NON-UNIQUE ERGODICITY FOR DETERMINISTIC AND STOCHASTIC 3D NAVIER–STOKES AND EULER EQUATIONS

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Abstract. We establish existence of infinitely many stationary solutions as well as ergodic stationary solutions to the three dimensional Navier–Stokes and Euler equations in the deterministic as well as stochastic setting, driven by an additive noise. The solutions belong to the regularity class $C(\mathbb{R}; H^{\vartheta}) \cap C^{\vartheta}(\mathbb{R}; L^2)$ for some $\vartheta > 0$ and satisfy the equations in an analytically weak sense. Moreover, we are able to make conclusions regarding the vanishing viscosity limit and the anomalous dissipation. The result is based on a new stochastic version of the convex integration method which provides uniform moment bounds locally in the aforementioned function spaces.

Contents

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1. Introduction

1.1. **Motivation.** Hydrodynamic turbulence is omnipresent in engineering applications and nature. And yet developing a rigorous mathematical understanding remains one of the big challenges in contemporary fluid dynamics research. Up to date, the results providing reliable predictions are very limited. On the physical side, the understanding has been driven by well-accepted theoretical hypotheses such as those of the celebrated Kolmogorov's theory [Kol41a, Kol41b, Kol41c], see also [Fri95]. These hypotheses have been confirmed to a large extent by experiments. However, their rigorous verification from the basic physical principles and in particular from the incompressible Navier–Stokes equations remains an outstanding open problem. In what follows, we briefly recall some of the key aspects of the physical theories of turbulence, namely those that mainly motivated our investigation, while leaving out many others. Due to the complexity of the subject, we encourage the reader to discuss e.g. [Fri95, MY13] for thorough expositions.

The Navier–Stokes equations describe the time evolution of the velocity $u : [0,\infty) \times D \to \mathbb{R}^3$ of a viscous fluid confined in a domain $D \subset \mathbb{R}^3$. They read as

$$\begin{aligned} \partial_t u + \mathrm{div}(u \otimes u) + \nabla P &= \nu \Delta u + f, \\ \mathrm{div} u &= 0, \end{aligned} \tag{1.1}$$

where $P : [0,\infty) \times D \to \mathbb{R}$ denotes the associated pressure, $\nu > 0$ the kinematic viscosity of the fluid and $f : [0,\infty) \times D \to \mathbb{R}^3$ is a given external force. The equations are further supplemented by initial and boundary conditions. Particularly relevant for the study of turbulence is the regime of high Reynolds number which corresponds to the vanishing viscosity limit $\nu \to 0$. On the formal level, the Navier–Stokes equations then converge to the Euler equations

$$\begin{aligned} \partial_t u + \mathrm{div}(u \otimes u) + \nabla P &= f, \\ \mathrm{div} u &= 0, \end{aligned} \tag{1.2}$$

which represent an idealized model for the highly turbulent limit regime. Assumptions allowing for a rigorous passage to the limit $\nu \to 0$ are predicted by the physical theories of turbulence. However, it was shown in [CG12] and also [CV18] that already weaker assumptions allow for the vanishing viscosity limit, namely, proving convergence of the Navier–Stokes to Euler equations.

From experiments it became clear that exact realizations of turbulent trajectories are not suited for predictions due to high sensitivity to input data such as initial and boundary conditions. On the other hand, and rather surprisingly, statistical properties are universal and well-reproducible. Thus, a certain probabilistic description seems indispensable. Furthermore, one of the basic assumptions in turbulence theory is the so-called ergodic hypothesis taken for granted by physicists and engineers. It assures that time averages along trajectories coincide with ensemble averages taken with respect to some probability measure. This measure is then invariant, i.e., preserved by the flow. Accordingly, statistically stationary solutions (i.e. solutions whose probability law does not change with time) play a distinguished role in the modeling of turbulence and it is of essential interest to characterize these solutions as well as their attraction properties.

As the authors of the survey [DLS21] explain, another argument in favor of a probabilistic description of turbulence is the observed lack of uniqueness, which was recently established for the Euler as well as Navier–Stokes equation in the deterministic and stochastic setting, see e.g. [BCV18, BMS20, BV19b, CL19, CL20, DLS09, DLS10, DLS13, Luo19, HZZ19, HZZ22a, HZZ21, HZZ23]. More precisely, they say: *"It is worth emphasizing that the non-uniqueness in such examples is not a mathematical pathology, but seems to be a generic phenomenon, strongly suggesting that a probability measure on ensembles with restored symmetries may exist even without having to resort to stochastic modifications of the basic continuum equations."*

However, it may be possible to profit from the presence of stochastic perturbations of the equations and some properties of the Navier–Stokes system have indeed been shown to improve under the presence of a stochastic noise. Namely, the force f is considered to be a Gaussian noise white in time and colored in space, which models a large scale stirring driving turbulent fluids. In the deterministic setting, a selection of solutions depending continuously on the initial condition has not been obtained. But the probabilistic counterpart, i.e. the Feller property and even the strong Feller property which corresponds to a smoothing with respect to the initial condition, was established for a sufficiently non-degenerate noise in [DPD03] and [FR08]. This led in particular to uniqueness of the invariant measure associated to the constructed selection of a Markov semigroup in [DPD03, FR08].

Another fundamental principle of Kolmogorov's K41 theory, also called the zeroth law of turbulence, is the anomalous dissipation. Denoting u_ν a statistically stationary solution to the Navier–Stokes equations (1.1) with viscosity ν, it postulates that[1]

$$\lim_{\nu\to 0}\epsilon_\nu := \lim_{\nu\to 0}\nu\mathbf{E}\|\nabla u_\nu\|_{L^2}^2 = \epsilon > 0. \tag{1.3}$$

The quantity on the left hand side is the mean energy dissipation of u_ν and the expectation is taken with respect to the underlying randomness. Were u_ν converging to a sufficiently regular solution to the Euler equations (1.2) with $f = 0$, the anomalous dissipation would not take place as the Euler equations are energy conserving in that case. This is related to Onsager's conjecture [Ons49] which attracted a lot of attention lately with a number of groundbreaking results, see [DLS09, DLS10, DLS13, DS17, BDLIS15, Ise18, BDLSV19].

Finally, let us mention the Kolmogorov two-third's law, which predicts the behavior of the second order structure function as

$$S_2(h) := \mathbf{E}\|u_\nu(\cdot + h) - u_\nu(\cdot)\|_{L^2}^2 \simeq (\epsilon_\nu|h|)^{2/3},$$

where ϵ_ν is the dissipation rate from (1.3) and $|h|$ belongs to the inertial range $[\eta_\nu, L]$ with η_ν the dissipation and L the integral scale of turbulence. Moreover, if the statistics are translation invariant, the energy spectrum obeys

$$E(k) := |k|^2\mathbf{E}|\hat{u}_\nu(k)|^2 \simeq \epsilon_\nu^{2/3}k^{-5/3},$$

where $\hat{u}_\nu(k)$, $k \in \mathbb{Z}^3$, denotes the Fourier transform of $\hat{u}_\nu$. This is the so-called Kolmogorov-Obhukhov 5/3-power spectrum, which is related to $H^{1/3}$-regularity of solutions.

To summarize the above discussion, it is desired to investigate the validity of the following claims.

(i) Existence and (non)uniqueness of ergodic stationary solutions u_ν to the Navier–Stokes equations (1.1).
(ii) Relative compactness of the family of stationary solutions u_ν, $\nu > 0$, and the convergence towards a statistically stationary solution to the Euler equations (1.2).
(iii) Anomalous dissipation along the vanishing viscosity limit in the sense of (1.3).
(iv) Existence and (non)uniqueness of ergodic stationary solutions to the Euler equations (1.2).

[1] Here and in various expressions in the sequel, the expected value does not depend on time due to stationarity.

Up to now, these questions could only be answered in several simplified settings, such as certain shell models of turbulence [FGHV16] or passive scalar models of turbulence [BBPS19]. However, for the actual models of interest, i.e. the three dimensional stochastic Navier–Stokes and Euler equations, the available results are very limited. Mere existence of stationary solutions to the stochastic Navier–Stokes equations is classical and was proved in [FG95]. The only available result in the direction of uniqueness of the invariant measure in this context is the unique ergodicity from [DPD03] and [FR08]. The uniqueness here only relates to the Markov process constructed therein, but as shown in [HZZ23], there are other Markov processes associated to the same equation with possibly different invariant measures. Due to the lack of dissipation, even the existence of statistically stationary solutions to the three dimensional stochastic Euler equations is fully open and nothing is known about the vanishing viscosity limit in the framework of stationary solutions in three dimensions.

We also mention that nonunique ergodic measures in the Lorenz system were constructed when adding a noise in the last component in [CH21]. It is also asked in [CH21, Remark 1.3] whether a bifurcation of invariant measures appears at high Reynolds number for the Navier–Stokes system.

Recently the anomalous dissipation and the property (1.3) were studied for the deterministic forced 3D Navier–Stokes equations and the advection–diffusion equation in [BD22, CCS22].

1.2. **Main results.** Our aim is to provide some answers to the above problems (i), (ii), (iii), (iv) in the physically relevant context of the stochastic Navier–Stokes and Euler equations on $\mathbb{T}^3$ driven by an additive stochastic noise. The Navier–Stokes equations read as

$$\begin{aligned} \mathrm{d}u + \mathrm{div}(u\otimes u)\,\mathrm{d}t + \nabla P\,\mathrm{d}t &= \nu\Delta u\,\mathrm{d}t + \mathrm{d}B, \\ \mathrm{div} u &= 0, \end{aligned} \tag{1.4}$$

whereas the Euler equations are

$$\begin{aligned} \mathrm{d}u + \mathrm{div}(u\otimes u)\,\mathrm{d}t + \nabla P\,\mathrm{d}t &= \mathrm{d}B, \\ \mathrm{div} u &= 0. \end{aligned} \tag{1.5}$$

In the above, P is the associated pressure, $\nu > 0$ is the viscosity, B is a GG^*-Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and G is a Hilbert–Schmidt operator from U to L^2_σ for some Hilbert space U and L^2_σ the subspace of L^2 containing mean and divergence free functions.

As the main tool within this study, we apply a stochastic version of the convex integration method. Convex integration is an iterative procedure which has already permitted to establish a number of breakthrough results concerning the Navier–Stokes and Euler equations in the deterministic setting, see e.g. [BDLSV19, BCV18, BDLIS15, BMS20, BV19b, CL19, CL20, DLS09, DLS10, DLS13, DS17, Ise18, Luo19]. Unlike our previous works using convex integration for the Navier–Stokes and Euler equations in the stochastic setting [HZZ19, HZZ22a, HZZ23, HZZ21], we are now inspired by [CDZ22] and overcome the limitation originating from stopping times, previously used to control the noise terms in the iteration. More precisely, we no longer work with stopping times and instead we include expectations in the iterative estimates.

While this is a very natural idea to avoid the stopping times, making it possible is not obvious at all. Namely, it requires a very careful analysis of each bound, since due to the quadratic non-linearity, the estimates become superlinear. Accordingly, it is necessary to estimate all moments simultaneously and when aiming for a finite rth moment, the blow up of all mth moments for $m > r$ must be controlled. The main reason why this strategy is possible is that any fixed mth moment of the approximate velocity and the error at step q only depends on m and the parameters up to the step q of the iteration. Additionally, it is possible to choose the parameters at the level $q+1$ to guarantee smallness of the velocity perturbations and the error at step $q+1$.

Our key result is then the construction of solutions satisfying global-in-time H^ϑ-bounds for some $\vartheta > 0$ uniformly along the vanishing viscosity limit. In comparison to [CDZ22], this is achieved by carefully calculating how the mth moment at step q depends on m and on the stochastic terms. Then it is possible to choose suitable parameters to give an exact convergence rate of the L^2-norm as well as an exact divergence rate of the C^1-norm, leading to the desired uniform H^ϑ-bound. The interesting point is that this bound does not depend on the dissipation given by the Laplacian and consequently all the bounds hold uniformly for $\nu \geqslant 0$.

As we are interested in the long time behavior of solutions and particularly in the construction of statistically stationary solutions, we work with entire solutions solving the equations for all times $t \in \mathbb{R}$. Accordingly, the norms in the convex integration scheme must be chosen appropriately in order to provide the desired global-in-time estimates. This is achieved through bounds of the form

$$\sup_{\nu\geqslant 0}\sup_{t\in\mathbb{R}} \mathbf{E}\left[\sup_{t\leqslant s\leqslant t+1} \|u^\nu(s)\|_{H^\vartheta}^{2r}\right] < \infty,$$

with $r > 1$ and $\vartheta > 0$. Such bounds provide uniform moment estimates locally in $C(\mathbb{R}; H^\vartheta)$ and guarantee the convergence of the corresponding ergodic averages, even in the case of Euler equations. This leads to the existence of stationary solutions.

Within this study, we focus on analytically weak solutions which satisfy the equations in the following sense.

Definition 1.1. *We say that* $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbf{P}), u, B)$ *is an analytically weak solution to the Navier–Stokes system* (1.4) *provided*

1. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbf{P})$ *is a stochastic basis with a complete right-continuous filtration;*
2. B *is an* $\mathbb{R}^3$*-valued, spatial mean and divergence free, two-sided trace-class Brownian motion with respect to the filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}}$*;*
3. *the velocity* $u \in L^2_{\rm loc}(\mathbb{R}; L^2_\sigma) \cap C(\mathbb{R}; H^{-\delta})$ $\mathbf{P}$*-a.s. for some* $\delta > 0$ *and is and* $(\mathcal{F}_t)_{t\in\mathbb{R}}$*-adapted;*
4. *for every* $-\infty < s \leqslant t < \infty$ *it holds* $\mathbf{P}$*-a.s.*

$$\langle u(t), \psi\rangle + \int_s^t \langle {\rm div}(u\otimes u), \psi\rangle {\rm d}r = \langle u(s), \psi\rangle + \nu \int_s^t \langle \Delta u, \psi\rangle\, {\rm d}r + \langle B(t) - B(s), \psi\rangle$$

for all $\psi \in C^\infty(\mathbb{T}^3)$, ${\rm div}\psi = 0$.

We note that solutions are not required to belong to $L^2_{\rm loc}(\mathbb{R}, H^1)$ and they do not satisfy the corresponding energy inequality, obtained formally by testing the equation by the solution itself. In other words, our solutions are generally not the so-called Leray solutions, a feature common to all convex integration results treating the Navier–Stokes equations. Analytically weak solutions to the Euler equations (1.5) are defined exactly the same way, in particular using the same function spaces, the only difference is that $\nu = 0$.

As pathwise non-uniqueness, non-uniqueness in law and even non-uniqueness of Markov selections have been established in our previous works [HZZ19, HZZ22a, HZZ23], we understand stationarity in the sense of shift invariance of laws of solutions on the space of trajectories, see also [BFHM19, BFH20e, FFH21, HZZ22]. More precisely, we define the joint trajectory space for the solution and the driving Brownian motion as

$$\mathcal{T} = C(\mathbb{R}; L^2_\sigma) \times C(\mathbb{R}; L^2_\sigma)$$

and let S_t, $t \in \mathbb{R}$, be shifts on trajectories given by

$$S_t(u, B)(\cdot) = (u(\cdot + t), B(\cdot + t) - B(t)), \quad t \in \mathbb{R}, \quad (u, B) \in \mathcal{T}.$$

We note that the shift in the second component acts differently in order to guarantee that for a Brownian motion B the shift S_tB is again a Brownian motion.

Stationary solutions to the stochastic Navier–Stokes equations (1.4) are defined as follows.

Definition 1.2. *We say that* $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}},\mathbf{P}),u,B)$ *is a stationary solution to the stochastic Navier–Stokes equations* (1.4) *provided it satisfies* (1.4) *in the sense of Definition 1.1 and its law is shift invariant, that is,*

$$\mathcal{L}[S_t(u,B)]=\mathcal{L}[u,B]\qquad \text{for all}\quad t\in\mathbb{R}.$$

Note that this setting is different from the usual setting of invariance with respect to a Markov semigroup. The latter notion can be applied to problems with uniqueness, i.e. where the Markov property holds. The construction of invariant measures then additionally requires the Feller property which corresponds to continuous dependence on initial condition. Since non-uniqueness holds true for the above Navier–Stokes and Euler equations, we employ the more general notion of invariance with respect to shifts on trajectories. Another advantage is that continuity of the shift operators comes for free and therefore there is no need for any Feller property.

Every stationary solution (u,B) defines a dynamical system $(\mathcal{T},\mathcal{B}(\mathcal{T}),(S_t,t\in\mathbb{R}),\mathcal{L}[u,B])$ in the sense of e.g. [DPZ96, Chapter 1], where $\mathcal{B}(\mathcal{T})$ denotes the σ-algebra of Borel sets on $\mathcal{T}$. Accordingly, we may formulate ergodicity of stationary solutions as ergodicity of the associated dynamical system. Also with this notion of invariance, the existence of an ergodic stationary solution as defined below implies the validity of the so-called ergodic hypothesis, i.e. the fact that ergodic averages along trajectories of the ergodic solution converge to the ensemble average given by its law. This leads us to the following definition.

Definition 1.3. *A stationary solution* $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}},\mathbf{P}),u,B)$ *is ergodic provided*

$$\mathcal{L}[u,B](A)=1\quad \text{or}\quad \mathcal{L}[u,B](A)=0\quad \text{for all } A\subset\mathcal{T} \text{ Borel and shift invariant.}$$

The same definitions are also valid in the setting of the stochastic Euler equations (1.5). Particularly, the trajectory space and regularity of the solutions is the same as for the Navier–Stokes equations.

With these definitions at hand, we are able to state our first main result, tackling the problems (i) and (iv) above. Rather surprisingly, the result is independent of the value of the viscosity ν. In particular, it holds uniformly along the vanishing viscosity limit $\nu\to 0$ and we obtain the result for both the stochastic Navier–Stokes (1.4) as well as Euler equations (1.5). Only in the case of Euler equations we require more regularity of the Brownian motion, namely, we postulate $\mathrm{Tr}((-\Delta)^\sigma GG^*)<\infty$ for some $\sigma>0$. The result then reads as follows and is proved in Theorem 4.1, Theorem 4.2 and Theorem 5.1.

Theorem 1.4. *There exist*

(1) *infinitely many stationary solutions;*
(2) *infinitely many ergodic stationary solutions;*

to the stochastic Navier–Stokes (1.4) *and Euler* (1.5) *equations. Moreover, the solutions belong to* $C(\mathbb{R},H^\vartheta)\cap C^\vartheta(\mathbb{R},L^2)$ *a.s. for some* $\vartheta>0$.

Regarding the vanishing viscosity limit (ii) as formulated above, we are able to make the following conclusion which is proved in Theorem 5.1.

Theorem 1.5. *Assume* $\mathrm{Tr}((-\Delta)^\sigma GG^*)<\infty$ *for some* $\sigma>0$. *There exists* $K_0>0$ *so that for every* $K\geqslant K_0$ *the following holds: For an arbitrary sequence of vanishing viscosities* $\nu_n\to 0$, $n\in\mathbb{N}$, *there exist a sequence of stationary solutions u_n, $n \in \mathbb{N}$, to the following stochastic Navier–Stokes equations*

$$\mathrm{d}u_n + \mathrm{div}(u_n \otimes u_n)\,\mathrm{d}t + \nabla P_n\,\mathrm{d}t = \nu_n \Delta u_n\,\mathrm{d}t + \mathrm{d}B,$$

so that the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to the stochastic Euler equations (1.5) satisfying

$$\mathbf{E}\|u\|_{L^2}^2 = K.$$

Furthermore, we are able to prove a result related to anomalous dissipation in the spirit of (iii) along the vanishing viscosity limit in a stochastic Navier–Stokes–Reynolds system with vanishing Reynolds stresses. The proof of this result is given in Theorem 5.4.

Theorem 1.6. *Assume $\mathrm{Tr}((-\Delta)^{5/2+\sigma}GG^*) < \infty$ for some $\sigma > 0$. There exists $K_0 > 0$ so that for every $K \geqslant K_0$ the following holds: For any $\epsilon > 0$ there exists a sequence of viscosities $\nu_n \to 0$ and stationary processes $(u_n, \mathring{R}_n) \in C(\mathbb{R}; H^1) \times C(\mathbb{R}; L^1)$ satisfying the following stochastic Navier–Stokes–Reynolds equations*

$$\mathrm{d}u_n + \mathrm{div}(u_n \otimes u_n)\,\mathrm{d}t + \nabla P_n\,\mathrm{d}t = \nu_n \Delta u_n\,\mathrm{d}t + \mathrm{div}\mathring{R}_n\,\mathrm{d}t + \mathrm{d}B,$$

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{0\leqslant s\leqslant 1} \|\mathring{R}_n(s)\|_{L^1}\right] = 0,$$

and

$$\liminf_{n\to\infty} \nu_n \mathbf{E}\|\nabla u_n\|_{L^2}^2 \geqslant \epsilon + \frac{1}{2}\mathrm{Tr}(GG^*).$$

Furthermore, the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to the stochastic Euler equations (1.5) with

$$\mathbf{E}\|u\|_{L^2}^2 = K.$$

In particular, the solutions u_n, $n \in \mathbb{N}$, can be chosen as ergodic stationary solutions.

Remark 1.7. The above result related to anomalous dissipation is produced by the nonlinearity by means of the convex integration at the price of including an additional error term $\mathrm{div}\mathring{R}_n$, which vanishes in the limit. More precisely, for the stationary solution to the linear counterpart without the error term (which is mean and divergence free due to the assumptions on the noise)

$$\mathrm{d}z_n = \nu_n \Delta z_n\,\mathrm{d}t + \mathrm{d}B,$$

Itô's formula yields for all $n \in \mathbb{N}$

$$\nu_n \mathbf{E}\|\nabla z_n\|_{L^2}^2 = \frac{1}{2}\mathrm{Tr}(GG^*). \tag{1.6}$$

If $G = 0$, the above results Theorem 1.4, Theorem 1.5 and Theorem 1.6 apply to the case of deterministic Navier–Stokes (1.1) and Euler (1.2) equations with zero force. Furthermore, with a small modification in the proof, it is possible to show more. The proof is given in Theorem 6.3, Theorem 6.4 and Theorem 6.6. The latter result also contains further observations regarding the limit of vanishing viscosity and/or vanishing noise.

Theorem 1.8. *Let $G = 0$. Let $\varepsilon > 0$, $r > 1$ be given and let Z be a stationary stochastic process with smooth trajectories, vanishing mean and divergence and satisfying*

$$\left(\mathbf{E}\|Z\|_{L^2}^m + \mathbf{E}\|Z\|_{C^2_{t,x}}^m\right)^{1/m} \leqslant m^{1/2} L,$$

for any $m > 1$ and some $L \geqslant (2\pi)^3$. Then up to a change of probability space,

$$\mathbf{E}\left[\|u - Z\|_{C_t W^{1,1}}^r\right] \leqslant \varepsilon. \tag{1.7}$$

holds true for

(1) the stationary as well as ergodic stationary solutions u obtained in Theorem 1.4;
(2) the limit stationary solutions u to the Euler equations (1.2) obtained in Theorem 1.5 as well as in Theorem 1.6.

In particular, the solutions can be random and time dependent. If Z is uniformly bounded in ω in $C^2_{t,x}$ then (1.7) holds pathwise, not only in expectation.

In the proof of the above result we make use of our stochastic convex integration construction. The added value lies particularly in the claim (1.7). It shows that the solutions can possess certain statistics that are close to those of the prescribed process Z. In particular, Z can be chosen Gaussian or non-Gaussian. If we dropped this requirement in the case of *deterministic Euler equations*, no new convex integration construction would be necessary. Precisely, we can use some of the explicit smooth steady state solutions (i.e. time independent), together with a Krein–Milman argument as in our proof to deduce the result of Theorem 1.4 in this Euler case with $G = 0$.

