Convergence in Wasserstein Distance for Empirical Measures of Non-Symmetric Subordinated Diffusion Processes*

Feng-Yu Wang

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

wangfy@tju.edu.cn

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Abstract

By using the spectrum of the underlying symmetric diffusion operator, the convergence in L^p-Wasserstein distance $\mathbb{W}_p(p \geq 1)$ is characterized for the empirical measure μ_t of non-symmetric subordinated diffusion processes in an abstract framework. The main results are applied to the subordinations of several typical models, which include the (reflecting) diffusion processes on compact manifolds, the conditional diffusion processes, the Wright-Fisher diffusion process, and hypoelliptic diffusion processes on $\mathbf{SU}(2)$. In particular, for the (reflecting) diffusion processes on a compact Riemannian manifold with invariant probability measure μ:

(1) the sharp limit of $t\mathbb{W}_2(\mu_t, \mu)^2$ is derived in $L^q(\mathbb{P})$ for concrete $q \geq 1$, which provides a precise characterization on the physical observation that a divergence-free perturbation accelerates the convergence in $\mathbb{W}_2$;

(2) the sharp convergence rates are presented for $(\mathbb{E}[\mathbb{W}_{2p}(\mu_t, \mu)^q])^{\frac{1}{q}}(p, q \geq 1)$, where a critical phenomenon appears with the critical rate $t^{-1} \log t$ as $t \to \infty$.

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1 Introduction

A. Background of the study. In statistical physics, the empirical measure is a fundamental object to simulate the stationary distribution (Gibbs measure). Since the ground breaking

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series work [12] (1975-1983) where Donsker and Varadhan developed their celebrated larger deviation principle, the long time behavior of empirical measures has become a key research topic in the study of Markov processes, see [39, 40] for criteria on the central limit theorem and large deviations for hyperbounded Markov processes.

On the other hand, the Wasserstein distance is intrinsic in the theory of optimal transport and calculus on Wasserstein space, see [1, 28] and references therein. So, it is crucial and interesting to study the convergence in Wasserstein distance for the empirical measure of Markov processes.

Moreover, it was observed in [16] that a divergence-free perturbation to symmetric stochastic systems may accelerate the algorithm of Gibbs measures. This has been confirmed in several papers for the convergence of Markov semigroups to stationary distributions, see [17, 19, 20] and references therein. It is interesting to provide a sharp characterization on the acceleration for the convergence of empirical measures in Wasserstein distance.

In recent years, the sharp convergence rate in the second moment of the L^2-Wasserstein distance has been derived in [33, 36, 34, 35, 38] for empirical measures of symmetric diffusion processes. In particular, in lower dimensions the precise limit is explicitly formulated by using eigenvalues and eigenfunctions of the generator. These results have been extended to subordinated processes in [37, 22, 23, 24] and the fractional Brownian motion on torus in [18], see also [13] for the study of McKean-Vlasov SDEs.

B. Purpose of the present work. Based on the above background, this paper investigates the convergence of empirical measures for *non-symmetric subordinated* diffusion processes in an *abstract framework*, describes the *acceleration of the convergence* for divergence-free perturbations to symmetric systems, and illustrates the main results by typical examples.

To figure out a clear picture of our general results (see Section 2 for details), in the following we only consider non-symmetric diffusion processes on a compact manifold. See Section 5 for applications of the general results to three more examples including the subordinated conditional diffusion process, the subordinated Wright-Fisher process, and the subordinated subelliptic diffusion process on $\mathbf{SU}(2)$.

C. A picture for non-symmetric diffusion processes on compact manifolds. Let M be an n-dimensional compact connected Riemannian manifold possibly with a boundary ∂M. Let $\mathscr{P}$ be the space of all probability measures on M, let ρ be the Riemannian distance, and for any $p \geq 1$, let $\mathbb{W}_p$ be the L^p-Wasserstein distance induced by ρ, cf. (2.1) below.

Let $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x \in \mathscr{P}$, where $V \in C^2(M)$ and $\mathrm{d}x$ is the volume measure on M, and let Z be a C^1-vector field with $\mathrm{div}_\mu Z = 0$, i.e.

$$\mu(Zf) := \int_M \langle Z, \nabla f\rangle \mathrm{d}\mu = 0, \quad f \in C^1(M).$$

Then the spectrum of $\hat{L} := \Delta + \nabla V$ (with Neumann boundary if ∂M exists) is discrete, and all eigenvalues $\{\lambda_i\}_{i\geq 0}$ of $-\hat{L}$ listed in the increasing order counting multiplicities satisfy

$$c_1 i^{\frac{n}{2}} \leq \lambda_i \leq c_2 i^{\frac{n}{2}}, \quad i \geq 0$$

for some constants $c_1, c_2 > 0$, see for instance [9]. Let $\{\phi_i\}_{i\geq 0}$ with $\phi_0 \equiv 1$ being the corresponding unitary eigenfunctions in $L^2(\mu)$.

Let X_t be the diffusion process on M generated by

$$L := \Delta + \nabla V + Z,$$

with reflecting boundary if ∂M exists. We consider the empirical measure

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s, \quad t > 0,$$

where δ_{X_s} is the Dirac measure at X_s. By the central limit theorem (see [39]), for any

$$f \in L_0^2(\mu) := \big\{f \in L^2(\mu) :\ \mu(f) = 0\big\},$$

we have

$$\lim_{t\to\infty} \sqrt{t}\mu_t(f) = \lim_{t\to\infty} \frac{1}{\sqrt{t}}\int_0^t f(X_s)\mathrm{d}\mathbf{s} = N(0, \mathbf{V}(f)) \ \text{ in law,} \tag{1.1}$$

where $N(0, \mathbf{V}(f))$ is the centered normal distribution on $\mathbb{R}$ with variance

$$\mathbf{V}(f) := \int_0^\infty \mu(fP_s f)\mathrm{d}s = \mu(|\nabla L^{-1} f|^2). \tag{1.2}$$

For any $k, R \geq 1$, let

$$\mathscr{P}_{k,R} := \big\{\nu \in \mathscr{P} :\ \mathrm{d}\nu = h\mathrm{d}\mu, \|h\|_k \leq R\big\}, \tag{1.3}$$

where $\|\cdot\|_k$ is the norm in $L^k(\mu)$. For any $\nu \in \mathscr{P}$, let $\mathbb{E}^\nu$ be the expectation for the diffusion process X_t with initial distribution ν.

We first consider the long time behavior of $\mathbb{W}_2(\mu_t, \mu)$. The following result shows that when $n \leq 3$ and ∂M is either empty or convex, for long time $t\mathbb{W}_2(\mu_t, \mu)^2$ behaves as

$$\Xi(t) := \sum_{i=1}^\infty \frac{1}{\lambda_i}|\psi_i(t)|^2, \quad \psi_i(t) := \frac{1}{\sqrt{t}}\int_0^t \phi_i(X_s)\mathrm{d}s,$$

so that uniformly in $\nu \in \mathscr{P}$, $t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu)^2]$ converges to

$$\eta_Z := \sum_{i=1}^\infty \frac{2\mathbf{V}(\phi_i)}{\lambda_i} = \sum_{i=1}^\infty \frac{2}{\lambda_i^2}\Big(1 - \frac{1}{\lambda_i}\mathbf{V}(Z\phi_i)\Big), \tag{1.4}$$

where the second equality follows from Lemma 4.2 below.

Theorem 1.1. *There exists a constant $\kappa \geq 1$ with $\kappa = 1$ when ∂M is either empty or convex, such that the following assertions hold.*

(1) *When* $n \le 2$, *for any* $q \in [1, \frac{2n}{(3n-4)^+})$,

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu\Big[\big|\big\{t\mathbb{W}_2(\mu_t,\mu)^2 - \Xi(t)\big\}^+ + \big\{\Xi(t) - \kappa t\mathbb{W}_2(\mu_t,\mu)^2\big\}^+\big|^q\Big] = 0, \tag{1.5}$$

so that when ∂M *is either empty or convex,*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu\Big[\big|t\mathbb{W}_2(\mu_t,\mu)^2 - \Xi(t)\big|^q\Big] = 0. \tag{1.6}$$

(2) *When* $n = 3$, *for any* $R \in [1,\infty)$, $k \in (\frac{3}{2},\infty]$ *and* $q \in [1, \frac{6}{5})$,

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\Big[\big|\big\{t\mathbb{W}_2(\mu_t,\mu)^2 - \Xi(t)\big\}^+ + \big\{\Xi(t) - \kappa t\mathbb{W}_2(\mu_t,\mu)^2\big\}^+\big|^q\Big] = 0, \tag{1.7}$$

so that when ∂M *is either empty or convex,*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\Big[\big|t\mathbb{W}_2(\mu_t,\mu)^2 - \Xi(t)\big|^q\Big] = 0. \tag{1.8}$$

(3) *For* $n \le 3$, *we have* $\eta_Z \in (0,\infty)$ *and*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}} \Big(\big\{t\mathbb{E}^\nu[W_2(\mu_t,\mu)^2] - \eta_Z\big\}^+ + \big\{\eta_Z - \kappa t\mathbb{E}^\nu[W_2(\mu_t,\mu)^2]\big\}^+\Big) = 0. \tag{1.9}$$

In particular, when ∂M *is either empty or convex,*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}} \big|t\mathbb{E}^\nu[W_2(\mu_t,\mu)^2] - \eta_Z\big| = 0. \tag{1.10}$$

(4) *For* $n = 4$, *there exist constants* $c_1, c_2, t_0 > 0$ *such that*

$$\frac{c_1}{t}\log(1+t) \le \inf_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu)^2] \le \sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu)^2] \le \frac{c_2}{t}\log(1+t), \quad t \ge t_0. \tag{1.11}$$

(5) *For* $n \ge 5$, *there exist constants* $c_1, c_2, t_0 > 0$ *such that*

$$c_1 t^{-\frac{2}{d-2}} \le \inf_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t,\mu)]\big)^2 \le \sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu)^2] \le c_2 t^{-\frac{2}{d-2}}, \quad t \ge 1. \tag{1.12}$$

Remark 1.1. (1) By (1.4) we have $\eta_Z < \eta_0$ for $Z \ne 0$, so (1.9) and (1.10) provide a precise characterization on the acceleration of a divergence-free perturbation Z for the convergence of empirical measures in $\mathbb{W}_2$.

(2) When $Z = 0$ (i.e. the symmetric case), (1.9), (1.10), (1.12) and the upper bound in (1.11) have been presented in [38], which are covered by Theorem 1.1. The L^q-convergence (1.5)-(1.8) appear here for the first time, which together with the lower bound in (1.11) are new also in the symmetric case.

(3) It is proved in [38] that for M being the 4-dimensional torus and $L = \Delta$, there exists a constant $c > 0$ such that

$$\inf_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t,\mu)]\big)^2 \ge ct^{-1}\log(1+t),\quad t\ge 1.$$

We hope that this estimate also holds for general non-symmetric diffusions on 4-dimensional compact manifolds, such that the lower bound estimate in (1.11) is strengthened with $\mathbb{W}_1$ replacing $\mathbb{W}_2$.

In the next result, we estimate $\big(\mathbb{E}[\mathbb{W}_{2p}(\mu_t,\mu)^{2q}]\big)^{\frac{1}{q}}$ for all $p,q\in[1,\infty)$. Besides the critical phenomenon in Theorem 1.1 with the critical convergence rate $t^{-1}\log t$ for dimension $n=4$, the critical rate also appears to dimensions $n=2,3$ with different (p,q).

Theorem 1.2. *There exist $c, t_0 \in (0,\infty)$ and $\kappa : [1,\infty)\times[1,\infty)\to(0,\infty)$, such that the following assertions hold.*

(1) *When $n=1$, for any $(p,q)\in[1,\infty)\times[1,\infty)$,*

$$\frac{c}{t} \le \inf_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t,\mu)]\big)^2 \le \sup_{\nu\in\mathscr{P}} \big\{\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t,\mu)^{2q}]\big\}^{\frac{1}{q}} \le \frac{\kappa_{p,q}}{t},\quad t\ge t_0. \tag{1.13}$$

(2) *Let $n=2$. Then (1.13) holds for any $p\in[1,\infty)$ and $q\in[1,\frac{p}{p-1})$. Next, for any $p\in(1,\infty)$ and $q=\frac{p}{p-1}$,*

$$\sup_{\nu\in\mathscr{P}} \big\{\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t,\mu)^{2q}]\big\}^{\frac{1}{q}} \le \frac{\kappa_{p,q}}{t}\log(1+t),\quad t\ge t_0. \tag{1.14}$$

Finally, for any $p\in(1,\infty)$ and $q\in(\frac{p}{p-1},\infty)$,

$$\sup_{\nu\in\mathscr{P}} \big\{\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t,\mu)^{2q}]\big\}^{\frac{1}{q}} \le \kappa_{p,q}t^{-\frac{2}{n(3-p^{-1}-q^{-1})-2}},\quad t\ge 1. \tag{1.15}$$

(3) *Let $n=3$. Then (1.13) holds for any $p\in[1,\frac{3}{2})$ and $q\in[1,\frac{3p}{5p-3})$; (1.14) holds for $p\in[1,\frac{3}{2})$ and $q=\frac{3p}{5p-3}$; and (1.15) holds for any $p\in[1,\infty)$ and $q\in(\frac{3p}{5p-3},\infty)\cap[1,\infty)$.*

(4) *Let $n=4$. Then (1.14) holds for $p=q=1$, and (1.15) holds for any $(p,q)\in[1,\infty)\times[1,\infty)\setminus\{(1,1)\}$.*

(5) *When $n\ge 5$, (1.15) holds for any $(p,q)\in[1,\infty)\times[1,\infty)$.*

D. Structure of the paper. In Section 2, we state our main results for non-symmetric subordinated diffusion processes in an abstract framework. In Sections 3 and 4, we prove the main results on upper and lower bound estimates respectively. In Section 5, we apply the main results to some concrete models, where the result for the first model covers Theorems 1.1 and 1.2 as direct consequences with $B(\lambda)=\lambda$ (hence, $\alpha=1$).

2 Main results in an abstract framework

We first introduce the framework of the study, then state the main results on the Wasserstein distance of the empirical measures for non-symmetric subordinated diffusion processes.

2.1 The framework

A. State space. Let (M, ρ) be a length space, let $\mathscr{P}$ be the set of all probability measures on M, let $\mathscr{B}_b(M)$ be the class of bounded measurable functions on M, and let $C_{b,L}(M)$ be the set of all bounded Lipschitz continuous functions on M. For any $p \in [1, \infty)$, the L^p-Wasserstein distance is defined as

$$\mathbb{W}_p(\nu_1, \nu_2) := \inf_{\pi \in \mathscr{C}(\nu,\gamma)} \left(\int_{M \times M} \rho(x, y)^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}}, \quad \nu_1, \nu_2 \in \mathscr{P}, \tag{2.1}$$

where $\mathscr{C}(\nu_1, \nu_2)$ is the set of all couplings for ν_1 and ν_2.

B. Symmetric diffusion process. Let $\hat{X}_t$ be a reversible Markov process on M with the unique invariant probability measure $\mu \in \mathscr{P}$ having full support. For any $q \geq p \in [1, \infty]$, let $\|\cdot\|_p$ be the norm in $L^p(\mu)$, and let $\|\cdot\|_{p \to q}$ the operator norm from $L^p(\mu)$ to $L^q(\mu)$. Throughout the paper, we simply denote $\mu(f) = \int_M f \mathrm{d}\mu$ for $f \in L^1(\mu)$.

The Markov semigroup $\hat{P}_t$ is formulated as

$$\hat{P}_t f(x) = \mathbb{E}^x[f(\hat{X}_t)], \quad t \geq 0, x \in M, f \in \mathscr{B}_b(M),$$

where and in the sequel, $\mathbb{E}^x$ stands for the expectation for the underlying Markov process starting at point x. In general, for any $\nu \in \mathscr{P}$, $\mathbb{E}^\nu$ is the expectation for the underlying Markov process with initial distribution ν.

Let $(\hat{\mathscr{E}}, \mathscr{D}(\hat{\mathscr{E}}))$ and $(\hat{L}, \mathscr{D}(\hat{L}))$ be, respectively, the associated symmetric Dirichlet form and self-adjoint generator in $L^2(\mu)$. We assume that $C_{b,L}(M)$ is a dense subset of $\mathscr{D}(\hat{\mathscr{E}})$ under the $\hat{\mathscr{E}}_1$-norm $\|f\|_{\hat{\mathscr{E}}_1} := \sqrt{\mu(f^2) + \hat{\mathscr{E}}(f, f)}$, and

$$\hat{\mathscr{E}}(f, g) = \int_M \Gamma(f, g) \mathrm{d}\mu, \quad f, g \in C_{b,L}(M)$$

holds for a symmetric local square field (champ de carré)

$$\Gamma : C_{b,L}(M) \times C_{b,L}(M) \to \mathscr{B}_b(M),$$

such that for any $f, g, h \in C_{b,L}(M)$ and $\phi \in C_b^1(\mathbb{R})$, we have

$$\sqrt{\Gamma(f, f)(x)} = |\nabla f(x)| := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\rho(x, y)}, \quad x \in M,$$
$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h), \quad \Gamma(\phi(f), h) = \phi'(f)\Gamma(f, h).$$

We also assume that $\hat{L}$ satisfies the chain rule

$$\hat{L}\Phi(f) = \Phi'(f)\hat{L}f + \Phi''(f)|\nabla f|^2, \quad f \in \mathscr{D}(\hat{L}) \cap C_{b,l}(M), \Phi \in C^2(\mathbb{R}).$$

C. Non-symmetric perturbation. Let

$$Z : C_{b,L}(M) \to \mathscr{B}_b(M)$$

be a bounded vector field with $\mathrm{div}_\mu Z = 0$, i.e. it satisfies

$$\begin{aligned}
&Z(fg) = fZg + gZf, \quad Z(\phi(f)) = \phi'(f)Zf, \quad f, g \in C_{b,L}(M), \ \phi \in C^1(\mathbb{R}), \\
&\|Z\|_\infty := \inf\big\{K \geq 0 : \ |Zf| \leq K|\nabla f|, \ f \in C_{b,L}(M)\big\} < \infty, \\
&\mu(Zf) := \int_M (Zf)\mathrm{d}\mu = 0, \quad f \in C_{b,L}(M).
\end{aligned}$$

Consequently, Z uniquely extends to a bounded linear operator from $\mathscr{D}(\hat{\mathscr{E}})$ to $L^2(\mu)$ with

$$\mu(Zf) = 0, \quad f \in \mathscr{D}(\hat{\mathscr{E}}), \tag{2.2}$$

and

$$\mathscr{E}(f,g) := \hat{\mathscr{E}}(f,g) + \mu(fZg), \quad f, g \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\hat{\mathscr{E}})$$

is a (non-symmetric) conservative Dirichlet form with generator

$$L := \hat{L} + Z, \quad \mathscr{D}(L) = \mathscr{D}(\hat{L}),$$

which satisfies the chain rule

$$L\Phi(f) = \Phi'(f)Lf + \Phi''(f)|\nabla f|^2, \quad f \in \mathscr{D}(\hat{L}) \cap C_{b,l}(M), \Phi \in C^2(\mathbb{R}).$$

Assume that L generates a unique diffusion process X_t on M, such that the associated Markov semigroup is given by

$$P_t f(x) = \mathbb{E}^x[f(X_t)], \quad x \in M, t \geq 0, f \in \mathscr{B}_b(M).$$

By Duhamel's formula,

$$P_t f = \hat{P}_t f + \int_0^t P_s\{Z\hat{P}_{t-s}f\}\mathrm{d}s, \quad f \in \mathscr{D}(\hat{\mathscr{E}}), t \geq 0. \tag{2.3}$$

C. Subordination. Let $\mathbf{B}$ be the set of Bernstein functions B satisfying $B(0) = 0$ and $B(r) > 0$ for $r > 0$. For each $B \in \mathbf{B}$, there exists a unique stable increasing process S_t^B on $[0, \infty)$ with Laplace transform

$$\mathbb{E}[\mathrm{e}^{-rS_t^B}] = \mathrm{e}^{-B(r)t}, \quad t, r \geq 0. \tag{2.4}$$

Let S_t^B be independent of X_t. We consider the subordinated diffusion process

$$X_t^B := X_{S_t^B}, \quad t \geq 0,$$

and study the convergence to μ in $\mathbb{W}_p$ ($p \geq 1$) for the empirical measure

$$\mu_t^B := \frac{1}{t}\int_0^t \delta_{X_s^B} \mathrm{d}s, \quad t > 0.$$

We will mainly consider α-stable type time change for $\alpha \in [0,1]$, i.e. the Bernstein function B is in the classes

$$\mathbf{B}^\alpha := \Big\{B \in \mathbf{B}: \liminf_{r\to\infty} B(r)r^{-\alpha} > 0\Big\}, \quad \mathbf{B}_\alpha := \Big\{B \in \mathbf{B}: \limsup_{r\to\infty} B(r)r^{-\alpha} < \infty\Big\}.$$

2.2 Upper bound estimates

We make the following assumption, where (2.5) implies that the spectrum of $-\hat{L}$ is discrete and all eigenvalues $\{\lambda_i\}_{i\geq 0}$ listed in the increasing order counting multiplicities satisfy

$$\lambda_i \geq c i^{\frac{2}{d}}, \quad i \in \mathbb{Z}_+$$

for some constant $c > 0$, where $\lambda_1 \geq \lambda$, see for instance [11]. In general, λ_i may increase faster than $i^{\frac{2}{d}}$, see for instance Subsection 5.2 where $d = n+2$ but $\lambda_i \sim i^{\frac{2}{n}}$, we make the additional assumption (2.6).

(A_1) Let $B \in \mathbf{B}^\alpha$ for some $\alpha \in [0,1]$. There exist constants $c, \lambda > 0, d \geq d' \geq 1$ and a map $k: (1,\infty) \to (0,\infty)$ such that

$$\|\hat{P}_t - \mu\|_{1\to\infty} \leq c t^{-\frac{d}{2}} \mathrm{e}^{-\lambda t}, \quad t > 0, \tag{2.5}$$

$$\lambda_i \geq c i^{\frac{2}{d'}}, \quad i \in \mathbb{Z}_+, \tag{2.6}$$

$$|\nabla \hat{P}_t f| \leq k(p)(\hat{P}_t|\nabla f|^p)^{\frac{1}{p}}, \quad t \in [0,1], p \in (1,\infty), f \in C_{b,L}(M). \tag{2.7}$$

We will also need the following condition on the continuity of $\hat{X}_t$.

(A_2) For any $p \in [1,\infty)$ there exists a constant $c(p) > 0$ such that

$$\mathbb{E}^\mu\big[\rho(\hat{X}_0, \hat{X}_t)^p\big] \leq c(p) t^{\frac{p}{2}}, \quad t \in [0,1]. \tag{2.8}$$

Let $\{\phi_i\}_{i\geq 0}$ with $\phi_0 \equiv 1$ be the unitary eigenfunctions for $\{\lambda_i\}_{i\geq 0}$, i.e.

$$\hat{L}\phi_i = -\lambda_i\phi_i, \quad \hat{P}_t\phi_i = \mathrm{e}^{-\lambda_i t}\phi_i, \quad \mu(\phi_i\phi_j) = 1_{\{i=j\}}, \quad i,j \in \mathbb{Z}_+, t \geq 0. \tag{2.9}$$

Let

$$\Xi^B(t) := \sum_{i=1}^\infty \frac{\psi_i^B(t)^2}{\lambda_i}, \quad \psi_i^B(t) := \frac{1}{\sqrt{t}}\int_0^t \phi_i(X_s^B)\mathrm{d}s, \quad t > 0, i \in \mathbb{N}. \tag{2.10}$$

Let $\mathscr{P}_{k,R}$ be in (1.3), let

$$q_\alpha := \frac{2d}{(2d+d'-2-2\alpha)^+}, \tag{2.11}$$

and denote the integer parts of $q \in [1,\infty)$ by

$$\mathrm{i}(q) := \sup\big\{i \in \mathbb{N} : i \le q\big\}.$$

The first main result of the paper is the following.

