}_\tau|)ds = \mu_0(\Psi^{\vartheta}_\tau) - \mu_t(\Psi^{\vartheta}_\tau) + \int_0^t \mu_s(L^+\Psi^{\vartheta}_\tau)ds \\ &\leq \mu_0(\Psi^{\vartheta}_\tau) + \int_0^t \mu_s(L^+\Psi^{\vartheta}_\tau)ds \leq \mu_0(\Psi^{\vartheta}) + \int_0^t Q^+_2(s)ds. \end{aligned}$$

This completes the proof. □

4.3. **Localizing the solutions.** The aim of this subsection is to prove that any solution of the Fokker-Planch equation for $(L, \mathcal{F}, \mu_0)$ with $\mu_0 \in \mathcal{P}_{\rm exp}$ lies in $\mathcal{P}_{\rm exp}$, that can be achieved with the help of Lemma 4.2.

For $\vartheta \subset \Theta$, $|\vartheta| = n$, define

$$F^{\vartheta}(\gamma) = \sum_{x_1\in\gamma} \theta_1(x_1) \sum_{x_2\in\gamma\setminus x_1} \theta_2(x_2) \cdots \sum_{x_n\in\gamma\setminus\{x_1,\dots,x_{n-1}\}} \theta_n(x_n). \tag{4.27}$$

By (2.9) and (2.10) it follows that $F^{\vartheta} = n!KG^{\vartheta}$, which by (2.14) yields

$$\mu(F^{\vartheta}) = \int_{X^n} k^{(n)}_\mu(x_1,\dots,x_n)\theta_1(x_1)\cdots\theta_n(x)dx_1\cdots dx_n \leq \varkappa^n_\mu\langle\theta_1\rangle\cdots\langle\theta_n\rangle, \tag{4.28}$$

holding for all $\mu \in \mathcal{P}_{\rm exp}$. At the same time, by (2.4) we get

$$F^{\vartheta} = \sum_{\mathbb{G}\subset\mathbb{K}_n} (-1)^{l_{\mathbb{G}}}\Psi^{\widehat{\vartheta}}, \qquad \widehat{\vartheta} = \{\widehat{\theta}_1, \dots \widehat{\theta}_{n_{\mathbb{G}}}\}, \tag{4.29}$$

where $\widehat{\theta}_j(x) = \prod_{i\in \mathbb{V}_j} \theta_i(x)$, $\mathbb{V}_j$ being the vertex set of the j-th connected component of $\mathbb{G}$.

Remark 4.3. Each $\widehat{\theta}_j$ that appears in (4.29) is in Θ, see (3.9). Then F^ϑ is a linear combination of $\Psi^{\widehat{\vartheta}}$ with $\widehat{\theta} \in \Theta$. Hence, by in Lemma 4.2 a solution, μ_t, solves (1.2) also with $F = F^\vartheta$ for any $\vartheta \subset \Theta$.

Then similarly as in (4.20) we get

$$\mu_t(F^\vartheta) \leq \mu_0(F^\vartheta) + \sum_{\theta\in\vartheta} \|b\| \langle \theta\rangle \int_0^t \mu_t(L^+ F^{\vartheta\setminus\theta}) ds.$$

The latter and (4.28) readily yield

$$\mu_t(F^\vartheta) \leq (\varkappa_{\mu_0} + \|b\| t)^n \, \langle \theta_1\rangle \cdots \langle \theta_n\rangle. \tag{4.30}$$

Thereby, we have proved the following statement.

Lemma 4.4. *For every $\mu_0 \in \mathcal{P}_{\rm exp}$, each solution μ_t of the Fokker-Planck equation* (1.2) *for $(L, \mathcal{F}, \mu_0)$ lies in $\mathcal{P}_{\rm exp}$ for all $t > 0$. Moreover, its type satisfies $\varkappa_{\mu_t} \leq \varkappa_{\mu_0} + \|b\| t$.*

For μ_t as in Lemma 4.4, by (2.25) it follows that $\mu_t(\Gamma_*) = 1$ for all $t \geq 0$, which by Remark 2.7 yields that each μ_t can be redefined as an element of $\mathcal{P}(\Gamma_*)$. At the same time, for $\gamma \in \Gamma_*$ and $\vartheta \subset \Theta$, by (4.17) and (4.18) one gets

$$\begin{aligned} |L^- \Psi^\vartheta_\tau(\gamma)| &\leq H_\vartheta(\gamma) \sum_{\varnothing \neq \vartheta' \subset \vartheta} \Psi_\tau^{\vartheta\setminus\vartheta'}(\gamma), \\ H_\vartheta(\gamma) &= \sum_{x\in\gamma} \theta_*(x) \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y)\right) \\ &\leq \sum_{x\in\gamma} \theta_*(x) \left(m(x) + \gamma(\psi) \|a\| \ell_a(x)\right) < \infty, \end{aligned}$$

see also (3.4). Here $\theta_* \in C^+_{\rm cs}(X)$ is such that $\theta(x) \leq \theta_*(x)$ for all $\theta \in \vartheta$ and $x \in X$.

5. Solving the Fokker-Planck equation

In this section, we prove Theorem 3.3. Here we essentially use the results of [13].

5.1. **Evolution of correlation functions.**

By means of the parameters of L, see Assumption 3.1, we introduce

$$E(\eta) = \sum_{x\in\eta} m(x) + \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a(x-y), \qquad \eta \in \Gamma_{\rm fin}. \tag{5.1}$$

Then we define an operator acting in the Banach space $\mathcal{K}_\alpha$, see (2.29), (2.30), by means of the following expression

$$(L^\Delta k)(\eta) = \sum_{x\in\eta} b(x) k(\eta\setminus x) - E(\eta) k(\eta) - \int_X \left(\sum_{y\in\eta} a(x-y)\right) k(\eta\cup x) dx. \tag{5.2}$$

The reason for this can be seen from the formula

$$\mu(LKG) = \langle\!\langle L^\Delta k_\mu, G\rangle\!\rangle, \tag{5.3}$$

valid for $\mu \in \mathcal{P}_{\rm exp}$ and appropriate functions G. Its more precise meaning will be clarified in Lemma 5.4 below.

Set

$$\mathcal{D}_\alpha = \{k \in \mathcal{K}_\alpha : L^\Delta k \in \mathcal{K}_\alpha\}, \qquad \alpha \in \mathbb{R}. \tag{5.4}$$

Keeping in mind the embedding as in (2.30), for $\alpha' < \alpha$ we introduce now a linear operator $L^{\Delta}_{\alpha\alpha'} : \mathcal{K}_{\alpha'} \to \mathcal{K}_{\alpha}$ which acts according to (5.2). It turns out that this operator is bounded as its operator norm satisfies, see [13, eq. (2.26)],

$$\|L^{\Delta}_{\alpha\alpha'}\| \leq \frac{4\|a\|}{e^2(\alpha-\alpha')^2} + \frac{\|b\|e^{-\alpha'} + \|m\| + \langle a\rangle e^{\alpha}}{e(\alpha-\alpha')}, \tag{5.5}$$

where $\langle a\rangle$ is as in (3.2). Let $\mathcal{L}_{\alpha\alpha'}$ stand for the Banach space of bounded linear operators $A : \mathcal{K}_{\alpha'} \to \mathcal{K}_{\alpha}$. Then

$$L^{\Delta}_{\alpha\alpha'} \in \mathcal{L}_{\alpha\alpha'} \qquad \text{and} \qquad L^{\Delta}_{\alpha\alpha'}|_{\mathcal{K}_{\alpha''}} = L^{\Delta}_{\alpha\alpha''} \qquad \text{for} \qquad \alpha'' < \alpha'. \tag{5.6}$$

In view of (2.30), we also have that, see (5.4),

$$\forall \alpha' < \alpha \qquad \mathcal{K}_{\alpha'} \subset \mathcal{D}_{\alpha}. \tag{5.7}$$

Let us consider the following Cauchy problem in $\mathcal{K}_{\alpha}$ for the operator $(L^{\Delta}, \mathcal{D}_{\alpha})$

$$\frac{d}{dt}k_t = L^{\Delta}k_t, \qquad k_t|_{t=0} = k_0 \in \mathcal{D}_{\alpha}. \tag{5.8}$$

In this general setting, it is barely possible to solve (5.8), e.g., by applying C_0-semigroup methods. Recall that we deal here with L^{∞}-type spaces, see (2.29). However, if one takes $k_0 \in \mathcal{K}_{\alpha'}$ for some $\alpha' < \alpha$, i.e., from a subset of the domain, see (5.7), a solution can be obtained in the following way. Define

$$(S(t)k)(\eta) = \exp\left(-tE(\eta)\right)k(\eta), \qquad t > 0, \tag{5.9}$$

where $E(\eta)$ is as in (5.1). Note that this is one of the steps where we properly take into account the sign of the quadratic term in L^{-}. By means of (5.9) we then define $S_{\alpha\alpha'}(t) \in \mathcal{L}_{\alpha\alpha'}$. One can show that the map $t \mapsto S_{\alpha\alpha'}(t) \in \mathcal{L}_{\alpha\alpha'}$ is continuous. Obviously, for each t, $S(t)$ as in (5.9) defines a bounded operator acting from $\mathcal{K}_{\alpha}$ to $\mathcal{K}_{\alpha}$. We use it as $S_{\alpha\alpha'}(t)$ (i.e., acting to a bigger space) to secure the continuity just mentioned. Let $A_{\alpha\alpha'} \in \mathcal{L}_{\alpha\alpha'}$ be defined by the expression

$$(Ak)(\eta) = -E(\eta)k(\eta), \tag{5.10}$$

i.e., it is the multiplication operator by $-E(\eta)$. Clearly, the map $t \mapsto S_{\alpha\alpha'}(t)$ is differentiable and the following holds

$$\frac{d}{dt}S_{\alpha\alpha'}(t) = A_{\alpha\alpha''}S_{\alpha''\alpha'}(t) = S_{\alpha\alpha''}(t)A_{\alpha''\alpha'},$$

for each $\alpha'' \in (\alpha', \alpha)$. Now we set $B = L^{\Delta} - A$ and define the corresponding linear operator $B_{\alpha\alpha'} \in \mathcal{L}_{\alpha\alpha'}$, the norm of which satisfies, cf. (5.5),

$$\|B_{\alpha\alpha'}\| \leq \frac{\|b\|e^{-\alpha'} + \langle a\rangle e^{\alpha}}{e(\alpha-\alpha')}.$$

It is crucial that $\alpha - \alpha'$ appears here in the first power. Define

$$T(\alpha, \alpha') = \frac{\alpha - \alpha'}{\|b\|e^{-\alpha'} + \langle a\rangle e^{\alpha}}. \tag{5.11}$$

Fix now some $\delta < \alpha - \alpha'$ and $l \in \mathbb{N}$, and then set

$$\alpha^{2s} = \alpha' + \frac{s}{l+1}\delta + s\epsilon, \qquad \epsilon = (\alpha - \alpha' - \delta)/l,$$

$$\alpha^{2s+1} = \alpha' + \frac{s+1}{l+1}\delta + s\epsilon, \qquad s = 0, 1, \ldots, l.$$

Note that $\alpha^0 = \alpha'$ and $\alpha^{2l+1} = \alpha$. For $t > 0$, set

$$\mathcal{T}_l = \{(t, t_1, \ldots, t_l) : 0 \leq t_l \leq t_{l-1} \leq \cdots \leq t_1 \leq t\} \subset (\mathbb{R}_+)^{l+1},$$

and then

$$\Pi^l_{\alpha\alpha'}(t, t_1, \ldots, t_l) = S_{\alpha\alpha^{2l}}(t - t_1)B_{\alpha^{2l}\alpha^{2l-1}} \cdots \times$$

$$\times S_{\alpha^3\alpha^2}(t - t_1)(t_{l-1} - t_l)B_{\alpha^2\alpha^1}S_{\alpha^1\alpha'}(t_t), \quad (t, t_1, \ldots, t_l) \in \mathcal{T}_l.$$

It is known, see [13, Proposition 3.1], that the map

$$\mathcal{T}_l \ni (t, t_1, \ldots, t_l) \mapsto \Pi^l_{\alpha\alpha'}(t, t_1, \ldots, t_l) \in \mathcal{L}_{\alpha\alpha'}$$

is continuous. Moreover, for each $\delta \in (0, \alpha - \alpha')$, the operator norm satisfies

$$\|\Pi^l_{\alpha\alpha'}(t, t_1, \ldots, t_l)\| \leq \left(\frac{l}{eT(\alpha - \delta, \alpha')}\right)^l, \tag{5.12}$$

see (5.11). Set

$$Q_{\alpha\alpha'}(t) = S_{\alpha\alpha'}(t) + \sum_{l=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{l-1}} dt_l \Pi^l_{\alpha\alpha'}(t, t_1, \ldots, t_l). \tag{5.13}$$

By means of (5.12) one can prove the following.

Proposition 5.1. [13, Proposition 3.2] *For each $\alpha, \alpha' \in \mathbb{R}$, $\alpha' < \alpha$, and $t < T(\alpha, \alpha')$, the series in (5.13) converges in the norm of $\mathcal{L}_{\alpha\alpha'}$ in such a way that*

(i) *the map $[0, T(\alpha, \alpha')) \ni t \mapsto Q_{\alpha\alpha'}(t) \in \mathcal{L}_{\alpha\alpha'}$ is continuous and $Q_{\alpha\alpha'}(0)$ is the embedding as in (2.30);*

(ii) *for each $\alpha'' \in (\alpha', \alpha)$ and $t < \min\{T(\alpha'', \alpha'); T(\alpha, \alpha'')\}$, the following holds*

$$\frac{d}{dt}Q_{\alpha\alpha'}(t) = L^\Delta_{\alpha\alpha''}Q_{\alpha''\alpha'}(t) = Q_{\alpha\alpha''}(t)L^\Delta_{\alpha''\alpha'}. \tag{5.14}$$

(iii) *the operators $Q_{\alpha\alpha'}(t)$ enjoy the semigroup property*

$$Q_{\alpha\alpha'}(t + s) = Q_{\alpha\alpha''}(t)Q_{\alpha''\alpha'}(s), \tag{5.15}$$

that holds provided $t < T(\alpha, \alpha'')$, $s < T(\alpha'', \alpha')$ and $t + s < T(\alpha, \alpha')$.

This assertion allows one to solve the Cauchy problem in (5.8) in the following form.

Proposition 5.2. [13, Lemma 3.3] *For each $k_0 \in \mathcal{K}_{\alpha'}$, the problem in (5.8) has a unique classical solution $k_t \in \mathcal{K}_\alpha$, $t < T(\alpha, \alpha')$, given by the formula*

$$k_t = Q_{\alpha\alpha'}(t)k_0. \tag{5.16}$$

This solution has the properties: (a) $k_t(\varnothing) = k_0(\varnothing)$, $t < T(\alpha, \alpha')$; (b) its norm in $\mathcal{K}_\alpha$, see (2.29)*, satisfies*

$$\|k_t\|_\alpha \leq \frac{T(\alpha, \alpha')}{T(\alpha, \alpha') - t}\|k_0\|_{\alpha'}. \tag{5.17}$$

At this point, one ought to stress that the aforementioned solution k_t need not be related to any state $\mu \in \mathcal{P}_{\rm cor}(\Gamma)$. In particular, k_t need not be positive, cf. (2.20), which means that Proposition 5.2 says not too much concerning the evolution of states of the model we consider. This drawback is overcome by a method elaborated in [13, subsect. 3.2], based on Proposition 2.8 and certain approximations of the solution k_t. One of its outcomes is proving the positivity of k_t that allows for continuing the evolution $t \mapsto k_t$ to all $t > 0$. The corresponding result can be formulated as follows.

