Diffusion Processes on p-Wasserstein Space over Banach Space *

Panpan Ren [b], **Feng-Yu Wang** [a], **Simon Wittmann** [c]

[a] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[b] Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Hong Kong, China

[c] Department of Mathematics, The Hong Kong Polytechnic University, 11 Yuk Chio Road, Hong Kong, China

panparen@cityu.edu.hk wangfy@tju.edu.cn, simon.wittmann@polyu.edu.hk

Abstract

To study diffusion processes on the p-Wasserstein space $\mathscr{P}_p$ for $p \in [1, \infty)$ over a separable, reflexive Banach space X, we present a criterion on the quasi-regularity of Dirichlet forms in $L^2(\mathscr{P}_p, \Lambda)$ for a reference probability Λ on $\mathscr{P}_p$. It is formulated in terms of an upper bound condition with the uniform norm of the intrinsic derivative. The condition is easy to check in relevant applications and allows to construct a type of Ornstein-Uhlenbeck process on $\mathscr{P}_p$. We find a versatile class of quasi-regular local Dirichlet forms on $\mathscr{P}_p$ by using images of Dirichlet forms on the tangent space $L^p(X \to X, \mu_0)$ at a reference point $\mu_0 \in \mathscr{P}_p$. The Ornstein-Uhlenbeck type Dirichlet form is an important example in this class. An L^2-estimate for the corresponding heat kernel is derived, based on the eigenvalues of the covariance operator of the underlying Gaussian measure.

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Contents

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1 Introduction

As a crucial topic in the crossed field of probability theory, optimal transport and partial differential equations, stochastic analysis on the Wasserstein space has received much attention. Some measure-valued diffusion processes have been constructed by using the theory of Dirichlet forms, see [18, 21, 13, 28, 24] and references therein. The pre-Dirichlet forms are defined by integrating a square field operator with respect to a reference Borel probability measure Λ on a topological space, whose points are measures over a Riemannian manifold, or $\mathbb{R}^d$. The square field operators are determined by the intrinsic or extrinsic derivatives, which describe the stochastic motion and birth-death of particles respectively. In order to establish the integration by parts formula ensuring the closability of the pre-Dirichlet form, the selection of reference measures Λ found in the literature are typically supported on the class of singular measures. Hence, these do not provide natural options when looking for a suitable substitute for a volume measure or a Gaussian measure on the set of probability measures. On the other hand, for stochastic analysis on the Wasserstein space, it is essential to construct a diffusion process which plays a role of Brownian motion in finite-dimensions, or the Ornstein-Uhlenbeck (O-U for short) process on a separable Hilbert space. This has been a long standing open problem due to the lack of a volume or Gaussian measure on such a state space, which could serve as an invariant measure. As a solution to this problem, [23] presents a general technique to construct an abundance of 'Gaussian like' probability measures on $\mathscr{P}_2(\mathbb{R}^d)$ together with the related O-U type Dirichlet forms. The construction is very natural as the Gaussian measure and the related Dirichlet form are obtained as images of the corresponding objects form the tangent space $T_{\mu_0,2} := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu_0)$ at a fixed element $\mu_0 \in \mathscr{P}_2$. Here, μ_0 is chosen as being absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

The main idea of [23] is based on the following fact from the theory of optimal transport, which can be found in [29] or [4], for example. The set $\mathscr{P}_2$ coincides with the image set of

$$\Psi : T_{\mu_0,2} \ni h \mapsto \mu_0 \circ h^{-1} \in \mathscr{P}_2.$$

The map Ψ is 1-Lipschitz continuous with respect to the 2-Wasserstein distance

$$\mathbb{W}_2(\mu,\nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \,\mathrm{d}\pi(x,y) \right)^{\frac{1}{2}}, \quad \mu,\nu \in \mathscr{P}_2,$$

where $\mathscr{C}(\mu, \nu)$ denotes the set of all couplings of μ and ν.

More precisely, let G be a non-degenerate Gaussian measure on the Hilbert space $T_{\mu_0,2}$ with trace-class covariance operator A^{-1}, where $(A, \mathscr{D}(A))$ is a positive definite self-adjoint operator in $T_{\mu_0,2}$. The associated O-U process on $T_{\mu_0,2}$ is generated by

$$L^{\mathrm{OU}}u(h) := \Delta u(h) - \langle A\nabla u(h), h\rangle_{T_{\mu_0,2}}, \quad h \in T_{\mu_0,2},\ u \in \mathscr{D}(L^{\mathrm{OU}}) \subseteq L^2(T_{\mu_0,2}, G). \tag{1.1}$$

Here, ∇ and Δ denote the gradient and Lapalcian on $T_{\mu_0,2}$ respectively. The O-U process $(X_t)_{t\geq 0}$ on the tangent space can be constructed as the mild solution of the corresponding semi-linear SPDE, i.e.

$$X_t = \mathrm{e}^{-At}X_0 + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)A}\mathrm{d}W_s, \quad t \geq 0,$$

where W_t is the standard cylindrical Brownian motion on $T_{\mu_0,2}$ (see e.g. [11, Chap. 6]). The associated O-U Dirichlet form $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is the closure of

$$\widetilde{\mathscr{E}}(u, v) := \int_{T_{\mu_0,2}} \langle \nabla u, \nabla v\rangle \mathrm{d}G, \quad u, v \in C_b^1(T_{\mu_0,2}).$$

Now, under the map Ψ, the image of the Gaussian measure G gives a reference measure

$$N_G := G \circ \Psi^{-1}$$

on $\mathscr{P}_2$, which is called the Gaussian measure induced by G. It is proved in [23] that the Ψ-image $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ of $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is a symmetric conservative local Dirichlet form in $L^2(\mathscr{P}_2, N_G)$ satisfying

$$\mathscr{E}(u, v) = \int_{\mathscr{P}_2} \langle Df, Dg\rangle_{T_{\mu_0,2}}\mathrm{d}N_G, \quad f, g \in C_b^1(\mathscr{P}_2), \tag{1.2}$$

where D is the intrinsic derivative on $\mathscr{P}_2$, which is first introduced in [1] on the configuration space over Riemannian manifolds, and see [5] or Definition 2.2 below for the class $C_b^1(\mathscr{P}_p), p \geq 1$.

The form $\mathscr{E}$ in (1.2) has the same type as the O-U Dirichlet form on a Hilbert space. Moreover, as shown in [23], it inherits several nice properties from the O-U Dirichlet form $\widetilde{\mathscr{E}}$ on the tangent space $T_{\mu_0,2}$, including the log-Sobolev inequality and compactness of its semigroup. The generator of $\mathscr{E}$ can be formally represented as the intrinsic Laplacian with a drift. So, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in [23] is called an O-U type Dirichlet form in $L^2(\mathscr{P}_2, N_G)$. However, the quasi-regularity of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is still open, up to now. Since quasi-regularity is the key to construct Markov processes using Dirichlet forms, the existence of the corresponding OU type stochastic process is still an open problem. As such a process is of wide interest, we prove a handy, general criterion (see Theorem 2.1 below) for the quasi-regularity of Dirichlet forms on the Wasserstein space. In particular, we verify the existence of the OU type process.

We will work in a more general framework to construct diffusion processes on the p-Wasserstein space $\mathscr{P}_p$ over a separable, reflexive Banach space for $p \in [1, \infty)$. Applications to the O-U type process for $p = 2$ then serve as a typical and highly relevant example. The proof of quasi-regularity is inspired by the methods developed in [25] and [26]. The latter of these two only presumes a Polish state space. Nevertheless, application of the techniques and verification of conditions require a detailed analysis which takes into account the special nature and properties of the metric respectively topology involved. An adaptation to the p-Wasserstein distance, as realized in this article, is completely new. For the configuration space equipped with the vague topology, a similar result has been achieved in [19]. Regarding the weak topology on the set of Borel probability measures over a Polish space, [21] provides a quasi-regularity result. The latter, however, focusses on Dirichlet forms linked to the extrinsic derivative instead of the intrinsic. Our main results, Theorems 2.1 & 3.1, relating to the intrinsic derivative and the p-Wasserstein distance, show quasi-regularity for a wide class of Dirichlet forms with state space $\mathscr{P}_p$. They imply the existence of a versatile class of diffusion processes on $\mathscr{P}_p$ and open up the door to further stochastic analysis via the theory of Dirichlet forms. The methods of this survey should also be applicable for the Wasserstein space over non-linear metric spaces like Riemannian manifolds. To save space we leave this for a future study.

Throughout this text, let $(X, \|\cdot\|_X)$ be a separable, reflexive Banach space and $\mathscr{P}$ be the space of probability measures on X. For fixed $p \in [1, \infty)$, we consider the p-Wasserstein space

$$\mathscr{P}_p := \big\{\mu \in \mathscr{P} : \ \mu(\|\cdot\|_X^p) < \infty\big\}. \tag{1.3}$$

As stated in [29, Thm. 6.18], the p-Wasserstein distance

$$\mathbb{W}_p(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left(\int_{X\times X} \|x - y\|_X^p \, \mathrm{d}\pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}}, \quad \mu, \nu \in \mathscr{P}_p, \tag{1.4}$$

yields a complete, separable metric on $\mathscr{P}_p$. Hence, its induced topology is second countable and in particular Lindelöf. It is worth mentioning, that the metric space $(\mathscr{P}_p, \mathbb{W}_p)$ is not locally compact, not even in case $X = \mathbb{R}^d$. We study the quasi-regularity of Dirihlet forms in $L^2(\mathscr{P}_p, \Lambda)$ for a reference probability measure Λ on $\mathscr{P}_p$. A typical example for Λ is the above mentioned Gaussian measure N_G. In particular, the quasi-regularity of O-U type Dirichlet forms is confirmed.

In Section 2, we present a general condition on the quasi-regularity of Dirichlet forms in $L^2(\mathscr{P}_p, \Lambda)$ by finding a comparison criterion involving the uniform norm of the intrinsic derivative. In Section 3, we apply this criterion to construct a class of quasi-regular local Dirichlet forms. These are obtained as images of Dirichlet forms on the tangent space at a fixed point of the Wasserstein space. Finally, in Section 4 we confirm the quasi-regularity of the O-U type Dirichlet form and give an L^2-estimate for the heat kernel, based on the eigenvalues of the covariance operator for the underlying Gaussian measure.

2 Quasi-regular Dirichlet forms on $\mathscr{P}_p$

We first recall some notions on Dirichlet forms which can be found in [20].

Let (E, ρ) be a Polish (or slightly more general, Lusin) space and Λ be a σ-finite measure on the Borel σ-algebra $\mathscr{B}(E)$. A Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on $L^2(E, \Lambda)$ is a densely defined, closed bilinear form, which is Markovian, see for instance [20, Chapt. I]. We denote by $\Lambda(f)$ the integral of a function f with respect to the measure Λ and set

$$\mathscr{E}_1(f,g) := \Lambda(fg) + \mathscr{E}(f,g), \quad f \in \mathscr{D}(\mathscr{E}).$$

For an open set $O \subseteq E$ the 1-Capacity associated to $\mathscr{E}$ is defined as

$$\mathrm{Cap}_1(O) := \inf\big\{\mathscr{E}_1(f,f) \,:\, f \in \mathscr{D}(\mathscr{E}),\, f(z) \geq 1 \text{ for } \Lambda\text{-a.e. } z \in O\big\}$$

with the convention of $\inf(\emptyset) := \infty$. For an arbitrary set $A \subseteq E$, let

$$\mathrm{Cap}_1(A) := \inf\big\{\mathrm{Cap}_1(O) \,:\, A \subseteq O,\, O \text{ is an open set in } E\big\}.$$

An $\mathscr{E}$-nest (or nest for short) is a sequence of closed subsets $\{K_n\}_{n\in\mathbb{N}}$ of E such that

$$\lim_{n\to\infty} \mathrm{Cap}_1(E \setminus K_n) = 0.$$

A measurable function $f : E \to \mathbb{R}$ is called quasi-continuous, if there exists a nest $\{K_n\}_{n\in\mathbb{N}}$ such that the restriction $f|_{K_n}$ is continuous for each $n \in \mathbb{N}$.

A sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions is said to converge quasi-uniformly to a function $f : E \to \mathbb{R}$, if there exists a nest $\{K_n\}_{n\in\mathbb{N}}$ such that the sequence of restricted functions $f_k|_{K_n}$, $k \in \mathbb{N}$, converge to $f|_{K_n}$ uniformly on K_n as $k \to \infty$ for each $n \in \mathbb{N}$.

If a property, which an element $z \in E$ either has or doesn't, holds for all z in the complement of a set $N \subseteq E$ with $\mathrm{Cap}_1(N) = 0$, then this property is said to hold quasi-everywhere (q.-e.) on E.

Definition 2.1 (Definition 3.1 in [20])**.** The Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called quasi-regular, if the following three conditions are met.

1) There exists an $\mathscr{E}$-nest of compact sets (i.e. $\mathrm{Cap}_1(\,\cdot\,)$ is tight).

2) The Hilbert space $\mathscr{D}(\mathscr{E})$ has a dense subspace consisting of quasi-continuous functions.

3) There exists $N \subseteq E$ with $\mathrm{Cap}_1(N) = 0$ and a sequence $\{f_i\}_{i\geq 1} \subseteq \mathscr{D}(\mathscr{E})$ of quasi-continuous functions which separate points in $E \setminus N$, i.e. for any two different points $z_1, z_2 \in E \setminus N$ there exists $i \in \mathbb{N}$ such that $f_i(z_1) \neq f_i(z_2)$.

2.1 A criterion of the quasi-regularity

From now on, we consider $(E,\rho) = (\mathscr{P}_p, \mathbb{W}_p)$ given in (1.3) and (1.4) for some $p \in [1,\infty)$. To prove the quasi-regularity of a Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in $L^2(\mathscr{P}_p, \Lambda)$, we look at the class $C_b^1(\mathscr{P}_p)$ defined as follows by using the intrinsic derivative. This derivative is first introduced in [1] on the configuration space over a Riemannian manifold, and has been extended in [5] to $\mathscr{P}_p$ over a Banach space.

Let X^* be the dual space of X, i.e. X^* is the Banach space of bounded linear functionals $X \to \mathbb{R}$. We write

$${}_{X^*}\langle x', x\rangle_X := x'(x), \quad x' \in X^*,\ x \in X.$$

Let $p^* = \frac{p}{p-1}$ which is ∞ if $p = 1$. The tangent space of $\mathscr{P}_p$ at a point $\mu \in \mathscr{P}_p$ is defined as

$$T_{\mu,p} := L^p(X \to X, \mu).$$

By virtue of [14, Thm. IV 1.1 & Cor. III 3.4], its dual space $T_{\mu,p}^*$ can be identified with the space $L^{p^*}(X \to X^*, \mu)$. Accordingly, we write

$$_{T_{\mu,p}^*}\langle \phi', \phi \rangle_{T_{\mu,p}} := \int_X {}_{X^*}\langle \phi'(x), \phi(x) \rangle_X \, \mu(\mathrm{d}x), \quad \phi' \in T_{\mu,p}^*, \ \phi \in T_{\mu,p}.$$

Definition 2.2. Let f be a continuous function on $\mathscr{P}_p$.

- f is called intrinsically differentiable, if for every $\mu \in \mathscr{P}_p$,

  $$T_{\mu,p} \ni \phi \mapsto D_\phi f(\mu) := \lim_{\varepsilon \to 0} \frac{f(\mu \circ (id + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon}$$

  is a bounded linear functional, and the intrinsic derivative of f at μ is defined as the unique element $Df(\mu) \in T_{\mu,p}^*$ such that

  $$D_\phi f(\mu) = {}_{T_{\mu,p}^*}\langle Df(\mu), \phi \rangle_{T_{\mu,p}} := \int_X {}_{X^*}\langle Df(\mu)(x), \phi(x) \rangle_X \, \mu(\mathrm{d}x), \quad \phi \in T_{\mu,p}.$$

- We denote $f \in C^1(\mathscr{P}_p)$, if f is intrinsically differentiable such that

  $$\lim_{\|\phi\|_{T_{\mu,p}} \downarrow 0} \frac{|f(\mu \circ (id + \phi)^{-1}) - f(\mu) - D_\phi f(\mu)|}{\|\phi\|_{T_{\mu,p}}} = 0, \quad \mu \in \mathscr{P}_p,$$

  and $Df(\mu)(x)$ has a continuous version in $(\mu, x) \in \mathscr{P}_p \times X$, i.e. there exists a continuous map $g : \mathscr{P}_p \times X \to X^*$ such that $g(\mu, \cdot)$ is a μ-version of $Df(\mu)$ for each $\mu \in \mathscr{P}_p$. In this case, we always take Df to be its continuous version, which is unique.

- We write $f \in C_b^1(\mathscr{P}_p)$, if $f \in C^1(\mathscr{P}_p)$ and $|f| + \|Df\|_{X^*}$ is bounded on $\mathscr{P}_p \times X$.

A typical subspace of $C_b^1(\mathscr{P}_p)$ is the class of cylindrical functions introduced as follows. First, we recall the continuously differentiable cylindrical functions on the Banach space X:

$$\mathscr{F}C_b^1(X) := \big\{ g(x_1', \cdots, x_n') : \ n \in \mathbb{N}, \, x_i' \in X^*, \, g \in C_b^1(\mathbb{R}^n) \big\},$$

where each $x_i' : X \to \mathbb{R}$ is a bounded linear functional. It is well known that each

$$\psi := g(x_1', \cdots, x_n') \in \mathscr{F}C_b^1(X)$$

belongs to $C_b^1(X)$, since ψ is bounded and Fréchet differentiable on X with bounded and continuous derivative

$$\nabla\psi(x) = \sum_{i=1}^n (\partial_i g)\big({}_{X^*}\langle x_1', x \rangle_X, \cdots, {}_{X^*}\langle x_n', x \rangle_X \big) x_i', \qquad x \in X.$$

Next, we consider the space of continuously differentiable cylindrical functions on $\mathscr{P}_p$:

$$\mathscr{F}C_b^1(\mathscr{P}) := \big\{\mathscr{P} \ni \mu \mapsto g(\mu(\psi_1), \cdots, \mu(\psi_n)) : \ n \in \mathbb{N}, \psi_i \in \mathscr{F}C_b^1(X), g \in C_b^1(\mathbb{R}^n)\big\}. \tag{2.1}$$

It is clear that for a function $f \in \mathscr{F}C_b^1(\mathscr{P})$ with $f(\mu) := g(\mu(\psi_1), \cdots, \mu(\psi_n))$, its restriction to $\mathscr{P}_p$ which is also denoted by f for simplicity, i.e. the map $\mathscr{P}_p \ni \mu \mapsto f(\mu) \in \mathbb{R}$, is an element of $C_b^1(\mathscr{P}_p)$ with

$$Df(\mu)(x) = \sum_{i=1}^n (\partial_i g)(\mu(\psi_1), \cdots, \mu(\psi_n))\nabla\psi_i(x) \in X^*, \quad (\mu, x) \in \mathscr{P}_p \times X. \tag{2.2}$$

In the following, we sometimes write $\mathscr{F}C_b^1(\mathscr{P}_p)$ for the restrictions of functions in $\mathscr{F}C_b^1(\mathscr{P})$ to $\mathscr{P}_p$. We will see that for any probability measure Λ on $\mathscr{P}_p$, $\mathscr{F}C_b^1(\mathscr{P}_p)$ is dense in $L^2(\mathscr{P}_p, \Lambda)$ (Lemma 2.3 below).

From now on, we identify a Λ-square integrable measurable function f on $\mathscr{P}_p$ with its equivalent class in $L^2(\mathscr{P}_p, \Lambda)$ by denoting $f \in L^2(\mathscr{P}_p, \Lambda)$.

Theorem 2.1. *A Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in $L^2(\mathscr{P}_p, \Lambda)$ is quasi-regular if it satisfies the following two conditions:*

(C_1) *$\mathscr{F}C_b^1(\mathscr{P}) \subseteq \mathscr{D}(\mathscr{E})$ and there exists a constant $C \in (0, \infty)$ such that*

$$\mathscr{E}(f, f) \le C \sup_{\mu\in\mathscr{P}_p} \|Df(\mu)\|^2_{T^*_{\mu,p}}, \quad f \in \mathscr{F}C_b^1(\mathscr{P}).$$

(C_2) *The Hilbert space $\mathscr{D}(\mathscr{E})$ has a dense subspace consisting of quasi-continuous functions.*

We would like to indicate that the conditions in Theorem 2.1 are easy to check in applications. When the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is constructed as the closure (i.e. smallest closed extension) of a bilinear form defined on $C_b^1(\mathscr{P}_p)$ or $\mathscr{F}C_b^1(\mathscr{P})$, the second condition holds automatically, and the first condition holds if there exists a positive function $F \in L^1(\mathscr{P}_p, \Lambda)$ such that

$$\mathscr{E}(f, f) \le \int_{\mathscr{P}_p} F(\mu)\|Df(\mu)\|^2_{T^*_{\mu,p}}\Lambda(\mathrm{d}\mu), \quad f \in \mathscr{F}C_b^1(\mathscr{P}).$$

For example, $F(\cdot)$ may be a dominating function for some diffusion coefficient, see Section 3.2 below.

2.2 Proof of the criterion

We first present some lemmas.

Lemma 2.2. *There exists a sequence $\{\psi_n\}_{n\ge1} \subseteq \mathscr{F}C_b^1(X)$ with the following properties.*

(i) The family $\{\mathscr{P}(X) \ni \mu \mapsto \mu(\psi_n)\}_{n\in\mathbb{N}}$ separates the points on $\mathscr{P}$, and hence separates the points on $\mathscr{P}_p$.

(ii) The σ-algebra generated by the family $\{\mathscr{P}(X) \ni \mu \mapsto \mu(\psi_n)\}_{n\in\mathbb{N}}$ coincides with the Borel σ-algebra $\mathscr{B}(\mathscr{P}_p)$.

Proof. Since X is a separable Banach space, there exists a sequence $\{x_i'\}_{i\in\mathbb{N}}$ separating the points on X. Let $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}C_b^1(X)$ consist of all functions

$$\psi(x) := \prod_{j=1}^{m} \frac{x_{i_j}'(x)}{1+|x_{i_j}'(x)|} \in [0,1], \quad x \in X,$$

for $m \in \mathbb{N}$ and $i_1, \dots, i_m \in \mathbb{N}$. Since $\{\psi_n\}_{n\in\mathbb{N}}$ is closed under multiplication and separates the points on X, it satisfies (i) according to [6, Theorem 11(b)].