Theorem 2.1. *Assume (A_1) and (A_2) with $d' < 2(1+\alpha)$.*

(1) *If $q_\alpha > \frac{d}{2\alpha}$ and*

$$\alpha > \alpha(d,d') := \frac{1}{4}\Big(\sqrt{(2+d-d')^2+4d(d+d'-2)} + d' - d - 2\Big),$$

then

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu\Big[\Big|\big\{t\mathbb{W}_2(\mu_t^B,\mu)^2 - \Xi^B(t)\big\}^+\Big|^q\Big] = 0, \quad q \in [1,q_\alpha). \tag{2.12}$$

(2) *For any $q \in [1,q_\alpha)$ and $k \in (\frac{d}{2\alpha \mathrm{i}(q)},\infty] \cap [1,\infty]$, where we set $(\frac{d}{2\alpha \mathrm{i}(q)},\infty] = \{\infty\}$ if $\alpha = 0$,*

$$\lim_{t\to\infty} \sup_{\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\Big[\Big|\big\{t\mathbb{W}_2(\mu_t^B,\mu)^2 - \Xi^B(t)\big\}^+\Big|^q\Big] = 0, \quad R \in (0,\infty). \tag{2.13}$$

To estimate $\mathbb{E}[\mathbb{W}_2(\mu_t^B,\mu)^2]$, we let

$$\eta_Z^B := \sum_{i=1}^\infty \frac{2\mathbf{V}_B(\phi_i)}{\lambda_i}, \qquad \mathbf{V}_B(\phi_i) := \int_0^\infty \mu(\phi_i P_s^B \phi_i)\mathrm{d}s.$$

Theorem 2.2. *Assume (A_1) and (A_2). Let $q \in [1,\infty)$.*

(1) *If $d' < 2(1+\alpha)$, then $\eta_Z^B < \infty$ and*

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2] \le \eta_Z^B. \tag{2.14}$$

(2) *Let $d' \ge 2(1+\alpha)$. Then there exists a constant $c > 0$ such that for any $t \ge 1$,*

$$\sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2] \le \begin{cases} ct^{-1}\log(1+t), & \text{if } d' = 2(1+\alpha),\\ ct^{-\frac{2}{d'-2\alpha}}, & \text{if } d' > 2(1+\alpha).\end{cases} \tag{2.15}$$

To estimate $\mathbb{W}_p(\mu_t^B, \mu)$, we will use the L^p-boundedness of the Riesz transform $\nabla(a_0 - \hat{L})^{-\frac{1}{2}}$ for some $a_0 \geq 0$. According to [3], together with the non-degeneracy condition, the volume doubling condition and the scaled Poincaré inequality, (2.7) implies

$$\|\nabla(a_0 - \hat{L})^{-\frac{1}{2}}\|_p < \infty \text{ for some } a_0 \in [0, \infty) \text{ and all } p \in (2, \infty). \tag{2.16}$$

Under assumption (A_1), let

$$\gamma_{\alpha,p,q} := \frac{d'}{2} + \frac{d}{2}\big(2 - p^{-1} - q^{-1}\big) - \alpha - 1, \quad p, q \in [1, \infty), \alpha \in [0, 1].$$

Theorem 2.3. *Assume (A_1) and (A_2).*

(1) *If $\gamma_{\alpha,p,q} < 0$, then there exists a constant $c > 0$ such that*

$$\sup_{\nu \in \mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}\big)^{\frac{1}{q}} \leq ct^{-1}, \quad t \geq 1. \tag{2.17}$$

(2) *If $\gamma_{\alpha,p,q} \geq 0$, then for any $\gamma > \gamma_{\alpha,p,q}$, there exists a constant $c > 0$ such that*

$$\sup_{\nu \in \mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}]\big)^{\frac{1}{q}} \leq ct^{-\frac{1}{1+\gamma}}, \quad t \geq 1. \tag{2.18}$$

(3) *Let (2.16) hold. If $\gamma_{\alpha,p,q} \geq 0$, then there exists a constant $c > 0$ such that for any $t \geq 1$,*

$$\sup_{\nu \in \mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}]\big)^{\frac{1}{q}} \leq \begin{cases} ct^{-1}\log(1+t), & \text{if } \gamma_{\alpha,p,q} = 0, \\ ct^{-\frac{1}{1+\gamma_{\alpha,p,q}}}, & \text{if } \gamma_{\alpha,p,q} > 0. \end{cases} \tag{2.19}$$

2.3 Lower bound estimate

To derive sharp lower bound for $\mathbb{E}[\mathbb{W}_2(\mu_t^B, \mu)^2]$, we make the following assumption.

(B) (M, ρ) is a geodesic space, there exist constants $\theta, K > 0$ and $m \geq 1$ such that

$$|\nabla \hat{P}_t \mathrm{e}^f|^2 \leq (\hat{P}_t \mathrm{e}^f)\hat{P}_t(|\nabla f|^2 \mathrm{e}^f) + Kt^\theta \|\nabla f\|_\infty^2 (\hat{P}_t \mathrm{e}^{2mf})^{\frac{1}{m}}, \quad t \in [0, 1], f \in C_{b,L}(M), \tag{2.20}$$

and there exists a function $h \in C([0, 1]; [1, \infty))$ such that

$$\mathbb{W}_2(\nu \hat{P}_r, \mu)^2 \leq h(r)\mathbb{W}_2(\nu, \mu)^2, \quad \nu \in \mathscr{P}, r \in [0, 1]. \tag{2.21}$$

When M is a Riemannian manifold without boundary or with convex boundary, if the Bakry-Emery curvature of $\hat{L}$ is bounded below by a constant $-K$, then (B) holds for $m = 1$ and $h(r) = \mathrm{e}^{2Kr}$, see for instance [32, Theorem 2.3.3(2′)(9)] or [26].

Theorem 2.4. *Assume (A_1) and (B) with $d' < 2(1 + \alpha)$.*

(1) *If $\alpha > \alpha(d, d')$ and $q_\alpha > \frac{d}{2\alpha}$, then*

$$\lim_{t \to \infty} \sup_{\nu \in \mathscr{P}} \mathbb{E}^\nu\Big[\Big|\big\{th(0)\mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t)\big\}^-\Big|^q\Big] = 0, \quad q \in [1, q_\alpha).$$

(2) *For any* $q\in[1,q_\alpha)$ *and* $k\in(\frac{d}{2\alpha \mathrm{i}(q)},\infty]\cap[1,\infty)$, *where we set* $(\frac{d}{2\alpha \mathrm{i}(q)},\infty]=\{\infty\}$ *if* $\alpha=0$,

$$\lim_{t\to\infty}\sup_{\nu\in\mathscr{P}_{k,R}}\mathbb{E}^\nu\Big[\Big|\big\{th(0)\mathbb{W}_2(\mu_t^B,\mu)^2-\Xi^B(t)\big\}^-\Big|^q\Big]=0,\quad q\in[1,q_\alpha),R\in[1,\infty).$$

(3) $\liminf_{t\to\infty}\sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2]\ge h(0)^{-1}\eta_Z^B.$

The next result manages the critical case where the convergence rate of $\mathbb{E}[\mathbb{W}_2(\mu_t^B,\mu)^2]$ is at most $t^{-1}\log t$, correspondingly to (2.15) on the upper bound estimate.

Theorem 2.5. *Assume* (2.5), (2.7), (A_2), (B) *and that*

$$k' i^{\frac{2}{d'}}\le\lambda_i\le k i^{\frac{2}{d'}},\quad i\in\mathbb{N}\tag{2.22}$$

holds for some constants $k,k'>0$. *If* $\alpha':=\frac{d'}{2}-1\in[0,1]$ *and* $B\in\mathbf{B}^\alpha\cap\mathbf{B}_{\alpha'}$ *for some* $\alpha\in[0,\alpha']\cap(\alpha'-1,\alpha']$, *then there exist constants* $c,t_0>0$ *such that*

$$\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2]\ge ct^{-1}\log(1+t),\quad t\ge t_0.\tag{2.23}$$

Finally, we consider the lower bound estimate on $\mathbb{W}_1$.

Theorem 2.6. *Let* $B\in\mathbf{B}$. *Then the following assertions hold.*

(1) *Assume* (2.5), (2.7) *and that the completion* $\bar M$ *of* M *is a Polish space. Then there exist constants* $c,t_0>0$ *such that*

$$\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\ge ct^{-\frac{1}{2}},\quad t\ge t_0.\tag{2.24}$$

(2) *Assume that* (2.8) *holds for* $p=1$, *and there exist constants* $k,d''>0$ *such that*

$$\sup_{x\in M}\mu(B(x,r))\le kr^{d''},\quad r\ge0,\tag{2.25}$$

where $B(x,r):=\{y\in M:\rho(x,y)\le r\}$. *If* $B\in\mathbf{B}_\alpha$ *for some* $\alpha\in[0,1]$ *with* $d''>2(1+\alpha)$, *then there exist constants* $c,t_0>0$ *such that*

$$\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\ge ct^{-\frac{1}{d''-2\alpha}},\quad t\ge t_0.\tag{2.26}$$

3 Proofs of Theorems 2.1-2.3

For a density function f with respect to μ, let $(f\mu)(A):=\int_A f\mathrm{d}\mu$ for a measurable set $A\subset M$. Recall that for any probability density functions $f,f_1,f_2\in L^2(\mu)$, we have

$$\mathbb{W}_2(f\mu,\mu)^2\le\int_M\frac{|\nabla\hat L^{-1}(f-1)|^2}{\mathscr{M}(f)}\mathrm{d}\mu,\quad \mathscr{M}(f):=1_{\{f>0\}}\frac{f-1}{\log f},\tag{3.1}$$

$$\mathbb{W}_p(f_1\mu, f_2\mu)^p \le p^p \int_M \frac{|\nabla \hat{L}^{-1}(f_1 - f_2)|^p}{f_2^{p-1}}\,\mathrm{d}\mu. \tag{3.2}$$

These estimates have been presented in [2] and [21] respectively by using the Kantorovich dual formula and Hamilton-Jacobi equations, which are available when (M,ρ) is a length space as we assumed, see [28].

Since the empirical measure μ_t^B is singular with respect to μ, to apply these estimates we make the following regularization of μ_t^B:

$$\mu_{t,r}^B := f_{t,r}^B \mu, \quad f_{t,r}^B := 1 + \frac{1}{\sqrt{t}} \sum_{i=1}^\infty \mathrm{e}^{-\lambda_i r} \psi_i^B(t)\phi_i, \quad t, r > 0, \tag{3.3}$$

where $\psi_i^B(t) := \frac{1}{\sqrt{t}} \int_0^t \phi_i(X_s^B)\mathrm{d}s$. Letting $\nu \hat{P}_r$ being the distribution of $\hat{X}_r$ with initial distribution ν, by (2.9) and the spectral representation

$$\hat{p}_r(x,y) = 1 + \sum_{i=1}^\infty \mathrm{e}^{-\lambda_i r}\phi_i(x)\phi_i(y)$$

for the heat kernel $\hat{p}_r$ of $\hat{P}_r$ with respect to μ, we have

$$\mu_{t,r}^B = \mu_t^B \hat{P}_r, \quad t, r > 0. \tag{3.4}$$

So, (3.1) implies

$$\mathbb{W}_2(\mu_{t,r}^B, \mu)^2 \le \int_M \frac{|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)|^2}{\mathscr{M}(f_{t,r}^B)}\,\mathrm{d}\mu, \quad t, r > 0. \tag{3.5}$$

According to (2.5), we have $\lim_{t\to\infty} f_{t,r}^B \to 1$ so that $\lim_{t\to\infty} \mathscr{M}(f_{t,r}^B) = 1$. When the convergence is fast enough, (2.9) and (3.3) would imply that for large enough t, $t\mathbb{W}_2(\mu_{t,r}^B, \mu)^2$ is bounded above by

$$\Xi_r^B(t) := t\mu(|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)|^2) = \sum_{i=1}^\infty \frac{\mathrm{e}^{-2\lambda_i r}}{\lambda_i}\psi_i^B(t)^2, \quad t, r > 0. \tag{3.6}$$

On the other hand, by (3.2) we have

$$\mathbb{W}_p(\mu_{t,r}^B, \mu)^p \le p^p \mu\big(|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)|^p\big), \quad t, r > 0, p \in [1, \infty). \tag{3.7}$$

With the above observations, and noting that $\mathbb{W}_p(\mu_t^B, \mu) \le \mathbb{W}_p(\mu_{t,r}^B, \mu) + \mathbb{W}_p(\mu_t^B, \mu_{t,r}^B)$, to estimate $\mathbb{W}_p(\mu_t^B, \mu)$ we present some lemmas on $\Xi_r^B(t)$, $\mu\big(|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)|^p\big)$ and $\mathbb{W}_p(\mu_{t,r}^B, \mu_t^B)$ respectively.

3.1 Some lemmas

To apply (3.7), we need estimate $\|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)\|_p$, see (3.12) below. To this end, and also for later use, we first estimate $\|P_t^B - \mu\|_{p\to q}$ and $\|P_t Z\|_{2p}$ for $q \geq p \geq 1$, where P_t^B is the Markov semigroup for the subordinated diffusion process X_t^B given by

$$P_t^B f(x) := \mathbb{E}^x[f(X_t^B)] = \mathbb{E}[P_{S_t^B} f(x)], \quad t \geq 0, x \in M, f \in \mathscr{B}_b(M). \tag{3.8}$$

Lemma 3.1. *Assume* (2.5). *Then there exists a possibly different constant* $\lambda \in (0, \lambda_1]$ *such that the following assertions hold.*

(1) *Let* $B \in \mathbf{B}^\alpha$ *for some* $\alpha \in (0,1]$. *Then there exists a constant* $k > 0$ *such that*

$$\|P_t^B - \mu\|_{p\to q} \leq k t^{-\frac{d(q-p)}{2pq\alpha}} \mathrm{e}^{-\lambda t}, \quad t > 0, q \geq p \in [1, \infty]. \tag{3.9}$$

(2) *Let* (2.7) *hold. Then for any* $p \in [1, \infty)$ *there exists a constant* $c(p) > 0$ *such that*

$$\|\nabla P_t f\|_p \leq c(p) t^{-\frac{1}{2}} \mathrm{e}^{-\lambda t} \|f\|_p, \quad t > 0, f \in C_{b,L}(M), \tag{3.10}$$

$$\|P_t(Zf)\|_{2p} \leq c(p) \|Z\|_\infty t^{-\frac{1}{2}} \mathrm{e}^{-\lambda t} \|f\|_{2p}, \quad f \in \mathscr{D}(\hat{\mathscr{E}}) \cap L^{2p}(\mu), t > 0. \tag{3.11}$$

Moreover, for any $\kappa \in (0, \frac{1}{2})$ *there exists a constant* $c(p, \kappa) > 0$ *such that*

$$\|\nabla \hat{L}^{-1} f\|_{2p} \leq c(p,\kappa) \|(-\hat{L})^{\frac{d(p-1)}{4p} - \kappa} f\|_2, \quad \mu(f) = 0. \tag{3.12}$$

Proof. (a) We will use some known results on functional inequalities which can be found in e.g. [30]. Firstly, since $\lambda_1 > 0$, we have the Poincaré inequality

$$\mu(f^2) \leq \frac{1}{\lambda_1} \hat{\mathscr{E}}(f, f), \quad f \in \mathscr{D}(\hat{\mathscr{E}}), \mu(f) = 0. \tag{3.13}$$

By (2.2) we have $\mathscr{E}(f, f) = \hat{\mathscr{E}}(f, f)$. So, (3.13) implies

$$\|P_t - \mu\|_2 \leq \mathrm{e}^{-\lambda_1 t}, \quad t \geq 0. \tag{3.14}$$

Next, according to [30, Theorem 3.3.14 and 3.3.15], (2.5) implies the super Poincaré inequality

$$\mu(f^2) \leq r \hat{\mathscr{E}}(f, f) + c_1(1 + r^{-\frac{d}{2}}) \mu(|f|)^2, \quad r > 0, f \in \mathscr{D}(\hat{\mathscr{E}}) \tag{3.15}$$

for some constant $c_1 > 0$, which further yields

$$\|P_t\|_{1\to\infty} \leq c_2 (1 \wedge t)^{-\frac{d}{2}}, \quad t > 0 \tag{3.16}$$

for some constant $c_2 > 0$. Noting that $B \in \mathbf{B}^\alpha$ implies

$$B(r) \geq k 1_{\{r \geq \lambda_1\}} r^\alpha, \quad r \geq 0 \tag{3.17}$$

for some constant $k>0$, by (2.4) we find a constant $c_3>0$ such that

$$\mathbb{E}[(S_t^B)^{-d}] = \frac{\mathbb{E}\int_0^\infty r^{d-1}\mathrm{e}^{-rS_t^B}\mathrm{d}r}{\int_0^\infty r^{d-1}\mathrm{e}^{-r}\mathrm{d}r} \le c_3(1\wedge t)^{-\frac{d}{\alpha}}, \quad t>0.$$

Combining this with (2.4), (3.8), (3.14) and (3.16), we find constants $c_4, c_5 \ge 1$ such that

$$\begin{aligned}\|P_t^B-\mu\|_{1\to\infty} &\le \mathbb{E}[\|P_{S_t^B}-\mu\|_{1\to\infty}] \le c_4\mathbb{E}\Big[\big\{1+(S_t^B)^{-\frac{d}{2}}\big\}\mathrm{e}^{-\lambda_1 S_t^B}\Big]\\ &\le 2c_4\big(\mathbb{E}[1+(S_t^B)^{-d}]\big)^{\frac12}\big(\mathbb{E}[\mathrm{e}^{-2\lambda_1 S_t^B}]\big)^{\frac12} \le c_5(1\wedge t)^{-\frac{d}{2\alpha}}\mathrm{e}^{-B(2\lambda_1)t/2}, \quad t>0.\end{aligned}$$

By the interpolation theorem, this and $\|P_t^B-\mu\|_p \le 2$ for $p\in[1,\infty]$ imply (3.9) for some constants $c,\lambda>0$. In particular, for $B(r)=r$ and $Z=0$ or $Z\neq 0$, (3.9) implies to

$$\|\hat P_t-\mu\|_{p\to q}\vee\|P_t-\mu\|_{p\to q} \le kt^{-\frac{d(q-p)}{2pq}}\mathrm{e}^{-\lambda t}, \quad t>0, q\ge p\ge 1. \tag{3.18}$$

(b) To prove (3.10), we first prove that for some decreasing $c:(1,\infty)\to(0,\infty)$,

$$|\nabla\hat P_t f| \le \frac{c(p)}{\sqrt t}(\hat P_t|f|^p)^{\frac1p}, \quad t\in(0,1], f\in C_{b,L}(M). \tag{3.19}$$

By Hölder's inequality and $f=f^+-f^-$, it suffices to prove for $p\in(1,2]$ and $f\ge 0$. Moreover, by first using $f+\varepsilon$ replacing f for $\varepsilon>0$ then letting $\varepsilon\downarrow 0$, we may and do assume that $\inf f>0$.

By (2.7) we have $\hat P_{t-s}f\in\mathscr{D}(\hat L)\cap C_{b,L}(M)$ for $s\in[0,t)$. By the chain rule, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\hat P_s(\hat P_{t-s}f)^p = \hat P_s\hat L(\hat P_{t-s}f)^p - \hat P_s\big\{p(\hat P_{t-s}f)^{p-1}\hat L\hat P_{t-s}f\big\}\\ &= p(p-1)\hat P_s\big\{(\hat P_{t-s}f)^{p-2}|\nabla\hat P_{t-s}f|^2\big\}, \quad s\in[0,t).\end{aligned}$$

So, for $p\in(1,2]$, we have

$$\begin{aligned}I &:= \hat P_t f^p - (\hat P_t f)^p = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\hat P_s(\hat P_{t-s}f)^p\mathrm{d}s\\ &= p(p-1)\int_0^t \hat P_s\big\{(\hat P_{t-s}f)^{p-2}|\nabla\hat P_{t-s}f|^2\big\}\mathrm{d}s.\end{aligned} \tag{3.20}$$

By the Hölder/Jensen inequalities, we obtain

$$\begin{aligned}\big[\hat P_s|\nabla\hat P_{t-s}f|^p\big]^{\frac2p} &\le \big[\hat P_s\{(P_{t-s}f)^{p-2}|\nabla\hat P_{t-s}f|^2\}\big]\big\{\hat P_s(\hat P_{t-s}f)^p\big\}^{\frac{2-p}{p}}\\ &\le \big[\hat P_s\{(P_{t-s}f)^{p-2}|\nabla\hat P_{t-s}f|^2\}\big]\big(\hat P_t f^p\big)^{\frac{2-p}{p}}.\end{aligned}$$

Combining this with (3.20) and (2.7) where we may assume that $k(p)$ is decreasing in p due to Jensen's inequality, we find increasing $C:(1,\infty)\to(0,\infty)$ such that

$$I \ge p(p-1)\int_0^t \big(\hat P_s|\nabla\hat P_{t-s}f|^p\big)^{\frac2p}(\hat P_t f^p)^{\frac{p-2}{p}}\mathrm{d}s$$

$$\geq C(p)\int_0^t |\nabla \hat P_t f|^2(\hat P_t f^p)^{\frac{p-2}{p}}\mathrm{d}s = C(p)t|\nabla \hat P_t f|^2(\hat P_t f^p)^{\frac{p-2}{p}}.$$

This implies (3.19) for some decreasing $c:(1,\infty)\to(0,\infty)$.

Next, we intend to prove that for some constant $c>0$,

$$\|\nabla P_t f\|_\infty \leq c\|f\|_\infty t^{-\frac{1}{2}},\quad t\in(0,1], f\in\mathscr{B}_b(M). \tag{3.21}$$

For any $x\neq y\in M$, let

$$h_t(x,y) := \sup_{\|g\|_\infty\leq 1}\frac{|P_t g(x)-P_t g(y)|}{\rho(x,y)},\quad t\geq 0.$$

By (2.3) and (3.19), we find a constant $c_1>0$ such that

$$h_t(x,y)\leq c_1 t^{-\frac{1}{2}}+\int_0^t c_1(t-s)^{-\frac{1}{2}}h_s(x,y)\mathrm{d}s,\quad t\in(0,1].$$

By the generalized Gronwall inequality, see [41], this implies (3.21).

Moreover, by (3.19), the L^p-contraction of P_t and $\hat P_t$, and the Duhamel's formula

$$P_t f = \hat P_t f + \int_0^t \hat P_s(ZP_{t-s}f)\mathrm{d}s,$$

we obtain

$$\begin{aligned}\|\nabla P_t f\|_p &\leq \|\nabla \hat P_t f\|_p + \int_0^t \|\nabla \hat P_s(ZP_{t-s}f)\|_p\mathrm{d}s\\ &\leq c(p)t^{-\frac{1}{2}}\|f\|_p + \int_0^t c(p)\|Z\|_\infty s^{-\frac{1}{2}}\|\nabla P_{t-s}f\|_p\mathrm{d}s,\quad t>0.\end{aligned}$$

When $f\in\mathscr{B}_b(M)$, by (3.21) and the generalized Gronwall inequality, this imply (3.10) for $t\in(0,1]$.