Proposition 5.3. [13, Theorem 2.4] *Let $\mu_0 \in \mathcal{P}_{\rm exp}$ be such that its correlation function lies in $\mathcal{K}_{\alpha_0}$. Then the solution of the problem in (5.8) with $\alpha > \alpha_0$ can uniquely be continued to all $t > 0$ in such a way that the following holds*

$$0 \leq k_t(\eta) \leq \sum_{\xi \subset \eta} e(\xi; \varrho_t)e(\eta \setminus \xi; q_t)k_{\mu_0}(\eta \setminus \xi), \tag{5.18}$$

where ϱ_t and q_t are as in (3.5). Moreover, for each $t > 0$, this solution k_t is the correlation function of a unique $\mu_t \in \mathcal{P}_{\rm exp}$ which satisfies (3.14).

Note that the trajectory $t \mapsto k_t$ mentioned in Proposition 5.3 has the following property: if $k_s \in \mathcal{K}_{\alpha'}$ for some $s \geq 0$ and $\alpha' \in \mathbb{R}$, which can be established by (5.18) and the type of μ_0, then by (5.15) and (5.16) it follows that

$$k_{t+s} = Q_{\alpha\alpha'}(t)k_s, \tag{5.19}$$

holding for some $\alpha > \alpha'$ and $t < T(\alpha, \alpha')$. Then the stated uniqueness of this trajectory follows by its local (in t) uniqueness established in Proposition 5.2.

5.2. **The proof of Theorem 3.3.** In this subsection, we prove that the map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ obtained according to Proposition 5.3 solves (1.2), and that it is a unique solution in the sense of Definition 3.2.

We begin by recalling Lemma 4.1 and the definition of $\mathcal{G}_\alpha$ in (2.32). Now we set, cf. (5.10),

$$\breve{L} = \breve{A} + \breve{B}, \tag{5.20}$$

$$(\breve{A}G)(\eta) = -E(\eta)G(\eta),$$

$$(\breve{B}G)(\eta) = -\sum_{x\in\eta}\left(\sum_{y\in\eta\setminus x} a(x-y)\right)G(\eta\setminus x) + \int_X b(x)G(\eta\cup x)dx.$$

By means of (5.20) we now define bounded linear operators $\breve{L}_{\alpha'\alpha} : \mathcal{G}_\alpha \to \mathcal{G}_{\alpha'}$, $\alpha' < \alpha$, see (2.33). One can show, see [13, eq. (3.19)], that they satisfy

$$\|\breve{L}_{\alpha'\alpha}\| \leq {\rm RHS}(5.5). \tag{5.21}$$

At the same time, for $G \in B_{\rm bs}(X)$, see Definition 2.1, $(\breve{L}G)(\eta)$ is defined pointwise for all $\eta \in \Gamma_{\rm fin}$ since the sums in (5.20) are finite for such G. By direct calculations it follows that, see (2.19),

$$\langle\!\langle L^\Delta_{\alpha\alpha'}k, G\rangle\!\rangle = \langle\!\langle k, \breve{L}_{\alpha'\alpha}G\rangle\!\rangle, \tag{5.22}$$

holding for each $k \in \mathcal{K}_{\alpha'}$, $G \in B_{\rm bs}(X)$ and $\alpha > \alpha'$, see (5.21). This can be extended to $G \in \mathcal{G}_\alpha$, see Remark 2.9. Moreover, it is possible to construct maps $\breve{Q}_{\alpha'\alpha}(t) : \mathcal{G}_\alpha \to \mathcal{G}_{\alpha'}$, dual to those described in Propositions 5.1 and 5.2, that satisfy

$$\langle\!\langle Q_{\alpha\alpha'}(t)k, G\rangle\!\rangle = \langle\!\langle k, \breve{Q}_{\alpha'\alpha}(t)G\rangle\!\rangle, \qquad t < T(\alpha,\alpha'), \tag{5.23}$$

whereas the norm (2.31) of $G_t := \breve{Q}_{\alpha'\alpha}(t)G$ in $\mathcal{G}_{\alpha'}$, cf. (5.17), satisfies

$$|G_t|_{\alpha'} \leq \frac{T(\alpha,\alpha')}{T(\alpha,\alpha') - t}|G|_\alpha, \tag{5.24}$$

see [13, eqs. (3.20), (3.21)] for more detail.

Lemma 5.4. *For a given $t > 0$, let $\mu_t \in \mathcal{P}_{\rm exp}$ be as in Proposition 5.3 and $e^{\alpha'}$, $\alpha' \in \mathbb{R}$, be its type, see Definition 2.3. Then for each $\alpha > \alpha'$ and $G \in \mathcal{G}_\alpha$, the following holds*

$$\mu_t(LKG) = \mu_t(K\breve{L}_{\alpha'\alpha}G) = \langle\!\langle k_t, \breve{L}_{\alpha'\alpha}G\rangle\!\rangle = \langle\!\langle L^\Delta_{\alpha\alpha'}k_t, G\rangle\!\rangle. \tag{5.25}$$

Proof. It is possible to show, cf. (5.3) and (5.22), that

$$(LKG)(\gamma) = (K\breve{L}G)(\gamma), \qquad G \in B_{\rm bs}(X), \quad \gamma \in \Gamma. \tag{5.26}$$

Then for $G \in B_{\rm bs}(X)$, the first equality in (5.25) follows by (5.26). The second one follows by (2.19), whereas that last equality is just (5.22). □

Proof of Theorem 3.3. The stated in claim (c) inclusion $\mathcal{F} \subset \mathcal{F}_{\max}$ has been proved in Lemma 4.1. Let us prove that the map $t \mapsto \mu_t$ as in Proposition 5.3 solves the Fokker-Planck equation (1.2) in the sense of Definition 3.2 for each $F \in \mathcal{F}_{\max}$. Thus, take $F = KG$ with $G \in \mathcal{G}_\alpha$ with $\alpha > \alpha'$, where the latter is as in Lemma 5.4. By (5.25) it follows that, see (2.20),

$$\begin{aligned} \mu_t(|LF|) &= \mu_t(|LKG|) = \mu_t(|K\breve{L}G|) \leq \mu_t(K|\breve{L}G|) \qquad (5.27) \\ &\leq \langle\!\langle k_t, |\breve{L}_{\alpha'\alpha}G|\rangle\!\rangle = \int_{\Gamma_{\rm fin}} k_t(\eta)|\breve{L}_{\alpha'\alpha}G(\eta)|\lambda(d\eta) \\ &\leq \int_{\Gamma_{\rm fin}} e^{\alpha'|\eta|}|\breve{L}_{\alpha'\alpha}G(\eta)|\lambda(d\eta) \leq \|\breve{L}_{\alpha'\alpha}\||G|_\alpha. \end{aligned}$$

Thus, $|LF|$ is μ_t-absolutely integrable, see item (i) of Definition 3.2. Now for t, s and α, α' as in (5.19) by (5.14) we have

$$k_{t+s} = k_s + \int_0^t L^\Delta_{\alpha\alpha''} k_{s+u} du, \qquad (5.28)$$

where $\alpha'' \in (\alpha', \alpha)$ is chosen in such a way that $t < T(\alpha'', \alpha')$, see (5.11), and hence $k_{t+s} \in \mathcal{K}_{\alpha''}$. Now we take G as in (5.27), apply (5.25) and obtain from (5.28) the following

$$\begin{aligned} \mu_{t+s}(KG) &= \langle\!\langle k_{t+s}, G\rangle\!\rangle = \langle\!\langle k_s, G\rangle\!\rangle + \langle\!\langle \int_0^t L^\Delta_{\alpha\alpha''} k_{s+u} du, G\rangle\!\rangle \qquad (5.29) \\ &= \langle\!\langle k_s, G\rangle\!\rangle + \int_0^t \langle\!\langle k_{s+u}, \breve{L}_{\alpha''\alpha} G\rangle\!\rangle du = \mu_s(KG) + \int_0^t \mu_{s+u}(LKG) du, \end{aligned}$$

which yields the proof that the map $t \mapsto \mu_t$ as in Proposition 5.3 solves (1.2) with any $F \in \mathcal{F}_{\max}$. The proof of the bound in (3.13) readily follows by (5.18), whereas the bound in (3.14) was proved in [13, Theorem 2.5].

Thus, it remains to prove uniqueness. To this end, we use the following arguments. First we write (1.2) for μ'_t

$$\mu'_t(F) = \mu_0(F) + \int_0^t \mu'_s(LF) ds, \qquad (5.30)$$

which has to be satisfied also by μ_t mentioned in Proposition 5.3, with the same $\mu_0 \in \mathcal{P}_{\rm exp}$. By Lemma 4.4, μ'_t lies in $\mathcal{P}_{\rm exp}$ and its correlation function satisfies, cf. (4.30) and (2.29), (2.30),

$$\begin{aligned} &0 \leq k_{\mu'_t}(\eta) \leq (\varkappa_{\mu_0} + \|b\|t)^{|\eta|}, \qquad , \qquad (5.31) \\ &k_{\mu'_t} \in \mathcal{K}_{\alpha_t}, \qquad \text{for} \quad \alpha_t := \ln(\varkappa_{\mu_0} + \|b\|t), \end{aligned}$$

that has to hold for all $t \geq 0$. By (5.30) and Remark 4.3 we then get that $k_{\mu'_t}$ satisfies, cf. (5.25),

$$\begin{aligned} \langle\!\langle k_{\mu'_t}, G^\vartheta\rangle\!\rangle &= \langle\!\langle k_{\mu_0}, G^\vartheta\rangle\!\rangle + \int_0^t \langle\!\langle k_{\mu'_s}, \breve{L}_{\alpha\alpha'} G^\vartheta\rangle\!\rangle ds \qquad (5.32) \\ &= \langle\!\langle k_{\mu_0}, G^\vartheta\rangle\!\rangle + \langle\!\langle L^\Delta_{\alpha'\alpha} \int_0^t k_{\mu'_s} ds, G^\vartheta\rangle\!\rangle, \end{aligned}$$

holding for all $\vartheta \subset \Theta$. Here G^ϑ is such that $F^\vartheta = KG^\vartheta$, i.e., it is given in (2.9). The integral $\int_0^t k_{\mu'_s} ds$ is considered in the Banach space $\mathcal{K}_\alpha$, $\alpha \geq \alpha_t$, see (5.31) and (5.6), whereas $\alpha' > \alpha$ can be arbitrary since $G \in \mathcal{G}_{\alpha'}$ for any α'. To interchange the integrations in (5.32) we used the absolute integrability as in (5.27), cf. (5.29).

The set $\{G^\vartheta : \vartheta \in \Theta\}$ is clearly separating for the σ-finite positive measures on $\Gamma_{\rm fin}$, including λ, for it is closed under multiplication and separates points. Assume that an absolutely λ-integrable function $H : \Gamma_{\rm fin} \to \mathbb{R}$ is such that $\int_{\Gamma_{\rm fin}} H G^\vartheta d\lambda = 0$ for all $\vartheta \subset \Theta$. Then $H(\eta) = 0$ for λ-almost all η. Indeed, write $H = H^+ - H^-$, $H^\pm \geq 0$, and define $\lambda^\pm = H^\pm \lambda$. Then the assumed equality implies $\lambda^+ = \lambda^-$ and hence $H^+(\eta) = H^-(\eta)$ holding for λ-almost all η. We use this argument in (5.32) and thereby get

$$k_{\mu'_t} = k_{\mu_0} + L^\Delta_{\alpha'\alpha} \int_0^t k_{\mu'_s} ds, \tag{5.33}$$

being the equality of vectors in $\mathcal{K}_{\alpha'}$. Thus, $k_{\mu'_t}$ is a mild solution of the Cauchy problem in (5.8). To prove that this fact implies $k_{\mu'_t} = k_{\mu_t}$ (hence $\mu'_t = \mu_t$) we adapt standard arguments by which uniqueness of mild solutions is proved when one deals with C_0-semigroups, see [10, Proposition 6.4, page 146]. Fix $t > 0$ and choose α_0 according to $e^{\alpha_0} = \varkappa_{\mu_0}$, see Lemma 4.4 and (5.31). For $u \leq t$, define $q_u = k_{\mu_u} - k_{\mu'_u}$. Since k_{μ_t} also satisfies (5.31), then both q_u and $\int_0^u q_s ds$ lie in $\mathcal{K}_{\alpha_u} \subset \mathcal{K}_{\alpha_t}$. By (5.33) and (5.6) it then follows

$$q_u = L^\Delta_{\alpha'\alpha_t} \int_0^u q_s ds, \tag{5.34}$$

that holds in $\mathcal{K}_{\alpha'}$ for all $\alpha' > \alpha_t$, see (2.30). The latter means that we consider each $q_u \in \mathcal{K}_{\alpha_u}$ as an element of $\mathcal{K}_{\alpha'}$ with such α'. In this sence, $q_u = Q_{\alpha'\alpha_u}(0) q_u$, see (5.15). Now we take $u < t$ and then $v \in (u, t]$ such that

$$v < T(\alpha', \alpha_t) = \frac{\alpha' - \alpha_t}{\|b\| e^{-\alpha_t} + \langle a \rangle e^{\alpha'}}, \tag{5.35}$$

see (5.11), and consider, see (5.14),

$$\begin{aligned} \frac{d}{du} Q_{\alpha'\alpha_t}(v-u) \int_0^u q_s ds &= Q_{\alpha'\alpha_t}(v-u) q_u - Q_{\alpha'\alpha}(v-u) L^\Delta_{\alpha\alpha_t} \int_0^u q_s ds \qquad (5.36) \\ &= Q_{\alpha'\alpha}(v-u) \left[Q_{\alpha\alpha_t}(0) q_u - L^\Delta_{\alpha\alpha_t} \int_0^u q_s ds \right] \\ &= Q_{\alpha'\alpha}(v-u) \left[q_u - L^\Delta_{\alpha\alpha_t} \int_0^u q_s ds \right] = 0, \end{aligned}$$

in view of (5.34). At the same time, integration over u of both sides of (5.36) yields

$$Q_{\alpha'\alpha_t}(0) \int_0^v q_s ds = 0, \quad \text{hence} \quad \int_0^v q_s ds = 0, \quad \text{for all} \quad v \leq t. \tag{5.37}$$

By (5.34) the latter yields $q_v = 0$, hence $k_{\mu_v} = k_{\mu'_v}$, holding for all $v \leq t$. Now we use the fact that α' in the latter formulas can be chosen arbitrarely, and that $e^{\alpha_t} = e^{\alpha_0} + \|b\| t$. In view of this, we take $\alpha' = \alpha_t + 1$, which yields, see (5.11),

$$T(\alpha_t + 1, \alpha_t) > \tau(t) := \frac{1}{\|b\| e^{-\alpha_0} + \langle a \rangle e^{\alpha_0 + 1} + \langle a \rangle e \|b\| t}, \quad t > 0. \tag{5.38}$$

Then for (5.35) to hold for all $v \leq t$, the latter should satisfy $t \leq t_1$ with $t_1 > 0$ being the unique solution of the equation $t = \tau(t)$. Thus, by (5.37) we get that $k_{\mu_v} = k_{\mu'_v}$ for all $v \leq t_1$. To futher extend this equality, we rewrite (5.33) in the form

$$k_{\mu'_{t_1+v}} = k_{\mu_{t_1}} + L^\Delta_{\alpha'\alpha_{t_1+v}} \int_0^v k_{\mu'_{t_1+s}} ds,$$

and then introduce $q_u = k_{\mu_{t_1+u}} - k_{\mu'_{t_1+u}}$, for which we get, cf. (5.34),

$$q_u = L^\Delta_{\alpha'\alpha_{t_1+v}} \int_0^u q_s ds, \qquad u \leq v.$$

To use the arguments as in (5.36) with $\alpha' = \alpha_{t_1+v} + 1$, one should impose the restriction $v < T(\alpha_{t_1+v}+1, \alpha_{t_1+v})$, which can be satisfied if $v \leq t_2$, where t_2 is the unique solution of the equation $t = \tau(t_1 + t)$, see (5.38). As a result, we obtain the equality $k_{\mu_t} = k_{\mu'_t}$ for all $t \leq t_1 + t_2$. Repeating this procedure due times we obtain the just mentioned equality for $t \leq t_1 + t_2 + \cdots + t_n$, where $t_n > 0$ is the unique solution of the corresponding equation, cf. (5.38), i.e., verifies

$$t_n = \frac{1}{\|b\|e^{-\alpha_0} + \langle a\rangle e^{\alpha_0+1} + \langle a\rangle e\|b\|(t_1 + \cdots + t_{n-1} + t_n)}. \tag{5.39}$$

If the series $t_1 + \cdots + t_{n-1} + t_n + \cdots$ is convergent, then the left-hands side of (5.39) tends to zero as $n \to +\infty$, whereas its right-hand side remains separated away from zero. Hence, $t_1 + \cdots + t_{n-1} + t_n + \cdots = +\infty$, which yields $\mu_t = \mu'_t$ holding for all $t > 0$. This completes the whole proof of the theorem. □

6. Constructing the Markov Process: Auxiliary Models

The aim of this section is to prepare the proof of Theorem 3.10. As mentioned above, the proof will be done by approximating the initial model described by L given in (1.1) by a family of modeles described by their Kolmogorov operators $\{L_\sigma : \sigma \in [0,1]\}$ such that L_0 coincides with L given in (1.1) whereas each L_σ, $\sigma \in (0,1]$ can be used to construct a Markov transition function, p_t^σ, by means of which one defines a Markov process, corresponding to L_σ. Then the process in question is obtained as the weak limit of the Markov processes obtained in this way.