Due to the dissipation, following the same strategy for the *deterministic Navier–Stokes equations* is more delicate. The existence of nontrivial steady state solutions was established in [CL19] by convex integration. But it is only proved in [CL19] that these solutions belong to L^2, no higher regularity is shown. Hence, these solutions are not suitable for our proof of ergodicity based on Krein–Milman's theorem, because of the lacking compactness. Nevertheless, the seminal paper [BV19a] in particular permits to construct time periodic solutions taking values in H^ϑ for some $\vartheta > 0$. These can be obtained by first prescribing a compactly supported kinetic energy and then repeating periodically. Consequently, these solutions can be used to prove the existence and non-uniqueness of ergodic stationary solutions as in the Navier–Stokes part of Theorem 1.4 with $G = 0$. The periodicity is used in particular to obtain nontrivial stationary solutions and their non-uniqueness.

We also remark that in [Luo19] existence and non-uniqueness of H^ϑ-steady state solutions for every $\vartheta < 1/200$ was to the Navier–Stokes system in $d = 4$ was proved. A corollary of this result is therefore the existence of non-unique ergodic statistically stationary solutions.

Organization of the paper. In Section 2, we collect the basic notations used throughout the paper. Section 3 is the core of our proofs: here, the stochastic convex integration is developed and employed to construct entire non-unique analytically weak and probabilistically strong solutions with a prescribed kinetic energy. This is then used in Section 4 together with a Krylov–Bogoliubov's argument to obtain existence of non-unique stationary solutions to the stochastic Navier–Stokes equations. The results concerning stationary solutions to the stochastic Euler equations, the vanishing viscosity limit and the result related to anomalous dissipation can be found in Section 5, whereas the results for the deterministic systems are proved in Section 6. In Appendix A, we recall the construction of intermittent jets from [BCV18, BV19a] and in Appendix B we give estimates on amplitude functions used in the convex integration construction.

2. Notations

2.1. **Function spaces.** Throughout the paper, we employ the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given a Banach space E with a norm $\|\cdot\|_E$ and $t \in \mathbb{R}$, we write $C_tE = C([t, t+1]; E)$ for the space of continuous functions from $[t, t+1]$ to E, equipped with the supremum norm $\|f\|_{C_tE} = \sup_{s\in[t,t+1]} \|f(s)\|_E$. For $\alpha \in (0, 1)$ we define $C^\alpha_t E$ as the space of α-Hölder continuous functions from $[t, t+1]$ to E, endowed with the norm $\|f\|_{C^\alpha_t E} = \sup_{s,r\in[t,t+1],s\neq r} \frac{\|f(s)-f(r)\|_E}{|r-s|^\alpha} + \sup_{s\in[t,t+1]} \|f(s)\|_E$. Here we use C^α_t to denote the case when $E = \mathbb{R}$. We also write $C_b(\mathbb{R}; E)$ for functions in $C(\mathbb{R}; E)$ such that $\|f\|_{C_b(\mathbb{R};E)} := \sup_{t\in\mathbb{R}} \|f(t)\|_E < \infty$. For $\beta \in (0, 1]$ we define $C^\beta_b(\mathbb{R}; E)$ as functions in $C^\beta(\mathbb{R}; E)$ such that $\|f\|_{C^\beta_b(\mathbb{R};E)} := \sup_{t\in\mathbb{R}} \|f\|_{C^\beta_t E} < \infty$. We use L^p to denote the set of standard L^p-integrable functions from $\mathbb{T}^3$ to $\mathbb{R}^3$. For $s > 0$, $p > 1$ we set $W^{s,p} := \{f \in L^p; \|(I-\Delta)^{s/2} f\|_{L^p} < \infty\}$ with the norm $\|f\|_{W^{s,p}} = \|(I-\Delta)^{s/2} f\|_{L^p}$. Set $L^2_\sigma = \{f \in L^2; \int_{\mathbb{T}^3} f \,\mathrm{d}x = 0, \mathrm{div} f = 0\}$. For $s > 0$, we define $H^s := W^{s,2} \cap L^2_\sigma$. For $s < 0$ we define H^s to be the dual space of H^{-s}. For $t \in \mathbb{R}$ and a domain $D \subset \mathbb{R}^+$ we denote by $C^N_{t,x}$ and $C^N_{D,x}$, respectively, the space of C^N-functions on $[t, t+1] \times \mathbb{T}^3$ and on $D \times \mathbb{T}^3$, respectively, $N \in \mathbb{N}_0$. The spaces are equipped with the norms

$$\|f\|_{C^N_{t,x}} = \sum_{\substack{0\leqslant n+|\alpha|\leqslant N\\ n\in\mathbb{N}_0,\alpha\in\mathbb{N}^3_0}} \|\partial^n_t D^\alpha f\|_{L^\infty_{[t,t+1]}L^\infty}, \qquad \|f\|_{C^N_{D,x}} = \sum_{\substack{0\leqslant n+|\alpha|\leqslant N\\ n\in\mathbb{N}_0,\alpha\in\mathbb{N}^3_0}} \sup_{t\in D} \|\partial^n_t D^\alpha f\|_{L^\infty}.$$

For a Polish space H we denote by $\mathcal{B}(H)$ the σ-algebra of Borel sets in H. We also use $\mathring{\otimes}$ to denote the trace-free part of the tensor product.

By $\mathbb{P}$ we denote the Helmholtz projection. We recall the inverse divergence operator $\mathcal{R}$ from [BV19a, Section 5.6], which acts on vector fields v with $\int_{\mathbb{T}^3} v\mathrm{d}x = 0$ as

$$(\mathcal{R}v)^{kl} = (\partial_k\Delta^{-1}v^l + \partial_l\Delta^{-1}v^k) - \frac{1}{2}(\delta_{kl} + \partial_k\partial_l\Delta^{-1})\mathrm{div}\Delta^{-1}v,$$

for $k, l \in \{1, 2, 3\}$. Then $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$, and $\mathcal{R}$ is a right inverse of the div operator, i.e. $\mathrm{div}(\mathcal{R}v) = v$. By [CL20, Theorem B.3] we know

$$\|\mathcal{R}f(\sigma\cdot)\|_{L^p} \lesssim \sigma^{-1}\|f\|_{L^p} \quad \text{for } \sigma \in \mathbb{N}. \tag{2.1}$$

By $\mathcal{S}^{3\times3}$ we denote the set of symmetric 3×3 matrices and by $\mathcal{S}^{3\times3}_0$ the set of symmetric trace-free matrices. Let $C^\infty_0(\mathbb{T}^3, \mathbb{R}^{3\times3})$ be the set of periodic smooth matrix valued functions with zero mean. We also introduce the bilinear version $\mathcal{B} : C^\infty(\mathbb{T}^3, \mathbb{R}^3) \times C^\infty_0(\mathbb{T}^3, \mathbb{R}^{3\times3}) \to C^\infty(\mathbb{T}^3, \mathcal{S}^{3\times3}_0)$ as in [CL20, Section B.3] by

$$\mathcal{B}(v, A) = v\mathcal{R}A - \mathcal{R}(\nabla v\mathcal{R}A).$$

Then by [CL20, Theorem B.4] we have $\mathrm{div}(\mathcal{B}(v, A)) = vA - \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} vA\mathrm{d}x$ and

$$\|\mathcal{B}(v, A)\|_{L^p} \lesssim \|v\|_{C^1}\|\mathcal{R}A\|_{L^p}. \tag{2.2}$$

2.2. **Probabilistic elements.** For a given probability measure P we denote by $\mathbf{E}^P$ the expectation under P. Regarding the driving noise, we assume that B is an $\mathbb{R}^3$-valued two-sided GG^*-Wiener process with zero spatial mean and zero divergence, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and G is a Hilbert–Schmidt operator from U to L^2_σ for some Hilbert space U.

For $p \in [1, \infty)$ we denote

$$\|\!|u|\!\|^p_{L^2,p} := \sup_{t\in\mathbb{R}} \mathbf{E}\left[\sup_{t\leqslant s\leqslant t+1} \|u(s)\|^p_{L^2}\right], \quad \|\!|u|\!\|^p_{C^1_{t,x},p} := \sup_{t\in\mathbb{R}} \mathbf{E}\left[\|u(s)\|^p_{C^1_{[t,t+1],x}}\right].$$

These norms define function spaces of random variables on Ω taking values in $C(\mathbb{R}, L^2)$ and $C^1(\mathbb{R} \times \mathbb{T}^3)$, respectively, with bounds in $L^p(\Omega; C(I, L^2))$ and $L^p(\Omega; C^1(I \times \mathbb{T}^3))$ for any bounded interval $I \subset \mathbb{R}$. Furthermore, the bounds only depend on the length of the interval I, not on its location within $\mathbb{R}$. In the sequel, we simply say that u has a uniform moment of order p locally in $C(\mathbb{R}; L^2)$ provided $\|\!|u|\!\|_{L^2,p} < \infty$. Similarly, we define the corresponding norms with L^2 replaced by L^p, H^ϑ or $C^1_{t,x}$ replaced by $C_t^{\frac{1}{2}-2\delta}L^\infty$, $C_t^\vartheta L^2$ and $C_t W^{1,p}$.

3. Stochastic convex integration

The previous works using convex integration in the stochastic setting always reduced the problem to the deterministic setting by introducing suitable stopping times.[2] This permitted to control the noise uniformly in ω so that the convex integration could proceed pathwise up to the stopping time. The stopping times were then removed a posteriori by a suitable extension of solutions. Such an approach is not suitable for the construction of stationary solutions. Hence, inspired by [CDZ22] we present an honest stochastic convex integration, constructing directly solutions on the whole time line $\mathbb{R}$. This is achieved by introducing expectations to the iterative estimates in convex integration. The main difficulty lies in the fact that due to the quadratic nonlinearity the estimates are superlinear. More precisely, the estimate of any pth moment at the level $q+1$ necessarily contains higher moments at the level q. Accordingly, all the estimates need to be tracked down very carefully, paying a particular attention to the appearing constants, what they depend on and how precisely. Otherwise, it would not be possible to close the estimates. The key observation is that the superlinear terms always contain a small constant which may be used to absorb the bounds.

As the first step, we decompose a solution to the Navier–Stokes system (1.4) with $\nu = 1$ into the sum $u = z + v$ where z is the unique stationary solution to the linear stochastic heat equation

$$\mathrm{d}z - (\Delta - 1)z\,\mathrm{d}t = \mathrm{d}B, \tag{3.1}$$

where B is a $\mathbb{R}^3$-valued two-sided trace-class Wiener process with spatial zero mean (see e.g. [PR07, page 99]), and v solves the nonlinear deterministic equation

$$\begin{aligned} \partial_t v - \Delta v - z + \operatorname{div}((v+z)\otimes(v+z)) + \nabla P &= 0, \\ \operatorname{div} v &= 0. \end{aligned} \tag{3.2}$$

Here, z is divergence free by the assumptions on the noise and by P we denote the pressure term associated to v.

Remark 3.1. For notational simplicity, we work in this section as well as in Section 4 with the unit viscosity $\nu = 1$. This fact is used only in Proposition 3.2 below, which profits from the smoothing effect of the Laplacian. More precisely, the spatial regularity is needed for the convergence rate in the convex integration in order to deduce the H^ϑ-estimate. Only a bound in L^2 would not be enough. For a general ν, the bound in Proposition 3.2 would depend on ν^{-p}. Hence, for the results regarding stationary solutions to the Euler equations, the vanishing viscosity limit and the anomalous dissipation in Section 5 and Section 6, it is necessary to increase the regularity of the noise in order to compensate for the lack of smoothing of the linear part.

[2] The first exception was our previous work on a class of supercritical/critical SPDEs with an irregular spatial perturbation [HZZ22]. Due to the time independence of the noise, no stopping times were necessary.

We included the linear zero order term z on the left hand side of (3.1) in order to obtain a stabilization of the equation needed for the necessary global in time estimates. It will be seen in the course of the construction that the corresponding counter term in (3.2) will not cause any difficulties. Using the factorization method it is standard to derive regularity of the stochastic convolution z on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbf{P})$ with $(\mathcal{F}_t)_{t\in\mathbb{R}}$ canonical filtration given in [PR07, page 99]. In particular, the following result follows from [DPZ92, Theorem 5.16] together with the Kolmogorov continuity criterion.

Proposition 3.2. *Suppose that* $\mathrm{Tr}(GG^*) < \infty$. *Then for any* $\delta \in (0, 1/2)$, $p \geqslant 2$

$$\sup_{t\in\mathbb{R}} \mathbf{E}\left[\sup_{t\leqslant s\leqslant t+1} \|z(s)\|^p_{H^{1-\delta}} + \|z\|^p_{C_t^{1/2-\delta}L^2}\right] \leqslant (p-1)^{p/2} L^p, \tag{3.3}$$

where $L \geqslant 1$ *depends on* $\mathrm{Tr}(GG^*)$, δ *and is independent of* p.

Proof. We recall that the unique stationary solution to (3.1) has the explicit form $z(t) = \int_{-\infty}^t S(t-s)\mathrm{d}B(s)$ where $S(t) = e^{t(\Delta - I)}$, $t \geqslant 0$. The Wiener process B is given by $B = \sum_{k\in\mathbb{N}} c_k e_k \beta_k$ for an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of L^2_σ, a sequence of mutually independent standard two-sided Brownian motions $\{\beta_k\}_{k\in\mathbb{N}}$ and the coefficients satisfy $\sum_{k\in\mathbb{N}} c_k^2 < \infty$. Then it holds for $\gamma \in (0, 1/2)$, $t \geqslant s$

$$\begin{aligned}
\mathbf{E}\|z(t) - z(s)\|^2_{L^2} &= \sum_{k=1}^\infty c_k^2 \int_s^t \|S(t-\sigma)e_k\|^2_{L^2}\mathrm{d}\sigma + \sum_{k=1}^\infty \int_{-\infty}^s c_k^2 \|[S(t-\sigma) - S(s-\sigma)]e_k\|^2_{L^2}\mathrm{d}\sigma \\
&\leqslant M\mathrm{Tr}(GG^*)\Big[(t-s) + \int_{-\infty}^s e^{-2(s-\sigma)}\Big|\int_{s-\sigma}^{t-\sigma} \frac{1}{r}\mathrm{d}r\Big|^2 \mathrm{d}\sigma\Big] \\
&\leqslant M\mathrm{Tr}(GG^*)\Big[(t-s) + \int_{-\infty}^s e^{-2(s-\sigma)}(s-\sigma)^{-2\gamma}\Big|\int_{s-\sigma}^{t-\sigma} r^{\gamma-1}\mathrm{d}r\Big|^2 \mathrm{d}\sigma\Big] \\
&\leqslant M\mathrm{Tr}(GG^*)[(t-s) + (t-s)^{2\gamma}],
\end{aligned}$$

where the constant M depends only on the semigroup and γ but is independent of time. Using Gaussianity we have

$$\mathbf{E}\|z(t) - z(s)\|^p_{L^2} \leqslant (p-1)^{p/2}\Big(\mathbf{E}\|z(t) - z(s)\|^2_{L^2}\Big)^{p/2}.$$

Similar computations can be performed for the $H^{1-\delta}$-norm as well, only the resulting time regularity is lower and depends on δ. The result then follows from Kolmogorov's continuity criterion. □

In the following we choose $L \geqslant (2\pi)^{3/2}$ for simplicity.

Global in time estimates of the form (3.3) are well-suited for the application of a Krylov–Bogoliubov argument leading to existence of stationary solutions as limits of ergodic averages. Our goal in this section is to construct solutions to the Navier–Stokes system (1.4) satisfying similar bounds. To this end, we use the norms $|\!|\!| \cdot |\!|\!|_{L^2,p}$ and $|\!|\!| \cdot |\!|\!|_{C^1_{t,x},p}$ introduced in Section 2.2, which play the essential role in the construction.

Let us now explain how the convex integration iteration is set up. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}_0} \subset \mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q\in\mathbb{N}} \subset (0, 1)$ which is decreasing to 0. We choose $a \in \mathbb{N}$, $b \in \mathbb{N}$, $\beta \in (0, 1]$ and let

$$\lambda_q = a^{(b^q)}, \quad \delta_1 = 1, \quad \delta_q = \frac{1}{2}\lambda_2^{2\beta}\lambda_q^{-2\beta}, \quad q \geqslant 2.$$

Here β will be chosen sufficiently small and a as well as b will be chosen sufficiently large. We assume $\sum_{r\geqslant 1}\delta_r^{1/2}\leqslant 1+\sum_{r\geqslant 2}a^{b^2\beta-(r-1)b^2\beta}=1+\frac{1}{1-a^{-\beta b^2}}\leqslant 3$ which boils down to

$$a^{\beta b^2}\geqslant 2. \tag{3.4}$$

Here we require more than is necessary for the later use and we keep this assumption from now on. More details on the choice of these parameters will be given below in the course of the construction. The iteration is indexed by a parameter $q\in\mathbb{N}_0$. At each step q, a pair $(v_q,\mathring{R}_q)$ is constructed solving the following system

$$\begin{aligned}\partial_t v_q - z_q - \Delta v_q + \mathrm{div}((v_q+z_q)\otimes(v_q+z_q)) + \nabla p_q &= \mathrm{div}\mathring{R}_q,\\ \mathrm{div} v_q &= 0.\end{aligned} \tag{3.5}$$

In the above we define $z_q=\mathbb{P}_{\leqslant f(q)}z$ with $f(q)=\lambda_{q+1}^{\alpha/8}$ and $\mathring{R}_q$ is trace-free and we put the trace part into the pressure. Thanks to this approximation of z, we are able to lower the assumption on the spatial regularity of the noise B, namely, to cover the case of any trace-class noise. We observe that

$$\|z_q\|_{L^\infty,p}\leqslant (p-1)^{1/2}L\lambda_{q+1}^{\alpha/8},\quad \|z_q\|_{C_t^{\frac{1}{2}-2\delta}L^\infty,p}\leqslant (p-1)^{1/2}L\lambda_{q+1}^{\alpha/4}. \tag{3.6}$$

We intend to construct approximations v_q with a uniform moment of order $2r$ for a given $r>1$ locally in $C(\mathbb{R},H^\vartheta)$ and $C^\vartheta(\mathbb{R},L^2)$ for some $\vartheta>0$, in the sense of the norms $\|\cdot\|_{H^\vartheta,2r}$ and $\|\cdot\|_{C_t^\vartheta L^2,2r}$. But to this end, it is necessary to quantify the blow up of the higher moments, as these also appear in the estimates.

Under the above assumptions, our main iteration reads as follows, the proof of this result is presented in Section 3.1 below.

Proposition 3.3. *Assume (3.3) and let $r>1$ be fixed. Given smooth function $e:\mathbb{R}\to(0,\infty)$ so that $\bar{e}\geqslant e(t)\geqslant \underline{e}\geqslant 1$ with $\|e'\|_{C^0}\leqslant\tilde{e}$ for some constants $\bar{e},\underline{e},\tilde{e}>0$, there exists a choice of parameters a,b,β and $\alpha\in(0,1/49)$ with $\alpha b>32/7$ such that the following holds true: Let $(v_q,\mathring{R}_q)$ for some $q\in\mathbb{N}_0$ be an $(\mathcal{F}_t)_{t\geqslant 0}$-adapted solution to (3.5) satisfying*

$$\|v_q\|_{L^2,2r}\leqslant M_0\bar{e}^{1/2}\sum_{k=1}^{q}\delta_k^{1/2} \tag{3.7}$$

for a universal constant M_0, and for $m\geqslant 1$

$$\|v_q\|_{L^2,m}\leqslant M_0(6^{q-1}\cdot 12mL^2)^{3(6^{q-1})}+M_0\bar{e}^{1/2}\sum_{r=1}^{q}\delta_r^{1/2}, \tag{3.8}$$

and

$$\|v_q\|_{C^1_{t,x},m}\leqslant \lambda_q^{23/7}(6^{q-1}\cdot 16mL^2)^{4(6^{q-1})},\quad \|v_q\|_{C^2_{t,x},m}\leqslant \lambda_q^{37/7}(6^{q-1}\cdot 20mL^2)^{5(6^{q-1})}, \tag{3.9}$$

$$\|\mathring{R}_q\|_{L^1,r}\leqslant \frac{1}{48}\delta_{q+2}\underline{e}, \tag{3.10}$$

and for $m\geqslant 1$

$$\|\mathring{R}_q\|_{L^1,m}\leqslant (6^q\cdot 4mL^2)^{(6^q)}. \tag{3.11}$$

Moreover, for any $t\in\mathbb{R}$

$$\frac{3}{4}\delta_{q+1}e(t)\leqslant e(t)-\mathbf{E}\|(v_q+z_q)(t)\|_{L^2}^2\leqslant \frac{5}{4}\delta_{q+1}e(t), \tag{3.12}$$

Then there exists an $(\mathcal{F}_t)_{t\geqslant 0}$-adapted process $(v_{q+1}, \mathring{R}_{q+1})$ which solves (3.5), obeys (3.7)– (3.12) at the level $q+1$ and satisfies

$$\|v_{q+1} - v_q\|_{L^2,2r} \leqslant M_0 \bar{e}^{1/2} \delta_{q+1}^{1/2}. \tag{3.13}$$

and for $p = \frac{32}{32-7\alpha}$

$$\|v_{q+1} - v_q\|_{C_t W^{1,p},r} \leqslant \lambda_{q+1}^{-\alpha/2} + \lambda_{q+1}^{-1/7+6\alpha}. \tag{3.14}$$

Using (3.9) and the choice of the parameters in Section 3.1.1 we have

$$\|v_q\|_{C^1_{t,x},2r} \leqslant \lambda_q^4, \quad \|v_q\|_{C^2_{t,x},r} \leqslant \lambda_q^6. \tag{3.15}$$

We start the iteration from $v_0 \equiv 0$ on $\mathbb{R}$. In that case, we have $\mathring{R}_0 = z_0 \mathring{\otimes} z_0 - \mathcal{R} z_0$ so that

$$\|\mathring{R}_0\|_{L^1,m} \leqslant \|z\|^2_{L^2,2m} + (2\pi)^{3/2}\|z\|_{L^2,m} \leqslant 4mL^2$$

and (3.10), (3.11) are satisfied on the level $q=0$, since $\delta_2 = 1/2$ and provided $8\cdot 48 r L^2 \leqslant \underline{e}$. Here, we used $L \geqslant (2\pi)^{3/2}$. For (3.12) we require

$$\frac{3}{4} e(t) \leqslant e(t) - \mathbf{E}\|z_0(t)\|^2_{L^2} \leqslant \frac{5}{4} e(t),$$

which is satisfied provided $e(t) \geqslant 4L^2$.

We deduce the following result.

