

MEAN-FIELD LIMIT OF NON-EXCHANGEABLE INTERACTING DIFFUSIONS WITH SINGULAR KERNELS

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ABSTRACT. The mean-field limit of interacting diffusions without exchangeability, caused by weighted interactions and non-i.i.d. initial values, are investigated. The weights could be signed and unbounded. The result applies to a large class of singular kernels including the Biot-Savart law. We demonstrate a flexible type of mean-field convergence, in contrast to the typical convergence of $\frac{1}{N} \sum_{i=1}^N \delta_{X_i}$. More specifically, the sequence of signed empirical measure processes with arbitrary uniform l^r -weights, $r > 1$, weakly converges to a coupled PDE's, such as the dynamics describing the passive scalar advected by the 2D Navier-Stokes equation.

Our method is based on a tightness/compactness argument and makes use of the systems' uniform Fisher information. The main difficulty is to determine how to propagate the regularity properties of the limits of empirical measures in the absence of the DeFinetti-Hewitt-Savage theorem for the non-exchangeable case. To this end, a sequence of random measures, which merges weakly with a sequence of weighted empirical measures and has uniform Sobolev regularity, is constructed through the disintegration of the joint laws of particles.

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1. INTRODUCTION

In this article we consider non-exchangeable interacting particle systems with singular kernels in the Euclidean space \mathbb{R}^d . Given random initial data $\{X_i(0)\}_{i=1}^N$, the position of each particle X_i is characterized by the following SDEs

$$dX_i = \frac{1}{N} \sum_{j \neq i} w_j^N K(X_i - X_j) dt + \sqrt{2} dB_t^i, \quad i = 1, \dots, N, \quad (1.1)$$

where K denotes the interaction kernel, (B^i) are the independent standard Brownian motions on \mathbb{R}^d , $d \geq 2$, and those $\{w_j^N\} \subset \mathbb{R}$ are non-identical deterministic weights that satisfy the following assumption for some $r \in (1, \infty]$

$$(\mathbb{W}_r) : \quad \frac{1}{N} \sum_{j=1}^N |w_j^N|^r = O(1), \quad \text{for } r < \infty; \quad \max_{1 \leq j \leq N} |w_j^N| = O(1), \quad \text{for } r = \infty, \quad \text{as } N \rightarrow \infty, \quad (1.2)$$

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where $O(\cdot)$ means “proportional to”. Here $r > 1$ ensures that the system is weakly interacting, indeed $|w_j^N|/N \rightarrow 0$.

One guiding example of the interacting particle system (1.1) is the famous stochastic vortex model with general intensities, i.e. the kernel K in the system (1.1) is the Biot-Savart law defined by

$$K = \nabla^\perp G = (-\partial_2 G, \partial_1 G), \quad (1.3)$$

where G is the Green function of the Laplacian on \mathbb{R}^2 . Note in particular that

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$$

where $x^\perp = (x_1, x_2)^\perp = (-x_2, x_1) \in \mathbb{R}^2$. Now the weights w_j^N denotes the intensity/magnitude of the j -th point vortex at position X_j . One may expect that now the (weighted) empirical measure defined as

$$\mu_N(t) = \frac{1}{N} \sum_{j=1}^N w_j^N \delta_{X_j(t)}$$

will converge to the solutions of the 2D Navier-Stokes equations in the vorticity form

$$\partial_t v(t, x) + \operatorname{div}(v(t, x) K * v_t(x)) = \Delta v(t, x). \quad (1.4)$$

Our main results validate the above mean-field approximation for 2D Navier-Stokes equation under very general assumptions on the intensities w_j^N : now those (w_j^N) can be of mixed-sign and unbounded. See in particular Theorem 1.9.

Many-particle systems written in the canonical form (1.1) or its variant are now quite ubiquitous. Such systems are usually formulated by the first-principle agent-based models which are conceptually simple. For instance, in physics those particles X_i can represent ions and electrons in plasma physics [Dob79], or molecules in a fluid [JO04] or even large scale galaxies [Jea15] in some cosmological models; in biological sciences, they typically model the collective behavior of animals or micro-organisms (for instance flocking, swarming and chemotaxis and other aggregation phenomena [CCH14]); in economics or social sciences particles are usually individual “agents” or “players” for instance in opinion dynamics [FJ90] or in the study of mean-field games [LL07, HMC⁺06]. Motivation even extends to the analysis of large biological [BFT15] or artificial [MMN18] neural networks in neuroscience or in machine learning.

The classical and more recent investigations on the topic of mean-field approximation have mainly focused on the case $w_j^N \equiv 1$ for all $1 \leq j \leq N$. In this case, under mild assumptions, it is well-known (see for instance [MJ, BH77, Dob79, Osa86, Szn91, FHM14, JW18, Ser20, Jab14, BJW20]) that the (usual) empirical measure $\frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$ of the particle system (1.1) converges to the solution v_t to the nonlinear mean-field PDE (1.4) as $N \rightarrow \infty$. In particular, mean-field limit on exchangeable systems is equivalent to *propagation of chaos*, i.e. the k -marginal distribution of the particle system converges to the tensor product of the limit law $g^{\otimes k}$ as N goes to infinity, given for instance the i.i.d. initial data.

Classically, Mean-field limit implies that a continuum model can be found to approximate the associated particle system when N is large. In this article, we not only establish mean-field limit for systems with general weights $w^N := (w_1^N, \dots, w_N^N)$ as in the system (1.1), as a byproduct, we demonstrate a more flexible mean-field convergence. Indeed, we can consider the following continuum model given by two coupled PDE's

$$\begin{cases} \partial_t g_t = \Delta g_t - \operatorname{div}(g_t K * v_t), \\ \partial_t v_t = \Delta v_t - \operatorname{div}(v_t K * v_t). \end{cases} \quad (1.5)$$

The continuum model (1.5) turns out to be a suitable mean-field system for the linear statistics of the interacting diffusions (1.1). Formally, let $\{\tilde{w}^N\}$ be any other sequence of weights that satisfies the

assumption (\mathbb{W}_r) , then the system (1.5) is a continuous approximation to (1.1), in the sense that as $N \rightarrow \infty$,

$$\frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i} \stackrel{d}{=} v + o(1), \quad \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \delta_{X_i} \stackrel{d}{=} g + o(1),$$

where $\stackrel{d}{=}$ means that the approximation holds in the sense of distribution. The novelty of this approximation is that now the choice of weights \tilde{w}^N can be quite flexible, including the classical average type, i.e. $\tilde{w}_i^N = 1$ for all i , or more general choice based on the relative importance of each particle. When K is the Biot-Savart law, our main results not only provides viscous vortex model approximation to the vorticity formulation of the 2D Navier-Stokes equation but now the vorticity is of mixed sign, but also establish a particle approximation to the related passive scalar equation where the flow is given by the Navier-Stokes equation.

Remark 1.1. (1) *If we just consider the deterministic setting, there is no Brownian motion term in (1.1), and set for instance $\mu_N = \frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i}$ and $\tilde{\mu}_N = \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \delta_{X_i}$, then it is easy to check that μ_N and $\tilde{\mu}_N$ solves*

$$\partial_t \mu_N + \operatorname{div}_x (\mu_N K \star \mu_N) = 0,$$

and

$$\partial_t \tilde{\mu}_N + \operatorname{div}_x (\tilde{\mu}_N K \star \mu_N) = 0,$$

respectively. To derive the above PDE or the continuum model (1.5), the interacting particle system (1.1) is the most general form we can expect. Indeed, if the weights depend both on i and j , i.e. $w = (w_{ij}^N)$ as in [JPS21], then there is no simple way to define an empirical measure, let alone to study its corresponding PDEs.

(2) *If originally we write that $Z_i = (X_i, w_i^N)$, then the expected mean field limit for the extended space reads that*

$$\partial_t f(t, x, w) + \operatorname{div}_x \left(f(t, x, w) \int_{\mathbb{R}^d \times \mathbb{R}} w' K(x - y) f(t, dy, dw') \right) = \frac{1}{2} \Delta_x f(t, x, w).$$

The 2nd equation in (1.5) is related to the above one by

$$v(t, x) = \int_{\mathbb{R}} w f(t, x, dw).$$

Then a natural idea to establish the mean field limit from (1.1) towards (1.5) is to do so in the extended phase space first then project to the observable (v or g) in a lower dimensional space as what have been done for systems with smooth interaction kernels in [Cre19] by the classical Dobrushin's estimate. See also the examples in [PR14, RJBVE19] for particle systems with even evolutionary weights. We leave the study of the system (1.1) with singular K and time-varying $w_j^N = w_j^N(t)$ for future work.

1.1. Main results. To state our main result in a concrete way, we first give the definitions of solutions to the particle system (1.1) and the mean-field PDEs (1.5).

We shall use the (non-normalized) Boltzmann entropy functional on $\mathcal{P}_\gamma(\mathbb{R}^{dN})$, which is the subspace of the probability measure space $\mathcal{P}(\mathbb{R}^{dN})$ under the constraint of finite γ -th moment, $\gamma \in (0, 1)$. The entropy functional is given by

$$H(f) := \int_{\mathbb{R}^{dN}} f \log f dx^N, \quad f \in \mathcal{P}_\gamma(\mathbb{R}^{dN}) \cap L^1(\mathbb{R}^d),$$

with x^N denoting (x_1, \dots, x_N) . If f has no density, then we set $H(f) = +\infty$. A well-known fact is that the negative part of $H(f)$ is bounded by a universal constant plus the γ -th moment of f . Thus the entropy functional is well-defined on $\mathcal{P}_\gamma(\mathbb{R}^{dN})$.

We assume the following conditions on the initial value and the interaction kernel.

(H): Let F_0^N be the joint distribution of $X^N(0) := (X_1(0), \dots, X_N(0))$. There exists some constant $\gamma \in (0, 1)$ such that

$$H(F_0^N) + \sum_{i=1}^N \mathbb{E} \langle X_i(0) \rangle^\gamma = O(N),$$

where $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$.

(\mathbb{K}_r): Given $r \in (1, \infty]$ from (\mathbb{W}_r) and $d \geq 2$. The kernel K is of the form $K = K_1 + K_2$, with K_1, K_2 satisfying

- (1) $\operatorname{div} K_1$, $K_1 \in L^{q_1}([0, T], L^{p_1}(\mathbb{R}^d))$ with $\frac{d}{p_1} + \frac{2}{q_1} + \frac{2}{r} < 2$, where the equality can be attached when $q_1, r < \infty$;
- (2) $K_2 \in L^{q_2}([0, T], L^{p_2}(\mathbb{R}^d))$ with $\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} < 1$, where the equality can be attached when $q_2, r < \infty$.

In the following we give typical examples satisfying condition (\mathbb{K}_r).

Examples

- (1) The Biot-Savart law in dimension 2 as in (1.3). It is divergence free and belongs to $L^p + L^\infty$ with $1 < p < 2$, so that it satisfies (\mathbb{K}_r) with $r > 2$.
- (2) $K(x) = \frac{x}{|x|^\alpha}$ or $-\frac{x}{|x|^\alpha}$ with $\alpha \in (1, 2)$ and $d \geq 2$. Then $K \in L^p + L^\infty$ with $1 < p < \frac{d}{\alpha-1}$ and (\mathbb{K}_r) holds with $r > \frac{1}{2-\alpha}$. Since the weights are allowed to be of mixed signs, the force K could be attractive or repulsive.

For the particle system (1.1) on $[0, T]$, $T > 0$, we define the notion of entropy solutions.

Definition 1.2 (Entropy Solutions). *Let $X^N = (X_1, \dots, X_N)$ be a $C([0, T], \mathbb{R}^{dN})$ -valued random variable satisfying the initial condition (H), and denote the law of $X^N(t)$ by F_t^N .*

For the system (1.1) with the condition (\mathbb{K}_r), we call X^N is an entropy solution if there exists a universal constant $C > 0$ and a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with a standard dN -dimensional Brownian motions (B_1, \dots, B_N) such that X^N satisfies the system (1.1) \mathbb{P} -almost surely and for $t \in [0, T]$, it holds

$$H(F_t^N) + \sum_{i=1}^N \mathbb{E} \langle X_i(t) \rangle^\gamma + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{dN}} \frac{|\nabla F_t^N|^2}{F_t^N} dx^N dt \leq H(F_0^N) + \sum_{i=1}^N \mathbb{E} \langle X_i(0) \rangle^\gamma + CN. \quad (1.6)$$

Clearly, each entropy solution to (1.1) is a probabilistically weak solution. The next result gives the existence of entropy solutions.

Proposition 1.3 (Proposition 3.5 below). *Under the conditions (H), (\mathbb{K}_r), and (\mathbb{W}_r) for some $r \in (1, \infty]$, for each $N \in \mathbb{N}$, there exists an entropy solution X^N to the particle system (1.1) such that the entropy dissipation inequality (1.6) holds with some universal constant C that is independent of N .*

The regularity of entropy solutions with a uniform constant would enable us to find the mean-field limits. The entropy solution has been shown useful for studying interacting diffusions, and Proposition 1.3 is indeed analogous to [FHM14, Proposition 5.1] and [JW18, Proposition 1], but we do not need the divergence free or bounded-like conditions on the kernel.

The well-posedness of the singular interacting system (1.1) with general weights is a fascinating and challenging problem. To the best of the authors' knowledge, the existing results concern specific kernels, such as [FM07] and [FGP11] on the Biot-Savart law. There are a lot of results for identical weights, for example [Osa85, Tak85, MP12] on the Biot-Savart law and the recent result [HRZ22] on $L^q([0, T], L^p(\mathbb{R}^d))$ -kernels with $d/p + 2/q < 1$. However, under the condition (\mathbb{K}_∞), even the well-posedness result for associated SDE: $X_t = X_0 + B_t + \int_0^t K(X_s) ds$ remains open, where the difficulty comes from the singular kernel K_1 . The existence of probabilistically weak solutions to the SDE with kernel K_1 in the assumption (\mathbb{K}_∞) has been shown in [ZZ21].

We consider the solution in the space $C([0, T], \mathcal{M}(\mathbb{R}^d))$ for the mean-field PDE system (1.5). Here $\mathcal{M}(\mathbb{R}^d)$ stands for the space of finite signed measures on \mathbb{R}^d with the topology induced by bounded and continuous (test) functions. We use it as the state space for the convergence in our main results below. The solutions to (1.5) are defined as follows.

Definition 1.4. We call $(v, g) \in C([0, T], \mathcal{M}(\mathbb{R}^d))^{\otimes 2} \cap L^\infty([0, T], L^1(\mathbb{R}^d))^{\otimes 2}$ is a solution to the system (1.5) if (v, g) satisfies (1.5) in the distributional sense and the following estimate holds

$$\mathbb{E}\|v\|_{L_q^p} + \mathbb{E}\|g\|_{L_q^p} < \infty, \quad \frac{d}{p} + \frac{2(r-1)}{r} \geq d, \quad \frac{d}{p} + \frac{2}{q} \geq d, \quad 1 \leq p, q < \infty, \quad (1.7)$$

where $\|\cdot\|_{L_q^p} := \|\cdot\|_{L^q([0, T], L^p(\mathbb{R}^d))}$ and $r \in (1, \infty]$. We denote $\frac{r-1}{r} := 1$ when $r = \infty$.

Remark 1.5. The conditions on p, q, r in the above definition ensures that the nonlinear term in the coupled PDEs (1.5) is well-defined, and also enables us to deduce uniqueness later.

The first main result shows that (1.5) characterizes the mean-field limits of the interacting system (1.1).

Theorem 1.6. Let $\{w^N\}$ and $\{\tilde{w}^N\}$ be two sequences of weights satisfying the condition (\mathbb{W}_r) with $r \in (1, \infty]$ and suppose that the conditions (\mathbb{H}) , (\mathbb{K}_r) hold. Let X^N be an entropy solution to (1.1) given by Proposition 1.3. Assume that there exist $v_0, g_0 \in L^1(\mathbb{R}^d)$ such that

$$\frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i(0)} \rightharpoonup v_0, \quad \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \delta_{X_i(0)} \rightharpoonup g_0 \quad (1.8)$$

in $\mathcal{M}(\mathbb{R}^d)$ almost surely, where \rightharpoonup means the weak convergence in $\mathcal{M}(\mathbb{R}^d)$.

It holds that the corresponding family of laws for the weighted empirical measures $(\mu_N, \tilde{\mu}_N)$ defined by

$$\mu_N(t) := \frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i(t)}, \quad \tilde{\mu}_N(t) := \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \delta_{X_i(t)}, \quad (1.9)$$

is tight in $C([0, T], \mathcal{M}(\mathbb{R}^d))^{\otimes 2}$ and every accumulation point is a solution to (1.5) with initial value (v_0, g_0) .