6.1. **The models.** We begin by recalling that each $\mu \in \mathcal{P}_{\rm exp}$ has the property $\mu(\Gamma_*) = 1$, see (2.24), (2.25). Keeping this in mind we set, cf. (2.23),

$$\psi_\sigma(x) = \frac{1}{1 + \sigma|x|^{d+1}}, \qquad \sigma \in [0,1]. \tag{6.1}$$

and

$$b_\sigma(x) = b(x)\psi_\sigma(x), \qquad m_\sigma(x) = m(x)\psi_\sigma(x), \tag{6.2}$$
$$a_\sigma(x,y) = a(x-y)\psi_\sigma(x)\psi_\sigma(y).$$

Clearly,

$$\psi(x) \leq \psi_\sigma(x) \leq \sigma^{-1}\psi(x), \qquad \sigma \in (0,1], \tag{6.3}$$

which yields, cf. (2.24),

$$\gamma(\psi) = \varPhi(\gamma) \leq \varPhi_\sigma(\gamma) \leq \sigma^{-1}\varPhi(\gamma), \quad \Phi_\sigma(\gamma) := \gamma(\psi_\sigma). \tag{6.4}$$

Then we define

$$L_\sigma F(\gamma) = \int_X b_\sigma(x)\nabla_x F(\gamma)dx - \sum_{x\in\gamma}\left(m_\sigma(x) + \sum_{y\in\gamma\setminus x} a_\sigma(x,y)\right)\nabla_x F(\gamma\setminus x), \tag{6.5}$$

where $\nabla_x F(\gamma)$ is as in (4.15). Clearly, L_0 coincides with the generator defined in (1.1). By (3.1) and (6.4) we have that

$$\sum_{x\in\gamma} m_\sigma(x) \leq \|m\|\varPhi_\sigma(\gamma) \leq \|m\|\sigma^{-1}\varPhi(\gamma),$$

and also, see (6.2) and (3.1),

$$\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} a_\sigma(x,y) \leq \|a\|\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\psi_\sigma(x)\psi_\sigma(y), \qquad \gamma \in \Gamma_*. \tag{6.6}$$

Thereby we set, cf. (5.1), (5.2),

$$E_\sigma(\eta) = \sum_{x\in\eta} m_\sigma(x) + \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a_\sigma(x,y), \qquad \eta\in\Gamma_{\rm fin}, \tag{6.7}$$

$$(L^\Delta_\sigma k)(\eta) = \sum_{x\in\eta} b_\sigma(x)k(\eta\setminus x) - E_\sigma(\eta)k(\eta) - \int_X \left(\sum_{y\in\eta} a_\sigma(x,y)\right) k(\eta\cup x)dx.$$

Similarly as in (5.3) we then have

$$\mu(L_\sigma KG) = \langle\!\langle L^\Delta_\sigma k_\mu, G\rangle\!\rangle, \qquad \sigma\in(0,1],$$

which holds for all $\mu\in\mathcal{P}_{\rm exp}$ and appropriate $G:\Gamma_{\rm fin}\to\mathbb{R}$. In view of (6.2), L^Δ_σ can be used to define bounded linear operators $(L^\Delta_\sigma)_{\alpha\alpha'}:\mathcal{K}_{\alpha'}\to\mathcal{K}_\alpha$ the norms of which satisfy

$$\|(L^\Delta_\sigma)_{\alpha\alpha'}\| \leq {\rm RHS}(5.5),$$

Hence, the Cauchy problem in $\mathcal{K}_\alpha$ for $(L^\Delta_\sigma,\mathcal{D}_\alpha)$, with the same domain as in (5.8),

$$\frac{d}{dt}k^\sigma_t = L^\Delta_\sigma k^\sigma_t, \qquad k^\sigma_t|_{t=0} = k_0\in\mathcal{K}_\alpha, \tag{6.8}$$

has a unique solution, see Proposition 5.2, given by the formula

$$k^\sigma_t = Q^\sigma_{\alpha\alpha'}(t)k_0, \qquad t<T(\alpha,\alpha'), \tag{6.9}$$

with $T(\alpha,\alpha')$ given in (5.11) and the family of operators $\{Q^\sigma_{\alpha\alpha'}(t) : t\in[0,T(\alpha,\alpha')\}$ possessing all the properties established in Proposition 5.1.

Remark 6.1. Similarly as in Proposition 5.3 the evolution $t\mapsto k^\sigma_t$ described by (6.9) determines the evolution of states $t\mapsto\mu^\sigma_t\in\mathcal{P}_{\rm exp}$, $t>0$, the type of which satisfies, cf. (4.30),

$$\varkappa_{\mu^\sigma_t} \leq e^{\alpha_t} := \varkappa_{\mu_0} + \|b\|t. \tag{6.10}$$

These states μ^σ_t solve the Fokker-Planck equation for $(L_\sigma,\mathcal{F},\mu_0)$ with the same domain given in (3.10). Similarly as in Proposition 5.3 this solution is unique.

6.2. **The Markov transition functions.** The abovementioned transition functions p^σ_t will be obtained in the form

$$p^\sigma_t(\gamma,\cdot) = S^\sigma(t)\delta_\gamma, \qquad t\geq 0, \quad \sigma\in(0,1], \tag{6.11}$$

where δ_γ is the Dirac measure on Γ_* centered at $\gamma\in\Gamma_*$ and $S^\sigma(t)$ is a bounded positive operator acting in the Banach space of finite signed measures on Γ_*, such that $S^\sigma = \{S^\sigma(t)\}_{t\geq0}$ is a stochastic semigroup related to L_σ given in (6.5). This semigroup will be constructed (see Lemma 6.5 below) by means of the Thieme-Voigt technique developed in [22] which proved effective in the problems like the one considered here. Its detailed presentation in the form adapted to the present context can be found in [15, Sect. 7]. Here we just briefly outline the main aspects.

6.2.1. *The Thieme-Voigt theory.* Let $\mathcal{X}$ be an ordered real Banach space with a generating cone $\mathcal{X}^+$ such that the norm of $\mathcal{X}$ is additive on the cone, i.e., $\|x+y\|_\mathcal{X} = \|x\|_\mathcal{X}+\|y\|_\mathcal{X}$, whenever $x,y\in\mathcal{X}^+$. By the latter fact there exists a linear positive functional, $\varphi_\mathcal{X}$, such that

$$\varphi_\mathcal{X}(x) = \|x\|_\mathcal{X}, \qquad \text{for} \quad x\in\mathcal{X}^+. \tag{6.12}$$

A C_0-semigroup $S=\{S(t)\}_{t\geq0}$ of bounded linear operators on $\mathcal{X}$ is said to be *stochastic* (resp. *substochastic*) if the following holds $\|S(t)x\|_\mathcal{X} = \|x\|_\mathcal{X}$ (resp. $\|S(t)x\|_\mathcal{X}\leq\|x\|_\mathcal{X}$) for all $x\in\mathcal{X}^+$ and $t>0$. For a dense linear subset $\mathcal{D}\subset\mathcal{X}$, set $\mathcal{D}^+=\mathcal{D}\cap\mathcal{X}^+$ and assume that $(A,\mathcal{D})$ and $(B,\mathcal{D})$ are linear operators on $\mathcal{X}$. The Thieme-Voigt theory gives sufficient conditions on this pair of operators under which the closure of $(A+B,\mathcal{D})$ is the generator of a stochastic semigroup. Its key aspect is the use of a subspace $\widetilde{\mathcal{X}} \subset \mathcal{X}$ with a specific set of properties listed below.

Assumption 6.2. *The linear subspace $\widetilde{\mathcal{X}} \subset \mathcal{X}$ has the following properties:*

(a) *$\widetilde{\mathcal{X}}$ is dense in $\mathcal{X}$;*
(b) *there exists a norm, $\|\cdot\|_{\widetilde{\mathcal{X}}}$, that makes $\widetilde{\mathcal{X}}$ a Banach space;*
(c) *$\widetilde{\mathcal{X}}^+ := \widetilde{\mathcal{X}} \cap \mathcal{X}^+$ is the generating cone in $\widetilde{\mathcal{X}}$, the norm $\|\cdot\|_{\widetilde{\mathcal{X}}}$ is additive on $\widetilde{\mathcal{X}}^+$;*
(d) *the cone $\widetilde{\mathcal{X}}^+$ is dense in $\mathcal{X}^+$.*

By item (c) of Assumption 6.2 there exists a linear positive functional, $\varphi_{\widetilde{\mathcal{X}}}$, cf (6.12), such that

$$\varphi_{\widetilde{\mathcal{X}}}(x) = \|x\|_{\widetilde{\mathcal{X}}}, \qquad \text{for} \quad x \in \widetilde{\mathcal{X}}^+. \tag{6.13}$$

For a dense linear subset $\mathcal{D} \subset \mathcal{X}$, let $(A, \mathcal{D})$ be a linear operator on $\mathcal{X}$. Define $\widetilde{\mathcal{D}} = \{x \in \mathcal{D} \cap \widetilde{\mathcal{X}} : Ax \in \widetilde{\mathcal{X}}\}$ Then the operator $(A, \widetilde{\mathcal{D}})$ is said to be the *trace* of $(A, \mathcal{D})$ in $\widetilde{\mathcal{X}}$. The next statement is an adaptation of [22, Theorem 2.7], see also [15, Proposition 7.2].

Proposition 6.3. *Let $(A, \mathcal{D})$ and $(B, \mathcal{D})$ be linear operator on $\mathcal{X}$ which have the following properties*

(i) *$-A : \mathcal{D}^+ \to \mathcal{X}^+$ and $B : \mathcal{D}^+ \to \mathcal{X}^+$;*
(ii) *$(A, \mathcal{D})$ is the generator of a substochastic semigroup, $S_0 = \{S_0(t)\}_{t\geq 0}$, on $\mathcal{X}$ such that $S_0(t) : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$, holding for all $t \geq 0$, and the restrictions $S_0(t)|_{\widetilde{\mathcal{X}}}$, $t \geq 0$, constitute a C_0-semigroup on $\widetilde{\mathcal{X}}$ genereted by $(A, \widetilde{\mathcal{D}})$;*
(iii) *$B : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{X}}$ and*
$$\varphi_{\mathcal{X}}((A + B)x) = 0, \qquad \text{for all} \quad x \in \mathcal{D}^+;$$
(iv) *there exist positive c and ϵ such that*
$$\varphi_{\widetilde{\mathcal{X}}}((A + B)x) \leq c\varphi_{\widetilde{\mathcal{X}}}(x) - \epsilon\|Ax\|_{\mathcal{X}}, \qquad \text{for all} \quad x \in \widetilde{\mathcal{D}} \cap \mathcal{X}^+.$$

Then the closure of $(A + B, \mathcal{D})$ in $\mathcal{X}$ is the generator of a stochastic semigroup, $S = \{S(t)\}_{t\geq 0}$, on $\mathcal{X}$ which leaves $\widetilde{\mathcal{X}}$ invariant.

6.2.2. *The Banach spaces of measures.* Now we turn to constructing the semigroups S^σ that appear in (6.11). Let $\mathcal{M}$ stand for the set of all finite signed measures on Γ_*, see [7, Chapt. 12]. Set $\mathcal{M}^+ = \{\mu \in \mathcal{M} : \mu(\mathbb{A}) \geq 0, \ \mathbb{A} \in \mathcal{B}(\Gamma_*)\}$. Each $\mu \in \mathcal{M}$ can uniquely be decomposed $\mu = \mu^+ - \mu^-$ with $\mu^\pm \in \mathcal{M}^+$, which means that the latter is the generating cone in $\mathcal{M}$. Set $|\mu| = \mu^+ + \mu^-$. Then

$$\|\mu\| := |\mu|(\Gamma_*)$$

is a norm, which is additive on $\mathcal{M}^+$. With this norm $\mathcal{M}$ is a Banach space, see [7, Proposition 4.1.8, page 119]. For $n \in \mathbb{N}$, let now F_n stand for F^ϑ, $\vartheta = \{\psi, \ldots, \psi\}$, $|\vartheta| = n$, see (4.27). Also set $F_0 \equiv 1$. Since $\psi(x) \leq 1$, see (6.1), these functions satisfy

$$F_n(\gamma \cup x) = F_n(\gamma) + n\psi(x)F_{n-1}(\gamma), \tag{6.14}$$

$$F_1(\gamma)F_n(\gamma) \leq F_{n+1}(\gamma) + nF_n(\gamma),$$

$$F_2(\gamma)F_n(\gamma) \leq F_{n+2}(\gamma) + 2nF_{n+1}(\gamma) + n(n-1)F_n(\gamma).$$

Define

$$\|\mu\|_n = \sum_{k=0}^{n} \frac{1}{k!}|\mu|(F_k), \qquad \mathcal{M}_n = \{\mu \in \mathcal{M} : \|\mu\|_n < \infty\}, \quad n \in \mathbb{N}. \tag{6.15}$$

Clearly, each $\mathcal{M}_n$ is a Banach space. Let $\mathcal{M}_n^+$ denote the corresponding cones of positive measures. Then

$$\mathcal{M}_{n+1} \subset \mathcal{M}_n \subset \mathcal{M}, \qquad \mathcal{M}_{n+1}^+ \subset \mathcal{M}_n^+ \subset \mathcal{M}^+, \quad n \in \mathbb{N},$$

where all the inclusions are dense in the corresponding topologies, cf. [15, Lemma 7.4]. Let $\mathcal{M}^{1,+}$ consist of all $\mu \in \mathcal{M}^+$ for which $\|\mu\| = \mu(\Gamma_*) = 1$, i.e, of all probability measures. Then, for each $n \in \mathbb{N}$, it follows that

$$\mathcal{P}_{\rm exp} \subset \mathcal{M}_n \cap \mathcal{M}^{1,+}. \tag{6.16}$$

Indeed, by (4.28) for $\mu \in \mathcal{P}_{\rm exp}$ one gets

$$\mu(F_k) \leq [\varkappa_\mu \langle \psi \rangle]^k, \qquad k \in \mathbb{N}.$$

We conclude this part by noting that, for each $n \in \mathbb{N}$, the Banach spaces $\mathcal{M}$ and $\mathcal{M}_n$ satisfy all the conditions of Assumption 6.2, whereas the linear functionals as in (6.12) and (6.13) are

$$\varphi(\mu) := \varphi_{\mathcal{M}}(\mu) = \mu(\Gamma_*), \qquad \varphi_n(\mu) := \varphi_{\mathcal{M}_n}(\mu) = \sum_{k=1}^n \frac{1}{k!}\mu(F_k), \tag{6.17}$$

respectively.