Theorem 3.4. *Let $r > 1$ and a smooth function $e : \mathbb{R} \to (0,\infty)$ satisfying $\bar{e} \geqslant e(t) \geqslant \underline{e} \geqslant 8{\cdot}48 r L^2$ be given. There exists an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted process u which belongs to $C(\mathbb{R}, H^\vartheta) \cap C^\vartheta(\mathbb{R}, L^2)$ $\mathbf{P}$-a.s. for some $\vartheta > 0$ and is an analytically weak solution to (1.4) in the sense of Definition 1.1. Moreover, the solution satisfies*

$$\|u\|_{H^\vartheta,2r} + \|u\|_{C_t^\vartheta L^2,2r} < \infty, \tag{3.16}$$

and for all $t \in \mathbb{R}$

$$\mathbf{E}\|u(t)\|^2_{L^2} = e(t). \tag{3.17}$$

There are infinitely many such solutions by choosing different energies e. Furthermore, for every $\varepsilon > 0$ one may find solution such that $v = u - z$ satisfying

$$\|v\|_{C_t W^{1,1},r} \leqslant \varepsilon. \tag{3.18}$$

Proof. By interpolation we deduce for $\vartheta \in (0, \frac{\beta}{4+\beta})$,

$$\sum_{q\geqslant 0} \|v_{q+1} - v_q\|_{H^\vartheta,2} \lesssim \sum_{q\geqslant 0} \|v_{q+1} - v_q\|^{1-\vartheta}_{L^2,2} \|v_{q+1} - v_q\|^{\vartheta}_{H^1,2} \lesssim \sum_{q\geqslant 0} \delta_{q+1}^{\frac{1-\vartheta}{2}} \lambda_{q+1}^{4\vartheta} < \infty.$$

Similarly we could change H^ϑ to $C_t^\vartheta L^2$. As a consequence, a limit $v = \lim_{q\to\infty} v_q$ exists and lies in $L^2(\Omega, C(\mathbb{R}, H^\vartheta) \cap C^\vartheta(\mathbb{R}, L^2))$. Since v_q is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted for every $q \in \mathbb{N}_0$, the limit v is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted as well. Furthermore, it follows from (3.10) that $\lim_{q\to\infty} \mathring{R}_q = 0$ in $L^1(\Omega, C(\mathbb{R}; L^1))$ and $\lim_{q\to\infty} z_q = z$ in $L^p(\Omega, C(\mathbb{R}; L^2))$ for any $p \geqslant 1$. Thus v is an analytically weak solution to (3.2). Hence letting $u = v + z$ we obtain an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted analytically weak solution to (1.4). Moreover, the estimate for u holds. Finally, (3.17) follows from (3.12).

For the last result, we use (3.14) and conditions on α to have

$$\|v\|_{C_t W^{1,p},r} \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{C_t W^{1,p},r}$$

$$
\begin{aligned}
&\leqslant \sum_{q=0}^{\infty}(\lambda_{q+1}^{-\alpha/2}+\lambda_{q+1}^{-1/7+6\alpha}) \lesssim \frac{a^{-\alpha b/2}}{1-a^{-\alpha b/2}}\\
&= \frac{1}{a^{\alpha b/2}-1} \leqslant \frac{1}{a^{16/7}-1} \leqslant \varepsilon,
\end{aligned}
$$

where we use $\alpha b > 32/7$ and $-1/7+6\alpha < -\alpha/2$ and we may choose a large enough such that the last inequality holds. □

3.1. **Proof of Proposition 3.3.** The proof proceeds in several main steps which are the same in many convex integration schemes. First of all, we start the construction by fixing the parameters in Section 3.1.1 and proceed with a mollification step in Section 3.1.2. Section 3.1.3 introduces the new iteration v_{q+1}. This is the main part of the construction which differs in each convex integration scheme. Here, we construct new amplitude functions $a_{(\xi)}$ similarly to [HZZ23] but we replace the pathwise construction by a stochastic variant, namely, we work explicitly with expectations of v_q+z_q. Section 3.1.4 contains the inductive estimates of v_{q+1}, especially the moment estimates, whereas in Section 3.1.5 we show how the energy is controlled. Finally, in Section 3.1.6, we define the new stress $\mathring{R}_{q+1}$ and establish the inductive moment estimate on $\mathring{R}_{q+1}$ in Section 3.1.7.

3.1.1. *Choice of parameters.* In the sequel, additional parameters will be indispensable and their value has to be carefully chosen in order to respect all the compatibility conditions appearing in the estimations below. First, for a sufficiently small $\alpha \in (0,1)$ to be chosen below, we let $\ell \in (0,1)$ be a small parameter satisfying

$$\ell\lambda_q^4 \leqslant \lambda_{q+1}^{-\alpha}, \quad \ell^{-1} \leqslant \lambda_{q+1}^{2\alpha}, \quad \bar{e} \leqslant \ell^{-1}. \tag{3.19}$$

In particular, we define

$$\ell := \lambda_{q+1}^{-3\alpha/2}\lambda_q^{-2}. \tag{3.20}$$

In the sequel, we use the following bounds

$$\alpha b > 32/7, \quad 43\alpha < 1/14, \quad \alpha > 40\beta b^2, \quad 6/b+2\beta b^2 < 1/14, \quad 2\beta b < \frac{1}{7} - 127\alpha,$$

which can be obtained by choosing α small such that $\frac{1}{128\cdot 7} > \alpha$, and choosing $b \in \mathbb{N}$ large enough such that $\alpha b > 32/7$ and finally choosing β small such that $\alpha > 40\beta b^2$. Hence, we shall choose rational α small first and b large, then β small enough. The last free parameter is a which satisfies the lower bounds given through (3.4) and the last bound in (3.19). Let c satisfy $q6^q \leqslant c7^q$. We then choose $a \geqslant (192L^2 r)^c \vee (252L^2)^{3c}$. In the sequel, we increase a in order to absorb various implicit and universal constants.

3.1.2. *Mollification.* We intend to replace v_q by a mollified velocity field v_ℓ. To this end, let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^3$, and let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers with support in $(0,1)$. The one-sided mollifier here is used in order to preserve adaptedness. We define a mollification of v_q, $\mathring{R}_q$ and z_q in space and time by convolution as follows

$$v_\ell = (v_q *_x \phi_\ell) *_t \varphi_\ell, \qquad \mathring{R}_\ell = (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell, \qquad z_\ell = (z_q *_x \phi_\ell) *_t \varphi_\ell,$$

where $\phi_\ell = \frac{1}{\ell^3}\phi(\frac{\cdot}{\ell})$ and $\varphi_\ell = \frac{1}{\ell}\varphi(\frac{\cdot}{\ell})$. Since the mollifier φ_ℓ is supported on $(0,1)$, it is easy to see that z_ℓ is $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted and so are v_ℓ and $\mathring{R}_\ell$. Since (3.5) holds, it follows that $(v_\ell, \mathring{R}_\ell)$ satisfies

$$
\begin{aligned}
\partial_t v_\ell - z_\ell - \Delta v_\ell + \operatorname{div}((v_\ell+z_\ell)\otimes(v_\ell+z_\ell)) + \nabla p_\ell &= \operatorname{div}(\mathring{R}_\ell + R_{\mathrm{com}})\\
\operatorname{div} v_\ell &= 0,
\end{aligned} \tag{3.21}
$$

where

$$R_{\text{com}} = (v_\ell + z_\ell)\mathring{\otimes}(v_\ell + z_\ell) - ((v_q + z_q)\mathring{\otimes}(v_q + z_q)) *_x \phi_\ell *_t \varphi_\ell,$$
$$p_\ell = (p_q *_x \phi_\ell) *_t \varphi_\ell - \frac{1}{3}\big(|v_\ell + z_\ell|^2 - (|v_q + z_q|^2 *_x \phi_\ell) *_t \varphi_\ell\big).$$

We have

$$\|v_q(t) - v_\ell(t)\|_{L^2} \lesssim \ell \|v_q\|_{C^1_{[t-1,t+1],x}}, \tag{3.22}$$

which by (3.15) implies that

$$\|\!|v_q - v_\ell|\!\|_{L^2,2r} \lesssim \ell \|\!|v_q|\!\|_{C^1_{t,x},2r} \leqslant \ell\lambda_q^4 \leqslant \frac{1}{4}\bar{e}^{1/2}\delta_{q+1}^{1/2}, \tag{3.23}$$

where we used the fact that $\ell\lambda_q^4 < \lambda_{q+1}^{-\beta}$. In addition,

$$\|v_\ell\|_{C_t L^2} \leqslant \|v_q\|_{C_{[t-1,t+1]}L^2}. \tag{3.24}$$

3.1.3. *Construction of* v_{q+1}. Let us now proceed with the construction of the perturbation w_{q+1} which then defines the next iteration by $v_{q+1} := v_\ell + w_{q+1}$. To this end, we employ the intermittent jets introduced in [BCV18] and presented in [BV19a, Section 7.4], which we recall in Appendix A. In particular, the building blocks $W_{(\xi)} = W_{\xi,r_\perp,r_\|,\lambda,\mu}$ for $\xi \in \Lambda$ are defined in (A.3) and the set Λ is introduced in Lemma A.1. The necessary estimates are collected in (A.7). We choose the following parameters

$$\lambda = \lambda_{q+1}, \qquad r_\| = \lambda_{q+1}^{-4/7}, \qquad r_\perp = r_\|^{-1/4}\lambda_{q+1}^{-1} = \lambda_{q+1}^{-6/7}, \qquad \mu = \lambda_{q+1} r_\| r_\perp^{-1} = \lambda_{q+1}^{9/7}. \tag{3.25}$$

It is required that b is a multiple of 7 to ensure that $\lambda_{q+1} r_\perp = a^{(b^{q+1})/7} \in \mathbb{N}$.

Now we define ρ as follows

$$\rho := 2\sqrt{\ell^2 + |\mathring{R}_\ell|^2} + \gamma_\ell, \tag{3.26}$$
$$\gamma_q(t) := \frac{1}{3\cdot(2\pi)^3}\Big[e(t)(1-\delta_{q+2}) - \mathbf{E}\|v_q(t) + z_q(t)\|_{L^2}^2\Big],$$

and

$$\gamma_\ell := \gamma_q *_t \varphi_\ell.$$

We observe that (3.4) and $b \geqslant 2$ implies in particular $\frac{4}{3} \leqslant a^{2\beta b(b-1)}$, i.e. $\frac{3}{4}\delta_{q+1} \geqslant \delta_{q+2}$, it follows that $\gamma_q \geqslant 0$. In view of the definition of ρ in (3.26), we obtain for any $p \in [1,\infty]$,

$$\|\rho\|_{L^p} \leqslant 2\ell(2\pi)^{3/p} + 2\|\mathring{R}_\ell\|_{L^p} + \frac{1}{2}\delta_{q+1}\bar{e}. \tag{3.27}$$

Furthermore, by mollification estimates, the embedding $W^{4,1} \subset L^\infty$ we obtain for $N \geqslant 0$

$$\|\mathring{R}_\ell\|_{C^N_{t,x}} \lesssim \ell^{-4-N}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1},$$

which in particular leads to

$$\|\rho\|_{C^0_{t,x}} \lesssim \ell + \ell^{-4}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \delta_{q+1}\bar{e}. \tag{3.28}$$

We put further details on the $C^N_{t,x}$-estimates of ρ in Appendix B and by (B.1) we obtain

$$\|\rho\|_{C^N_{t,x}} \lesssim \ell^{-4-N}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \ell^{-6N+1}\|\mathring{R}_q\|^N_{C_{[t-1,t+1]}L^1} + \frac{1}{2}\ell^{-N}\delta_{q+1}\bar{e}. \tag{3.29}$$

Now, we define the amplitude functions

$$a_{(\xi)}(\omega,t,x) := a_{\xi,q+1}(\omega,t,x) := \rho(\omega,t,x)^{1/2}\gamma_\xi\left(\mathrm{Id} - \frac{\mathring{R}_\ell(\omega,t,x)}{\rho(\omega,t,x)}\right), \tag{3.30}$$

where γ_ξ is introduced in Lemma A.1. By (A.5) we have

$$(2\pi)^{-3}\sum_{\xi\in\Lambda}a_{(\xi)}^2\int_{\mathbb{T}^3}W_{(\xi)}\otimes W_{(\xi)}\mathrm{d}x=\rho\mathrm{Id}-\mathring{R}_\ell, \tag{3.31}$$

and using (B.2)

$$\|a_{(\xi)}\|_{C_tL^2}\leqslant\frac{M}{4|\Lambda|}\left(2\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+\frac{1}{2}\delta_{q+1}\bar{e}\right)^{1/2}, \tag{3.32}$$

where M denotes the universal constant from Lemma A.1. Moreover, we could get the following $C^N_{t,x}$-norm of $a_{(\xi)}$. Since the calculation is similar as in [HZZ23] except the explicit dependence on $\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}$, we put the main part in Appendix B. In particular, by (B.9)-(B.8) we obtain for $N\geqslant 1$

$$\|a_{(\xi)}\|_{C^N_{t,x}}\lesssim\ell^{-7-6N}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{N+1}, \tag{3.33}$$

and

$$\|a_{(\xi)}\|_{C^0_{t,x}}\lesssim\ell^{-2}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{1/2}. \tag{3.34}$$

Here, the implicit constant depends on N and in the following we only used $N\leqslant 9$.

With these preparations in hand, we define the principal part $w^{(p)}_{q+1}$ of the perturbation w_{q+1} as

$$w^{(p)}_{q+1}:=\sum_{\xi\in\Lambda}a_{(\xi)}W_{(\xi)}. \tag{3.35}$$

Since the coefficients $a_{(\xi)}$ are $(\mathcal{F}_t)_{t\geqslant0}$-adapted and $W_{(\xi)}$ is a deterministic function we deduce that $w^{(p)}_{q+1}$ is also $(\mathcal{F}_t)_{t\geqslant0}$-adapted. Moreover, according to (3.31) and (A.4) it follows that

$$w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}+\mathring{R}_\ell=\sum_{\xi\in\Lambda}a_{(\xi)}^2\mathbb{P}_{\neq0}(W_{(\xi)}\otimes W_{(\xi)})+\rho\mathrm{Id}, \tag{3.36}$$

where we use the notation $\mathbb{P}_{\neq0}f:=f-\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}f\mathrm{d}x$.

We also define the incompressibility corrector by

$$w^{(c)}_{q+1}:=\sum_{\xi\in\Lambda}\mathrm{curl}(\nabla a_{(\xi)}\times V_{(\xi)})+\nabla a_{(\xi)}\times\mathrm{curl}V_{(\xi)}+a_{(\xi)}W^{(c)}_{(\xi)}, \tag{3.37}$$

with $W^{(c)}_{(\xi)}$ and $V_{(\xi)}$ being given in (A.6). Since $a_{(\xi)}$ is $(\mathcal{F}_t)_{t\geqslant0}$-adapted and $W_{(\xi)},W^{(c)}_{(\xi)}$ and $V_{(\xi)}$ are deterministic it follows that $w^{(c)}_{q+1}$ is also $(\mathcal{F}_t)_{t\geqslant0}$-adapted. By a direct computation we deduce that

$$w^{(p)}_{q+1}+w^{(c)}_{q+1}=\sum_{\xi\in\Lambda}\mathrm{curl}\,\mathrm{curl}(a_{(\xi)}V_{(\xi)}),$$

hence

$$\mathrm{div}(w^{(p)}_{q+1}+w^{(c)}_{q+1})=0.$$

Next, we introduce the temporal corrector

$$w^{(t)}_{q+1}:=-\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}\mathbb{P}_{\neq0}\left(a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\xi\right), \tag{3.38}$$

where $\mathbb{P}$ is the Helmholtz projection. Similarly as above, $w_{q+1}^{(t)}$ is $(\mathcal{F}_t)_{t\geqslant 0}$-adapted and by a direct computation (see [BV19a, (7.20)]) we obtain

$$\begin{aligned}\partial_t w_{q+1}^{(t)} &+ \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(a_{(\xi)}^2 \mathrm{div}(W_{(\xi)}\otimes W_{(\xi)})\right)\\ &= -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}\mathbb{P}_{\neq 0}\partial_t\left(a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\xi\right) + \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(a_{(\xi)}^2\partial_t(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right)\\ &= (\mathrm{Id}-\mathbb{P})\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\partial_t\left(a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\xi\right) - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t a_{(\xi)}^2(\phi_{(\xi)}^2\psi_{(\xi)}^2\xi)\right).\end{aligned} \tag{3.39}$$

Note that the first term on the right hand side can be viewed as a pressure term ∇p_1.

Finally, the total perturbation w_{q+1} is defined by

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}, \tag{3.40}$$

which is mean zero, divergence free and $(\mathcal{F}_t)_{t\geqslant 0}$-adapted. The new velocity v_{q+1} is defined as

$$v_{q+1} := v_\ell + w_{q+1}. \tag{3.41}$$

Thus, it is also $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted.

3.1.4. *Inductive estimates for v_{q+1}.* Next, we verify the inductive estimates (3.7) on the level $q+1$ for v and we prove (3.13).

In the following we use [CL20, Lemma B.1]. This result is applied to bound $w_{q+1}^{(p)}$ in L^2 whereas for the other L^p-norms we use a different approach. By (3.32), (B.2) and (3.33) we obtain

$$\begin{aligned}\|w_{q+1}^{(p)}\|_{C_tL^2} &\lesssim \sum_{\xi\in\Lambda}\|a_{(\xi)}\|_{C_tL^2}\|W_{(\xi)}\|_{C_tL^2} + \frac{1}{(\lambda_{q+1}r_\perp)^{1/2}}\|a_{(\xi)}\|_{C^1_{t,x}}\|W_{(\xi)}\|_{C_tL^2}\\ &\leqslant \frac{M_0}{8}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \bar{e}\delta_{q+1})^{1/2} + \frac{1}{(\lambda_{q+1}r_\perp)^{1/2}}\ell^{-13}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^2\end{aligned} \tag{3.42}$$

where we used the fact that due to (A.3) together with the normalizations (A.1), (A.2) it holds $\|W_{(\xi)}\|_{L^2}\simeq 1$ uniformly in all the involved parameters. Here, we may choose $M_0 = cM \geqslant 1$ with a universal constant c.

For a general L^p-norm we apply (A.7) and (3.34) to deduce for $p\in(1,\infty)$

$$\|w_{q+1}^{(p)}\|_{C_tL^p} \lesssim \sum_{\xi\in\Lambda}\|a_{(\xi)}\|_{C^0_{t,x}}\|W_{(\xi)}\|_{C_tL^p} \lesssim \ell^{-2}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{1/2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2}, \tag{3.43}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{C_tL^p} &\lesssim \sum_{\xi\in\Lambda}\left(\|a_{(\xi)}\|_{C^0_{t,x}}\|W_{(\xi)}^{(c)}\|_{C_tL^p} + \|a_{(\xi)}\|_{C^2_{t,x}}\|V_{(\xi)}\|_{C_tW^{1,p}}\right)\\ &\lesssim \ell^{-19}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3 r_\perp^{2/p-1} r_\parallel^{1/p-1/2}\left(r_\perp r_\parallel^{-1} + \lambda_{q+1}^{-1}\right)\\ &\lesssim \ell^{-19}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3 r_\perp^{2/p} r_\parallel^{1/p-3/2},\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{C_tL^p} &\lesssim \mu^{-1}\sum_{\xi\in\Lambda}\|a_{(\xi)}\|_{C^0_{t,x}}^2\|\phi_{(\xi)}\|_{L^{2p}}^2\|\psi_{(\xi)}\|_{C_tL^{2p}}^2\\ &\lesssim \ell^{-4}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1) r_\perp^{2/p-1} r_\parallel^{1/p-2}(\mu^{-1}r_\perp^{-1}r_\parallel)\\ &= \ell^{-4}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1) r_\perp^{2/p-1} r_\parallel^{1/p-2}\lambda_{q+1}^{-1}.\end{aligned} \tag{3.45}$$

We note that for $p=2$ (3.43) provides a worse bound than (3.42). Hence, we obtain for $p=\frac{32}{32-7\alpha}>1$ so that $r_\perp^{2/p-2}r_\parallel^{1/p-1}\leqslant\lambda_{q+1}^{\alpha}$ it holds that

$$\|w_{q+1}\|_{C_tL^p}\lesssim\lambda_{q+1}^{-8/7+5\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3, \tag{3.46}$$

where we use (3.19) and the fact that $\lambda_{q+1}^{19\alpha-\frac{1}{7}}<1$ by our choice of α. The bound (3.46) will be used below in the estimation of the Reynolds stress.

Combining (3.42), (3.44) and (3.45) we obtain

$$\begin{aligned}\|w_{q+1}\|_{C_tL^2}\leqslant&\frac{M_0}{8}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}^{1/2}+\bar{e}^{1/2}\delta_{q+1}^{1/2}\frac{M_0}{4}+\lambda_{q+1}^{-1/14}\ell^{-13}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^2\\&+\lambda_{q+1}^{-2/7}\ell^{-19}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3+\lambda_{q+1}^{-1/7}\ell^{-4}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)\\\leqslant&\frac{M_0}{8}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}^{1/2}+\bar{e}^{1/2}\delta_{q+1}^{1/2}\frac{M_0}{4}+\lambda_{q+1}^{-1/14+26\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}^3+1)\frac{M_0}{8}.\end{aligned} \tag{3.47}$$

Thus by (3.10) and (3.11) we obtain

$$\begin{aligned}\|\!|w_{q+1}|\!\|_{L^2,2r}\leqslant&\frac{M_0}{4}\|\!|\mathring{R}_q|\!\|_{L^1,r}^{1/2}+\bar{e}^{1/2}\delta_{q+1}^{1/2}\frac{M_0}{4}+\frac{M_0}{4}\lambda_{q+1}^{-1/14+26\alpha}(\|\!|\mathring{R}_q|\!\|_{L^1,6r}^3+1)\\\leqslant&\frac{M_0}{4}\bar{e}^{1/2}\delta_{q+1}^{1/2}+\frac{1}{4}M_0\bar{e}^{1/2}\delta_{q+1}^{1/2}+\frac{M_0}{4}\lambda_{q+1}^{-1/14+26\alpha}((6^qr\cdot24L^2)^{3(6^q)}+1)\\\leqslant&\frac{M_0}{2}\bar{e}^{1/2}\delta_{q+1}^{1/2}+\frac{M_0}{4}\lambda_{q+1}^{-1/14+26\alpha}(\lambda_q^3+1)\\\leqslant&\frac{3M_0}{4}\bar{e}^{1/2}\delta_{q+1}^{1/2}.\end{aligned} \tag{3.48}$$

Here we used $a>(144L^2r)^c$ and $-1/14+26\alpha+3/b<-\beta$ by the choice of parameters in Section 3.1.1. The bound (3.48) can be directly combined with (3.23) and the definition of the velocity v_{q+1} (3.41) to deduce

$$\|\!|v_{q+1}-v_q|\!\|_{L^2,2r}\leqslant\|\!|w_{q+1}|\!\|_{L^2,2r}+\|\!|v_\ell-v_q|\!\|_{L^2,2r}\leqslant M_0\bar{e}^{1/2}\delta_{q+1}^{1/2},$$

hence (3.13) holds and (3.7) follows at the level of $q+1$. Moreover, by (3.47) and (3.11) we obtain

$$\begin{aligned}\|\!|w_{q+1}|\!\|_{L^2,m}\leqslant&\frac{M_0}{4}\|\!|\mathring{R}_q|\!\|_{L^1,m/2}^{1/2}+\bar{e}^{1/2}\delta_{q+1}^{1/2}\frac{M_0}{4}+\frac{M_0}{4}\lambda_{q+1}^{-1/14+26\alpha}(\|\!|\mathring{R}_q|\!\|_{L^1,3m}^3+1)\\\leqslant&\frac{M_0}{4}(6^q2mL^2)^{\frac{1}{2}6^q}+M_0\bar{e}^{1/2}\delta_{q+1}^{1/2}+\frac{M_0}{4}\lambda_{q+1}^{-1/14+26\alpha}(6^q\cdot12mL^2)^{3(6^q)}\\\leqslant&\frac{M_0}{2}(6^q\cdot12mL^2)^{3(6^q)}+M_0\bar{e}^{1/2}\delta_{q+1}^{1/2}.\end{aligned} \tag{3.49}$$

Thus combined with (3.24) and (3.8) we deduce that (3.8) holds at level of $q+1$.