To verify (ii), let $\bar{\sigma}$ be the σ-algebra on $\mathscr{P}_p$ induced by $\{\mu \mapsto \mu(\psi_n)\}_{n\in\mathbb{N}}$, and let $\sigma(\tau_\rho)$ denote the σ-algebra generated by the family of all open sets τ_ρ w.r.t. to the metric

$$\rho(\mu,\nu) := \sum_{n=1}^{\infty} 2^{-n} |\mu(\psi_n) - \nu(\psi_n)|, \quad \mu,\nu \in \mathscr{P}_p.$$

Then $\sigma(\tau_\rho) \subseteq \bar{\sigma}$. Noting that $\{\psi_n\}_{n\in\mathbb{N}}$ are continuous and uniformly bounded, according to [6, Lemma 3(a)], property (i) of this lemma implies $\mathscr{B}(\mathscr{P}_p) = \sigma(\tau_\rho)$, so that $\mathscr{B}(\mathscr{P}_p) \subseteq \bar{\sigma}$. On the other hand, each ψ_n is a bounded continuous function on X, so that $\mu \mapsto \mu(\psi_n)$ is continuous in $\mathscr{P}_p$, and hence $\bar{\sigma} \subseteq \mathscr{B}(\mathscr{P}_p)$. Therefore, (ii) is satisfied. □

Lemma 2.3. *The linear space $\mathscr{F}C_b^1(\mathscr{P}_p)$ is dense in $L^2(\mathscr{P}_p, \Lambda)$.*

Proof. Let $\mathscr{A}$ be the class of all subsets $A \subseteq \mathscr{P}_p$ given by

$$A := \bigcap_{i=1}^{m} \{\mu \in \mathscr{P}_p \,:\, \mu(\psi_i) \in B_i\}$$

for some $m \in \mathbb{N}$, $B_1, \dots B_m \in \mathscr{B}(\mathbb{R})$ and $\psi_1, \dots, \psi_m \in \mathscr{F}C_b^1(X)$. Obviously, $\mathscr{A}$ is $\cap$-stable, i.e. $A_1 \cap A_2 \in \mathscr{A}$ for $A_1, A_2 \in \mathscr{A}$. Furthermore, Lemma 2.2(ii) implies that $\sigma(\mathscr{A}) = \mathscr{B}(\mathscr{P}_p)$. Now, let V denote the vector space of real-valued, measurable functions on $\mathscr{P}_p$ which coincide in Λ-a.e. sense with some element of $\overline{\mathscr{C}^{\mathrm{cyl}}}$, the topological closure of $\mathscr{F}C_b^1(\mathscr{P}_p)$ in $L^2(\mathscr{P}_p, \Lambda)$. The claim of this Lemma reads $\overline{\mathscr{C}^{\mathrm{cyl}}} = L^2(\mathscr{P}_p, \Lambda)$. It suffices to show that V contains every bounded, measurable function. The monotone class theorem for functions (see [7, Preliminaries, Theorem 2.3]) yields exactly the desired statement if V meets the following properties:

1) V contains the indicator function $\mathbf{1}_A$ for any set $A \in \mathscr{A}$.

2) If $\{f_n\}_{n\in\mathbb{N}} \subseteq V$ is an increasing sequence of non-negative functions such that $f(\,\cdot\,) := \lim_{n\to\infty} f_n(\,\cdot\,)$ is bounded on $\mathscr{P}_p$, then $f \in V$.

To show 1), let $A \in \mathscr{A}$, and $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathscr{B}(\mathbb{R})$ and $\psi_1, \dots, \psi_m \in \mathscr{F}C_b^1(X)$ such that

$$\mathbf{1}_A(\mu) = \prod_{i=1}^{m} \mathbf{1}_{B_i}(\mu(\psi_i)), \quad \mu \in \mathscr{P}_p.$$

If we denote the image measure of Λ under $\mathscr{P}_p \ni \mu \mapsto (\mu(\psi_1), \dots, \mu(\psi_m)) \in \mathbb{R}^m$ by Λ_m, then

$$\int_{\mathscr{P}_p} \left| g(\mu(\psi_1), \dots, \mu(\psi_m)) - \mathbf{1}_A \right|^2 \Lambda(\mathrm{d}\mu) = \int_{\mathbb{R}^m} \left| g(x) - \prod_{i=1}^{m} \mathbf{1}_{B_i}(x_i) \right|^2 \mathrm{d}\Lambda_m(\mathrm{d}x)$$

holds for any $g \in C_b^1(\mathbb{R}^m)$. This implies $\mathbf{1}_A \in V$, since $C_b^1(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m, \Lambda_m)$,

Now, let $\{f_n\}_{n\in\mathbb{N}} \subseteq V$ and f be in 2). It remains to show that $f \in V$. For $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $(\int_{\mathscr{P}_p} |f - f_n|^2 \mathrm{d}\Lambda)^{1/2} \le \varepsilon/2$. Since $f_n \in V$, we find $u_n \in \mathscr{F}C_b^1(\mathscr{P}_p)$ such that $(\int_{\mathscr{P}_p} |f_n - u_n|^2 \mathrm{d}\Lambda)^{1/2} \le \varepsilon/2$. Then the Minkowski inequality yields $(\int_{\mathscr{P}_p} |u_n - f|^2 \mathrm{d}\Lambda)^{1/2} \le \varepsilon$. Hence, $f \in V$ as desired. □

To verify the tightness of capacity, we shall construct a class of reference functions. For $a, b \in \mathbb{R}$ let $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$ and $a^+ := a \vee 0$. For any $l \in \mathbb{N}$, let

$$\chi_l(s) := -\frac{3l}{2} + \int_{-\infty}^{s} \Big[\Big(\frac{t}{l} + 2\Big)^+ \wedge 1\Big] \wedge \Big[\Big(2 - \frac{t}{l}\Big)^+ \wedge 1\Big] \,\mathrm{d}t, \quad t \in \mathbb{R}. \tag{2.3}$$

Then $\chi_l \in C_b^1(\mathbb{R})$ with

$$\chi_l(s) = s \quad \text{for } s \in [-l, l], \quad \text{and} \quad \mathbf{1}_{[-l,l]}(s) \le \chi_l'(s) \le \mathbf{1}_{[-2l,2l]}(s), \quad s \in \mathbb{R}.$$

We recall the notation $p^* := \frac{p}{p-1}$ for $p \in [1, \infty)$ as introduced above.

Lemma 2.4. *Assume that Condition* (1) *in Theorem* 2.1 *holds. Let* $\Phi \in C_b^1(\mathbb{R})$ *and* $\gamma \in C^1(\mathbb{R}, [0, \infty))$ *such that for some constants* $a, b \in (0, \infty)$ *it holds*

$$\sup_{s\in[0,\infty)} |\Phi'(s)|(1+s)^{\frac{1}{p^*}} \le a \quad \text{and} \quad \sup_{s\in\mathbb{R}} |\gamma'(s)|(1+\gamma(s))^{-\frac{1}{p^*}} \le b.$$

Then for any $m \in \mathbb{N}$*, Lipschitz function* f *on* $\mathbb{R}^m$*, and* $x_i' \in X^*$ *with* $\|x_i'\|_{X^*} = 1$ *for* $i = 1, \dots, m$*, the function*

$$u(\mu) := \Phi\big(\mu\big(\gamma \circ f(x_1', \dots, x_m')\big)\big) \quad \text{for } \mu \in \mathscr{P}_p$$

belongs to $\mathscr{D}(\mathscr{E})$ *with*

$$\mathscr{E}(u, u) \le C(ab)^2 \Big\| \sum_{i=1}^m |\partial_i f| \Big\|_{L^\infty(\mathbb{R}^m)}^2.$$

Proof. (a) We first prove under the additional assumption that f is a bounded function, i.e. $f \in C_b^1(\mathbb{R}^m)$. In this case, $\gamma \circ f \in C_b^1(\mathbb{R}^m)$ and the function

$$u(\mu) := \Phi\big(\mu\big(\gamma \circ f(x_1', \dots, x_m')\big)\big), \quad \mu \in \mathscr{P}_p$$

is in $u \in \mathscr{F}C_b^1(\mathscr{P}_p)$. By (2.2), with

$$T : X \ni x \mapsto \big({}_{X^*}\langle x_1', x\rangle_X, \dots, {}_{X^*}\langle x_m', x\rangle_X\big) \in \mathbb{R}^m$$

it holds

$$Du(\mu)(x) = \Phi'\big(\mu(\gamma \circ f \circ T)\big)(\gamma' \circ f)(Tx) \sum_{i=1}^m \partial_i f(Tx)\, x_i'$$

for $\mu \in \mathscr{P}_p$ and $x \in X$. So,

$$\sup_{\mu\in\mathscr{P}_p} \|Du(\mu)\|_{T_{\mu,p}^*}^2 \le (ab)^2 \Big\| \sum_{i=1}^m |\partial_i f| \Big\|_{L^\infty(\mathbb{R}^m)}^2,$$

and the desired assertion follows from Condition (1) in Theorem 2.1.

(b) Next, let $f \in C^1(\mathbb{R}^m)$. For $l \in \mathbb{N}$ the composition $f_l := \chi_l \circ f$ yields an element of $C_b^1(\mathbb{R}^m)$ with $|\partial_i f_l(z)| \le |\partial_i f(z)|$ for $i = 1, \dots, m$, $z \in \mathbb{R}^m$. By what has been shown above, for each l the function

$$u_l(\mu) := \Phi\big(\mu\big(\gamma \circ f_l(x'_1, \dots, x'_m)\big)\big) \quad \text{for } \mu \in \mathscr{P}_p$$

is in $\mathscr{D}(\mathscr{E})$ for $l \in \mathbb{N}$ with

$$\mathscr{E}(u_l, u_l) \le C(ab)^2 \sup_{z \in \mathbb{R}^m} \Big(\sum_{i=1}^m |\partial_i f(z)|\Big)^2. \tag{2.4}$$

Clearly, $\lim_{l\to\infty} \gamma \circ f_l(Tx) = \gamma \circ f(Tx)$ for $x \in X$. By the condition on γ we find constants $A, B \in (0, \infty)$ such that

$$\big|\gamma \circ f_l(Tx)\big| \le A\big(1 + |f_l(Tx)|^p\big) \le B\big(1 + \|x\|_X)^p\big), \quad x \in X. \tag{2.5}$$

Then Lebesgue's dominated convergence applies and

$$\lim_{l\to\infty} \mu(\gamma \circ f_l \circ T) = \mu(\gamma \circ f \circ T), \quad \mu \in \mathscr{P}_p.$$

Consequently, again by dominated convergence, we conclude $\lim_{l\to\infty} u_l = u$ in $L^2(\mathscr{P}_p, \Lambda)$. Now, [20, Lemma I.2.12] yields $u \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(u,u) \le \liminf_{l\to\infty} \mathscr{E}(u_l, u_l)$ as the sequence $\{u_l\}_l$ is bounded w.r.t. $\mathscr{E}_1^{1/2}$-norm in view of (2.4). Since (2.4) delivers the desired upper bound, the proof is complete.

(c) Finally, let f be a Lipschitz continuous function on $\mathbb{R}^m$. Then, f is weakly differentiable on $\mathbb{R}^m$ with weak partial derivatives $\partial_i f \in L^\infty(\mathbb{R}^m)$. Let h be a non-negative smooth function on $\mathbb{R}^m$ with compact support and $\int h(z)\mathrm{d}z = 1$. Then the mollifying approximations $\{f^{(n)}\}_{n\in\mathbb{N}}$ of f defined by

$$f^{(n)}(x) := n^{-m} \int_{\mathbb{R}^m} f(z) h\big(n(z-x)\big)\mathrm{d}z = \int_{\mathbb{R}^m} f(x + n^{-1}z)h(z)\mathrm{d}z, \quad x \in \mathbb{R}^m$$

satisfies $f^{(n)} \in C^1(\mathbb{R}^m)$, $\lim_{n\to\infty} f^{(n)} = f$ and

$$\sup_{n\in\mathbb{N}} \sup_{z\in\mathbb{R}^m} \sum_{i=1}^m |\partial_i f^{(n)}(z)| \le \Big\| \sum_{i=1}^m |\partial_i f(z)| \Big\|_{L^\infty(\mathbb{R}^n)},$$

so that

$$\lim_{n\to\infty} \gamma \circ f^{(n)}(x'_1, \dots, x'_m) = \gamma \circ f(x'_1, \dots, x'_m), \tag{2.6}$$

and by step (b), the functions

$$u^{(n)}(\mu) := \Phi\big(\mu\big(\gamma \circ f^{(n)}(x'_1, \dots, x'_m)\big)\big), \quad \mu \in \mathscr{P}_p,\ n \in \mathbb{N}$$

belong to $\mathscr{D}(\mathscr{E})$ with

$$\sup_{n\in\mathbb{N}} \mathscr{E}(u^{(n)}, u^{(n)}) \le C(ab)^2 \Big\| \sum_{i=1}^m |\partial_i f(z)| \Big\|^2_{L^\infty(\mathbb{R}^m)}. \tag{2.7}$$

Moreover, since f is Lipschitz continuous and h is smooth with compact support, we find constants $A, B \in (0,\infty)$ such that (2.5) holds. Thus, by the dominated convergence theorem, (2.6) implies

$$\lim_{n\to\infty} \mu\big(\gamma \circ f^{(n)}(x_1',\ldots,x_m')\big) = \mu\big(\gamma \circ f(x_1',\ldots,x_m')\big), \quad \mu \in \mathscr{P}_p,$$

while the definition of u and $u^{(n)}$ with $\Phi \in C_b^1(\mathbb{R})$ yields

$$\lim_{n\to\infty} \|u^{(n)} - u\|_{L^2(\mathscr{P}_p,\Lambda)} = 0.$$

According to [20, Lemma I.2.12], this together with (2.7) finishes the proof. □

As an application of Lemma 2.4, we have the following assertion.