6.2.3. *The semigroup.* For L_σ introduced in (6.5), we define $L^\dagger_\sigma$ by the formula

$$(L^\dagger_\sigma \mu)(F) = \mu(L_\sigma F), \tag{6.18}$$

where F is a suitable function. To this end, we first define the following measure kernel

$$\Omega^\gamma_\sigma(\mathbb{A}) = \int_X b_\sigma(x)\mathbb{1}_{\mathbb{A}}(\gamma \cup x)dx + \sum_{x\in\gamma}\left(m_\sigma(x) + \sum_{y\in\gamma\setminus x} a_\sigma(x,y)\right)\mathbb{1}_{\mathbb{A}}(\gamma \setminus x), \tag{6.19}$$

and the function

$$R_\sigma(\gamma) = \Omega^\gamma_\sigma(\Gamma_*) = \langle b_\sigma \rangle + E_\sigma(\gamma) =: \int_X b_\sigma(x)dx + \sum_{x\in\gamma}\left(m_\sigma(x) + \sum_{y\in\gamma\setminus x} a_\sigma(x,y)\right). \tag{6.20}$$

By (6.4) and (6.6) one gets that $R_\sigma(\gamma) < \infty$ for all $\gamma \in \Gamma_*$. Then $L^\dagger_\sigma$ can be presented in the form

$$L^\dagger_\sigma = A + B, \tag{6.21}$$

$$(A\mu)(d\gamma) = -R_\sigma(\gamma)\mu(d\gamma), \qquad (B\mu)(d\gamma) = \int_{\Gamma_*} \Omega^{\gamma'}_\sigma(d\gamma)\mu(d\gamma').$$

Let us show that both $-A$ and B introduced in (6.21) define positive (unbounded) operators in each of $\mathcal{M}_n$. Set

$$\mathrm{Dom}_n(A) := \{\mu \in \mathcal{M} : R_\sigma|\mu| \in \mathcal{M}_n\}, \quad n \in \mathbb{N}_0, \tag{6.22}$$

where $\mathcal{M}_0$ is just $\mathcal{M}$. By (6.3) and (6.20) it follows that

$$R_\sigma(\gamma) \leq \langle b_\sigma \rangle + \frac{\|m\|}{\sigma}F_1(\gamma) + \frac{\|a\|}{\sigma^2}F_2(\gamma). \tag{6.23}$$

Then by (6.14) we obtain

$$R_\sigma(\gamma)F_k(\gamma) \leq \langle b_\sigma \rangle F_k(\gamma) + \frac{\|m\|}{\sigma}kF_k(\gamma) + \frac{\|m\|}{\sigma}F_{k+1}(\gamma) \tag{6.24}$$

$$+\frac{\|a\|}{\sigma^2}k(k-1)F_k(\gamma) + \frac{2\|a\|}{\sigma^2}kF_{k+1}(\gamma) + \frac{\|a\|}{\sigma^2}F_{k+2}(\gamma).$$

We apply this estimate in (6.15) and get

$$\|A\mu\|_n \leq \langle b_\sigma \rangle \|\mu\|_n + \frac{2(n+1)\|m\|}{\sigma}\|\mu\|_{n+1} + \frac{4(n+1)(n+2)\|a\|}{\sigma^2}\|\mu\|_{n+2}.$$

By (6.22) this yields

$$\mathcal{M}_{n+2} \subset \mathrm{Dom}_n(A), \quad n \in \mathbb{N}_0. \tag{6.25}$$

Remark 6.4. For each $n \geq 1$, the operator $(A, \mathrm{Dom}_n(A))$ is the trace of $(A, \mathrm{Dom}_0(A))$ in the Banach space $\mathcal{M}_n$.

For an appropriate $\mu \in \mathcal{M}$, by (6.19) one gets

$$\begin{aligned}\int_{\Gamma_*} F_n(\gamma)(B\mu)(d\gamma) &= \int_{\Gamma_*^2} F_n(\gamma')\Omega_\sigma^\gamma(d\gamma')\mu(d\gamma) \\ &= \int_{\Gamma_*} \left(\int_X b_\sigma(x) F_n(\gamma \cup x) dx \right) \mu(d\gamma) \\ &+ \int_{\Gamma_*} \left(\sum_{x \in \gamma} \left[m_\sigma(x) + \sum_{y \in \gamma \setminus x} a_\sigma(x,y) \right] F_n(\gamma \setminus x) \right) \mu(d\gamma).\end{aligned} \tag{6.26}$$

For $n = 0$ and $\mu \in \mathcal{M}^+$, this yields

$$\|B\mu\| = \mu(R_\sigma), \quad \text{hence} \quad (L_\sigma^\dagger \mu)(\Gamma_*) = \mu(R_\sigma) - \mu(R_\sigma) = 0, \tag{6.27}$$

see (6.21). For $\mu \in \mathcal{M}^+$ and $n \geq 1$, by (6.14) and the evident estimate $F_n(\gamma \setminus x) \leq F_n(\gamma)$ we get from (6.26) the following

$$\int_{\Gamma_*} F_n(\gamma)(B\mu)(d\gamma) \leq n\|b\|\langle\psi\rangle\mu(F_{n-1}) + \int_{\Gamma_*} F_n(\gamma)R_\sigma(\gamma)\mu(d\gamma),$$

which readily yields that

$$B : \mathrm{Dom}_n(A) \to \mathcal{M}_n, \qquad n \geq 2. \tag{6.28}$$

Now we set, see (6.25),

$$\mathrm{Dom}(L_\sigma^\dagger) = \mathrm{Dom}_0(A) = \{\mu \in \mathcal{M} : |\mu|(R_\sigma) < \infty\}. \tag{6.29}$$

Lemma 6.5. *For each $\sigma \in (0,1]$, the closure of the operator $(L_\sigma^\dagger, \mathrm{Dom}(L_\sigma^\dagger))$ in $\mathcal{M}$ is the generator of a stochastic semigroup, $S^\sigma = \{S^\sigma(t)\}_{t \geq 0}$, such that $S^\sigma(t) : \mathcal{M}_n \to \mathcal{M}_n$ for each $n \geq 2$ and $t \geq 0$.*

Proof. Our aim is to show that the operator defined in (6.21) and (6.29) satisfies the conditions of Proposition 6.3. By the very definition of A and B in (6.21), and then by (6.22), (6.28) and (6.29), it readily follows that condition (i) is met. Define

$$S_0(t)\mu(d\gamma) = \exp\left(-tR_\sigma(\gamma)\right)\mu(d\gamma), \qquad \mu \in \mathcal{M}, \quad t \geq 0. \tag{6.30}$$

Clearly, for each $n \in \mathbb{N}_0$, $S_0(t) : \mathcal{M}_n \to \mathcal{M}_n$, acting as a multiplication operator, and $S_0 = \{S_0(t)\}_{t \geq 0}$ constitue a semigroup. Certainly,

$$\|S_0(t)\mu\| \leq \|\mu\|, \qquad \mu \in \mathcal{M}^+. \tag{6.31}$$

To show the strong continuity of S_0 in $\mathcal{M}$, for fixed $\mu \in \mathcal{M}$ and $\varepsilon > 0$, we have to find $\delta > 0$ such that $\|S_0(t)\mu - \mu\| < \varepsilon$ whenever $t < \delta$. Since $\mathcal{M}_2$ is dense in $\mathcal{M}$, see (6.22) and (6.25), one finds $\mu' \in \mathcal{M}_2$ such that $\|(\mu - \mu')^\pm\| < \varepsilon/6$. Then by (6.30) and (6.31) we have

$$\begin{aligned}\|S_0(t)\mu - \mu\| &\leq \|\mu - \mu'\| + \|S_0(t)(\mu - \mu')\| + \|S_0(t)\mu' - \mu'\| \\ &\leq t\|A\mu'\| + 2\varepsilon/3 = t|\mu'|(R_\sigma) + 2\varepsilon/3 \leq c_\sigma t\|\mu'\|_2 + 2\varepsilon/3,\end{aligned}$$

where $c_\sigma = \max\{\langle b_\sigma\rangle; \|m\|/\sigma; 2\|a\|/\sigma^2\}$, see (6.23). This estimate yields the strong continuity in question. The strong continuity of S_0 in $\mathcal{M}_n$ can be shown in a similar way by employing (6.24) and (6.25). Now we take into account Remark 6.4 and thus conclude that condition (ii) of Proposition 6.3 is also met. The validity of (iii) follows by (6.28) (concerning B) and (6.17), (6.27) (concerning φ). To complete the whole proof, we have to show that, for each $n \geq 2$, there exist positive c and ϵ such that, for each $\mu \in \mathrm{Dom}_n(A) \cap \mathcal{M}_n^+$, the following holds

$$\varphi_n(L_\sigma^\dagger \mu) \leq c\varphi_n(\mu) - \epsilon\mu(R_\sigma),$$

where φ_n is as in (6.17). In view of (6.23), to this end it is enough to show that

$$\sum_{k=0}^{n} \frac{1}{k!} \int_{\Gamma_*} F_k(\gamma)(L_\sigma^\dagger \mu)(d\gamma) \leq C\varphi_n(\mu), \qquad \mu \in \mathrm{Dom}_n(A) \cap \mathcal{M}_n^+, \tag{6.32}$$

holding for some $C > 0$. By (6.18) we have

$$\int_{\Gamma_*} F_k(\gamma)(L_\sigma^\dagger \mu)(d\gamma) = \mu(L_\sigma F_k).$$

At the same time, by (6.14) and similarly as in (4.15), see also (4.18), we conclude that

$$\mu(L_\sigma F_k) \leq \mu(L_\sigma^+ F_k) \leq \|b\| \langle \psi \rangle k\mu(F_{k-1}),$$

which by (7.7) yields the validity of (6.32) with $C = n\|b\|\langle\psi\rangle$. Now the proof follows by Proposition 6.3. □

Lemma 6.5 establishes the existence of the transition function (6.11). It is streightforward that p_t^σ defined in this way satisfies the standard conditions and thus determines finite dimensional distributions of a Markov process, see [11, pages 156, 157]. The next step is to prove that such processes have cadlag versions.

6.2.4. *The Cauchy problem.* Now we use the semigroup constructed in Lemma 6.5 to solve the following Cauchy problem in $\mathcal{M}$

$$\frac{d}{dt}\mu_t = L_\sigma^\dagger \mu_t, \qquad \mu_t|_{t=0} = \mu_0, \quad \sigma \in (0, 1]. \tag{6.33}$$

By [10, Proposition 6.2, page 145] this problem has a unique solution given by the formula

$$\hat{\mu}_t^\sigma = S^\sigma(t)\mu_0 = \int_{\Gamma_*} p_t^\sigma(\gamma, \cdot)\mu_0(d\gamma), \tag{6.34}$$

whenever $\mu_0 \in \mathrm{Dom}(L_\sigma^\dagger)$, see (6.29). Here we use notations $\hat{\mu}_t^\sigma$ in order not to mix this solution with the measures μ_t^σ mentioned in Remark 6.1. By (6.16) and (6.25) $\mu_0 \in \mathcal{P}_{\exp}$ lies in $\mathrm{Dom}(L_\sigma^\dagger)$. Hence, $\hat{\mu}_t^\sigma$ lies in each $\mathcal{M}_n$, but a priori not in $\mathcal{P}_{\exp}$.

Lemma 6.6. *For a given $\mu_0 \in \mathcal{P}_{\exp}$ and $\sigma \in (0, 1]$, let the map $t \mapsto \mu_t^\sigma$ be the unique solution of the Fokker-Planck equation for $(L_\sigma, \mathcal{F}, \mu_0)$ mentioned in Remark 6.1. Then for each $t > 0$, it follows that $\mu_t^\sigma = \hat{\mu}_t^\sigma$, where the latter is obtained as in* (6.34) *with the same μ_0.*

Proof. Since $\hat{\mu}_t^\sigma$ solves (6.33), for each $F \in \mathcal{F}$ one has

$$\hat{\mu}_t^\sigma(F) = \hat{\mu}_0^\sigma(F) + \int_0^t (L_\sigma^\dagger \hat{\mu}_s^\sigma)(F)ds,$$

which by (6.18) yields that $\hat{\mu}_t^\sigma$ solves the Fokker-Planck equation for $(L_\sigma, \mathcal{F}, \mu_0)$; hence, $\hat{\mu}_t^\sigma = \mu_t^\sigma$ since the solution is unique. □

7. The Markov Process

7.1. **The cadlag paths.** In this subsection, we use Chentsov's theorem [6] in the version formulated below as Proposition 7.1. Its presentation is preceded by recalling that the metric ρ introduced in (2.27) makes Γ_* a Polish space, see Proposition 2.6. By means of this metric and the transition function as in (6.11) we define

$$\begin{aligned} w^\sigma_u(\gamma) &= \int_{\Gamma_*} \rho(\gamma,\gamma') p^\sigma_u(\gamma, d\gamma'), \\ W^\sigma_{u,v}(\gamma) &= \int_{\Gamma_*} \rho(\gamma,\gamma') w^\sigma_u(\gamma') p^\sigma_v(\gamma, d\gamma'). \end{aligned} \tag{7.1}$$

Now for a certain $\mu \in \mathcal{P}_{\rm exp}$ and a triple (t_1, t_2, t_3), $0 \leq t_1 < t_2 < t_3$, we set

$$\widehat{W}^\sigma(t_1,t_2,t_3) = \int_{\Gamma_*} W^\sigma_{t_3-t_2, t_2-t_1}(\gamma)\mu^\sigma_{t_1}(d\gamma), \qquad \mu^\sigma_t = S^\sigma(t)\mu. \tag{7.2}$$

see Lemma 6.6. By means of the main result of [6] and [11, Theorems 7.2 and 8.6 – 8.8, pages 128 and 137–139] one can state the following, cf. [15, Proposition 7.8].

Proposition 7.1. *Assume that: (a) for each $t > 0$, the family $\{\mu^\sigma_t : \sigma \in (0,1]\} \subset \mathcal{P}(\Gamma_*)$ is weakly relatively compact; (b) for each $T > 0$, there exists $C(T) > 0$ such that for each triple (t_1,t_2,t_3) satisfying $t_3 \leq T$ the following holds*

$$\widehat{W}^\sigma(t_1,t_2,t_3) \leq C(T)(t_3 - t_1)^2. \tag{7.3}$$

Then:

(i) *For each $\sigma \in (0,1]$, the transition function p^σ and $\mu \in \mathcal{P}_{\rm exp}$, see (7.2), determine a probability measure P^σ_μ on $\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*)$.*

(ii) *The family $\{P^\sigma_\mu : \sigma \in (0,1]\}$ of path measures just mentioned is tight, hence possesses accumulation points in the weak topology of $\mathcal{P}(\mathfrak{D}_{\mathbb{R}_+}(\Gamma_*))$.*

Remark 7.2. For each $s > 0$, one can consider $\widehat{W}^\sigma(t_1,t_2,t_3)$ with $t_1 \geq s$ and $\mu^\sigma_t = S^\sigma(t-s)\mu$. Then by Proposition 7.1 p^σ and $\mu \in \mathcal{P}_{\rm exp}$ determine a probability measure $P^\sigma_{s,\mu}$ on $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$, and the family $\{P^\sigma_{s,\mu} : \sigma \in (0,1]\}$ is tight in the weak topology of $\mathcal{P}(\mathfrak{D}_{[s,+\infty)}(\Gamma_*))$.