As the next step, we shall verify the first bound in (3.9). The $C^1_{t,x}$-bound follows similarly as in [HZZ23] but with an explicit dependence on $\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}$. Hence, we omit most details for these estimates. Using (3.33)-(3.34) and (A.7) we have

$$\|w_{q+1}^{(p)}\|_{C^1_{t,x}}\lesssim\ell^{-13}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^2r_\perp^{-1}r_\parallel^{-1/2}\lambda_{q+1}^2, \tag{3.50}$$

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x}}\lesssim\ell^{-25}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4r_\parallel^{-3/2}\lambda_{q+1}^2, \tag{3.51}$$

and

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{C^1_{t,x}} &\leqslant \frac{1}{\mu}\sum_{\xi\in\Lambda}[\|a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\|_{C_tW^{1+\alpha,p}}+\|a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\|_{C_t^1W^{\alpha,p}}]\\ &\lesssim \ell^{-21}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3 r_\perp^{-1}r_\parallel^{-2}\lambda_{q+1}^{1+\alpha},\end{aligned}\tag{3.52}$$

where we chose p large enough and applied the Sobolev embedding in the first inequality. This was needed because $\mathbb{P}\mathbb{P}_{\neq 0}$ is not a bounded operator on C^0. In the last inequality in (3.52), we used interpolation and an extra λ_{q+1}^{α} appeared. Combining (3.50), (3.51), (3.52) with (3.19) we obtain

$$\begin{aligned}\|v_{q+1}\|_{C^1_{t,x}} &\leqslant \|v_\ell\|_{C^1_{t,x}}+\|w_{q+1}\|_{C^1_{t,x}}\\ &\leqslant (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4\left(C\lambda_{q+1}^{26\alpha+22/7}+C\lambda_{q+1}^{50\alpha+20/7}+C\lambda_{q+1}^{43\alpha+3}\right)+\|v_q\|_{C^1_{[t-1,t+1],x}}.\end{aligned}$$

Thus,

$$\begin{aligned}|\!|\!| v_{q+1}|\!|\!|_{C^1_{t,x},m} &\lesssim (|\!|\!|\mathring{R}_q|\!|\!|_{L^1,4m}^4+1)\lambda_{q+1}^{26\alpha+22/7}+|\!|\!| v_q|\!|\!|_{C^1_{t,x},m}\\ &\leqslant \lambda_{q+1}^{23/7}(6^q\cdot 16mL^2)^{4(6^q)},\end{aligned}$$

which implies the first inequality in (3.9) holds true on the level $q+1$.

Similarly, we have

$$\begin{aligned}\|w_{q+1}^{(p)}\|_{C^2_{t,x}} &\lesssim \ell^{-19}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3 r_\perp^{-1}r_\parallel^{-1/2}\lambda_{q+1}^2\left(1+\frac{r_\perp\mu}{r_\parallel}\right)^2\\ &\lesssim \lambda_{q+1}^{38\alpha+36/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3,\end{aligned}\tag{3.53}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{C^2_{t,x}} &\lesssim \ell^{-31}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^5 r_\parallel^{-3/2}\Big(\lambda_{q+1}\frac{r_\perp\mu}{r_\parallel}\Big)^2\\ &\lesssim \lambda_{q+1}^{34/7+62\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^5,\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{C^2_{t,x}} &\lesssim \ell^{-27}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^5 r_\perp^{-2}r_\parallel^{-1}\lambda_{q+1}^{2+\alpha}\mu^{-1}\left(1+\frac{r_\perp\mu}{r_\parallel}\right)^2\\ &\lesssim (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^5\lambda_{q+1}^{55\alpha+5}.\end{aligned}\tag{3.55}$$

Hence, we obtain

$$\|v_{q+1}\|_{C^2_{t,x}} \lesssim (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^5\lambda_{q+1}^{38\alpha+36/7}+\|v_q\|_{C^2_{t,x}},$$

which implies

$$\begin{aligned}|\!|\!| v_{q+1}|\!|\!|_{C^2_{t,x},m} &\lesssim (|\!|\!|\mathring{R}_q|\!|\!|_{C_{[t-1,t+1]}L^1,5m}^5+1)\lambda_{q+1}^{38\alpha+36/7}+|\!|\!| v_q|\!|\!|_{C^2_{t,x},m}\\ &\leqslant \lambda_{q+1}^{37/7}\cdot(6^q\cdot 20mL^2)^{5\cdot 6^q}.\end{aligned}$$

Hence, the second inequality in (3.9) holds true on the level $q+1$.

We conclude this part with further estimates of the perturbations $w_{q+1}^{(p)}$, $w_{q+1}^{(c)}$ and $w_{q+1}^{(t)}$, which will be used below in order to bound the Reynolds stress $\mathring{R}_{q+1}$ and to establish (3.14). These estimates follow similarly as in [HZZ23] with an explicit dependence on $\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}$. We omit most details and derive the following estimates: by using (3.19), (3.33), (3.34) and (A.7)

$$\begin{aligned}
&\|w_{q+1}^{(p)}+w_{q+1}^{(c)}\|_{C_tW^{1,p}}\\
&\leqslant \sum_{\xi\in\Lambda}\|\mathrm{curl}\,\mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tW^{1,p}}\\
&\lesssim r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\left(\ell^{-25}\lambda_{q+1}^{-2}+\ell^{-19}\lambda_{q+1}^{-1}+\ell^{-13}+\ell^{-2}\lambda_{q+1}\right)(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4\\
&\lesssim r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\ell^{-2}\lambda_{q+1}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4,
\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
\|w_{q+1}^{(t)}\|_{C_tW^{1,p}}&\lesssim \frac{1}{\mu}r_\perp^{2/p-2}r_\parallel^{1/p-1}\left(\ell^{-15}+\ell^{-4}\lambda_{q+1}\right)(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{5/2}\\
&\lesssim r_\perp^{2/p-2}r_\parallel^{1/p-1}\ell^{-4}\lambda_{q+1}^{-2/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{5/2}.
\end{aligned} \tag{3.57}$$

We then obtain for $p=\frac{32}{32-7\alpha}$

$$\begin{aligned}
\|w_{q+1}\|_{C_tW^{1,p}}&\lesssim (r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\ell^{-2}\lambda_{q+1}+r_\perp^{2/p-2}r_\parallel^{1/p-1}\ell^{-4}\lambda_{q+1}^{-2/7})(1+\|R_q\|_{C_tL^1})^4\\
&\lesssim (\lambda_{q+1}^{-2/7+9\alpha}+\lambda_{q+1}^{-1/7+5\alpha})(1+\|R_q\|_{C_tL^1})^4.
\end{aligned} \tag{3.58}$$

Taking expectation we obtain

$$\mathbf{E}\|w_{q+1}\|_{C_tW^{1,p}}^r\lesssim \lambda_{q+1}^{-r/7+5r\alpha}(1+\mathbf{E}\|R_q\|_{C_tL^1}^{4r})\lesssim \lambda_{q+1}^{-r/7+5r\alpha}(6^q\cdot 16rL^2)^{6^q\cdot 4r}\leqslant \lambda_{q+1}^{-r/7+6r\alpha},$$

where we used $(6^q\cdot 16rL^2)^{6^q}\leqslant\lambda_q$, $\lambda_q^4\leqslant\lambda_{q+1}^{\alpha}$ and we chose a large enough to absorb the constant. Moreover, by (3.15) we obtain

$$\mathbf{E}\|v_\ell-v_q\|_{C_tW^{1,p}}^r\leqslant \ell^r\lambda_q^{6r}\leqslant\lambda_{q+1}^{-\alpha r/2}.$$

Now, (3.14) follows.

3.1.5. *Proof of* (3.12). We define

$$\delta E(t):=\Big|e(t)(1-\delta_{q+2})-\mathbf{E}\|v_{q+1}(t)+z_{q+1}(t)\|_{L^2}^2\Big|.$$

Proposition 3.5. *It holds for $t\in\mathbb{R}$*

$$\delta E(t)\leqslant\frac{1}{4}\delta_{q+2}e(t). \tag{3.59}$$

Proof. By definition of γ_q we find

$$\begin{aligned}
\delta E(t)\leqslant{}&\mathbf{E}\big|\|w_{q+1}^{(p)}\|_{L^2}^2-3\gamma_q(2\pi)^3\big|+\mathbf{E}\|w_{q+1}^{(c)}+w_{q+1}^{(t)}\|_{L^2}^2+2\mathbf{E}\|(v_\ell+z_{q+1})(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}\\
&+2\mathbf{E}\|(v_\ell+z_{q+1})w_{q+1}^{(p)}\|_{L^1}+2\mathbf{E}\|w_{q+1}^{(p)}(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}\\
&+\mathbf{E}\|v_\ell-v_q+z_{q+1}-z_q\|_{L^2}^2+2\mathbf{E}\|(v_\ell-v_q+z_{q+1}-z_q)(v_q+z_q)\|_{L^1},
\end{aligned} \tag{3.60}$$

which shall be estimated. Let us begin with the bound of the first term on the right hand side of (3.60). We use (3.36) and the fact that $\mathring{R}_\ell$ is traceless to deduce for $t\in\mathbb{R}$

$$|w_{q+1}^{(p)}|^2-3\gamma_q=6\sqrt{\ell^2+|\mathring{R}_\ell|^2}+3(\gamma_\ell-\gamma_q)+\sum_{\xi\in\Lambda}a_{(\xi)}^2P_{\neq 0}|W_{(\xi)}|^2,$$

hence

$$\begin{aligned}&\mathbf{E}|\|w_{q+1}^{(p)}\|_{L^2}^2-3\gamma_{q+1}(2\pi)^3|\\&\leqslant 6\cdot(2\pi)^3\ell+6\mathbf{E}\|\mathring{R}_\ell\|_{L^1}+3\cdot(2\pi)^3|\gamma_\ell-\gamma_q|+\mathbf{E}\sum_{\xi\in\Lambda}\Big|\int a_{(\xi)}^2P_{\neq0}|W_{(\xi)}|^2\Big|.\end{aligned}\tag{3.61}$$

Here we estimate each term separately. Using (3.20) we find

$$6\cdot(2\pi)^3\ell\leqslant 6\cdot(2\pi)^3\lambda_{q+1}^{-3\alpha/2}\leqslant\frac{1}{48}\lambda_{q+1}^{-2\beta b}e(t)\leqslant\frac{1}{48}\delta_{q+2}e(t),$$

which requires $2\beta b<3\alpha/2$ and choosing a large to absorb the constant. Using (3.10) on $\mathring{R}_q$ and $\operatorname{supp}\varphi_\ell\subset[0,\ell]$ we know for $t\in\mathbb{R}$

$$6\mathbf{E}\|\mathring{R}_\ell(t)\|_{L^1}\leqslant\frac{1}{8}\delta_{q+2}e(t).$$

For the third term in (3.61) we use (3.3) and (3.7), (3.15) to have for $0\leqslant\delta\leqslant1/6$

$$\begin{aligned}3\cdot(2\pi)^3|\gamma_\ell-\gamma_q|&\lesssim\ell\|e'\|_{C^0_{t-1}}+\ell\mathbf{E}\|v_q\|_{C^1_{t-1,x}}(\|v_q\|_{C_{t-1}L^2}+\|z_q\|_{C_{t-1}L^2})\\&\quad+\ell^{1/2-\delta}\mathbf{E}\|z_q\|_{C^{1/2-\delta}_{t-1}L^2}(\|v_q\|_{C_{t-1}L^2}+\|z_q\|_{C_{t-1}L^2})\\&\lesssim\ell\tilde{e}+\ell\lambda_q^4(\bar{e}^{1/2}M_0+L)+\ell^{1/2-\delta}L(M_0\bar{e}^{1/2}+L)\\&\lesssim\lambda_{q+1}^{-3\alpha/2}\tilde{e}+\lambda_{q+1}^{-\alpha}(M_0\bar{e}^{1/2}+L)+\lambda_{q+1}^{-\frac{3\alpha}{2}(1/2-\delta)}L(M_0\bar{e}^{1/2}+L)\\&\lesssim\lambda_{q+1}^{-\alpha/2}(\bar{e}+\tilde{e}+L^2)\leqslant\frac{1}{48}\delta_{q+2}e(t),\end{aligned}$$

where we choose a large to absorb the constant.

For the last term in (3.61) we apply (3.33), (3.34) and $\|a_{(\xi)}^2\|_{C^N}\lesssim\|a_{(\xi)}\|_{C^0}\|a_{(\xi)}\|_{C^N}$ to bound

$$\begin{aligned}&\sum_{\xi\in\Lambda}\Big|\int_{\mathbb{T}^3}a_{(\xi)}^2\mathbb{P}_{\neq0}|W_{(\xi)}|^2\mathrm{d}x\Big|=\sum_{\xi\in\Lambda}\Big|\int_{\mathbb{T}^3}a_{(\xi)}^2P_{\geqslant r_\perp\lambda_{q+1}/2}|W_{(\xi)}|^2\mathrm{d}x\Big|\\&=\sum_{\xi\in\Lambda}\Big|\int_{\mathbb{T}^3}|\nabla|^Na_{(\xi)}^2|\nabla|^{-N}\mathbb{P}_{\geqslant r_\perp\lambda_{q+1}/2}|W_{(\xi)}|^2\mathrm{d}x\Big|\\&\lesssim\|a_{(\xi)}^2\|_{C^N}(r_\perp\lambda_{q+1})^{-N}\||W_{(\xi)}|^2\|_{L^2}\lesssim\ell^{-6N-9}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{N+3/2}(r_\perp\lambda_{q+1})^{-N}r_\perp^{-1}r_\parallel^{-\frac12}\\&\leqslant\ell^{-6N-9}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{N+3/2}\lambda_{q+1}^{\frac{8-N}{7}}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{E}\sum_{\xi\in\Lambda}\Big|\int_{\mathbb{T}^3}a_{(\xi)}^2\mathbb{P}_{\neq0}|W_{(\xi)}|^2\mathrm{d}x\Big|&\lesssim\lambda_{q+1}^{(12N+18)\alpha+\frac{8-N}{7}}(6^q\cdot4(N+3/2)L^2)^{6^q(N+3/2)}\\&\leqslant\lambda_{q+1}^{127\alpha-1/7}\leqslant\frac{1}{48}\delta_{q+2}e(t).\end{aligned}$$

Here we may choose $N=9$, $a>[252L^2]^{3c}$ such that $(6^q\cdot4(N+3/2)L^2)^{6^q(N+3/2)}<\lambda_q^4<\lambda_{q+1}^\alpha$ and use $2\beta b<1/7-127\alpha$. This completes the bound for (3.61).

Going back to (3.60), it remains to control

$$\begin{aligned}&\mathbf{E}\|w_{q+1}^{(c)}+w_{q+1}^{(t)}\|_{L^2}^2+2\mathbf{E}\|(v_\ell+z_{q+1})(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}\\&\quad+2\mathbf{E}\|(v_\ell+z_{q+1})w_{q+1}^{(p)}\|_{L^1}+2\mathbf{E}\|w_{q+1}^{(p)}(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}\end{aligned}$$

$$+\mathbf{E}\|v_\ell-v_q+z_{q+1}-z_q\|_{L^2}^2+2\mathbf{E}\|(v_\ell-v_q+z_{q+1}-z_q)(v_q+z_q)\|_{L^1}.$$

Using the estimates (3.44), (3.45) and (3.19) we have

$$\begin{aligned}\mathbf{E}\|w_{q+1}^{(c)}+w_{q+1}^{(t)}\|_{L^2}^2 &\lesssim (1+\|\mathring{R}_q\|_{L^1,6}^6)\lambda_{q+1}^{76\alpha-4/7}+(1+\|\mathring{R}_q\|_{L^1,2}^2)\lambda_{q+1}^{16\alpha-2/7}\\ &\lesssim (6^q\cdot 24L^2)^{6^{q+1}}\lambda_{q+1}^{16\alpha-2/7}\leqslant \frac{\delta_{q+2}}{48}e(t),\end{aligned}$$

where we use a similar bound for the parameters as above. Next, we use (3.24) together with (3.42) to have

$$\begin{aligned}&\mathbf{E}[2\|(v_\ell+z_{q+1})(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}+2\|w_{q+1}^{(p)}(w_{q+1}^{(c)}+w_{q+1}^{(t)})\|_{L^1}]\lesssim (M_0\bar{e}^{1/2}+L)\|w_{q+1}^{(c)}+w_{q+1}^{(t)}\|_{L^2,2}\\ &\quad\lesssim (M_0\bar{e}^{1/2}+L)(1+\|\mathring{R}_q\|_{L^1,6}^3)\lambda_{q+1}^{8\alpha-1/7}\\ &\quad\leqslant \frac{1}{48}\lambda_{q+1}^{-2\beta b}\leqslant \frac{\delta_{q+2}}{48}e(t),\end{aligned}$$

with similar arguments for the second last inequality as above. We employ (3.19), (3.15), (3.43) as well as $\|v_\ell\|_{C^1_{t,x}}\leqslant \|v_q\|_{C^1_{[t-1,t+1],x}}$ to have for every $\varepsilon>0$

$$\begin{aligned}2\mathbf{E}\|(v_\ell+z_{q+1})w_{q+1}^{(p)}\|_{L^1} &\lesssim (\|v_\ell\|_{L^\infty,2}+\|z_q\|_{L^\infty,2})\|w_{q+1}^{(p)}\|_{L^1,2}+\|z_{q+1}-z_q\|_{L^4,2}\|w_{q+1}^{(p)}\|_{L^{4/3},2}\\ &\lesssim (\lambda_q^4+\lambda_{q+1}^{\alpha/8}L)\ell^{-2}\delta_{q+2}^{1/2}r_\perp^{1-\varepsilon}r_\parallel^{\frac12(1-\varepsilon)}+\lambda_{q+1}^{-\frac{\alpha}{8}(\frac14-\delta)}L\ell^{-2}\delta_{q+2}^{1/2}r_\perp^{1/2}r_\parallel^{1/4}\\ &\lesssim L\lambda_{q+1}^{5\alpha-\frac87(1-\varepsilon)}+L\lambda_{q+1}^{4\alpha-\frac47}\leqslant \frac{1}{96}\lambda_{q+1}^{-2\beta b}\leqslant \frac{\delta_{q+2}}{96}e(t).\end{aligned}\tag{3.62}$$

For the last terms, we apply (3.23) and obtain for $0<\delta<1/9$

$$\begin{aligned}&\mathbf{E}\|v_\ell-v_q+z_{q+1}-z_q\|_{L^2}^2+2\mathbf{E}\|(v_\ell-v_q+z_{q+1}-z_q)(v_q+z_q)\|_{L^1}\\ &\quad\lesssim \|v_\ell-v_q\|_{L^2,2}(\|v_q\|_{L^2,2}+\|z_q\|_{L^2,2}+1)+\mathbf{E}\|z_{q+1}-z_q\|_{L^2}^2\\ &\qquad+\|z_{q+1}-z_q\|_{L^2,2}(\|v_q\|_{L^2,2}+\|z_q\|_{L^2,2})\\ &\quad\lesssim \ell\lambda_q^4(M_0\bar{e}^{1/2}+L)+\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(M_0\bar{e}^{1/2}+L)\\ &\quad\leqslant \lambda_{q+1}^{-\alpha/2}(M_0\bar{e}^{1/2}+L)+\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(M_0\bar{e}^{1/2}+L)\\ &\quad\leqslant \frac{1}{96}\lambda_{q+1}^{-2\beta b}\leqslant \frac{\delta_{q+2}}{96}e(t).\end{aligned}\tag{3.63}$$

Here, we choose again a large enough to absorb the extra constant.

Combining the above estimates, (3.12) follows on the level $q+1$. □

3.1.6. *Definition of the Reynolds stress* $\mathring{R}_{q+1}$. Subtracting from (3.5) at level $q+1$ the system (3.21), we obtain

$$
\begin{aligned}
&\mathrm{div}\mathring{R}_{q+1} - \nabla p_{q+1} \\
&= \underbrace{-\Delta w_{q+1} + \partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \mathrm{div}((v_\ell + z_\ell) \otimes w_{q+1} + w_{q+1} \otimes (v_\ell + z_\ell))}_{\mathrm{div}(R_{\mathrm{lin}})+\nabla p_{\mathrm{lin}}} \\
&\quad + \underbrace{\mathrm{div}\left((w^{(c)}_{q+1} + w^{(t)}_{q+1}) \otimes w_{q+1} + w^{(p)}_{q+1} \otimes (w^{(c)}_{q+1} + w^{(t)}_{q+1})\right)}_{\mathrm{div}(R_{\mathrm{cor}})+\nabla p_{\mathrm{cor}}} \\
&\quad + \underbrace{\mathrm{div}(w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \mathring{R}_\ell) + \partial_t w^{(t)}_{q+1}}_{\mathrm{div}(R_{\mathrm{osc}})+\nabla p_{\mathrm{osc}}} \\
&\quad + \underbrace{\mathrm{div}\left(v_{q+1}\otimes z_{q+1} - v_{q+1}\otimes z_\ell + z_{q+1}\otimes v_{q+1} - z_\ell\otimes v_{q+1} + z_{q+1}\otimes z_{q+1} - z_\ell\otimes z_\ell\right) + (z_\ell - z_{q+1})}_{\mathrm{div}(R_{\mathrm{com1}})+\nabla p_{\mathrm{com1}}} \\
&\quad + \mathrm{div}(R_{\mathrm{com}}) - \nabla p_\ell.
\end{aligned}
\tag{3.64}
$$

By using $\mathcal{R}$ introduced in Section 2 we define

$$R_{\mathrm{lin}} := -\mathcal{R}\Delta w_{q+1} + \mathcal{R}\partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1}) + (v_\ell + z_\ell)\mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes}(v_\ell + z_\ell),$$

$$R_{\mathrm{cor}} := (w^{(c)}_{q+1} + w^{(t)}_{q+1})\mathring{\otimes} w_{q+1} + w^{(p)}_{q+1}\mathring{\otimes}(w^{(c)}_{q+1} + w^{(t)}_{q+1}),$$

$$R_{\mathrm{com1}} := v_{q+1}\mathring{\otimes} z_{q+1} - v_{q+1}\mathring{\otimes} z_\ell + z_{q+1}\mathring{\otimes} v_{q+1} - z_\ell\mathring{\otimes} v_{q+1} + z_{q+1}\mathring{\otimes} z_{q+1} - z_\ell\mathring{\otimes} z_\ell + \mathcal{R}(z_\ell - z_{q+1}).$$

In order to define the remaining oscillation error from the third line in (3.64), we apply (3.36) and (3.39) to obtain

$$
\begin{aligned}
&\mathrm{div}(w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \mathring{R}_\ell) + \partial_t w^{(t)}_{q+1} \\
&\quad = \sum_{\xi\in\Lambda} \mathrm{div}\left(a^2_{(\xi)}\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right) + \nabla\rho + \partial_t w^{(t)}_{q+1} \\
&\quad = \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(\nabla a^2_{(\xi)}\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right) + \nabla\rho + \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(a^2_{(\xi)}\mathrm{div}(W_{(\xi)} \otimes W_{(\xi)})\right) + \partial_t w^{(t)}_{q+1} \\
&\quad = \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(\nabla a^2_{(\xi)}\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right) + \nabla\rho + \nabla p_1 - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\left(\partial_t a^2_{(\xi)}(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi)\right)
\end{aligned}
$$

Therefore,

$$R_{\mathrm{osc}} := \sum_{\xi\in\Lambda} \mathcal{B}\left(\nabla a^2_{(\xi)}, \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathcal{R}\left(\partial_t a^2_{(\xi)}(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi)\right) =: R^{(x)}_{\mathrm{osc}} + R^{(t)}_{\mathrm{osc}},$$

with $\mathcal{B}$ given in Section 2.

Finally we define the Reynolds stress on the level $q+1$ by

$$\mathring{R}_{q+1} := R_{\mathrm{lin}} + R_{\mathrm{cor}} + R_{\mathrm{osc}} + R_{\mathrm{com}} + R_{\mathrm{com1}}.$$

3.1.7. *Inductive estimate for* $\mathring{R}_{q+1}$. To conclude the proof of Proposition 3.3, we shall verify the estimates in (3.10) and (3.11). To this end, we estimate each term in the definition of $\mathring{R}_{q+1}$ separately.