Finally, by (3.18) for $p=q$ such that

$$\|P_t-\mu\|_p\leq k\mathrm{e}^{-\lambda t}.$$

Combining this with the semigroup property and (3.10) for $t\in(0,1]$, for any $t>1$ we have

$$\|\nabla P_t f\|_p = \|\nabla P_1(P_{t-1}f-1)\|_p\leq c(p)\|P_{t-1}(f-1)\|_p\leq c(p)k\mathrm{e}^{-\lambda(t-1)}\|f\|_p.$$

So, (3.10) also holds for $t>1$ and some constant $\lambda>0$.

(c) Let P_t^* be the $L^2(\mu)$-adjoint operator of P_t. By (2.2), P_t^* is the diffusion semigroup generated by $L^*:=\hat L - Z$, and satisfies

$$P_t^* g = \hat P_t g - \int_0^t \hat P_s(ZP_{t-s}^* g)\mathrm{d}s,\quad t>0,\quad g\in L^2(\mu). \tag{3.22}$$

Let $g \in \mathscr{D}(\hat{\mathscr{E}})$ with $\|g\|_{\frac{2p}{2p-1}} \le 1$. We have

$$\|\nabla P_t^* g\|_{\frac{2p}{2p-1}}^2 \le \|\nabla P_t^* g\|_2^2 = \hat{\mathscr{E}}(P_t^* g, P_t^* g) \le \hat{\mathscr{E}}(g,g) < \infty, \quad t \ge 0,$$

so that (2.2), (3.10) and (3.22) yield that for some constant $c_1 > 0$,

$$\|\nabla P_t^* g\|_{\frac{2p}{2p-1}} \le c_1 t^{-\frac{1}{2}} + c_1 \int_0^t s^{-\frac{1}{2}} \|\nabla P_{t-s}^* g\|_{\frac{2p}{2p-1}} \mathrm{d}s < \infty, \quad t \in (0,1].$$

By the generalized Gronwall inequality, see [41], we find a constant $c_2 > 0$ such that

$$\sup_{g \in \mathscr{D}(\hat{\mathscr{E}}), \|g\|_{\frac{2p}{2p-1}} \le 1} \|\nabla P_t^* g\|_{\frac{2p}{2p-1}} \le c_2 t^{-\frac{1}{2}}, \quad t \in (0,1].$$

Combining this with the semigroup property and (3.18), we find a constant $c_3 > 0$ such that

$$\sup_{g \in \mathscr{D}(\hat{\mathscr{E}}), \|g\|_{\frac{2p}{2p-1}} \le 1} \|\nabla P_t^* g\|_{\frac{2p}{2p-1}} \le c_3 t^{-\frac{1}{2}} \mathrm{e}^{-\lambda t}, \quad t > 0.$$

Thus, by (2.2), for any $f \in \mathscr{D}(\hat{\mathscr{E}}) \cap L^{2p}(\mu)$ we have

$$\begin{aligned}
\|P_t(Zf)\|_{2p} &= \sup_{g \in \mathscr{D}(\hat{\mathscr{E}}), \|g\|_{\frac{2p}{2p-1}} \le 1} \big|\mu((P_t^* g)(Zf))\big| \\
&= \sup_{g \in \mathscr{D}(\hat{\mathscr{E}}), \|g\|_{\frac{2p}{2p-1}} \le 1} \big|\mu(f(ZP_t^* g))\big| \le c_3 \|Z\|_\infty t^{-\frac{1}{2}} \mathrm{e}^{-\lambda t} \|f\|_{2p}, \quad t > 0.
\end{aligned}$$

Therefore, (3.11) holds for some constants $c(p), \lambda > 0$.

(d) Noting that

$$(-\hat{L})^{-(1+\frac{d(p-1)}{4p}-\kappa)} = \frac{1}{\Gamma(1+\frac{d(p-1)}{4p}-\kappa)} \int_0^\infty s^{\frac{d(p-1)}{4p}-\kappa} \hat{P}_s \mathrm{d}s,$$

by (3.10) and (3.18), we find constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}
&\|\nabla \hat{L}^{-1} f\|_{2p} = \|\nabla(-\hat{L})^{-(1+\frac{d(p-1)}{4p}-\kappa)} (-\hat{L})^{\frac{d(p-1)}{4p}-\kappa} f\|_{2p} \\
&\le \frac{1}{\Gamma(1+\frac{d(p-1)}{4p}-\kappa)} \int_0^\infty s^{\frac{d(p-1)}{4p}-\kappa} \|\nabla \hat{P}_{s/2}\{\hat{P}_{s/2}(-\hat{L})^{\frac{d(p-1)}{4p}-\kappa} f\}\|_{2p} \mathrm{d}s \\
&\le c_1 \int_0^\infty s^{\frac{d(p-1)}{4p}-\kappa-\frac{1}{2}} \mathrm{e}^{-\lambda s/2} \|\hat{P}_{s/2}(-\hat{L})^{\frac{d(p-1)}{4p}-\kappa} f\|_{2p} \mathrm{d}s \\
&\le c_2 \int_0^\infty s^{-(\kappa+\frac{1}{2})} \mathrm{e}^{-\lambda s} \|(-\hat{L})^{\frac{d(p-1)}{4p}-\kappa} f\|_2 \mathrm{d}s \le c_3 \|(-\hat{L})^{\frac{d(p-1)}{4p}-\kappa} f\|_2,
\end{aligned}$$

where the last step is due to $\frac{1}{2} + \kappa < 1$. Thus, (3.12) holds.

□

Next, we present some consequence of (A_1) and (A_2).

Lemma 3.2. *We have the following assertions.*

(1) *If* (2.5) *holds, then* (M,ρ) *is bounded, i.e.*

$$D := \sup_{x,y\in M} \rho(x,y) < \infty. \tag{3.23}$$

(2) *If* (2.8) *holds, then for any* $p,q \in [1,\infty)$*, there exists a constant* $c>0$ *such that*

$$\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_t^B,\mu_{t,r}^B)^{2q}]\big)^{\frac{1}{q}} \le cr, \quad r\in(0,1]. \tag{3.24}$$

(3) *If* (2.7) *and* (2.20) *hold, then there exist constants* $\kappa_0,\kappa_1>0$ *such that*

$$\begin{aligned} &|\nabla P_t \mathrm{e}^f|^2 \le (P_t\mathrm{e}^f)P_t(|\nabla f|^2\mathrm{e}^f) + \kappa_1 t^\theta \|\nabla f\|_\infty^2 (P_t\mathrm{e}^f)^2,\\ &\qquad \textit{for } t\in[0,1],\ f\in C_{b,L}(M) \textit{ with } t\|\nabla f\|_\infty^2 \le \kappa_0. \end{aligned} \tag{3.25}$$

Proof. (1) According to [30, Theorem 3.3.15(2)], (2.5) implies the super Poincaré inequality

$$\mu(f^2) \le r\hat{\mathscr{E}}(f,f) + (1+r^{-\frac{d}{2}})\mu(|f|)^2, \quad r>0, f\in\mathscr{D}(\hat{\mathscr{E}}),$$

which further implies (3.23) due to [30, Theorem 3.3.20].

(2) By Jensen's inequality, we only need to prove (3.24) for $q\ge p\ge 1$. Recall that $\delta_{X_s^B}\hat{P}_r$ is the distribution of $\hat{X}_r$ with initial value X_s^B, we have

$$\pi_t := \frac{1}{t}\int_0^t \big\{\delta_{X_s^B}\times(\delta_{X_s^B}\hat{P}_r)\big\}\mathrm{d}s \in \mathscr{C}(\mu_t^B,\mu_{t,r}^B),$$

so that

$$\mathbb{W}_{2p}(\mu_t^B,\mu_{t,r}^B)^{2p} \le \int_{M\times M}\rho(x,y)^{2p}\pi_t(\mathrm{d}x,\mathrm{d}y) = \frac{1}{t}\int_0^t \mathbb{E}^x[\rho(x,\hat{X}_r)^{2p}]\big|_{x=X_s^B}\mathrm{d}s.$$

Noting that $\mathbb{E}^\mu = \int_M \mathbb{E}^x\mu(\mathrm{d}x)$ and $\hat{X}_t$ is stationary with initial distribution μ, by combining this with Jensen's inequality and (2.8), we obtain

$$\begin{aligned}\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_t^B,\mu_{t,r}^B)^{2q}] &\le \mathbb{E}^\mu\bigg[\frac{1}{t}\int_0^t \mathbb{E}^x[\rho(x,\hat{X}_r)^{2q}]\big|_{x=X_s^B}\mathrm{d}s\bigg]\\ &= \frac{1}{t}\int_0^t \mathbb{E}^\mu[\rho(\hat{X}_0,\hat{X}_r)^{2q}]\mathrm{d}s \le c(2q)r^q, \quad t>0, r\in(0,1].\end{aligned}$$

So, (3.24) holds.

(3) Let $f\in C_{b,L}(M)$. By (3.21), we have $(P_{t-s}\mathrm{e}^f)^{2m}\in\mathscr{D}(L)\cap C_{b,L}(M)$ for $s\in[0,t)$, so that the chain rule implies

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}\mathrm{e}^f)^{2m} = P_sL(P_{t-s}\mathrm{e}^f)^{2m} - P_s\big\{2m(P_{t-s}\mathrm{e}^f)^{2m-1}LP_{t-s}\mathrm{e}^f\big\}$$

$$= 2m(2m-1)P_s\big\{(P_{t-s}e^f)^{2m-2}|\nabla P_{t-s}e^f|^2\big\},\quad s\in[0,t).$$

By combining this with (2.7) for $p=2$ and Jensen's inequality, we find a constant $c_1>0$ such that

$$\begin{aligned}P_t e^{2mf} - (P_t e^f)^{2m} &= \int_0^t \frac{d}{ds}P_s(P_{t-s}e^f)^{2m}ds\\ &= \int_0^t P_s\big\{2m(2m-1)(P_{t-s}e^f)^{2m-2}|\nabla P_{t-s}e^f|^2\big\}ds\\ &\le c_1\|\nabla f\|_\infty^2\int_0^t P_s\big\{(P_{t-s}e^f)^{2m-2}P_{t-s}e^{2f}\big\}ds \le c_1 t\|\nabla f\|_\infty^2 P_t e^{2mf}.\end{aligned}$$

Taking $\kappa_0=\frac{1}{2c_1}$ such that $t\|\nabla f\|_\infty^2\le\kappa_0$ implies $c_1t\|\nabla f\|_\infty^2\le\frac12$, we derive

$$P_t e^{2mf}\le 2(P_t e^f)^{2m}. \tag{3.26}$$

Combining this with (2.20), we obtain (3.25) for some constant $\kappa_1>0$. □

Noting that (2.9) and (3.3) imply

$$\|(-\hat L)^\beta(f_{t,r}^B-1)\|_2^2 = \frac1t\sum_{i=1}^\infty \lambda_i^{2\beta}e^{-2\lambda_i r}\psi_i^B(t)^2,\quad r,t>0,\beta\in\mathbb{R}, \tag{3.27}$$

to bound $\mathbb{E}^\nu[\mathbb{W}_p(\mu_{t,r}^B,\mu)^{2q}]$ from above using (3.7) and (3.12), we estimate $\mathbb{E}^\nu[|\psi_i^B|^{2q}]$ as follows.

Lemma 3.3. *Assume* (2.5) *and let $B\in\mathbf{B}^\alpha$ for some $\alpha\in[0,1]$. Then:*

(1) *For any $q\in[1,\infty)$, there exists a constant $c(q)>0$ such that*

$$\sup_{t>0}\mathbb{E}^\nu[|\psi_i^B(t)|^{2q}]\le c(q)\|h\|_\infty\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\in\mathbb{N},\nu=h\mu. \tag{3.28}$$

(2) *For any $q\in[1,\infty)$ and $k\in(\frac{d}{2\alpha\mathrm{i}(q)},\infty]\cap[1,\infty]$, there exists a constant $c(q,k)>0$ such that*

$$\mathbb{E}^\nu[|\psi_i^B(t)|^{2q}]\le c(q,k)\|h\|_k(1\wedge t)^{-\frac{d}{2\alpha k}}\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\in\mathbb{N},\nu=h\mu,t>0. \tag{3.29}$$

Moreover, if $\mathrm{i}(q)>\frac{d}{2\alpha}$, then there exists a constant $c(q)>0$ such that

$$\sup_{\nu\in\mathscr{P}}\mathbb{E}^\nu[|\psi_i^B(t)|^{2q}]\le c(q)(1\wedge t)^{-\frac{d}{2\alpha}}\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\in\mathbb{N},t>0. \tag{3.30}$$

Proof. (1) Let $h_{i,\alpha}(t):=\min\big\{(\frac12\wedge t)^{-\frac{1}{2\alpha}},\ \lambda_i^{\frac12}\big\}$. When $\alpha>0$, for any $k>0$ there exist constants $a_1,a_2>0$ such that

$$\begin{aligned}\int_0^\infty h_{i,\alpha}(t)e^{-kt}dt &\le \int_0^{\lambda_i^{-\alpha}}\lambda_i^{\frac12}dt+\int_{\lambda_i^{-\alpha}}^\infty\Big(t^{-\frac{1}{2\alpha}}+2^{\frac{1}{2\alpha}}\Big)e^{-kt}dt\\ &\le \lambda_i^{\frac12-\alpha}+a_1\lambda_i^{(\frac12-\alpha)^+}\big[1+1_{\{\alpha=\frac12\}}\log(1+\lambda_i)\big]\\ &\le a_2\lambda_i^{(\frac12-\alpha)^+}\big[1+1_{\{\alpha=\frac12\}}\log(1+\lambda_i)\big],\quad i\in\mathbb{N}.\end{aligned} \tag{3.31}$$

When $\alpha=0$ we have $h_{i,\alpha}(t)=\lambda_i^{\frac{1}{2}}$ so that this estimate holds as well.

We first prove the following estimate for some constants $k_1,k_2>0$:

$$\|P_t^B\phi_i-\mathrm{e}^{-B(\lambda_i)t}\phi_i\|_{2q}\le k_1\|Z\|_\infty\lambda_i^{\frac{d(q-1)}{4q}-1}h_{i,\alpha}(t)\mathrm{e}^{-k_2t},\quad t>0, i\in\mathbb{N}, q\in[1,\infty]. \tag{3.32}$$

By (2.9) and (3.18), we find constants $c_1,c_2>0$ such that

$$\|\phi_i\|_{2q}=\inf_{s>0}\|\hat{P}_s\phi_i\|_{2q}\mathrm{e}^{\lambda_i s}\le c_1\inf_{s\in(0,1]}s^{-\frac{d(q-1)}{4q}}\mathrm{e}^{\lambda_i s}\le c_2\lambda_i^{\frac{d(q-1)}{4q}},\quad i\in\mathbb{N}, q\in[1,\infty]. \tag{3.33}$$

By (2.3), (3.8) and (2.9), we obtain

$$P_t^B\phi_i=\mathbb{E}\bigg[\mathrm{e}^{-\lambda_i S_t^B}\phi_i+\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}P_s(Z\phi_i)\,\mathrm{d}s\bigg],\quad t>0. \tag{3.34}$$

Combining this with (2.4), (3.11) and (3.33), we derive

$$\|P_t^B\phi_i-\mathrm{e}^{-B(\lambda_i)t}\phi_i\|_{2q}\le c_2c(q)\|Z\|_\infty\lambda_i^{\frac{d(q-1)}{4q}}\,\mathbb{E}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s}\mathrm{d}s. \tag{3.35}$$

Noting that $\lambda_i\ge\lambda_1\ge\lambda$ implies

$$-\lambda_i(S_t^B-s)-\lambda s\le-\frac{\lambda_i}{2}(S_t^B-s)-\frac{\lambda}{2}(S_t^B-s)-\lambda s=-\frac{\lambda_i}{2}(S_t^B-s)-\frac{\lambda}{2}S_t^B-\frac{\lambda}{2}s,$$

by the FKG inequality, we find a constant $c_3>0$ such that

$$\begin{aligned}&\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s}\mathrm{d}s\le\mathrm{e}^{-\lambda S_t^B/2}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)/2}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s/2}\mathrm{d}s\\&\le\mathrm{e}^{-\lambda S_t^B/2}\bigg(\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)/2}\mathrm{d}s\bigg)\frac{1}{S_t^B}\int_0^{S_t^B}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s}\mathrm{d}s\le\frac{c_3}{\lambda_i}\mathrm{e}^{-\lambda S_t^B/2}(S_t^B)^{-\frac{1}{2}}.\end{aligned} \tag{3.36}$$

Moreover, by (2.4) and (3.17), we find constants $c_4,c_5,c_6>0$ such that

$$\begin{aligned}&\mathbb{E}\big[(S_t^B)^{-\frac{1}{2}}\mathrm{e}^{-\lambda S_t^B/2}\big]=\mathbb{E}\bigg[\frac{\mathrm{e}^{-\lambda S_t^B/2}}{\Gamma(1/2)}\int_0^\infty u^{-\frac{1}{2}}\mathrm{e}^{-uS_t^B}\mathrm{d}u\bigg]\\&=\frac{1}{\Gamma(1/2)}\int_0^\infty u^{-\frac{1}{2}}\mathrm{e}^{-B(u+\lambda/2)t}\mathrm{d}u\le\frac{1}{\Gamma(1/2)}\int_0^\infty u^{-\frac{1}{2}}\mathrm{e}^{-B(u)t/2-B(\lambda/2)t/2}\mathrm{d}u\\&\le c_4\mathrm{e}^{-B(\lambda/2)t/2}\bigg[\int_0^{\lambda_1}u^{-\frac{1}{2}}\mathrm{d}u+\int_{\lambda_1}^\infty u^{-\frac{1}{2}}\mathrm{e}^{-ku^\alpha t}\mathrm{d}u\bigg]\le c_5\Big(\frac{1}{2}\wedge t\Big)^{-\frac{1}{2\alpha}}\mathrm{e}^{-c_6t}.\end{aligned} \tag{3.37}$$

Combining this with (3.35) and (3.36), we find constants $k_1,k_2>0$ such that

$$\big\|P_t^B\phi_i-\mathrm{e}^{-B(\lambda_i)t}\phi_i\big\|_{2q}\le k_1\|Z\|_\infty\lambda_i^{\frac{d(q-1)}{4q}-1}\Big(\frac{1}{2}\wedge t\Big)^{-\frac{1}{2\alpha}}\mathrm{e}^{-k_2t}. \tag{3.38}$$

On the other hand, by (3.10) and (3.33), we find constants $c_1', c_2' > 0$ such that

$$\|\nabla\phi_i\|_{2q} = \inf_{s>0} \|\nabla \hat{P}_s \phi_i\|_{2q} \mathrm{e}^{\lambda_i s} \le \inf_{s>0} c_1' s^{-\frac{1}{2}} \mathrm{e}^{\lambda_i s} \|\phi_i\|_{2q} \le c_2' \lambda_i^{\frac{d(q-1)}{4q}+\frac{1}{2}}. \tag{3.39}$$

So, instead of (3.35), this and (3.18) imply

$$\big\|P_t^B\phi_i - \mathrm{e}^{-B(\lambda_i)t}\phi_i\big\|_{2q} \le \|Z\|_\infty \|\nabla\phi\|_{2q} \mathbb{E}\int_0^{S_t^B} \mathrm{e}^{-\lambda_i(S_t^B-s)-\lambda s}\mathrm{d}s. \tag{3.40}$$

By $\lambda_i \ge \lambda$ and (2.4), we obtain

$$\mathbb{E}\int_0^{S_t^B} \mathrm{e}^{-\lambda_i(S_t^B-s)-\lambda s}\mathrm{d}s \le \mathbb{E}\bigg[\mathrm{e}^{-\lambda S_t^B/2}\int_0^{S_t^B} \mathrm{e}^{-\lambda_i(S_t^B-s)/2}\mathrm{d}s\bigg] \le \frac{2}{\lambda_i}\mathrm{e}^{-B(\lambda/2)t}.$$

Combining this with (3.39) and (3.40), we find constants $k_1, k_2 > 0$ such that

$$\big\|P_t^B\phi_i - \mathrm{e}^{-B(\lambda_i)t}\phi_i\big\|_{2q}] \le k_1\|Z\|_\infty \lambda_i^{\frac{d(q-1)}{4q}-\frac{1}{2}}\mathrm{e}^{-k_2 t}.$$

This together with (3.38) implies (3.32).

Next, we prove (3.28) for $q\in\mathbb{N}$. By [37, (2.14)] for $f=\phi_i$, we find a constant $k_0>0$ such that

$$\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] \le k_0\bigg(\frac{1}{t}\int_0^t \mathrm{d}s_1\int_0^{s_1} \big\{\mathbb{E}^\nu[|\phi_i P_{s_1-s}^B\phi_i|^q](X_s^B)\big\}^{\frac{1}{q}}\mathrm{d}s\bigg)^q. \tag{3.41}$$

By $\nu = h\mu$ and the Markov property, we obtain

$$\begin{aligned}\mathbb{E}^\nu\big[|\phi_i P_{s_1-s}^B\phi_i|^q(X_s^B)\big] &= \mu\big(hP_s^B|\phi_i P_{s_1-s}^B\phi_i|^q\big) \le \|h\|_k \big\|P_s^B|\phi_i P_{s_1-s}^B\phi_i|^q\big\|_{\frac{k}{k-1}}\\ &\le \|h\|_k\|P_s^B\|_{1\to\frac{k}{k-1}}\|\phi_iP_{s_1-s}^B\phi_i\|_q^q \le \|h\|_k\|P_s^B\|_{1\to\frac{k}{k-1}}\|\phi_i\|_{2q}^q\|P_{s_1-s}^B\phi_i\|_{2q}^q,\quad k\in[1,\infty].\end{aligned} \tag{3.42}$$

Taking $k=\infty$ and combining with (3.32), (3.33) and (3.41), we find constants $k_1,k_2>0$ such that

$$\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] \le k_1\|h\|_\infty\lambda_i^{\frac{d(q-1)}{2}}\bigg(\frac{1}{t}\int_0^t\mathrm{d}s_1\int_0^{s_1}\big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1}h_{i,\alpha}(s_1-s)\mathrm{e}^{-k_2(s_1-s)}\big)\mathrm{d}s\bigg)^q.$$

Combining this with (3.17), which together with (3.31) implies

$$\int_0^\infty\Big[\mathrm{e}^{-B(\lambda_i)t} + \lambda_i^{-1}h_{i,\alpha}(t)\mathrm{e}^{-k_2t}\Big]\mathrm{d}t \le c\lambda_i^{-\alpha},\quad i\in\mathbb{N} \tag{3.43}$$

for some constant $c>0$, we derive (3.28) for $q\in\mathbb{N}$.

Finally, for any $q\in(1,\infty)$, let $\mathrm{i}(q)$ be the integer part of q. By (3.28) for $\mathrm{i}(q)$ and $1+\mathrm{i}(q)$ replacing q which have just been proved, and using Hölder's inequality, we find a constant $c(q)>0$ such that

$$\begin{aligned}\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] &\le \big(\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2\mathrm{i}(q)}\big]\big)^{\mathrm{i}(q)+1-q}\big(\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2+2\mathrm{i}(q)}\big]\big)^{q-\mathrm{i}(q)}\\ &\le c(q)\|h\|_\infty\lambda_i^{\frac{1}{2}\{d(\mathrm{i}(q)-1)(\mathrm{i}(q)+1-q)+d\mathrm{i}(q)(q-\mathrm{i}(q))\}-\alpha\{\mathrm{i}(q)(\mathrm{i}(q)+1-q)+(1+\mathrm{i}(q))(q-\mathrm{i}(q))\}}\\ &= c(q)\|h\|_\infty\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad t>0, i\in\mathbb{N}.\end{aligned} \tag{3.44}$$

Then (3.28) is proved.