Proposition 2.5. *Let $\Phi \in C_b^1(\mathbb{R})$ and $\gamma \in C^1(\mathbb{R},[0,\infty))$ be as in Lemma 2.4 with constants $a, b \in (0,\infty)$. For $M \in \mathbb{N}$ and $y_1,\ldots,y_M \in X$, define*

$$\begin{aligned} &u(\mu) := \Phi(\mu(\gamma \circ g)), \quad \mu \in \mathscr{P}_p,\\ &g(x) := \min\big\{\|x-y_j\|_X \,:\, 1 \le j \le M\big\}, \quad x \in X. \end{aligned}$$

Then $u \in \mathscr{D}(\mathscr{E})$ with $\mathscr{E}(u,u) \le C(ab)^2$.

Proof. By Lemma 2.4, we need to approximate the distance function by using $f(x_1',\cdots,x_m')$ for $m \in \mathbb{N}, x_1',\cdots,x_m' \in X^*$ and Lipschitz functions f on $\mathbb{R}^m$.

(a) For $m \in \mathbb{N}, r \in \mathbb{R}^m$ and subsets $I_1,\ldots,I_k$ of $\{1,\ldots,m\}$, the function

$$f(z) := \min\Big\{\max_{i\in I_j}(z_i + r_i) \,:\, j = 1,\ldots,k\Big\}, \quad z \in \mathbb{R}^m$$

is Lipschitz continuous with

$$\sum_{i=1}^m |\partial_i f(z)| = 1 \quad \text{for } \mathrm{d}z\text{-a.e. } z \in \mathbb{R}^m.$$

By Lemma 2.4, the associated function $u : \mathscr{P}_p \to \mathbb{R}$ belongs to $\mathscr{D}(\mathscr{E})$ with $\mathscr{E}(u,u) \le C(ab)^2$.

(b) Let $\{x_k : k \in \mathbb{N}\}$ be a dense subset of X. For each $k \in \mathbb{N}$ we choose $x_k' \in X^*$ with $\|x_k'\|_{X^*} = 1$ and ${}_{X^*}\langle x_k', x_k\rangle_X = \|x_k\|_X$. Moreover, let $M \in \mathbb{N}$, $y_1,\ldots,y_M \in X$ and

$$g_l(x) := \min_{j=1,\ldots,M}\Big(\max_{k=1,\ldots,l} {}_{X^*}\langle x_k', x - y_j\rangle_X\Big), \quad x \in X,\, l \in \mathbb{N}.$$

As shown in step (a) in the present proof, the function

$$u_l(\mu) := \Phi(\mu(\gamma \circ g_l)) \quad \text{for } \ \mu \in \mathscr{P}_p$$

is in $\mathscr{D}(\mathscr{E})$ with $\mathscr{E}(u_l,u_l) \le C(ab)^2$. By construction of x_k', $k \in \mathbb{N}$, it holds $\sup_{k\in\mathbb{N}} {}_{X^*}\langle x_k', x\rangle_X = \|x\|_X$ for $x \in X$. Consequently,

$$\lim_{l\to\infty} g_l(x) = \min\big\{\|x-y_i\|_X \,:\, j \in \{1,\ldots,M\}\big\} =: g(x) \quad \text{for } x \in X.$$

Since $\sup_{l\in\mathbb{N}} |g_l(x)| \le \max_{j=1,\dots,M} \|x-y_j\|_X$, we find constants $c_1, c_2 \in (0,\infty)$ such that

$$\big|\gamma\circ g(x)\big| \le c_1\big(1+|g(x)|^p\big) \le c_2\big(1+\|x\|_X)^p\big), \quad x \in X.$$

So, by Lebesgue's dominated convergence twice as in the proof of Lemma 2.4, we obtain

$$\lim_{l\to\infty} \|u_l - u\|_{L^2(\mathscr{P}_p,\Lambda)} = 0.$$

By [20, Lemma I.2.12], this together with $\mathscr{E}(u_l,u_l) \le C(ab)^2$ finishes the proof. $\square$

Proof of Theorem 2.1. Lemma 2.2(i) together with (C_1) yields the existence of a countable set $\{f_i\}_{i\in\mathbb{N}}$ of quasi-continuous functions in $\mathscr{D}(\mathscr{E})$ which separate points in $\mathscr{P}_p$. So, it suffices to find a $\mathscr{E}$-nest of compact sets in terms of (C_2).

First, we recall a characterization of precompact sets in $\mathscr{P}_p$ as stated in [22, Proposition 2.2.3]. The closure w.r.t. $\mathbb{W}_p$ of a set $\mathscr{A} \subseteq \mathscr{P}_p$ is compact if and only

(1) (Uniform Integrability) $\lim_{R\to\infty} \sup_{\mu\in\mathscr{A}} \mu\big(\|\cdot\|_X^p \mathbf{1}_{[R,\infty)}(\|\cdot\|_X)\big) = 0,$

(2) (Tightness) for every $\varepsilon > 0$ there is a compact set $Y \subseteq X$ such that $\sup_{\mu\in\mathscr{A}} \mu(Y^c) \le \varepsilon$.

With this characterization, we only need to find an $\mathscr{E}$-nests of closed sets $\{K_n^{(i)}\}_{n\in\mathbb{N}}$ for $i = 1,2$, such that (1) holds for $\mathscr{A} = K_n^{(1)}$ and (2) holds for $\mathscr{A} = K_n^{(2)}$. When this is achieved, $\{K_n := K_n^{(1)} \cap K_n^{(2)}\}_{n\in\mathbb{N}}$ is a $\mathscr{E}$-nest of compact sets, and hence the proof is finished. Indeed, as Cap_1 is a Choquet capacity on $\mathscr{P}_p$, it holds

$$\begin{aligned}\mathrm{Cap}_1(X\setminus K_n) &= \mathrm{Cap}_1\big((X\setminus K_n^{(1)}) \cup (X\setminus K_n^{(2)})\big)\\ &\le \mathrm{Cap}_1\big(X\setminus K_n^{(1)}\big) + \mathrm{Cap}_1\big(X\setminus K_n^{(2)}\big) \stackrel{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

(a) Construction of $\{K_n^{(1)}\}_{n\in\mathbb{N}}$. For $k \in \mathbb{N}$ we let

$$u_k(\mu) := \chi_1\big(\mu\big(\gamma_k(\|\cdot\|_X)\big)\big), \quad \mu \in \mathscr{P}_p, \tag{2.8}$$

where χ_1 is the function from (2.3) with $l = 1$ and

$$\gamma_k(s) := \big(1 + [(s-k)^+]^2\big)^{\frac{p}{2}} - 1, \quad s \in \mathbb{R}.$$

It is easy to find constants $a, b \in (0,\infty)$ independent of k such that conditions in Lemma 2.4 holds for $(\Phi,\gamma) = (\chi_1,\gamma_k), k \in \mathbb{N}$. So, Proposition 2.5 implies

$$\sup_{k\in\mathbb{N}} \mathscr{E}(u_k,u_k) < \infty,$$

and by Lebesgue's dominated convergence theorem twice as in the proof of Lemma 2.4, we have $u_k(\mu) \to 0$ as $k \to \infty$ for each $\mu \in \mathscr{P}_p$, and

$$\lim_{k\to\infty} \Lambda(|u_k|^2) = 0.$$

By [20, Lemma I.2.12], there exists a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ such that

$$v_m := \frac{1}{m}\sum_{k=1}^{m} u_{l_k} \overset{m\to\infty}{\longrightarrow} 0 \tag{2.9}$$

strongly in the Hilbert space $(\mathscr{D}(\mathscr{E}), \mathscr{E}_1)$. Then, by [20, Proposition III.3.5], there exists an $\mathscr{E}$-nest of closed sets $\{K_n^{(1)}\}_{n\geq 1}$ and a subsequence $\{v_{m_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{\mu\in K_n^{(1)}} v_{m_k}(\mu) = 0 \quad \text{for every } n \in \mathbb{N}.$$

Since $\{u_k\}_{k\in\mathbb{N}}$ is a decreasing sequence, it holds $v_m(\mu) \geq u_{l_m}(\mu)$ for every $m \in \mathbb{N}$ and $\mu \in \mathscr{P}_p$. In particular, $v_{m_k}(\mu) \geq u_{l_{m_k}}(\mu)$ for $\mu \in \mathscr{P}_p$ and $k \in \mathbb{N}$. Now, it follows

$$\lim_{k\to\infty} \sup_{\mu\in K_n^{(1)}} u_{l_{m_k}}(\mu) = 0 \quad \text{for every } n \in \mathbb{N},$$

which in turn implies

$$\lim_{k\to\infty} \sup_{\mu\in K_n^{(1)}} \mu\big(\gamma_{l_{m_k}}(\|\cdot\|_X)\big) = 0 \quad \text{for every } n \in \mathbb{N}. \tag{2.10}$$

Next, by the definition of γ_k we can choose $R_k \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that

$$\mu\big(\gamma_k(\|\cdot\|_X)\big) \geq \mu\big(\|\cdot\|_X^p\, \mathbf{1}_{[R_k,\infty)}(\|\cdot\|_X)\big), \quad \mu \in \mathscr{P}_p.$$

This together with (2.10) yields that for every $n \in \mathbb{N}$,

$$\liminf_{k\to\infty} \sup_{\mu\in K_n^{(1)}} \mu\big(\|\cdot\|_X^p \mathbf{1}_{[R_k,\infty)}(\|\cdot\|_X)\big) \leq \liminf_{k\to\infty} \sup_{\mu\in K_n^{(1)}} \mu\big(\gamma_k(\|\cdot\|_X)\big) = 0.$$

Thus, (1) holds for $\mathscr{A} = K_n^{(1)}$ as desired.