In the following we choose $p = \frac{32}{32-7\alpha} > 1$ so that it holds in particular that $r_\perp^{2/p-2} r_\parallel^{1/p-1} \leqslant \lambda_{q+1}^\alpha$. For the linear error we obtain

$$\begin{aligned}\|R_{\rm lin}\|_{C_tL^p} &\lesssim \|\mathcal{R}\Delta w_{q+1}\|_{C_tL^p} + \|\mathcal{R}\partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_tL^p}\\ &\quad + \|(v_\ell + z_\ell)\mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes}(v_\ell+z_\ell)\|_{C_tL^p}\\ &\lesssim \|w_{q+1}\|_{C_tW^{1,p}} + \sum_{\xi\in\Lambda}\|\partial_t \mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tL^p}\\ &\quad + (\|v_q\|_{C_{[t-1,t+1]}L^\infty} + \|z_q\|_{C_{[t-1,t+1]}L^\infty})\|w_{q+1}\|_{C_tL^p},\end{aligned}$$

where by (A.7) and (3.33)

$$\begin{aligned}\sum_{\xi\in\Lambda}\|\partial_t \mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_tL^p} &\leqslant \sum_{\xi\in\Lambda}\big(\|a_{(\xi)}\|_{C_tC^1_x}\|\partial_t V_{(\xi)}\|_{C_tW^{1,p}} + \|\partial_t a_{(\xi)}\|_{C_tC^1_x}\|V_{(\xi)}\|_{C_tW^{1,p}}\big)\\ &\lesssim (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^2\ell^{-13} r_\perp^{2/p} r_\parallel^{1/p-3/2}\mu\\ &\qquad + (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3\ell^{-19} r_\perp^{2/p-1} r_\parallel^{1/p-1/2}\lambda_{q+1}^{-1}.\end{aligned}$$

In view of (3.58) as well as (3.46), we deduce

$$\begin{aligned}\|R_{\rm lin}\|_{C_tL^p} &\lesssim \Big(\lambda_{q+1}^{5\alpha-1/7} + \lambda_{q+1}^{9\alpha-2/7} + \lambda_{q+1}^{27\alpha-1/7} + \lambda_{q+1}^{39\alpha-15/7}\Big)(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4\\ &\quad + (\|v_q\|_{C_{[t-1,t+1]}L^\infty} + \|z_q\|_{C_{[t-1,t+1]}L^\infty})\lambda_{q+1}^{5\alpha-8/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3.\end{aligned}$$

The corrector error is estimated using (3.43), (3.44), (3.45) as

$$\begin{aligned}\|R_{\rm cor}\|_{C_tL^p} &\leqslant \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}}\|w_{q+1}\|_{C_tL^{2p}} + \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}}\|w_{q+1}^{(p)}\|_{C_tL^{2p}}\\ &\lesssim \Big(\ell^{-19} r_\perp^{1/p} r_\parallel^{1/(2p)-3/2} + \ell^{-4} r_\perp^{1/p-1} r_\parallel^{1/(2p)-2}\lambda_{q+1}^{-1}\Big)\ell^{-2} r_\perp^{1/p-1} r_\parallel^{1/(2p)-1/2}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{7/2}\\ &\quad + \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|^2_{C_tL^{2p}}\\ &\lesssim \lambda_{q+1}^{-1/7+13\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{7/2} + \lambda_{q+1}^{-2/7+17\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^6.\end{aligned}$$

We continue with the oscillation error $R_{\rm osc}$. Using (2.1), (2.2) and the definition of $W_{(\xi)}$ we have

$$\begin{aligned}\|R_{\rm osc}^{(x)}\|_{C_tL^p} &\leqslant \sum_{\xi\in\Lambda}\big\|\mathcal{B}\big(\nabla a_{(\xi)}^2, \mathbb{P}_{\geqslant r_\perp\lambda_{q+1}/2}(W_{(\xi)}\otimes W_{(\xi)})\big)\big\|_{C_tL^p}\\ &\lesssim \|\nabla a_{(\xi)}^2\|_{C_tC^1}\|\mathcal{R}(W_{(\xi)}\otimes W_{(\xi)})\|_{C_tL^p} \lesssim \|\nabla a_{(\xi)}^2\|_{C_tC^1}\frac{\|W_{(\xi)}\otimes W_{(\xi)}\|_{C_tL^p}}{r_\perp\lambda_{q+1}}\\ &\lesssim \|\nabla a_{(\xi)}^2\|_{C_tC^1}\frac{\|W_{(\xi)}\|^2_{C_tL^{2p}}}{r_\perp\lambda_{q+1}} \lesssim \ell^{-21}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4 r_\perp^{2/p-2} r_\parallel^{1/p-1}(r_\perp^{-1}\lambda_{q+1}^{-1})\\ &\lesssim \lambda_{q+1}^{43\alpha-1/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4.\end{aligned}$$

For the second term $R_{\rm osc}^{(t)}$ we use Fubini's theorem to integrate along the orthogonal directions of $\phi_{(\xi)}$ and $\psi_{(\xi)}$ and apply (A.7) to deduce

$$\begin{aligned}\|R_{\rm osc}^{(t)}\|_{C_tL^p} &\leqslant \mu^{-1}\sum_{\xi\in\Lambda}\|\partial_t a_{(\xi)}^2\|_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{C_tL^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}}\\&\lesssim (\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{5/2}\mu^{-1}\ell^{-15}r_\perp^{2/p-2}r_\parallel^{1/p-1}\lesssim \lambda_{q+1}^{31\alpha-9/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^{5/2}.\end{aligned}$$

In view of the standard mollification estimates and (3.7) it holds

$$\begin{aligned}\|R_{\rm com}\|_{C_tL^1} &\lesssim \ell\|v_q\|_{C^1_{[t-1,t+1],x}}(\|v_q\|_{C_{[t-1,t+1]}L^2}+\|z_q\|_{C_{[t-1,t+1]}L^2})\\&\quad+\ell^{1/2-\delta}(\|z_q\|_{C^{1/2-\delta}_{[t-1,t+1]}L^2}+\|z_q\|_{C_{[t-1,t+1]}H^{1-\delta}})(\|v_q\|_{C_{[t-1,t+1]}L^2}+\|z_q\|_{C_{[t-1,t+1]}L^2}),\end{aligned}$$

where $\delta<\frac{1}{12}$. Finally, we use (3.6) to obtain

$$\begin{aligned}\|R_{\rm com1}\|_{C_tL^1} &\lesssim (\|v_{q+1}\|_{C_tL^2}+\|z_{q+1}\|_{C_{[t-1,t+1]}L^2}+\|z_q\|_{C_{[t-1,t+1]}L^2}+1)\|z_\ell-z_{q+1}\|_{C_tL^2}\\&\leqslant (\ell^{\frac12-\delta}\|z\|_{C^{1/2-\delta}_{[t-1,t+1]}L^2}+\|z\|_{C_{[t-1,t+1]}H^{1-\delta}}\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)})(\|v_{q+1}\|_{C_tL^2}+\|z\|_{C_{[t-1,t+1]}L^2}+1)\\&\leqslant M_0|\!|\!|z|\!|\!|\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(\|v_{q+1}\|_{C_tL^2}+\|z\|_{C_{[t-1,t+1]}L^2}+1).\end{aligned}$$

Here $|\!|\!|z|\!|\!|=\|z\|_{C^{1/2-\delta}_{[t-1,t+1]}L^2}+\|z\|_{C_{[t-1,t+1]}H^{1-\delta}}$. Summing up all the above estimates, we obtain

$$\begin{aligned}\|\mathring{R}_{q+1}\|_{C_tL^1} &\lesssim \lambda_{q+1}^{43\alpha-1/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^4+\lambda_{q+1}^{-2/7+17\alpha}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^6\\&\quad+(\|v_q\|_{C_{[t-1,t+1]}L^\infty}+\|z_q\|_{C_{[t-1,t+1]}L^\infty})\lambda_{q+1}^{5\alpha-8/7}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+1)^3\\&\quad+|\!|\!|z|\!|\!|\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(\|v_{q+1}\|_{C_tL^2}+\|z\|_{C_{[t-1,t+1]}L^2}+1)\\&\quad+(\ell\|v_q\|_{C^1_{[t-1,t+1],x}}+\ell^{\frac12-\delta}|\!|\!|z|\!|\!|)(\|v_q\|_{C_{[t-1,t+1]}L^2}+\|z\|_{C_{[t-1,t+1]}L^2}).\end{aligned}$$

Thus taking the r-th moment, using Hölder's inequality and (3.15), (3.11), (3.6) we obtain

$$\begin{aligned}|\!|\!|\mathring{R}_{q+1}|\!|\!|_{L^1,r} &\lesssim \lambda_{q+1}^{43\alpha-1/7}(|\!|\!|\mathring{R}_q|\!|\!|^4_{L^1,4r}+1)+\lambda_{q+1}^{-2/7+17\alpha}(|\!|\!|\mathring{R}_q|\!|\!|^6_{L^1,6r}+1)\\&\quad+(|\!|\!|v_q|\!|\!|_{C^1_{t,x},2r}+|\!|\!|z_q|\!|\!|_{L^\infty,2r})\lambda_{q+1}^{5\alpha-8/7}(|\!|\!|\mathring{R}_q|\!|\!|^3_{L^1,6r}+1)\\&\quad+\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(|\!|\!|v_{q+1}|\!|\!|_{L^2,2r}+(2r-1)^{1/2}L)L(2r-1)^{1/2}\\&\quad+(\ell|\!|\!|v_q|\!|\!|_{C^1_{t,x},2r}+\ell^{\frac12-\delta}L(2r-1)^{1/2})(|\!|\!|v_q|\!|\!|_{L^2,2r}+L(2r-1)^{1/2})\\&\lesssim \lambda_{q+1}^{43\alpha-1/7}(6^q\cdot 24L^2r)^{6^{q+1}}+\lambda_{q+1}^{5\alpha-8/7}(6^q\cdot 24L^2r)^{3(6^q)}(\lambda_q^4+\lambda_{q+1}^{\alpha/8}L)\\&\quad+(M_0\bar{e}^{1/2}+L)(L\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}+\ell\lambda_q^4+\ell^{\frac12-\delta}L)\\&\lesssim \lambda_{q+1}^{43\alpha-1/7}\lambda_q^6+(\lambda_{q+1}^{5\alpha-8/7}\lambda_q^3+\lambda_{q+1}^{-3\alpha/2})(\lambda_q^4+\lambda_{q+1}^{\alpha/8})+\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}\\&\leqslant \frac{1}{48}\delta_{q+3}\underline{e}.\end{aligned}$$

Here in the third inequality we used $(6^q\cdot 24L^2r)^{6^q}\leqslant\lambda_q$ and in the last inequality we used $43\alpha<1/14$ and $6/b+2\beta b^2<1/14$ and $\alpha>40\beta b^2$ and $\alpha b>32/7$. Hence (3.10) holds on the level $q+1$.

Similarly, for $m \geqslant 1$ we use the first inequality as above with r replaced by m and instead of (3.7) we use (3.9) and (3.8) to obtain

$$
\begin{aligned}
\interleave \mathring{R}_{q+1} \interleave_{L^1,m} &\lesssim \lambda_{q+1}^{43\alpha-1/7}(\interleave \mathring{R}_q \interleave_{L^1,4m}^4 + 1) + \lambda_{q+1}^{-2/7+17\alpha}(\interleave \mathring{R}_q \interleave_{L^1,6m}^6 + 1) \\
&\quad + (\interleave v_q \interleave_{C^1_{t,x},2m} + \interleave z_q \interleave_{L^\infty,2m})\lambda_{q+1}^{5\alpha-8/7}(\interleave \mathring{R}_q \interleave_{L^1,6m}^3 + 1) \\
&\quad + \lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}(\interleave v_{q+1} \interleave_{L^2,2m} + (2m)^{1/2}L)(2m)^{1/2}L \\
&\quad + \Big(\ell \interleave v_q \interleave_{C^1_{t,x},2m} + \ell^{\frac{1}{2}-\delta}L(2m)^{1/2}\Big)(\interleave v_q \interleave_{L^2,2m} + L(2m)^{1/2}) \\
&\lesssim \lambda_{q+1}^{43\alpha-1/7}(6^q \cdot 24mL^2)^{6^{q+1}} \\
&\quad + \lambda_{q+1}^{5\alpha-8/7}(6^q \cdot 24mL^2)^{3(6^q)}\Big(\lambda_q^{23/7}(6^{q-1} \cdot 32mL^2)^{4(6^{q-1})} + \lambda_{q+1}^{\alpha/8}(2m)^{1/2}L\Big) \\
&\quad + (2m)^{1/2}L\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)}\Big((6^q \cdot 24mL^2)^{3(6^q)} + \bar{e}^{1/2} + (2m)^{1/2}L\Big) \\
&\quad + \Big((6^q \cdot 4mL^2)^{3(6^{q-1})} + \bar{e}^{1/2} + (2m)^{1/2}L\Big) \\
&\quad \times \Big(\ell\lambda_q^{23/7}(6^{q-1} \cdot 32mL^2)^{4(6^{q-1})} + \ell^{\frac{1}{2}-\delta}L(2m)^{1/2}\Big) \\
&\leqslant (6^{q+1} \cdot 4mL^2)^{6^{q+1}}.
\end{aligned}
$$

The proof of Proposition 3.3 is therefore complete.

4. Stationary solutions to the stochastic Navier–Stokes system

We recall that the trajectory space is $\mathcal{T} = C(\mathbb{R}; L^2_\sigma) \times C(\mathbb{R}; L^2_\sigma)$ and the corresponding shifts S_t, $t \in \mathbb{R}$, on trajectories are given by

$$S_t(u, B)(\cdot) = (u(\cdot + t), B(\cdot + t) - B(t)), \quad t \in \mathbb{R}, \quad (u, B) \in \mathcal{T}.$$

The notion of stationary solution was introduced in Definition 1.2. Our first result of this section is existence of stationary solutions as limits of ergodic averages of solutions constructed in the previous section. This in particular implies their non-uniqueness.

Theorem 4.1. *Let u be a solution obtained in Theorem 3.4 with $e(t) = K$ for some $K \geqslant 8 \cdot 48L^2 r$ and satisfying (3.16) and (3.18) with given $\varepsilon > 0$ and $r > 1$. Then there exists a sequence $T_n \to \infty$ and a stationary solution $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}), \tilde{u}, \tilde{B})$ to (1.4) such that*

$$\frac{1}{T_n}\int_0^{T_n} \mathcal{L}[S_t(u, B)]\mathrm{d}t \to \mathcal{L}[\tilde{u}, \tilde{B}]$$

weakly in the sense of probability measures on $\mathcal{T}$ as $n \to \infty$. Moreover, it holds true that

$$\tilde{\mathbf{E}}\|\tilde{u}\|_{L^2}^2 = K, \tag{4.1}$$

and for $\varepsilon > 0$

$$\interleave \tilde{u} - \tilde{z} \interleave_{W^{1,1},r} \leqslant \varepsilon, \tag{4.2}$$

for $\tilde{z}(t) = \mathbb{P}\int_{-\infty}^t e^{(t-s)(\Delta-1)}\mathrm{d}\tilde{B}_s$ and for some $\vartheta > 0$ and for every $N \in \mathbb{N}$

$$\tilde{\mathbf{E}} \sup_{t \in [-N,N]} \|\tilde{u}(t)\|_{H^\vartheta}^{2r} + \mathbf{E}\|\tilde{u}\|_{C^\vartheta([-N,N],L^2)}^{2r} \lesssim N. \tag{4.3}$$

Proof. We define the ergodic averages of the solution (u, B) as the probability measures on the space of trajectories $\mathcal{T}$

$$\nu_T = \frac{1}{T}\int_0^T \mathcal{L}[S_t(u,B)]\mathrm{d}t, \qquad T \geqslant 0.$$

By Theorem 3.4 we obtain

$$\begin{aligned}\sup_{s\in\mathbb{R}}\mathbf{E}\sup_{t\in[-N,N]}\|u(t+s)\|_{H^\vartheta}^{2r} &\leqslant \sup_{s\in\mathbb{R}}\sum_{i=-N}^{N-1}\mathbf{E}\sup_{t\in[i,i+1]}\|u(t+s)\|_{H^\vartheta}^{2r}\\ &\leqslant 2N\sup_{s\geqslant 0}\mathbf{E}\sup_{t\in[0,1]}\|u(t+s)\|_{H^\vartheta}^{2r}\lesssim N,\end{aligned}$$

and similarly

$$\sup_{s\in\mathbb{R}}\mathbf{E}\|u(\cdot+s)\|^2_{C^\vartheta([-N,N],L^2)}\lesssim N.$$

For $R_N > 0$, $N\in\mathbb{N}$, we note that the set

$$K_M := \cap_{N=M}^\infty\left\{g_1; \|g_1\|_{C^\vartheta_{[-N,N]}L^2} + \sup_{t\in[-N,N]}\|g_1(t)\|_{H^\vartheta}\leqslant R_N\right\}$$

is relatively compact in $C(\mathbb{R};L^2_\sigma)$. As a consequence, we deduce that the time shifts $S_t u$, $t\in\mathbb{R}$, are tight on $C(\mathbb{R};L^2_\sigma)$. Since S_tB is a Wiener process for every $t\in\mathbb{R}$, the law of S_tB does not change with $t\in\mathbb{R}$ and is tight. Accordingly, for any $\varepsilon>0$ there is a compact set $\bar{K}_\varepsilon$ in $\mathcal{T}$ such that

$$\sup_{t\in\mathbb{R}}\mathbf{P}(S_t(u,B)\in\bar{K}_\varepsilon^c)<\varepsilon.$$

This implies

$$\nu_T(\bar{K}^c_\varepsilon) = \frac{1}{T}\int_0^T\mathbf{P}(S_t(u,B)\in\bar{K}^c_\varepsilon)\mathrm{d}t<\varepsilon$$

and therefore there is a weakly converging subsequence of the probability measures ν_T, $T\geqslant 0$. That is, there is a subsequence $T_n\to\infty$ and $\nu\in\mathcal{P}(\mathcal{T})$ such that $\nu_{T_n}\to\nu$ weakly in $\mathcal{P}(\mathcal{T})$.

Define the set

$$\begin{aligned}A = \Big\{(u,B)\in\mathcal{T};\ &\langle u(t),\psi\rangle + \int_s^t\langle \mathrm{div}(u\otimes u),\psi\rangle\mathrm{d}r\\ &= \langle u(s),\psi\rangle + \int_s^t\langle\Delta u,\psi\rangle\mathrm{d}r + \langle B(t)-B(s),\psi\rangle, \quad \forall\psi\in C^\infty(\mathbb{T}^3),\ \mathrm{div}\psi=0,\ t\geqslant s\Big\}.\end{aligned}$$

Since (u, B) in the statement of the theorem satisfies the equation, we have for all $t\in\mathbb{R}$

$$\mathcal{L}[S_t(u,B)](A)=1.$$

Hence, also $\nu_{T_n}(A) = 1$ for all $n\in\mathbb{N}$. By Jakubowski–Skorokhod representation theorem, there is a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbf{P}})$ with a sequence of random variables $(\tilde u^n,\tilde B^n)$, $n\in\mathbb{N}$, such that $\mathcal{L}[\tilde u^n,\tilde B^n]=\nu_{T_n}$ and $(\tilde u^n,\tilde B^n)$ satisfy equation (1.4) on $\mathbb{R}$. By (3.17) we know

$$\tilde{\mathbf{E}}\|\tilde u^n(t)\|^2_{L^2} = \frac{1}{T_n}\int_0^{T_n}\mathbf{E}\|S_su(t)\|^2_{L^2}\mathrm{d}s = K. \tag{4.4}$$

By (3.16)

$$\sup_n\tilde{\mathbf{E}}\|\tilde u^n(t)\|^{2r}_{L^2} = \sup_n\frac{1}{T_n}\int_0^{T_n}\mathbf{E}\|S_su(t)\|^{2r}_{L^2}\mathrm{d}s<\infty. \tag{4.5}$$

Moreover, there is a random variable $(\tilde{u}, \tilde{B})$ having the law $\mathcal{L}[\tilde{u}, \tilde{B}] = \nu$ so that

$$(\tilde{u}^n, \tilde{B}^n) \to (\tilde{u}, \tilde{B}) \qquad \tilde{\mathbf{P}}\text{-a.s. in } \mathcal{T}.$$

Thus, we can pass to the limit in the equation to deduce that ν is a law of a solution on $\mathbb{R}$.

The shift invariance follows from the same argument as in [BFH20e, Lemma 5.2]. Namely, it holds for $G \in C_b(\mathcal{T})$ and $r \in \mathbb{R}$

$$\int_{\mathcal{T}} G \circ S_r(u, B) \mathrm{d}\nu(u, B) = \int_{\mathcal{T}} G(u, B) \mathrm{d}\nu(u, B).$$

Finally, (4.1) follows from (4.4), (4.5) and (4.2), (4.3) follow from a lower-semicontinuity argument and the related bound for the approximations. In fact, we define

$$\tilde{z}^n(t) := \int_{-\infty}^t e^{(t-s)(\Delta - I)} \mathrm{d}\tilde{B}^n = \tilde{B}^n(t) + \int_{-\infty}^t (\Delta - I) e^{(t-s)(\Delta - I)} \tilde{B}^n \mathrm{d}s.$$

We know that $\tilde{z}(t)^n \to \tilde{z}(t) = \int_{-\infty}^t e^{(t-s)(\Delta - I)} \mathrm{d}\tilde{B}$ in $C(\mathbb{R}, H^{-2})$ $\tilde{\mathbf{P}}$ a.s. Thus (4.2) follows from lower-semicontinuity. □

Using the above result and choosing different K, the first claim in Theorem 1.4 follows.

By a general result applied also in [BFH20e, FFH21, HZZ22] and using Theorem 4.1 we obtain existence of infinitely many ergodic stationary solutions as follows.

Theorem 4.2. *Let $r > 1$. For $K \geqslant 8 \cdot 48 L^2 r$ there exists $C > 0$ and an ergodic stationary solution $((\Omega, \mathcal{F}, \mathbf{P}), u, B)$ satisfying*

$$\mathbf{E} \|u\|_{L^2}^2 = K, \tag{4.6}$$

and for some $\vartheta > 0$ and for every $N \in \mathbb{N}$

$$\mathbf{E} \sup_{t \in [-N, N]} \|u(t)\|_{H^\vartheta}^{2r} + \mathbf{E} \|u\|_{C^\vartheta([-N, N], L^2)}^{2r} \leqslant C N. \tag{4.7}$$

Proof. In view of Theorem 4.1, this is a consequence of a Krein–Milman argument. In particular, we observe that the set of all laws of stationary solutions satisfying (4.6) and (4.7) is non-empty, convex, tight and closed which follows from the arguments in the proof of Theorem 4.1. Hence there exist an extremal point. By a classical contradiction argument, it is the law of an ergodic stationary solution.

Non-uniqueness of ergodic stationary solutions follows from choosing different K. □

5. Stationary solutions to the stochastic Euler equations

We proceed with a construction of stationary solutions to the stochastic Euler equations (1.5).

Theorem 5.1. *Assume that $\mathrm{Tr}((-\Delta)^\sigma G G^*) < \infty$ for some $\sigma > 0$. There exist infinitely many stationary solutions $((\Omega, \mathcal{F}, \mathbf{P}), u, B)$ to stochastic Euler equations (1.5) on $\mathbb{R} \times \mathbb{T}^3$. In particular, let $r > 1$ and for a given $K \geqslant 8 \cdot 48 L^2 r$ with L being the bound for the noise in Proposition 3.2, there exists a stationary solution $((\Omega, \mathcal{F}, \mathbf{P}), u, B)$ to (1.5) satisfying*

$$\mathbf{E} \|u\|_{L^2}^2 = K,$$

as well as (4.3) for some $\vartheta > 0$.