Theorem 1.6 gives the mean-field limits of interacting diffusions when the kernel is singular and the system is non-exchangeable at the same time. Our results extend classical mean-field limits for singular interacting diffusions to non-exchangeable cases, additionally, the weights for the interaction can be unbounded, i.e., $r < \infty$. The system (1.1) with the Biot-Savart law and bounded weights has been studied in [FHM14, Wyn21]. However, the result in [Wyn21] only applies to the so-called two pieces interaction, which is basically $w_i^N = a_1 > 0$ when i is odd and $w_i^N = a_2 < 0$ when i is even. Under that restrictive condition, the pairs of particles (X_{2i-1}, X_{2i}) are exchangeable. We also mention that the result in [FHM14] falls in the category of exchangeable N -particle systems if we treat $Z_i = (X_i, w_i^N)$ as a single particle in the extended phase space, where X_i and w_i^N denote the position and the magnitude of the i -th point vortex, respectively.

As particles/agents in applications are not always identical, more natural assumptions on weights are required. Let us now look at a specific example. Let $K(X_i - X_j)$ be the interaction in a N -particles system described by the system of SDEs (1.1). The interaction could be viewed as a function of how X_i is influenced by X_j , with w_j^N representing the intensity associated with X_j . Let the weights be $w^{5N} = (w_1^N, \dots, w_N^N, 0, \dots, 0)$ with (w_1^N, \dots, w_N^N) satisfying

$$|w_i^N| = O(N^{\frac{1}{3}}), \quad \forall 1 \leq i \leq N^{\frac{1}{3}}; \quad |w_i^N| = O(1), \quad N^{\frac{1}{3}} < i \leq N.$$

Now the model is

$$\begin{cases} dY_i = \frac{1}{5N} \sum_{j \neq i} w_j^N K(Y_i - Y_j) dt + \sqrt{2} dB_i, & i = 1, \dots, N, \\ dZ_m = \frac{1}{5N} \sum_{j=1}^N w_j^N K(Z_m - Y_j) dt + \sqrt{2} dB_m, & m = 1, \dots, 4N. \end{cases}$$

Here (Y^N, Z^{4N}) plays the role of X^N in (1.1) and (\mathbb{W}_r) holds for $r = 2$. Notice that every particle only interacts with the particles of the type Y , in other words, only the particles (Y_i) contribute to the dynamics of the system. Furthermore, there are still differences among the particles of the type Y . The particles (Y_i) belongs to two groups, the majority of the number $N - N^{\frac{1}{3}}$ and the minority of the number $N^{\frac{1}{3}}$. Each particle in the majority contributes to the system in a normal way that $|w_j^N| = O(1)$. In contrast, those particles from the minority make significant contributions, at the scale of $N^{\frac{1}{3}}$.

As mentioned earlier, the mean-field convergence of $(\mu_N, \tilde{\mu}_N)$ is much more general than the convergence of (μ_N) , since it gives the convergence for any possible weights (\tilde{w}_j^N) in l^r . Formally, one may think that our result fully recovers the linear statistical information of (X_1, \dots, X_N) rather than just the average statistics $\frac{1}{N} \sum_{i=1}^N \varphi(X_i)$ or the specific weighted one $\frac{1}{N} \sum_{i=1}^N w_i^N \varphi(X_i)$. In particular, when K is the Biot-Savart law, let $u_t = K * v_t$ denote the velocity of the fluid, then Theorem 1.6 gives a mean-field approximation to the dynamics of a passive tracer undergoing advection-diffusion in the fluid described by the Navier-Stokes equation,

$$\begin{cases} \partial_t u_t = \Delta u_t - u_t \cdot \nabla u_t + \nabla p, & \operatorname{div} u = 0, \\ \partial_t g_t = \Delta g_t - u_t \cdot \nabla g_t, \end{cases} \quad (1.10)$$

with p being the associated pressure. For more information on the physics behind passive scalars, we refer to [FGV01, SS00, War00] and references therein, and for mathematical studies, see e.g. [ACM19, BBPS22a, BBPS22b, Sei13, ZDE20] for instance.

Particle systems on graphs have attracted increasing attention in recent years, including interacting diffusions with more general w_{ij}^N replacing w_j^N . Our work can be thought of as a special case (the weight w_{ij}^N depends only on j) of interacting diffusions on graphs with weights in l^r space. The mean-field convergence for interacting diffusions on dense graphs has been studied by [BCW20, BCN20, JPS21, Luc20, OR19] in different settings. However, to the best of our knowledge their results do not cover singular kernels. We will do detailed comparisons on the methods of [BCW20, JPS21] in Section 1.2 below. Our work takes a different approach in this direction. Motivated by the general form of the stochastic 2D vortex model, we simply consider the weights given as (w_j^N) in this work, where w_j^N means the intensity of the vortex localized in X_i . However, the hypothesis $w_{ij}^N = w_j^N$ for all i is not essential, our analysis would extend in an analogous manner to the interacting diffusions on general l^r -graphons ($r > 1$), which is left as future work. Beyond the setting of the mean-field convergence for systems on dense graphs, the asymptotic behavior of interacting diffusions on sparse graphs, which involves strong interactions, is another related active topic. The local weak convergence for the sparse case has been obtained in [ORS20, LRW19] recently.

With additional constraints on the kernel and weights, we can demonstrate the uniqueness of the limiting points of converging subsequences and thus obtain the convergence of the entire sequence.

Theorem 1.7. *If the kernel K belongs to either of the following two cases,*

- (1) K is the 2D Biot-Savart law and $r \in [3, \infty]$ (since the definition of entropy solutions depends on r).
- (2) Given $r \in (1, \infty]$, and K belongs to $L^{q_2}([0, T], L^{p_2}(\mathbb{R}^d))$ with

$$\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} \leq 1, \quad \frac{d}{p_2} + \frac{2}{q_2} < 1. \quad (1.11)$$

Then there exists a unique solution (v, g) to (1.5) for each given initial value from $L^1(\mathbb{R}^d)^{\otimes 2}$.

Corollary 1.8. *Given two sequences of weights $\{w^N\}$ and $\{\tilde{w}^N\}$ satisfying the condition (\mathbb{W}_r) with $r \in [3, \infty]$. Let X^N be an entropy solution to the stochastic vortex model (i.e. Eq. (1.1) with K the*

Biot-Savart law) given by Proposition 1.3. Assume that there exist $v_0, g_0 \in L^1(\mathbb{R}^d)$ such that

$$\frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i(0)} \rightharpoonup v_0, \quad \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \delta_{X_i(0)} \rightharpoonup g_0$$

in $\mathcal{M}(\mathbb{R}^2)$ almost surely. Then $(\mu_N, \tilde{\mu}_N)$ defined in Theorem 1.6 converges in law to (v, g) in $C([0, T], \mathcal{M}(\mathbb{R}^2))^{\otimes 2}$. Here (v, g) uniquely solves (1.5) in the sense of Definition 1.4. In particular, $(\nabla^\perp(-\Delta)^{-1}v, g)$ solves the system (1.10) of the passive scalar advected by the 2D Navier-Stokes equation.

Finally, simply by choosing the same weight sequences $\tilde{w}_j^N = w_j^N$ as the intensities of the j -th point vortex, one arrives at the following theorem.

Theorem 1.9 (Mean field limit for stochastic vortex model with general intensities). *Given a sequence of intensities $w^N = (w_j^N)_{1 \leq j \leq N}$ that satisfies the condition (\mathbb{W}_r) with $r \in [3, \infty]$. Let X^N be an entropy solution to the stochastic vortex model (1.1) with K the Biot-Savart law. Assume that the initial data for the 2D Navier-Stokes equation (1.4) $v_0 \in L^1(\mathbb{R}^d)$ and*

$$\frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i(0)} \rightharpoonup v_0,$$

in $\mathcal{M}(\mathbb{R}^2)$ almost surely. Then the empirical measure $\mu_N = \frac{1}{N} \sum_{i=1}^N w_i^N \delta_{X_i}$ converges in law to v in $C([0, T], \mathcal{M}(\mathbb{R}^2))$, where v is the unique solution to (1.5) in the sense of Definition 1.4.

Mean field limit and propagation of chaos for the 1st order system given in the form (1.1) with $w_j^N = 1$ have been extensively studied over the last decade. The basic idea of deriving some effective PDE describing the large scale behaviour of interacting particle systems dates back to Maxwell and Boltzmann. But in our setting, the very first mathematical investigation can be traced back to McKean in [MJ]. See also the classical mean field limit from Newton dynamics towards Vlasov Kinetic PDEs in [Dob79, BH77, JH15, Laz16] and the review [Jab14]. Recently much progress has been made in the mean field limit for systems as (1.1) with $w_j^N = 1$ and singular interaction kernels, including those results focusing on the vortex model [Osa86, FHM14] and very recently quantitative convergence results on general singular kernels for example as in [JW18, BJW20] and [Ser20, Due16, Ros20, NRS21]. See also the references therein for more complete development on mean field limit.

In particular, as we studied in Corollary 1.9, the point vortex approximation towards the 2D Navier-Stokes equation has aroused much interest since the 1980s. Osada [Osa86] firstly obtained a propagation of chaos result with bounded initial distribution and large viscosity. More recently, Fournier, Hauray, and Mischler [FHM14] obtained entropic propagation of chaos by the compactness argument, and their result applies to all viscosity and all initial distributions with finite γ -th moment ($\gamma > 0$) and finite Boltzmann entropy. Jabin and Wang have established a quantitative estimate of the propagation of chaos in [JW18] by evolving the relative entropy between the joint distribution of X^N and the tensorized law at the limit. Very recently, the authors proved the Gaussian fluctuations in [WZZ21] by a tightness argument. We also mention the recent large deviation result obtained by Chen and Ge [CG22]. However, we are not aware of any mean-field approximations to the system of the passive scalar (1.10). Given that $K \in L_q^p$ with $d/p + 2/q < 1$, the interacting diffusions (1.1) (with $w_j^N = 1$) has been extensively studied, particularly the mean-field convergence obtained by various approaches; we specifically refer to [HHMT20] for the study of large deviations.

1.2. Difficulties and Methodology. The property of exchangeability is crucial among scaling limits of interacting diffusions, particularly those with singular interactions. The difficulty is to compare interacting particle systems with the mean-field limits in the absence of exchangeability. As mentioned before, the results in [BCW20, JPS21] can be applied to the weights given by the general form w_{ij}^N , and the symmetry of w_{ij}^N is not required in [JPS21]. In both papers the coupling method and graph theory are used. The basic idea in the proof of [BCW20] is comparing the SDEs of the particles and the SDEs of the limiting system, using the convergence of graphons (referring to [Lov12, BCCZ19]).

In contrast, the coupling method was used in [JPS21] to obtain a coupled PDEs of the McKean-Vlasov type by propagation of independence first. Then a class of new observables was constructed through the combinations of weights and laws of independent particles via a family of labeled trees. They then transformed the problem into the Vlasov/mean-field hierarchy likewise. Instead of directly using the convergence of graphons, a similar version of Szemerédi’s regularity lemma was established in [JPS21, Lemma 4.7] for non-exchangeable systems. However, the coupling method leads to the bounded and Lipschitz continuity restriction on the interactions K . As to other approaches based on weak-strong type arguments, like the relative entropy method in [JW18] and the modulated energy method in [Ser20], the problem here is that μ_N is no longer a positive measure. We do not know yet how to extend these methods which have been shown powerful for exchangeable systems to non-exchangeable ones. Indeed, it would be interesting to combine relative entropy/energy method with the limiting graphon structures of the weights w_{ij}^N .

Our idea of the proof is to apply the compactness/tightness argument, consisting of the classical three steps: the tightness, characterizing the limits, and the uniqueness of the limit equation. For particle systems with singular interactions, the regularity of the joint law is crucial to compensate the singularity. The article by Fournier, Hauray, and Mischler [FHM14] is probably the one closest to our article regarding tackling the singularity, where they also heavily exploited the Fisher information of the joint laws of particles. Many features of the Fisher information will also be used in our proof, for instance sub-additivity, the chain rule, and notably the Sobolev regularity estimates from it. Unfortunately, it is not apparent how to apply the compactness argument, since the non-exchangeability and the singularity cause two difficulties. The first one is to derive uniform estimates (about the Fisher information in our case) for the non-exchangeable system (1.1). For instance, when K is the Biot-Savart law, the following frequently used inequality requires exchangeability,

$$\mathbb{E}|K(X_i - X_j)| \lesssim 1 + I(\mathcal{L}(X_i, X_j)) \lesssim 1 + \frac{2}{N}I(\mathcal{L}(X^N)),$$

where $I(\mu) : \mathcal{P}(\mathbb{R}^{dk}) \rightarrow [0, +\infty]$, $k \in \mathbb{N}$, denotes the Fisher information functional to be defined in Section 2. This indicates that the interaction between any two particles can be controlled by $\frac{1}{N}$ of the total Fisher information of the system. However, when the particles are *not* indistinguishable from each other, the joint law is no longer symmetric, and there might be a pair of particles such that $I(\mathcal{L}(X_i, X_j)) > \frac{2}{N}I(\mathcal{L}(X^N))$. To overcome this difficulty, we build a technical lemma (Lemma 3.1) concerning the average of the interactions, which allows us to derive uniform Fisher information for non-exchangeable systems and estimate singular interactions. Investigating averaging statistics is a key step to study non-exchangeable systems. Similarly, observables with averaged information of particles also play a crucial role in [JPS21]. Note that the presence of noise is crucial in our analysis since we effectively use the control given by the Fisher information. In the deterministic setting, for instance in the vortex approximation towards the 2D incompressible Euler equation, since now those w_j^N in general can be distinct to each other, the symmetrization trick used in [JW18] does not work anymore.

The other major difficulty is to show the regularity of the limiting points of the empirical measures $\{\mu_N, N \in \mathbb{N}\}$. This is closely related to the exchangeability. With compactness argument, one usually find μ_N converges to μ in the distributional sense. To make the mean-field equation with singular coefficients well-defined and show the convergence of the nonlinear interaction term, one must propagate the regularities. Clearly, the empirical measure μ_N enjoys no regularity at all. Again, we find this difficulty is not problematic for the symmetric case since there are well-established tools based on the famous DeFinetti–Hewitt–Savage theorem [DF37, HS55] to propagate the Fisher information, c.f. [HM14] and [FHM14]. For the non-exchangeable case, we introduce a sequence of random measures (see g^N defined in (4.2)) constructed via disintegration of the joint laws $\{F^N, N \in \mathbb{N}\}$. This sequence of random measures would merge with the sequence of empirical measures as N goes to infinity, thus playing a similar role as 1-marginal distribution in the symmetric case. An analogous construction can be found in the proof of Laplace principle via weak convergence method, see e.g. [DE11, Section 2.5]. The difference is that the proof in [DE11] studies the relative entropy while we focus on the Fisher

information functional of probability measures. In the end, uniform Sobolev regularity estimates for the random measures are obtained in Section 4. Consequently, we obtain the required regularity of the limiting points.

The remainder of the compactness argument is standard, except that we shall work on the space of finite signed measures instead of the space of probability measures. It is worth mentioning here that the proof of Theorem 1.7 for the Biot-Savart law relies on the uniqueness result for the 2D Navier-Stokes equation in [FHM14], which is based on [BA94, Bre94].

1.3. Organization of the paper. This paper is organized as follows. We shall state the notations and auxiliary estimates related to the Fisher information in Section 2. Section 3 is devoted to obtaining the main estimate in this article, which gives a uniform control on the Fisher information of the joint laws of N -particles. The proof is based on an averaging estimate for the Fisher information when the probability measure is asymmetric. In Section 4, we study a sequence of random measures, which turns out to be close to the sequence of weighted empirical measures and enjoys certain Sobolev regularity estimates uniformly. Lastly, we finish the proofs of Theorem 1.6 and Theorem 1.7 by the compactness argument in Section 5.

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2. PRELIMINARIES

2.1. Notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$. During the computations, the universal constant may change from line to line, we will point out the dependence on the parameters when it is necessary. Recall that we have used the notations: $X^N := (X_1, \dots, X_N)$, $x^N := (x_1, \dots, x_N)$, $w^N := (w_1^N, \dots, w_N^N)$, and $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. The γ -th moment, $\gamma > 0$, of a positive measure μ on \mathbb{R}^d is represented by $\int_{\mathbb{R}^d} \langle x \rangle^\gamma \mu(dx)$. As usual, $C_0(\mathbb{R}^d)$ stands for the space of continuous functions vanishing at infinity, and $C_0^\infty(\mathbb{R}^d)$ stands for the space of smooth functions vanishing at infinity. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space. We also use $C_b^k(\mathbb{R}^d)$ to denote the space of bounded continuous functions with bounded k -th derivative. We use $\|\cdot\|_{L_b^p}$ to denote the $L^p([0, T], L^p(\mathbb{R}^d))$ -norm. For the notation’s simplicity we shall not distinguish the space and the norm for the vector valued functions and the scalar valued functions. For $r \in [1, \infty)$, we define

$$\|w^N\|_{l^r} := \left(\frac{1}{N} \sum_{j=1}^N |w_j^N|^r \right)^{\frac{1}{r}},$$

and $\|w^N\|_{l^\infty} := \max_{1 \leq j \leq N} |w_j^N|$. Obviously, $\|w^N\|_{l^{r_1}} \leq \|w^N\|_{l^{r_2}}$ when $r_1 \leq r_2$.