The measure $P^\sigma_{s,\mu}$ is defined by its finite dimensional marginals, which in turn are defined by p^σ and μ as follows, see eq. (1.10), page 157 in [11]. For a given $m \in \mathbb{N}$, the m-dimensional marginal is the following measure

$$\begin{aligned} P^\sigma_{s,\mu}((\mathbb{1}_{\mathbb{A}_1} \circ \varpi_{t_1}) \cdots (\mathbb{1}_{\mathbb{A}_m} \circ \varpi_{t_m})) &= \int_{\Gamma_*^{m+1}} \mathbb{1}_{\mathbb{A}_m}(\gamma_m) p^\sigma_{t_m - t_{m-1}}(\gamma_{m-1}, d\gamma_m) \\ &\times \mathbb{1}_{\mathbb{A}_{m-1}}(\gamma_{m-1}) p^\sigma_{t_{m-1}-t_{m-2}}(\gamma_{m-2}, d\gamma_{m-1}) \cdots \mathbb{1}_{\mathbb{A}_1}(\gamma_1) p^\sigma_{t_1 - s}(\gamma_0, d\gamma_1)\mu(d\gamma_0), \end{aligned} \tag{7.4}$$

where $\mathbb{A}_1, \ldots, \mathbb{A}_m$ are in $\mathcal{B}(\Gamma_*)$.

Lemma 7.3. *For each $\mu \in \mathcal{P}_{\rm exp}$ and $T > 0$, the estimate in (7.3) holds true for all $\sigma \in (0,1]$ with a σ-independent $C(T) > 0$.*

Proof. For each $\gamma \in \Gamma_*$ and $\sigma \in (0,1]$, we have that $\delta_\gamma \in {\rm Dom}(L^\dagger_\sigma)$, see (6.29) and (6.23). By standard semigroup formulas, e.g., [11, page 9], it then follows

$$p^\sigma_u(\gamma, \cdot) = \delta_\gamma + \int_0^u L^\dagger_\sigma p^\sigma_v(\gamma, \cdot) dv. \tag{7.5}$$

By (7.1) and the latter one gets

$$w_u^\sigma(\gamma) = w_0^\sigma(\gamma) + \int_0^u \left(\int_{\Gamma_*} \rho(\gamma,\gamma')(L_\sigma^\dagger p_v^\sigma(\gamma, d\gamma')) \right) dv \tag{7.6}$$
$$= \int_0^u \left(\int_{\Gamma_*} (L_\sigma \rho(\gamma,\gamma')) p_v^\sigma(\gamma, d\gamma') \right) dv,$$

since $w_0^\sigma(\gamma) = \rho(\gamma,\gamma) = 0$. With the help of the triangle inequality one then obtains, cf. (6.5) and (2.27),

$$|\nabla_x \rho(\gamma,\gamma')| = |\rho(\gamma,\gamma'\cup x) - \rho(\gamma,\gamma')| \leq \rho(\gamma'\cup x, \gamma') \leq \psi(x), \tag{7.7}$$

which yields

$$|L_\sigma \rho(\gamma,\gamma')| \leq \|b\|\langle\psi\rangle + \|m\| F_1(\gamma') + \sum_{x\in\gamma'} \psi(x) \sum_{y\in\gamma'\setminus x} a_\sigma(x,y) =: V_\sigma(\gamma'). \tag{7.8}$$

Then similarly as in (7.6) by (7.5) and the latter one gets

$$h_v^\sigma(\gamma) := \int_{\Gamma_*} (L_\sigma \rho(\gamma,\gamma')) p_v^\sigma(\gamma, d\gamma') \leq \int_{\Gamma_*} V_\sigma(\gamma') p_v^\sigma(\gamma, d\gamma') \tag{7.9}$$
$$= V_\sigma(\gamma) + \int_0^v (L_\sigma V_\sigma(\gamma')) p_s^\sigma(\gamma, d\gamma') ds.$$

By (7.8) it follows that

$$\nabla_x V_\sigma(\gamma') = \|m\|\psi(x) + \omega_\sigma(x,\gamma'), \tag{7.10}$$

where

$$\omega_\sigma(x,\gamma') = \sum_{z\in\gamma'} \left(\psi(x) a_\sigma(x,z) + \psi(z) a_\sigma(z,x)\right). \tag{7.11}$$

Since $\nabla_x V_\sigma(\gamma') \geq 0$, cf. (4.16), (4.18), then

$$L_\sigma V_\sigma(\gamma') \leq L_\sigma^+ V_\sigma(\gamma') = \|m\| \int_X b_\sigma(x)\psi(x) dx \tag{7.12}$$
$$+ \int_X b_\sigma(x) \left(\sum_{z\in\gamma'} \psi(z) a_\sigma(z,x) + \psi(x) \sum_{y\in\gamma'} a_\sigma(x,y) \right) dx$$
$$\leq \|m\|\|b\|\langle\psi\rangle + \|b\|\langle a\rangle F_1(\psi) + \|b\| \sum_{x\in\gamma'} f_\sigma(x) =: \widetilde{V}_\sigma(\gamma'),$$

where

$$f_\sigma(x) = \int_X a_\sigma(z,x)\psi(z) dz. \tag{7.13}$$

We apply (7.12) in (7.9) and get

$$h_v^\sigma(\gamma) \leq V_\sigma(\gamma) + v\widetilde{V}_\sigma(\gamma) + \int_0^v \int_0^s \int_{\Gamma_*} \left(L_\sigma \widetilde{V}_\sigma(\gamma') \right) p_t^\sigma(\gamma, d\gamma') dt ds. \tag{7.14}$$

Furthermore, by (7.12), (7.13) we have

$$L_\sigma \widetilde{V}_\sigma(\gamma') \leq L_\sigma^+ \widetilde{V}_\sigma(\gamma') \leq \|b\|^2 \left(\langle a\rangle\langle\psi\rangle + \int_X f_\sigma(x) dx \right) \leq 2\|b\|^2 \langle\psi\rangle\langle a\rangle =: C_V,$$

which we use in (7.14) and then in (7.6), and thereafter obtain

$$w_u^\sigma(\gamma) \leq \tilde{w}_u^\sigma(\gamma) := u V_\sigma(\gamma) + \frac{u^2}{2}\widetilde{V}_\sigma(\gamma) + \frac{u^3}{6} C_V. \tag{7.15}$$

Now we turn to estimating $W^\sigma_{u,v}$. First we write, see the second line in (7.1) and (7.15),

$$\begin{aligned} W^\sigma_{u,v}(\gamma) &\leq u\Upsilon^\sigma_{1,v}(\gamma) + \frac{u^2}{2}\Upsilon^\sigma_{2,v}(\gamma) + \frac{u^3}{6}\Upsilon^\sigma_{3,v}(\gamma), \qquad (7.16) \\ \Upsilon^\sigma_{1,v}(\gamma) &= \int_{\Gamma_*} \rho(\gamma,\gamma')V_\sigma(\gamma')p^\sigma_v(\gamma, d\gamma') = \int_0^v \int_{\Gamma_*} L_\sigma \rho(\gamma,\gamma')V_\sigma(\gamma')p^\sigma_s(\gamma, d\gamma')ds, \\ \Upsilon^\sigma_{2,v}(\gamma) &= \int_{\Gamma_*} \rho(\gamma,\gamma')\widetilde{V}_\sigma(\gamma')p^\sigma_v(\gamma, d\gamma'), \\ \Upsilon^\sigma_{3,v}(\gamma) &= C_V \int_{\Gamma_*} \rho(\gamma,\gamma')p^\sigma_v(\gamma, d\gamma'). \end{aligned}$$

In the same way as in (7.7) by (7.10) we get

$$\begin{aligned} |\nabla_x \rho(\gamma,\gamma')V_\sigma(\gamma')| &\leq \rho(\gamma, \gamma'\cup x)\nabla_x V_\sigma(\gamma') + |\nabla_x \rho(\gamma,\gamma')|V_\sigma(\gamma') \\ &\leq \psi(x)V_\sigma(\gamma') + \|m\|\psi(x) + \omega_\sigma(x,\gamma'). \end{aligned}$$

Thereafter, similarly as in (7.7), (7.8) it follows that

$$\begin{aligned} |L_\sigma \rho(\gamma,\gamma')V_\sigma(\gamma')| &\leq \big(V_\sigma(\gamma') + \|m\|\big)\int_X b_\sigma(x)\psi(x)dx \qquad (7.17) \\ &+ \big(V_\sigma(\gamma') + \|m\|\big)\sum_{x\in\gamma'}\psi(x)\left[m_\sigma(x) + \sum_{y\in\gamma'\setminus x} a_\sigma(x,y)\right] \\ &+ \int_X b_\sigma(x)\omega_\sigma(x,\gamma')dx \\ &+ \sum_{x\in\gamma'}\left(m_\sigma(x) + \sum_{y\in\gamma'\setminus x} a_\sigma(x,y)\right)\omega_\sigma(x,\gamma'\setminus x), \\ &\leq V_\sigma(\gamma')[\|m\| + V_\sigma(\gamma')] + \Delta_\sigma(\gamma') =: V_{1,\sigma}(\gamma'), \end{aligned}$$

where $\omega_\sigma(x,\gamma')$ is given in (7.11) and

$$\Delta_\sigma(\gamma') = \int_X b_\sigma(x)\omega_\sigma(x,\gamma')dx + \sum_{x\in\gamma'}\left(m_\sigma(x) + \sum_{y\in\gamma'\setminus x} a_\sigma(x,y)\right)\omega_\sigma(x,\gamma'\setminus x). \qquad (7.18)$$

Then, see (7.16),

$$\Upsilon^\sigma_{1,v}(\gamma) \leq \int_0^v \int_{\Gamma_*} V_{1,\sigma}(\gamma')p^\sigma_s(\gamma', d\gamma)ds. \qquad (7.19)$$

Now we recall that μ^σ_t lies in $\mathcal{P}_{\rm exp}$ and thus, see Lemma 4.4, its correlation functions satisfy

$$0 \leq k^{(n)}_{\mu^\sigma_t}(x_1,\dots,x_n) \leq \varkappa(t) := \varkappa_{\mu_0} + \|b\|t, \qquad (7.20)$$

where the right-hand side is independent of σ. By (7.2) and then by (7.16) one can come to the conclusion that proving (7.3) now amounts to estimating the integrals

$$I^\sigma_j(t_2 - t_1) = \int_{\Gamma_*} \Upsilon^\sigma_{j,t_2-t_1}(\gamma)\mu^\sigma_{t_1}(d\gamma), \qquad j = 1,2,3. \qquad (7.21)$$

Indeed, in this setting we have

$$\begin{aligned}
\widehat{W}^\sigma(t_1,t_2,t_3) &\leq (t_3-t_2)\Big(I_1^\sigma(t_2-t_1) \\
&+ \frac{(t_3-t_2)}{2} I_2^\sigma(t_2-t_1) + \frac{(t_3-t_2)^2}{6} I_3^\sigma(t_2-t_1)\Big) \\
&\leq (t_3-t_2) I_1^\sigma(t_2-t_1) + \frac{(t_3-t_1)^2}{2}\left(I_2^\sigma(t_2-t_1) + \frac{t_3-t_2}{3} I_3^\sigma(t_2-t_1)\right).
\end{aligned} \tag{7.22}$$

Note that the second summand in the last line of (7.22) already has the desired power of $t_3 - t_1$. Thus, it is enough for us just to get σ-independent bounds for $I_2^\sigma(t_2-t_1)$ and $I_3^\sigma(t_2-t_1)$. At the same time, $I_1^\sigma(t_2-t_1)$ requires a more accurate estimating.

By the semigroup property it follows that

$$\int_{\Gamma_*} p_s^\sigma(\gamma, d\gamma')\mu_{t_1}^\sigma(d\gamma) = \mu_{t_1+s}^\sigma(d\gamma'). \tag{7.23}$$

We take this into account and by (7.21), (7.17), (7.18) and (7.19) obtain

$$\begin{aligned}
I_1^\sigma(t_2-t_1) &\leq \int_0^{t_2-t_1}\int_{\Gamma_*} V_{1,\sigma}(\gamma)\mu_{t_1+s}^\sigma(d\gamma)ds \\
&= I_{1a}^\sigma(t_2-t_1) + I_{1b}^\sigma(t_2-t_1) + I_{1c}^\sigma(t_2-t_1),
\end{aligned} \tag{7.24}$$

where

$$\begin{aligned}
I_{1a}^\sigma(t_2-t_1) &= \|m\|\int_0^{t_2-t_1}\int_{\Gamma_*} V_\sigma(\gamma)\mu_{t_1+s}^\sigma(d\gamma)ds \\
I_{1b}^\sigma(t_2-t_1) &= \int_0^{t_2-t_1}\int_{\Gamma_*} [V_\sigma(\gamma)]^2\mu_{t_1+s}^\sigma(d\gamma)ds \\
I_{1c}^\sigma(t_2-t_1) &= \int_0^{t_2-t_1}\int_{\Gamma_*} \Delta_\sigma(\gamma)\mu_{t_1+s}^\sigma(d\gamma)ds.
\end{aligned} \tag{7.25}$$

To estimate the integrals in (7.25) we employ the fact that each of the integrands can be presented as KG with an appropriate G, and then use (2.19) and (7.20). By (7.8) it follows that

$$\begin{gathered}
V_\sigma = KG_\sigma, \quad G_\sigma^{(0)} = \|b\|\langle\psi\rangle, \quad G_\sigma^{(1)}(x) = \|m\|\psi(x), \\
G_\sigma^{(2)}(x,y) = \psi(x)a_\sigma(x,y) + \psi(y)a_\sigma(y,x),
\end{gathered} \tag{7.26}$$

and $G_\sigma^{(k)} \equiv 0$ for $k > 2$. Then

$$\begin{aligned}
I_{1a}^\sigma(t_2-t_1) = \|m\|\int_0^{t_2-t_1}&\Big(\|b\|\langle\psi\rangle + \|m\|\int_X \psi(x)k^{(1)}_{\mu^\sigma_{t_1+s}}(x)dx \\
&+ \int_{X^2}\psi(x)a_\sigma(x,y)k^{(2)}_{\mu^\sigma_{t_1+s}}(x,y)dxdy\Big)ds \\
\leq (t_2-t_1)\|m\|\langle\psi\rangle&\Big(\|b\| + \|m\|\varkappa(T) + \langle a\rangle[\varkappa(T)]^2\Big) := (t_2-t_1)C_{1a}(T).
\end{aligned} \tag{7.27}$$