Moreover, for an arbitrary sequence of vanishing viscosities $\nu_n \to 0$, $n \in \mathbb{N}$, there exist a sequence of stationary solutions u_n, $n \in \mathbb{N}$, to the following stochastic Navier–Stokes equations

$$\mathrm{d}u_n + \mathrm{div}(u_n \otimes u_n)\,\mathrm{d}t + \nabla P_n\,\mathrm{d}t = \nu_n \Delta u_n\,\mathrm{d}t + \mathrm{d}B, \tag{5.1}$$

so that the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to (1.5).

Finally, there exist infinitely many ergodic stationary solutions to (1.5).

Proof. In order to construct stationary solutions to the stochastic Euler equations (1.5), we decompose its solution u as $v + z$ where z solves the linear stochastic problem

$$\mathrm{d}z + z\,\mathrm{d}t = \mathrm{d}B, \tag{5.2}$$

whereas v is a solution to the nonlinear equation with random coefficients

$$\begin{aligned} \partial_t v - z + \mathrm{div}((v+z)\otimes(v+z)) + \nabla P = 0,\\ \mathrm{div} v = 0. \end{aligned} \tag{5.3}$$

Suppose that z is the unique stationary solution to (5.2) such that the bound from Proposition 3.2 is changed to: for any $\delta \in (0,1/2)$, $p \geqslant 1$

$$\sup_{t\in\mathbb{R}} \mathbf{E}\Big[\|z\|^p_{C_t^{1/2-\delta}H^\sigma}\Big] \leqslant (p-1)^{p/2} L^p. \tag{5.4}$$

Here, unlike in Proposition 3.2, we cannot use the smoothing effect of the Laplacian and hence we get the spatial regularity of z from the strengthened assumption on the Wiener process.

Accordingly, (3.6) changes to

$$\|z_q\|_{L^\infty,p} \leqslant \|z_q\|_{C_t^{1/2-2\delta}L^\infty,p} \leqslant \lambda_{q+1}^{\alpha/4}(p-1)^{1/2}L. \tag{5.5}$$

In Section 3.1.5, we need to modify the bounds (3.62) and (3.63). The former one now reads for every $\varepsilon > 0$ as

$$\begin{aligned} 2\mathbf{E}\|(v_\ell + z_{q+1})w^{(p)}_{q+1}\|_{L^1} &\lesssim (\|v_\ell\|_{L^\infty,2} + \|z_q\|_{L^\infty,2})\|w^{(p)}_{q+1}\|_{L^1,2} + \|z_{q+1} - z_q\|_{L^2,2}\|w^{(p)}_{q+1}\|_{L^2,2}\\ &\lesssim (\lambda_q^4 + \lambda_{q+1}^{\alpha/8}L)\ell^{-2}\delta_{q+2}^{1/2} r_\perp^{1-\varepsilon} r_\parallel^{\frac12(1-\varepsilon)} + \lambda_{q+1}^{-\alpha\sigma/8} M_0 \bar{e}^{1/2}\\ &\lesssim L\lambda_{q+1}^{5\alpha - \frac{8}{7}(1-\varepsilon)} + \lambda_{q+1}^{-\alpha\sigma/8} M_0 \bar{e}^{1/2} \leqslant \frac{1}{96}\lambda_{q+1}^{-2\beta b} \leqslant \frac{\delta_{q+2}}{96} e(t), \end{aligned}$$

where we use (3.42) and (3.48) to control $\|w^{(p)}_{q+1}\|_{L^2,2}$ and we need $\alpha\sigma > 16\beta b$. The latter one now relies on

$$\mathbf{E}\|z_{q+1} - z_q\|^2_{L^2} \leqslant L\lambda_{q+1}^{-\alpha\sigma/8},$$

which requires $M_0(\bar{e}^{1/2} + L) \leqslant \lambda_{q+1}^{\alpha\sigma/8 - 2\beta b}$, i.e. $\alpha\sigma > 16\beta b$ and a large enough to absorb the extra constant.

For the control of $\mathring{R}_{q+1}$ we can use (5.5) to derive the same bounds for most of the terms as in Section 3.1.7. The main change comes from the following two parts in R_{com} and R_{com1}, namely,

$$\ell^\sigma \|z_q\|_{C_{[t-1,t+1]}H^\sigma}(\|v_q\|_{C_{[t-1,t+1]}L^2} + \|z_q\|_{C_{[t-1,t+1]}L^2} + \|v_{q+1}\|_{C_{[t-1,t+1]}L^2}),$$

and

$$\|z\|_{C_{[t-1,t+1]}H^\sigma}\lambda_{q+1}^{-\alpha\sigma/8}(\|v_q\|_{C_{[t-1,t+1]}L^2} + \|z_q\|_{C_{[t-1,t+1]}L^2} + 1).$$

Then, when we estimate $\|\mathring{R}_{q+1}\|_{L^1,r}$ we have the following extra term

$$(M_0\bar{e}^{1/2}+L)L(\lambda_{q+1}^{-\alpha\sigma/8}+\ell^{\sigma}),$$

which requires $(M_0\bar{e}^{1/2}+L)L\leqslant \lambda_{q+1}^{\alpha\sigma/8-2b\beta^2}$, i.e. $\alpha\sigma>16\beta b^2$. We obtain this additional bound by choosing β small enough.

Consequently, we deduce existence and non-uniqueness of solutions to stochastic Euler equations as in Theorem 3.4. Furthermore, existence and non-uniqueness of stationary solutions follow by the same argument as in Theorem 4.1. This completes the proof of the first claim in Theorem 5.1.

For the second result in Theorem 5.1, we first apply Theorem 4.1 to derive the existence of stationary solutions u_n, $n\in\mathbb{N}$, to equations (5.1) with u_n satisfying (4.3) uniformly in n. More precisely, we choose z_n satisfying

$$\mathrm{d}z_n+z_n\,\mathrm{d}t=\nu_n\Delta z_n\,\mathrm{d}t+\mathrm{d}B. \tag{5.6}$$

Then z_n satisfies (5.4) uniformly in n, using again the regularity of the noise instead of the smoothing effect of the Laplacian. Hence, by exactly the same argument as above and Theorem 4.1 we know that (4.3) holds uniformly in n, which implies tightness of u_n, $n\in\mathbb{N}$, in $C(\mathbb{R};L^2_\sigma)$. By Jakubowski–Skorokhod representation theorem, we can modify the stochastic basis and pass to the limit in the approximate Navier–Stokes equations (5.1) and derive the claim.

Existence and non-uniqueness of ergodic stationary solutions follows from the same argument as in Theorem 4.2. □

Motivated by the recent work [BD22], we note that our construction can be employed to give a result related to anomalous dissipation along a vanishing viscosity limit in a Navier–Stokes–Reynolds system. Unlike [BD22], our result holds in the context of statistically stationary solutions, bringing our theory even closer to the fundamental principles of the Kolmogorov's 1941 theory of homogeneous, isotropic turbulence [Kol41a, Kol41b, Kol41c]. To this end, we first recall the following version of the geometric lemma from [DK20, Lemma 3.2], which we use as a replacement for Lemma A.1. It permits to derive a lower bound on the L^2-norm of the gradient of the perturbations v_{q+1} (see Lemma 5.3 below) and leads to the anomalous dissipation produced by convex integration. In particular, we point out that the anomalous dissipation is not a consequence of the behavior of the linear part of the equations as discussed in Remark 1.7.

Lemma 5.2. *Let $\Lambda=\{\xi_i\}_{i=1}^6$ be a set of vectors in $\mathbb{Q}^3$ and $C>0$ such that*

$$\sum_{i=1}^6\xi_i\otimes\xi_i=C\mathrm{Id},\qquad \{\xi_i\otimes\xi_i\}_{i=1}^6 \text{ forms a basis of } \mathcal{S}^{3\times3}. \tag{5.7}$$

Then there exists a positive constant N_0 such that for any $N\leqslant N_0$, there are functions $\{\gamma_{\xi_i}\}_{i=1}^6\subset C^\infty(\overline{B_{N_0}}(\mathrm{Id}))$ satisfying for $K\in\overline{B_N}(\mathrm{Id})$

$$K=\sum_{i=1}^6\gamma_{\xi_i}^2(K)(\xi_i\otimes\xi_i).$$

Moreover, $\gamma^2_{\xi_i}(K)\geqslant\frac{1}{2C}$ for $K\in\overline{B_{N_0}}(\mathrm{Id})$.

Now, we choose $\xi_i\in\mathbb{S}^2\cap\mathbb{Q}^3$ as $(\frac35,\pm\frac45,0)$, $(\frac45,0,\pm\frac35)$ and $(0,\frac35,\pm\frac45)$. They satisfy the condition (5.7) with $C=2$. In this case we could find N_0 such that for every matrix $R\in\overline{B_{N_0}}(\mathrm{Id})$ the result in Lemma 5.2 holds with $\gamma_\xi^2\geqslant 1/4$. This is the key point in the following lemma.

Lemma 5.3. *In the setting of Proposition 3.3 it holds that for $q \in \mathbb{N}_0$ and every $t \in \mathbb{R}$*

$$\mathbf{E}\|\nabla v_{q+1}(t)\|_{L^2}^2 \geqslant \frac{1}{4N_0}\ell\lambda_{q+1}^2 - \lambda_{q+1}^{13/7+14\alpha}.$$

Proof. Since we use a modified geometric lemma we change the definition of ρ with 2 replaced by $1/N_0$. Since N_0 is a universal constant we only need to change the bound (3.10) as

$$\|\mathring{R}_q\|_{L^1,r} \leqslant \frac{1}{48}N_0\underline{e}\delta_{q+2},$$

and we choose M_0 in (3.7) depending on N_0. We calculate $\|\nabla v_{q+1}\|_{L^2}^2$ and have

$$\begin{aligned}\|\nabla v_{q+1}\|_{L^2}^2 = \|\nabla v_\ell\|_{L^2}^2 + \|\nabla w_{q+1}^{(p)}\|_{L^2}^2 + \|\nabla(w_{q+1}^{(c)} + w_{q+1}^{(t)})\|_{L^2}^2 + 2\langle\nabla v_\ell, \nabla(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)})\rangle\\ + 2\langle\nabla w_{q+1}^{(p)}, \nabla(w_{q+1}^{(c)} + w_{q+1}^{(t)})\rangle.\end{aligned}$$

We use the lower bound of γ_ξ from Lemma 5.2, the fact that $\rho \geqslant N_0^{-1}\ell$, a direct calculation showing $\|\nabla W_{(\xi)}\|_{L^2} \sim \lambda_{q+1}$ and (3.33) and (A.7) to have

$$\begin{aligned}\|\nabla w_{q+1}^{(p)}\|_{L^2}^2 &\geqslant \sum_\xi \frac{1}{4N_0}\ell\|\nabla W_{(\xi)}\|_{L^2}^2 - \sum_\xi \|a_{(\xi)}\|_{C_{t,x}^1}^2\|W_{(\xi)}\|_{L^2}^2\\ &\gtrsim \frac{1}{4N_0}\ell\lambda_{q+1}^2 - \ell^{-26}(\|\mathring{R}_q\|_{L^1}^4 + 1).\end{aligned}$$

Taking expectation we use (3.11) to get

$$\begin{aligned}\mathbf{E}\|\nabla w_{q+1}^{(p)}\|_{L^2}^2 &\gtrsim \frac{1}{4N_0}\ell\lambda_{q+1}^2 - \ell^{-26}\mathbf{E}(\|\mathring{R}_q\|_{L^1}^4 + 1)\\ &\gtrsim \frac{1}{4N_0}\ell\lambda_{q+1}^2 - \lambda_{q+1}^{52\alpha}\lambda_q^4.\end{aligned}$$

For the rest thems we first use (3.15) to have

$$\sup_{t\in\mathbb{R}}\mathbf{E}\|\nabla v_\ell\|_{L^2}^2 \leqslant \lambda_q^8.$$

For the other terms we apply (A.7) and (3.33)-(3.34) to have

$$\begin{aligned}\|\nabla w_{q+1}^{(c)}\|_{L^2}^2 &\lesssim \|a_{(\xi)}\|_{C_{t,x}^1}^2\|\nabla W_{(\xi)}^{(c)}\|_{L^2}^2 + \|a_{(\xi)}\|_{C_{t,x}^3}^2\|V_{(\xi)}\|_{C_tH^2}^2\\ &\lesssim \ell^{-26}\lambda_{q+1}^2 r_\perp^2 r_\parallel^{-2}(\|\mathring{R}_q\|_{L^1}^4 + 1) + \ell^{-50}(\|\mathring{R}_q\|_{L^1}^8 + 1),\end{aligned}$$

and

$$\begin{aligned}\|\nabla w_{q+1}^{(t)}\|_{L^2}^2 \lesssim \frac{1}{\mu^2}\sum_{\xi\in\Lambda}\Big(&\|a_{(\xi)}\|_{C_{t,x}^0}\|a_{(\xi)}\|_{C_{t,x}^1}\|\phi_{(\xi)}\|_{L^4}^2\|\psi_{(\xi)}\|_{C_tL^4}^2\\ &+ \|a_{(\xi)}\|_{C_{t,x}^0}^2\|\phi_{(\xi)}\|_{L^4}\|\nabla\phi_{(\xi)}\|_{L^4}\|\psi_{(\xi)}\|_{C_tL^4}^2\\ &+ \|a_{(\xi)}\|_{C_{t,x}^0}^2\|\phi_{(\xi)}\|_{L^4}^2\|\nabla\psi_{(\xi)}\|_{C_tL^4}\|\psi_{(\xi)}\|_{C_tL^4}\Big)^2\\ \lesssim \frac{1}{\mu^2}\Big(&\ell^{-15}(\|\mathring{R}_q\|_{L^1}^{5/2} + 1)r_\perp^{-1}r_\parallel^{-1/2}\\ &+ \ell^{-4}(\|\mathring{R}_q\|_{L^1} + 1)\lambda_{q+1}r_\perp^{-1}r_\parallel^{-1/2}(1 + \frac{r_\perp}{r_\parallel})\Big)^2.\end{aligned}$$

Hence, in view of (3.11) we use $a > (192rL^2)^c$ to have $\|\mathring{R}_q\|_{L^1,8} \leqslant \lambda_q$ to obtain

$$\begin{aligned}\mathbf{E}\|\nabla w_{q+1}^{(c)}\|_{L^2}^2 + \mathbf{E}\|\nabla w_{q+1}^{(t)}\|_{L^2}^2 &\lesssim \lambda_{q+1}^{10/7+52\alpha}\lambda_q^4 + \lambda_{q+1}^{100\alpha}\lambda_q^8 + \lambda_{q+1}^{60\alpha-2/7}\lambda_q^5 + \lambda_{q+1}^{12/7+16\alpha}\lambda_q^2 \\ &\lesssim \lambda_{q+1}^{12/7+17\alpha}.\end{aligned}$$

We use (3.33), (A.7) and have

$$\begin{aligned}\mathbf{E}\|\nabla w_{q+1}^{(p)}\|_{L^2}^2 &\lesssim \ell^{-4}\lambda_{q+1}^2\lambda_q + \ell^{-26}\mathbf{E}(\|\mathring{R}_q\|_{L^1}^4 + 1) \\ &\lesssim \ell^{-4}\lambda_{q+1}^2\lambda_q.\end{aligned}$$

As a result, we apply Hölder's inequality for the cross term and get

$$\begin{aligned}&2\mathbf{E}|\langle \nabla v_\ell, \nabla(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)})\rangle| + 2\mathbf{E}|\langle \nabla w_{q+1}^{(p)}, \nabla(w_{q+1}^{(c)} + w_{q+1}^{(t)})\rangle| \\ &\lesssim \lambda_q^4\lambda_{q+1}^{1+4\alpha}\lambda_q^{1/2} + \lambda_{q+1}^{1+4\alpha}\lambda_{q+1}^{6/7+9\alpha} \lesssim \lambda_{q+1}^{13/7+13\alpha}.\end{aligned}$$

Summing up the above estimates the result follows. □

Our result related to anomalous dissipation then reads as follows.

Theorem 5.4. *Suppose that* $\mathrm{Tr}((-\Delta)^{5/2+\sigma}GG^*) < \infty$ *for some* $\sigma > 0$. *Let* $\epsilon > 0$, $r > 1$ *and* $K \geqslant 48 \cdot 8L^2r/N_0$ *be given with* L *being the bound for the noise in Proposition 3.2. There exists a sequence of viscosities* $\nu_n \to 0$ *and stationary processes* $(u_n, \mathring{R}_n) \in C(\mathbb{R}; H^1) \times C(\mathbb{R}; L^1)$ *satisfying the following stochastic Navier–Stokes–Reynolds equations*

$$\mathrm{d}u_n + \mathrm{div}(u_n \otimes u_n)\,\mathrm{d}t + \nabla P_n\,\mathrm{d}t = \nu_n\Delta u_n\,\mathrm{d}t + \mathrm{div}\mathring{R}_n\,\mathrm{d}t + \mathrm{d}B, \tag{5.8}$$

$$\lim_{n\to\infty}\|\mathring{R}_n\|_{L^1,r} = 0,$$

and

$$\liminf_{n\to\infty}\nu_n\mathbf{E}\|\nabla u_n\|_{L^2}^2 \geqslant \epsilon + \frac{1}{2}\mathrm{Tr}(GG^*). \tag{5.9}$$

Furthermore, the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, *is tight in* $C(\mathbb{R}; L^2_\sigma)$ *and every accumulation point is a stationary solution to the stochastic Euler equations* (1.5) *with* $\mathbf{E}\|u\|_{L^2}^2 = K$ *and satisfying* (4.3) *for some* $\vartheta > 0$. *In particular, the solutions* u_n, $n \in \mathbb{N}$, *can be chosen as ergodic stationary solutions.*

Proof. Let z_n to be the stationary solution to (5.6). Here, we suppose that G satisfies the stronger condition from the statement of the theorem so that for $\delta \in (0, 1/2)$ and $n \in \mathbb{N}$

$$\|z_n\|_{C_t^{1/2-\delta}C_x^1,p} + \|z_n\|_{H^{5/2},p} \leqslant (p-1)^{1/2}L.$$

For a fixed $n \in \mathbb{N}$, we run the stochastic convex integration as in Section 3 but we do not project z_n as before. More precisely, we construct a sequence of solutions $(v_{n,q}, \mathring{R}_{n,q})$ satisfying the following equations

$$\begin{aligned}\partial_t v_{n,q} - z_n + \mathrm{div}((v_{n,q} + z_n) \otimes (v_{n,q} + z_n)) + \nabla p_{n,q} &= \nu_n\Delta v_{n,q} + \mathrm{div}\mathring{R}_{n,q}, \\ \mathrm{div}v_{n,q} &= 0.\end{aligned}$$

Set $u_n = v_{n,n} + z_n$ and $\mathring{R}_n = \mathring{R}_{n,n}$. Using the estimates in Proposition 3.3 with $e(t) = K$ we obtain

$$\|u_n\|_{H^\vartheta,2r} + \|u_n\|_{C_t^\vartheta L^2,2r} \lesssim 1, \tag{5.10}$$

with the proportional constant independent of n and

$$\|u_n - z_n\|_{C^1_{t,x},2r} \leqslant \lambda_n^4, \quad \|u_n - z_n\|_{C^2_{t,x},r} \leqslant \lambda_n^6, \quad \|\mathring{R}_n\|_{L^1,r} \leqslant \frac{1}{48} N_0 \delta_{n+2} K, \tag{5.11}$$

and for any $t \in \mathbb{R}$

$$\frac{3}{4}\delta_{n+1} K \leqslant K - \mathbf{E}\|u_n(t)\|_{L^2}^2 \leqslant \frac{5}{4}\delta_{n+1} K. \tag{5.12}$$

In order to derive (5.9) we apply Lemma 5.3 to have

$$\mathbf{E}\|\nabla v_{n,n}\|_{L^2}^2 \geqslant \frac{1}{4N_0}\lambda_n^{2-3\alpha/2}\lambda_{n-1}^{-2} - \lambda_n^{13/7+14\alpha}.$$

On the other hand, we estimate using (3.14)

$$\begin{aligned}\mathbf{E}\langle \nabla v_{n,n}, \nabla z_n\rangle &\leqslant \mathbf{E}\|v_{n,n}\|_{W^{1,1}}\|\nabla z_n\|_{L^\infty} \leqslant (\mathbf{E}\|v_{n,n}\|_{W^{1,1}}^2)^{1/2}(\mathbf{E}\|\nabla z_n\|_{L^\infty}^2)^{1/2} \\ &\leqslant \sum_{q=0}^{n}(\lambda_q^{-\alpha/2} + \lambda_q^{-1/7+6\alpha})L \leqslant L,\end{aligned}$$

where we could choose a large enough as in the proof of Theorem 3.4. We then get

$$\mathbf{E}\|\nabla u_n\|_{L^2}^2 \gtrsim \frac{1}{4N_0}\lambda_n^{2-3\alpha/2}\lambda_{n-1}^{-2} - \lambda_n^{13/7+14\alpha} - 2L + \mathbf{E}\|\nabla z_n\|_{L^2}^2,$$

with a proportional constant independent of n. We denote this constant by c and let

$$\nu_n = \left(\epsilon + \frac{1}{2}\mathrm{Tr}(GG^*)\right) c^{-1} 4N_0 \lambda_n^{-2+3\alpha/2}\lambda_{n-1}^2, \tag{5.13}$$

we deduce for $t \in \mathbb{R}$

$$\nu_n \mathbf{E}\|\nabla u_n(t)\|_{L^2}^2 \geqslant \epsilon + \frac{1}{2}\mathrm{Tr}(GG^*) + \nu_n c\left(-\lambda_n^{13/7+14\alpha} - 2L + \mathbf{E}\|\nabla z_n\|_{L^2}^2\right). \tag{5.14}$$

Next, we recall that z_n is a stationary solution to (5.6). Hence it follows from Itô's formula that

$$\mathbf{E}\|z_n\|_{L^2}^2 + \nu_n \mathbf{E}\|\nabla z_n\|_{L^2}^2 = \frac{1}{2}\mathrm{Tr}(GG^*).$$

On the other hand, since

$$z_n(t) = \int_{-\infty}^{t} e^{(t-s)(\nu_n\Delta - I)} \mathrm{d}B,$$

a direct computation yields

$$\lim_{n\to 0} \mathbf{E}\|z_n\|_{L^2}^2 = \frac{1}{2}\mathrm{Tr}(GG^*),$$

which further implies

$$\lim_{n\to\infty} \nu_n \mathbf{E}\|\nabla z_n\|_{L^2}^2 = 0.$$

Therefore, in view of (5.13) and (5.14), and the choice of the parameter α, (5.9) follows for $u_n(t), t \in \mathbb{R}$.

Furthermore, we use (3.53)-(3.55) to have for $\delta > 0$

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|\mathring{R}_n\|_{C_t^\delta W^{\delta,1}}^r \lesssim C(n). \tag{5.15}$$

We need the above estimate for the tightness in the trajectory space for $\mathring{R}_n$ in order to construct a stationary solution below. Here, the constant may depend on n.