We use $\mathcal{P}(\mathbb{R}^d)$ to denote the probability space on \mathbb{R}^d and for a given Polish space \mathcal{X} we use $\mathcal{B}(\mathcal{X})$ to denote the Borel σ -algebra on \mathcal{X} and use $\mathcal{M}(\mathcal{X})$ to denote the space of finite signed measures endowed with the weak topology induced by bounded, continuous functions on \mathcal{X} , i.e. the convergence in $\mathcal{M}(\mathcal{X})$ is equivalent to the convergence testing with bounded continuous functions on \mathcal{X} . Given $\mu \in \mathcal{M}(\mathcal{X})$, its absolute value is denoted by $|\mu|$, i.e. $|\mu| := \mu^+ + \mu^-$. We use $\|\cdot\|_{TV}$ to denote the total variation norm of elements in $\mathcal{M}(\mathcal{X})$. The notation $L_w^p(\mathcal{X})$, $p \geq 1$, denotes the $L^p(\mathcal{X})$ space endowed with the weak topology induced by its dual space.

2.2. Entropy and Fisher information functionals. In this section, we define the Fisher information functional for N -particle distribution functions, and collect some related auxiliary estimates.

For $F \in \mathcal{P}(\mathbb{R}^{dN})$, the (non-normalized) Fisher information functional is defined as follows

$$I(F) := \int_{\mathbb{R}^{dN}} \frac{|\nabla F(x^N)|^2}{F(x^N)} dx^N.$$

When F has no density, we set $I(F) = +\infty$.

The following lemma (a modification of [HM14, Lemma 3.7]) shows the Fisher information of probability measures is sub-additive. The general study of this topic can be found in [HM14]. See also [Car91, Theorem 3] for an analytic proof.

Lemma 2.1. *Let $X^N := (X_1, \dots, X_N)$ be a random variable on \mathbb{R}^{dN} with joint law F^N , and denote by F_i the law of X_i , or more precisely*

$$F_i(dx_i) = \int_{\mathbb{R}^{d(N-1)}} F^N(dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_N).$$

Then it holds that

$$\sum_{i=1}^N I(F_i) \leq I(F^N)$$

where $I(F_i)$ is the Fisher information for distributions in $\mathcal{P}(\mathbb{R}^d)$, while $I(F^N)$ is the one for the joint law $F^N \in \mathcal{P}(\mathbb{R}^{dN})$.

Proof. One could assume $I(F^N)$ is finite, and use the variational formulation of Fisher information (see [HM14, Lemma 3.5]) to obtain

$$\begin{aligned} I(F^N) &= \sup_{\varphi \in C_b^1(\mathbb{R}^{dN}; \mathbb{R}^{dN})} \left\langle F^N, -\frac{|\varphi|^2}{4} - \operatorname{div} \varphi \right\rangle \\ &\geq \sup_{\varphi_i \in C_b^1(\mathbb{R}^d; \mathbb{R}^d), 1 \leq i \leq N} \left\langle F^N, -\sum_{i=1}^N \left(\frac{|\varphi_i|^2}{4} + \operatorname{div}_i \varphi_i \right) \right\rangle \\ &= \sum_{i=1}^N \sup_{\varphi_i \in C_b^1(\mathbb{R}^d; \mathbb{R}^d)} \left\langle F_i, -\left(\frac{|\varphi_i|^2}{4} + \operatorname{div}_i \varphi_i \right) \right\rangle \\ &= \sum_{i=1}^N I(F_i), \end{aligned}$$

where φ_i depends only on the i -th variable. This completes the proof. \square

One can control L^p norms and $W^{1,p}$ norms of a probability density function by its Fisher information. More precisely

Lemma 2.2. *For $d \geq 3$, a probability measure F on \mathbb{R}^d with finite Fisher information, one has*

- (1) For all $p \in [1, \frac{d}{d-2}]$, it holds that $\|F\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} I(F)^{\frac{d}{2}(1-\frac{1}{p})}$.
- (2) For all $q \in [1, \frac{d}{d-1}]$, it holds that $\|\nabla F\|_{L^q(\mathbb{R}^d)} \leq C_{q,d} I(F)^{\frac{d+1}{2}-\frac{d}{2q}}$.

Note that if $d = 2$, then the 1st control holds for all $p \in [1, +\infty)$, while the 2nd estimate holds for $q \in [1, 2)$.

Proof. These estimates are quite standard. We refer the interested readers to Lemma 3.2 in [FHM14] for the 2-dimensional case and also Lemma 2.4 in [LLY19] for the general case $d \geq 3$. The proof is essentially based on the interpolation inequality, Sobolev inequality, and also the fact that $\|F\|_{L^1} = 1$ since it is a probability density. \square

Lemma 2.3. For $d \geq 2$, consider an \mathbb{R}^d -valued function $K \in L^q([0, T], L^p(\mathbb{R}^d))$ with

$$\frac{d}{p} + \frac{2}{q} + \frac{2}{r} \leq 2, \quad \frac{d}{p} + \frac{2}{r} < 2, \quad r \in (1, \infty],$$

and probability measures $F(t, \cdot)$ on \mathbb{R}^d with finite Fisher information for a.e. $t \in [0, T]$. Then for any $\varepsilon > 0$, we have

$$\int_0^T \int_{\mathbb{R}^d} |K(t, x)|^{\frac{r}{r-1}} F(t, x) dx dt \leq \|K\|_{L_q^{\frac{r}{r-1}}} \left(C_{\varepsilon, p, q, r, d} + \varepsilon \int_0^T I(F(t, \cdot)) dt \right). \quad (2.1)$$

Proof. When $p = +\infty$, the result is trivial. So we prove only for the case when $p < \infty$, and we then have $1/q + 1/r < 1$. Notice that when $d \geq 2$, the condition $d/p + 2/r < 2$ implies that $p > r/(r-1)$.

Repeatedly applying Hölder's inequality gives

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} |K(t, x)|^{\frac{r}{r-1}} F(t, x) dx dt &\leq \|K\|_{L_q^{\frac{r}{r-1}}} \left(\int_0^T \left(\|F(t, \cdot)\|_{L^{\frac{p}{p-r-1}}} \right)^{\frac{q}{q-r-1}} dt \right)^{\frac{q-r-1}{q}} \\ &\leq C_{p, r, d} \|K\|_{L_q^{\frac{r}{r-1}}} \left(\int_0^T I(F(t, \cdot))^{\frac{dq}{2p(q-r-1)}} dt \right)^{\frac{q-r-1}{q}}, \end{aligned}$$

where the constant $C_{p, r, d}$ is from applying Lemma 2.2.

The conditions $d/p + 2/q + 2/r \leq 2$ and $1/q + 1/r < 1$ imply

$$\frac{dq \frac{r}{r-1}}{2p(q - \frac{r}{r-1})} = \frac{\frac{d}{p}}{2\left(1 - \frac{1}{r} - \frac{1}{q}\right)} \leq 1, \quad \frac{r}{r-1} < q.$$

Therefore,

$$\int_0^T \int_{\mathbb{R}^d} |K(t, x)|^{\frac{r}{r-1}} F(t, x) dx dt \leq C_{p, r, d} \|K\|_{L_q^{\frac{r}{r-1}}} \left(\int_0^T I(F(t, \cdot))^{\alpha_1} dt \right)^{\alpha_2},$$

with $0 < \alpha_1 \leq 1$ and $0 < \alpha_2 < 1$. The result is then concluded by Young's inequality. \square

3. UNIFORM FISHER INFORMATION

Uniform Fisher information for N -particles system is quite useful when the interaction is singular, see for instance applications in [FHM14] on the Biot-Savart law and [FH16] on the homogenous Landau equation with moderate soft potential. The key observation is that Fisher information provides Sobolev regularities, see Lemma 2.2, and controls the singularity of interaction, see Lemma 2.3. In this section, we derive uniform Fisher information of the joint laws $\{F^N, N \in \mathbb{N}\}$. As mentioned in the introduction, the difficulty is the lack of symmetry. However, we are fortunate enough to establish the following estimate for the average of singular interactions, which will be applied to derive the main estimate Proposition 3.5 and to identify the limits in the subsequent section.

Lemma 3.1. Assume that the function $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ satisfies the following property

$$\int_0^T \int_{\mathbb{R}^d} f(t, x) F(t, dx) dt \leq \alpha + \beta \int_0^T I(F(t, \cdot)) dt,$$

for some constants $\alpha, \beta > 0$ and any probability measures $F(t, \cdot)$ on \mathbb{R}^d with finite Fisher information for a.e. $t \in [0, T]$.

Then given $F^N(t, \cdot)$ the joint distribution of $(X_i(t))$, one has the estimate

$$\frac{1}{N^2} \sum_{i \neq j} \int_0^T \mathbb{E} \left[f \left(t, \frac{1}{\sqrt{2}} (X_i(t) - X_j(t)) \right) \right] dt \leq \alpha + \frac{2\beta}{N} \int_0^T I(F^N(t, \cdot)) dt. \quad (3.1)$$

Note that the typical choices of f are of the forms $f(x) = |K(x)|^\theta$ as in Lemma 2.3.

Remark 3.2. *If the joint distribution F^N is symmetric/exchangeable, then the conclusion in Lemma 3.1 is almost trivial. The novelty of this lemma is that we do not impose the symmetry constraint on F_N . It would be an interesting topic to study the tensorized property of entropy and Fisher information without the usual symmetry assumption.*

Remark 3.3. *The static version of this lemma holds as well. More precisely, if the function/kernel above doesn't depend on t , i.e. $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$, and $\int_{\mathbb{R}^d} f(x)F(dx) \leq \alpha + \beta I(F)$ for all $F \in \mathcal{P}(\mathbb{R}^d)$, then it holds*

$$\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[f \left(\frac{1}{\sqrt{2}} (X_i - X_j) \right) \right] \leq \alpha + \frac{2\beta}{N} I(F^N).$$

Proof. Since we study the 2-body interactions, the case $N = 2$ has nothing different compared to the exchangeable case (for any N) and its proof has been essentially verified in the proof of Lemma 3.3 in [FHM14]. The only difference here is that we now write an abstract function f , instead of a particular form $f(x) = 1/|x|^\theta$ as in [FHM14].

Now we consider the general case $N \geq 3$, where the proof is quite non-trivial and the key point is the following novel decomposition strategy for some average statistics (dating back to [Hoe94]).

We start with rewriting the right-hand side of Eq. (3.1). Let σ be a partition which divides the set $\{1, \dots, N\}$ into $\frac{N}{2}$ groups of pairs of distinct numbers when N is even, or $\frac{N+1}{2}$ groups with one group containing a single number and the other $\frac{N-1}{2}$ groups consisting pairs of distinct numbers when N is odd. Denote the collection of such partitions by S_N . Indeed, when N is even, then up to a permutation, any partition in S_N can be reduced to the following canonical form

$$\{(1, 2), (3, 4), \dots, (N-1, N)\}.$$

When N is odd, then again any partition in S_N can be reduce to the canonical form

$$\{(1, 2), (3, 4), \dots, (N-2, N-1), \{N\}\}.$$

Note that we keep the order in those pairs in $\sigma \in S_N$, i.e. when we write that $(i, j) \in \sigma$, by default we mean $i < j$.

Given non-negative variables $(x_{i,j})_{i \neq j}$, one has

$$\sum_{i \neq j} x_{i,j} = \sum_{i > j} x_{i,j} + \sum_{i < j} x_{i,j}. \quad (3.2)$$

Below we focus on the summation of $i < j$, the case $i > j$ can be dealt in the same manner. When $x_{i,j} = x_{j,i}$, the two summations are identical. We further find

$$\sum_{i < j} x_{i,j} = \frac{1}{|S_{N-2}|} \sum_{i < j} |S_{N-2}| x_{i,j} = \frac{1}{|S_{N-2}|} \sum_{\sigma \in S_N} \sum_{(i,j) \in \sigma} x_{i,j}, \quad (3.3)$$

where the last equality follows by the fact that for each pair (i, j) with $i < j$, it appears exactly at $|S_{N-2}|$ times in the summation $\sum_{\sigma \in S_N}$. This is more evident by regarding $\{x_{i,j}\}$ as variables and comparing the coefficient of each $x_{i,j}$.

Furthermore, when counting $|S_N|$, the cardinality of S_N , one can proceed by first arranging a number j for the number 1 to get a pair $(1, j)$, then there are $|S_{N-2}|$ possible ways to do the partition of the remaining numbers $\{1, 2, \dots, N\} \setminus \{1, j\}$ when N is even. This reasoning gives the conclusion that

$$|S_N| = (N-1)|S_{N-2}|, \quad N = 4, 6, \dots, \quad (3.4)$$

When N is odd, we have $|S_N| = (N-1)|S_{N-2}| + |S_{N-1}|$, since now we can first arrange a pair like $(1, 2)$ or simply the single one $\{1\}$. By induction, one has for even N , $|S_N| = |S_{N-1}|$. Consequently,

$$|S_N| = N|S_{N-2}|, \quad N = 3, 5, \dots. \quad (3.5)$$

Combining Eq. (3.3), (3.4) and (3.5), one obtains that

$$\begin{aligned} \frac{1}{N^2} \sum_{i < j} x_{i,j} &= \frac{1}{|S_N|} \sum_{\sigma \in S_N} \frac{|S_N|}{N^2 |S_{N-2}|} \sum_{(i,j) \in \sigma} x_{i,j} \\ &\leq \frac{1}{|S_N|} \sum_{\sigma \in S_N} \frac{1}{N} \sum_{(i,j) \in \sigma} x_{i,j}. \end{aligned} \quad (3.6)$$

This holds as well for the summation of $i > j$ pairs. Now letting $\int_0^T \mathbb{E}[f(\frac{1}{\sqrt{2}}(X_i - X_j))]$ play the role of $x_{i,j}$, it thus suffices to show for every partition σ ,

$$\frac{1}{N} \sum_{(i,j) \in \sigma} \int_0^T \mathbb{E} \left[f \left(\frac{1}{\sqrt{2}} (X_i(t) - X_j(t)) \right) \right] dt \leq \frac{\alpha}{2} + \frac{\beta}{N} \int_0^T I(F^N(t, \cdot)) dt.$$

To this end, for each partition $\sigma \in S_N$, we define $(Y_i)_{1 \leq i \leq N}$ as

$$\begin{aligned} Y_i &:= \frac{1}{\sqrt{2}}(X_i - X_j), \quad Y_j := \frac{1}{\sqrt{2}}(X_i + X_j), \quad \text{for } (i, j) \in \sigma \text{ (with } i < j); \\ Y_i &:= X_i, \quad \text{for } \{i\} \in \sigma. \end{aligned} \quad (3.7)$$

Indeed, without loss of generality, one can always reduce all σ to the canonical form by a permutation of N indices, for instance in the following let us assume that N is even and $\sigma = \{(1, 2), (3, 4), \dots, (X_{N-1}, X_N)\}$. We denote $Y^N = (Y_1, \dots, Y_N)$ as a function of $X^N = (X_1, \dots, X_N)$, or simply $Y^N = \Phi(X^N)$, according to the definition in Eq. (3.7).

Consequently, by change of variables,

$$\frac{1}{N} \sum_{(i,j) \in \sigma} \int_0^T \mathbb{E} \left[f \left(t, \frac{1}{\sqrt{2}} (X_i(t) - X_j(t)) \right) \right] dt = \frac{1}{N} \sum_{k=1}^{N/2} \int_0^T \mathbb{E} \left[f(t, Y_{2k-1}(t)) \right] dt. \quad (3.8)$$

Denote that $\bar{F}^N = F^N \circ \Phi^{-1}$. Then \bar{F}^N is nothing but the law of the random variable Y^N , and in particular $I(\bar{F}^N) = I(F^N)$ since the determinant of the Jacobian matrix of Φ is 1. Furthermore, let \bar{F}_i be the distribution of Y_i . Then recalling our assumption on the function f and applying Lemma 2.1, the right-hand side of Eq. (3.8) can be further bounded by

$$\begin{aligned} \sum_{k=1}^{N/2} \int_0^T \mathbb{E} \left[f(t, Y_{2k-1}(t)) \right] dt &= \sum_{k=1}^{N/2} \int_0^T \int_{\mathbb{R}^d} f(t, y) \bar{F}_{2k-1}(t, dy) dt \\ &\leq \sum_{k=1}^{N/2} (\alpha + \beta \int_0^T I(\bar{F}_{2k-1}(t, \cdot)) dt) \\ &\leq \frac{N\alpha}{2} + \beta \int_0^T I(\bar{F}^N(t, \cdot)) dt = \frac{N\alpha}{2} + \beta \int_0^T I(F^N(t, \cdot)) dt. \end{aligned}$$

When N is odd, the bound is simply replacing $N\alpha/2$ by $(N-1)\alpha/2$.