(2) By the same reason leading to (3.44), we only need to prove (3.29) and (3.30) for $q \in \mathbb{N}$ so that $\mathrm{i}(q) = q$. Let $k \in (\frac{d}{2\alpha q}, \infty] \cap [1, \infty]$. By (3.32), (3.41) and (3.42), we find a constant $c_1 > 0$ such that

$$\begin{aligned} \mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] &\le c_1 \|h\|_k \lambda_i^{\frac{d(q-1)}{2}} \\ &\times \bigg(\frac{1}{t}\int_0^t \mathrm{d}s_1 \int_0^{s_1} (1\wedge s)^{-\frac{d}{2kq\alpha}} \Big[\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \frac{h_{i,\alpha}(s_1-s)}{\lambda_i \mathrm{e}^{k_2(s_1-s)}}\Big]\mathrm{d}s\bigg)^q. \end{aligned} \tag{3.45}$$

By the FKG inequality and (3.43), we find a constant $c_2 > 0$ such that

$$\begin{aligned} &\int_0^{s_1} (1\wedge s)^{-\frac{d}{2kq\alpha}} \big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1} h_{i,\alpha}(s_1-s)\mathrm{e}^{-k_2(s_1-s)}\big)\mathrm{d}s \\ &\le \bigg(\frac{1}{s_1}\int_0^{s_1}(1\wedge s)^{-\frac{d}{2kq\alpha}}\mathrm{d}s\bigg)\int_0^{s_1} \big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1} h_{i,\alpha}\mathrm{e}^{-k_2(s_1-s)}\big)\mathrm{d}s \\ &\le c_2 (1\wedge s_1)^{-\frac{d}{2kq\alpha}} \lambda_i^{-\alpha}, \quad t > 0, i \in \mathbb{N}. \end{aligned} \tag{3.46}$$

This together with (3.45) yields

$$\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] \le c\|h\|_k (1\wedge t)^{-\frac{d}{2\alpha k}} \lambda_i^{\frac{d(q-1)}{2} - q\alpha}, \quad s_1 > 0, i \in \mathbb{N}$$

for some constant $c > 0$. Therefore, (3.29) holds for $q \in \mathbb{N}$.

It remains to prove (3.30) for $\frac{d}{2\alpha} < q \in \mathbb{N}$. By (3.9), (3.32) and (3.33), we find constants $c_1 > 0$ such that

$$\begin{aligned} &\sup_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu\big[|\phi_i P_{s_1-s}^B \phi_i|^q (X_s^B)\big]\big)^{\frac{1}{q}} = \sup_{\nu\in\mathscr{P}} \Big\{\nu\big(P_s^B |\phi_i P_{s_1-s}^B \phi_i|^q\big)\Big\}^{\frac{1}{q}} \\ &\le \|P_s^B\|_{1\to\infty}^{\frac{1}{q}} \|\phi_i P_{s_1-s}^B\phi_i\|_q \le c_1 (1\wedge s)^{-\frac{d}{2q\alpha}} \|\phi_i\|_{2q} \|P_{s_1-s}^B \phi_i\|_{2q} \\ &\le c_1 (1\wedge s)^{-\frac{d}{2q\alpha}} \lambda_i^{\frac{d(q-1)}{2q}} \big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1} h_{i,\alpha}(s_1-s)\mathrm{e}^{-k_2(s_1-s)}\big). \end{aligned}$$

Combining this with (3.41) and (3.46) for $k = 1$, we get (3.30) for $\frac{d}{2\alpha} < q \in \mathbb{N}$. □

Lemma 3.4. *Assume (A_1), (A_2). Let $a_0 \in [0,\infty), q \in [1,\infty), \beta \in \mathbb{R}$.*

1. *For any $k \in (\frac{d}{2\alpha \mathrm{i}(q)}, \infty] \cap [1,\infty]$ where $k = \infty$ if $\alpha = 0$, and for any $R \in [1,\infty)$, there exists a constant $c > 0$ such that*

$$\begin{aligned} &\sup_{t\ge 1, \nu\in\mathscr{P}_{k,R}} t^q \mathbb{E}^\nu\Big[\big\|(a_0 - \hat{L})^{\frac{1}{2}}(-\hat{L})^{\beta-\frac{1}{2}}(f_{t,r}^B - 1)\big\|_2^{2q}\Big] \\ &\le c\big[r^{-(2\beta + \frac{d'}{2} + \frac{d(q-1)}{2q} - \alpha)^+} + 1_{\{\alpha = 2\beta + \frac{d'}{2} + \frac{d(q-1)}{2q}\}} \log(1 + r^{-1})\big]^q, \quad r \in (0,1]. \end{aligned} \tag{3.47}$$

2. *If $\mathrm{i}(q) > \frac{d}{2\alpha}$, then there exists a constant $c > 0$ such that*

$$\begin{aligned} &\sup_{t\ge 1, \nu\in\mathscr{P}} t^q \mathbb{E}^\nu\Big[\big\|(a_0 - \hat{L})^{\frac{1}{2}}(-\hat{L})^{\beta-\frac{1}{2}}(f_{t,r}^B - 1)\big\|_2^{2q}\Big] \\ &\le c\big[r^{-(2\beta + \frac{d'}{2} + \frac{d(q-1)}{2q} - \alpha)^+} + 1_{\{\alpha = 2\beta + \frac{d'}{2} + \frac{d(q-1)}{2q}\}} \log(1 + r^{-1})\big]^q, \quad r \in (0,1]. \end{aligned} \tag{3.48}$$

Proof. By (3.17), (3.27), Hölder's inequality and (3.29), we find a constant $c_1 > 0$ such that

$$\begin{aligned}
I &:= \sup_{t\geq 1,\nu\in\mathscr{P}_{k,R}} t^q \mathbb{E}^\nu\Big[\big\|(a_0-\hat{L})^{\frac{1}{2}}(-\hat{L})^{\beta-\frac{1}{2}}(f_{t,r}^B-1)\big\|_2^{2q}\Big] \\
&= \sup_{t\geq 1,\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\bigg[\bigg|\sum_{i=1}^\infty\Big(\frac{\lambda_i+a_0}{\lambda_i}\Big)\lambda_i^{2\beta}\mathrm{e}^{-2\lambda_i r}\psi_i^B(t)^2\bigg|^q\bigg] \\
&\leq \Big(\frac{\lambda_1+a_0}{\lambda_1}\Big)^q \sup_{t\geq 1,\nu\in\mathscr{P}_{k,R}}\Big(\sum_{i=1}^\infty \lambda_i^\theta \mathrm{e}^{-2\lambda_i r}\Big)^{q-1}\sum_{i=1}^\infty \lambda_i^{2q\beta-\theta(q-1)}\mathrm{e}^{-2\lambda_i r}\mathbb{E}^\nu[|\psi_i^B(t)|^{2q}] \\
&\leq c_1\Big(\sum_{i=1}^\infty \lambda_i^\theta \mathrm{e}^{-2\lambda_i r}\Big)^{q-1}\sum_{i=1}^\infty \lambda_i^{2q\beta-\theta(q-1)+\frac{d(q-1)}{2}-q\alpha}\mathrm{e}^{-2\lambda_i r}, \quad \theta\in\mathbb{R}.
\end{aligned}$$

Taking

$$\theta := 2\beta + \frac{d(q-1)}{2q} - \alpha, \tag{3.49}$$

so that $\theta = 2q\beta - \theta(q-1) + \frac{d(q-1)}{2} - q\alpha$, and noting that

$$\lambda_i^{\theta^+}\mathrm{e}^{-\lambda_i r} \leq \sup_{s>0} s^{\theta^+}\mathrm{e}^{-sr} \leq cr^{-\theta^+}, \quad r\in(0,1]$$

holds for some constant $c > 0$ depending on θ^+, we find a constant $c_2 > 0$ such that

$$I \leq c_1\Big(\sum_{i=1}^\infty \lambda_i^\theta \mathrm{e}^{-2\lambda_i r}\Big)^q \leq c_2 r^{-q\theta^+}\Big(\sum_{i=1}^\infty \lambda_i^{-\theta^-}\mathrm{e}^{-\lambda_i r}\Big)^q.$$

On the other hand, by (2.6) and the integral transform $t = rs^{\frac{2}{d'}}$, we find constants $c_3, c_4, c_5 > 0$ such that

$$\begin{aligned}
\sum_{i=1}^\infty \lambda_i^{-\theta^-}\mathrm{e}^{-\lambda_i r} &\leq c_3\int_1^\infty s^{-\frac{2\theta^-}{d'}}\mathrm{e}^{-c_3 rs^{\frac{2}{d'}}}\mathrm{d}s = \frac{c_3 d'}{2}\int_r^\infty r^{\theta^- -\frac{d'}{2}}t^{\frac{d'}{2}-\theta^- -1}\mathrm{e}^{-c_3 t}\mathrm{d}t \\
&\leq c_5\Big\{r^{-(\frac{d'}{2}-\theta^-)^+} + 1_{\{\frac{d'}{2}=\theta^-\}}\log(1+r^{-1})\Big\}, \quad r\in(0,1].
\end{aligned} \tag{3.50}$$

Thus, we find a constant $c_6 > 0$ such that

$$\begin{aligned}
I &\leq c_6 r^{-q\theta^+}\big[r^{-(\frac{d'}{2}-\theta^-)^+} + 1_{\{\frac{d'}{2}=\theta^-\}}\log(1+r^{-1})\big]^q \\
&= c_6\big[r^{-(\frac{d'}{2}+\theta)^+} + 1_{\{\frac{d'}{2}+\theta=0\}}\log(1+r^{-1})\big]^q, \quad r\in(0,1].
\end{aligned}$$

This together with (3.49) implies (3.47).

When $\mathrm{i}(q) > \frac{d}{2\alpha}$, (3.48) can be proved in the same way by using (3.30) replacing (3.29). □

We are now ready to show that as $t \to \infty$, $\mathbb{E}[\Xi_r^B(t)]$ converges to

$$\eta_{Z,r}^B := \sum_{i=1}^{\infty} \frac{2e^{-2\lambda_i r}}{\lambda_i} \mathbf{V}_B(\phi_i), \quad r > 0. \tag{3.51}$$

Lemma 3.5. *Assume (A_1), and let $R \in [1, \infty)$.*

(1) *There exists a constant $c > 0$ such that*

$$\eta_{Z,r}^B \le c \sum_{i=1}^{\infty} \lambda_i^{-1-\alpha} e^{-2r\lambda_i}, \quad r > 0. \tag{3.52}$$

Consequently, $\eta_Z^B < \infty$ provided $\sum_{i=1}^{\infty} \lambda_i^{-1-\alpha} < \infty$.

(2) *There exists a constant $c > 0$ such that*

$$\sup_{\nu \in \mathscr{P}_{\infty,R}} \left| \mathbb{E}^\nu[\Xi_r^B(t)] - \eta_{Z,r}^B \right| \le \frac{c}{t} \sum_{i=1}^{\infty} \lambda_i^{-1-\alpha} e^{-2\lambda_i r}, \quad t, r > 0. \tag{3.53}$$

Consequently, when $\sum_{i=1}^{\infty} \lambda_i^{-1-\alpha} < \infty$, there exists a constant $c' > 0$ such that

$$\sup_{\nu \in \mathscr{P}_{\infty,R}} \mathbb{E}^\nu[\Xi^B(t)] \le \eta_Z^B + \frac{c'}{t} < \infty, \quad t > 0. \tag{3.54}$$

(3) *For any $k > \frac{d}{2\alpha}$, there exists a constant $c > 0$ such that*

$$\sup_{t \ge 1, \nu \in \mathscr{P}_{k,R}} \mathbb{E}^\nu[\Xi_r^B(t)] \le c_2 \left\{ r^{-(\frac{d'}{2} - 1 - \alpha)^+} + 1_{\{d' = 2(1+\alpha)\}} \log(1 + r^{-1}) \right\}, \quad r \in (0, 1]. \tag{3.55}$$

Proof. (1) By (3.32) for $q = 1$ and (3.43), we find constants $c_1, c_2 > 0$ such that

$$\mathbf{V}_B(\phi_i) := \int_0^\infty \mu(\phi_i P_t^B \phi_i) dt \le c_1 \int_0^\infty (e^{-B(\lambda_i)t} + \lambda_i^{-1} h_{i,\alpha}(t) e^{-k_2 t}) dt \le c_2 \lambda_i^{-\alpha}, \quad i \ge 1.$$

This together with (3.51) implies (3.52). By the dominated convergence theorem with $r \to 0$, the claimed consequence follows from (3.52).

(2) By (3.3) and the Markov property, we obtain

$$\begin{aligned} \mathbb{E}^\nu[\psi_i^B(t)^2] &= \frac{2}{t} \int_0^t ds_1 \int_0^{s_1} \mathbb{E}^\nu[\phi_i(X_{s_1}^B) \phi_i(X_s^B)] ds \\ &= \frac{2}{t} \int_0^t ds_1 \int_0^{s_1} \nu\big(P_s^B\{\phi_i P_{s_1-s}^B \phi_i\}\big) ds. \end{aligned} \tag{3.56}$$

Since $\nu \in \mathscr{P}_{\infty,R}$ implies $\nu = h\mu$ with $\|h - 1\|_\infty \le R + 1$, by (3.9) and (3.32), we find constants $c_1, c_2 > 0$ such that

$$\sup_{\nu \in \mathscr{P}_{\infty,R}} \left| \nu\big(P_s^B\{\phi_i P_{s_1-s}^B \phi_i\}\big) - \mu(\phi_i P_{s_1-s}^B \phi_i) \right| = \sup_{\nu \in \mathscr{P}_{\infty,R}} \left| \mu\big(\{(P_s^B)^* h - 1\} \phi_i P_{s_1-s}^B \phi_i\big) \right|$$

$$\leq c_1 \mathrm{e}^{-c_2 s}\|\phi_i P^B_{s_1-s}\phi_i\|_1 \leq c_1 \mathrm{e}^{-c_2 s}\|P^B_{s_1-s}\phi_i\|_2 \leq c_1 \mathrm{e}^{-c_2 s}\big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1} h_{i,\alpha}(s_1-s)\mathrm{e}^{-c_2(s_1-s)}\big).$$

Combining this with (3.56) and (3.43) for $k_2 = \frac{1}{2}c_2$, and noting that $c_2 s + c_2(s_2 - s) \geq \frac{1}{2}c_2 s_1 + \frac{1}{2}c_2(s_1 - s)$, we find a constant $c_3 > 0$ such that

$$\begin{aligned}
&\sup_{\nu\in\mathscr{P}_{\infty,R}} \bigg| t\mathbb{E}^\nu[\Xi^B_r(t)] - \sum_{i=1}^\infty \frac{2\mathrm{e}^{-2\lambda_i r}}{\lambda_i t}\int_0^t \mathrm{d}s_1 \int_0^{s_1} \mu(\phi_i P^B_{s_1-s}\phi_i)\mathrm{d}s\bigg| \\
&\leq \sum_{i=1}^\infty \frac{2\mathrm{e}^{-2\lambda_i r}}{\lambda_i t}\int_0^t \mathrm{d}s_1 \int_0^{s_1} c_1 \mathrm{e}^{-c_2 s}\big(\mathrm{e}^{-B(\lambda_i)(s_1-s)} + \lambda_i^{-1} h_{i,\alpha}(s_1-s)\mathrm{e}^{-c_2(s_1-s)}\big)\mathrm{d}s \qquad (3.57)\\
&\leq \frac{c_3}{t}\sum_{i=1}^\infty \lambda_i^{-1-\alpha}\mathrm{e}^{-2\lambda_i r}, \quad t, r > 0.
\end{aligned}$$

Similarly, by (3.32) for $q = 1$ and (3.43), we find constants $c_4, c_5, c_6 > 0$ such that

$$\begin{aligned}
&\bigg|\int_0^{s_1} \mu\big(\phi_i P^B_{s_1-s}\phi_i\big)\mathrm{d}s - \mathbf{V}_B(\phi_i)\bigg| \leq \int_{s_1}^\infty \big|\mu\big(\phi_i P^B_s \phi_i\big)\big| \leq \int_{s_1}^\infty \|P^B_s\phi_i\|_2 \mathrm{d}s \\
&\leq c_4 \int_{s_1}^\infty \big(\mathrm{e}^{-B(\lambda_i)s} + \lambda_i^{-1} h_{i,\alpha}(s_1-s)\mathrm{e}^{-k_2 s}\big)\mathrm{d}s \leq c_6 \lambda_i^{-\alpha}\mathrm{e}^{-c_5 s_1}, \quad s_1 > 0, i \in \mathbb{N}.
\end{aligned}$$

This together with (3.57) implies (3.53) for some constant $c > 0$.

(3) Let $k > \frac{d}{2\alpha}$. By (3.29) for $q = 1$, and (3.50) for $\theta^- = 1 + \alpha$, we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\sup_{t\geq 1,\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu[\Xi^B_r(t)] = \sup_{t\geq 1,\nu\in\mathscr{P}_{k,R}} \sum_{i=1}^\infty \frac{\mathrm{e}^{-2\lambda_i r}}{\lambda_i}\mathbb{E}^\nu[\psi^B_i(t)^2] \\
&\leq c_1 \sum_{i=1}^\infty \frac{\mathrm{e}^{-2\lambda_i r}}{\lambda_i^{1+\alpha}} \leq c_2\big\{r^{-(\frac{d'}{2}-1-\alpha)^+} + 1_{\{d'=2(1+\alpha)\}}\log(1+r^{-1})\big\}, \quad r \in (0,1].
\end{aligned}$$

Then the proof is finished. □

Finally, to get rid of the term $\mathscr{M}(f^B_{t,r})$ from (3.5), we present one more lemma.

Lemma 3.6. *Assume* (A_1) *and* (A_2) *with* $d' < 2(1+\alpha)$. *Then the following assertions hold.*

(1) *There exists a constant* $c > 0$ *and* $\sigma \in (0,1)$ *such that*

$$\mathbb{E}^\mu[\mu(|f^B_{t,r} - 1|^2)] \leq ct^{-1}r^{-\sigma}, \quad t \geq 1, r \in (0,1]. \qquad (3.58)$$

(2) *There exists a constant* $\gamma > 1$ *such that*

$$\lim_{t\to\infty} \mathbb{E}^\mu\big[\mu\big(|\mathscr{M}(f^B_{t,t^{-\gamma}})^{-1} - 1|^q\big)\big] = 0, \quad q \in [1,\infty). \qquad (3.59)$$

(3) *We have* $q_\alpha > 1$ *and*

$$\sup_{t,r>0} t^q \mathbb{E}^\mu\big[\mu\big(|\nabla \hat{L}^{-1}(f^B_{t,r} - 1)|^{2q}\big)\big] < \infty, \quad q \in [1, q_\alpha). \qquad (3.60)$$

Proof. (1) By (3.3), (3.28) and (3.17), we find constants $c_1, c_2 > 0$ such that

$$t\mathbb{E}^\mu[\mu(|f_{t,r}^B - 1|^2)] \le c_1 \sum_{i=1}^\infty \mathrm{e}^{-2\lambda_i r}\mathbb{E}^\mu[\psi_i^B(t)^2] \le c_2 \sum_{i=1}^\infty \mathrm{e}^{-2\lambda_i r}\lambda_i^{-\alpha}, \quad t, r > 0.$$

Since $d' < 2(1+\alpha)$ implies $\frac{d'}{2} - \alpha < 1$, combining this with (3.50) we derive (3.58) for any $\sigma \in (\frac{d'}{2} - \alpha, 1)$.

(2) Let $\sigma \in (0,1)$ be in (3.58) and take $\theta \in (0, q^{-1}(1-\sigma))$ for fixed $q \in [1,\infty)$. According to [37, Lemma 3.2], for any $\eta \in (0,1)$ and $\delta(\eta) := |(1-\eta)^{-\frac{1}{2}} - \frac{2}{\eta+2}|$, we have

$$\begin{aligned}&\mathbb{E}^\mu\big[|\mathscr{M}(f_{t,r}^B(y))^{-1} - 1|^q\big] \\ &\le \delta(\eta) + (1 + \theta^{-1}r^{-\frac{\theta}{2}})^q \mathbb{E}^\mu[1_{\{|f_{t,r}^B(y)-1|>\eta\}}], \quad t \ge 1, r \in (0,1], y \in M.\end{aligned}$$

Next, by (3.58) and Chebyshev's inequality, we obtain

$$\int_M \mathbb{E}^\mu[1_{\{|f_{t,r}^B(y)-1|>\eta\}}]\mu(\mathrm{d}y) \le \eta^{-2}\mathbb{E}^\mu[\mu(|f_{t,r}^B - 1|^2)] \le c\eta^{-2}t^{-1}r^{-\sigma}.$$

Putting these two estimates together, we find a function $C : (0,1) \to (0,\infty)$ such that

$$\mathbb{E}^\mu\big[\mu\big(|\mathscr{M}(f_{t,r}^B)^{-1} - 1|^q\big)\big] \le \delta(\eta) + C(\eta)t^{-1}r^{-(\sigma + \frac{\theta q}{2})}, \quad t \ge 1, r, \eta \in (0,1).$$

Noting that $\sigma \in (0,1)$ and $\theta \in (0, q^{-1}(1-\sigma))$ imply $\theta' := \sigma + \frac{\theta q}{2} \in (0,1)$, by taking $r = t^{-\gamma}$ for $\gamma \in (1, \frac{1}{\theta'})$, and letting first $t \to \infty$ then $\eta \to 0$, we derive (3.59).

(3) By (2.11), $d' < 2(1+\alpha)$ implies $q_\alpha > 1$. For any $q \in [1, q_\alpha)$, we have

$$\frac{d(q-1)}{2q} - 1 < \alpha - \frac{d'}{2} - \frac{d(q-1)}{2q}.$$

So, there exists $\kappa \in (0, \frac{1}{2})$ such that $\beta := \frac{d(q-1)}{4q} - \kappa$ satisfies

$$2\beta < \alpha - \frac{d'}{2} - \frac{d(q-1)}{2q}. \tag{3.61}$$

By (3.12), (3.3), (3.17) and (3.47), we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}&t^q\mathbb{E}^\mu\big[\mu\big(|\nabla\hat{L}^{-1}(f_{t,r}^B - 1)|^{2q}\big)\big] \le c_1 t^q \mathbb{E}^\mu\big[\|(-\hat{L})^{\frac{d(q-1)}{4q} - \kappa}(f_{t,r}^B - 1)\|_2^{2q}\big] \\ &\le c_2 r^{-(2\beta + \frac{d'}{2} + \frac{d(q-1)}{2q} - \alpha)^+} = c_2,\end{aligned}$$

where the last step follows from (3.61). Then (3.60) holds.