Based on these estimates, we consider $(u_n, B, \mathring{R}_n)$ on the trajectory space $\mathcal{T}_1 = C(\mathbb{R}; L^2_\sigma) \times C(\mathbb{R}; L^2_\sigma) \times C(\mathbb{R}; L^1)$. The corresponding shifts $S_t, t \in \mathbb{R}$ are given by

$$S_t(u, B, \mathring{R})(\cdot) = (u(\cdot + t), B(\cdot + t) - B(t), \mathring{R}(\cdot + t)), \quad t \in \mathbb{R}, \quad (u, B, \mathring{R}) \in \mathcal{T}_1.$$

Using (5.10) and the last estimate in (5.11) and (5.15) it follows that for a fixed n, the ergodic averages

$$\nu_{n,T} = \frac{1}{T}\int_0^T \mathcal{L}[S_t(u_n, B, \mathring{R}_n)]\mathrm{d}t, \qquad T \geqslant 0,$$

form a tight set in $\mathcal{T}_1$. Furthermore, using the second estimate in (5.11) the probability measures $\frac{1}{T}\int_0^T \mathcal{L}[S_t u_n]\mathrm{d}t$, $T \geqslant 0$, form a tight set in $C(\mathbb{R}; H^1)$. Hence similar arguments as in the proof of Theorem 4.1 lead to the existence of a stationary solution to (5.8), which is still denoted by $(u_n, B, \mathring{R}_n)$ and satisfies (5.9)-(5.12).

Now, we apply a tightness argument to the stationary solutions $(u_n, B, \mathring{R}_n)$, $n \in \mathbb{N}$, and the third estimate in (5.11) to obtain $\|\mathring{R}_n\|_{L^1,r} \to 0$. Letting $n \to \infty$ we obtain that the tight limit of u_n is a stationary solution to the stochastic Euler equations.

Furthermore, Krein–Milman's theorem permits to choose u_n as ergodic stationary solutions and the final result follows. □

6. Stationary solutions to the deterministic Navier–Stokes/Euler equations

In this section, we construct random statistically stationary solutions to the deterministic Navier–Stokes/Euler equations on $\mathbb{R}\times\mathbb{T}^3$. As mentioned in the introduction, a lot has been already achieved by using the known deterministic results about Euler and Navier–Stokes equations. Furthermore, also the results of Section 3, Section 4 and Section 5 can be applied with $G = 0$. Therefore, here we focus on proving that the constructed stationary solutions may be genuinely random as well as time dependent. Precisely, we show that the solutions can be close in a certain sense to a given stationary stochastic process. This further highlights the fact how arbitrary the stationary solutions to the deterministic Navier–Stokes/Euler equations can be. In particular, the constructed stationary solutions can possess "almost" Gaussian or non-Gaussian statistics.

In the sequel, we make a few modifications in the construction of Section 3. We consider the iteration

$$\partial_t u_q + \mathrm{div}(u_q \otimes u_q) + \nabla p_q = \nu\Delta u_q + \mathrm{div}\mathring{R}_q \tag{6.1}$$

with $\nu = 1$ or 0, which corresponds to the Navier–Stokes and Euler equations, respectively. In this case Proposition 3.3 holds for $(u_q, \mathring{R}_q)$ satisfying (6.1). Its proof simplifies as we do not need to include the process z_q anymore. Based on this, we obtain the following result.

Theorem 6.1. *Let $r > 1$ be fixed and Z be an $\mathcal{F}$-measurable stationary stochastic process with smooth trajectories and vanishing mean and divergence and satisfying*

$$\left(\mathbf{E}\|Z\|^m_{L^2} + \mathbf{E}\|Z\|^m_{C^2_{t,x}}\right)^{1/m} \leqslant m^{1/2}L, \tag{6.2}$$

for any $m > 1$ and some $L \geqslant (2\pi)^3$. Let a smooth function $e : \mathbb{R} \to (0, \infty)$ satisfying $\bar{e} \geqslant e(t) \geqslant \underline{e} > 192rL^2$ and $\varepsilon > 0$ be given. There exists an $\mathcal{F}$-measurable process u which belongs to $C(\mathbb{R}; H^\vartheta) \cap C^\vartheta(\mathbb{R}; L^2)$ $\mathbf{P}$-a.s. and is an analytically weak solution to the deterministic Navier–Stokes/Euler equations on $\mathbb{R} \times \mathbb{T}^3$. Moreover, there exists $\vartheta > 0$ such that

$$\|u\|_{H^\vartheta,2r} + \|u\|_{C^\vartheta_t L^2,2r} < \infty,$$

and for $t \in \mathbb{R}$

$$\mathbf{E}\|u(t)\|_{L^2}^2 = e(t), \tag{6.3}$$

and

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|u - Z\|_{C_t W^{1,1}}^r \leqslant \varepsilon. \tag{6.4}$$

Proof. We start the iteration with $u_0 = Z$, so (6.1) on the level $q = 0$ reads as

$$\partial_t Z + \operatorname{div}(Z \otimes Z) + \nabla p_0 - \nu\Delta Z =: \operatorname{div} \mathring{R}_0$$

with

$$\mathring{R}_0 = Z \mathring{\otimes} Z - \mathcal{R}(\nu\Delta Z - \partial_t Z).$$

The last term is the reason why we require smoothness of trajectories of the process Z and we also need a bound for the $C^2_{t,x}$-norm of $v_0 = Z$. Hence, for some $r > 1$ such that $\underline{e} \geqslant 192rL^2$

$$|\!|\!|\mathring{R}_0|\!|\!|_{L^1,r} \leqslant 2rL^2 + (2\pi)^3 \cdot 2rL \leqslant \frac{1}{48}\underline{e},$$

and for $m \geqslant 1$

$$|\!|\!|\mathring{R}_0|\!|\!|_{L^1,m} \leqslant 2mL^2 + (2\pi)^3 \cdot 2mL \leqslant 4mL^2.$$

Now, we run the convex integration from Proposition 3.3 and we get a limit $u = \lim_{q\to\infty} u_q$ in $C(\mathbb{R}, H^\vartheta) \cap C^\vartheta(\mathbb{R}, H^\vartheta)$ which solves the deterministic Navier–Stokes/Euler equations. Moreover, it holds

$$u - Z = \sum_{q=0}^{\infty} (u_{q+1} - u_q).$$

Thus, as in Proposition 3.3 we could choose a large enough so that (3.14) implies (6.4). The rest of the proof follows exactly the same arguments as in the proof of Theorem 3.4. □

Remark 6.2. From the proof it can be seen that the stochastic convex integration is not necessary provided Z satisfies certain stronger assumptions. For instance, if Z possesses a uniform in ω bound in $C^2_b(\mathbb{R} \times \mathbb{T}^3)$, a deterministic convex integration à la [BV19b] can be applied pathwise, an ω-dependent energy can be prescribed pathwise and the expectation in (6.4) can be dropped. Furthermore, the stochastic convex integration is also not necessary provided the trajectories of Z belong to $C^2_b(\mathbb{R} \times \mathbb{T}^3)$ a.s. In this case, we can apply the deterministic convex integration on each of the sets

$$\Omega_L = \left\{\omega \in \Omega;\ L - 1 \leqslant \|Z(\omega)\|_{C^2_b(\mathbb{R}\times\mathbb{T}^3)} < L\right\}$$

and glue the solutions together similarly to [HZZ22, Theorem 3.2]. Restricting to the sets Ω_L permits to obtain the $\mathcal{F}$-measurability of the solutions, since the parameters in the convex integration only differ for different L but not for each different ω.

Combining the above with the proof of Theorem 5.1 we obtain the following result.

Theorem 6.3. *Let $r > 1$, $\varepsilon > 0$, let Z be as in Theorem 6.1 and let $K \geqslant 192rL^2$. There exist a random, time dependent, stationary solution $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{u})$ to the deterministic Navier–Stokes/Euler equations on $\mathbb{R} \times \mathbb{T}^3$ satisfying*

$$\tilde{\mathbf{E}}\|\tilde{u}\|_{L^2}^2 = K,$$

and a stochastic process $\tilde{Z}$ defined on the same probability space, $\mathcal{L}[\tilde{Z}] = \mathcal{L}[Z]$, so that

$$\tilde{\mathbf{E}}\|\tilde{u} - \tilde{Z}\|_{C_t W^{1,1}}^r \leqslant \varepsilon. \tag{6.5}$$

Furthermore, there exist non-unique ergodic stationary solutions.

Proof. The proof proceeds similarly to the proof of Theorem 4.1. The main difference is that now we consider the ergodic averages

$$\nu_T := \frac{1}{T}\int_0^T \mathcal{L}[S_t(u,Z)]\mathrm{d}t := \frac{1}{T}\int_0^T \mathcal{L}[(u,Z)(t+\cdot)]\mathrm{d}t, \qquad T \geqslant 0.$$

The uniform bounds of u and Z imply tightness of ν_T in $C(\mathbb{R};L^2_\sigma)\times C(\mathbb{R};C^1)$. We could find a stationary solution $(\tilde{\Omega},\tilde{F},\tilde{P},\tilde{u},\tilde{Z})$ such that $\tilde{\mathbf{E}}\|\tilde{u}\|^2_{L^2}=K$ for some given K and

$$\sup_{t\in\mathbb{R}}\tilde{\mathbf{E}}\|\tilde{u}-\tilde{Z}\|_{C_tW^{1,1}}\leqslant\varepsilon.$$

This also implies that the stationary solution depends on time and is genuinely random provided Z is time dependent and genuinely random. For different K we have different stationary solutions which also gives non-unique ergodic stationary solutions. □

The above results offer a lot of freedom in the choice of the process Z, showing how arbitrary the law of constructed stationary solutions can be. For instance, the construction can be performed with a Gaussian stationary process Z with smooth trajectories. A simple example is given by

$$Z(t)=(\cos(t)\xi_1+\sin(t)\xi_2)\,e^{ik\cdot x}e_k,$$

where ξ_1, ξ_2 are independent standard Gaussians and $k,e_k\in\mathbb{R}^3\setminus\{0\}$ are orthogonal. In this case, we could find $L>0$ such that for every $m\geqslant 2$, $\|Z\|_{C^2_{t,x},m}+\|Z\|_{L^2,m}\leqslant(m-1)^{1/2}L$. Hence, (6.2) is satisfied. Note that this corresponds to the second situation discussed in Remark 6.2, namely, where the trajectories of Z belong to $C^2_b(\mathbb{R}\times\mathbb{T}^3)$ a.s.

To construct a Gaussian process Z outside of the simplified setting of Remark 6.2, we may consider z as in Section 3, i.e. a stationary solution to (3.1), and define Z to be its space-time mollification. Then (6.2) follows from Proposition 3.2 and the stochastic convex integration as used in Theorem 6.1 can be applied.

But we may as well define Z to be non-Gaussian. For example, we let $Z(t)=X(t)=\cos(t+Y)e^{ik\cdot x}e_k$, where k, $e_k\in\mathbb{R}^3\setminus\{0\}$ are orthogonal and Y is uniformly distributed on $(0,2\pi]$. Alternatively, we use the process X given above and define $Z(t)=\int_0^\infty e^{s(\nu\Delta-I)}X(t-s)\mathrm{d}s$ which is a stationary solution to the following equation

$$\partial_t Z+Z=\nu\Delta Z+X, \tag{6.6}$$

which is also of zero mean and divergence free. Both these examples have uniform in ω bounds as discussed in Remark 6.2 and they are non-Gaussian.

Theorem 6.4. *Let $r>1$, $\epsilon>0$, $\varepsilon>0$, let Z be as in Theorem 6.1 and let $K>48\cdot 4rL^2$. Up to a change of probability space, there exist $\nu_n\to 0$ and stationary processes $(u_n,\mathring{R}_n)\in C(\mathbb{R};H^1)\times C(\mathbb{R};L^1)$, satisfying the following random Navier–Stokes–Reynolds equations*

$$\partial_t u_n+\mathrm{div}(u_n\otimes u_n)+\nabla P_n=\nu_n\Delta u_n+\mathrm{div}\mathring{R}_n, \tag{6.7}$$

$$\lim_{n\to\infty}\|\mathring{R}_n\|_{L^1,1}=0,$$

and

$$\liminf_{n\to\infty}\nu_n\mathbf{E}\|\nabla u_n\|^2_{L^2}\geqslant\epsilon. \tag{6.8}$$

Furthermore, the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to the deterministic Euler equations

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0,$$

with $\mathbf{E}\|u\|_{L^2}^2 = K$ and satisfying (4.3) for some $\vartheta > 0$ and

$$\mathbf{E}\|u - Z\|^r_{C_t W^{1,1}} \leqslant \varepsilon. \tag{6.9}$$

This implies that the solution u can be random and time dependent. In particular, the solutions u_n, $n \in \mathbb{N}$, can be chosen as ergodic stationary solutions.

Proof. For a fixed $n \in \mathbb{N}$, we run the stochastic convex integration as in Section 3 starting at $u_{n,0} = Z$ and we obtain a sequence of solutions $(u_{n,q}, \mathring{R}_{n,q})$ satisfying the following equations

$$\begin{aligned}\partial_t u_{n,q} + \operatorname{div}(u_{n,q} \otimes u_{n,q}) + \nabla p_{n,q} &= \nu_n \Delta u_{n,q} + \operatorname{div}\mathring{R}_{n,q},\\ \operatorname{div} v_{n,q} &= 0.\end{aligned}$$

Similarly as in the proof of Theorem 6.3 it holds

$$|\!|\!|\mathring{R}_{n,0}|\!|\!|_{L^1,r} \leqslant 2rL^2 + (2\pi)^3 \cdot 2rL \leqslant \frac{1}{48}\underline{e},$$

and for $m \geqslant 1$

$$|\!|\!|\mathring{R}_0|\!|\!|_{L^1,m} \leqslant 2mL^2 + (2\pi)^3 \cdot 2mL \leqslant 4mL^2.$$

Set $u_n = u_{n,n}$ and $\mathring{R}_n = \mathring{R}_{n,n}$. Using the estimates in Proposition 3.3 with $e(t) = K$ we obtain

$$|\!|\!|u_n|\!|\!|_{H^\vartheta,2r} + |\!|\!|u_n|\!|\!|_{C_t^\vartheta L^2,2r} \lesssim 1, \qquad \sup_{t\in\mathbb{R}} \mathbf{E}\|u_n - Z\|^r_{C_t W^{1,1}} \leqslant \varepsilon. \tag{6.10}$$

with the proportional constant independent of n and

$$|\!|\!|u_n|\!|\!|_{C^1_{t,x},2r} \leqslant \lambda_n^4, \quad |\!|\!|u_n|\!|\!|_{C^2_{t,x},r} \leqslant \lambda_n^6, \quad |\!|\!|\mathring{R}_n|\!|\!|_{L^1,r} \leqslant \frac{1}{48}\delta_{n+2}K, \tag{6.11}$$

and for any $t \in \mathbb{R}$

$$\frac{3}{4}\delta_{n+1}K \leqslant K - \mathbf{E}\|u_n(t)\|^2_{L^2} \leqslant \frac{5}{4}\delta_{n+1}K. \tag{6.12}$$

Similarly as in (5.14) and applying Lemma 5.3 we get

$$\nu_n \mathbf{E}\|\nabla u_n(t)\|^2_{L^2} > \epsilon, \tag{6.13}$$

for n large enough. Furthermore, we use (3.53)-(3.55) to have for $\delta > 0$

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|\mathring{R}_n\|^r_{C_t^\delta W^{\delta,1}} \lesssim C(n). \tag{6.14}$$

We need the above estimate for tightness in trajectory space for $\mathring{R}_n$ in order to construct a stationary solution below. Here, the constant may depend on n.

Based on these estimates, we consider $(u_n, \mathring{R}_n, Z)$ on the trajectory space $\mathcal{T}_1 = C(\mathbb{R}; L^2_\sigma) \times C(\mathbb{R}; L^1) \times C(\mathbb{R}; L^2_\sigma)$. The corresponding shifts $S_t, t \in \mathbb{R}$ are given by

$$S_t(u, \mathring{R}, Z)(\cdot) = (u(\cdot + t), \mathring{R}(\cdot + t), Z(\cdot + t)), \quad t \in \mathbb{R}, \quad (u, \mathring{R}, Z) \in \mathcal{T}_1.$$

Using (6.10) and the second estimate in (6.11) and (6.14) it follows that for a fixed n, the ergodic averages

$$\nu_{n,T} = \frac{1}{T}\int_0^T \mathcal{L}[S_t(u_n, \mathring{R}_n, Z)]\mathrm{d}t, \qquad T \geqslant 0,$$

form a tight set in $\mathcal{T}_1$. Furthermore, $\frac{1}{T}\int_0^T \mathcal{L}[S_t(u_n)]\mathrm{d}t$ form a tight set in $C(\mathbb{R}; H^1)$. Hence similar arguments as in the proof of Theorem 4.1 lead to the existence of a stationary solution to (6.7), which is still denoted by $(u_n, \mathring{R}_n, Z_n)$ and satisfies (6.10)-(6.14).

Now, we apply a tightness argument together with a Skorokhod representation theorem for the stationary solution $(u_n, \mathring{R}_n, Z_n)$. For notational simplicity, we do not rename the objects after changing the probability space. Applying the third estimate in (6.11) we obtain $\|\mathring{R}_n\|_{L^1,r} \to 0$. The strong convergence of u_n in $C(\mathbb{R}; L^2_\sigma)$ follows from (6.10). Letting $n \to \infty$, we see that the tight limit is a stationary solution to the deterministic Euler equations.

Thus, we use (6.13) to deduce that (6.8) holds and

$$\liminf_{n\to\infty} \nu_n \mathbf{E}\|\nabla u_n\|_{L^2}^2 \geqslant \epsilon.$$

The claim (6.9) as well as randomness and time dependendence of the limit solutions follow from similar arguments as in Theorem 6.3.

Furthermore, Krein–Milman's theorem permits to choose u_n as ergodic stationary solutions and the final result follows. □

Remark 6.5. The regularity corresponding to the Kolmogorov 2/3-law, implying the decay of the energy spectrum as discussed in the introduction, remains out of reach for our solutions. Nevertheless, our solutions satisfy a weaker version of the corresponding Kolmogorov hypothesis in the spirit of [CG12, CV18] which is sufficient to perform rigorously the vanishing viscosity limit and obtain stationary solutions to the Euler equations.

We conclude this section by an observation that stationary solutions to the deterministic Navier–Stokes/Euler equations can also be obtained as limits of stationary solutions to the stochastic counterparts of the equations, possibly combining with a vanishing viscosity limit. More precisely, we have the following result.

Theorem 6.6. *Suppose that* $\mathrm{Tr}((-\Delta)^{3/2+\sigma}GG^*) < \infty$ *for some* $\sigma > 0$. *Let* $r > 1$ *fixed,* Z *be as in Theorem 6.1,* $K \geqslant 384rL^2$ *and* $\varepsilon > 0$. *For an arbitrary sequence of vanishing constants* $\gamma_{1,n}, \gamma_{2,n} \geqslant 0$, $n \in \mathbb{N}$, $\gamma_{1,n}, \gamma_{2,n} \to 0$, *up to a change of probability space, there exist a sequence of stationary solutions* u_n, $n \in \mathbb{N}$, *to the following stochastic Navier–Stokes/Euler equations (for* $\nu = 1$ *or* $\nu = 0$*)*

$$\mathrm{d}u_n + \mathrm{div}(u_n \otimes u_n)\,\mathrm{d}t + \nabla P_n\,\mathrm{d}t = (\nu + \gamma_{1,n})\Delta u_n\,\mathrm{d}t + \gamma_{2,n}\mathrm{d}B, \tag{6.15}$$

so that the corresponding family of laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, *is tight in* $C(\mathbb{R}; L^2_\sigma)$ *and every accumulation point is a stationary solution to the deterministic Navier–Stokes/Euler equations on* $\mathbb{R} \times \mathbb{T}^3$. *Furthermore, every accumulation point* u *satisfies*

$$\mathbf{E}\|u\|_{L^2}^2 = K,$$

and

$$\mathbf{E}\|u - Z\|_{C_t W^{1,1}}^r \leqslant \varepsilon. \tag{6.16}$$

Remark 6.7. The formulation of the above theorem in particular permits a simultaneous limit of vanishing viscosity and vanishing noise. We recall that the particular choice of $\gamma_{2,n} = \sqrt{\gamma_{1,n}}$ was treated in [Kuk04] (see also [GHSV15]) in two spatial dimensions. It gave rise to the so-called Kuksin measures, a genuinely random statistically stationary solutions to the deterministic Euler equations on $\mathbb{T}^2$. It was argued on page 472 in [Kuk04] that in three dimensions, the correct scaling is $\gamma_{2,n} = 1$ in order to be consistent with Kolmogorov's prediction of anomalous dissipation. This is indeed what is suggested by the formal energy equality, but in this setting the limit satisfies the stochastic Euler system. For the approximation (6.15) we are not able to obtain the anomalous dissipation. More precisely, we cannot prove (1.3) due to the worse spatial regularity of our solutions. This issue can be overcome by including an additional vanishing Reynolds stress as shown in Theorem 5.4.

Proof of Theorem 6.6. We decompose (6.15) with z_n a stationary solution to

$$\begin{aligned} \mathrm{d}z_n + z_n\mathrm{d}t + \nabla P_{1,n} &= (\nu+\gamma_{1,n})\Delta z_n \mathrm{d}t + \gamma_{2,n}\mathrm{d}B, \\ \mathrm{div} z_n &= 0. \end{aligned}$$

Then we have for any $p \geqslant 2$, $0<\delta<1/2$

$$\| z_n \|_{C_t^{1/2-\delta}L^\infty,p} + \| z_n \|_{C_tH^{3/2},p} \leqslant \gamma_{2,n}(p-1)^{1/2}L_1,$$

for some $L_1>0$. For each fixed n, we run the convex integration iteration as in Proposition 3.3 indexed by q and starting from $v_0 = Z$, which gives

$$\| \mathring{R}_0 \|_{L^1,r} \leqslant 4rL^2 + 2\cdot(2\pi)^3 rL + \gamma_{2,n}L_1 + 2\gamma_{2,n}^2 rL_1^2 \leqslant 8rL^2 \leqslant \frac{1}{48}\underline{e}.$$

Here we may choose n large enough such that $\gamma_{2,n}4rL_1^2+\gamma_{2,n}L_1$ is small. We then could use similar argument as in Theorem 6.1 and Theorem 6.3 to construct, up to a change of probability space, stationary solutions (u_n, z_n) so that

$$\mathbf{E}\|u_n\|_{L^2}^2 = K,$$

and

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|u_n - z_n - Z\|_{C_tW^{1,1}}^r \leqslant \varepsilon. \tag{6.17}$$

Furthermore, for some $\vartheta>0$ we obtain

$$\| u_n \|_{H^\vartheta,2r} + \| u_n \|_{C_t^\vartheta L^2,2r} \lesssim 1,$$

with the proportional constant independent of n. Hence, we obtain tightness of u_n in $C(\mathbb{R};L^2_\sigma)$ as in the proof of Theorem 5.1 and conclude that the tight limit is a stationary solution to the deterministic Navier–Stokes/Euler equations. Since $z_n\to 0$ in $C(\mathbb{R};H^1)$, (6.17) leads to (6.16). □

Appendix A. Intermittent jets

In this part we recall the construction of intermittent jets from [BV19a, Section 7.4]. We point out that the construction is entirely deterministic, that is, none of the functions below depends on ω. Let us begin with the following geometric lemma which can be found in [BV19a, Lemma 6.6].