This completes the proof. \square

Applying Lemma 3.1 with $|\tilde{K}|^{\frac{r}{r-1}}$ in Lemma 2.3 playing the role of f , we arrive at the following result.

Corollary 3.4. *For $d \geq 2$, consider an \mathbb{R}^d -valued function $\tilde{K} \in L^q([0, T], L^p(\mathbb{R}^d))$ with*

$$\frac{d}{p} + \frac{2}{q} + \frac{2}{r} \leq 2, \quad \frac{d}{p} + \frac{2}{r} < 2, \quad r \in (1, \infty].$$

Then for any $\varepsilon > 0$ and any $F^N \in C([0, T], \mathcal{P}(\mathbb{R}^{dN}))$, we have

$$\frac{1}{N^2} \sum_{i \neq j} \int_0^T \int_{\mathbb{R}^d} |\tilde{K}(t, x_i - x_j)|^{\frac{r}{r-1}} F^N(t, x) dx^N dt \leq \|\tilde{K}\|_{L_q^{\frac{r}{r-1}}} \left(C_{\varepsilon, p, q, r, d} + \frac{\varepsilon}{N} \int_0^T I(F^N(t, \cdot)) dt \right).$$

Now we are in the position to show the main estimate of this article. Due to the previously established technical lemmas 2.3 and 3.1, the proof is quite neat.

Proposition 3.5. *Suppose that (\mathbb{K}_r) and (\mathbb{H}) hold for some $r \in (1, \infty]$. For each $N \in \mathbb{N}$ and $T \geq 0$, there exists an entropy solution to (1.1). Furthermore, let $\{w^N\}$ be a bounded sequence in l^r , there exists a positive constant C_T such that for all $t \in [0, T]$, $N \in \mathbb{N}$ and $\gamma \in (0, 1)$,*

$$H(F_t^N) + \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle x_i \rangle^\gamma F_t^N dx^N + \frac{1}{2} \int_0^t I(F_s^N) ds \leq H(F_0^N) + \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle x_i \rangle^\gamma F_0^N dx^N + C_T N. \quad (3.9)$$

Proof. We start with showing the a priori estimate uniformly in N .

For any $\varphi \in C^2(\mathbb{R}^{dN})$ vanishing at infinity, applying Itô's formula to $\varphi(X^N)$ and taking expectation, we arrive at the Liouville equation of F^N as

$$\partial_t F^N = \Delta F^N - \sum_{i=1}^N \operatorname{div}_{x_i} \left(F^N \frac{1}{N} \sum_{j \neq i} w_j^N K(x_i - x_j) \right) \quad (3.10)$$

in the distributional sense. We then do some formal computations that can be made rigorous by approximating the singular kernel K by smooth functions. Writing $K = K_1 + K_2$ as in (\mathbb{K}_r) , we have

$$\begin{aligned} \frac{d}{dt} H(F^N) &= -I(F^N) + \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{R}^{dN}} \nabla_i F^N \cdot w_j^N K(x_i - x_j) dx^N \\ &= -I(F^N) - \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{R}^{dN}} F^N w_j^N \operatorname{div} K_1(x_i - x_j) dx^N \\ &\quad + \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{R}^{dN}} \nabla_i F^N \cdot w_j^N K_2(x_i - x_j) dx^N \\ &:= -I(F^N) + J_1 + J_2. \end{aligned} \quad (3.11)$$

In the following we apply Corollary 3.4 to handle the interaction terms J_1 and J_2 . Applying Hölder's inequality, we obtain

$$\begin{aligned} |J_1| &\leq N \int_{\mathbb{R}^{dN}} F^N \|w^N\|_{l^r} \left(\frac{1}{N^2} \sum_{i \neq j} |\operatorname{div} K_1(x_i - x_j)|^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} dx^N \\ &\lesssim N \|w^N\|_{l^r} \int_{\mathbb{R}^{dN}} F^N \left(1 + \frac{1}{N^2} \sum_{i \neq j} |\operatorname{div} K_1(x_i - x_j)|^{\frac{r}{r-1}} \right) dx^N \\ &\lesssim N \|w^N\|_{l^r} + \|w^N\|_{l^r} \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{R}^{dN}} F^N |\operatorname{div} K_1(x_i - x_j)|^{\frac{r}{r-1}} dx^N. \end{aligned}$$

By the condition (\mathbb{K}_r) , we are allowed to apply Corollary 3.4 with $\operatorname{div} K_1$ playing the role of \tilde{K} . That means, there exists a positive constant independent of N such that

$$\frac{1}{N} \sum_{i \neq j} \int_0^t \int_{\mathbb{R}^{dN}} F^N |\operatorname{div} K_1(x_i - x_j)|^{\frac{r}{r-1}} dx^N ds \leq CN + \frac{1}{8 \|w^N\|_{l^r}} \int_0^t I(F^N) ds.$$

Therefore, integrating $|J_1|$ w.r.t. time then gives

$$\int_0^t |J_1| ds \leq CN + \frac{1}{8} \int_0^t I(F^N) ds. \quad (3.12)$$

The procedure for K_2 is similar. We first apply the Young's inequality to find

$$\begin{aligned} |J_2| &\leq \frac{1}{N} \sum_{i \neq j} |w_j^N| \left(\epsilon \int_{\mathbb{R}^{dN}} \frac{|\nabla_i F^N|^2}{F^N} dx^N + C_\epsilon \int_{\mathbb{R}^{dN}} F^N |K_2(x_i - x_j)|^2 dx^N \right) \\ &\leq \epsilon \|w^N\|_{l^1} I(F^N) + \frac{C_\epsilon}{N} \sum_{i \neq j} |w_j^N| \int_{\mathbb{R}^{dN}} F^N |K_2(x_i - x_j)|^2 dx^N. \end{aligned}$$

Similarly, let $|K_2|^2$ play the role of \tilde{K} in Corollary 3.4, there exists a constant $C'_\epsilon > 0$, depending on ϵ only, such that

$$\frac{C_\epsilon}{N} \sum_{i \neq j} |w_j^N| \int_0^t \int_{\mathbb{R}^{dN}} F^N |K_2(x_i - x_j)|^2 dx^N ds \leq C'_\epsilon N + \frac{1}{8} \int_0^t I(F^N) ds.$$

Choosing ϵ such that $\epsilon \sup_N \|w^N\|_{l^1}$ less than $1/8$, we have

$$\int_0^t |J_2| ds \leq CN + \frac{1}{8} \int_0^t I(F^N) ds. \quad (3.13)$$

Combining (3.11), (3.12), and (3.13) then yields that

$$\begin{aligned} H(F_t^N) - H(F_0^N) &\leq - \int_0^t I(F_s^N) ds + CN + \frac{1}{4} \int_0^t I(F_s^N) ds \\ &\leq - \frac{3}{4} \int_0^t I(F_s^N) ds + C_\Theta N. \end{aligned} \quad (3.14)$$

Here the constant C_Θ depends on $\Theta = \{w^N, K_2, \text{div} K_1, p_1, q_1, p_2, q_2, r, d\}$.

On the other hand, testing $\partial_t F_t^N$ with $\sum_i |x_i|^\gamma$ gives

$$\frac{d}{dt} \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle x_i \rangle^\gamma F^N dx^N = \sum_{i=1}^N \int_{\mathbb{R}^{dN}} F^N \Delta_i \langle x_i \rangle^\gamma dx^N + \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{R}^{dN}} F^N \nabla_i \langle x_i \rangle^\gamma w_j^N K(x_i - x_j) dx^N.$$

Since $\gamma \in (0, 1)$, the functions $\Delta(\cdot)^\gamma$ and $\nabla(\cdot)^\gamma$ are bounded. This implies

$$\sum_{i=1}^N \left| \int_{\mathbb{R}^{dN}} F^N \Delta_i \langle x_i \rangle^\gamma dx^N \right| \leq CN,$$

and

$$\frac{1}{N} \sum_{i \neq j} \left| \int_{\mathbb{R}^{dN}} F^N \nabla_i \langle x_i \rangle^\gamma w_j^N K(x_i - x_j) dx^N \right| \leq \frac{C}{N} \sum_{i \neq j} |w_j^N| \int_{\mathbb{R}^{dN}} F^N |K(x_i - x_j)| dx^N, \quad (3.15)$$

with a constant C depending on γ only. One may find the right hand side of (3.15) familiar, which enjoys the same formulation as J_1 and J_2 . Similarly, by the condition (\mathbb{K}_r) , we can apply Corollary 3.4 with $|K|$ playing the role of \tilde{K} , and obtain

$$\frac{1}{N} \sum_{i \neq j} \int_0^t \left| \int_{\mathbb{R}^{dN}} F^N \nabla_i \langle x_i \rangle^\gamma w_j^N K(x_i - x_j) dx^N \right| ds \leq CN + \frac{1}{4} \int_0^t I(F^N) ds. \quad (3.16)$$

Therefore, we have

$$\sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle x_i \rangle^\gamma F_t^N dx^N \leq \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \langle x_i \rangle^\gamma F_0^N dx^N + C_T N + \frac{1}{4} \int_0^t I(F_s^N) ds. \quad (3.17)$$

Now that we conclude the uniform estimate (3.9) by summing up (3.14) and (3.17).

In the following we prove the existence of entropy solutions to the particle systems (1.1). We consider the approximating systems to (1.1) with K in (1.1) replaced by regularized kernels $\{K_\epsilon\}$.

When $p, q < \infty$, we can construct $K_\varepsilon := K * \rho_\varepsilon \chi_{1/\varepsilon}$ with $\rho_\varepsilon = \varepsilon^{-d} \rho(\varepsilon^{-1}x)$ being the mollifiers and $\chi_R \in C_c^\infty(\mathbb{R}^d)$ with $\chi_R = 1$ for $|x| \leq R$ and $\chi_R = 0$ for $|x| > 2R$. We then have

$$\|K_{1,\varepsilon} - K_1\|_{L^{p_1}} + \|\operatorname{div} K_{1,\varepsilon} - \operatorname{div} K_1\|_{L^{q_1}} + \|K_{2,\varepsilon} - K_2\|_{L^{p_2}} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.18)$$

When $p_1 = \infty$ or $p_2 = \infty$, i.e. the bounded case, it is intuitively less singular in our setting, but requires additionally truncations in the approximating procedure. For instance $p_2 = \infty$, we decompose K_2 into $K_2 1_{|x| \leq R}$ plus the reminder $K_2 1_{|x| > R}$. Thus $K_2 1_{|x| \leq R}$ is L^p -integrable for any $p > 1$, we then proceed the approximations $\{K_{2,\varepsilon}^R\}$ for $K_2 1_{|x| < R}$ as the case when $p < \infty$.

Since for the approximating system the coefficients K^ε are smooth and have compact support, there exist unique solutions $X^{\varepsilon,N}$ to the approximating system. Moreover, the related infinitesimal generator for the approximating system is uniform elliptic and has smooth coefficients, which implies that the law of $X^{\varepsilon,N}$ has a smooth density $F^{\varepsilon,N} \in C([0, T], C^\infty(\mathbb{R}^d))$. Furthermore, the above computation for (3.9) holds for $F^{\varepsilon,N}$ with C_T independent of ε and N .

To pass the limit $\varepsilon \rightarrow 0$ and construct a entropy solutions to (1.1), we could use a standard tightness argument, which is similar as the tightness argument in Section 5 below. Here we only give a sketch of the proof and refer the readers to Section 5 for more details. We could obtain uniform in ε estimates for $X^{\varepsilon,N}$ as in (5.3) below, which gives that the sequence of $\{X^{\varepsilon,N}, \varepsilon > 0\}$ is tight in $C([0, T], \mathbb{R}^{dN})$. Extracting a subsequence of $\{X^{\varepsilon,N}\}$, using the Skorohod theorem to modify the stochastic basis, we obtain a limiting point X^N . We then show that the limiting point solves (1.1). As usual, due to the singularity of the kernel, one needs to regularize the kernel when identifying the limits, the error term produced by regularizing eventually vanishes. More precisely, notice that

$$\begin{aligned} & K_\varepsilon(X^{\varepsilon,N}) - K(X^N) \\ &= (K_\delta(X^{\varepsilon,N}) - K_\delta(X^N)) + (K_\varepsilon(X^{\varepsilon,N}) - K_\delta(X^{\varepsilon,N})) + (K_\delta(X^N) - K(X^N)), \end{aligned} \quad (3.19)$$

where $K_\varepsilon(X^{\varepsilon,N})$ is short for $\sum_j w_j^N K_\varepsilon(X_j^\varepsilon - X_j^\varepsilon)$, and other abbreviations are analogous. By (3.18), the uniform in ε estimate (3.9), and similar calculation as in the proof of (3.12), (3.13), we then have that the time integrals of the second term and the third term on the right hand side of (3.19) converge to zero as $\varepsilon, \delta \rightarrow 0$. For fixed small δ , we could also have the first term on the right hand side of (3.19) go to zero as $\varepsilon \rightarrow 0$. Hence, we get the convergence of the interacting term. By Lévy's characterization theorem, X^N satisfies (1.1). Finally, since the Boltzmann entropy, the γ -th moment, and the Fisher information functionals are lower semicontinuous with respect to the weak convergence, the estimate (3.9) holds uniformly for F^N . Therefore, X^N is an entropy solution. \square

4. RANDOM MEASURES WITH SOBOLEV REGULARITY

In this section, we investigate a sequence of random measures g^N in order to propagate the regularities.

When the systems are exchangeable, every accumulation point of $\{\frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)} := \nu_N\}$ enjoys finite Fisher information once the normalized Fisher information of the joint laws F^N , i.e. $\frac{1}{N} I(F_N)$, is uniformly bounded, see [HM14, Theorem 5.7]. However, the exchangeability plays a crucial role in the above argument, so it cannot be applied in our setting. In order to propagate the regularity of empirical measures for non-exchangeable systems, we introduce a sequence of auxiliary random measures $\{g^N\}$ as follows.

We use the disintegrate theorem from [AGS08, Theorem 5.3.1] to write the product measure $dt \times F_t^N(dx_1, \dots, dx_N)$ as

$$\begin{aligned} F_t^N(dx_1, \dots, dx_N) dt &= dt \times f_t^1(dx_1) f_t^2(x_1, dx_2) \dots f_t^N(x_1, \dots, x_{N-1}, dx_N) \\ &:= dt \times \prod_{i=1}^N f_t^i(x^{i-1,N}, dx_i), \end{aligned}$$

where $f_t^i(x^{i-1,N}, dx_i)$ is a transition probability kernel from $[0, T] \times \mathbb{R}^{d(i-1)}$ to $\mathcal{B}(\mathbb{R}^d)$, i.e. for every $A \in \mathcal{B}(\mathbb{R}^d)$, $(t, x^{i-1,N}) \rightarrow f_t^i(x^{i-1,N}, A)$ is $\mathcal{B}([0, T] \times \mathbb{R}^{d(i-1)})$ -measurable and for every $t \in [0, T]$, $x^{i-1,N} \in$

$\mathbb{R}^{d(i-1)}$, $f_t^i(x^{i-1,N}, dx_i)$ is a probability on \mathbb{R}^d . Furthermore, there exists a zero measure set $\mathcal{N} \subset [0, T]$ such that for $t \in \mathcal{N}^c$

$$f_t^i(X^{i-1,N}(t), dx_i) = \mathcal{L}(X_i(t)|X^{i-1,N}(t)) \quad \mathbb{P} - a.s., \quad (4.1)$$

where $\mathcal{L}(X_i(t)|X^{i-1,N}(t))$ is the conditional probability of $X_i(t)$ w.r.t. the σ -algebra generated by $X^{i-1,N}(t)$.

Given a set of deterministic weights $\{\tilde{w}_i^N, 1 \leq i \leq N\}$, we define the random measures $\{g^N\}$ as

$$g^N(t, dx) := \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N f_t^i(X_t^{i-1,N}, dx), \quad (4.2)$$

where $X_t^{i-1,N} = (X_1(t), \dots, X_{i-1}(t))$. Since $f_t^i(x^{i-1,N}, dx_i)$ is a transition probability kernel and $t \rightarrow X_t^{i-1,N}$ is continuous a.s., $g^N(t, dx)$ is also a transition kernel from $[0, T]$ to $\mathcal{B}(\mathbb{R}^d)$ a.s..

The main results in this section are Lemma 4.2 and Lemma 4.5. Lemma 4.2 tells us that the sequence $\{g^N\}$ merges with the sequence of empirical measures $\{\tilde{\mu}_N\}$ as $N \rightarrow \infty$. We use Lemma 4.5 to obtain the uniform regularities of g^N .