(b) Construction of $\{K_n^{(2)}\}_{n\in\mathbb{N}}$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a countable dense subset of X, and for each $k \in \mathbb{N}$, let

$$u_k(\mu) := \mu\Big(\chi_1\big(\min\big\{\|\cdot - y_i\|_X \,:\, i = 1,\ldots,k\big\}\big)\Big), \quad \mu \in \mathscr{P}_p.$$

Again, χ_1 is the function from (2.3) with $l = 1$. Noting that $\chi_1\big(\min\big\{\|\cdot - y_i\|_X \,:\, i = 1,\ldots,k\big\}\big)$ decreases to 0 as $k \to \infty$, by Proposition 2.5 and the dominated convergence theorem, we have

$$\sup_{k\in\mathbb{N}} \mathscr{E}(u_k, u_k) < \infty, \quad \lim_{k\to\infty} \Lambda(|u_k|^2) = 0.$$

So, as shown above [20, Lemma I.2.12] and [20, Proposition III.3.5] imply the existence of a subsequence of $\{u_k\}_k$ which converges to zero quasi-uniformly. The arguments, which include the strong convergence w.r.t. $\mathscr{E}_1^{1/2}$ of the Césaro means for a suitable subsequence together with the fact that $\{u_k\}_k$ is a decreasing in k, are completely analogous to step (a). Hence, there exists a nest $\{K_n^{(2)}\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ the following property holds:

$$\text{For } \varepsilon > 0 \text{ there exists } \{k_m\}_m \subseteq \mathbb{N} \text{ with } \sup_{\mu\in K_n^{(2)}} u_{k_m}(\mu) \leq \frac{\varepsilon}{m2^m}, \quad m \in \mathbb{N}. \tag{2.11}$$

It remains to show that $K_n^{(2)}$ satisfies property (2) for fixed $n \in \mathbb{N}$. Let $m \in \mathbb{N}$, $\varepsilon > 0$ be arbitrary and k_m be chosen according to (2.11). Since for $r \leq 1$ and $s \in [0, \infty)$ it holds

$$\chi_1(s) \geq r \quad \text{if and only if} \quad s \geq r,$$

we estimate

$$\begin{aligned} &\mu\Big(\Big\{x \in X \,:\, \min\big(\{\| x - y_i\|_X \,:\, i = 1, \dots, k_m\} \geq \frac{1}{m}\Big\}\Big) \\ &= \mu\Big(\Big\{x \in X \,:\, \chi_1\big(\min\{\| x - y_i\|_X \,:\, i = 1, \dots, k_m\}\big) \geq \frac{1}{m}\Big\}\Big) \\ &\leq m\, u_{k_m}(\mu) \leq \frac{\varepsilon}{2^m}, \quad m \in \mathbb{N},\ \mu \in K_n^{(2)}. \end{aligned} \tag{2.12}$$

Now we define $Y := \bigcap_{m \in \mathbb{N}} Y_m$ for

$$Y_m := \Big\{x \in X \,:\, \min\big\{\| x - y_i\|_X \,:\, i = 1, \dots, k_m\big\} < \frac{1}{m}\Big\}.$$

Obviously, Y is a totally bounded set in X and hence the closure $\overline{Y}$ a compact set. This proves the tightness of $K_n^{(2)}$, because for any $\mu \in K_n^{(2)}$ using (2.12) it holds

$$\mu(X \setminus \overline{Y}) \leq \mu\Big(\bigcup_{m \in \mathbb{N}} X \setminus Y_m\Big) \leq \sum_{m=1}^{\infty} \mu(X \setminus Y_m) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon$$

and the choice of ε above is arbitrary. □

3 Quasi-regular image Dirichlet forms on $\mathscr{P}_p$

In this section, we prove the quasi-regularity for image Dirichlet forms on $\mathscr{P}_p$ under the map

$$\Psi : T_{\mu_0,p} \ni \phi \mapsto \mu_0 \circ \phi^{-1} \in \mathscr{P}_p \tag{3.1}$$

from the tangent space $T_{\mu_0,p}$ for a fixed element $\mu_0 \in \mathscr{P}_p$. To shorten notation, we denote

$$T_0 := T_{\mu_0,p} = L^p(X \to X, \mu_0), \quad T_0^* := T_{\mu_0,p}^* = L^{p^*}(X \to X^*, \mu_0).$$

The map Ψ is Lipschitz continuous, since

$$\begin{aligned} \mathbb{W}_p(\Psi(\phi_1), \Psi(\phi_2))^p &\leq \int_{X \times X} \|x - y\|_X^p \,\mathrm{d}\pi(x, y) \\ &= \int_X \|\phi_1(x) - \phi_2(x)\|_X^p \,\mathrm{d}\mu_0(x) = \|\phi_1 - \phi_2\|_{T_0}^p, \quad \phi_1, \phi_2 \in T_0. \end{aligned}$$

In the case, where X is a separable Hilbert space and $\mu_0 \in \mathscr{P}_p$ is absolutely continuous with respect to a non-degenerate Gaussian measure, the theory of optimal transport provides that Ψ is surjective and that for any $\mu \in \mathscr{P}_p$, there exists a unique $\phi_\mu \in T_0$ such that

$$\mu_0 \circ \phi_\mu^{-1} = \mu \quad \text{and} \quad \mathbb{W}_p(\mu_0, \mu) = \|\mathrm{id} - \phi_\mu\|_{T_0}$$

(see [4, Theorem 6.2.10]). Here, ϕ_μ is called the optimal map from μ_0 to μ. In particular, this holds if $X = \mathbb{R}^d$ and $\mu_0 \in \mathscr{P}_p$ is absolutely continuous w.r.t. the Lebesgue measure.

Let Λ_0 be a probability measure on T_0. Then the image measure

$$\Lambda := \Lambda_0 \circ \Psi^{-1}$$

is a probability measure on $\mathscr{P}_p$. The map

$$L^2(\mathscr{P}_p, \Lambda) \ni u \mapsto u \circ \Psi \in L^2(T_0, \Lambda_0)$$

is isometric, i.e.

$$\Lambda(uv) = \Lambda_0\big((u \circ \Psi)(v \circ \Psi)\big), \quad u, v \in L^2(\mathscr{P}_p, \Lambda). \tag{3.2}$$

We would like to infer that, if Ψ is surjective, then choosing Λ_0 such that

$$\Lambda_0(U) > 0 \quad \text{for any non-empty open set} \quad U \subseteq T_0$$

results in Λ bearing the analogous property, i.e. takes a strictly positive value on each non-empty open set in $\mathscr{P}_p$. We refer to this property of Λ (resp. Λ_0) as full support.

In the following, a class of quasi-regular Dirichlet forms in $L^2(\mathscr{P}_p, \Lambda)$ are constructed by using the image of Dirichlet forms in $L^2(T_0, \Lambda)$. This provides quasi-regular local Dirichlet forms associated with diffusion processes on $\mathscr{P}_p$.

3.1 Main result

From here on, the Dirichlet forms we consider are assumed to be symmetric. Let $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ be a symmetric Dirichlet form in $L^2(T_0, \Lambda_0)$, where $\mathscr{D}(\widetilde{\mathscr{E}})$ contains the following defined class $C_b^1(T_0)$ of functions on T_0.

Definition 3.1. $C_b^1(T_0)$ consists of all bounded Fréchet differentiable functions f on T_0 with

$$T_0 \ni \phi \mapsto \nabla f(\phi) \in L^1(X \to X^*, \mu_0)$$

continuous, and

$$\sup_{\phi \in T_0} \|\nabla f(\phi)\|_{L^\infty(X \to X^*, \mu_0)} < \infty.$$

Note that by the dominated convergence theorem, if $f \in C_b^1(T_0)$ then

$$T_0 \ni \phi \mapsto \nabla f(\phi) \in L^q(X \to X^*, \mu_0)$$

is continuous for all $q \in [1, \infty)$. The main result of this section is the following.

Theorem 3.1. *Let Λ_0 be Borel probability measure on T_0, and let $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ be a symmetric Dirichlet form in $L^2(T_0, \Lambda_0)$ such that $C_b^1(T_0) \subseteq \mathscr{D}(\widetilde{\mathscr{E}})$. We have the following assertions.*

(1) *It holds*

$$C_b^1(\mathscr{P}_p)\circ\Psi := \big\{u\circ\Psi\ :\ u\in C_b^1(\mathscr{P}_p)\big\}\subseteq C_b^1(T_0), \tag{3.3}$$

and the bilinear form

$$\mathscr{E}(u,v) := \widetilde{\mathscr{E}}(u\circ\Psi, v\circ\Psi),\quad u,v\in C_b^1(\mathscr{P}_p)$$

is closable in $L^2(\mathscr{P}_p,\Lambda)$*, where* $\Lambda := \Lambda_0\circ\Psi^{-1}$*. Its closure* $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ *is a Dirichlet form satisfying*

$$\begin{aligned}&\mathscr{D}(\mathscr{E})\circ\Psi := \big\{u\circ\Psi :\ u\in\mathscr{D}(\mathscr{E})\big\}\subseteq\mathscr{D}(\widetilde{\mathscr{E}}),\\ &\mathscr{E}(u,v) = \widetilde{\mathscr{E}}(u\circ\Psi, v\circ\Psi),\quad u,v\in\mathscr{D}(\mathscr{E}).\end{aligned} \tag{3.4}$$

(2) *If the generator* $(\widetilde{L},\mathscr{D}(\widetilde{L}))$ *of* $(\widetilde{\mathscr{E}},\mathscr{D}(\widetilde{\mathscr{E}}))$ *has purely discrete spectrum, let* $\{\sigma_n\}_{n\geq1}$ *be all eigenvalues of* $-\widetilde{L}$ *listed in the increasing order with multiplicities. Then the generator* $(L,\mathscr{D}(L))$ *of* $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ *also has purely discrete spectrum, and eigenvalues* $\{\lambda_n\}_{n\geq1}$ *of* $-L$ *listed in the same order satisfies* $\lambda_n\geq\sigma_n, n\geq1$*. If moreover* $\sum_{n=1}^\infty \mathrm{e}^{-\sigma_n t}<\infty$ *for* $t>0$*, then the diffusion semigroup* $P_t := \mathrm{e}^{tL}$ *has heat kernel* $p_t(\mu,\nu)$ *with respect to* Λ *satisfying*

$$\int_{\mathscr{P}_p\times\mathscr{P}_p} p_t(\mu,\nu)^2\Lambda(\mathrm{d}\mu)\Lambda(\mathrm{d}\nu)\leq\sum_{n=1}^\infty \mathrm{e}^{-2\lambda_n t}<\infty,\quad t>0.$$

(3) *If there exists a constant* $C>0$ *such that*

$$\widetilde{\mathscr{E}}(f,f)\leq C\sup_{\phi\in T_0}\|\nabla f(\phi)\|_{T_0^*}^2,\quad f\in C_b^1(T_0), \tag{3.5}$$

then $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ *is quasi-regular.*

To prove (3.3), we need the following chain rule.

Lemma 3.2. *If* $u\in C_b^1(\mathscr{P}_p)$*, then* $u\circ\Psi\in C_b^1(T_0)$ *with*

$$\nabla(u\circ\Psi)(\phi) = Du(\Psi(\phi))\circ\phi,\quad \phi\in T_0.$$

In particular,

$$\big\|\nabla(u\circ\Psi)(\phi)\big\|_{T_0^*} = \big\|Du(\Psi(\phi))\big\|_{T^*_{\Psi(\phi),p}},\quad \phi\in T_0.$$

Proof. Let $\phi,\xi\in T_0$. On the probability space $(X,\mathscr{B}(X),\mu_0)$ [5, Theorem 2.1] implies

$$\begin{aligned}(u\circ\Psi)(\phi+\xi)-(u\circ\Psi)(\phi) &= \int_0^1\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(u\circ\Psi)(\phi+\varepsilon\xi)\,\mathrm{d}\varepsilon\\ &= \int_0^1 {}_{T_0^*}\big\langle Du(\Psi(\phi+\varepsilon\xi))\circ(\phi+\varepsilon\xi),\xi\big\rangle_{T_0}.\end{aligned}$$

Combining this with the boundedness and continuity of Df on $\mathscr{P}_p \times X$, we may apply the dominated convergence theorem to deduce

$$\lim_{\|\xi\|_{T_0}\downarrow 0} \left| \frac{(u\circ\Psi)(\phi+\xi) - (u\circ\Psi)(\phi) - {}_{T_0^*}\langle Du(\Psi(\phi))\circ\phi, \xi\rangle_{T_0}}{\|\xi\|_{T_0}} \right| = 0, \tag{3.6}$$

hence $u\circ\Psi$ is Fréchet differentiable on T_0 with derivative

$$\nabla(u\circ\Psi)(\phi) = Du(\Psi(\phi))\circ\phi \in T_0^* \tag{3.7}$$

satisfying

$$\|\nabla(u\circ\Psi)(\phi)\|_{L^\infty(X\to X^*,\mu_0)} \le \|Du\|_\infty < \infty.$$

Finally, let $\{\phi_n\}_n \subseteq T_0$ and $\phi \in T_0$ such that as $n\to\infty$,

$$\|\phi_n - \phi\|_{T_0} := \left(\int_X \|\phi_n-\phi\|_X^p \mathrm{d}\mu_0\right)^{\frac{1}{p}} \to 0.$$

By the continuity of $\Psi : T_0 \to \mathscr{P}_p$, and the boundedness and continuity of $Du : \mathscr{P}_p\times X \to X^*$, we may apply the dominated convergence theorem to derive

$$\lim_{n\to\infty} \int_X \|Du(\Psi(\phi_n))(\phi_n) - Du(\Psi(\phi))(\phi)\|_{X^*}\mathrm{d}\mu_0 = 0,$$

which together with (3.7) yields the continuity of $T_0 \ni \phi \mapsto \nabla u(\phi) \in L^1(X\to X^*,\mu_0)$.