□

3.2 Proof of Theorem 2.1

We first consider the stationary case where the initial distribution is the invariant measure μ, then extend to more general setting by using an approximation argument. To this end, we need the following further modification of the empirical measure:

$$\mathrm{d}\tilde{\mu}_{t,r}^B = \tilde{f}_{t,r}^B \mathrm{d}\mu, \quad \tilde{f}_{t,r}^B := (1-r) f_{t,r}^B + r, \quad r \in (0,1], t > 0. \tag{3.62}$$

By (3.2), we have

$$\mathbb{W}_2(\mu_{t,r}^B, \tilde{\mu}_{t,r}^B)^2 \le 4r \Xi_r^B(t), \quad t > 0, r \in (0,1]. \tag{3.63}$$

Proposition 3.7. *Assume* (A_1) *and* (A_2) *with* $d' < 2(1+\alpha)$. *Then*

$$\lim_{t\to\infty} \mathbb{E}^\mu \Big[\big| \{ t\mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t) \}^+ \big|^q \Big] = 0, \quad q \in [1, q_\alpha). \tag{3.64}$$

Proof. Let $t \ge 1$ and take $r_t = t^{-\gamma}$ for $\gamma > 1$ in (3.59). By (3.6) and (3.62), we have

$$t\mu(|\nabla \hat{L}^{-1}(\tilde{f}_{t,r_t}^B - 1)|^2) = (1-r_t)^2 \Xi_{r_t}^B(t),$$

so that (3.5), (3.60) and (3.59) yield

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}^\mu \Big[\big| \{ t\mathbb{W}_2(\tilde{\mu}_{t,r_t}^B, \mu)^2 - (1-r_t)^2 \Xi_{r_t}^B(t) \}^+ \big|^q \Big] \\
&= \lim_{t\to\infty} \mathbb{E}^\mu \Big[\big| \{ t\mathbb{W}_2(\tilde{\mu}_{t,r_t}^B, \mu)^2 - t\mu(|\nabla \hat{L}^{-1}(\tilde{f}_{t,r_t}^B - 1)|^2) \}^+ \big|^q \Big] \\
&\le \lim_{t\to\infty} t^q \mathbb{E}^\mu \Big[\big\{ \mu\big(|\nabla \hat{L}^{-1}(\tilde{f}_{t,r_t}^B - 1)|^2 |\mathscr{M}(\tilde{f}_{t,r_t}^B)^{-1} - 1| \big) \big\}^q \Big] \\
&\le \lim_{t\to\infty} t^q \mathbb{E}^\mu \Big[\mu\big(|\nabla \hat{L}^{-1}(\tilde{f}_{t,r_t}^B - 1)|^{2q} |\mathscr{M}(\tilde{f}_{t,r_t}^B)^{-1} - 1|^q \big) \Big] \\
&\le \lim_{t\to\infty} \Big(t^{q'} \mathbb{E}^\mu \Big[\mu\big(|\nabla \hat{L}^{-1}(\tilde{f}_{t,r_t}^B - 1)|^{2q'} \big) \Big] \Big)^{\frac{q}{q'}} \Big(\mathbb{E}^\mu \Big[\mu\big(|\mathscr{M}(\tilde{f}_{t,r_t}^B)^{-1} - 1|^{\frac{qq'}{q'-q}} \big) \Big] \Big)^{\frac{q'-q}{q'}} \\
&= 0, \quad q' \in (q, q_\alpha).
\end{aligned}$$

Noting that (2.10) and (3.6) imply

$$\Xi^B(t) \ge \Xi_{r_t}^B(t) \ge (1-r_t)^2 \Xi_{r_t}^B(t),$$

we derive

$$\lim_{t\to\infty} \mathbb{E}^\mu \Big[\big| \{ t\mathbb{W}_2(\tilde{\mu}_{t,r_t}^B, \mu)^2 - \Xi^B(t) \}^+ \big|^q \Big] = 0, \quad q \in [1, q_\alpha). \tag{3.65}$$

On the other hand, noting that $q \in [1, \frac{d}{(d+d'-2-2\alpha)^+})$ implies $1 + \alpha - \frac{d'}{2} - \frac{d(q-1)}{2q} > 0$, by (3.6), (3.7) and (3.47) with $a_0 = 0$ and $\beta = -\frac{1}{2}$, we obtain

$$\begin{aligned}
&\sup_{r>0, t\ge 1} 4^{-q} t^q \mathbb{E}^\mu [\mathbb{W}_2(\mu_{t,r}^B, \mu)^{2q}] \le \sup_{r>0, t\ge 1} t^q \mathbb{E}^\mu \big[\|(-\hat{L})^{-\frac{1}{2}} (f_{t,r}^B - 1)\|_2^{2q} \big] \\
&= \sup_{r>0, t\ge 1} \mathbb{E}^\mu \big[\Xi_r^B(t)^q \big] < \infty, \quad 1 \le q < \frac{d}{(d+d'-2-2\alpha)^+}.
\end{aligned} \tag{3.66}$$

Since $q \in [1, q_\alpha)$ implies $d' < 2(1+\alpha)$ and $q \in [1, \frac{d}{(d+d'-2-2\alpha)^+})$, by combining this with (3.24), (3.63), the triangle inequality and $r_t = t^{-\gamma}$ with $\gamma > 1$, we find a constant $c > 0$ such that

$$\begin{aligned}&\lim_{t\to\infty} t^q \mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B, \tilde{\mu}_{t,r_t}^B)^{2q}] \\ &\le 2^q \lim_{t\to\infty} t^q \mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r_t}^B, \tilde{\mu}_{t,r_t}^B)^{2q} + \mathbb{W}_2(\mu_{t,r_t}^B, \mu_t^B)^{2q}] \\ &\le c \lim_{t\to\infty} (r_t t)^q \big(1 + \mathbb{E}^\mu[\Xi_{r_t}^B(t)^q]\big) = 0.\end{aligned} \tag{3.67}$$

This together with (3.65) and (3.66) implies (3.64). □

Next, we consider arbitrary initial distribution $\nu \in \mathscr{P}$. Let

$$\nu_\varepsilon := \nu P_\varepsilon^B, \quad \varepsilon \in (0,1).$$

By (2.5), there exists a constant $c > 0$ such that

$$\nu_\varepsilon \le c\varepsilon^{-\frac{d}{2\alpha}}\mu, \quad \mathbb{E}^{\nu_\varepsilon} \le c\varepsilon^{-\frac{d}{2\alpha}}\mathbb{E}^\mu, \quad \varepsilon \in (0,1). \tag{3.68}$$

Let

$$\mu_t^{B,\varepsilon} := \frac{1}{t}\int_\varepsilon^{t+\varepsilon} \delta_{X_s^B} \mathrm{d}s, \quad t, \varepsilon > 0. \tag{3.69}$$

By the Markov property, $\mu_t^{B,\varepsilon}$ is the empirical measure with initial distribution ν_ε, so that for any nonnegative measurable function F on $\mathscr{P}$,

$$\mathbb{E}^\nu[F(\mu_t^{B,\varepsilon})] = \mathbb{E}^{\nu_\varepsilon}[F(\mu_t^B)], \quad t, \varepsilon > 0. \tag{3.70}$$

To estimate $\mathbb{W}_2(\mu_t^{B,\varepsilon}, \mu)$, we take

$$\Xi^{B,\varepsilon}(t) := \sum_{i=1}^\infty \frac{1}{\lambda_i} \psi_i^{B,\varepsilon}(t)^2, \quad \psi_i^{B,\varepsilon}(t) := \frac{1}{\sqrt{t}}\int_\varepsilon^{t+\varepsilon} \phi_i(X_s^B)\mathrm{d}s. \tag{3.71}$$

Proposition 3.8. *Assume (A_1) and (A_2).*

(1) *If $\alpha > 0$ such that $d + d' < 2 + 4\alpha$, then for any $q \in [1, \frac{d}{(d+d'-2-2\alpha)^+})$,*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\ge 1, \nu\in\mathscr{P}} \mathbb{E}^\nu\big[|t\mathbb{W}_2(\mu_t^B, \mu)^2 - t\mathbb{W}_2(\mu_t^{B,\varepsilon}, \mu)^2|^{2q}\big] = 0. \tag{3.72}$$

(2) *If $\alpha > 0$ and $d + d' \ge 2 + 4\alpha$, then for any $k > \frac{d+d'-2-2\alpha}{2\alpha}$ and $q \in [1, \frac{d}{(d+d'-2-2\alpha)^+})$,*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\ge 1, \nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\big[|t\mathbb{W}_2(\mu_t^B, \mu)^2 - t\mathbb{W}_2(\mu_t^{B,\varepsilon}, \mu)^2|^{2q}\big] = 0, \quad R \in [1,\infty). \tag{3.73}$$

(3) *For any* $q \in [1,\infty)$, $k=\infty$ *or* $k \in (\frac{d}{2\alpha \mathrm{i}(q)},\infty]\cap[1,\infty]$, *there exists a constant* $c>0$ *such that*

$$\sup_{\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\big[|\psi_i^{B,\varepsilon}(t)^2-\psi_i^B(t)^2|^q\big] \le cR\varepsilon^{\frac{q}{2}}t^{-\frac{q}{2}}\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\in\mathbb{N}, t\ge 1, \varepsilon\in(0,1). \tag{3.74}$$

Moreover, if $\mathrm{i}(q)>\frac{d}{2\alpha}$, *then there exists a constant* $c>0$ *such that*

$$\sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu\big[|\psi_i^{B,\varepsilon}(t)^2-\psi_i^B(t)^2|^q\big] \le c\varepsilon^{\frac{q}{2}}t^{-\frac{q}{2}}\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\in\mathbb{N}, t\ge 1, \varepsilon\in(0,1). \tag{3.75}$$

Proof. (1) By $d+d'<2+4\alpha$, we have $\frac{d}{2\alpha}<\frac{d}{(d+d'-2-2\alpha)^+}$. So, it suffices to prove for $1\le q\in(\frac{d}{2\alpha},\frac{d}{(d+d'-2-2\alpha)^+})$. It is easy to see that

$$\pi_{t,\varepsilon} := \frac{1}{t}\int_\varepsilon^t \delta_{(X_s^B,X_s^B)}\mathrm{d}s + \frac{1}{t}\int_0^\varepsilon \delta_{(X_s^B,X_{t+s}^B)}\mathrm{d}s \in \mathscr{C}(\mu_t^B,\mu_t^{B,\varepsilon}),\quad t>\varepsilon\ge 0.$$

So, (3.23) implies that for any $t>\varepsilon\ge0$ and $p\in[1,\infty)$,

$$\mathbb{W}_p(\mu_t^B,\mu_t^{B,\varepsilon})^p \le \int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\pi_{t,\varepsilon}(\mathrm{d}x,\mathrm{d}y) = \frac{1}{t}\int_0^\varepsilon \rho(X_s^B,X_{s+t}^B)^p\mathrm{d}s \le \frac{\varepsilon D^p}{t}. \tag{3.76}$$

On the other hand, by (3.2), (3.66), (3.68) and (3.70), we find a map

$$c:\ \Big[1,\frac{d}{(d+d'-2-2\alpha)^+}\Big)\to(0,\infty)$$

such that

$$\begin{aligned}&\sup_{t\ge1,\nu\in\mathscr{P}} t^q\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^{2q}] = \sup_{t\ge1,r\in(0,1],\nu\in\mathscr{P}} t^q\mathbb{E}^{\nu_\varepsilon}[\mathbb{W}_2(\mu_{t,r}^B,\mu)^{2q}]\\ &\le 4^q\sup_{t\ge1,r\in(0,1],\nu\in\mathscr{P}}\mathbb{E}^{\nu_\varepsilon}[\Xi_r^B(t)^q]\le c(q)\varepsilon^{-\frac{d}{2\alpha}},\quad \varepsilon\in(0,1),\ q\in\Big[1,\frac{d}{(d+d'-2-2\alpha)^+}\Big).\end{aligned} \tag{3.77}$$

Combining this with (3.76), (3.68) and $q>\frac{d}{2\alpha}$, we find constants $c_1,c_2>0$ such that

$$\begin{aligned}&\lim_{\varepsilon\downarrow0}\sup_{t\ge1,\nu\in\mathscr{P}}\mathbb{E}^\nu\big[|t\mathbb{W}_2(\mu_t^B,\mu)^2-t\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^2|^{2q}\big]\\ &\le\lim_{\varepsilon\downarrow0}\sup_{t\ge1,\nu\in\mathscr{P}}\mathbb{E}^\nu\big[|t\mathbb{W}_2(\mu_t^B,\mu_t^{B,\varepsilon})^2+2t\mathbb{W}_2(\mu_t^B,\mu_t^{B,\varepsilon})\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)|^{2q}\big]\\ &\le\lim_{\varepsilon\downarrow0}\sup_{t\ge1,\nu\in\mathscr{P}}c_1\Big(\varepsilon^{2q}+\varepsilon^q\sup_{\nu\in\mathscr{P}}\mathbb{E}^{\nu_\varepsilon}[\mathbb{W}_2(\mu_t^B,\mu)^{2q}]\Big)\le\lim_{\varepsilon\downarrow0}c_2\varepsilon^{q-\frac{d}{2\alpha}}=0.\end{aligned}$$

So, (3.72) holds.

(2) Let $\alpha>0$, $d+d'\ge2+4\alpha$ and $k>\frac{d+d'-2-2\alpha}{2\alpha}$. It suffices to prove (3.73) for $1\le q\in(\frac{d}{2\alpha k},\frac{d}{(d+d'-2-2\alpha)^+})$. By the same reason leading to (3.68), we find a constant $c>0$ such that

$$\sup_{\nu\in\mathscr{P}_{k,R}}\mathbb{E}^{\nu_\varepsilon}\le c\varepsilon^{-\frac{d}{2\alpha k}}\mathbb{E}^\mu,\quad\varepsilon\in(0,1).$$

Hence, as shown above that $q > \frac{d}{2\alpha k}$ implies

$$\lim_{\varepsilon\downarrow 0} \sup_{t\geq 1, \nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\big[|t\mathbb{W}_2(\mu_t^B,\mu)^2 - t\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^2|^{2q}\big] \leq \lim_{\varepsilon\downarrow 0} c_2\varepsilon^{q-\frac{d}{2\alpha k}} = 0.$$

(3) Let ψ_i^B and $\psi_i^{B,\varepsilon}$ be in (2.10) and (3.71). We have

$$\begin{aligned} &\big|\psi_i^B(t)^2 - \psi_i^{B,\varepsilon}(t)^2\big| \\ &= \frac{1}{t}\bigg|\int_0^\varepsilon \big\{\phi_i(X_{t+s}^B) - \phi_i(X_s^B)\big\}\mathrm{d}s\bigg| \cdot \bigg|\int_0^t \big\{\phi_i(X_{s+\varepsilon}^B) + \phi_i(X_s^B)\big\}\mathrm{d}s\bigg| \\ &\leq \frac{\sqrt{\varepsilon}}{\sqrt{t}}\big(|\psi_i^{B,t}(\varepsilon)| + |\psi_i^B(\varepsilon)|\big)\big(|\psi_i^{B,\varepsilon}(t)| + |\psi_i^B(t)|\big). \end{aligned} \tag{3.78}$$

Since (3.70) implies $\{\nu_t : \nu\in\mathscr{P}, t\geq 1\}\subset\mathscr{P}_{\infty,R}$ for some $R>0$, by (3.28) and (3.70) we find a constant $c_1>0$ such that

$$\sup_{\varepsilon>0, t\geq 1,\nu\in\mathscr{P}} \mathbb{E}^\nu[\psi_i^{B,t}(\varepsilon)^{2q}] = \sup_{\varepsilon>0,t\geq 1,\nu\in\mathscr{P}} \mathbb{E}^{\nu_t}[\psi_i^B(\varepsilon)^{2q}] \leq c_1\lambda_i^{\frac{d(q-1)}{2}-q\alpha}. \tag{3.79}$$

Moreover, by (3.28) for $k=\infty$, (3.29) for $\alpha>0$ and $k\in(\frac{d}{2\alpha \mathrm{i}(q)},\infty]\cap[1,\infty]$, and the fact that $\nu\in\mathscr{P}_{k,R}$ implies $\nu_\varepsilon\in\mathscr{P}_{k,R}$ for $\varepsilon>0$, we find a constant $c_2>0$ such that

$$\sup_{\varepsilon>0,t\geq 1,\nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu[\psi_i^{B,\varepsilon}(t)^{2q} + \psi_i^B(\varepsilon)^{2q}] \leq c_2\lambda_i^{\frac{d(q-1)}{2}-q\alpha},\quad i\geq 1. \tag{3.80}$$

Combining these estimates we derive (3.74).

Finally, when $\mathrm{i}(q) > \frac{d}{2\alpha}$, (3.75) can be proved in the same way by using (3.30) in place of (3.29). □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. (1) It suffices to prove for $q\in(\frac{d}{2\alpha}, q_\alpha)$. By $\alpha>\alpha(d,d')$, we have

$$\frac{d}{2\alpha} < \frac{2(1+\alpha)-d'}{(d+d'-2-2\alpha)^+}.$$

So, either $\mathrm{i}(q) > \frac{d}{2\alpha}$ or $\mathrm{i}(q) < \frac{2+2\alpha-d'}{(d+d'-2-2\alpha)^+}$. Below we consider these two situations respectively.

(1_a) Let $\mathrm{i}(q)>\frac{d}{2\alpha}$. By (3.70), (3.68) and (3.65), we obtain

$$\begin{aligned} &\lim_{t\to\infty}\sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu\Big[\Big|\big\{t\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^2 - \Xi^{B,\varepsilon}(t)\big\}^+\Big|^q\Big] \\ &= \lim_{t\to\infty}\sup_{\nu\in\mathscr{P}} \mathbb{E}^{\nu_\varepsilon}\Big[\Big|\big\{t\mathbb{W}_2(\mu_t^B,\mu)^2 - \Xi^B(t)\big\}^+\Big|^q\Big] \\ &\leq \lim_{t\to\infty} c\varepsilon^{-\frac{d}{2\alpha}}\mathbb{E}^\mu\Big[\Big|\big\{t\mathbb{W}_2(\mu_t^B,\mu)^2 - \Xi^B(t)\big\}^+\Big|^q\Big] = 0,\quad \varepsilon\in(0,1). \end{aligned} \tag{3.81}$$

Next, by (2.10) and (3.71),

$$|\Xi^B(t) - \Xi^{B,\varepsilon}(t)| \le \sum_{i=1}^\infty \frac{1}{\lambda_i} |\psi_i^{B,\varepsilon}(t)^2 - \psi_i^B(t)^2|.$$

Combining this with (3.75), when $\mathrm{i}(q) > \frac{d}{2\alpha}$ we find a constant $k_1 > 0$ such that

$$\begin{aligned}
&\sup_{t\ge1,\nu\in\mathscr{P}} t^{\frac{q}{2}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] \\
&\le \Big(\sum_{i=1}^\infty \lambda_i^{-\theta}\Big)^{q-1} \sum_{i=1}^\infty \lambda_i^{\theta(q-1)-q} \sup_{t\ge1,\nu\in\mathscr{P}} t^{\frac{q}{2}} \mathbb{E}^\nu\big[|\psi_i^{B,\varepsilon}(t)^2 - \psi_i^B(t)^2|^q\big] \\
&\le k_1 \varepsilon^{\frac{q}{2}} \Big(\sum_{i=1}^\infty \lambda_i^{-\theta}\Big)^{q-1} \sum_{i=1}^\infty \lambda_i^{\theta(q-1)+\frac{d(q-1)}{2}-q-\alpha q}, \quad \theta\in\mathbb{R},\ \varepsilon\in(0,1).
\end{aligned}$$

Taking

$$\theta = 1+\alpha - \frac{d(q-1)}{2q} \tag{3.82}$$

such that $-\theta = \theta(q-1) + \frac{d(q-1)}{2} - q - \alpha q$, we arrived at

$$\sup_{t\ge1,\varepsilon\in(0,1),\nu\in\mathscr{P}} \varepsilon^{-\frac{q}{2}} t^{\frac{q}{2}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] \le k_1 \Big(\sum_{i=1}^\infty \lambda_i^{-\theta}\Big)^q. \tag{3.83}$$

Noting that (2.11), (3.82) and $q < q_\alpha$ imply $\frac{2\theta}{d'} > 1$, by (2.6) we find a constant $k_2 > 0$ such that

$$\sum_{i=1}^\infty \lambda_i^{-\theta} \le k_2 \sum_{i=1}^\infty i^{-\frac{2\theta}{d'}} < \infty.$$

Thus,

$$\sup_{t\ge1,\varepsilon\in(0,1),\nu\in\mathscr{P}} \varepsilon^{-\frac{q}{2}} t^{\frac{q}{2}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] < \infty. \tag{3.84}$$

Consequently,

$$\lim_{t\to\infty} \sup_{\varepsilon\in(0,1),\nu\in\mathscr{P}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] = 0.$$

Combining this with (3.72) and (3.81), we derive (2.12).

(1_b) Let $\mathrm{i}(q) < \frac{2+2\alpha-d'}{(d+d'-2-2\alpha)^+}$. Since $q > \frac{d}{2\alpha}$ implies $\mathrm{i}(q)+1 > \frac{d}{2\alpha}$, by (3.30) and Hölder's inequality, we find a constant $c_1 > 0$ such that

$$\sup_{t\ge1,\nu\in\mathscr{P}} \mathbb{E}^\nu\big[|\psi_i^B(t)|^{2q}\big] \le \sup_{t\ge1,\nu\in\mathscr{P}} \big(\mathbb{E}^\nu\big[|\psi_i^B(t)|^{2\{\mathrm{i}(q)+1\}}\big]\big)^{\frac{q}{\mathrm{i}(q)+1}} \le c_1 \lambda_i^{\frac{dq\mathrm{i}(q)}{2(\mathrm{i}(q)+1)}-q\alpha}, \quad i\in\mathbb{N}.$$

By the calculations leading to (3.83), we derive the same estimate for

$$\theta := 1+\alpha - \frac{d\mathrm{i}(q)}{2(\mathrm{i}(q)+1)}.$$

We have $\theta > \frac{d'}{2}$ due to $\mathrm{i}(q) < \frac{2+2\alpha-d'}{(d+d'-2-2\alpha)^+}$. Hence, (3.84) holds, which together with (3.72) and (3.81) imply (2.12).

(2) Let $\alpha \in (0,1], q \in [1, q_\alpha)$ and $k \in (\frac{d}{2\alpha \mathrm{i}(q)}, \infty] \cap [1,\infty]$. By using (3.74) in place of (3.75), the proof of (3.84) implies

$$\sup_{t\geq 1, \varepsilon\in(0,1), \nu\in\mathscr{P}_{k,R}} \varepsilon^{-\frac{q}{2}} t^{\frac{q}{2}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] < \infty, \quad R\in[1,\infty),$$

so that

$$\lim_{t\to\infty} \sup_{\varepsilon\in(0,1), \nu\in\mathscr{P}_{k,R}} \mathbb{E}^\nu\big[|\Xi^B(t) - \Xi^{B,\varepsilon}(t)|^q\big] = 0, \quad R\in[1,\infty).$$

This together with (3.72) and (3.81) implies (2.13).

When $k=\infty$, (2.13) follows from (3.64) and that $\mathbb{E}^\nu \leq \|h\|_\infty \mathbb{E}^\mu$ for $\nu = h\mu$.

□

3.3 Proof of Theorem 2.2

(1) Let $d' < 2(1+\alpha)$. By Lemma 3.5, (3.54) holds with $\eta_Z^B < \infty$. Combining this with (3.64) and $\mathbb{E}^\nu \leq \|h\|_\infty \mathbb{E}^\mu$ for $\nu = h\mu$, we obtain

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}_{\infty,R}} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^B,\mu)^2\big] \leq \limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}_{\infty,R}} \mathbb{E}^\nu\big[\Xi^B(t)\big] \leq \eta_Z^B, \quad R\in[1,\infty). \tag{3.85}$$

On the other hand, by (3.68), for any $\varepsilon\in(0,1)$ there exists $R\in[1,\infty)$ such that $\nu_\varepsilon \in \mathscr{P}_{\infty,R}$ holds for all $\nu\in\mathscr{P}$. So, (3.85) together with (3.70) implies

$$\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^2\big] = \limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^{\nu_\varepsilon}\big[\mathbb{W}_2(\mu_t^B,\mu)^2\big] \leq \eta_Z^B, \quad \varepsilon\in(0,1). \tag{3.86}$$

Combining this with (3.76) for $p=2$ and using the triangle inequality, we find a constant $c>0$ such that

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^B,\mu)^2\big] \\
&\leq \limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu\big[(1+\varepsilon^{\frac12})\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu)^2 + (1+\varepsilon^{-\frac12})\mathbb{W}_2(\mu_t^{B,\varepsilon},\mu_t^B)^2\big] \\
&\leq (1+\varepsilon^{\frac12})\eta_Z^B + c(\varepsilon+\varepsilon^{\frac12}), \quad \varepsilon\in(0,1).
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we obtain (2.14).