4.1. Weakly merging sequences. We use the concept *weakly merging* to describe how “close” is g_t^N to $\tilde{\mu}_N(t)$.

Definition 4.1. *Two sequences of finite measure valued stochastic processes $\{\mu_N(t)\}_{t \in [0, T]}$ and $\{\nu_N(t)\}_{t \in [0, T]}$ on \mathbb{R}^d are called weakly merging if for each $\varphi \in C_b(\mathbb{R}^d)$, the sequence of random variables $\{\langle \varphi, \mu_N(t) - \nu_N(t) \rangle\}$ converges to zero for almost all $(t, \omega) \in [0, T] \times \Omega$.*

When the sequences are deterministic, Definition 4.1 agrees with classical version as in [DDF88, Bog07, Dud18]. The first result below shows that $\{g_t^N\}_{t \in [0, T]}$ and $\{\tilde{\mu}_N(t)\}_{t \in [0, T]}$ defined in Theorem 1.6 are weakly merging.

Lemma 4.2. *Given a family $\{\tilde{w}^N, N \in \mathbb{N}\}$ bounded in l^r for some $r \in (1, \infty]$. Then the sequences of finite measure valued stochastic processes $\{\tilde{\mu}_N, N \in \mathbb{N}\}$ and $\{g^N, N \in \mathbb{N}\}$ are weakly merging.*

Proof. We start with representing $\tilde{\mu}_N(t) - g_t^N$ via a martingale difference sequence for a.e. $t \in [0, T]$. For $t \in [0, T]$ we use \mathcal{F}_i , $i = 0, \dots, N-1$ to denote the σ -fields generated by $(X_1(t), \dots, X_i(t))$, where we omit the dependence of \mathcal{F}_i on t for simplicity. Observe that for each bounded Borel measurable function φ on \mathbb{R}^d and $t \in \mathcal{N}^c$

$$\mathbb{E}(\varphi(X_i(t)) | \mathcal{F}_{i-1}) = \int_{\mathbb{R}^d} \varphi(x) f_t^i(X_1(t), \dots, X_{i-1}(t) | dx) = \langle \varphi, f_t^i(X_t^{i-1,N}, \cdot) \rangle, \quad \mathbb{P} - a.s.,$$

which leads to for $t \in \mathcal{N}^c$

$$\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle = \frac{1}{N} \sum_{i=1}^N \tilde{w}_i^N \left(\varphi(X_i(t)) - \langle \varphi, f_t^i(X_t^{i-1,N}, \cdot) \rangle \right) := \frac{1}{N} \sum_{i=1}^N M_i, \quad (4.3)$$

where $\{M_i, i = 1, \dots, N\}$ is a martingale difference sequence with respect to (\mathcal{F}_i) . Applying the Azuma–Hoeffding inequality [AS16, Theorem 7.2.1] thus gives, for all $N \in \mathbb{N}$ and $\varepsilon > 0$

$$\mathbb{P}(|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon) \leq 2 \exp\left(-\frac{N^2 \varepsilon^2}{8 \|\varphi\|_{L^\infty}^2 \sum_{i=1}^N |\tilde{w}_i^N|^2}\right). \quad (4.4)$$

When $r \geq 2$, the fact that $\|\tilde{w}^N\|_{l^2} \leq \|\tilde{w}^N\|_{l^r}$ gives for $t \in \mathcal{N}^c$

$$\mathbb{P}(|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon) \leq 2 \exp\left(-CN \varepsilon^2\right).$$

When $r \in (1, 2)$, we use $\|\tilde{w}^N\|_{l^2}^2 \leq \|\tilde{w}^N\|_{l^r}^r \|\tilde{w}^N\|_{l^\infty}^{2-r}$ and $\|\tilde{w}^N\|_{l^\infty} \lesssim N^{\frac{1}{r}}$ to obtain for $t \in \mathcal{N}^c$

$$\mathbb{P}(|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon) \leq 2 \exp\left(-CN \|\tilde{w}^N\|_{l^\infty}^{r-2} \varepsilon^2\right) \leq 2 \exp\left(-CN^{2-\frac{2}{r}} \varepsilon^2\right).$$

We note that the universal constant C is independent of t and N , which yields that

$$\int_{[0,T] \times \Omega} \mathbb{1}_{\{|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon\}} dt \times d\mathbb{P} \leq \sup_{t \in \mathbb{N}^c} T \mathbb{P}(|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon) \\ \leq 2T \exp\left(-CN^{\theta_r} \varepsilon^2\right),$$

where $\theta_r > 0$. Therefore, for each φ , the sequence

$$\{\langle \varphi, \tilde{\mu}_N(\cdot) \rangle - \langle \varphi, g_t^N \rangle, N \in \mathbb{N}\}$$

converges to zero in measure. Furthermore, by the Borel-Cantelli lemma and

$$\sum_{N \geq 1} \int_0^T \mathbb{P}(|\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle| > \varepsilon) < \infty,$$

we conclude that $\langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, g_t^N \rangle$ converges to zero $dt \times d\mathbb{P}$ -almost everywhere for each $\varphi \in C_b(\mathbb{R}^d)$. \square

4.2. Regularity of g^N . In the subsequent lemmas, we shall study the regularity of g^N . Classically, we first justify the absolutely continuity of the random measures.

Lemma 4.3. *Assume that (\mathbb{H}) , (\mathbb{K}_r) and (\mathbb{W}_r) hold for some $r \in (1, \infty]$. For all $1 \leq i \leq N$, $f_t^i(X_t^{i-1, N}, dx)$ is absolutely continuous with respect to the Lebesgue measure for a.s. $(t, \omega) \in [0, T] \times \Omega$. Furthermore, we have for $\gamma \in (0, 1)$,*

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} \left(\int_{\mathbb{R}^d} |x|^\gamma f_t^i(X_t^{i-1, N}, x) dx + H(f_t^i(X_t^{i-1, N})) \right) \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}|X_i(t)|^\gamma + \frac{1}{N} H(F_t^N).$$

Proof. For each $\theta \in \mathcal{P}(\mathbb{R}^d)$, the chain rule for relative entropy [DE11, Theorem C.3.1] gives that

$$H(F_t^N | \theta^{\otimes N}) = \sum_{i=1}^N \int_{\mathbb{R}^{d(i-1)}} H(f_t^i(x^{i-1, N}, \cdot) | \theta) \prod_{k=1}^{i-1} f_t^k(x^{k-1, N}, dx_k) \\ = \sum_{i=1}^N \int_{\mathbb{R}^{dN}} H(f_t^i(x^{i-1, N}, \cdot) | \theta) F_t^N(dx^N) \\ = \sum_{i=1}^N \mathbb{E} \left[H(f_t^i(X_t^{i-1, N}, \cdot) | \theta) \right].$$

Then we choose θ to be $Ce^{-|x|^\gamma}$, where C is the normalizing constant such that $\|\theta\|_{L^1} = 1$. We thus find

$$\sum_{i=1}^N \mathbb{E} \left[H(f_t^i(X_t^{i-1, N}, \cdot) | \theta) \right] = H(F_t^N) - N \log C + \sum_{i=1}^N \mathbb{E}|X_i(t)|^\gamma < \infty, \quad (4.5)$$

which implies the absolute continuity of $f_t^i(X_t^{i-1, N}, \cdot)$ for each i .

On the other hand, we find

$$H(f_t^i(X_t^{i-1, N}, \cdot) | \theta) = H(f_t^i(X_t^{i-1, N}, \cdot)) - \log C + \int_{\mathbb{R}^d} |x|^\gamma f_t^i(X_t^{i-1, N}, x) dx. \quad (4.6)$$

The proof is thus completed by combining (4.5) and (4.6). \square

From Proposition 3.5 we know F_t^N is absolutely continuous w.r.t. the Lebesgue measure, which combined with Lemma 4.3 implies the absolute continuity of $f_t^i(x^{i-1, N}, dx_i)$, denoted as $f_t^i(x_1, \dots, x_i) dx_i$.

Lemma 4.3 also implies the absolute continuity of g^N .

Corollary 4.4. *For each N , g^N has a density, still denoted by g^N , with respect to the Lebesgue measure for a.s. $(t, \omega) \in [0, T] \times \Omega$,*

The following lemma ensures that the Fisher information does not increase under the construction of random measures.

Lemma 4.5. *For the conditional distributions $\{f_t^i, 1 \leq i \leq N\}$ constructed by disintegration as in (4.1), it holds that*

$$\sum_{i=1}^N \mathbb{E} \int_0^T I\left(f_t^i(X_t^{i-1, N}, \cdot)\right) dt \leq \int_0^T I(F_t^N) dt. \quad (4.7)$$

In particular, if $\tilde{w}_i^N = 1$ in (4.2) for all $1 \leq i \leq N$, then

$$\mathbb{E} \int_0^T I(g_t^N) dt \leq \frac{1}{N} \int_0^T I(F_t^N) dt.$$

Proof. We first rewrite the left side of (4.7) using the definition of the Fisher information, and find

$$\begin{aligned} \sum_{i=1}^N \mathbb{E} \int_0^T I\left(f_t^i(X_t^{i-1, N}, \cdot)\right) dt &= \sum_{i=1}^N \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla_x f_t^i(X_t^{i-1, N}, x)|^2}{f_t^i(X_t^{i-1, N}, x)} dx dt \\ &= \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} \int_{\mathbb{R}^d} \frac{|\nabla_x f_t^i(x_1, \dots, x_{i-1}, x)|^2}{f_t^i(x_1, \dots, x_{i-1}, x)} dx F_t^N(x^N) dx^N dt. \end{aligned}$$

Using the disintegration of F_t^N , we have

$$\begin{aligned} \sum_{i=1}^N \mathbb{E} \int_0^T I\left(f_t^i(X_t^{i-1, N})\right) dt &= \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} \int_{\mathbb{R}^d} \frac{|\nabla_x f_t^i(x_1, \dots, x_{i-1}, x)|^2}{f_t^i(x_1, \dots, x_{i-1}, x)} \Pi_{j=1}^N f_t^j(x_1, \dots, x_j) dx dx^N dt \\ &= \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{di}} \frac{|\nabla_{x_i} f_t^i(x_1, \dots, x_{i-1}, x_i)|^2}{f_t^i(x_1, \dots, x_{i-1}, x_i)} \Pi_{j=1}^{i-1} f_t^j(x_1, \dots, x_j) dx_1 \dots dx_i dt \\ &= \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} |\nabla_{x_i} \log f_t^i(x_1, \dots, x_{i-1}, x_i)|^2 F_t^N(x^N) dx^N dt, \end{aligned} \quad (4.8)$$

where we used $\int_{\mathbb{R}^d} f_t^i(x_1, \dots, x_{i-1}, x_i) dx_i = 1$. On the other hand, we find

$$\begin{aligned} I(F_t^N) &= \int_{\mathbb{R}^{dN}} \left| \frac{\nabla_{x^N} F_t^N(x^N)}{F_t^N(x^N)} \right|^2 F_t^N(x^N) dx^N \\ &= \int_{\mathbb{R}^{dN}} \left| \sum_{i=1}^N \nabla_{x^N} \log f_t^i(x_1, \dots, x_i) \right|^2 F_t^N(x^N) dx^N \\ &= \sum_{i=1}^N \int_{\mathbb{R}^{dN}} |\nabla_{x^N} \log f_t^i(x_1, \dots, x_i)|^2 F_t^N(x^N) dx^N. \end{aligned}$$

At the last equality we used the chain rule of Fisher information [Zam98], equivalent to the fact that the summation of cross terms equals to zero. More precisely, the summation of all the cross terms consists of the following summations with $k \leq i < j$ (we use abbreviated notations for simplicity),

$$\begin{aligned} &\sum_{j>i} \int_{\mathbb{R}^{dN}} \nabla_{x_k} \log f^i(x_1, \dots, x_i) \nabla_{x_k} \log f^j(x_1, \dots, x_i, \dots, x_j) F^N(x^N) \\ &= \sum_{j>i} \int_{\mathbb{R}^{di}} \nabla_{x_k} \log f^i(x_1, \dots, x_i) \Pi_{m \leq i} f^m \int_{\mathbb{R}^{d(N-i)}} \frac{\nabla_{x_k} f^j(x_1, \dots, x_i, \dots, x_j)}{f^j} \Pi_{l>i} f^l \\ &= \int_{\mathbb{R}^{di}} \nabla_{x_k} \log f^i(x_1, \dots, x_i) \Pi_{m \leq i} f^m \left(\sum_{j>i} \int_{\mathbb{R}^{d(N-i)}} \nabla_{x_k} f^j \Pi_{l>i, l \neq j} f^l \right) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^{di}} \nabla_{x_k} \log f^i(x_1, \dots, x_i) \Pi_{m \leq i} f^m \left(\nabla_{x_k} \int_{\mathbb{R}^{d(N-i)}} \Pi_{l > i} f^l \right) \\
&= \int_{\mathbb{R}^{di}} \nabla_{x_k} \log f^i(x_1, \dots, x_i) \Pi_{m \leq i} f^m (\nabla_{x_k} 1) = 0.
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E} \int_0^T I(f_t^i(X_t^{i-1, N})) dt &= \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} |\nabla_{x_i} \log f_t^i(x_1, \dots, x_i)|^2 F_t^N(x^N) dx^N dt \\
&\leq \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} |\nabla_{x_N} \log f_t^i(x_1, \dots, x_i)|^2 F_t^N(x^N) dx^N dt = \int_0^T I(F_t^N) dt,
\end{aligned}$$

which is exactly (4.7).

When $\tilde{w}_i^N = 1$ for all $1 \leq i \leq N$, the convexity of the Fisher information yields that

$$I(g_t^N) \leq \frac{1}{N} \sum_{i=1}^N I(f_t^i(X_t^{i-1, N}, \cdot)),$$

the result thus follows. \square

Since we are interested in non-identical and even unbounded weights, there would be loss of regularity for g^N defined in (4.2), due to the appearance of the weights and the corresponding condition (\mathbb{W}_r) . Fortunately, we can still obtain certain Sobolev regularity.

Lemma 4.6. *Given a family $\{\tilde{w}^N, N \in \mathbb{N}\}$ satisfying the condition (\mathbb{W}_r) (i.e. uniformly bounded in l^r) for some $r \in (1, \infty]$ and assume (\mathbb{H}) , (\mathbb{W}_r) and (\mathbb{K}_r) , then one has the following results:*

(1) *It holds that*

$$\mathbb{E} \|g_t^N\|_{L^\infty([0, T], L^1(\mathbb{R}^d))} \leq \|w^N\|_{l^1} \leq \|w^N\|_{l^r}. \quad (4.9)$$

(2) *For any $1 \leq p, q < \infty$ satisfying*

$$\frac{d}{p} + \frac{2(r-1)}{r} \geq d, \quad \frac{d}{p} + \frac{2}{q} \geq d, \quad (4.10)$$

it holds that

$$\mathbb{E} \int_0^T \|g_t^N\|_{L^p}^q dt \leq C \|\tilde{w}^N\|_{l^r}^q T + C \frac{\|\tilde{w}^N\|_{l^r}^q}{N} \mathbb{E} \int_0^T I(F_t^N) dt, \quad (4.11)$$

where $p \in [1, \frac{d}{d-2}]$ when $d \geq 3$ and $p \in [1, \infty)$ when $d = 2$.

(3) *When $d \geq 3$, for any $1 \leq p, q < \infty$ satisfying*

$$\frac{d}{p} + \frac{2(r-1)}{r} \geq d+1, \quad \frac{d}{p} + \frac{2}{q} \geq d+1, \quad (4.12)$$

it holds that

$$\mathbb{E} \int_0^T \|\nabla g_t^N\|_{L^p}^q dt \leq C \|\tilde{w}^N\|_{l^r}^q T + C \frac{\|\tilde{w}^N\|_{l^r}^q}{N} \mathbb{E} \int_0^T I(F_t^N) dt, \quad (4.13)$$

When $d = 2$, the result holds for $p \in [1, 2)$.

Here the proportional constants are independent of N .

Proof. The part (1) follows by the fact that f_t^i is a probability density for each i . More precisely,

$$\|g_t^N\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \int_{\mathbb{R}^d} f_t^i(X_t^{i-1, N}, x) dx = \|\tilde{w}^N\|_{l^1}.$$

The proof of other parts is based on Lemma 2.2, Lemma 4.5, and repeatedly applying Hölder's inequality.