Lemma A.1. *Denote by $\overline{B_{1/2}}(\mathrm{Id})$ the closed ball of radius $1/2$ around the identity matrix* Id*, in the space of 3×3 symmetric matrices. There exists $\Lambda\subset\mathbb{S}^2\cap\mathbb{Q}^3$ such that for each $\xi\in\Lambda$ there exists a C^∞-function $\gamma_\xi:\overline{B_{1/2}}(\mathrm{Id})\to\mathbb{R}$ such that*

$$R = \sum_{\xi\in\Lambda}\gamma_\xi^2(R)(\xi\otimes\xi)$$

for every symmetric matrix satisfying $|R-\mathrm{Id}|\leqslant 1/2$. For $C_\Lambda = 8|\Lambda|(1+8\pi^3)^{1/2}$, where $|\Lambda|$ is the cardinality of the set Λ, we define the constant

$$M = C_\Lambda \sup_{\xi\in\Lambda}(\|\gamma_\xi\|_{C^0} + \sum_{|j|\leqslant N}\|D^j\gamma_\xi\|_{C^0}).$$

For each $\xi \in \Lambda$ let us define $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ to be an orthogonal vector to ξ. Then for each $\xi \in \Lambda$ we have that $\{\xi, A_\xi, \xi \times A_\xi\} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ form an orthonormal basis for $\mathbb{R}^3$. We label by n_* the smallest natural such that

$$\{n_*\xi, n_*A_\xi, n_*\xi \times A_\xi\} \subset \mathbb{Z}^3$$

for every $\xi \in \Lambda$.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with support in a ball of radius 1. We normalize Φ such that $\phi = -\Delta\Phi$ obeys

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \phi^2(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 = 1. \tag{A.1}$$

By definition we know $\int_{\mathbb{R}^2} \phi dx = 0$. Define $\psi : \mathbb{R} \to \mathbb{R}$ to be a smooth, mean zero function with support in the ball of radius 1 satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}} \psi^2(x_3)\mathrm{d}x_3 = 1. \tag{A.2}$$

For parameters $r_\perp, r_\| > 0$ such that

$$r_\perp \ll r_\| \ll 1,$$

we define the rescaled cut-off functions

$$\phi_{r_\perp}(x_1, x_2) = \frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right), \quad \Phi_{r_\perp}(x_1, x_2) = \frac{1}{r_\perp}\Phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right), \quad \psi_{r_\|}(x_3) = \frac{1}{r_\|^{1/2}}\psi\left(\frac{x_3}{r_\|}\right).$$

We periodize $\phi_{r_\perp}, \Phi_{r_\perp}$ and $\psi_{r_\|}$ so that they are viewed as periodic functions on $\mathbb{T}^2, \mathbb{T}^2$ and $\mathbb{T}$ respectively.

Consider a large real number λ such that $\lambda r_\perp \in \mathbb{N}$, and a large time oscillation parameter $\mu > 0$. For every $\xi \in \Lambda$ we introduce

$$\begin{aligned}
\psi_{(\xi)}(t, x) &:= \psi_{\xi, r_\perp, r_\|, \lambda, \mu}(t, x) := \psi_{r_\|}(n_* r_\perp \lambda(x \cdot \xi + \mu t)) \\
\Phi_{(\xi)}(x) &:= \Phi_{\xi, r_\perp, \lambda}(x) := \Phi_{r_\perp}(n_* r_\perp \lambda(x - \alpha_\xi) \cdot A_\xi, n_* r_\perp \lambda(x - \alpha_\xi) \cdot (\xi \times A_\xi)) \\
\phi_{(\xi)}(x) &:= \phi_{\xi, r_\perp, \lambda}(x) := \phi_{r_\perp}(n_* r_\perp \lambda(x - \alpha_\xi) \cdot A_\xi, n_* r_\perp \lambda(x - \alpha_\xi) \cdot (\xi \times A_\xi)),
\end{aligned}$$

where $\alpha_\xi \in \mathbb{R}^3$ are shifts to ensure that $\{\Phi_{(\xi)}\}_{\xi \in \Lambda}$ have mutually disjoint support.

The intermittent jets $W_{(\xi)} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ are defined as in [BV19a, Section 7.4].

$$W_{(\xi)}(t, x) := W_{\xi, r_\perp, r_\|, \lambda, \mu}(t, x) := \xi \psi_{(\xi)}(t, x)\phi_{(\xi)}(x). \tag{A.3}$$

By the choice of α_ξ we have that

$$W_{(\xi)} \otimes W_{(\xi')} \equiv 0, \text{ for } \xi \neq \xi' \in \Lambda, \tag{A.4}$$

and by the normalizations (A.1) and (A.2) we obtain

$$\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} W_{(\xi)}(t, x) \otimes W_{(\xi)}(t, x)\mathrm{d}x = \xi \otimes \xi.$$

These facts combined with Lemma A.1 imply that

$$\frac{1}{(2\pi)^3}\sum_{\xi \in \Lambda} \gamma_\xi^2(R)\int_{\mathbb{T}^3} W_{(\xi)}(t, x) \otimes W_{(\xi)}(t, x)\mathrm{d}x = R, \tag{A.5}$$

for every symmetric matrix R satisfying $|R - \mathrm{Id}| \leqslant 1/2$. Since $W_{(\xi)}$ are not divergence free, we introduce the corrector term

$$W_{(\xi)}^{(c)} := \frac{1}{n_*^2\lambda^2}\nabla\psi_{(\xi)} \times \mathrm{curl}(\Phi_{(\xi)}\xi) = \mathrm{curl}\,\mathrm{curl}\, V_{(\xi)} - W_{(\xi)}. \tag{A.6}$$

with

$$V_{(\xi)}(t,x) := \frac{1}{n_*^2\lambda^2}\xi\psi_{(\xi)}(t,x)\Phi_{(\xi)}(x).$$

Thus we have

$$\operatorname{div}\left(W_{(\xi)} + W_{(\xi)}^{(c)}\right) \equiv 0.$$

Finally, we recall the key bounds from [BV19a, Section 7.4]. For $N, M \geqslant 0$ and $p \in [1,\infty]$ the following holds

$$\begin{aligned}
&\|\nabla^N \partial_t^M \psi_{(\xi)}\|_{C_t L^p} \lesssim r_\|^{1/p-1/2}\left(\frac{r_\perp \lambda}{r_\|}\right)^N \left(\frac{r_\perp \lambda\mu}{r_\|}\right)^M,\\
&\|\nabla^N \phi_{(\xi)}\|_{L^p} + \|\nabla^N \Phi_{(\xi)}\|_{L^p} \lesssim r_\perp^{2/p-1}\lambda^N,\\
&\|\nabla^N \partial_t^M W_{(\xi)}\|_{C_t L^p} + \frac{r_\|}{r_\perp}\|\nabla^N \partial_t^M W_{(\xi)}^{(c)}\|_{C_t L^p} + \lambda^2\|\nabla^N \partial_t^M V_{(\xi)}\|_{C_t L^p}\\
&\lesssim r_\perp^{2/p-1} r_\|^{1/p-1/2}\lambda^N\left(\frac{r_\perp\lambda\mu}{r_\|}\right)^M,
\end{aligned} \tag{A.7}$$

where the implicit constants may depend on p, N and M, but are independent of $\lambda, r_\perp, r_\|, \mu$.

Appendix B. Estimates of ρ and $a_{(\xi)}$

For completeness, we include here the detailed proof of the estimates (3.33) and (3.34) employed in Section 3.1.3.

We first aim at estimating the $C_{t,x}^N$-norm of ρ for $N \in \mathbb{N}$. To this end, we first apply the chain rule [BDLIS15, Proposition C.1] to the function $\Psi(z) = \sqrt{\ell^2 + z^2}$, $|D^m\Psi(z)| \lesssim \ell^{-m+1}$ to obtain

$$\begin{aligned}
\|\rho\|_{C_{t,x}^N} &\lesssim \left\|\sqrt{\ell^2 + |\mathring{R}_\ell|^2}\right\|_{C_{t,x}^0} + \|D\Psi\|_{C^0}\|\mathring{R}_\ell\|_{C_{t,x}^N} + \|D\Psi\|_{C^{N-1}}\|\mathring{R}_\ell\|_{C_{t,x}^1}^N + \|\gamma_\ell\|_{C_t^N}\\
&\lesssim \ell^{-4-N}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \ell^{-6N+1}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}^N + \frac{1}{2}\ell^{-N}\delta_{q+1}\bar{e}.
\end{aligned} \tag{B.1}$$

For the amplitude functions $a_{(\xi)}$ defined in (3.30) we deduce using (3.27)

$$\begin{aligned}
\|a_{(\xi)}\|_{C_t L^2} &\leqslant \|\rho\|_{C_t L^1}^{1/2}\|\gamma_\xi\|_{C^0(B_{1/2}(\mathrm{Id}))} \leqslant \frac{M}{8|\Lambda|(1+8\pi^3)^{1/2}}\left(2\ell(2\pi)^3 + 2\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \frac{1}{2}\delta_{q+1}\bar{e}\right)^{1/2}\\
&\leqslant \frac{M}{4|\Lambda|}\left(2\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \frac{1}{2}\delta_{q+1}\bar{e}\right)^{1/2},
\end{aligned} \tag{B.2}$$

where M denotes the universal constant from Lemma A.1.

Let us now estimate the $C_{t,x}^N$-norm of $a_{(\xi)}$. By Leibniz rule, we get

$$\|a_{(\xi)}\|_{C_{t,x}^N} \lesssim \sum_{m=0}^N \|\rho^{\frac12}\|_{C_{t,x}^m}\left\|\gamma_\xi\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho}\right)\right\|_{C_{t,x}^{N-m}} \tag{B.3}$$

and estimate each norm separately. First, by (3.28)

$$\|\rho^{1/2}\|_{C_{t,x}^0} \lesssim \ell^{-2}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}^{1/2} + \delta_{q+1}^{1/2}\bar{e}^{1/2},$$

and by Lemma A.1

$$\left\|\gamma_\xi\left(\mathrm{Id}-\frac{\mathring{R}_\ell}{\rho}\right)\right\|_{C^0_{t,x}}\lesssim 1.$$

Second, applying [BDLIS15, Proposition C.1] to the function $\Psi(z)=z^{1/2}$, $|D^m\Psi(z)|\lesssim|z|^{1/2-m}$, for $m=1,\dots,N$, and using (B.1) and $\rho\geqslant\ell$ and $\ell^{-1}\geqslant\bar e$ we obtain for $m\geqslant 1$

$$\begin{aligned}\|\rho^{1/2}\|_{C^m_{t,x}}&\lesssim\|\rho^{1/2}\|_{C^0_{t,x}}+\ell^{-1/2}\|\rho\|_{C^m_{t,x}}+\ell^{1/2-m}\|\rho\|^m_{C^1_{t,x}}\\&\lesssim\ell^{-3m+1/2}\delta^m_{q+1}+\ell^{-2}\|\mathring{R}_q\|^{1/2}_{C_{[t-1,t+1]}L^1}+\ell^{-9/2-m}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}\\&\quad+\ell^{-6m+1/2}\|\mathring{R}_q\|^m_{C_{[t-1,t+1]}L^1}+\ell^{-m-3/2}\delta_{q+1}+\ell^{-1/2}\delta^{1/2}_{q+1}.\end{aligned}\tag{B.4}$$

For $m\geqslant 1$ using $\delta_{q+1}\leqslant 1$ the above is bounded by

$$\ell^{-6m+1/2}(1+\|\mathring{R}_q\|^m_{C_{[t-1,t+1]}L^1}).$$

We proceed with a bound for $\left\|\gamma_\xi\left(\mathrm{Id}-\frac{\mathring{R}_\ell}{\rho}\right)\right\|_{C^{N-m}_{t,x}}$ for $m=0,\dots,N-1$. Keeping [BDLIS15, Proposition C.1] as well as Lemma A.1 in mind, we need to estimate

$$\left\|\frac{\mathring{R}_\ell}{\rho}\right\|_{C^{N-m}_{t,x}}+\left\|\frac{\nabla_{t,x}\mathring{R}_\ell}{\rho}\right\|^{N-m}_{C^0_{t,x}}+\left\|\frac{\mathring{R}_\ell}{\rho^2}\right\|^{N-m}_{C^0_{t,x}}\|\rho\|^{N-m}_{C^1_{t,x}}.\tag{B.5}$$

We use $\rho\geqslant\ell$ to have

$$\left\|\frac{\nabla_{t,x}\mathring{R}_\ell}{\rho}\right\|^{N-m}_{C^0_{t,x}}\lesssim\ell^{-(N-m)}\ell^{(-4-1)(N-m)}\|\mathring{R}_q\|^{N-m}_{C_{[t-1,t+1]}L^1}\lesssim\ell^{-6(N-m)}\|\mathring{R}_q\|^{N-m}_{C_{[t-1,t+1]}L^1},$$

and in view of $|\frac{\mathring{R}_\ell}{\rho}|\leqslant 1$

$$\left\|\frac{\mathring{R}_\ell}{\rho^2}\right\|^{N-m}_{C^0_{t,x}}\lesssim\left\|\frac{1}{\rho}\right\|^{N-m}_{C^0_{t,x}}\lesssim\ell^{-(N-m)},$$

and by (B.1) and $\bar e\delta_{q+1}\leqslant\ell^{-1}$

$$\|\rho\|^{N-m}_{C^1_{t,x}}\lesssim\ell^{-5(N-m)}\|\mathring{R}_q\|^{N-m}_{C_{[t-1,t+1]}L^1}+\ell^{-2(N-m)}.$$

To estimate the first term in (B.5), we write

$$\left\|\frac{\mathring{R}_\ell}{\rho}\right\|_{C^{N-m}_{t,x}}\lesssim\sum_{k=0}^{N-m}\|\mathring{R}_\ell\|_{C^k_{t,x}}\left\|\frac{1}{\rho}\right\|_{C^{N-m-k}_{t,x}},\tag{B.6}$$

where for $N-m-k=0$ we have

$$\left\|\frac{1}{\rho}\right\|_{C^0_{t,x}}\lesssim\ell^{-1}$$

and for $k=0,\dots,N-m-1$ using (B.1)

$$\begin{aligned}\left\|\frac{1}{\rho}\right\|_{C^{N-m-k}_{t,x}}&\lesssim\left\|\frac{1}{\rho}\right\|_{C^0_{t,x}}+\ell^{-2}\|\rho\|_{C^{N-m-k}_{t,x}}+\ell^{-(N-m-k)-1}\|\rho\|^{N-m-k}_{C^1_{t,x}}\\&\lesssim\ell^{-4(N-m-k)-1}+\ell^{-6-(N-m-k)}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1}+\ell^{-6(N-m-k)-1}\|\mathring{R}_q\|^{N-m-k}_{C_{[t-1,t+1]}L^1}.\end{aligned}$$

Altogether, we therefore bound (B.6) as

$$\left\|\frac{\mathring{R}_\ell}{\rho}\right\|_{C^{N-m}_{t,x}} \lesssim \ell^{-5-4(N-m)}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \ell^{-10-(N-m)}\|\mathring{R}_q\|^2_{C_{[t-1,t+1]}L^1} + \ell^{-5-6(N-m)}(\|\mathring{R}_q\|^{N-m+1}_{C_{[t-1,t+1]}L^1} + 1). \tag{B.7}$$

Finally, plugging (B.7), and the other bounds into (B.5) leads to

$$\left\|\gamma_\xi\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho}\right)\right\|_{C^{N-m}_{t,x}} \lesssim \ell^{-5-4(N-m)}\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + \ell^{-10-(N-m)}\|\mathring{R}_q\|^2_{C_{[t-1,t+1]}L^1} + \ell^{-5-6(N-m)}(\|\mathring{R}_q\|^{N-m+1}_{C_{[t-1,t+1]}L^1}+1) + \ell^{-3(N-m)}.$$

For $N-m \geqslant 1$ the above is bounded by

$$\ell^{-5-6(N-m)}(\|\mathring{R}_q\|^{N-m+1}_{C_{[t-1,t+1]}L^1} + \|\mathring{R}_q\|^2_{C_{[t-1,t+1]}L^1} + 1).$$

Combining this with the bounds for $\rho^{1/2}$ above and plugging into (B.3) yields for $N \geqslant 2$

$$\|a_{(\xi)}\|_{C^N_{t,x}} \lesssim \ell^{-7-6N}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + 1)^{N+3/2}.$$

It is easy to see that

$$\|a_{(\xi)}\|_{C^0_{t,x}} \lesssim \ell^{-2}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + 1)^{1/2}. \tag{B.8}$$

By interpolation $\|a_{(\xi)}\|_{C^N_{t,x}} \lesssim \|a_{(\xi)}\|^{1/2}_{C^0_{t,x}}\|a_{(\xi)}\|^{1/2}_{C^{2N}_{t,x}}$, the following estimate

$$\|a_{(\xi)}\|_{C^N_{t,x}} \lesssim \ell^{-7-6N}(\|\mathring{R}_q\|_{C_{[t-1,t+1]}L^1} + 1)^{N+1}. \tag{B.9}$$

holds.

References

[BBPS19] J. Bedrossian, A. Blumenthal, S. Punshon-Smith, The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics at fixed Reynolds number, *arXiv:1911.11014*, 2019.

[BFHM19] D. Breit, E. Feireisl, M. Hofmanová, B. Maslowski. *Stationary solutions to the compressible Navier–Stokes system driven by stochastic forcing. Probab. Theory Relat. Fields* 174, 981–1032, 2019.

[BFH20e] D. Breit, E. Feireisl, M. Hofmanová. On the long time behavior of compressible fluid flows excited by random forcing. *arXiv:2012.07476*, 2020.

[BD22] E. Brué, C. De Lellis. Anomalous dissipation for the forced 3D Navier–Stokes equations. *arXiv:2207.06301*, 2022.

[BCV18] T. Buckmaster, M. Colombo, V. Vicol. Wild solutions of the navier-stokes equations whose singular sets in time have hausdorff dimension strictly less than 1. *arXiv:1809.00600*, 2018.

[BDLIS15] T. Buckmaster, C. De Lellis, P. Isett, L. Székelyhidi, Jr. Anomalous dissipation for 1/5-Hölder Euler flows. *Ann. of Math. (2)*, 182(1):127–172, 2015.

[BDLSV19] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.

[BMS20] J. Burczak, S. Modena, L. Székelyhidi. Non uniqueness of power-law flows. *arXiv:2007.08011*, 2020.

[BV19a] T. Buckmaster, V. Vicol. Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.*, 6(1-2):173–263, 2019.

[BV19b] T. Buckmaster, V. Vicol. Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. of Math. (2)*, 189(1):101–144, 2019.

[CDZ22] W. Chen, Z. Dong, X. Zhu, Ill-posedness of d-dimensional incompressible stochastic Navier-Stokes equations, preprint, 2022.

[CG12] G. Q. Chen, J. Glimm. Kolmogorov's Theory of Turbulence and Inviscid Limit of the Navier-Stokes Equations in $\mathbb{R}^3$. *Commun. Math. Phys.* 310, 267–283, 2012.

[CL19] A. Cheskidov, X. Luo. Stationary and discontinuous weak solutions of the Navier-Stokes equations. *arXiv:1901.07485*, 2019.

[CL20] A. Cheskidov, X. Luo. Sharp nonuniqueness for the Navier-Stokes equations. *arXiv:2009.06596*, 2020.

[CCS22] M. Colombo, G. Crippa, M. Sorella. Anomalous dissipation and lack of selection in the Obukhov-Corrsin theory of scalar turbulence. *arXiv:2207.06833*, 2022.

[CV18] P. Constantin, V. Vicol. Remarks on High Reynolds Numbers Hydrodynamics and the Inviscid Limit. *J. Nonlinear Sci.* 28, 711–724, 2018.

[CH21] M. Coti Zelati, M. Hairer, A Noise-Induced Transition in the Lorenz System. *Commun. Math. Phys.* 383, 2243–2274, 2021.

[DK20] C. De Lellis, , H. Kwon, On non-uniqueness of Hölder continuous globally dissipative Euler flows, http://arxiv.org/abs/2006.06482v2, 2020.

[DS17] S. Daneri, L. Székelyhidi, Jr. Non-uniqueness and h-Principle for Hölder-Continuous Weak Solutions of the Euler Equations. *Arch. Rational Mech. Anal.* 224, 471–514, 2017.

[DPD03] G. Da Prato and A. Debussche. Ergodicity for the 3D stochastic Navier–Stokes equations. *J. Math. Pures Appl.*, 82:877–947, 2003.

[DPZ96] G. Da Prato, J. Zabczyk. Ergodicity for infinite dimensional systems. Vol. 229. Cambridge University Press, 1996.

[DPZ92] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[DLS09] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.

[DLS10] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[DLS13] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.

[DLS21] C. De Lellis and L. Székelyhidi, Jr. Weak stablity and closure in turbulence. *arxiv.2108.01597*, 2021.

[FFH21] F. Fanelli, E. Feireisl, M. Hofmanová, Ergodic theory for energetically open compressible fluid flows, *Physica D: Nonlinear Phenomena*, 423, 2021.

[FG95] F. Flandoli and D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations *Probab. Theory Related Fields*, 102:367–391, 1995.

[FR08] F. Flandoli and M. Romito. Markov selections for the 3D stochastic Navier-Stokes equations. *Probab. Theory Related Fields*, 140(3-4):407–458, 2008.

[Fri95] U. Frisch. *Turbulence.* Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.

[FGHV16] S. Friedlander, N. Glatt-Holtz, V. Vicol, Inviscid limits for a stochastically forced shell model of turbulent flow *Ann. Inst. H. Poincaré Probab. Statist.* 52(3), 1217–1247, 2016.

[GHSV15] N. Glatt-Holtz, V. Sverák, V. Vicol. On inviscid limits for the stochastic Navier–Stokes equations and related models. *Arch. Rational Mech. Anal.* 217(2) 619–649, 2015.

[HM06] M. Hairer, J. C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) 164, 993–1032, 2006.

[HZZ19] M. Hofmanová, R. Zhu, X. Zhu. Non-uniqueness in law of stochastic 3D Navier–Stokes equations. *to appear in Journal of the European Mathematical Society, arXiv:1912.11841v2*, 2019.

[HZZ21] M. Hofmanová, R. Zhu, X. Zhu. Global existence and non-uniqueness for 3D Navier–Stokes equations with space-time white noise. *to appear in Arch. Ration. Mech. Anal., arXiv:2112.14093*, 2021.

[HZZ22a] M. Hofmanová, R. Zhu, X. Zhu. On ill- and well-posedness of dissipative martingale solutions to stochastic 3D Euler equations. *Communications on Pure and Applied Mathematics,* , Vol. LXXV, 2446–2510, 2022.

[HZZ22] M. Hofmanová, R. Zhu, X. Zhu. A class of supercritical/critical singular stochastic PDEs: existence, non-uniqueness, non-Gaussianity, non-unique ergodicity. *arXiv:2205.13378*, 2022.

[HZZ23] M. Hofmanová, R. Zhu, X. Zhu. Global-in-time probabilistically strong and Markov solutions to stochastic 3D Navier–Stokes equations: existence and non-uniqueness. *The Annals of Probability*, Vol. 51, No. 2, 524–579, 2023.

[Ise18] P. Isett. A proof of Onsager's conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.

[Kol41a] A. N. Kolmogorov. Dissipation of energy in the locally isotropic turbulence. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 32:16–18, 1941.

[Kol41b] A. N. Kolmogorov. On degeneration of isotropic turbulence in an incompressible viscous liquid. *C. R. (Doklady) Acad. Sci. URSS (N. S.)*, 31:538–540, 1941.

[Kol41c] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynold's numbers. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 30:301–305, 1941.

[Kuk04] S. B. Kuksin, The Eulerian limit for 2D statistical hydrodynamics. *Journal of Statistical Physics* 115(1), 469–492, 2004.

[Luo19] X. Luo. Stationary Solutions and Nonuniqueness of Weak Solutions for the Navier?Stokes Equations in High Dimensions. *Arch. Rational Mech. Anal.* 233, 701–747, 2019.

[MY13] A. Monin and A. Yaglom. Statistical fluid mechanics, volume II: mechanics of turbulence, volume 2. Courier Corporation, 2013.

[Ons49] L. Onsager. Statistical hydrodynamics. *Nuovo Cim. Suppl.* 6, 279–287, 1949.

[PR07] C. Prévôt, M. Röckner, A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007

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