(2) Let $d' \geq 2(1+\alpha)$. By (3.6), (3.7) and (3.47) with $a_0 = 0, q=1$ and $\beta = -\frac12$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2] \leq 4\mathbb{E}^\mu[\mu(|\nabla \hat{L}^{-1}(f_{t,r}^B - 1)|^2)] = 4\mathbb{E}^\mu[\mu(|(-\hat{L})^{-\frac12}(f_{t,r}^B-1)|^2)] \\
&\leq \frac{c_1}{t}\big(r^{1+\alpha-\frac{d'}{2}} + 1_{\{d'=2+2\alpha\}}\log(1+r^{-1})\big), \quad t\geq 1, r\in(0,1].
\end{aligned} \tag{3.87}$$

Combining this with (3.24) and the triangle inequality, we find a constant $c_2>0$ such that

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B,\mu)^2] \leq 2\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2] + 2\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B,\mu_t^B)^2]$$

$$\leq \frac{c_1}{t}\big\{r^{1+\alpha-\frac{d'}{2}} + 1_{\{d'=2+2\alpha\}}\log(1+r^{-1})\big\} + c_2 r, \quad t\geq 1, r\in(0,1].$$

Taking $r = t^{-\frac{2}{d'-2\alpha}}$ when $d' > 2+2\alpha$, and $r = t^{-1}$ when $d' = 2+2\alpha$, we find a constant $c_3 > 0$ such that

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B,\mu)^2] \leq \begin{cases} c_3 t^{-1}\log(1+t), & \text{if } d' = 2(1+\alpha), \\ c_3 t^{-\frac{2}{d'-2\alpha}}, & \text{if } d' > 2(1+\alpha). \end{cases}$$

By combining this with (3.68) and (3.70), we find a constant $c_4 > 0$ such that

$$\begin{aligned} &\sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^{B,1},\mu)^2] = \sup_{\nu\in\mathscr{P}} \mathbb{E}^{\nu_1}[\mathbb{W}_2(\mu_t^B,\mu)^2] \leq c_4 \mathbb{E}^\mu[\mathbb{W}_2(\mu_t^B,\mu)^2] \\ &\leq c_3c_4\Big\{1_{\{d'=2(1+\alpha)\}}t^{-1}\log(1+t) + t^{-\frac{2}{d'-2\alpha}}\Big\}, \quad t\geq 1. \end{aligned} \tag{3.88}$$

Noting that the triangle inequality implies

$$\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2] \leq 2\mathbb{W}_2(\mu_t^{B,1},\mu)^2 + 2\mathbb{W}_2(\mu_t^{B,1},\mu_t^B)^2,$$

we deduce (2.15) from (3.76) and (3.88).

3.4 Proof of Theorem 2.3

By approximating μ_t^B using $\mu_t^{B,1}$ as in (3.76) and (3.88), we only need to prove for $\nu = \mu$.

By (3.7), (3.12) and (3.47), for any $\kappa \in (0,\frac{1}{2})$ we find constants $c_1, c_2 > 0$ such that for any $t \geq 1$ and $r \in (0,1]$,

$$\begin{aligned} &\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_{t,r}^B,\mu)^{2q}]\big)^{\frac{1}{q}} \leq c_1\big(\mathbb{E}^\mu[\|(-\hat{L})^{\frac{d(p-1)}{4p}-\kappa}(f_{t,r}^B-1)\|_2^{2q}\big)^{\frac{1}{q}} \\ &\leq \frac{c_2}{t}\Big\{r^{-\left(\frac{d(p-1)}{2p}-2\kappa+\frac{d'}{2}+\frac{d(q-1)}{2q}-\alpha\right)^+} + 1_{\left\{\frac{d(p-1)}{2p}-2\kappa+\frac{d'}{2}+\frac{d(q-1)}{2q}-\alpha=0\right\}}\log(1+r^{-1})\Big\}. \end{aligned} \tag{3.89}$$

Below we prove assertions (1)-(3) in Theorem 2.3 respectively.

(1) Let $\gamma_{\alpha,p,q} < 0$. We may take $\kappa \in (0,\frac{1}{2})$ such that

$$\frac{d(p-1)}{2p} - 2\kappa + \frac{d'}{2} + \frac{d(q-1)}{2q} - \alpha < 0,$$

so that (3.89) implies

$$\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_{t,r}^B,\mu)^{2q}]\big)^{\frac{1}{q}} \leq \frac{c_2}{t}, \quad r\in(0,1], t\geq 1.$$

By Fatou's lemma for $r \to 0$, we obtain (2.17) for $\nu = \mu$.

(2) Let $\gamma_{\alpha,p,q} \geq 0$. For any $\gamma > \gamma_{\alpha,p,q}$, we find $\kappa \in (0,\frac{1}{2})$ such that

$$\frac{d(p-1)}{2p} - 2\kappa + \frac{d'}{2} + \frac{d(q-1)}{2q} - \alpha \leq \gamma,$$

so that (3.89), (3.24) and the triangle inequality imply

$$\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac{1}{q}} \le c_1\big\{t^{-1}r^{-\gamma}+r\big\},\quad r\in(0,1], t\ge 1$$

for some constant $c_1>0$. Taking $r=t^{-\frac{1}{1+\gamma}}$ we obtain (2.19) for $\nu=\mu$.

(3) Let (2.16) hold and $\gamma_{\alpha,p,q}\ge 0$. By (3.7) we find constants $c_1,c_2>0$ such that

$$\mathbb{W}_{2p}(\mu_{t,r}^B,\mu)^{2p} \le c_1\|\nabla(-\hat{L})^{-1}(f_{t,r}^B-1)\|_{2p}^{2p} \le c_2\|(a_0-\hat{L})^{\frac{1}{2}}(-\hat{L})^{-1}(f_{t,r}^B-1)\|_{2p}^{2p}. \tag{3.90}$$

On the other hand, by the Sobolev embedding theorem, (2.5) implies that for any constants $k_2\ge k_1>-\infty$ and $q_1\ge q_2\ge 1$ with

$$\frac{1}{q_1}=\frac{1}{q_2}+\frac{k_1-k_2}{d}, \tag{3.91}$$

there exists a constant $C>0$ such that

$$\|(-\hat{L})^{\frac{k_1}{2}}f\|_{q_1}\le C\|(-\hat{L})^{\frac{k_2}{2}}f\|_{q_2},\quad \mu(f)=0.$$

Taking

$$k_1=-2,\quad k_2=\frac{d(p-1)}{2p}-2,\quad q_1=2p,\quad q_2=2$$

such that (3.91) holds, we find a constant $c_2>0$ such that

$$\|(a_0-\hat{L})^{\frac{1}{2}}(-\hat{L})^{-1}(f_{t,r}^B-1)\|_{2p}\le c_2\|(a_0-\hat{L})^{\frac{1}{2}}(-\hat{L})^{\frac{d(p-1)}{4p}-1}(f_{t,r}^B-1)\|_2.$$

Combining this with (3.47) and (3.90), we find a constant $c_3>0$ such that

$$\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_{t,r}^B,\mu)^{2q}]\big)^{\frac{1}{q}}\le\frac{c_3}{t}\big\{r^{-\gamma_{\alpha,p,q}}+1_{\{\gamma_{\alpha,p,q}=0\}}\log(1+r^{-1})\big\},\quad t\ge 1, r\in(0,1].$$

By this together with (3.24) and the triangle inequality, we find a constant $c_4>0$ such that

$$\big(\mathbb{E}^\mu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac{1}{q}}\le c_4\Big\{t^{-1}r^{-\gamma_{\alpha,p,q}}+t^{-1}1_{\{\gamma_{\alpha,p,q}=0\}}\log(1+r^{-1})+r\Big\},\quad t\ge 1, r\in(0,1].$$

Taking $r=t^{-\frac{1}{1+\gamma_{\alpha,p,q}}}$, we obtain (2.19).

4 Proofs of Theorems 2.4 and 2.5

We will follow the line of [38] to estimate the lower bound of $\mathbb{W}_2(\mu_t^B,\mu)$ by using an idea of [1]. For any $f\in\mathscr{D}(L)$ with $\|f\|_\infty+\|\nabla f\|_\infty+\|Lf\|_\infty<\infty$, let

$$T_t^\sigma f:=-\sigma\log\hat{P}_{\frac{\sigma t}{2}}\mathrm{e}^{\sigma^{-1}f},\quad \sigma>0, t\in[0,1].$$

Lemma 4.1. *Assume that* (M,ρ) *is a geodesic space. If* (2.7) *and* (2.20) *hold, then there exist constants* $k_1,k_2>0$ *such that* $\|\nabla f\|_\infty^2\le k_1\sigma$ *implies*

$$
\begin{aligned}
&T_1^\sigma f(y)-f(x)\le \frac{1}{2}\rho(x,y)^2+\sigma\|\hat{L}f\|_\infty^2+k_2\sigma^\theta\|\nabla f\|_\infty^2,\\
&\mu(f-T_1^\sigma f)\le \frac{1}{2}\mu(|\nabla f|^2)+k_2\sigma^{-1}\|\nabla f\|_\infty^4.
\end{aligned} \tag{4.1}
$$

Proof. Let $k_1=2\kappa_0$, then $\|\nabla f\|_\infty^2\le k_1\sigma$ implies

$$\frac{t\sigma}{2}\|\nabla\sigma^{-1}f\|_\infty^2\le\frac{1}{2}\sigma^{-1}\|\nabla f\|_\infty^2\le\kappa_0,\quad t\in[0,1],$$

so that (3.26) holds for $m=1$ and $(\frac{\sigma t}{2},\sigma^{-1}f)$ replacing (t,f), which together with (2.7) yields

$$
\begin{aligned}
|\nabla T_t^\sigma f|^2&=\frac{\sigma^2|\nabla\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}|^2}{(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})^2}\le\frac{(1+k(2))\sigma^2\hat{P}_{\frac{t\sigma}{2}}(|\nabla\sigma^{-1}f|^2\mathrm{e}^{2\sigma^{-1}f})}{(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})^2}\\
&\le\frac{(1+k(2))\|\nabla f\|_\infty^2\hat{P}_{\frac{t\sigma}{2}}(\mathrm{e}^{2\sigma^{-1}f})}{(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})^2}\le 2(1+k(2))\|\nabla f\|_\infty^2=:c_1\|\nabla f\|_\infty^2.
\end{aligned} \tag{4.2}
$$

Next, by (3.25), we find a constant $c_2>0$ such that

$$
\begin{aligned}
\hat{L}T_t^\sigma f&=-\frac{\sigma\hat{L}\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}}{\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}}+\frac{\sigma|\nabla\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}|^2}{(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})^2}\\
&=\frac{-\hat{P}_{\frac{t\sigma}{2}}\{\mathrm{e}^{\sigma^{-1}f}\hat{L}f\}}{\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}}+\frac{\sigma|\nabla\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}|^2-\sigma(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})\hat{P}_{\frac{t\sigma}{2}}(|\nabla\sigma^{-1}f|^2\mathrm{e}^{\sigma^{-1}f})}{(\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f})^2}\\
&\le\|\hat{L}f\|_\infty+c_2\sigma^{\theta-1}\|\nabla f\|_\infty^2,\quad\sigma,t\in(0,1],\|\nabla f\|_\infty^2\le k_1\sigma.
\end{aligned} \tag{4.3}
$$

Moreover, for any two points $x,y\in M$, let $\gamma:[0,1]\to M$ be the minimal geodesic from x to y with

$$|\dot{\gamma}_t|:=\limsup_{s\to t}\frac{\rho(\gamma_t,\gamma_s)}{|t-s|}=\rho(x,y),\quad\text{a.e. } t\in[0,1].$$

So,

$$\limsup_{s\to t}\frac{|f(\gamma_t)-f(\gamma_s)|}{|t-s|}\le|\nabla f(\gamma_t)|\rho(x,y),\quad t\in[0,1]. \tag{4.4}$$

By the backward Kolmogorov equation and the chain rule, we have

$$\partial_t T_t^\sigma f=-\frac{\sigma^2\hat{L}\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\sigma^{-1}f}}{2\hat{P}_{\frac{t\sigma}{2}}\mathrm{e}^{\mathrm{e}^{\sigma^{-1}f}}}=\frac{\sigma}{2}\hat{L}T_t^\sigma f-\frac{1}{2}|\nabla T_t^\sigma f|^2. \tag{4.5}$$

This together with (4.3) and (4.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t^\sigma f(\gamma_t)=(\partial_t T_t^\sigma f)(\gamma_t)+\frac{\mathrm{d}}{\mathrm{d}t}T_s^\sigma f(\gamma_t)\Big|_{s=t}$$

$$\le \frac{\sigma}{2}\hat{L}T_t^\sigma f(\gamma_t) - \frac{1}{2}|\nabla T_t^\sigma f(\gamma_t)|^2 + |\nabla T_t^\sigma f(\gamma_t)|\rho(x,y)$$
$$\le \frac{1}{2}\big[\rho(x,y)^2 + \sigma\|\hat{L}f\|_\infty + c_2\sigma^\theta\|\nabla f\|_\infty^2\big], \quad t\in[0,1],\ \|\nabla f\|_\infty^2 \le k_1\sigma.$$

Integrating over $t\in[0,1]$ and noting that $T_0^\sigma f = f$, we derive the first inequality in (4.1).

On the other hand, by (4.5), $T_0^\sigma f = f$ and $\mu(\hat{L}T_t^\sigma f) = 0$, we obtain

$$\begin{aligned}\mu(f - T_1^\sigma f) &= -\int_M \mathrm{d}\mu \int_0^1 (\partial_t T_t^\sigma f)\mathrm{d}t \\ &= \int_0^1 \mathrm{d}t \int_M \Big\{\frac{1}{2}|\nabla T_t^\sigma f|^2 - \frac{\sigma}{2}\hat{L}T_t^\sigma f\Big\}\mathrm{d}\mu = \frac{1}{2}\int_0^1 \mu(|\nabla T_t^\sigma f|^2)\mathrm{d}t.\end{aligned} \tag{4.6}$$

Moreover, by (4.5) and the integration by parts formula, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\mu(|\nabla T_s^\sigma f|^2) = -\frac{\mathrm{d}}{\mathrm{d}s}\int_M (T_s^\sigma f)\hat{L}T_s^\sigma f\mathrm{d}\mu \\ &= -\int_M (\hat{L}T_s^\sigma f)\partial_s T_s^\sigma f\mathrm{d}\mu - \int_M (T_s^\sigma f)\hat{L}(\partial_s T_s^\sigma f)\mathrm{d}\mu \\ &= -2\int_M (\hat{L}T_s^\sigma f)\partial_s T_s^\sigma f\,\mathrm{d}\mu = -2\int_M (\hat{L}T_s^\sigma f)\Big(\frac{\sigma}{2}\hat{L}T_s^\sigma f - \frac{1}{2}|\nabla T_s^\sigma f|^2\Big)\mathrm{d}\mu \\ &\le \frac{1}{4\sigma}\|\nabla T_s^\sigma f\|_\infty^4, \quad s\in(0,1], t\in[0,1].\end{aligned}$$

This together with (4.2) implies

$$\mu(|\nabla T_t^\sigma f|^2) - \mu(|\nabla f|^2) \le \frac{c_1^2}{4\sigma}\|\nabla f\|_\infty^4, \quad t\in[0,1], \sigma\in(0,1], \|\nabla f\|_\infty^2 \le k_1\sigma \tag{4.7}$$

Substituting this into (4.6), we derive the second estimate in (4.1).

□

Proof of Theorem 2.4. Similarly to the proof of Theorem 2.1 using Proposition 3.7 and the approximation argument with Proposition 3.8, the assertions Theorem 2.4(1) and (2) follow from

$$\lim_{t\to\infty} \mathbb{E}^\mu\big[\big|\{th(0)\mathbb{W}_2(\mu_t^B,\mu)^2 - \Xi^B(t)\}^-\big|^q\big] = 0, \quad q\in[1,q_\alpha). \tag{4.8}$$

Moreover, according to the proof of Theorem 2.2(1), Theorem 2.4(3) is implied by Theorem 2.4(1) and (2). So, it remains to verify (4.8). The main idea for the proof of (4.8) goes back to [1, 38], but we have to make suitable modifications for the present situation. Let $\hat{f}_{t,r} = (-\hat{L})^{-1}(1 - f_{t,r}^B)$.

Firstly, by (3.18), we find a constant $c_1 > 0$ such that

$$\|\hat{f}_{t,r}\|_\infty \le \int_0^\infty \|\hat{P}_s(f_{t,r}^B - 1)\|_\infty \mathrm{d}s \le c_1\|f_{t,r}^B - 1\|_\infty \int_0^\infty \mathrm{e}^{-\lambda_1 s}\mathrm{d}s = \frac{c_1}{\lambda_1}\|f_{t,r}^B - 1\|_\infty. \tag{4.9}$$

By (3.10) and (3.9), we find a constant $c_2 > 0$ such that

$$\|\nabla \hat{P}_s g\|_\infty \le c_2(s\wedge 1)^{-\frac{1}{2}}\mathrm{e}^{-\lambda_1 s}\|g\|_\infty, \quad s>0, g\in \mathscr{B}_b(M).$$

Noting that $\hat{f}_{t,r} := (-\hat{L})^{-1}(f_{t,r}^B - 1) = \int_0^\infty \hat{P}_s(f_{t,r}^B-1)\mathrm{d}s$, we obtain

$$\|\nabla \hat{f}_{t,r}\|_\infty \le \int_0^\infty \|\nabla \hat{P}_s(f_{t,r}^B-1)\|_\infty \mathrm{d}s \le c_2\|f_{t,r}^B-1\|_\infty \int_0^\infty (1\wedge s)^{-\frac{1}{2}}\mathrm{e}^{-\lambda_1 s}\mathrm{d}s.$$

Combining this with (4.9) and $|\hat{L}\hat{f}_{t,r}| = |f_{t,r}^B - 1|$, we find a constant $c>0$ such that

$$\|\hat{L}\hat{f}_{t,r}\|_\infty + \|\hat{f}_{t,r}\|_\infty + \|\nabla \hat{f}_{t,r}\|_\infty \le c\|f_{t,r}^B-1\|_\infty, \quad t,r>0. \tag{4.10}$$

Next, let

$$\begin{aligned} &C_1(\hat{f}_{t,r},\sigma) := \sigma\|\hat{L}\hat{f}_{t,r}\|_\infty^2 + k_2\sigma^\theta\|\nabla\hat{f}_{t,r}\|_\infty^2,\\ &C_2(\hat{f}_{t,r},\sigma) := k_2\sigma^{-1}\|\nabla\hat{f}_{t,r}\|_\infty^4,\\ &B_{t,r}(\sigma) := \{\|f_{t,r}^B-1\|_\infty^2 \le k_1c^{-1}\sigma^{1+\theta}\}, \quad \sigma\in(0,1], t,r>0. \end{aligned} \tag{4.11}$$

By (4.1), the integration by parts formula, and the Kantorovich dual formula, we obtain

$$\begin{aligned} &C_1(\hat{f}_{t,r},\sigma) + \frac{1}{2}\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 \ge \mu(T_1^\sigma\hat{f}_{t,r}) - \mu_{t,r}^B(\hat{f}_{t,r})\\ &= \mu\big((f_{t,r}^B-1)(-\hat{L})^{-1}(f_{t,r}^B-1)\big) - \mu(\hat{f}_{t,r}) + \mu(T_1^\sigma\hat{f}_{t,r})\\ &\ge \mu(|\nabla\hat{f}_{t,r}|^2) - \frac{1}{2}\mu(|\nabla\hat{f}_{t,r}|^2) - C_2(\hat{f}_{t,r},\sigma) = \frac{1}{2t}\Xi_r^B(t) - C_2(\hat{f}_{t,r},\sigma), \quad \sigma\in(0,1], t,r>0. \end{aligned}$$

This together with (4.10) and (4.11) yields

$$1_{B_{t,r}(\sigma)}\big\{t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 - \Xi_r^B(t)\big\} \ge -c_3t\sigma^{1+2\theta}, \quad \sigma\in(0,1], t,r>0 \tag{4.12}$$

for some constant $c_3>0$. On the other hand, by (2.5) we find a constant $c_4>0$ such that

$$\|f_{t,r}^B-1\|_\infty^2 = \|\hat{P}_{\frac{r}{2}}(f_{t,\frac{r}{2}}^B-1)\|_\infty^2 \le c_4r^{-\frac{d}{2}}\|f_{t,\frac{r}{2}}^B-1\|_2^2, \quad r\in(0,1],$$

so that by (3.58) with 1 replacing $\sigma\in(0,1)$, there exists a constant $c_5>0$ such that

$$\mathbb{E}^\mu[\|f_{t,r}^B-1\|_\infty^2] \le c_4r^{-\frac{d}{2}}t^{-1}\sum_{i=1}^\infty \mathrm{e}^{-\lambda_i r}\mathbb{E}^\mu[\psi_i^B(t)^2] \le c_5r^{-\frac{d}{2}-1}t^{-1}, \quad t\ge 1, r\in(0,1].$$

Hence, we find a constant $c_6>0$ such that

$$\mathbb{P}^\mu(B_{t,r}(\sigma)^c) = \mathbb{P}^\mu(\|f_{t,r}^B-1\|_\infty^2 > k_1c^{-1}\sigma^{1+\theta}) \le c_6r^{-\frac{d}{2}-1}t^{-1}\sigma^{-(1+\theta)}, \quad t\ge1, \sigma,r\in(0,1]. \tag{4.13}$$

Taking $\sigma=\sigma_t := t^{-\frac{1}{1+3\theta/2}}$ in (4.12) and (4.13), we arrive at

$$\limsup_{t\to\infty}\mathbb{P}^\mu\big(\{t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2 - \Xi_r^B(t)\}^- \ge \varepsilon\big) \le \limsup_{t\to\infty}\mathbb{P}^\mu(B_{t,r}(\sigma_t)^c) = 0, \quad \varepsilon>0, r\in(0,1].$$

On the other hand, by (2.10), (3.6), (3.28) and $\sum_{i=1}^\infty \lambda_i^{-1-\alpha}<\infty$ due to (2.6) and $d'<2(1+\alpha)$, we obtain

$$\begin{aligned}&\lim_{r\to0}\sup_{t\ge1}\mathbb{E}^\mu\big[|\Xi_r^B(t)-\Xi^B(t)|\big]\le\lim_{r\to0}\sum_{i=1}^\infty\frac{1-\mathrm{e}^{-2\lambda_ir}}{\lambda_i}\sup_{t\ge1}\mathbb{E}^\mu[\psi_i^B(t)^2]\\&\le c(1)\lim_{r\to0}\sum_{i=1}^\infty\big(1-\mathrm{e}^{-2\lambda_ir}\big)\lambda_i^{-1-\alpha}=0.\end{aligned}$$

Moreover, (2.21) and (3.66) imply

$$\lim_{r\to0}\sup_{t\ge1}\mathbb{E}^\mu[\{th(0)\mathbb{W}_2(\mu_t^B,\mu)^2-t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2\}^-]=0.$$

Therefore, for any $\varepsilon>0$,

$$\begin{aligned}&\limsup_{t\to\infty}\mathbb{P}^\mu\big(\{th(0)\mathbb{W}_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^-\ge3\varepsilon\big)\\&\le\lim_{r\to0}\limsup_{t\to\infty}\mathbb{P}^\mu\big(\{t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2-\Xi_r^B(t)\}^-\ge\varepsilon\big)\\&+\lim_{r\to0}\sup_{t\ge1}\Big[\mathbb{P}^\mu\big(|\Xi^B(t)-\Xi_r^B(t)|\ge\varepsilon\big)+\mathbb{P}^\mu\big(\{th(0)\mathbb{W}_2(\mu_t^B,\mu)^2-t\mathbb{W}_2(\mu_{t,r}^B,\mu)^2\}^-\ge\varepsilon\big)\Big]=0.\end{aligned}$$

Combining this with (3.66) and applying the dominated convergence theorem, we prove (4.8). □

To prove Theorem 2.5, we need the following lemma.