For the part (2), by Jensen's inequality and Hölder's inequality, we find

$$\begin{aligned} \mathbb{E} \int_0^T \|g_t^N\|_{L^p}^q dt &\leq \mathbb{E} \int_0^T \left(\frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \|f_t^i(X_t^{i-1, N})\|_{L^p} \right)^q dt \\ &\leq \mathbb{E} \int_0^T \left[\|\tilde{w}^N\|_{l^r} \left(\frac{1}{N} \sum_{i=1}^N \|f_t^i(X_t^{i-1, N})\|_{L^p}^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \right]^q dt. \end{aligned}$$

Then applying the first point of Lemma 2.2 gives that

$$\begin{aligned} \mathbb{E} \int_0^T \|g_t^N\|_{L^p}^q dt &\leq C \|\tilde{w}^N\|_{l^r}^q \mathbb{E} \int_0^T \left(\frac{1}{N} \sum_{i=1}^N I(f_t^i(X_t^{i-1, N})) \right)^{\frac{d}{2}(1-\frac{1}{p})\frac{r}{r-1}} \frac{q(r-1)}{r} dt \\ &\leq C \|\tilde{w}^N\|_{l^r}^q \mathbb{E} \int_0^T \left(1 + \frac{1}{N} \sum_{i=1}^N I(f_t^i(X_t^{i-1, N})) \right)^{\frac{d}{2}(1-\frac{1}{p})\max\{q, \frac{r}{r-1}\}} dt, \end{aligned}$$

where the constant term "1" comes from applying Young's inequality and appears only when $q < r/(r-1)$. Observe that the condition (4.10) is equivalent to

$$\frac{d}{2} \left(1 - \frac{1}{p} \right) \max\left\{ q, \frac{r}{r-1} \right\} \leq 1,$$

thus the estimate (4.11) is concluded by Lemma 4.5.

The proof of Part (3) is almost the same, except that we use the 2nd part of Lemma 2.2 to control ∇g_t^N and also $\nabla f_t^i(X_t^{i-1, N}, \cdot)$, instead of the first one. In this case, the condition (4.12) is equivalent to

$$\left(\frac{d+1}{2} - \frac{d}{2p} \right) \max\left\{ q, \frac{r}{r-1} \right\} \leq 1.$$

We omit the rest of the proof to avoid repeating. \square

Lemma 4.7. *Suppose the same setting as in Lemma 4.6 and that g_t^N converges to g_t in the space of distributions $\mathcal{S}'(\mathbb{R}^d)$, i.e. the dual space of Schwartz functions, $dt \times d\mathbb{P}$ almost everywhere. Then the Sobolev regularity estimates (4.9), (4.11) and (4.13) hold for g . In particular, $g_t(\omega)$ has a density w.r.t. the Lebesgue measure for a.e. (t, ω) .*

Proof. Let A be the set $\{\varphi \in \mathcal{S}(\mathbb{R}^d), \|\varphi\|_{L^{\frac{p}{p-1}}} \leq 1\}$. By the convergence of g_t^N to g_t in $\mathcal{S}'(\mathbb{R}^d)$ for almost every (t, ω) , we find

$$\begin{aligned} \mathbb{E} \int_0^T \|g_t\|_{L^p}^q dt &= \mathbb{E} \int_0^T \left(\sup_{\varphi \in A} \langle \varphi, g_t \rangle \right)^q dt \leq \mathbb{E} \int_0^T \left(\liminf_{N \rightarrow \infty} \sup_{\varphi \in A} \langle \varphi, g_t^N \rangle \right)^q dt \\ &= \mathbb{E} \int_0^T \left(\liminf_{N \rightarrow \infty} \|g_t^N\|_{L^p} \right)^q dt \leq \liminf_{N \rightarrow \infty} \mathbb{E} \int_0^T \|g_t^N\|_{L^p}^q dt, \end{aligned}$$

where we used the lower semi-continuity of supremum and Fatou's lemma. By (4.11), Proposition 3.5, and the condition (\mathbb{W}_r) , we conclude that

$$\mathbb{E} \int_0^T \|g_t\|_{L^p}^q dt \leq C \|\tilde{w}^N\|_{l^r}^q T + C \liminf_{N \rightarrow \infty} \frac{\|\tilde{w}^N\|_{l^r}^q}{N} \mathbb{E} \int_0^T I(F_t^N) dt < \infty.$$

Since ∇g_t^N converges to ∇g_t in $\mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$ for a.e. (t, ω) . Again, by lower semi-continuity we find the estimate (4.13) holds for the limit ∇g_t . \square

5. MEAN-FIELDS LIMITS

This section is devoted to show the mean-field limits of the interacting system (1.1), or more precisely the convergence of empirical measures $\mu_N(t)$ and $\tilde{\mu}_N(t)$ as in (1.9), and completes the proof of Theorem 1.6 and Theorem 1.7.

5.1. Tightness. To apply the classical tightness argument (also known as stochastic compactness method), it requires to find a suitable topology, which is weak enough to show the tightness of laws while sufficiently strong such that the equation (primarily the nonlinear singular part) as a functional of solutions is continuous. However, the topology for the convergence and the tightness of the empirical measures on \mathbb{R}^d are too weak to ensure the convergence of the nonlinear term. To solve this problem, we consider the joint laws of the random measures $\{g^N\}$ introduced in Section 4, together with the associated weighted empirical measures, to handle singular interacting kernels. More precisely, we show the tightness of joint laws of $\{(\tilde{Q}^N, g^N)\}$, where \tilde{Q}^N defined below is the path version of $\tilde{\mu}_N$.

The definition of tightness and the Prokhorov theorem is well-known for probability measures. We recall the generalizations for signed measures for convenience, which could be easily found in the textbook [Bog07].

Definition 5.1. *A family \mathcal{V} of Radon measures on a topological space \mathcal{Y} is called tight if for every $\varepsilon > 0$, there exists a compact set A_ε such that $|\nu|(\mathcal{Y} \setminus A_\varepsilon) < \varepsilon$ for all $\nu \in \mathcal{V}$.*

The following theorem due to Prokhorov connects tightness, weak convergence sequences, and compactness, c.f. [Bog07, Theorems 8.6.2 and 8.6.7].

Lemma 5.2. *Let \mathcal{Y} be a complete separable metric space and let \mathcal{V} be a family of Radon measures on \mathcal{Y} . Then the following two statements are equivalent:*

- (1) every sequence $\{\nu_N\} \subset \mathcal{V}$ contains a weakly convergent subsequence;
- (2) the family \mathcal{V} is tight and uniformly bounded in total variation norm.

Let $\mathcal{V} \subset \mathcal{M}(\mathcal{Y})$ be a uniformly bounded in total variation norm and tight family of Radon measures on \mathcal{Y} . Then \mathcal{V} has compact closure in the weak topology.

We first show the tightness of laws of the empirical measures on the path space $C([0, T], \mathbb{R}^d)$, which are defined by

$$\tilde{Q}^N(\cdot) = \frac{1}{N} \sum_i \tilde{w}_i^N \delta_{X_i} \in \mathcal{M}(C([0, T]; \mathbb{R}^d)).$$

Notice that $\tilde{\mu}_N(t) = \tilde{Q}^N \circ \pi_t^{-1}$, where π_t for $t \in [0, T]$ is the canonical projection from $C([0, T], \mathbb{R}^d)$ to \mathbb{R}^d defined by $\pi_t(X) = X(t)$ for $X \in C([0, T], \mathbb{R}^d)$. Let $\phi : C([0, T], \mathbb{R}^d) \rightarrow [0, \infty]$ be the function

$$\phi(X) := \sup_{0 \leq s < t \leq T} \frac{|X(t) - X(s)|}{(t-s)^{1-\alpha}} + |X(0)|^{\frac{(r-1)\gamma}{r}}, \quad (5.1)$$

where $\gamma \in (0, 1)$ and $\alpha \in (\max\{\frac{1}{2}, \frac{d}{2p_1} + \frac{1}{q_1}\}, 1)$. The choice for such α ensures that for $\alpha^* := \frac{1}{1-\alpha}$,

$$2(1-\alpha) < 1, \quad 1 < \alpha^*, \quad \frac{d}{p_1} + \frac{2}{q_1} + \frac{2}{\alpha^*} < 2, \quad \frac{d}{p_2} + \frac{2}{q_2} + \frac{2}{\alpha^*} < 2, \quad \frac{1}{\alpha} = \frac{\alpha^*}{\alpha^* - 1}.$$

We will apply Corollary 3.4 with (α^*, p_1, q_1) and (α^*, p_2, q_2) playing the role of (r, p, q) , under the condition (\mathbb{K}_r) to obtain the following key uniform estimate.

Lemma 5.3. *Suppose that (\mathbb{H}) , (\mathbb{W}_r) and (\mathbb{K}_r) hold for some $r \in (1, \infty]$. Given a family $\{\tilde{w}^N, N \in \mathbb{N}\}$ satisfying the condition (\mathbb{W}_r) , it holds that*

$$\sup_N \mathbb{E} \langle \phi, |\tilde{Q}^N| \rangle = \sup_N \mathbb{E} \int_{C([0, T], \mathbb{R}^d)} \phi(X) |\tilde{Q}^N|(dX) < \infty. \quad (5.2)$$

Proof. By the definition of ϕ , we indeed need to show

$$\sup_N \left(\frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} |X_i(0)|^{\frac{(r-1)\gamma}{r}} + \frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} \sup_{0 \leq s < t \leq T} \frac{|X_i(t) - X_i(s)|}{(t-s)^{1-\alpha}} \right) < \infty. \quad (5.3)$$

The first summation in the bracket concerns on the γ -th moments of the initial values. Using Hölder's inequality, we find

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} |X_i(0)|^{\frac{(r-1)\gamma}{r}} &\leq \|\tilde{w}^N\|_{l^r} \left(\frac{1}{N} \sum_{i=1}^N \left[\mathbb{E} |X_i(0)|^{\frac{(r-1)\gamma}{r}} \right]^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \\ &\leq \|\tilde{w}^N\|_{l^r} \left(\frac{1}{N} \sum_{i=1}^N \mathbb{E} |X_i(0)|^\gamma \right)^{\frac{r-1}{r}}, \end{aligned}$$

which is uniformly bounded under the condition (H).

We then investigate the second summation. Observe that

$$\begin{aligned} &\frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} \sup_{s < t} \frac{|X_i(t) - X_i(s)|}{(t-s)^{1-\alpha}} \\ &= \frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} \sup_{s < t} \frac{\left| \int_s^t \frac{1}{N} \sum_{j \neq i} w_j^N K(X_i(\tau) - X_j(\tau)) d\tau + \sqrt{2}(B_i(t) - B_i(s)) \right|}{(t-s)^{1-\alpha}} \\ &\leq J_1^N + J_2^N \end{aligned}$$

where J_i^N , $i = 1, 2$, are defined by

$$\begin{aligned} J_1^N &:= \frac{1}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} \sup_{s < t} \frac{\left| \int_s^t \frac{1}{N} \sum_{j \neq i} w_j^N K(X_i(\tau) - X_j(\tau)) d\tau \right|}{(t-s)^{1-\alpha}}, \\ J_2^N &:= \frac{\sqrt{2}}{N} \sum_{i=1}^N |\tilde{w}_i^N| \mathbb{E} \sup_{s < t} \frac{|(B_i(t) - B_i(s))|}{(t-s)^{1-\alpha}}. \end{aligned}$$

For J_1^N involving the interactions, we have

$$\begin{aligned} J_1^N &\leq \frac{1}{N^2} \sum_{i \neq j} |\tilde{w}_i^N w_j^N| \mathbb{E} \left(\sup_{s < t} \frac{\int_s^t |K(X_i - X_j)| d\tau}{(t-s)^{1-\alpha}} \right) \\ &\leq \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(\frac{1}{N^2} \sum_{i \neq j} \left[\mathbb{E} \left(\sup_{s < t} \frac{\int_s^t |K(X_i - X_j)| d\tau}{(t-s)^{1-\alpha}} \right) \right]^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \\ &\leq \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(\frac{1}{N^2} \sum_{i \neq j} \left[\mathbb{E} \int_0^T |K(X_i - X_j)|^{\frac{1}{\alpha}} dt \right]^{\frac{\alpha r}{r-1}} \right)^{\frac{r-1}{r}} \\ &\leq C \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(T + \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \int_0^T |K(X_i - X_j)|^{\max\{\frac{r}{r-1}, \frac{1}{\alpha}\}} dt \right)^{\frac{r-1}{r}}, \end{aligned}$$

where the constant T is given by Young's inequality $|x|^{\alpha r/(r-1)} \leq |x| + 1$ when $\alpha r/(r-1) \leq 1$. We thus obtain

$$J_1^N \leq C \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(T + \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \int_0^T (|K(X_i - X_j)|^{\frac{r}{r-1}} + |K(X_i - X_j)|^{\frac{1}{\alpha}}) dt \right)^{\frac{r-1}{r}}.$$

Using Corollary 3.4, we find J_1^N is bounded by the Fisher information. That is

$$J_1^N \leq C \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(C + \frac{1}{N} \int_0^T I(F_t^N) dt \right)^{\frac{r-1}{r}},$$

for all $N \in \mathbb{N}$. Proposition 3.5 thus implies that J_1^N is uniformly bounded for all $N \in \mathbb{N}$.

Next, $\sup_N J_2^N < \infty$ follows by the modulus continuity of Brownian motions,

$$J_2^N \leq \sqrt{2} \|\tilde{w}^N\|_{l^1} \left[\mathbb{E} \sup_{s < t} \frac{|B_1(t) - B_1(s)|}{(t-s)^{1-\alpha}} \right].$$

The proof of (5.2) is thus completed, and the result follows. \square

Lemma 5.4. *Suppose that (\mathbb{H}) , (\mathbb{W}_r) and (\mathbb{K}_r) hold for some $r \in (1, \infty]$. Given a family $\{\tilde{w}^N, N \in \mathbb{N}\}$ satisfying the condition (\mathbb{W}_r) , the laws of the sequence $\{\tilde{Q}^N, N \in \mathbb{N}\}$ are tight on $\mathcal{M}(C([0, T], \mathbb{R}^d))$.*

Proof. By the Arzelà–Ascoli theorem, the measurable function ϕ on $C([0, T], \mathbb{R}^d)$ is lower bounded and has precompact level sets, i.e. the set $\{X | \phi(X) \leq c\}$ is precompact for any positive number c . Such function with precompact level sets is called a tightness function in the literature, c.f. [DE11].

We claim that $\Phi : \mathcal{M}(C([0, T], \mathbb{R}^d)) \rightarrow [0, \infty]$ defined by

$$\Phi(\mu) := \langle \phi, |\mu| \rangle + \|\mu\|_{TV}$$

has precompact level sets. One may deduce by the Chebyshev’s inequality that

$$|\mu|(\{\phi > c\}) \leq \frac{1}{c} \langle \phi, |\mu| \rangle.$$

Thus for any given level set $A_R := \{\mu | \Phi(\mu) \leq R\}$, $R > 0$, the family A_R is tight in $\mathcal{M}(C([0, T], \mathbb{R}^d))$ and uniformly bounded in total variation norm. By the generalized Prokhorov’s theorem (Lemma 5.2), the closure of A_R is compact in the weak topology.

Furthermore, by (\mathbb{W}_r) and Lemma 5.3, we obtain

$$\begin{aligned} \mathbb{P}(\tilde{Q}^N \notin A_R) &= \mathbb{P}(\Phi(\tilde{Q}^N) > R) \leq \frac{1}{R} \mathbb{E}[\Phi(\tilde{Q}^N)] \\ &\leq \frac{1}{R} \left(\|\tilde{w}^N\|_{l^1} + \mathbb{E}\langle \phi, |\tilde{Q}^N| \rangle \right) \xrightarrow{R \rightarrow \infty} 0, \end{aligned}$$

which is uniformly in N . The tightness of the laws of $\{\tilde{Q}^N, N \in \mathbb{N}\}$ thus follows. \square

The next result concerns on tightness of laws of $\{g^N\}$.

Lemma 5.5. *Suppose that (\mathbb{H}) , (\mathbb{W}_r) and (\mathbb{K}_r) hold for some $r \in (1, \infty]$. There exists $p^* > 1$ such that the laws of $\{g^N, N \in \mathbb{N}\}$ are tight on $L_w^{p^*}([0, T] \times \mathbb{R}^d)$.*

Proof. By the part (1) of Lemma 4.6, one may choose $p^* > 1$ such that $p^* = p = q$ such that

$$\frac{d}{p^*} + \frac{2(r-1)}{r} \geq d, \quad \frac{d+2}{p^*} \geq d,$$

which equals to

$$1 < p^* \leq \min \left\{ \frac{d}{d - \frac{2(r-1)}{r}}, \frac{d+2}{d} \right\}, \quad p^* < \infty.$$

Thus Lemma 4.6 shows there exists such $p^* > 1$ such that

$$\sup_N \mathbb{E} \|g^N\|_{L^{p^*}([0, T] \times \mathbb{R}^d)}^{p^*} \leq C + C \left(\sup_N \|\tilde{w}^N\|_{l^r}^{p^*} \right) \left(\sup_N \frac{1}{N} \mathbb{E} \int_0^T I(F_t^N) dt \right) < \infty.$$

This uniform bound of Fisher information is ensured by Proposition 3.5. Furthermore, applying Chebyshev’s inequality yields that

$$\sup_N \mathbb{P} \left(g^N(t, x) \in L^{p^*}([0, T] \times \mathbb{R}^d), \|g^N\|_{L^{p^*}} > R \right) \leq \frac{1}{R^{p^*}} \sup_N \mathbb{E} \|g^N\|_{L^{p^*}([0, T] \times \mathbb{R}^d)}^{p^*} \xrightarrow{R \rightarrow \infty} 0.$$

The proof is thus completed. \square

5.2. Identify the limits. Now we extract a subsequence of $\{(Q^N, v^N, \tilde{Q}^N, g^N)\}$, where (Q^N, v^N) is defined by replacing \tilde{w}^N in the definition of (\tilde{Q}^N, g^N) by w^N , and identify the limiting point as a solution to (1.5).