□

Proof of Theorem 3.1. (1) The inclusion (3.3) is ensured by (3.2) and Lemma 3.2. Next, by Lemma 2.3 and $C_b^1(\mathscr{P}_p) \supseteq \mathscr{F}C_b^1(\mathscr{P}_p)$, $C_b^1(\mathscr{P}_p)$ is dense in $L^2(\mathscr{P}_p,\Lambda)$, which together with (3.3) and $\mathscr{D}(\widetilde{\mathscr{E}}) \supseteq C_b^1(T_0)$ implies that

$$\Psi^*\mathscr{D}(\widetilde{\mathscr{E}}) := \big\{u \,:\, u\circ\Psi \in \mathscr{D}(\widetilde{\mathscr{E}})\big\} \supseteq C_b^1(\mathscr{P}_p)$$

is a dense subset of $L^2(\mathscr{P}_p,\Lambda)$. So, by [9, Chapt. V],

$$\Psi^*\widetilde{\mathscr{E}}(u,v) := \widetilde{\mathscr{E}}(u\circ\Psi, v\circ\Psi), \quad u,v\in \Psi^*\mathscr{D}(\widetilde{\mathscr{E}})$$

is a Dirichlet form in $L^2(\mathscr{P}_p,\Lambda)$. Since $\mathscr{E}(u,v) = \widetilde{\mathscr{E}}(u\circ\Psi, v\circ\Psi)$ for $u,v\in C_b^1(\mathscr{P}_p)$, this implies that $(\mathscr{E}, C_b^1(\mathscr{P}_p))$ is a densely defined closable bilinear form in $L^2(\mathscr{P}_p,\Lambda)$, its closure $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is a Dirichlet form.

(2) As shown in the proof of [23, Theorem 3.2], if $\widetilde{L}$ has purely discrete spectrum, then Theorem 3.4(2) implies that so does L. Moreover, (3.4) and the Courant-Fisher min-max principle, for any $n\in\mathbb{N}$,

$$\begin{aligned}\lambda_n &= \inf_{\mathscr{C}:n\text{-dim. subspace of } \mathscr{D}(\mathscr{E})} \sup_{0\ne u\in\mathscr{C}} \frac{\mathscr{E}(u,u)}{\Lambda(u^2)} \\ &= \inf_{\mathscr{C}:n\text{-dim. subspace of } \mathscr{D}(\mathscr{E})} \sup_{0\ne u\in\mathscr{C}} \frac{\widetilde{\mathscr{E}}(u\circ\Psi, u\circ\Psi)}{\Lambda_0((u\circ\Psi)^2)}\end{aligned}$$

$$\geq \inf_{\widetilde{\mathscr{C}}:n\text{-dim. subspace of } \mathscr{D}(\widetilde{\mathscr{E}})} \sup_{0\neq\widetilde{u}\in\widetilde{\mathscr{C}}} \frac{\widetilde{\mathscr{E}}(\widetilde{u},\widetilde{u})}{\Lambda_0(\widetilde{u}^2)} = \sigma_n.$$

So, when $\sum_{n=1}^{\infty} \mathrm{e}^{-2\sigma_n t} < \infty$ for $t > 0$, by the spectral representation (see for instance [12]), P_t has heat kernel p_t with respect to Λ such that

$$p_t(\mu,\nu) := \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} u_n(\mu) u_n(\nu), \quad \mu,\nu \in \mathscr{P}_p,$$

and hence

$$\int_{\mathscr{P}_p\times\mathscr{P}_p} p_t(\mu,\nu)^2 \,\mathrm{d}\Lambda(\mathrm{d}\mu)\,\mathrm{d}\Lambda(\mathrm{d}\nu) = \sum_{n=1}^{\infty} \mathrm{e}^{-2\lambda_n t} < \infty.$$

(3) Since $C_b^1(\mathscr{P}_p)$ is a dense subspace of $\mathscr{D}(\mathscr{E})$, the proof of (1) implies Condition (C_2) in Theorem 2.1. Moreover, Lemma 3.2, (3.5) and (3.2) imply (C_1). So, the quasi-regularity of $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ follows from Theorem 2.1. □

3.2 Local Dirichlet forms and diffusion processes

In classic theory, the Dirichlet form for a symmetric diffusion process on $\mathbb{R}^d$ is of gradient type

$$\mathscr{E}(f,g) = \int_{\mathbb{R}^d} \langle A\nabla f, \nabla g\rangle_{\mathbb{R}^d} \,\mathrm{d}\Lambda$$

for a nice probability measure Λ on $\mathbb{R}^d$ and a diffusion coefficient $A = (a_{ij})_{1\leq i,j\leq d}$.

In the following, we develop an analogous concept for the state space $\mathscr{P}_p$ with $p \in [1,2]$. We assume that Λ_0 has full support. Let $\{x_i\}_{i\in\mathbb{N}} \subset X$ be fixed such that

$$\sum_{i=1}^{\infty} {}_{X^*}\langle x', x_i\rangle_X^2 \leq M\|x'\|_{X^*}^2 \quad x' \in X^*, \tag{3.8}$$

for some constant $M \in (0,\infty)$. We denote by $\mathscr{L}_+(l^2)$ the set of symmetric, non-negative definite, bounded linear operators on $l^2 := L^2(\mathbb{N})$ and fix a measurable map

$$K : \mathscr{P}_p \times X \to \mathscr{L}_+(l^2) \subset \big(\mathscr{L}(l^2), \|\cdot\|_{\mathscr{L}(l^2)}\big)$$

such that

$$C_K := \int_{T_0} \Big\| \|K(\Psi(\phi),\phi(\cdot))\|_{\mathscr{L}(l^2)} \Big\|_{L^\infty(X,\mu_0)} \mathrm{d}\Lambda_0(\phi) < \infty. \tag{3.9}$$

Here, and $\|\cdot\|_{\mathscr{L}(l^2)}$ is the operator norm on the space of bounded operators $l^2 \to l^2$. Let e_i for $i \in \mathbb{N}$ denote the i-th unit vector of l^2. We may think of

$$\kappa_{i,j} := \langle Ke_i, e_j\rangle_{l^2} : \mathscr{P}_p \times X \to \mathbb{R}, \quad i,j \in \mathbb{N},$$

as the coefficients of K.

For $f, g \in C_b^1(T_0)$ we can define $\widetilde{\Gamma}(f,g) \in L^1(T_0, \Lambda_0)$ by

$$\widetilde{\Gamma}(f,g)(\phi) := \sum_{i,j=1}^{\infty} \int_X \kappa_{i,j}(\Psi(\phi), \phi(x)) \big({}_{X^*}\langle \nabla f(\phi)(x), x_i \rangle_X \big) \big({}_{X^*}\langle \nabla g(\phi)(x), x_j \rangle_X \big) \mathrm{d}\mu_0(x).$$

Indeed, we can use (3.8), (3.9) and Hölder inequality to estimate

(3.10)

$$\begin{aligned}
&\Lambda_0(|\widetilde{\Gamma}(f,g)|) \\
&\leq M \int_{T_0} \int_X \|K(\Psi(\phi), \phi(x))\|_{\mathscr{L}(l^2)} \|\nabla f(\phi)(x)\|_{X^*} \|\nabla g(\phi)(x)\|_{X^*} \mathrm{d}\mu_0(x) \mathrm{d}\Lambda_0(\phi) \\
&\leq M \int_{T_0} \Big\| \|K(\Psi(\phi), \phi(\,\cdot\,))\|_{\mathscr{L}(l^2)} \mathrm{d}\mu_0 \Big\|_{L^\infty(X,\mu_0)} \|\nabla f(\phi)\|_{L^2(X \to X^*, \mu_0)} \|\nabla g(\phi)\|_{L^2(X \to X^*, \mu_0)} \mathrm{d}\Lambda_0(\phi) \\
&\leq M C_K \Big(\sup_{\phi \in T_0} \|\nabla f(\phi)\|_{L^{p^*}(X \to X^*, \mu_0)} \Big) \Big(\sup_{\phi \in T_0} \|\nabla g(\phi)\|_{L^{p^*}(X \to X^*, \mu_0)} \Big) < \infty,
\end{aligned}$$

since $p^* \in [2, \infty)$. Hence, the non-negative definite bilinear form

$$\widetilde{\mathscr{E}}(f,g) := \Lambda_0(\widetilde{\Gamma}(f,g)), \quad f, g \in C_b^1(T_0), \tag{3.11}$$

is well-defined.

Remark 3.3. Let $u, v \in C_b^1(\mathscr{P}_p)$ and $\phi \in T_0$, $\mu := \Psi(\phi)$. From the chain rule (as stated in Lemma 3.2) and a transformation of integrals it follows

$$\begin{aligned}
&\widetilde{\Gamma}(u \circ \Psi, v \circ \Psi)(\phi) \\
&= \sum_{i,j=1}^{\infty} \int_X \kappa_{i,j}(\Psi(\phi), \phi(x)) \big({}_{X^*}\langle Duf(\Psi(\phi))(\phi(x)), x_i \rangle_X \big) \big({}_{X^*}\langle Dv(\Psi(\phi))(\phi(x)), x_j \rangle_X \big) \mathrm{d}\mu_0(x) \\
&= \sum_{i,j=1}^{\infty} \int_X \kappa_{i,j}(\mu, y) \big({}_{X^*}\langle Du(\mu)(y), x_i \rangle_X \big) \big({}_{X^*}\langle Dv(\mu)(y), x_j \rangle_X \big) \mathrm{d}\mu(y).
\end{aligned}$$

Hence, we may define a non-negative definite bilinear form

$$\begin{aligned}
&\mathscr{E}(u,v) := \Lambda(\Gamma(u,v)), \quad u, v \in C_b^1(\mathscr{P}_p), \\
&\Gamma(u,v)(\mu) := \sum_{i,j=1}^{\infty} \int_X \kappa_{i,j}(\mu, x) \big({}_{X^*}\langle Du(\mu)(x), x_i \rangle_X \big) \big({}_{X^*}\langle Dv(\mu)(x), x_j \rangle_X \big) \mathrm{d}\mu(x).
\end{aligned} \tag{3.12}$$

Due to the above remark it holds

$$\mathscr{E}(u,v) = \widetilde{\mathscr{E}}(u \circ \Psi, v \circ \Psi), \quad u, v \in C_b^1(T_0). \tag{3.13}$$

Theorem 3.4. *Assume that Λ_0 has full support and the above defined bilinear form $(\widetilde{\mathscr{E}}, C_b^1(T_0))$ is closable in $L^2(T_0, \Lambda_0)$ such that its closure $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is a local Dirichlet form. Then the bilinear form $(\mathscr{E}, C_b^1(\mathscr{P}_p))$ defined in* (3.12) *is closable in $L^2(\mathscr{P}_p, \Lambda)$, and its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular local Dirichlet form in $L^2(\mathscr{P}_p, \Lambda)$ satisfying* (3.4).