To proceeed futher, we write

$$V_\sigma = KG_\sigma \star G_\sigma =: KH_\sigma, \quad H_\sigma(\eta) = \sum_{\xi_1\subset\eta}\sum_{\xi_2\subset\eta\setminus\xi_1} G_\sigma(\xi_1\cup\xi_2)G_\sigma(\eta\setminus\xi_2),$$

see (4.7), which by (7.26) yields

$$H_\sigma^{(0)} = [\|b\|\langle\psi\rangle]^2, \quad H_\sigma^{(1)}(x) = 2\|b\|\|m\|\langle\psi\rangle\psi(x) + [\|m\|\psi(x)]^2,$$

$$H_\sigma^{(2)}(x_1,x_2) = 2\|b\|\langle\psi\rangle G_\sigma^{(2)}(x_1,x_2) + 2\|m\|^2\psi(x_1)\psi(x_2)$$

$$+2\|m\|(\psi(x_1)+\psi(x_2))G_\sigma^{(2)}(x_1,x_2) + [G_\sigma^{(2)}(x_1,x_2)]^2,$$

$$H_\sigma^{(3)}(x_1,x_2,x_3) = 2\|m\|\psi(x_1)G_\sigma^{(2)}(x_2,x_3) + 2\|m\|\psi(x_2)G_\sigma^{(2)}(x_1,x_3)$$

$$+2\|m\|\psi(x_3)G_\sigma^{(2)}(x_1,x_2) + 2G_\sigma^{(2)}(x_1,x_2)G_\sigma^{(2)}(x_1,x_3)$$

$$+2G_\sigma^{(2)}(x_1,x_2)G_\sigma^{(2)}(x_2,x_3) + 2G_\sigma^{(2)}(x_1,x_3)G_\sigma^{(2)}(x_2,x_3),$$

and

$$H_\sigma^{(4)}(x_1,x_2,x_3,x_4) = G_\sigma^{(2)}(x_1,x_2)G_\sigma^{(2)}(x_3,x_4) + G_\sigma^{(2)}(x_1,x_3)G_\sigma^{(2)}(x_2,x_4)$$

$$+G_\sigma^{(2)}(x_1,x_4)G_\sigma^{(2)}(x_2,x_3).$$

Then similarly as in (7.27) by means of the estimates $G_\sigma^{(2)}(x_1,x_2) \le 2\|a\|$ and

$$\int_{X^2} G_\sigma^{(2)}(x_1,x_2)dx_1dx_2 \le 2\int_{X^2}\psi(x_1)a(x_1-x_2)dx_1dx_2 = 2\langle\psi\rangle\langle a\rangle,$$

$$\int_{X^3} G_\sigma^{(2)}(x,y)G_\sigma^{(2)}(y,z)dxdydz \le 4\langle a\rangle^2\langle\psi\rangle,$$

we get

$$\begin{aligned} I_{1b}^\sigma(t_2-t_1) &= \int_0^{t_2-t_1}\int_{\Gamma_{\rm fin}} H_\sigma(\eta)k_{\mu_{t_1+s}^\sigma}(\eta)\lambda(d\eta) \\ &\le (t_2-t_1)\bigg((\|b\|\langle\psi\rangle)^2 + \|m\|\langle\psi\rangle\left(2\|b\|\langle\psi\rangle + \|m\|\right)\varkappa(T) \\ &+ \langle\psi\rangle\left(2\|b\|\langle a\rangle\langle\psi\rangle + \|m\|^2\langle\psi\rangle + 4\|m\|\langle a\rangle + 2\|a\|\langle a\rangle\right)[\varkappa(T)]^2 \\ &+ 2\langle\psi\rangle\langle a\rangle\left(\|m\|\langle\psi\rangle + 2\langle a\rangle\right)[\varkappa(T)]^3 + \frac{1}{2}\left(\langle\psi\rangle\langle a\rangle\right)^2[\varkappa(T)]^4\bigg) \\ &=: (t_2-t_1)C_{1b}(T). \end{aligned} \tag{7.28}$$

By (7.26), (7.11) and (7.18) one can write

$$\Delta_\sigma(\gamma) = \int_X b_\sigma(x)\sum_{y\in\gamma}G_\sigma^{(2)}(x,y)dx + \sum_{x\in\gamma}m_\sigma(x)\sum_{y\in\gamma\setminus x}G_\sigma^{(2)}(x,y)$$

$$+\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}a_\sigma(x,y)G_\sigma^{(2)}(x,y) + \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\sum_{z\in\gamma\setminus\{x,y\}}a_\sigma(x,y)G_\sigma^{(2)}(x,z).$$

Then similarly as in (7.28) one obtains

$$\begin{aligned} I_{1c}^\sigma(t_2-t_1) &\le (t_2-t_1)2\langle\psi\rangle\langle a\rangle\varkappa(t_2)\bigg(\|b\| + [\|m\|+\|a\|]\varkappa(T) + \langle a\rangle[\varkappa(T)]^2\bigg) \\ &=: (t_2-t_1)C_{1c}(T). \end{aligned} \tag{7.29}$$

By (7.24), (7.27), (7.28) and (7.29) we then get

$$I_1^\sigma(t_2-t_1) \le (t_2-t_1)(C_{1a}(T)+C_{1b}(T)+C_{1c}(T)) =: (t_2-t_1)C_1(T). \tag{7.30}$$

Now similarly as in (7.21) and (7.16), (7.12) we get

$$\begin{aligned} I_2^\sigma(t_2-t_1) &= \int_{\Gamma_*^2} \rho(\gamma,\gamma')\widetilde{V}_\sigma(\gamma')p^\sigma_{t_2-t_1}(\gamma,d\gamma')\mu^\sigma_{t_1}(d\gamma) \\ &\le \int_{\Gamma_*} \widetilde{V}_\sigma(\gamma)\mu^\sigma_{t_2}(d\gamma) \le \|b\|\langle\psi\rangle\left(\|m\| + 2\langle a\rangle \varkappa(T)\right) =: C_2(T), \end{aligned} \tag{7.31}$$

where we used (7.23) and the fact that $\rho(\gamma,\gamma') \le 1$, see (2.27), and the estimate

$$\int_X f_\sigma(x)dx \le \langle\psi\rangle\langle a\rangle,$$

see (7.13). Similarly,

$$I_3^\sigma(t_2-t_1) = C_V \int_{\Gamma_*^2} \rho(\gamma,\gamma')p^\sigma_{t_2-t_1}(\gamma,d\gamma')\mu^\sigma_{t_1}(d\gamma) \le C_V. \tag{7.32}$$

Now we employ (7.32), (7.31) and (7.30) in (7.22) and thereby come to the conclusion that the desired inequality in (7.3) holds true with

$$C(T) = \frac{1}{4}C_1(T) + \frac{1}{2}C_2(T) + \frac{T}{6}C_V,$$

which completes the proof. □

7.2. **The weak convergence.** The aim of this subsection is to prove that the aforementioned states $\hat{\mu}^\sigma_t = \mu^\sigma_t$, see Lemma 6.6, weakly converge to the corresponding states μ_t which solve the Fokker-Planck equation for $(L,\mathcal{F},\mu_0)$. By this we also demonstrate that condition (a) of Proposition 7.1 is met. We begin by proving the following lemma in which α_t is as in (6.10). Recall also that the Banach spaces $\mathcal{G}_\alpha$ are defined in (2.32).

Lemma 7.4. *For some α_0 and all $\sigma \in (0,1]$, let μ_0 and μ_0^σ be in $\mathcal{P}^{\alpha_0}_{\rm exp}$. Assume also that*

$$\forall \alpha \in \mathbb{R} \ \ \forall G \in \mathcal{G}_\alpha \qquad \langle\!\langle k_{\mu_0^\sigma}, G\rangle\!\rangle \to \langle\!\langle k_{\mu_0}, G\rangle\!\rangle \qquad \text{as} \ \ \sigma \to 0. \tag{7.33}$$

Next, for these μ_0^σ and μ_0, let μ_t^σ and μ_t be the unique solutions of the Fokker-Planck equation for $(L_\sigma,\mathcal{F},\mu_0^\sigma)$ and $(L,\mathcal{F},\mu_0)$, respectively. Then for each $t > 0$, the following holds

$$\forall \alpha \in \mathbb{R} \ \ \forall G \in \mathcal{G}_\alpha \qquad \langle\!\langle k_{\mu_t^\sigma}, G\rangle\!\rangle \to \langle\!\langle k_{\mu_t}, G\rangle\!\rangle \qquad \text{as} \ \ \sigma \to 0. \tag{7.34}$$

Proof. Recall that $\alpha_t = \ln(e^{\alpha_0} + \|b\|t)$, see (6.10). Also, by Proposition 5.3 it follows that $k_{\mu_t} = k_t$, where the latter solves the Cauchy problem in (5.8), and hence

$$k_{\mu_{t+s}} = k_{t+s} = Q_{\alpha\alpha_t}k_t, \qquad s < T(\alpha,\alpha_t), \quad \alpha > \alpha_t. \tag{7.35}$$

A similar representation holds also for $k_{\mu^\sigma_{t+s}} = k^\sigma_{t+s}$, see (6.8), (6.9).

Assume that the convergence stated in (7.34) holds for a given $t \ge 0$, see (7.33). Let us then prove that there exists a possibly t-dependent $s_0 > 0$ such that this convergence holds also for $t+s$ with $s \le s_0$. Write, cf. (7.35),

$$k_{t+s} - k^\sigma_{t+s} = Q_{\bar{\alpha}_t\alpha_t}k_t - Q^\sigma_{\bar{\alpha}_t\alpha_t}k^\sigma_t,$$

where $\bar{\alpha}_t = \alpha_t + 1$ and the equality is written in $\mathcal{K}_{\bar{\alpha}_t}$. In view of (5.14), we then obtain, cf. (5.36),

$$\begin{aligned} k_{t+s} - k^\sigma_{t+s} &= Q_{\bar{\alpha}_t \alpha_t}(s)(k_t - k^\sigma_t) - \left(\int_0^s \frac{d}{du} [Q_{\bar{\alpha}_t \alpha_1}(s-u) Q^\sigma_{\alpha_1 \alpha_t}(u)] du \right) k^\sigma_t \quad (7.36) \\ &= Q_{\bar{\alpha}_t \alpha_t}(s)(k_t - k^\sigma_t) + \int_0^s Q_{\bar{\alpha}_t \alpha_2}(s-u) L^\Delta_{\alpha_2 \alpha_1} Q^\sigma_{\alpha_1 \alpha_t}(u) k^\sigma_t du \\ &- \int_0^s Q_{\bar{\alpha}_t \alpha_2}(s-u) (L^\Delta_\sigma)_{\alpha_2 \alpha_1} Q^\sigma_{\alpha_1 \alpha_t}(u) k^\sigma_t du \\ &= Q_{\bar{\alpha}_t \alpha_t}(s)(k_t - k^\sigma_t) + \int_0^s Q_{\bar{\alpha}_t \alpha_2}(s-u) \widetilde{L}^{\Delta,\sigma}_{\alpha_2 \alpha_1} Q^\sigma_{\alpha_1 \alpha_t}(u) k^\sigma_t du \end{aligned}$$

where, see (5.2), (6.2) and (6.7),

$$\begin{aligned} \widetilde{L}^{\Delta,\sigma}_{\alpha' \alpha} k(\eta) &:= [L^\Delta_{\alpha'\alpha} - (L^\Delta_\sigma)_{\alpha'\alpha}] k(\eta) = \sum_{x \in \eta} \tilde{b}_\sigma(x) k(\eta \setminus x) \quad (7.37) \\ &- \widetilde{E}_\sigma(\eta) k(\eta) - \int_X \sum_{y \in \eta} \tilde{a}_\sigma(x,y) k(\eta \cup x) dx, \end{aligned}$$

and

$$\begin{aligned} \tilde{b}_\sigma(x) &= b(x) - b_\sigma(x) = b(x) \tilde{\psi}_\sigma(x), \qquad \tilde{\psi}_\sigma(x) = \frac{\sigma |x|^{d+1}}{1 + \sigma |x|^{d+1}} \quad (7.38) \\ \tilde{a}_\sigma(x,y) &= a(x-y)[1 - \psi_\sigma(x)\psi_\sigma(y)], \\ \widetilde{E}_\sigma(\eta) &= \sum_{x \in \eta} m(x) \tilde{\psi}_\sigma(x) + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} \tilde{a}_\sigma(x,y). \end{aligned}$$

In the last line in (7.36), $\alpha_1 \in (\alpha_t, \bar{\alpha}_t)$ and $\alpha_2 \in (\alpha_1, \bar{\alpha}_t)$ should be such that

$$s < \min \{ T(\bar{\alpha}_t, \alpha_2); T(\alpha_1, \alpha_t) \}.$$

Set $\alpha_1 = \alpha_t + \varepsilon$ and find $\varepsilon \in (0,1)$ from the condition

$$T(\alpha_t + \varepsilon, \alpha_t) = T(\alpha_t + 1, \alpha_t + \varepsilon). \quad (7.39)$$

Recall that $\alpha_t = \ln(e^{\alpha_0} + \|b\| t)$. The equation in (7.39) has a unique solution which defines a continuous decreasing function $\varepsilon : \mathbb{R}_+ \to (0,1)$ such that $\lim_{t \to +\infty} \varepsilon(t) = \varepsilon_* \in (0,1)$, where the latter is a unique solution of the equation $\varepsilon = (1-\varepsilon) e^{-(1-\varepsilon)}$. Then we define

$$\upsilon(t) = T(\alpha_t + \varepsilon(t), \alpha_t), \quad (7.40)$$

take some $\epsilon \in (0,1)$, and set

$$s_0 = \epsilon \upsilon(t). \quad (7.41)$$

It is possible to show that

$$\alpha_{t+\upsilon(t)} < \alpha_t + \varepsilon(t), \quad \text{hence} \quad k_{t+s} \in \mathcal{K}_{\alpha_{t+\upsilon(t)}} \subset \mathcal{K}_{\alpha_1}, \quad \text{for} \quad s \le s_0. \quad (7.42)$$

As the map $\alpha \mapsto T(\alpha', \alpha)$ is continuous, see (5.11), one finds $\alpha_2 \in (\alpha_t + \varepsilon(t), \alpha_t + 1)$ such that $s_0 < T(\alpha_t + 1, \alpha_2)$. Thus, for the chosen in this way α_1 and α_2, the operators $Q_{\bar{\alpha}_t \alpha_2}(s-u)$ and $Q^\sigma_{\alpha_1 \alpha_t}(u)$ in the last line of (7.36) make sense for all $s \le s_0$ and $u \le s$. Since G lies in $\mathcal{G}_\alpha$ with any α, we take $G \in \mathcal{G}_{\bar{\alpha}_t}$ and set $G_s = \breve{Q}_{\alpha_2 \bar{\alpha}_t}(s) G$, $s \le s_0$, see (5.23).