Observe that $\{w^N\}$ is just a specific example of $\{\tilde{w}^N\}$, by Lemma 5.4 and Lemma 5.5, one may deduce that the sequence of laws of $\{(Q^N, v^N, \tilde{Q}^N, g^N), N \in \mathbb{N}\}$ is tight on the space \mathcal{X} defined by

$$\mathcal{X} := \mathcal{M}\left(C([0, T], \mathbb{R}^d)\right) \times L_w^{p^*}([0, T] \times \mathbb{R}^d) \times \mathcal{M}\left(C([0, T], \mathbb{R}^d)\right) \times L_w^{p^*}([0, T] \times \mathbb{R}^d)$$

By the generalized Skorokhod Representation Theorem/ Jakubowski Theorem (c.f. [BFH18, Theorem 2.7.1]), we deduce the following result.

Proposition 5.6. *There exists a subsequence of $\{(Q^N, v^N, \tilde{Q}^N, g^N), N \in \mathbb{N}\}$, without relabeling for simplicity, and a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ with \mathcal{X} -valued random variables $\{(Q^{*N}, v^{*N}, \tilde{Q}^{*N}, g^{*N}), N \in \mathbb{N}\}$ and (Q, v, \tilde{Q}, g) such that*

- (1) *For each N , the law of $(Q^N, v^N, \tilde{Q}^N, g^N)$ coincides with the law of $(Q^{*N}, v^{*N}, \tilde{Q}^{*N}, g^{*N})$.*
- (2) *The sequence of random variables $(Q^{*N}, v^{*N}, \tilde{Q}^{*N}, g^{*N})$ converges to (Q, v, \tilde{Q}, g) in \mathcal{X} \mathbb{P}^* -almost surely.*

Remark 5.7. *To apply the Jakubowski theorem, one needs to check a topological property, that is the space \mathcal{X} is countably separated. This property is closely related to submetrizability (or metrizability for compact spaces). For our case, the weak topology of the Polish space L^{p^*} is clearly countably separated since its dual space is separable. As to $\mathcal{M}(C([0, T], \mathbb{R}^d))$, the required topological property follows by Koumoullis and Sapounakis [KS84, Theorem 4.1].*

For simplicity, we omit the superscript $*$ in the following text.

Now we are able to deduce the convergence of $\{\tilde{\mu}_N\}$ (and the specific sequence $\{\mu_N\}$).

Corollary 5.8. *The sequence of the empirical measure processes $\tilde{\mu}_N$ converges to $\tilde{\mu}$ in $C([0, T], \mathcal{M}(\mathbb{R}^d))$ almost surely, where $\tilde{\mu} := (\tilde{Q} \circ \pi_t^{-1})_{t \in [0, T]}$.*

Proof. Recall that the canonical projection π_t from $C([0, T], \mathbb{R}^d)$ to \mathbb{R}^d . Clearly, $\tilde{\mu}$ belongs to the space $C([0, T], \mathcal{M}(\mathbb{R}^d))$. Given any function $\varphi \in C_b(\mathbb{R}^d)$, it is straightforward to check that the family of functions $\{\Phi_t, t \in [0, T]\}$ on $C([0, T], \mathbb{R}^d)$, defined by $\Phi_t(X) := \varphi(X_t)$ for $X \in C([0, T], \mathbb{R}^d)$, is uniformly bounded and pointwise equicontinuous. Therefore, we have

$$\begin{aligned} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \varphi(x) \left(\tilde{\mu}_N(t)(dx) - \tilde{\mu}_t(dx) \right) \right| &= \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \varphi(x) \left(\tilde{Q}^N \circ \pi_t^{-1}(dx) - \tilde{Q} \circ \pi_t^{-1}(dx) \right) \right| \\ &= \sup_{t \in [0, T]} \left| \int_{C([0, T], \mathbb{R}^d)} \Phi_t(X) \left(\tilde{Q}^N(dX) - \tilde{Q}(dX) \right) \right| \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

where the convergence follows by the convergence of \tilde{Q}^N and applying [Bog07, Exercise 8.10.134]. \square

The next lemma connects the limiting points of weakly merging sequences $\{\tilde{\mu}_N\}$ and g^N .

Lemma 5.9. *The subsequence $\{g_i^N\}$ converges to $\tilde{\mu}_t$ in $\mathcal{S}'(\mathbb{R}^d)$ for almost every (t, ω) . Furthermore, g_t is a density of $\tilde{\mu}_t$ for almost every (t, ω) .*

Proof. For a.e. $(t, \omega) \in [0, T] \times \Omega$ and any $\varphi \in C_0(\mathbb{R}^d)$, we have

$$\begin{aligned} \langle \varphi, \tilde{\mu}_t \rangle &= \lim_{N \rightarrow \infty} \langle \varphi, \tilde{\mu}_N(t) \rangle \\ &= \lim_{N \rightarrow \infty} \left(\langle \varphi, \tilde{\mu}_N(t) - g_i^N \rangle + \langle \varphi, g_i^N \rangle \right) \\ &= \lim_{N \rightarrow \infty} \langle \varphi, g_i^N \rangle. \end{aligned}$$

The last equality follows by the fact that $\{\tilde{\mu}_N(t)\}$ and $\{g_t^N\}$ are weakly merging, proved in Lemma 4.2.

On the other hand, since g^N converges to g in $L_w^p([0, T] \times \mathbb{R}^d)$, we have

$$\int_{[0, T] \times \mathbb{R}^d} \varphi(t, x) g_t(x) dx dt = \int_{[0, T] \times \mathbb{R}^d} \varphi(t, x) \tilde{\mu}_t(dx) dt, \quad \forall \varphi \in C_0([0, T] \times \mathbb{R}^d),$$

almost surely. Choosing a countable dense subset of $C_0(\mathbb{R}^d)$ leads to $g_t(x) dx \times dt = \tilde{\mu}_t(dx) \times dt$ in $\mathcal{M}([0, T] \times \mathbb{R}^d)$ almost surely. Here we used the Riesz-Markov-Kakutani representation theorem (see [Bog07, Section 7.10]), which characterizes $\mathcal{M}(\mathbb{R}^d)$ as the space of bounded linear functionals on $C_0(\mathbb{R}^d)$. Furthermore, by the uniqueness of disintegration (we refer to [Bog07, Lemma 10.4.3] for the result on signed measures), we conclude that $g_t(x) dx = \tilde{\mu}_t(dx)$ for almost every (t, ω) . \square

In the following, we shall not distinguish g and $\tilde{\mu}$. The previous lemma together with Lemma 4.7 gives the following.

Corollary 5.10. *The Sobolev regularity estimates (4.9), (4.11) and (4.13) hold for g .*

Now we are in the position to identify the limiting point (v, g) .

Proposition 5.11. *Suppose that (H) and (\mathbb{K}_r) hold for some $r \in (1, \infty]$. Given two sequences $\{\tilde{w}^N, N \in \mathbb{N}\}$ and $\{w^N, N \in \mathbb{N}\}$ satisfying the condition (\mathbb{W}_r) , each limiting point (v, g) obtained from Proposition 5.6 is a solution to (1.5) in the sense of Definition 1.4.*

Proof. Clearly

$$M_t^N(\varphi) = \langle \varphi, \tilde{\mu}_N(t) \rangle - \langle \varphi, \tilde{\mu}_N(0) \rangle - \int_0^t \langle \Delta \varphi, \tilde{\mu}_N(s) \rangle ds - \int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds \quad (5.4)$$

is a martingale w.r.t. the filtration generated by $\tilde{\mu}_N$ and μ_N for $\varphi \in C_0^\infty(\mathbb{R}^d)$. Observe that the covariance of martingale $M_t^N(\varphi)$ is of $O(\frac{1}{N})$ by the independence of Brownian motions, we thus have

$$M_t^N(\varphi) \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

in probability. Up to a subsequence, the martingale converges to zero almost surely.

Since $\tilde{\mu}_N$ converges to g (equivalently to $\tilde{\mu}$) in $C([0, T], \mathcal{M}(\mathbb{R}^d))$, letting $N \rightarrow \infty$ on the both sides of the equality (5.4) leads to

$$\begin{aligned} \langle \varphi, g_t \rangle &= \lim_{N \rightarrow \infty} \langle \varphi, \tilde{\mu}_N(t) \rangle \\ &= \langle \varphi, g_0 \rangle + \int_0^t \langle \Delta \varphi, g_s \rangle ds + \lim_{N \rightarrow \infty} \int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds. \end{aligned} \quad (5.5)$$

It suffices to identify the limits of the interacting term. The main difficulty is the lack of continuity of the singular interacting term with respect to the weak topology in $\mathcal{M}(\mathbb{R}^d)$. Fortunately, with the estimates Proposition 3.5 and Corollary 5.10, later we shall show

$$\lim_{N \rightarrow \infty} \int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds = \int_0^t \langle \nabla \varphi \cdot K * v_s, g_s \rangle ds. \quad (5.6)$$

To obtain (5.6), we approximate $K = K_1 + K_2$ by $K_\varepsilon = K_{1,\varepsilon} + K_{2,\varepsilon}$ as in the proof of Proposition 3.5, where $K_{1,\varepsilon}$ and $K_{2,\varepsilon}$, $\varepsilon \in (0, 1)$, are smooth and compactly supported functions satisfying

$$\|K_{1,\varepsilon} - K_1\|_{L_{q_1}^{p_1}} \rightarrow 0; \quad \|K_{2,\varepsilon} - K_2\|_{L_{q_2}^{p_2}} \rightarrow 0,$$

for $p_1, p_2 < \infty$. When $p_1 = \infty$ or $p_2 = \infty$, we first truncate K by letting $K = K1_{|\cdot| \leq R} + K1_{|\cdot| > R}$, then proceed the regularization on the local term $K1_{|\cdot| \leq R}$. The term $K1_{|\cdot| > R}$ is controlled by finite moments of particles and causes no difficulty in singularity, we ignore this term in the following. Therefore,

one may divide the singular interacting term into a continuous functional on $C([0, T], \mathcal{M}(\mathbb{R}^d))$ and a correction. More precisely,

$$\int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds = \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * \mu_N(s), \tilde{\mu}_N(s) \rangle ds + R_{\varepsilon, \varphi}^N,$$

where the correction $R_{\varepsilon, \varphi}^N$ is taken as

$$R_{\varepsilon, \varphi}^N = \int_0^t \langle \nabla \varphi \cdot [K_1 - K_{1, \varepsilon}] * \mu_N(s), \tilde{\mu}_N(s) \rangle ds + \int_0^t \langle \nabla \varphi \cdot [K_2 - K_{2, \varepsilon}] * \mu_N(s), \tilde{\mu}_N(s) \rangle ds.$$

Similarly, the notation $R_{\varepsilon, \varphi}$ stands for

$$\begin{aligned} R_{\varepsilon, \varphi} &:= \int_0^t \langle \nabla \varphi \cdot K * v_s, g_s \rangle ds - \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * v_s, g_s \rangle ds \\ &= \int_0^t \langle \nabla \varphi \cdot [K_1 - K_{1, \varepsilon}] * v_s, g_s \rangle ds + \int_0^t \langle \nabla \varphi \cdot [K_2 - K_{2, \varepsilon}] * v_s, g_s \rangle ds. \end{aligned}$$

We now claim that for each $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\mathbb{E} \left(\sup_{t \in [0, T]} |R_{\varepsilon, \varphi}(t)| \right) \xrightarrow{\varepsilon \rightarrow 0} 0; \quad \sup_N \mathbb{E} \left(\sup_{t \in [0, T]} |R_{\varepsilon, \varphi}^N(t)| \right) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (5.7)$$

This uniform convergence of the corrections is the key ingredient to deduce (5.6). Indeed, the approximations for the kernel K implies that

$$\begin{aligned} & \left| \int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds - \int_0^t \langle \nabla \varphi \cdot K * v_s, g_s \rangle ds \right| \\ & \leq \left| \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * \mu_N(s), \tilde{\mu}_N(s) \rangle ds - \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * v_s, g_s \rangle ds \right| + |R_{\varepsilon, \varphi}^N(t)| + |R_{\varepsilon, \varphi}(t)|. \end{aligned}$$

By the convergence of $(\mu_N, \tilde{\mu}_N)$ to (v, g) in $C([0, T]; \mathcal{M}(\mathbb{R}^d))^{\otimes 2}$, the first absolute value at the second line vanishes almost surely as N goes to infinity. Thus for $\varepsilon > 0$

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E} \left| \int_0^t \langle \nabla \varphi \cdot K * \mu_N(s), \tilde{\mu}_N(s) \rangle ds - \int_0^t \langle \nabla \varphi \cdot K * v_s, g_s \rangle ds \right| \\ & \leq \lim_{N \rightarrow \infty} \left| \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * \mu_N(s), \tilde{\mu}_N(s) \rangle ds - \int_0^t \langle \nabla \varphi \cdot K_\varepsilon * v_s, g_s \rangle ds \right| \\ & \quad + \sup_N \mathbb{E} |R_{\varepsilon, \varphi}^N(t)| + \mathbb{E} |R_{\varepsilon, \varphi}(t)| \\ & \leq \sup_N \mathbb{E} |R_{\varepsilon, \varphi}^N(t)| + \mathbb{E} |R_{\varepsilon, \varphi}(t)|. \end{aligned}$$

Choosing ε sufficient small and applying (5.7), we arrive at (5.6).

Now that it remains to prove the claim (5.7).

Recall the definition of $R_{\varepsilon, \varphi}$, we have

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} |R_{\varepsilon, \varphi}(t)| \right) & \leq \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left(\int_0^T \int_{\mathbb{R}^d} |[K_1 - K_{1, \varepsilon}] * v_t(x) g_t(x)| dx dt \right) \\ & \quad + \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left(\int_0^T \int_{\mathbb{R}^d} |[K_2 - K_{2, \varepsilon}] * v_t(x) g_t(x)| dx dt \right) \\ & := J_1^\varepsilon + J_2^\varepsilon. \end{aligned}$$

For J_1^ε , applying Young's inequality for the convolution of two functions and Hölder's inequality gives

$$J_1^\varepsilon \leq C \mathbb{E} \left(\int_0^T \|K_1 - K_{1, \varepsilon}\|_{L^{p_1}} \|v_t\|_{L^1} \|g_t\|_{L^p} dt \right)$$

$$\leq C \|K_1 - K_{1,\varepsilon}\|_{L_{q_1}^{p_1}} \mathbb{E} \|g\|_{L_q^p}, \quad \left(\frac{1}{p_1} + \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q} = 1 \right).$$

The condition (\mathbb{K}_r) together with the relationship between (p, q) and (p_1, q_1) exactly leads to the condition (4.10), so that we can apply Corollary 5.10 to find $\mathbb{E} \|g\|_{L_q^p} < \infty$. We thus have

$$J_1^\varepsilon \leq C \|K_1 - K_{1,\varepsilon}\|_{L_{q_1}^{p_1}} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Since it does not involve the divergence of K_1 , the computation for K_1 applies to the less singular part K_2 as well, with p_1, q_1 replaced by (p_2, q_2) . We obtain the convergence of J_2^ε and arrive at

$$\mathbb{E} \left(\sup_{t \in [0, T]} |R_{\varepsilon, \varphi}(t)| \right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The second uniform convergence in (5.7) is similar. Since it concerns on N -particles, the regularity result we shall apply is Proposition 3.5 instead of Corollary 5.10. Again, the technical result Corollary 3.4 will be used to handle non-exchangeability. We start with a simple bound for $|R_{\varepsilon, \varphi}^N|$,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} |R_{\varepsilon, \varphi}^N| \right) &\leq \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left(\int_0^T \frac{1}{N^2} \sum_{i \neq j} |\tilde{w}_i^N| |w_j^N| \left| [K_1 - K_{1,\varepsilon}](X_i - X_j) \right| dt \right) \\ &\quad + \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left(\int_0^T \frac{1}{N^2} \sum_{i \neq j} |\tilde{w}_i^N| |w_j^N| \left| [K_2 - K_{2,\varepsilon}](X_i - X_j) \right| dt \right) \\ &:= J_1^{\varepsilon, N} + J_2^{\varepsilon, N}. \end{aligned}$$

We only give the details for the bound of $J_1^{\varepsilon, N}$ explicitly and the required bound for $J_2^{\varepsilon, N}$ follows similarly. First, applying Hölder's inequality w.r.t. the sum over i, j leads to

$$\begin{aligned} J_1^{\varepsilon, N} &= \|\nabla \varphi\|_{L^\infty} \int_0^T \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left(|\tilde{w}_i^N| |w_j^N| \left| [K_1 - K_{1,\varepsilon}](X_i - X_j) \right| \right) dt \\ &\leq C_\varphi \int_0^T \|w^N\|_{l^r} \|\tilde{w}^N\|_{l^r} \left(\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left| [K_1 - K_{1,\varepsilon}](X_i - X_j) \right|^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} dt. \end{aligned}$$

Since the two sequences $\{w^N\}$ and $\{\tilde{w}^N\}$ are uniformly bounded in l^r , there exists a universal constant $C > 0$ such that

$$J_1^{\varepsilon, N} \leq C \int_0^T \left(\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left| [K_1 - K_{1,\varepsilon}](X_i - X_j) \right|^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} dt.$$

Applying Corollary 3.4 with $|K_1 - K_{1,\varepsilon}|$ playing the role of \tilde{K} , we get

$$J_1^{\varepsilon, N} \leq C \|K_1 - K_{1,\varepsilon}\|_{L_{q_1}^{p_1}} \int_0^T \left(1 + \frac{1}{N} I(F_t^N) \right) dt.$$

By Proposition 3.5, Assumptions (\mathbb{H}) , (\mathbb{K}_r) and the convergence of $K_{1,\varepsilon}$ to K_1 , we conclude that

$$J_1^{\varepsilon, N} \leq C \|K_1 - K_{1,\varepsilon}\|_{L_{q_1}^{p_1}} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The claim (5.7) is thus proved. \square

Proof of Theorem 1.6. The result follows by combining Corollary 5.8 and Proposition 5.11. The regularity estimates in Definition 1.4 are obtained by Corollary 5.10. \square

5.3. Uniqueness. In this section, we prove the uniqueness of the mean-field system (1.5). We divide Theorem 1.7 into the following Theorem 5.12 and Theorem 5.13.