Proof. (a) By (3.10),

$$\widetilde{\mathscr{E}}(f,f) = \Lambda_0\big(\widetilde{\Gamma}(f,f)\big) \le MC_K \sup_{\phi\in T_0} \|\nabla f(\phi)\|^2_{T_0^*}, \quad f \in C_b^1(T_0).$$

Combining this with (3.13) and applying (1) from Theorem 3.1, the first assertion except the locality follows.

(b) To prove the locality of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, it suffices to show that

$$\text{supp}[u\circ\Psi] \subseteq \Psi^{-1}\big(\text{supp}[u]\big) \quad \text{for } u \in L^2(\mathscr{P}_p, \Lambda). \tag{3.14}$$

If so, then $\text{supp}[u] \cap \text{supp}[v] = \emptyset$, $u, v \in \mathscr{D}(\mathscr{E})$, implies $\text{supp}[u\circ\Psi] \cap \text{supp}[v\circ\Psi] = \emptyset$, so that (3.4) and the local property of $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ yields

$$\mathscr{E}(u,v) = \widetilde{\mathscr{E}}(u\circ\Psi, v\circ\Psi) = 0.$$

Since Ψ is continuous, for any $\phi \in \text{supp}[u\circ\Psi]$ and any open set $U \subseteq \mathscr{P}_p$ containing $\Psi(\phi)$, the set $\Psi^{-1}(U) \subseteq T_0$ is open and contains ϕ. Thus, $\phi \in \text{supp}[u\circ\Psi]$ and that Λ_0 has full support imply

$$\int_U |u|\,\mathrm{d}\Lambda = \int_{\Psi^{-1}(U)} |u\circ\Psi|\,\mathrm{d}\Lambda_0 > 0.$$

Hence, $\Psi(\phi) \in \text{supp}[u]$, i.e. $\phi \in \Psi^{-1}(\text{supp}[u])$. Therefore, (3.14) holds. □

Example 3.5. Let H be a separable Hilbert space continuously embedded into X, such that we have the Gelfand triple:

$$X^* \subseteq H^* = H \subseteq X.$$

When $p \in [1,2]$, this implies

$$L^{p^*}(X \to X^*, \mu) \subseteq L^2(X \to H, \mu) \subseteq L^p(X \to X, \mu)$$

for $\mu \in \mathscr{P}_p$. Let $\{x_i\}_{i\ge1}$ be an orthonormal basis of H. Then (3.8) holds. A simple choice of K is the constant field of identity operators, so that (3.11) and (3.12) reduce to

$$\widetilde{\mathscr{E}}(f,g) := \int_{T_0} \langle \nabla f(\phi), \nabla g(\phi)\rangle_{L^2(X\to H,\mu_0)} \mathrm{d}\Lambda_0(\phi), \quad f,g \in C_b^1(T_0),$$
$$\mathscr{E}(u,v) := \int_{\mathscr{P}_p} \langle Du(\mu), Dv(\mu)\rangle_{L^2(X\to H,\mu)} \mathrm{d}\Lambda(\mu), \quad u,v \in C_b^1(\mathscr{P}_p).$$

The closability of $(\widetilde{\mathscr{E}}, C_b^1(T_0))$ can be verified in many relevant cases using results from [3], see Subsection 4.2 for an example. So, Theorem 3.4 applies.

Theorem 3.4 enables us to construct diffusion processes on $\mathscr{P}_p$. In [16] a correspondence between regular Dirichlet forms and strong Markov processes is built , see [17] for a complete theory and more references. This is extended in [2], [20] to the quasi regular setting. According to [10], a quasi regular Dirichlet form becomes regular under one-point compactification.

According to [20, Definitions IV.1.8, IV.1.13, V.1.10], a standard Markov process

$$\mathbf{M} = (\Omega, \mathscr{F}, (X_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathscr{P}_p})$$

with natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is called a non-terminating diffusion process on $\mathscr{P}_p$ if

$$\mathbb{P}_\mu\big(X_\cdot \in C([0,\infty), \mathscr{P}_p)\big) = 1 \quad \text{for } \mu \in \mathscr{P}_p.$$

It is called Λ-tight if there exists a sequence $\{K_n\}_n$ of compact sets in $\mathscr{P}_p$ such that stopping times

$$\tau_n := \inf\{t \geq 0 : X_t \notin K_n\}, \quad n \in \mathbb{N}$$

satisfy

$$\mathbb{P}_\mu\big(\lim_{n\to\infty} \tau_n = \infty\big) = 1 \quad \text{for } \Lambda\text{-a.e. } \mu \in \mathscr{P}_p.$$

The diffusion process is called properly associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, if for any bounded measurable function $u : \mathscr{P}_p \to \mathbb{R}$ and $t > 0$,

$$\mathscr{P}_p \ni \mu \mapsto \int_\Omega u(X_t)\, \mathrm{d}\mathbb{P}_\mu$$

is a quasi-continuous Λ-version of $P_t u$, where $(P_t)_{t\geq 0}$ is the associated Markov semigroup on $L^2(\mathscr{P}_p, \Lambda)$ associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Corollary 3.6. *In the situation of Theorem* 3.4(1)*, we have the following assertions.*

(1) *There exists a non-terminating diffusion process* $\mathbf{M} = (\Omega, \mathscr{F}, (X_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathscr{P}_p})$ *on* $\mathscr{P}_p$ *which is properly associated with* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$*. In particular,* Λ *is an invariant probability measure of* $\mathbf{M}$*.*

(2) $\mathbf{M}$ *solves the martingale problem for* $(L, \mathscr{D}(L))$*, i.e. for* $u \in \mathscr{D}(L)$*, the additive functional*

$$u(X_t) - u(X_0) - \int_0^t Lu(X_s)\mathrm{d}s, \quad t \geq 0,$$

is an $\{\mathscr{F}_t\}_t$*-martingale under* $\mathbb{P}_\mu$ *for q.e.* $\mu \in \mathscr{P}_p$*.*

Proof. (1) By [20, Theorem IV.3.5 & Theorem V.1.11], the locality and quasi regularity ensured by Theorem 3.4 imply the existence of a Λ-tight special standard process

$$\mathbf{M} = (\Omega, \mathscr{F}, (X_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathscr{P}_p\cup\{\Delta\}})$$

with state space $(\mathscr{P}_p, \mathbb{W}_p)$, life time ζ and filtration $\{\mathscr{F}_t\}_{t\geq 0}$ (as defined in [20, Chap. IV, Definition IV.1.5, IV.1.8, IV.1.13, V.1.10]) which meets

$$\mathbb{P}_\mu(\{\omega \in \Omega \,:\, [0, \zeta(\omega)) \ni t \mapsto X_t(\omega) \text{ is continuous}\}) = 1 \quad \text{for } \mu \in \mathscr{P}_p.$$

and is properly associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in the sense of [20, Definition IV.2.5].

Since $\mathbf{1}_{\mathscr{P}_p} \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(\mathbf{1}_{\mathscr{P}_p}, \mathbf{1}_{\mathscr{P}_p}) = 0$, it holds $T_t \mathbf{1}_{\mathscr{P}_p}, = \mathbf{1}_{\mathscr{P}_p}$ for $t \geq 0$. This means there exists a set $N \subseteq \mathscr{P}_p$ of zero capacity (referring to the 1-capacity associated with $\mathscr{E}$) such that $\mathbb{P}_\mu(\{\zeta = \infty\}) = 1$ for $\mu \in \mathscr{P}_p \setminus N$. Without loss of generality, $\mathscr{P}_p \setminus N$ may be assumed to be $\mathbf{M}$-invariant, by virtue of [20, Corollariy IV.6.5]. Considering the restriction $\mathbf{M}|_{\mathscr{P}_p \setminus N}$ (see [20, Remark IV.6.2(i)]) and then applying the procedure described in [20, Chapt.IV, Sect.3, pp. 117f.], re-defining $\mathbf{M}$ in such way that each element from N is a trap, we may assume $\mathbb{P}_\mu(\{\zeta = \infty\}) = 1$ for all $\mu \in \mathscr{P}_p$. Furthermore, after the procedure of weeding (restricting the sample space to a subset of Ω), as explained in [15, Chap. III, Paragraph 2, pp. 86f.], we may assume that $\mathbf{M}$ is non-terminating and continuous, i.e. $\zeta(\omega) = \infty$ and $[0, \infty) \mapsto X_t(\omega)$ a continuous map for every $\omega \in \Omega$.

(2) Let $u \in \mathscr{D}(L)$ and

$$A_t : \Omega \ni \omega \mapsto \int_0^t Lu(X_s(\omega))\mathrm{d}s, \quad t \geq 0.$$

Then, $\{A_t\}_{t\geq 0}$ is an continuous additive functional of $\mathbf{M}$ with zero energy. Moreover,

$$\mathbb{E}_\mu(A_t) = \int_0^t (T_s Lu)(\mu)\mathrm{d}s = (T_t u - u)(\mu) \quad \text{for } \Lambda\text{-a.e. } \mu \in \mathscr{P}_p.$$

Now, the claim follows from [20, Theorem VI.2.5], resp. [17, Theorem 5.2.2], in combination with [17, Theorem 5.2.4] and regularization of the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ as explained in [20, Chap. VI]. □

4 Ornstein-Uhlenbeck type processes

In this section, we study O-U type Dirichlet forms as constructed in Section 3 for Λ_0 being a non-degenerate Gaussian measure on the tangent space T_0. We first consider the case that $X = H$ is a separable Hilbert space and $p = 2$, so that $T_0 := L^2(H \to H, \mu_0)$ is a Hilbert space, which covers the framework in [23] where $H = \mathbb{R}^d$ is concerned; then extend to the more general setting where X is a separable Banach space and $p \in [1, \infty)$.

4.1 O-U type process on $\mathscr{P}_2$ over Hilbert space

Let $X = H$ be a separable Hilbert space and consider the quadratic Wasserstein space $\mathscr{P}_2$. For fixed $\mu_0 \in \mathscr{P}_2$, the tangent space is $T_0 := T_{\mu_0,2} := L^2(H \to H, \mu_0)$. Let $(A, \mathscr{D}(A))$ be a strictly positive definite self-adjoint linear operator on T_0 with pure point spectrum. We denote its eigenvalues in increasing order with multiplicities by $0 < \alpha_1 \leq \alpha_2, \ldots$ and the corresponding unitary eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of T_0, which is called the eigenbasis of $(A, \mathscr{D}(A))$. We assume that

$$\sum_{n=1}^\infty \alpha_n^{-1} < \infty,$$

which ensures the existence of a centred Gaussian measure G on T_0 whose covariance operator is given by the inverse of A. In following, we identify T_0 with ℓ^2 using the coordinate representation w.r.t. $\{\phi_n\}_{n\in\mathbb{N}}$, i.e.

$$T_0 \ni \phi \overset{\simeq}{\longmapsto} \big(\langle \phi_n, \phi\rangle_{T_0}\big)_{n\in\mathbb{N}} \in \ell^2.$$

Then, the Gaussian measure G is represented as the product measure

$$G(\mathrm{d}\phi) := \prod_{n=1}^{\infty} m_n(\mathrm{d}\langle \phi_n, \phi\rangle_{T_0}) \quad \text{with} \quad m_n(\mathrm{d}r) := \Big(\frac{\alpha_n}{2\pi}\Big)^{\frac{1}{2}} \exp\Big[-\frac{\alpha_n r^2}{2}\Big]\mathrm{d}r. \tag{4.1}$$

According to [23], the corresponding non-degenerate Gaussian measure on $\mathscr{P}_2$ is defined as

$$N_G := G \circ \Psi^{-1}$$

with $\Psi : T_0 \to \mathscr{P}_2$ as in (3.1), which has full support. Moreover, by [3, Theorem 3.10], the bilinear form

$$\widetilde{\mathscr{E}}(f, g) := \Lambda_0(\langle \nabla f, \nabla g\rangle_{T_0}), \quad f, g \in C_b^1(T_0)$$

is closable in $L^2(T_0, \Lambda_0)$ and its closure $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is a local Dirichlet form. Moreover, by [27, Proposition 3.2], the class of smooth cylindrical functions

$$\mathscr{F}C_b^\infty(T_0) := \big\{g(\langle\cdot, \psi_1\rangle_{T_0}, \cdots, \langle\cdot, \psi_n\rangle_{T_0}) : \ n \in \mathbb{N}, \ g \in C_b^\infty(\mathbb{R}^n), \ \psi_1, \ldots, \psi_n \in T_0\big\}$$

is dense in $\mathscr{D}(\widetilde{\mathscr{E}})$ w.r.t. $\widetilde{\mathscr{E}}_1^{1/2}$-norm, so $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is also the closure of $(\widetilde{\mathscr{E}}, \mathscr{F}C_b^\infty(T_0))$.