This G_s lies in $\mathcal{G}_{\alpha_2} \subset \mathcal{G}_{\alpha_t}$, cf. (2.33), and then get from (7.36) the following formula

$$\langle\!\langle k_{t+s} - k_{t+s}^\sigma, G \rangle\!\rangle = \langle\!\langle k_t - k_t^\sigma, G_s \rangle\!\rangle + \Upsilon_\sigma(s), \tag{7.43}$$

$$\Upsilon_\sigma(s) := \int_0^s \langle\!\langle \widetilde{L}^{\Delta,\sigma}_{\alpha_2\alpha_1} k^\sigma_{t+u}, G_{s-u} \rangle\!\rangle du.$$

Our aim now is to prove that $\Upsilon_\sigma(s) \to 0$ as $\sigma \to 0$. In view of (7.37) and (7.38), we then write

$$\begin{aligned} \Upsilon_\sigma(s) &= \Upsilon^{(1)}_\sigma(s) + \Upsilon^{(2)}_\sigma(s) + \Upsilon^{(3)}_\sigma(s) + \Upsilon^{(4)}_\sigma(s), \qquad (7.44) \\ \Upsilon^{(1)}_\sigma(s) &= \int_0^s \left(\int_{\Gamma_{\rm fin}} \sum_{x\in\xi} \tilde{b}_\sigma(x) k^\sigma_{t+u}(\xi\setminus x) G_{s-u}(\xi)\lambda(d\xi) \right) du, \\ \Upsilon^{(2)}_\sigma(s) &= -\int_0^s \left(\int_{\Gamma_{\rm fin}} \sum_{x\in\xi} m(x)\tilde{\psi}_\sigma(x) k^\sigma_{t+u}(\xi) G_{s-u}(\xi)\lambda(d\xi) \right) du, \end{aligned}$$

and

$$\begin{aligned} \Upsilon^{(3)}_\sigma(s) &= -\int_0^s \left(\int_{\Gamma_{\rm fin}} \sum_{x\in\xi} \sum_{y\in\xi\setminus x} \tilde{a}_\sigma(x,y) k^\sigma_{t+u}(\xi) G_{s-u}(\xi)\lambda(d\xi) \right) du \qquad (7.45) \\ \Upsilon^{(4)}_\sigma(s) &= -\int_0^s \left(\int_{\Gamma_{\rm fin}} \int_X \sum_{y\in\xi} \tilde{a}_\sigma(x,y) k^\sigma_{t+u}(\xi\cup x) G_{s-u}(\xi) dx \lambda(d\xi) \right) du. \end{aligned}$$

By Lemma 4.4 it follows that $k^\sigma_{t+u}(\xi) \leq e^{\alpha_{t+u}|\xi|} \leq e^{\alpha_1|\xi|}$, see (7.42). We take this into account in (7.44), (7.45) and then get

$$|\Upsilon^{(i)}_\sigma(s)| \leq \int_0^s \int_{\Gamma_{\rm fin}} \widetilde{H}^{(i)}_\sigma(\xi) |G_{s-u}(\xi)| \lambda(d\xi) du, \qquad i = 1,2,3,4. \tag{7.46}$$

Here, see (7.38),

$$\begin{aligned} \widetilde{H}^{(1)}_\sigma(\xi) &= e^{\alpha_1|\xi| - \alpha_1} \sum_{x\in\xi} \tilde{\psi}_\sigma(x) b(x) \leq e^{-\alpha_1}\|b\||\xi| e^{\alpha_1|\xi|} =: C^{(1)}|\xi|e^{\alpha_1|\xi|}, \qquad (7.47) \\ \widetilde{H}^{(2)}_\sigma(\xi) &= e^{\alpha_1|\xi|} \sum_{x\in\xi} \tilde{\psi}_\sigma(x) m(x) \leq \|m\||\xi| e^{\alpha_1|\xi|} =: C^{(2)}|\xi|e^{\alpha_1|\xi|}, \\ \widetilde{H}^{(3)}_\sigma(\xi) &= e^{\alpha_1|\xi|} \sum_{x\in\xi} \sum_{y\in\xi\setminus x} \tilde{a}_\sigma(x,y) \leq \|a\||\xi|^2 e^{\alpha_1|\xi|} =: C^{(3)}|\xi|^2 e^{\alpha_1|\xi|}, \\ \widetilde{H}^{(4)}_\sigma(\xi) &= e^{\alpha_1|\xi| + \alpha_1} \int_X \sum_{y\in\xi} \tilde{a}_\sigma(x,y) dx \leq e^{\alpha_1}\langle a \rangle |\xi| e^{\alpha_1|\xi|} =: C^{(4)}|\xi|e^{\alpha_1|\xi|}. \end{aligned}$$

By the elementary inequality $se^{-\beta s} \leq 1/\beta e$, $s, \beta > 0$, we obtain

$$\begin{aligned} \mathrm{RHS}(7.46) &\leq \frac{C^{(i)}}{e(\alpha_2 - \alpha_1)} \int_0^s \int_{\Gamma_{\rm fin}} e^{\alpha_2|\xi|} |G_{s-u}(\xi)| \lambda(d\xi) du \qquad (7.48) \\ &= \frac{C^{(i)}}{e(\alpha_2 - \alpha_1)} \int_0^s |G_{s-u}|_{\alpha_2} du \\ &\leq \frac{sC^{(i)}}{e(\alpha_2 - \alpha_1)} \frac{T(\bar{\alpha}_t, \alpha_2)|G|_{\bar{\alpha}_t}}{T(\bar{\alpha}_t, \alpha_2) - s_0}, \qquad i = 1,2,4, \end{aligned}$$

where we have used also (5.24). For $i = 3$, by the same calculations we have

$$\mathrm{RHS}(7.46) \leq \frac{4s\|a\|}{[e(\alpha_2 - \alpha_1)]^2} \frac{T(\bar{\alpha}_t, \alpha_2)|G|_{\bar{\alpha}_t}}{T(\bar{\alpha}_t, \alpha_2) - s_0}. \tag{7.49}$$

In view of (7.38), each of $\widetilde{H}^{(i)}_\sigma(\xi)$, $i = 1, \dots, 4$ decreases to zero as $\sigma \to 0$. By (7.46), (7.48), (7.49) and the monotone convergence theorem this yields

$$\Upsilon^{(i)}_\sigma(s) \to 0, \qquad \text{as} \quad \sigma \to 0, \tag{7.50}$$

holding for all $i = 1, \dots, 4$. By (7.50) and (7.43) we then get that the convergence as in (7.34) can be prolonged from t to $t + s$, $s \leq s_0$, see (7.41). Our aim now is to prolong this ad infinitum. Define

$$t_l = t_{l-1} + s_l, \quad t_0 = 0, \quad s_l = \epsilon v(t_{l-1}), \quad l \in \mathbb{N}. \tag{7.51}$$

Since $k_0 = k_0^\sigma = k_{\mu_0}$, the developed above arguments yields the stated convergence for $t \leq \sup_l t_l = \lim_{l \to +\infty} t_l$. Then to complete the proof we have to show that $t_l \to +\infty$. Assume that $\sup_l t_l = t_* < +\infty$. By (7.51) it follows that $t_l = s_1 + \cdots + s_l$; hence, $s_l \to 0$ in this case. The function $v(t)$ defined in (7.40) is continuous – for both $\varepsilon(t)$ and α_t are continuous. By (7.51) we would then have

$$v(t_*) = \frac{\varepsilon(t_*)}{\|b\| e^{-\alpha_{t_*}} + \langle a \rangle e^{\alpha_{t_*} + \varepsilon(t_*)}} = 0,$$

which is obviously impossible. This completes the whole proof. □

The result just proved allows us to achive the main goal of this subsection.

Lemma 7.5. *Let μ_0, μ_0^σ, μ_t and μ_t^σ be as in Lemma 7.4. Then for each $t \geq 0$, it follows that $\mu_t^\sigma \Rightarrow \mu_t$ as $\sigma \to 0$, where we mean the weak convergence of measures on Γ_*.*

Proof. By Proposition 3.5, see (3.22), it follows that $\widetilde{F}_v = Ke(\cdot; h_v)$ with $e(\cdot; h_v) \in \mathcal{G}_\alpha$ for an arbitrary $\alpha \in \mathbb{R}$. Then we apply (2.19) and obtain by Lemma 7.4 that

$$\mu_t^\sigma(\widetilde{F}_v) = \langle\!\langle k_{\mu_t^\sigma}, e(\cdot; h_v) \rangle\!\rangle \to \langle\!\langle k_{\mu_t}, e(\cdot; h_v) \rangle\!\rangle = \mu_t(\widetilde{F}_v), \quad \sigma \to 0,$$

holding for all $t \geq 0$ and $v \in \mathcal{V}$. Then the proof follows by statement (ii) of Proposition 3.4. □

7.3. **The proof of Theorem 3.10.** By Lemmas 7.4 and 7.3 both conditions (a) and (b) of Proposition 7.1 are met. Therefore, for each $\sigma \in (0, 1]$, by statement (i) of the latter, see also Remark 7.2, a given measure $\mu \in \mathcal{P}_{\exp}$ determines probability measures $P^\sigma_{s,\mu}$, $s \geq 0$, on the cadlag path space $\mathfrak{D}_{[s,+\infty)}(\Gamma_*)$ through their marginals given in (7.4). In particular, this yields

$$P^\sigma_{s,\mu} \circ \varpi_t^{-1} = S^\sigma(t - s)\mu, \qquad t \geq s. \tag{7.52}$$

For each $\mu \in \mathcal{P}_{\exp}$, by Lemma 6.6 it follows that $S^\sigma(t - s)\mu = \mu_t$ also lies in $\mathcal{P}_{\exp}$, and $\varkappa_{\mu_t} \leq \varkappa_\mu + \|b\|(t - s)$. This yields that, for each fixed $\sigma \in (0, 1]$, the family $\{P^\sigma_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\exp}\}$ satisfies conditions (a), (b) and (c) of Definition 3.8. To prove that condition (d) is also met we take J as in (3.30) with given fixed m, $J_1, \dots J_m$ in $\widetilde{\mathcal{F}}$, see (3.16), and $s \leq s_1 < s_2 < \cdots < s_m \leq t_1$. Then define

$$\varsigma^\sigma_{1,s_1} = c_{1,\sigma} J_1 \mu^\sigma_{s_1} = c_{1,\sigma} J_1 S^\sigma(s_1 - s)\mu, \qquad c^{-1}_{1,\sigma} = \mu^\sigma_{s_1}(J_1) > 0. \tag{7.53}$$

The latter inequality follows by the fact that all J_l in J are strictly positive. Since $\mu^\sigma_{s_1} \in \mathcal{P}^{\alpha_{s_1}}_{\exp}$, $e^{\alpha_{s_1}} = \varkappa_\mu + \|b\|(s_1 - s)$, by item (c) of Remark 2.4 it follows that ς^σ_{1,s_1} is in $\mathcal{P}^{\tilde{\alpha}_{s_1}}_{\exp}$ with $\tilde{\alpha}_{s_1} = \alpha_{s_1} + \alpha_1$, where $\alpha_1 \geq 0$ may be taken σ-independent since the correlation functions of $\tilde{\varsigma}^\sigma_{1,s_1}$ obey estimates uniform in σ. Clearly, the choice of $\alpha_1 \geq 0$ dependes on J_1. Next we set

$$\varsigma^\sigma_{2,s_2} = S^\sigma(s_2 - s_1)\tilde{\varsigma}^\sigma_{1,s_1}, \qquad c^{-1}_{2,\sigma} = \varsigma^\sigma_{2,s_2}(J_2) > 0, \tag{7.54}$$
$$\tilde{\varsigma}^\sigma_{2,s_2} = c_{2,\sigma} J_2 \varsigma^\sigma_{2,s_2}.$$

Similarly as above, $\varsigma^\sigma_{2,s_2} \in \mathcal{P}^{\hat{\alpha}_{s_2}}_{\rm exp}$ and $\tilde{\varsigma}^\sigma_{2,s_2} \in \mathcal{P}^{\tilde{\alpha}_{s_2}}_{\rm exp}$ with $e^{\hat{\alpha}_{s_2}} = e^{\tilde{\alpha}_{s_1}} + \|b\|(s_2 - s_1)$ and $\tilde{\alpha}_{s_2} = \hat{\alpha}_{s_2} + \alpha_2$, where α_2 depends only on J_2 and is σ-independent. Then we continue proceeding in this way and thus obtain

$$\varsigma^\sigma_{l,s_l} = S^\sigma(s_l - s_{l-1})\tilde{\varsigma}^\sigma_{l-1,s_{l-1}}, \qquad c^{-1}_{l,\sigma} = \varsigma^\sigma_{l,s_l}(J_l) > 0, \tag{7.55}$$
$$\tilde{\varsigma}^\sigma_{l,s_l} = c_{l,\sigma} J_l \varsigma^\sigma_{l,s_l}, \qquad \tilde{\varsigma}^\sigma_{l,s_l} \in \mathcal{P}^{\tilde{\alpha}_{s_l}}_{\rm exp} \qquad l = 2, 3, \dots, m,$$

with $\tilde{\alpha}_{s_l} = \hat{\alpha}_{s_l} + \alpha_l$. As mentioned above, the choice of all $\tilde{\alpha}_{s_l}$ is σ-independent. Finally, we get

$$\varsigma^\sigma_u = S^\sigma(u - s_m)\tilde{\varsigma}^\sigma_{m,s_m}, \qquad u \geq s_m, \tag{7.56}$$

which lies in $\mathcal{P}^{\hat{\alpha}_m}_{\rm exp}$, $\hat{\alpha}_m \in \mathbb{R}$ is σ-independent. By (7.55) and Lemma 6.6 each $\varsigma^\sigma_{l,u}$ solves the Fokker-Planck equation for $(L_\sigma, \mathcal{F}_{\max}, \tilde{\varsigma}^\sigma_{l-1,s_{l-1}})$ on the time interval $[s_{l-1}, s_l]$, whereas ς^σ_u solves it for $(L_\sigma, \mathcal{F}_{\max}, \tilde{\varsigma}^\sigma_{m,s_m})$ on the time interval $[s_m, +\infty)$.

Now we take $F \in \widetilde{\mathcal{F}}$ and then write

$$\mathsf{F}_u = F \circ \varpi_u, \quad \mathsf{L}_u = LF \circ \varpi_u, \quad \mathsf{L}^\sigma_u = L_\sigma F \circ \varpi_u, \quad u \in [s_m, t_2]. \tag{7.57}$$

By (7.4) and (7.55), (7.56) and (7.57) it follows that

$$P^\sigma_{s,\mu}(\mathsf{F}_u \mathsf{J}) = C_\sigma P^\sigma_{s_m, \varsigma^\sigma_{m,s_m}}(\mathsf{F}_u) = C_\sigma \varsigma^\sigma_u(F), \qquad u \geq s_m, \tag{7.58}$$

$C_\sigma = P^\sigma_{s,\mu}(\mathsf{J}) = c^{-1}_{1,\sigma} \cdots c^{-1}_{m,\sigma}$. Thereby, we get

$$P^\sigma_{s,\mu}(\mathsf{H}) = P^\sigma_{s,\mu}(\mathsf{F}_{t_2}\mathsf{J}) - P^\sigma_{s,\mu}(\mathsf{F}_{t_1}\mathsf{J}) - P^\sigma_{s,\mu}\left(\int_{t_1}^{t_2} \mathsf{L}^\sigma_u \mathsf{J} du\right) \tag{7.59}$$
$$= C_\sigma \left(P^\sigma_{s_m,\varsigma^\sigma_{m,s_m}}(\mathsf{F}_{t_2}) - P^\sigma_{s_m,\varsigma^\sigma_{m,s_m}}(\mathsf{F}_{t_1}) - \int_{t_1}^{t_2} P^\sigma_{s_m,\varsigma^\sigma_{m,s_m}}(\mathsf{L}^\sigma_u) du \right)$$
$$= C_\sigma \left(\varsigma^\sigma_{t_2}(F) - \varsigma^\sigma_{t_1}(F) - \int_{t_1}^{t_2} \varsigma^\sigma_u (L_\sigma F) du \right) = 0.$$

While passing to the second line in (7.59), we have interchanged the integrations as the family $\{P^\sigma_{s,\mu} \circ \varpi_u^{-1} : u \in [s, t_2]\}$ satisfies the condition of Proposition 3.7. The last equality in (7.59) follows by the aforementioned fact that ς^σ_u solves the corresponding Fokker-Planck equation.