Theorem 5.12. *There exists a unique solution $(v, g) \in C([0, T], \mathcal{M}(\mathbb{R}^d))$ to (1.5) in the sense of Definition 1.4, if the kernel K belongs to $L^{q_2}([0, T], L^{p_2}(\mathbb{R}^d))$ with*

$$\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} \leq 1, \quad \frac{d}{p_2} + \frac{2}{q_2} < 1. \quad (5.8)$$

Proof. The proof consists of two parts: the uniqueness of the solution v to the first equation in (1.5) and the uniqueness of the solution g to the second equation (1.5) in the sense of Definition 1.4. Observe that g solves a linear equation depending on v , then it is natural to study the equation of v first.

Uniqueness of v :

For general $L^{p_2}_{q_2}$ -type kernel, the proof is through the mild formulation of (1.5). We consider the equation for v ,

$$v_t = \Gamma_t * v_0 - \int_0^t \nabla \Gamma_{t-s} * (K * v_s v_s) ds.$$

Let $\kappa > 0$ be a positive number satisfying

$$0 < \kappa < \min \left\{ \frac{1}{p_2}, \frac{2}{d} \left(1 - \frac{1}{r}\right), \frac{1}{2d} \left(\frac{1}{r} + 1 - \left[\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r}\right]\right) \right\} < \frac{1}{d}. \quad (5.9)$$

The constraint (5.8) implies that it happens either $\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} < 1$ or $r < \infty$, which together with $r > 1$ ensures the existence of κ .

Suppose that there exist two solutions v^1 and v^2 starting from the same initial data. Since $\kappa < \frac{2}{d} \left(1 - \frac{1}{r}\right)$, we deduce from Definition 1.4 that v^1 and v^2 belong to $L^{\frac{2}{d\kappa}}([0, T], L^{\frac{1}{1-\kappa}}(\mathbb{R}^d))$. Computing the $L^{\frac{1}{1-\kappa}}$ -norm of $v^1 - v^2$ then leads to

$$\begin{aligned} \|v_t^1 - v_t^2\|_{L^{\frac{1}{1-\kappa}}} &\leq \int_0^t \left\| \nabla \Gamma_{t-s} * (K * v_s^1 v_s^1 - K * v_s^2 v_s^2) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\leq \int_0^t \left\| \nabla \Gamma_{t-s} * (K * v_s^1 [v_s^1 - v_s^2]) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\quad + \int_0^t \left\| \nabla \Gamma_{t-s} * (K * [v_s^1 - v_s^2] v_s^2) \right\|_{L^{\frac{1}{1-\kappa}}} ds, \end{aligned} \quad (5.10)$$

where Γ_t is the the heat kernel of Δ . Using Young's convolution inequality, we have

$$\begin{aligned} &\int_0^t \left\| \nabla \Gamma_{t-s} * (K * v_s^1 [v_s^1 - v_s^2]) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\leq \int_0^t \|\nabla \Gamma_{t-s}\|_{L^{\frac{1}{1-\kappa^2}}} \|K * v_s^1 [v_s^1 - v_s^2]\|_{L^{\frac{1}{1-\kappa(1-\kappa)}}} ds. \end{aligned}$$

Furthermore, by the property of heat kernel $\|\nabla \Gamma_t\|_{L^q} \lesssim t^{\frac{d}{2q} - \frac{d+1}{2}}$ for $q \geq 1$, taking $q = \frac{1}{1-\kappa^2}$ gives

$$\begin{aligned} &\int_0^t \left\| \nabla \Gamma_{t-s} * (K * v_s^1 [v_s^1 - v_s^2]) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\lesssim \int_0^t (t-s)^{\frac{d}{2}(1-\kappa^2) - \frac{d+1}{2}} \|K * v_s^1\|_{L^{\frac{1}{\kappa^2}}} \|v_s^1 - v_s^2\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{1+d\kappa^2}{2}} \|K\|_{L^{p_2}} \|v_s^1\|_{L^{p_3}} \|v_s^1 - v_s^2\|_{L^{\frac{1}{1-\kappa}}} ds, \quad \frac{1}{p_3} = 1 + \kappa^2 - \frac{1}{p_2}. \end{aligned} \quad (5.11)$$

In the Definition 1.4, the maximal spatial integrability we obtained for v_t^1 and v_t^2 is $L^p(\mathbb{R}^d)$ with $p = \frac{d}{d-2+\frac{2}{r}} < \infty$; this excludes the case $d = 2$ and $r = \infty$, where the range of p is $[1, \infty)$. Now we

check that $p_3 \in (1, p)$. On one hand, $p_3 > 1$ follows by

$$1 + \kappa^2 - \frac{1}{p_2} < 1 + \kappa - \frac{1}{p_2} < 1,$$

where we used (5.9). On the other hand, we find the upper bound $p_3 < p$ by noticing

$$1 + \kappa^2 - \frac{1}{p_2} \geq 1 + \kappa^2 - \frac{1}{d}\left(1 - \frac{1}{r}\right) = 1 - \frac{2}{d}\left(1 - \frac{1}{r}\right) + \kappa^2 + \frac{1}{d}\left(1 - \frac{1}{r}\right) > \frac{1}{p},$$

where the first inequality follows by (5.8), while the last inequality is given by $\frac{1}{p} = 1 - \frac{2}{d}\left(1 - \frac{1}{r}\right)$.

By Corollary 5.10, we can take $q_3 > 1$ such that $\frac{1}{q_3} = \frac{d}{2}\left(1 - \frac{1}{p_3}\right)$. Let $m \geq 1$ such that $\frac{1}{q_3} + \frac{1}{q_2} + \frac{1}{m} + \frac{d\kappa}{2} = 1$, we find

$$\begin{aligned} \frac{1}{m} &= 1 - \frac{1}{q_3} - \frac{1}{q_2} - \frac{d\kappa}{2} \\ &= 1 + \frac{d}{2}\left(\frac{1}{p_3} - 1\right) - \frac{1}{2}\left(\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} - 1\right) + \frac{d}{2p_2} + \frac{1}{2r} - \frac{1}{2} - \frac{d\kappa}{2} \\ &= \frac{1}{2} + \frac{d}{2}\left(\frac{1}{p_3} + \frac{1}{p_2} - 1\right) + \frac{1}{2r} - \frac{1}{2}\left(\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} - 1\right) - \frac{d\kappa}{2} \\ &= \frac{1}{2} + \frac{d\kappa^2}{2} + \frac{1}{2r} - \frac{1}{2}\left(\frac{d}{p_2} + \frac{2}{q_2} + \frac{1}{r} - 1\right) - \frac{d\kappa}{2}, \end{aligned}$$

where we used the condition on (κ, p_3, p_2) in (5.11) to find the last equality. Recall the condition on κ , we have

$$\frac{1}{m} > \frac{1 + d\kappa^2}{2}. \quad (5.12)$$

Applying Hölder's inequality to (5.11), we find

$$\begin{aligned} &\int_0^t \left\| \nabla \Gamma_{t-s} * \left(K * v_s^1 [v_s^1 - v_s^2] \right) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\lesssim \|K\|_{L^{\frac{p_2}{q_2}}} \|v^1\|_{L^{\frac{p_3}{q_3}}} \|v^1 - v^2\|_{L^{\frac{1}{\frac{2}{d\kappa}}}} \left(\int_0^t (t-s)^{-\frac{1+d\kappa^2}{2}m} ds \right)^{\frac{1}{m}}. \end{aligned}$$

By (5.12) and the regularity estimate in the Definition 1.4, we conclude that

$$\int_0^t \left\| \nabla \Gamma_{t-s} * \left(K * v_s^1 [v_s^1 - v_s^2] \right) \right\|_{L^{\frac{1}{1-\kappa}}} ds \lesssim \|v^1 - v^2\|_{L^{\frac{1}{\frac{2}{d\kappa}}}}. \quad (5.13)$$

Similarly, we have

$$\begin{aligned} &\int_0^t \left\| \nabla \Gamma_{t-s} * \left(K * [v_s^1 - v_s^2] v_s^2 \right) \right\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{1+d\kappa^2}{2}} \left\| K * [v_s^1 - v_s^2] v_s^2 \right\|_{L^{\frac{1}{1-\kappa(1-\kappa)}}} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{1+d\kappa^2}{2}} \|K\|_{L^{p_2}} \|v_s^2\|_{L^{p_3}} \|v_s^1 - v_s^2\|_{L^{\frac{1}{1-\kappa}}} ds \\ &\lesssim \|v^1 - v^2\|_{L^{\frac{1}{\frac{2}{d\kappa}}}}. \end{aligned} \quad (5.14)$$

Combining (5.10)-(5.14), we arrive at

$$\|v_t^1 - v_t^2\|_{L^{\frac{2}{1-\kappa}}} \lesssim \int_0^t \|v_s^1 - v_s^2\|_{L^{\frac{2}{1-\kappa}}} ds. \quad (5.15)$$

Therefore we obtain the $v^1 = v^2$ in $L^{\frac{1}{1-\kappa}}(\mathbb{R}^d)$ for all $t \in (0, T]$ by applying Gronwall's inequality. We then conclude the uniqueness.

Uniqueness of g :

Now we consider the mild formulation of g ,

$$g_t = \Gamma_t * g_0 - \int_0^t \nabla \Gamma_{t-s} * (K * v_s g_s) ds.$$

Observe that this is a linearized version of the equation for g . Similarly, suppose that there exist two solutions g^1 and g^2 , then studying the $L^{\frac{1}{1-\kappa}}$ -norm of $g^1 - g^2$ leads to

$$\|g_t^1 - g_t^2\|_{L^{\frac{1}{1-\kappa}}} \leq \int_0^t \left\| \nabla \Gamma_{t-s} * (K * v_s [g_s^1 - g_s^2]) \right\|_{L^{\frac{1}{1-\kappa}}} ds.$$

Similar to (5.13), we find

$$\|g_t^1 - g_t^2\|_{L^{\frac{1}{1-\kappa}}} \lesssim \left(\int_0^t \|g_s^1 - g_s^2\|_{L^{\frac{1}{1-\kappa}}}^{\frac{2}{1-\kappa}} ds \right)^{\frac{1-\kappa}{2}},$$

the proof is thus completed by applying Gronwall's inequality. \square

When K is the Biot-Savart law on dimension two, Theorem 1.7 is indeed the uniqueness of solutions to the passive scalar advected by the 2D Navier-Stokes equation.

Theorem 5.13. *There exists a unique solution $(v, g) \in C([0, T], \mathcal{M}(\mathbb{R}^2))$ to (1.5) in the sense of Definition 1.4 if the kernel K is the 2D Biot-Savart law (1.3) and $r \in [3, \infty]$.*

Proof. When K is the Biot-Savart law, v solves the vorticity formulation of 2D Navier-Stokes equation. For this case, the uniqueness of solutions with the regularity properties [FHM14, (2.6)] (i.e. Corollary 5.10 with $r = \infty$) is already obtained in [FHM14], using the well-posedness result in the space $C([0, T], L^1(\mathbb{R}^2)) \cap C((0, T), L^\infty(\mathbb{R}^d))$ from [BA94] and the remark [Bre94]. The strategy in [FHM14] is to improve the regularity of solutions by the DiPerna-Lions' renormalized solution and the maximal regularity of the heat equation so that the solution v meets the conditions in [BA94] and [Bre94].

The regularity result Corollary 5.10 is in fact a generalization of [FHM14, (2.6)]. In particular, Corollary 5.10 implies

$$v \in L^\infty([0, T], L^1(\mathbb{R}^2)) \cap L^{\frac{p}{p-1}}([0, T], L^p(\mathbb{R}^2)), \quad p \leq r \text{ and } 1 < p < \infty;$$

and

$$\nabla v \in L^{\frac{2q}{3q-2}}([0, T], L^q(\mathbb{R}^2)), \quad q \leq \frac{2r}{r+2}, \quad q < 2, \text{ and } 1 \leq q < 2.$$

Although here we have the extra restrictions $p \leq r$ and $q \leq 2r/(r+2)$, by letting $r \geq 3$, one can track the proof [FHM14, Theorem 2.5] to get the uniqueness of the solutions to the vorticity form of the 2D Navier-Stokes equation. Therefore, v is the unique solution in the sense of Definition 1.4. Furthermore, the remark [Bre94] by Brezis shows that for L^1 -valued initial data,

$$\lim_{t \rightarrow 0} t \|v_t\|_{L^\infty} = 0. \tag{5.16}$$

We then use $|K(y)| \lesssim |y|^{-1}$ to obtain

$$\begin{aligned} |K * v_t(x)| &\leq \left| \int_{|y| \leq \sqrt{\frac{t}{c(t)}}} K(y) v_t(x-y) dy \right| + \left| \int_{|y| > \sqrt{\frac{t}{c(t)}}} K(y) v_t(x-y) dy \right| \\ &\leq \frac{1}{2\pi} \|v_t\|_{L^\infty} \int_{|y| \leq \sqrt{\frac{t}{c(t)}}} \frac{1}{|y|} dy + \frac{1}{2\pi} \sqrt{\frac{c(t)}{t}} \|v_t\|_{L^1} \\ &\lesssim \sqrt{\frac{t}{c(t)}} \|v_t\|_{L^\infty} + \sqrt{\frac{c(t)}{t}}, \end{aligned}$$

for $c(t) > 0$ and all $x \in \mathbb{R}^2$. Letting $c(t) = t\|v_t\|_{L^\infty}$ and applying (5.16), we arrive at

$$\lim_{t \rightarrow 0} t^{\frac{1}{2}} \|K * v_t\|_{L^\infty} \lesssim \lim_{t \rightarrow 0} \sqrt{t} \|v_t\|_{L^\infty} = 0. \quad (5.17)$$

Suppose there exist two solutions g^1 and g^2 to the second equation in (1.5). By the mild formulations of solutions, we have

$$\begin{aligned} \|g_t^1 - g_t^2\|_{L^1} &\leq \int_0^t \left\| \nabla \Gamma_{t-s} * \left(K * v_s [g_s^1 - g_s^2] \right) \right\|_{L^1} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \|K * v_s\|_{L^\infty} \|g_s^1 - g_s^2\|_{L^1} ds. \end{aligned}$$

Using the time regularity result (5.17), we find

$$\|g_t^1 - g_t^2\|_{L^1} \lesssim c_0(t) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \|g_s^1 - g_s^2\|_{L^1} ds,$$

where $c_0(t) = \sup_{s \in [0, t]} s^{\frac{1}{2}} \|K * v_s\|_{L^\infty} \rightarrow 0$ as $t \rightarrow 0$. We then deduce $g_t^1 = g_t^2$ up to a short time $t_0 > 0$ by Gronwall's inequality of Volterra type, see for instance [Zha10, Example 2.4]. Applying this argument for finite times, we conclude the uniqueness for all $t \in [0, T]$. \square

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