Now, By Theorem 3.4, the bilinear form

$$\mathscr{E}(u, v) := \int_{\mathscr{P}_2} \big\langle Du(\mu), Dv(\mu)\big\rangle_{T_{\mu,2}} \mathrm{d}\Lambda(\mu), \quad u, v \in \widetilde{C_b^1}(\mathscr{P}_2)$$

is closable in in $L^2(\mathscr{P}_p, \Lambda)$, and the closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular local Dirichlet form. Moreover, as shown in [23, Theorem 3.2] that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ satisfies the log-Sobolev inequality has a semigroup of compact operators. These results are already implied by the arguments from [23, Theorem 3.2]. Moreover, we have the following consequence of Theorem 3.1(2).

Corollary 4.1. *$(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular, local Dirichlet form on $L^2(\mathscr{P}_2, P)$. Its generator L has purely discrete spectrum with eigenvalues $0 > \lambda_1 \geq \lambda_2 \ldots$, listed in decreasing order containing multiplicities. The associated Markov semigroup $\{T_t\}_{t\geq 0}$ has density $\{p_t\}_{t\geq 0}$ with respect to Λ and the estimate*

$$\int_{\mathscr{P}_2\times\mathscr{P}_2} p_t(\mu, \nu)^2 \,\mathrm{d}\Lambda(\mu)\,\mathrm{d}\Lambda(\nu) = \sum_{n=1}^{\infty} \mathrm{e}^{2\lambda_n t} \leq \prod_{n\in\mathbb{N}} \Big(1 + \frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t) \wedge 1}\Big) < \infty, \quad t > 0,$$

holds true.

Proof. It suffices to verify the estimate for $\{p_t\}_{t\geq 0}$. For any $n \in \mathbb{N}$, let P_t^n be the O-U process on $\mathbb{R}$ generated by

$$L_n := \Delta - \alpha_n x \cdot \nabla.$$

It is well known that $-L_n$ has eigenvalues $\{k\alpha_n\}_{k\geq 0}$ with Hermit polynomials as eigenfunctions:

$$H_k(x) := \mathrm{e}^{\alpha_n x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-\alpha_n x^2/2}.$$

Let $p_t^n(x,y)$ be the heat kernel w.r.t. m_n in (4.1). Then for any $n\in\mathbb{N}$ and $t>0$,

$$\begin{aligned}(4.2)\qquad &\int_{\mathbb{R}\times\mathbb{R}} p_t^n(x,y)^2 m_n(\mathrm{d}x)m_n(\mathrm{d}y) = \sum_{k=0}^\infty \mathrm{e}^{-2k\alpha_n t}\\ &\le 1+\mathrm{e}^{-2\alpha_n t}+\int_1^\infty \mathrm{e}^{-2\alpha_n ts}\mathrm{d}s \le 1+\frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t)\wedge 1}.\end{aligned}$$

Noting that $\sum_{n=1}^\infty \alpha_n^{-1}<\infty$ implies

$$\sum_{n=1}^\infty \log\Big(1+\frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t)\wedge 1}\Big)\le \sum_{n=1}^\infty \frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t)\wedge 1}<\infty,$$

we conclude that

$$p_t^\infty(\mathbf{x},\mathbf{y}):=\prod_{n=1}^\infty p_t^n(x_n,y_n),\quad \mathbf{x}=(x_n),\mathbf{y}=(y_n)\in\mathbb{R}^{\mathbb{N}}$$

is a well defined measurable function in $L^2(m^\infty\times m^\infty)$, where $m^\infty:=\prod_{n=1}^\infty m_n$, and

$$\int p_t^\infty(\mathbf{x},\mathbf{y})^2 m^\infty(\mathrm{d}\mathbf{x})m^\infty(\mathrm{d}\mathbf{y})=\prod_{n=1}^\infty\int_{\mathbb{R}\times\mathbb{R}} p_t^n(x,y)^2 m_n(\mathrm{d}x)m_n(\mathrm{d}y)\le\tilde\xi_t,$$

holds for

$$\tilde\xi_t:=\prod_{n\in\mathbb{N}}\Big(1+\frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t)\wedge 1}\Big)<\infty,\quad t>0.$$

Let $\widetilde{T}_t$ be the O-U semigroup associated with $(\widetilde{\mathscr{E}},\mathscr{D}(\widetilde{\mathscr{E}}))$. Then for every $t>0$, $\widetilde{T}_t$ has the following density with respect to G:

$$\widetilde{p}_t(\phi,\phi')=p_t^\infty(\mathbf{x}(\phi),\mathbf{x}(\phi')),\quad \mathbf{x}(\phi):=(\langle\phi,\phi_n\rangle_{T_0})_{n\in\mathbb{N}},$$

so that by the spectral representation, see for instance [12], the eigenvalues $\{\sigma_n\}_{n\in\mathbb{N}}$ of $-\widetilde{L}$ satisfies

$$\sum_{n=1}^\infty \mathrm{e}^{-2t\sigma_n}=\int_{T_0\times T_0}\widetilde{p}_t(\phi,\phi')^2\,\mathrm{d}G(\mathrm{d}\phi)\,\mathrm{d}G(\phi')\le\tilde\xi_t.$$

Then the desired assertion is implied by Theorem 3.1(2). □

4.2 O-U type process on $\mathscr{P}_p$ over Banach space

We go back to the general setting of Example 3.5, where X is a separable Banach space and H is a separable Hilbert space densely and continuously included in X.

On the tangent space $T_0:=L^p(X\to X,\mu_0)$ at $\mu_0\in\mathscr{P}_p$, let G be a non-degenerate (not necessarily centred) Gaussian measure on T_0. We consider the set S of all G-shifted bounded linear functionals on T_0:

$$S:=\Big\{\psi-G\big({}_{T_0^*}\langle\psi,\,\cdot\,\rangle_{T_0}\big)\ :\ \psi\in T_0^*\Big\}.$$

Then, S is a subspace of $L^2(T_0, G)$ and we denote its closure w.r.t. $\|\cdot\|_{L^2(T_0,G)}$ by $\overline{S}^G$. Next, we define $\mathscr{H}_G$ as the subspace of T_0 comprising all elements $\phi \in T_0$ for which there exists an element $\widehat{\phi} \in \overline{S}^G$ with

$$\int_{T_0} \Big({}_{T_0^*}\langle \psi, \xi \rangle_{T_0} - G\big({}_{T_0^*}\langle \psi, \cdot \rangle_{T_0}\big)\Big)\widehat{\phi}(\xi)\, \mathrm{d}G(\xi) = {}_{T_0^*}\langle \psi, \phi \rangle_{T_0} \quad \text{for all } \psi \in T_0^*.$$

The space $\mathscr{H}_G$ is the Cameron-Martin space of G (see [8, Sect.'s 2.2 & 2.4, in part. Lem. 2.4.1]). It is a Hilbert space equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathscr{H}_G} := \langle \widehat{\phi}_1, \widehat{\phi}_2 \rangle_{L^2(T_0,G)}, \quad \phi_1, \phi_2 \in \mathscr{H}_G,$$

which is densely and continuously included in T_0.

Theorem 4.2. *Let $p \in [1, 2]$, H be a separable Hilbert space densely and continuously embedded into X. Let G be a non-degenerate (not necessarily centred) Gaussian measure on T_0 such that*

$$\mathcal{H}_G \cap L^2(X \to H, \mu_0) \text{ is dense in } L^2(X \to H, \mu_0), \tag{4.3}$$

and let $\Lambda = G \circ \Psi^{-1}$. Then the pre-Dirichlet form given by

$$\mathscr{E}(u, v) := \int_{\mathscr{P}_p} \langle Du(\mu), Dv(\mu)\rangle_{L^2(X\to H,\mu)}\, \mathrm{d}\Lambda(\mu), \quad u, v \in C_b^1(\mathscr{P}_p)$$

is closable in $L^2(\mathscr{P}_p, \Lambda)$, and its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular, local Dirichlet form. In particular, there exists a non-terminating diffusion process

$$\mathbf{M} = (\Omega, \mathscr{F}, (X_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathscr{P}_p})$$

on $\mathscr{P}_p$ with invariant measure Λ.

Proof. In view of Example 3.5 together with Theorem 3.4, we only need to verify the closability of the bilinear form

$$\widetilde{\mathscr{E}}(f, g) := \int_{T_0} \langle \nabla f, \nabla g \rangle_{L^2(X\to H,\mu_0)} \mathrm{d}G, \quad f, g \in C_b^1(T_0)$$

in $L^2(T_0, \Lambda_0)$, and its closure $(\widetilde{\mathscr{E}}, \mathscr{D}(\widetilde{\mathscr{E}}))$ is a local Dirichlet form. According to [3], it suffices to show that every $\phi \in \mathcal{H}_G \setminus \{0\}$ is admissible in the sense of [3, Definition 3.4].

By [8, Corollary 2.4.3] the Gaussian G is quasi shift invariant under any element $\phi \in \mathscr{H}_G \setminus \{0\}$ and

$$\frac{\mathrm{d}G \circ (id + s\phi)^{-1}}{\mathrm{d}G}(\psi) = \exp\Big(s\widehat{\phi}(\psi) - \frac{1}{2}s^2\langle \phi, \phi\rangle_{\mathcal{H}_G}\Big), \quad G\text{-a.e. } \psi \in T_0.$$

Then

$$\chi_\phi(\psi) := \Big(\int_{\mathbb{R}} \frac{\mathrm{d}G \circ (id + s\phi)^{-1}}{\mathrm{d}G}(\psi)\mathrm{d}s\Big)^{-1} = \Big(\frac{1}{2\pi}\langle \phi, \phi\rangle_{\mathcal{H}_G}\Big)^{\frac{1}{2}} \exp\Big(- \frac{\widehat{\phi}(\psi)^2}{2\langle \phi, \phi\rangle_{\mathcal{H}_G}}\Big),$$

and for any $\xi \in T_0$,

$$\rho_{\xi,\phi}(t) := \chi_\phi(\xi + t\phi) = \Big(\frac{1}{2\pi}\langle \phi, \phi\rangle_{\mathcal{H}_G}\Big)^{\frac{1}{2}} \exp\Big(-\frac{\widehat{\phi}(\xi + t\phi)^2}{2\langle \phi, \phi\rangle_{\mathcal{H}_G}}\Big), \quad t \in \mathbb{R}.$$

Noting that

$$R(\rho_{\xi,\phi}) := \Big\{ t \in \mathbb{R} \, : \, \int_{t-\varepsilon}^{t+\varepsilon} \rho_{\xi,\phi}(s)^{-1}\, \mathrm{d}s < \infty \text{ for some } \varepsilon > 0 \Big\} = \mathbb{R},$$

we have

$$(\rho_{\xi,\phi} 1_{\mathbb{R}\setminus R(\rho_{\xi,\phi})})(t) = 0 \quad \text{for a.e. } t,$$

so that by [3, Theorem 2.2], ϕ is admissible. □

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