By Lemma 7.5 it follows that $\mu^\sigma_{s_1} \Rightarrow \mu_{s_1}$ as $\sigma \to 0$, where μ_{s_1} solves (1.2) for $(L, \mathcal{F}_{\max}, \mu)$ on $[s, +\infty)$. And also $c^{-1}_{1,\sigma} \to c^{-1}_1 = \mu_{s_1}(J_1)$, $\tilde{\varsigma}^\sigma_{1,s_1} \Rightarrow \tilde{\varsigma}_{1,s_1} := c_1 J_1 \mu_{s_1}$, see (7.53). By Proposition 7.4 it follows that

$$\langle\!\langle k_{\mu^\sigma_{s_1}}, G \rangle\!\rangle \to \langle\!\langle k_{\mu_{s_1}}, G \rangle\!\rangle, \qquad \sigma \to 0, \tag{7.60}$$

holding for all $G \in \mathcal{G}_\alpha$, $\alpha \in \mathbb{R}$. Let us prove that

$$\langle\!\langle k_{\tilde{\varsigma}^\sigma_{1,s_1}}, G \rangle\!\rangle \to \langle\!\langle k_{\tilde{\varsigma}_{1,s_1}}, G \rangle\!\rangle, \qquad \sigma \to 0, \tag{7.61}$$

holding for all $G \in \mathcal{G}_\alpha$, $\alpha \in \mathbb{R}$. To this end, we use (3.19), which yields

$$\langle\!\langle k_{\tilde{\varsigma}^\sigma_{1,s_1}}, G \rangle\!\rangle = c_{1,\sigma} \langle\!\langle k_{\mu^\sigma_{1,s_1}}, G_{v_1} \rangle\!\rangle,$$

where v_1 is such that $J_1 = \widetilde{F}_{v_1}$. By (3.23) it follows that $G_{v_1} \in \mathcal{G}_\alpha$ with any real α. By (3.18) and (7.53) we then get that the convergence in (7.61) follows by (7.60). Now by

(7.54) and Lemma 7.5 it follows that $\varsigma^\sigma_{2,s_2} \Rightarrow \varsigma_{2,s_2} \in \mathcal{P}^{\hat{\alpha}_2}_{\rm exp}$, where the latter solves the Fokker-Planck equation for $(L, \mathcal{F}_{\max}, \tilde{\varsigma}_{1,s_1})$ on $[s_1, +\infty)$. And also

$$\tilde{\varsigma}^\sigma_{2,s_2} \Rightarrow \tilde{\varsigma}_{2,s_2} := c_2 J_2 \varsigma_{2,s_2}, \qquad \langle\!\langle k_{\tilde{\varsigma}^\sigma_{2,s_2}}, G\rangle\!\rangle \to \langle\!\langle k_{\tilde{\varsigma}_{2,s_2}}, G\rangle\!\rangle, \qquad \sigma \to 0,$$

We continue this procedure and get

$$\varsigma^\sigma_{l,s_l} \Rightarrow \varsigma_{l,s_l}, \quad c^{-1}_{l,\sigma} \to c^{-1}_l = \varsigma_{l,s_l}(J_l), \quad \tilde{\varsigma}^\sigma_{l,s_l} \Rightarrow \tilde{\varsigma}_{l,s_l} = c_l J_l \varsigma^\sigma_{l,s_l}, \tag{7.62}$$

$$\langle\!\langle k_{\tilde{\varsigma}^\sigma_{l,s_l}}, G\rangle\!\rangle \to \langle\!\langle k_{\tilde{\varsigma}_{l,s_l}}, G\rangle\!\rangle, \qquad \sigma \to 0, \qquad l = 2, 3, \ldots, m.$$

Moreover, $\varsigma_{l,s_l} \in \mathcal{P}^{\tilde{\alpha}_{s_l}}_{\rm exp}$, where $\tilde{\alpha}_{s_l}$ is the same as in (7.55). It solves the Fokker-Planck equation for $(L, \mathcal{F}_{\max}, \tilde{\varsigma}^\sigma_{l-1,s_{l-1}})$ on $[s_{l-1}, +\infty)$. Finally, by (7.56) we get

$$\varsigma^\sigma_u \Rightarrow \varsigma_u, \qquad \langle\!\langle k_{\varsigma^\sigma_u}, G\rangle\!\rangle \to \langle\!\langle k_{\varsigma_u}, G\rangle\!\rangle, \qquad \sigma \to 0, \tag{7.63}$$

and ς_u solves the Fokker-Planck for $(L, \mathcal{F}_{\max}, \varsigma_{m,s_m})$ on $[s_m, +\infty)$. And also

$$\varsigma^\sigma_u, \varsigma_u \in \mathcal{P}^{\tilde{\alpha}_u}_{\rm exp}, \qquad \tilde{\alpha}_u = \ln[e^{\tilde{\alpha}_{s_m}} + \|b\|(u - s_m)]. \tag{7.64}$$

Note that the convergence of integrals involving correlation functions in (7.63) follows by the corresponding convergences in (7.62).

By Lemmas 7.3 and 7.4, and hence by Proposition 7.1, the set $\{P^\sigma_{s,\mu} : \sigma \in (0,1]\}$ has accumulations points. Let $P_{s,\mu}$ be one of them and $\{\sigma_n\}_{n\in\mathbb{N}}$ be such that $\sigma_n \to 0$ and $P^{\sigma_n}_{s,\mu} \Rightarrow P_{s,\mu}$ as $n \to +\infty$. Define

$$\hat{\varsigma}_u(\mathbb{A}) = C^{-1} P_{s,\mu}((\mathbb{1}_{\mathbb{A}} \circ \varpi_u)\mathsf{J}), \quad C = P_{s,\mu}(\mathsf{J}), \quad u \in [s_m, t_2], \quad \mathbb{A} \in \mathcal{B}(\Gamma_*). \tag{7.65}$$

Then C_{σ_n} and $\varsigma^{\sigma_n}_u$ defined in (7.58) satisfy, see Lemma 7.4,

$$C_{\sigma_n} \to C, \qquad \varsigma^{\sigma_n}_u \Rightarrow \hat{\varsigma}_u = \varsigma_u, \qquad n \to +\infty, \tag{7.66}$$

see (7.63). In view of (7.59), one can write

$$P_{s,\mu}(\mathsf{H}) = P_{s,\mu}(\mathsf{H}) - P^{\sigma_n}_{s,\mu}(\mathsf{H}) = a_n(t_2) - a_n(t_2) - \int_{t_1}^{t_2} b_n(u)du - \int_{t_1}^{t_2} c_n(u)du, \tag{7.67}$$

$$a_n(u) = P_{s,\mu}(\mathsf{F}_u\mathsf{J}) - P^{\sigma_n}_{s,\mu}(\mathsf{F}_u\mathsf{J}), \quad b_n(u) = P_{s,\mu}(\mathsf{L}_u\mathsf{J}) - P^{\sigma_n}_{s,\mu}(\mathsf{L}_u\mathsf{J}),$$

$$c_n(u) = P^{\sigma_n}_{s,\mu}((\mathsf{L}_u - \mathsf{L}^{\sigma_n}_u)\mathsf{J}), \quad u \in [s_m, t_2].$$

Since $\mathsf{F}_u\mathsf{J} \in C_{\rm b}(\mathfrak{D}_{[s,+\infty)}(\Gamma_*))$, $a_n(u) \to 0$ for each $u \in [s_m, t_2]$, that follows by the fact that $P^{\sigma_n}_{s,\mu} \Rightarrow P_{s,\mu}$ as $n \to +\infty$, see above. To proceed further, by (7.67), (7.65) and (7.58) we write

$$b_n(u) = C(\varsigma_u(LF) - \varsigma^{\sigma_n}_u(LF)) + (C - C_{\sigma_n})\varsigma^{\sigma_n}_u(LF). \tag{7.68}$$

By (3.11) it follows that $F = KG$ with $G \in \mathcal{G}_\alpha$, holding for each $\alpha \in \mathbb{R}$. Then, see (5.3), (5.22) and (7.64), we have

$$|\varsigma^{\sigma_n}_u(LF)| = |\langle\!\langle k_{\varsigma^{\sigma_n}_u}, \breve{L}_{\tilde{\alpha}_u\alpha}G\rangle\!\rangle| \le \int_{\Gamma_{\rm fin}} e^{\tilde{\alpha}_u|\eta|}|\breve{L}_{\tilde{\alpha}_u\alpha}G(\eta)|\lambda(d\eta) \le \|\breve{L}_{\tilde{\alpha}_u\alpha}\||G|_\alpha, \tag{7.69}$$

where the operator norm $\|\breve{L}_{\tilde{\alpha}_u\alpha}\|$ satisfies the same estimate as in (5.5). By (7.66) the second summand on the right-hand side of (7.68) vanishes as $n \to +\infty$ since the bound in (7.69) is independent of n. At the same time, similarly as in (7.69) we have

$$\varsigma_u(LF) - \varsigma^{\sigma_n}_u(LF) = \langle\!\langle k_{\varsigma_u}, \breve{L}_{\tilde{\alpha}_u\alpha}G\rangle\!\rangle - \langle\!\langle k_{\varsigma^{\sigma_n}_u}, \breve{L}_{\tilde{\alpha}_u\alpha}G\rangle\!\rangle \to 0, \quad n \to +\infty, \tag{7.70}$$

which follows by (7.63). Then $b_n(u) \to 0$ as $n \to +\infty$, holding for each $u \in [s_m, t_2]$. Finally, similarly as in (7.58), see also (7.37), we have

$$c_n(u) = C_{\sigma_n}\varsigma^{\sigma_n}_u((L - L_{\sigma_n})F) = C_{\sigma_n}\langle\!\langle \widetilde{L}^{\Delta,\sigma_n}_{\alpha\tilde{\alpha}_u} k_{\varsigma^{\sigma_n}_u}, G\rangle\!\rangle,$$

where G is the same as in (7.69), (7.70), i.e., such that $F = KG$. Recall that $G \in \mathcal{G}_\alpha$ for each $\alpha \in \mathbb{R}$, see (3.11). Then similarly as in (7.43), (7.44) we may write

$$\langle\!\langle \widetilde{L}^{\Delta,\sigma_n}_{\alpha\tilde{\alpha}_u} k_{\varsigma_u^{\sigma_n}}, G\rangle\!\rangle = \Upsilon_n^{(1)} + \Upsilon_n^{(2)} + \Upsilon_n^{(3)} + \Upsilon_n^{(4)}, \tag{7.71}$$

$$\Upsilon_n^{(1)} = \int_{\Gamma_{\rm fin}} \sum_{x\in\xi} \tilde{b}_{\sigma_n}(x) k_{\varsigma_u^{\sigma_n}}(\xi \setminus x)|G(\xi)|\lambda(d\xi),$$

$$\Upsilon_n^{(2)} = \int_{\Gamma_{\rm fin}} \sum_{x\in\xi} m(x)\tilde{\psi}_{\sigma_n}(x) k_{\varsigma_u^{\sigma_n}}(\xi)|G(\xi)|\lambda(d\xi),$$

$$\Upsilon_n^{(3)} = \int_{\Gamma_{\rm fin}} \sum_{x\in\xi}\sum_{y\in\xi\setminus x} \tilde{a}_{\sigma_n}(x,y) k_{\varsigma_u^{\sigma_n}}(\xi)|G(\xi)|\lambda(d\xi),$$

$$\Upsilon_n^{(4)} = \int_{\Gamma_{\rm fin}} \int_X \sum_{y\in\xi} \tilde{a}_{\sigma_n}(x,y) k_{\varsigma_u^{\sigma_n}}(\xi\cup x)|G(\xi)|\lambda(d\xi)dx.$$

Now similarly as in (7.46), (7.47) we estimate

$$\Upsilon_n^{(i)} \leq \int_{\Gamma_{\rm fin}} h^{(i)}(\xi)|G(\xi)|\lambda(d\xi), \qquad i = 1,2,3,4,$$

where

$$h^{(1)}(\xi) = \|b\|e^{-\tilde{\alpha}_u}|\xi|e^{\tilde{\alpha}_u|\xi|} =: c_1|\xi|e^{\tilde{\alpha}_u|\xi|}, \quad h^{(2)}(\xi) = \|m\||\xi|e^{\tilde{\alpha}_u|\xi|} =: c_2|\xi|e^{\tilde{\alpha}_u|\xi|},$$

$$h^{(3)}(\xi) = \|a\||\xi|^2e^{\tilde{\alpha}_u|\xi|} =: c_3|\xi|^2e^{\tilde{\alpha}_u|\xi|}, \quad h^{(4)}(\xi) = \langle a\rangle e^{\tilde{\alpha}_u}|\xi|e^{\tilde{\alpha}_u|\xi|} =: c_4|\xi|e^{\tilde{\alpha}_u|\xi|}.$$

By these estimates we then get

$$\Upsilon_n^{(i)} \leq \frac{c_i}{e(\alpha-\tilde{\alpha}_u)}|G|_\alpha, \quad i=1,2,4, \qquad \Upsilon_n^{(3)} \leq \frac{4c_3}{e(\alpha-\tilde{\alpha}_u)^2}|G|_\alpha, \tag{7.72}$$

holding with an arbitrary $\alpha > \tilde{\alpha}_u$ since $G \in \mathcal{G}_\alpha$ for each $\alpha \in \mathbb{R}$. By (7.72) each $\Upsilon_n^{(i)}$ is bounded. At the same time, $\tilde{b}_{\sigma_n}(x)$, $\tilde{\psi}_{\sigma_n}(x)$, $\tilde{a}_{\sigma_n}(x,y)$ are monitone functions decreasing to zero. By the monotone convergence theorem one then concludes that each of the summands in the first line of (7.71) converges to zero as $n \to +\infty$, which by (7.67) concludes the proof that $P_{s,\mu}(\mathsf{H}) = 0$; hence, the accumulation point under discussion solves the restricted martingale problem.

Assume now that there exists another accumulation point, say $\{P'_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$, which also solves the restricted martingale problem as we just have shown. Then the one dimentional marginals of $P_{s,\mu}$ and $P'_{s,\mu}$ should coincide, see Remark 3.9, due to the uniqueness stated in Theorem 3.3. Thus, to complete the whole proof we have to show that all finite dimensional marginals of these measures coincide, see Definition 3.8. Take $F_1 \in \widetilde{\mathcal{F}}$ and $t_1 > s$ and define the following measures on $\mathfrak{D}_{[t_1,+\infty)}(\Gamma_*)$ by setting

$$Q_{t_1}(\mathbb{A}) = \frac{P_{s,\mu}(\mathbb{1}_{\mathbb{A}}F_1\circ\varpi_{t_1})}{P_{s,\mu}(F_1\circ\varpi_{t_1})}, \quad Q'_{t_1}(\mathbb{A}) = \frac{P'_{s,\mu}(\mathbb{1}_{\mathbb{A}}F_1\circ\varpi_{t_1})}{P_{s,\mu}(F_1\circ\varpi_{t_1})}. \tag{7.73}$$

Recall that $F_1(\gamma) > 0$, see (3.16) and Proposition 3.5. By the inductive assumption it follows that

$$Q_{t_1}\circ\varpi_{t_1}^{-1} = Q'_{t_1}\circ\varpi_{t_1}^{-1} =: \varsigma_{t_1}.$$

By (7.52) and (7.53) it follows that $\varsigma^{\sigma}_{t_1} \Rightarrow \varsigma_{t_1}$ as $\sigma \to 0$, where is as in (7.53) with $J_1 = F_1$. For $t_2 > t_1$ and $F_2 \in \mathcal{F}_{\max}$, define, cf. (3.29),

$$\mathsf{H}_1(\bar{\gamma}) = F_2(\varpi_{t_2}(\bar{\gamma})) - F_2(\varpi_{t_2}(\bar{\gamma})) - \int_{t_1}^{t_2} (LF_2)(\varpi_u(\bar{\gamma}))du$$

$$\mathsf{H}_2(\bar{\gamma}) = \mathsf{H}_1(\bar{\gamma})F_1(\varpi_{t_1}(\bar{\gamma}))$$

By (7.73) and the assumed properties of $P_{s,\mu}$ and $P'_{s,\mu}$ it follows that

$$Q_{t_1}(\mathsf{H}_1) = \frac{P_{s,\mu}(\mathsf{H}_2)}{P_{s,\mu}(F_1 \circ \varpi_{t_1})} = 0, \qquad Q'_{t_1}(\mathsf{H}_1) = \frac{P'_{s,\mu}(\mathsf{H}_2)}{P_{s,\mu}(F_1 \circ \varpi_{t_1})} = 0,$$

by which both $Q_{t_1} \circ \varpi_t^{-1}$ and $Q'_{t_1} \circ \varpi_t^{-1}$, $t > t_1$, solve the Fokker-Planck equation for $(L, \mathcal{F}_{\max}, \varsigma_{t_1})$ on the time interval $[t_1, +\infty)$. By Theorem 3.3 we then conclude that $Q_{t_1} \circ \varpi_t^{-1} = Q'_{t_1} \circ \varpi_t^{-1} \in \mathcal{P}_{\exp}$, which implies in turn that the two dimensional marginals of $P_{s,\mu}$ and $P'_{s,\mu}$ also coincide. Then the extension of this to all finite dimensional marginals goes by induction. This completes the whole proof.

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