Lemma 4.2. *Assume* (2.5).

(1) *Let* $\mathbf{V}=\mathbf{V}_B$ *for* $B(\lambda)=\lambda$. *We have*

$$\mathbf{V}(\phi_i)=\lambda_i^{-1}-\lambda_i^{-2}\mathbf{V}(Z\phi_i),\quad i\ge1. \tag{4.14}$$

(2) *If* (2.7) *holds and* $B\in\mathbf{B}^\alpha\cap\mathbf{B}_{\alpha'}$ *for some* $\alpha,\alpha'\in[0,1]$, *then there exist constants* $c_1,c_2>0$ *such that*

$$\mathbf{V}_B(\phi_i)\ge c_1\lambda_i^{-\alpha'}-c_2\lambda_i^{-1-(\frac12\wedge\alpha)}\big[1+1_{\{\alpha=\frac12\}}\log(1+\lambda_i)\big],\quad i\ge1. \tag{4.15}$$

Proof. (1) By $\hat L\phi_i=-\lambda_i\phi_i$ and using the Kolmogorov equation, we obtain

$$P_s\phi_i=-\frac{1}{\lambda_i}P_s\hat L\phi_i=-\frac{1}{\lambda_i}\frac{\mathrm{d}}{\mathrm{d}s}P_s\phi_i+\frac{1}{\lambda_i}P_s(Z\phi_i). \tag{4.16}$$

This together with (2.2) and $\mu(\phi_i^2)=1$ implies

$$\begin{aligned}&\int_0^t\mu(\phi_iP_s\phi_i)\mathrm{d}s=-\frac{1}{\lambda_i}\int_0^t\Big(\frac{\mathrm{d}}{\mathrm{d}s}\mu(\phi_iP_s\phi_i)-\mu\big(\phi_i(ZP_s\phi_i)\big)\Big\}\mathrm{d}s\\&=\frac{1}{\lambda_i}\Big(1-\mu(\phi_iP_t\phi_i)\Big)-\frac{1}{\lambda_i}\int_0^t\mu\big(\{Z\phi_i\}P_s\phi_i\big)\mathrm{d}s\\&=\frac{1}{\lambda_i}\Big(1-\mu(\phi_iP_t\phi_i)\Big)+\frac{1}{\lambda_i^2}\int_0^t\Big\{\frac{\mathrm{d}}{\mathrm{d}s}\mu\big(\{Z\phi_i\}P_s\phi_i\big)-\mu\big(\{Z\phi_i\}P_s\{Z\phi_i\}\big)\Big\}\mathrm{d}s\\&=\frac{1}{\lambda_i}\Big(1-\mu(\phi_iP_t\phi_i)\Big)+\frac{1}{\lambda_i^2}\mu(\{Z\phi_i\}P_t\phi_i)-\frac{1}{\lambda_i^2}\int_0^t\mu\big(\{Z\phi_i\}P_s\{Z\phi_i\}\big)\mathrm{d}s.\end{aligned} \tag{4.17}$$

By (1.2) and $\|P_t-\mu\|_2\le \mathrm{e}^{-\lambda_1 t}$, we may let $t\to\infty$ to derive

$$\mathbf{V}(\phi_i)=\frac{1}{\lambda_i}-\frac{1}{\lambda_i^2}\mathbf{V}(Z\phi_i),\quad i\ge 1.$$

(2) Let (3.10) hold. By (2.9), (2.4) and (3.34) we obtain

$$\mathbf{V}_B(\phi_i):=\int_0^\infty \mu(\phi_i P_t^B\phi_i)\mathrm{d}t=\frac{1}{B(\lambda_i)}+\int_0^\infty\bigg\{\mathbb{E}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}\mu\big(\phi_i P_s(Z\phi_i)\big)\mathrm{d}s\bigg\}\mathrm{d}t. \tag{4.18}$$

By (2.9) and (2.3) for $(P_t^*,-Z)$ replacing (P_t,Z), we derive

$$P_s^*\phi_i=\hat{P}_s\phi_i-\int_0^s P_r^*\{Z\hat{P}_{s-r}\phi_i\}\mathrm{d}r=\mathrm{e}^{-\lambda_i s}\phi_i-\int_0^s\mathrm{e}^{-\lambda_i(s-r)}P_r^*(Z\phi_i)\mathrm{d}r.$$

This together with (2.2) implies

$$\mu(\phi_i P_s(Z\phi_i))=\mu((P_s^*\phi_i)Z\phi_i)=-\int_0^s\mathrm{e}^{-\lambda_i(s-r)}\mu((Z\phi_i)P_r(Z\phi_i))\mathrm{d}r.$$

Combining this with (3.11) and noting that (2.9) yields

$$\|Z\phi_i\|_2^2\le\|Z\|_\infty\|\nabla\phi_i\|_2^2=\|Z\|_\infty^2\lambda_i,$$

we derive

$$\mu(\phi_i P_s(Z\phi_i))\le c(1)\lambda_i^{\frac{1}{2}}\int_0^s\mathrm{e}^{-\lambda_i(s-r)}r^{-\frac{1}{2}}\mathrm{e}^{-\lambda r}\mathrm{d}r,\quad s\ge 0, i\in\mathbb{N}. \tag{4.19}$$

By (3.36) for s replacing S_t^B, we find a constant $a_1>0$ such that

$$\int_0^s\mathrm{e}^{-\lambda_i(s-r)-\lambda r}r^{-\frac{1}{2}}\mathrm{d}r\le a_1\lambda_i^{-1}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s/2},\quad s>0.$$

Combining this with (4.19), (3.36) and (3.37), we find constants $a_2,a_3,a_4>0$ such that

$$\begin{aligned}I_i(t)&:=\mathbb{E}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}\mu(\phi_i P_s(Z\phi_i))\mathrm{d}s\le\frac{a_1}{\sqrt{\lambda_i}}\mathbb{E}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}s^{-\frac{1}{2}}\mathrm{e}^{-\lambda s/2}\mathrm{d}s\\&\le a_2\lambda_i^{-\frac{3}{2}}\mathbb{E}\Big[(S_t^B)^{-\frac{1}{2}}\mathrm{e}^{-\lambda S_t^B/4}\Big]\le a_3\lambda_i^{-\frac{3}{2}}\Big(\frac{1}{2}\wedge t\Big)^{-\frac{1}{2\alpha}}\mathrm{e}^{-a_4 t},\quad t>0, i\ge 1.\end{aligned} \tag{4.20}$$

On the other hand, noting that

$$\big|\mu\big((Z\phi_i)P_r(Z\phi_i)\big)\big|\le\|Z\|_\infty^2\|\nabla\phi_i\|_2^2\mathrm{e}^{-\lambda_1 r}=\lambda_i\|Z\|_\infty^2\mathrm{e}^{-\lambda_1 r},$$

by (4.19) and (2.4), we find constants $a_5,a_6>0$ such that

$$I_i(t)\le\lambda_i\|Z\|_\infty^2\mathbb{E}\int_0^{S_t^B}\mathrm{e}^{-\lambda_i(S_t^B-s)}\mathrm{d}s\int_0^s\mathrm{e}^{-\lambda_i(s-r)-\lambda_1 r}\mathrm{d}r$$

$$\leq a_5\mathbb{E}\int_0^{S_t^B} \mathrm{e}^{-\lambda_i(S_t^B-s)-\lambda_1 s/2}\mathrm{d}s \leq a_5\lambda_i^{-1}\mathrm{e}^{-a_6 t}.$$

Combining this with (4.20) and (3.31), we find a constant $a_7 > 0$ such that

$$\int_0^\infty I_i(t)\mathrm{d}t \leq (a_3\vee a_5)\lambda_i^{-\frac{3}{2}}\int_0^\infty h_{i,\alpha}(t)\mathrm{e}^{-(a_4\wedge a_6)t}\mathrm{d}t$$
$$\leq a_7\lambda_i^{-1-(\frac{1}{2}\wedge\alpha)}\big[1+1_{\{\alpha=\frac{1}{2}\}}\log(1+\lambda_i)\big].$$

This together with (4.18) and $B\in\mathbf{B}_{\alpha'}$ implies (4.15). □

Proof of Theorem 2.5. Since $\alpha'\in[0,1]$ and $\alpha\in[0,\alpha']\cap(\alpha'-1,\alpha']$ imply $2+(\frac{1}{2}\wedge\alpha)>1+\alpha'$, by (4.15) we find constants $a_0>1$ and $a_1,a_2\geq 0$ such that

$$\frac{\mathbf{V}_B(\phi_i)}{\lambda_i}\geq a_1\lambda_i^{-1-\alpha'}-a_2\lambda_i^{-a_0(1+\alpha')},\quad i\in\mathbb{N}.$$

Combining this with (2.22) and $d'=2(1+\alpha)$, we find constants $a_1',a_2',a_3,a_4>0$ such that

$$\begin{aligned}\eta_{Z,r}^B&=\sum_{i=1}^\infty \mathrm{e}^{-2\lambda_i r}\frac{\mathbf{V}_B(\phi_i)}{\lambda_i}\geq\sum_{i=1}^\infty \mathrm{e}^{-2\lambda_i r}\Big(a_1\lambda_i^{-1-\alpha'}-a_2\lambda_i^{-a_0(1+\alpha')}\Big)\\&\geq\sum_{i=1}^\infty \mathrm{e}^{-2rc_2 i^{\frac{1}{1+\alpha'}}}\Big(a_1' i^{-1}-a_2' i^{-a_0}\Big)\geq a_3\log(1+r^{-1})-a_4,\quad r\in(0,1].\end{aligned}\tag{4.21}$$

Next, by (4.12), we obtain

$$\begin{aligned}\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2]&\geq\mathbb{E}^\mu[1_{B_\sigma}\mathbb{W}_2(\mu_{t,r}^B,\mu)^2]\geq t^{-1}\mathbb{E}^\mu[1_{B_{t,r}(\sigma)}\Xi_r^B(t)]-c_3\sigma^{1+2\theta}\\&\geq t^{-1}\mathbb{E}^\mu[\Xi_r^B(t)]-t^{-1}\mathbb{E}^\mu[1_{B_{t,r}(\sigma)^c}\Xi_r^B(t)]-c_3\sigma^{1+2\theta},\quad t\geq 1, r,\sigma\in(0,1].\end{aligned}\tag{4.22}$$

By (3.53), (2.22) and $d'=2(1+\alpha')$, we find constants $k_1,k_2>0$ such that

$$\mathbb{E}^\mu[\Xi_r^B(t)]=\sum_{i=1}^\infty\frac{\mathrm{e}^{-2\lambda_i r}}{\lambda_i}\mathbb{E}^\mu[\psi_i^B(t)^2]\geq\eta_{Z,r}^B-k_1t^{-1}\sum_{i=1}^\infty\mathrm{e}^{-2r\lambda_i}\lambda_i^{-(1+\alpha)}$$
$$\geq\eta_{Z,r}^B-k_1t^{-1}\sum_{i=1}^\infty\mathrm{e}^{-2rc_2 i^{\frac{1}{1+\alpha'}}}c_2^{-1-\alpha}i^{-\frac{1+\alpha}{1+\alpha'}}\geq\eta_{Z,r}^B-k_2t^{-1}r^{-k_2},\quad t\geq 0, r\in(0,1].$$

Combining this with (4.21) and (4.22), we derive

$$\begin{aligned}\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B,\mu)^2]&\geq a_3t^{-1}\log(1+r^{-1})-a_4t^{-1}-k_2t^{-2}r^{-k_2}\\&\quad-t^{-1}\mathbb{E}^\mu[1_{B_{t,r}(\sigma)^c}\Xi_r^B(t)]-c_3\sigma^{1+2\theta},\quad t\geq 1, r,\sigma\in(0,1].\end{aligned}\tag{4.23}$$

On the other hand, by (3.47) and (4.13), we find constants $k_3,k_4>0$ such that

$$\mathbb{E}^\mu[\Xi_r^B(t)^2]=\mathbb{E}^\mu[\|(-\hat{L})^{-\frac{1}{2}}(f_{t,r}^B-1)\|_2^4]\leq k_3r^{-k_4},$$

$$\mathbb{P}^\mu(B_{t,r}(\sigma)^c) \le k_3 r^{-k_4} t^{-1} \sigma^{-(1+\theta)} \quad t \ge 1, r \in (0,1], \sigma \in (0,1],$$

where $\theta > 0$ is a constant. Thus,

$$\mathbb{E}^\mu[1_{B_{t,r}(\sigma)^c} \Xi_r^B(t)] \le \sqrt{\mathbb{P}^\mu(B_{t,r}(\sigma)^c)\mathbb{E}^\mu[\Xi_r^B(t)^2]} \le k_3 r^{-k_4} t^{-\frac{1}{2}} \sigma^{-\frac{1+\theta}{2}}, \quad t \ge 1, r, \sigma \in (0,1].$$

Combining this with (4.23), we arrive at

$$\begin{aligned}\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B, \mu)^2] \ge a_3 t^{-1} \log(1 + r^{-1}) - a_4 t^{-1} - k_2 t^{-2} r^{-k_2} \\ - k_3 r^{-k_4} t^{-\frac{3}{2}} \sigma^{-\frac{1+\theta}{2}} - c_3 \sigma^{1+2\theta}, \quad t \ge 1, r, \sigma \in (0,1].\end{aligned}$$

Taking $\sigma = t^{-\frac{1}{1+2\theta}}$ and $r = t^{-\frac{1}{k_2} \wedge \frac{\theta}{2k_4(1+2\theta)}}$, obtain

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_{t,r}^B, \mu)^2] \ge \frac{a_3}{1+2\theta} t^{-1} \log t - (k_2 + k_3 + c_3) t^{-1}, \quad t \ge 1.$$

Therefore, (2.23) holds for some constants $c, t_0 > 0$.

□

Finally, to prove Theorem 2.6, we present one more lemma.

Lemma 4.3. *Let (E, ρ) be a Polish space. Let X_t be a continuous time Markov process on E such that the associated semigroup P_t satisfies*

$$\|P_t - \mu\|_2 \le c_1 \mathrm{e}^{-\lambda_1 t}, \quad t \ge 0 \tag{4.24}$$

for some constants $c_1, \lambda_1 > 0$ and a probability measure μ on E. If there exists $\phi \in C_{b,L}(E)$ such that $\mu(\phi) = 0$ and

$$\mathbf{V}(\phi) := \int_0^\infty \mu(\phi P_t \phi) \mathrm{d}t > 0,$$

then there exist constants $c, t_0 > 0$ such that $\mu_t := \frac{1}{t} \int_0^t \delta_{X_s} \mathrm{d}s$ satisfies

$$\mathbb{E}^\mu[\mathbb{W}_1(\mu_t, \mu)] \ge c t^{-\frac{1}{2}}, \quad t \ge t_0. \tag{4.25}$$

If $M \subset E$ such that $\mu(M) = 1$ and $\|P_t\|_{1\to 2} < \infty$ for $t > 0$, then (4.25) holds for $\inf_{\nu \in \mathscr{P}} \mathbb{E}^\nu$ replacing $\mathbb{E}^\mu$, where $\mathscr{P}$ is the set of all probability measures on M.

Proof. By [39, Theorem 2.1(c)], we have

$$\lim_{t\to\infty} \sqrt{t} \mathbb{E}^\mu[|\mu_t(\phi)|] = \big(2\pi \mathbf{V}(\phi)\big)^{-\frac{1}{2}} \int_{-\infty}^\infty |r| \mathrm{e}^{-\frac{r^2}{2\mathbf{V}(\phi)}} \mathrm{d}r. \tag{4.26}$$

So, by the Kantorovich dual formula, there exist constants $c, t_0 > 0$ such that

$$\sqrt{t} \mathbb{E}^\mu[\mathbb{W}_1(\mu_t, \mu)] \ge \sqrt{t} \mathbb{E}^\mu[|\mu_t(\phi) - \mu(\phi)|] \ge c, \quad t \ge t_0.$$

Next, let $M \subset E$ such that $\mu(M) = 1$ and $\|P_t\|_{1\to 2} < \infty$ for $t > 0$. Then

$$\{\nu P_1 : \nu \in \mathscr{P}\} \subset \big\{\nu \in \mathscr{P} : \nu = h\mu, \|h\|_2 \le \|P_1\|_{1\to 2}\big\},$$

so that [39, Theorem 2.1(c)], $\hat{\mu}_t := \frac{1}{t}\int_1^{t+1}\delta_{X_s}\mathrm{d}s$ satisfies

$$\lim_{t\to\infty}\inf_{\nu\in\mathscr{P}}\sqrt{t}\mathbb{E}^\nu[|\hat{\mu}_t(\phi)|] > 0.$$

Noting that

$$|\hat{\mu}_t(\phi) - \mu_t(\phi)| \le \|\phi\|_\infty t^{-1}, \quad t > 1,$$

we obtain

$$\lim_{t\to\infty}\inf_{\nu\in\mathscr{P}}\sqrt{t}\mathbb{E}^\nu[|\mu_t(\phi)|] > 0.$$

By Kantorovich's dual formula and $\mu(\phi) = 0$, this implies (4.25) for $\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu$ replacing $\mathbb{E}^\mu$. □

Proof of Theorem 2.6. (1) By (2.5) and (2.7), for any $i \ge 1$ we have $\|\phi_i\|_\infty < \infty$ and there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\|\nabla\phi_i\|_\infty &= \mathrm{e}^{\lambda_i}\|\nabla\hat{P}_1\phi_i\|_\infty \le c_1\mathrm{e}^{\lambda_i}\big\|(P_1|\nabla\phi_i|^2)^{\frac{1}{2}}\big\|_\infty \\ &\le c_2\mathrm{e}^{\lambda_i}\|\nabla\phi_i\|_2 = c_2\mathrm{e}^{\lambda_i}\sqrt{\lambda_i} < \infty.\end{aligned}$$

So, $\phi_i \in C_{b,L}(M)$, and hence is uniquely extended to $\phi_i \in C_{b,L}(E)$ for $E := \bar{M}$. On the other hand, by (4.15) we have $\mathbf{V}_B(\phi_i) > 0$ for large enough i. Moreover, (3.9) implies $\|P_t^B\|_{1\to 2} < \infty$ for $t > 0$. So, the first assertion follows from Lemma 4.3 with $E = \bar{M}$.

(2) Let $B \in \mathbf{B}_\alpha$ for some $\alpha \in [0,1]$ with $d'' > 2(1+\alpha)$. By (2.8) for $p = 1$ we find a constant $c_1 > 0$ such that

$$\mathbb{E}^\mu[\rho(\hat{X}_t, \hat{X}_0)] \le c_1 t^{\frac{1}{2}}, \quad t \ge 0.$$

Combining this with (2.3) and (2.7), we find a constant $c_2 > 0$ such that

$$\begin{aligned}\mathbb{E}^\mu[\rho(X_t, X_0)] &= \int_M P_t\rho(x,\cdot)(x)\mu(\mathrm{d}x) \\ &= \mathbb{E}^\mu[\rho(\hat{X}_t, \hat{X}_0)] + \int_M \mu(\mathrm{d}x)\int_0^t P_s\{Z\hat{P}_{t-s}\rho(x,\cdot)\}(x)\mathrm{d}s \\ &\le c_1 t^{\frac{1}{2}} + c_2\int_0^t (t-s)^{-\frac{1}{2}}\mathrm{d}s \le c_3 t^{\frac{1}{2}}, \quad t \ge 0.\end{aligned}$$

According to the proof of [37, Theorem 1.1(2)], this and (2.25) imply

$$\mathbb{E}^\mu[\mathbb{W}_1(\mu_t^B, \mu)] \ge c_4 t^{-\frac{1}{d''-2\alpha}}, \quad t \ge t_1 \tag{4.27}$$

for some constants $c_4, t_1 > 0$.

Finally, by (3.9), we find a constant $t_2 > 0$ such that $\|P_{t_2} - \mu\|_{1\to\infty} \le \frac{1}{2}$, so that

$$\nu_{t_2} := \nu P_{t_2}^B \ge \frac{1}{2}\mu, \quad \nu \in \mathscr{P}.$$

Let μ_t^{B,t_2} be in (3.69) for $\varepsilon = t_2$. Then the Markov property and (4.27) yield

$$\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^{B,t_2}, \mu)] = \inf_{\nu\in\mathscr{P}}\mathbb{E}^{\nu_{t_2}}[\mathbb{W}_1(\mu_t^B, \mu)] \ge \frac{1}{2}c_4 t^{-\frac{1}{d''-2\alpha}}, \quad t \ge t_1.$$

Combining this with (3.76), the triangle inequality and $d''-2\alpha>2$, we find constants $c_5,c,t_0>t_1$ such that

$$\inf_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\ge\frac{1}{2}c_4t^{-\frac{1}{d''-2\alpha}}-c_5t^{-1}\ge ct^{-\frac{1}{d''-2\alpha}},\quad t\ge t_0.$$

□

5 Some concrete models

In this part, we apply our general results to some typical models including: 1) the (reflecting) subordinated diffusion process on a compact manifold; 2) the subordinated conditional diffusion process on a bounded open domain; 3) the subordinated Wright-Fisher diffusion process; 4) the subordinated subelliptic diffusion process on $\mathbf{SU}(2)$. It is also possible to consider more general hypoelliptic diffusion processes studied in [6, 7] under the generalized curvature-dimension conditions. For simplicity, throughout this section, we take

$$B\in\mathbf{B}^\alpha\cap\mathbf{B}^\alpha \text{ for some } \alpha\in(0,1].$$

5.1 Subordinated (Reflecting) diffusion process

In this part, we consider the model stated in Introduction, for which all conditions in Theorems 2.1-2.6 are satisfied for $d=d'=d''=n$.

Indeed, (2.5) and (2.22) are well known (see [9, 11]), (2.7) follows from [31, Lemma 2.1], and (2.21) with $h(r)=\kappa e^{Kr}$ is implied by [38, (3.36), (3.37)], where $\kappa\ge1$ and $K\ge0$ are constants with $\kappa=1$ when ∂M is empty or convex. Moreover, the following lemma confirms other conditions.

Lemma 5.1. *(2.20), (A_2) and (2.16) hold.*

Proof. (1) Let l_t be the local time of $\hat X_t$ on ∂M if ∂M exists, and let $l_t=0$ otherwise. By [31, (2.1)] and the proof of [31, Lemma 2.1], there exist constants $c_1,K,\delta>0$ such that

$$|\nabla\hat P_tf(x)|\le\mathbb{E}^x[|\nabla f(\hat X_t)|e^{Kt+\delta l_t}],\quad t\ge0,x\in M,f\in C_b^1(M),\tag{5.1}$$

$$\sup_{x\in M}\mathbb{E}^x[l_t^2]\le c_1t,\quad\sup_{x\in M}\mathbb{E}^x[e^{\lambda l_t}]<\infty,\quad\lambda,t\ge0.\tag{5.2}$$

By the Schwarz inequality, (5.1) implies

$$\begin{aligned}&|\nabla\hat P_te^f(x)|\le\hat P_t|\nabla e^f|(x)+\mathbb{E}^x\big[|\nabla e^f(\hat X_t)|(e^{Kt+\delta l_t}-1)\big]\\&\le\big\{\hat P_te^f(x)\big\}^{\frac12}\big\{\hat P_t(|\nabla f|^2e^f)(x)\big\}^{\frac12}+\|\nabla f\|_\infty(\hat P_te^{2f})^{\frac12}\big(\mathbb{E}^x[e^{2Kt+2\delta l_t}-1]\big)^{\frac12}.\end{aligned}$$

On the other hand, by (5.2) we find a constant $c_2>0$ such that

$$\begin{aligned}&\mathbb{E}^x[e^{2Kt+2\delta l_t}-1]\le\mathbb{E}^x[(2K+2\delta l_t)e^{2Kt+2\delta l_t}]\\&\le\big(\mathbb{E}^x[(2K+2\delta l_t)^2]\big)^{\frac12}\big(\mathbb{E}^x[e^{4Kt+4\delta l_t}]\big)^{\frac12}\le c_2t^{\frac12},\quad t\in[0,1].\end{aligned}$$

Therefore, (2.20) holds for $\theta = \frac{1}{2}$ and $m = 1$.

(2) When ∂M is empty or convex, we have

$$\langle \nabla \rho(\hat{X}_0, \cdot), \mathbf{N} \rangle (\hat{X}_t) \mathrm{d}l_t \le 0,$$

where $\mathbf{N}$ is the inward unit normal vector on ∂M. On the other hand, by the Laplacian comparison theorem, there exists a constant $c_0 > 0$ such that

$$\hat{L}\rho(\hat{X}_0, \cdot)^2 \le c_0.$$

So, by Itô's formula, we find a constant $c_1 > 0$ such that

$$\mathrm{d}\rho(\hat{X}_0, \hat{X}_t)^2 \le c_1 \mathrm{d}t + 2\sqrt{2}\rho(\hat{X}_0, \hat{X}_t)\mathrm{d}B_t, \quad t \ge 0,$$

where B_t is the one-dimensional Brownian motion. Thus, for any $p \ge 1$, there exists a constant $c(p) > 0$ such that

$$\mathrm{d}\rho(\hat{X}_0, \hat{X}_t)^{2p} \le c(p)\rho(\hat{X}_0, \hat{X}_t)^{2(p-1)}\mathrm{d}t + dM_t$$

holds for some martingale M_t, so that

$$\mathbb{E}[\rho(\hat{X}_0, \hat{X}_t)^{2p}] \le c(p) \int_0^t \mathbb{E}[\rho(\hat{X}_0, \hat{X}_s)^{2(p-1)}]\mathrm{d}s, \quad t \ge 0.$$

Consequently, for $p = 1$ we get

$$\mathbb{E}^x[\rho(\hat{X}_0, \hat{X}_t)^2] \le c(1)t,$$

and by inducting in $p \in \mathbb{N}$, we derive (2.8) for $p \in 2\mathbb{N}$. Therefore, (2.8) holds for all $p \ge 1$ due to Jensen's inequality.

When ∂M is non-convex, as explained in the proof of [32, Proposition 3.2.7], there exists a function $1 \le \phi \in C_b^\infty(M)$ such that ∂M is convex under the metric

$$\langle \cdot, \cdot \rangle' := \phi^{-1} \langle \cdot, \cdot \rangle,$$

and $\hat{L} = \phi^{-2}\Delta' + Z'$, where Δ' is the Laplacian induced by the new metric and Z' is a C_b^1 vector field. Let ρ' be the Riemannian distance induced by the new metric, we have $\rho \le \|\phi\|_\infty \rho'$, so that the above argument for convex ∂M leads to

$$\mathbb{E}[\rho(\hat{X}_0, \hat{X}_t)^{2p}] \le \|\phi\|_\infty^{2p} \mathbb{E}[\rho'(\hat{X}_0, \hat{X}_t)^{2p}] \le k(p)t^p, \quad t \in [0, 1].$$

(3) To verify (2.16), we follow the line of [3]. As explained in the end of page 12 in [3], see also the proof of [3, Theorem 1.5], under (3.10), it remains to verify the volume doubling condition and scaled Poincaré inequalities on balls. More precisely, we only need to find a distance $\tilde{\rho}$ and constants $c_1, c_2, c_3 > 0$ such that

$$c_1 \tilde{\rho} \le \rho \le c_2 \tilde{\rho},$$

and the balls $\tilde{B}(x, r) := \{y \in M : \tilde{\rho}(x, y) \le r\}$ for all $x \in M$ and $r > 0$ satisfy

$$\mu(\tilde{B}(x, 2r)) \le c_3 \mu(\tilde{B}(x, r)), \tag{5.3}$$

$$\mu(1_{\tilde B(x,r)}f^2)\le c_3r^2\mu(1_{\tilde B(x,r)}|\nabla f|^2),\quad f\in C_b^1(M),\mu(1_{\tilde B(x,r)}f)=0. \tag{5.4}$$

Since for the present model we have $\tilde D:=\|\tilde\rho\|_\infty<\infty$ and $kr^d\le\mu(\tilde B(x,r))\le Kr^d$ for some constants $K>k>0$ and all $r\in[0,\tilde D]$, (5.3) holds true. To verify (5.4), by the conformal change of metric used in step (2), we may and do assume that ∂M is either empty or convex. In this case, there exists a constant $r_0>0$ such that $B(x,r):=\{y\in M:\rho(x,y)\le r\}$ is convex for all $x\in M$ and $r\in(0,r_0]$. Then we take $\tilde\rho:=\rho\wedge r_0$, so that $\tilde B(x,r)$ is convex for all $r>0$, since $\tilde B(x,r)=B(x,r)$ for $r<r_0$ and $\tilde B(x,r)=M$ for $r\ge r_0$. Thus, (5.4) follows from [29, Theorem 1.4].

□

According to the above observations, we conclude that all assertions in Theorems 2.1-2.6 hold for $d=d'=d''=n$, i.e.

$$\begin{aligned} &q_\alpha=\frac{2n}{(3n-2-2\alpha)^+},\quad \alpha(d,d')=\alpha(n):=\frac12\sqrt{1+2n(n-1)}-\frac12,\\ &\gamma_{\alpha,p,q}=\frac n2(3-p^{-1}-q^{-1})-1-\alpha,\quad p,q\in[1,\infty).\end{aligned} \tag{5.5}$$

In this case, $\alpha>\alpha(n)$ implies $n<2(1+\alpha)$ and $q_\alpha>\frac{n}{2\alpha}$, so that in Theorem 2.1(1) we only need $\alpha>\alpha(n)$. Below we summarize these results, which in particular imply Theorems 1.1 and 1.2 stated in Introduction, according to $B(\lambda)=\lambda(\alpha=1)$, (4.14) and

$$\mathbf V(Z\phi_i):=\mu((Z\phi_i)(-L)^{-1}(Z\phi_i))=\mu(|\nabla L^{-1}(Z\phi_i)|^2).$$

Theorem 5.2. *Let $q_\alpha,\alpha(n),\gamma_{\alpha,p,q}$ for $p,q\in[1,\infty)$ be in (5.5). Then the following assertions hold for some constant $\kappa\in[1,\infty)$, where $\kappa=1$ when ∂M is either empty or convex.*

(1) *If $\alpha>\alpha(n)$, then*

$$\lim_{t\to\infty}\sup_{\nu\in\mathscr P}\mathbb E^\nu\big[\big|\{t\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^++\{t\kappa\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^-\big|^q\big]=0,\quad q\in[1,q_\alpha).$$

(2) *If $n<2(1+\alpha)$, then for any $q\in[1,q_\alpha)$ and $k\in(\frac{n}{2\alpha\mathrm i(q)},\infty]\cap[1,\infty]$,*

$$\lim_{t\to\infty}\sup_{\nu\in\mathscr P_{k,R}}\mathbb E^\nu\Big[\Big|\{t\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^++\{t\kappa\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^-\Big|^q\Big]=0,\quad R\in(0,\infty).$$

(3) *If $n<2(1+\alpha)$, then*

$$\kappa^{-1}\eta_Z^B\le\liminf_{t\to\infty}\inf_{\nu\in\mathscr P}t\mathbb E^\nu[\mathbb W_2(\mu_t^B,\mu)^2]\le\limsup_{t\to\infty}\sup_{\nu\in\mathscr P}t\mathbb E^\nu[\mathbb W_2(\mu_t^B,\mu)^2]\le\eta_Z^B<\infty.$$

(4) *If $n=2(1+\alpha)$, i.e. $(n,\alpha)\in\{(3,\frac12),(4,1)\}$, then there exist constants $c,t_0>1$ such that*

$$c^{-1}t^{-1}\log t\le\mathbb E^\nu[\mathbb W_2(\mu_t^B,\mu)^2]\le ct^{-1}\log t,\quad t\ge t_0,\nu\in\mathscr P.$$

(5) *If* $n > 2(1+\alpha)$*, then there exist constants* $c, t_0 > 1$ *such that*

$$c^{-1}t^{-\frac{2}{n-2\alpha}} \le \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\big)^2 \le \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2] \le ct^{-\frac{2}{n-2\alpha}}, \quad t\ge t_0, \nu\in\mathscr{P}.$$

(6) *If* $p,q\in[1,\infty)$ *with* $n(3-p^{-1}-q^{-1}) < 2(1+\alpha)$*, then there exist constants* $c, t_0 > 1$ *such that*

$$c^{-1}t^{-1} \le \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\big)^2 \le \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac{1}{q}} \le ct^{-1}, \quad t\ge t_0, \nu\in\mathscr{P}.$$

(7) *If* $p,q\in[1,\infty)$ *such that* $n(3-p^{-1}-q^{-1}) \ge 2(1+\alpha)$*, then there exists a constant* $c>0$ *such that for any* $t\ge 1$*,*

$$\sup_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac{1}{q}} \le \begin{cases} ct^{-1}\log(1+t), & \text{if } n(3-p^{-1}-q^{-1}) = 2(1+\alpha),\\ ct^{-\frac{1}{1+\gamma_{\alpha,p,q}}}, & \text{if } n(3-p^{-1}-q^{-1}) > 2(1+\alpha).\end{cases}$$

5.2 Subordinated conditional diffusion process

Let M be a bounded connected C^2 open domain in an n-dimensional complete Riemannian manifold, and let $V\in C_b^2(M)$ be such that $\mu_0(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M. Consider the Dirichlet eigenproblem for $\hat{L}_0 := \Delta + \nabla V$ in M:

$$\hat{L}_0 h_i = -\theta_i h_i, \quad h_i|_{\partial M} = 0, \quad i\ge 0,$$

where $\{\theta_i\}_{i\ge0}$ are listed in the increasing order counting multiplicities, and $\{h_i\}_{i\ge0}$ are the associated unitary eigenfunctions in $L^2(\mu_0)$ with $h_0>0$. Let

$$\hat{L} := \hat{L}_0 + 2\nabla\log h_0 = \Delta + \nabla(V + 2\log h_0), \quad \mu(\mathrm{d}x) := h_0(x)^2\mu_0(\mathrm{d}x).$$

Then the diffusion process $\hat{X}_t$ generated by $\hat{L}$ is non-explosive in M, whose distribution coincides with the conditional distribution of the $\hat{L}_0$-diffusion process $\hat{X}_t^0$ under the condition that

$$\tau := \inf\{t\ge 0 : \hat{X}_t^0 \in \partial M\} = \infty,$$

in the sense that for any $T>0$ and any $F\in C_b((C[0,T];M))$,

$$\mathbb{E}[F(\hat{X}_{[0,T]})] = \lim_{m\to\infty}\mathbb{E}[F(\hat{X}^0_{[0,T]})|\tau > m].$$

Let Z be a C_b^1-vector field on M satisfying (2.2).

It is well known that $\{\lambda_i := \theta_i - \theta_0\}_{i\ge0}$ are all eigenvalues of $-\hat{L}$ with unitary eigenfunctions $\{\phi_i := h_i h_0^{-1}\}_{i\ge0}$, and that (2.5), (2.22) and (2.25) hold for

$$d = n+2, \quad d' = d'' = n,$$

see for instance [9, 25]. Next, by [36, Lemma 4.6], (2.21) holds with $h(r) = \kappa\mathrm{e}^{Kr}$ for some constants $\kappa\ge1$ and $K\ge0$, where $\kappa=1$ when ∂M is convex. The following lemma confirms other conditions in Theorems 2.1-2.6, except (2.16) which is not yet verified.

Lemma 5.3. *For the present model, (2.7) and (A_2) hold. When ∂M is convex or $n \le 3$, (2.20) is satisfied.*

Proof. According to the proof of [36, Lemma 4.6], if ∂M is convex then the Bakry-Emery curvature $\hat{L}$ is bounded from below by a constant $-K$, so that

$$|\nabla P_t f| \le \mathrm{e}^{Kt} P_t |\nabla f|,$$

which implies (2.7) for $k(p) = \mathrm{e}^K$ as well as (2.20) for $\theta = 1$. So, in the following we only prove these conditions for non-convex ∂M, and verify (A_2).

(1) When ∂M is non-convex, let ρ' be the Riemannian distance induced by $\langle \cdot, \cdot \rangle'$ introduced in the proof of (5.1). According to the proof of [36, Lemma 4.6], for any $x, y \in M$, there exists a coupling $(\hat{X}_t, \hat{Y}_t)$ of the diffusion process generated by $\hat{L}$ starting from (x, y), such that for some constant $c_1 > 0$ we have

$$\mathrm{d}\rho'(\hat{X}_t, \hat{Y}_t) \le c_1 \rho'(\hat{X}_t, \hat{Y}_t)\mathrm{d}t + \mathrm{d}M_t,$$

where M_t is a martingale with $\mathrm{d}\langle M \rangle_t \le c_1 \rho'(\hat{X}_t, \hat{Y}_t)\mathrm{d}t$. Thus, for any $q \in (1, \infty)$, there exists a constant $K(q) > 0$ such that

$$\big(\mathbb{E}[\rho'(\hat{X}_t, \hat{Y}_t)^q]\big)^{\frac{1}{q}} \le K(q)\rho'(x, y), \quad t \in [0, 1].$$

Therefore, by $\rho' \le \rho \le \|\phi\|_\infty \rho'$,

$$\begin{aligned}
|\nabla \hat{P}_t f(x)| &:= \limsup_{y \to x} \frac{|\hat{P}_t f(x) - \hat{P}_t f(y)|}{\rho(x, y)} \le \limsup_{y \to x} \mathbb{E}\Big[\frac{|f(\hat{X}_t) - f(\hat{Y}_t)|}{\rho'(\hat{X}_t, \hat{Y}_t)} \cdot \frac{\rho'(\hat{X}_t, \hat{Y}_t)}{\rho(x, y)}\Big] \\
&\le \limsup_{y \to x} \Big(\mathbb{E}\Big[\frac{|f(\hat{X}_t) - f(\hat{Y}_t)|^p}{\rho'(\hat{X}_t, \hat{Y}_t)^p}\Big]\Big)^{\frac{1}{p}} \frac{\big(\mathbb{E}[\rho'(\hat{X}_t, \hat{Y}_t)^{\frac{p}{p-1}}]\big)^{\frac{p-1}{p}}}{\rho(x, y)} \le K\Big(\frac{p}{p-1}\Big)\|\phi\|_\infty (P_t |\nabla f|^p)^{\frac{1}{p}}.
\end{aligned}$$

So, (2.7) holds.

(2) Let ∂M be non-convex and $n \le 3$. Since $\nabla V \in C_b^1(M)$ and

$$-\mathrm{Hess}_{\log h_0} = -\frac{\mathrm{Hess}_{h_0}}{h_0} + \frac{(\nabla h_0) \otimes (\nabla h_0)}{h_0^2} \ge -\frac{\|\nabla^2 h_0\|_\infty}{h_0},$$

there exists a constant $c_1 > 0$ such that the Bakry-Emery curvature of $\hat{L}$ is bounded below by $-\frac{c_1}{2h_0}$, i.e.

$$\Gamma_2(g) := \frac{1}{2}\hat{L}|\nabla g|^2 - \langle \nabla g, \nabla \hat{L} g \rangle \ge -\frac{c_1}{2} h_0^{-1} |\nabla g|^2, \quad g \in C(M).$$

So, by (3.18) for $d = n + 2$, and applying Jensen's inequality, we find a constant $c_2 > 0$ such that

$$\begin{aligned}
|\nabla \hat{P}_t \mathrm{e}^f|^2 - \hat{P}_t |\nabla \mathrm{e}^f|^2 &= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \hat{P}_s |\nabla \hat{P}_{t-s} \mathrm{e}^f|^2 \mathrm{d}s \\
&\le c_1 \int_0^t \hat{P}_s \big\{h_0^{-1} |\nabla \hat{P}_{t-s} \mathrm{e}^f|^2\big\} \mathrm{d}s \le c_1 \int_0^t \big(\hat{P}_s h_0^{-\frac{m}{m-1}}\big)^{\frac{m-1}{m}} \big(\hat{P}_s |\nabla \hat{P}_{t-s} \mathrm{e}^f|^{2m}\big)^{\frac{1}{m}} \mathrm{d}s
\end{aligned}$$

$$\le c_2 \int_0^t s^{-\frac{(n+2)(m-1)}{2m}} \mu(h_0^{-\frac{m}{m-1}})^{\frac{m-1}{m}} (\hat P_s |\nabla \hat P_{t-s} \mathrm{e}^f|^{2m})^{\frac{1}{m}} \mathrm{d}s$$
$$\le c_2 \|\nabla f\|_\infty^2 (\hat P_t \mathrm{e}^{2mf})^{\frac{1}{m}} \int_0^t s^{-\frac{(n+2)(m-1)}{2m}} \mu(h_0^{-\frac{m}{m-1}})^{\frac{m-1}{m}} \mathrm{d}s.$$

Noting that $n+2 \le 5$ and $\mu(h_0^{-r}) = \mu_0(h_0^{2-r}) < \infty$ for $r < 3$, for any $m \in (\frac{3}{2}, \frac{5}{3})$, we have

$$\theta_1 := 1 - \frac{(n+2)(m-1)}{2m} \in (0,1), \quad \mu\big(h_0^{-\frac{m}{m-1}}\big) < \infty,$$

so that for some constant $c_3 > 0$ we have

$$|\nabla \hat P_t \mathrm{e}^f|^2 - \hat P_t |\nabla \mathrm{e}^f|^2 \le c_3 t^{\theta_1} \|\nabla f\|_\infty^2 (\hat P_t \mathrm{e}^{2m})^{\frac{1}{m}}. \tag{5.6}$$

Similarly, by (2.7) and its consequence

$$|\nabla \hat P_t g| \le \frac{c}{\sqrt{t}} (\hat P_t |\nabla g|^2)^{\frac{1}{2}},$$

we find a constant $c_4 > 0$ such that

$$\hat P_t |\nabla \mathrm{e}^f|^2 - (\hat P_t \mathrm{e}^f) \hat P_t(|\nabla f|^2 \mathrm{e}^f) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \hat P_{t-s}\big\{(\hat P_s \mathrm{e}^f) \hat P_s(|\nabla f|^2 \mathrm{e}^f)\big\} \mathrm{d}s$$
$$= -\int_0^t \hat P_{t-s} \langle \nabla \hat P_s \mathrm{e}^f, \nabla \hat P_s(|\nabla f|^2 \mathrm{e}^f)\rangle \mathrm{d}s$$
$$\le c_4 \|\nabla f\|_\infty^2 \int_0^t s^{-\frac{1}{2}} \hat P_{t-s}(\hat P_s \mathrm{e}^{2f}) \mathrm{d}s = 2c_4 t^{\frac{1}{2}} \|\nabla f\|_\infty^2 \hat P_t \mathrm{e}^{2f}.$$

This together with (5.6) implies (2.20) for $\theta = \theta_1 \wedge \frac{1}{2}$.

(3) It remains to verify (2.8). By the conformal change of metric as in the end of the proof of Lemma 5.1, we only consider the case where ∂M is convex, so that

$$\langle \mathbf{N}, \nabla \rho(x,\cdot)\rangle|_{\partial M} \le 0, \quad x \in M,$$

where $\mathbf{N}$ is the inward unit normal vector field of ∂M. Let ρ_∂ be the distance to ∂M. It is well known (see for instance [25]) that ∇h_0 is inward normal on the boundary and $c_1 := \|h_0^{-1}\rho_\partial\|_\infty < \infty$. So,

$$\rho_\partial \le c_1 h_0, \quad \langle \nabla h_0, \nabla \rho(x,\cdot)\rangle|_{\partial M} \le 0, \quad x \in M. \tag{5.7}$$

Moreover, by the Hessian comparison theorem, there exists a constant $c_2 > 0$ such that

$$\mathrm{Hess}_{\rho(x,\cdot)^2}(v,v) \le c_2 |v|^2, \quad x \in M,\ v \in TM. \tag{5.8}$$

We intend to show that these two estimates imply

$$\sup_{x,y \in M} \langle \nabla \log h_0, \nabla \rho(x,\cdot)^2\rangle(y) \le c \tag{5.9}$$

for some constant $c>0$. To see this, for any $x,y\in M$, let $z\in\partial M$ such that $\rho(y,z)=\rho_\partial(y)$. Let

$$\gamma:[0,1]\to M,\quad \gamma_0=z,\quad \gamma_1=y,\quad |\dot\gamma|=\rho_\partial(y)$$

be the minimal geodesic from z to y. Let $v_s=\nabla h_0(\gamma_s)$. We have

$$v_0=a_0\dot\gamma_0,\quad \|_{0\to s}\, v_0=a_0\dot\gamma_s,\quad s\in[0,1],$$

where $a_0:=\langle \mathbf{N},\nabla h_0\rangle(z)$ and $\|_{0,s}$ is the parallel displacement along the geodesic γ_s. Since $h_0\in C_b^2(M)$, we find a constant $c_3>0$ such that

$$|v_1-a_0\dot\gamma_1|=|v_1-\|_{0\to1}\, v_0|\le c_3\rho_\partial(y).$$

Combining this with (5.7) and (5.8), and noting that $|\nabla\rho^2|\le 2\|\rho\|_\infty<\infty$, we find a constant $c_4>0$ such that

$$\begin{aligned}&\langle\nabla h_0,\nabla\rho(x,\cdot)^2\rangle(y)=\langle v_1,\nabla\rho(x,\cdot)^2(\gamma_1)\rangle\le a_0\langle\dot\gamma_1,\nabla\rho(x,\cdot)^2(\gamma_1)\rangle+c_3\rho_\partial(y)\\&=a_0\langle\dot\gamma_0,\nabla\rho(x,\cdot)^2(\gamma_0)\rangle+a_0\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\langle\dot\gamma_s,\nabla\rho(x,\cdot)^2(\gamma_s)\rangle\mathrm{d}s+c_3\rho_\partial(y)\\&\le a_0\int_0^1 \mathrm{Hess}_{\rho(x,\cdot)^2}(\dot\gamma_s,\dot\gamma_s)\mathrm{d}s+c_3\rho_\partial(y)\le a_0c_2\rho_\partial(y)^2+c_3\rho_\partial(y)\le c_4h_0(y).\end{aligned}$$

Therefore, (5.9) holds for $c=c_4$.

By (5.9) and Itô's formula, we obtain

$$\mathrm{d}\rho(\hat X_0,\cdot)^2(\hat X_t)\le c\mathrm{d}t+2\sqrt2\rho(\hat X_0,\hat X_t)\mathrm{d}B_t,$$

where B_t is the one-dimensional Brownian motion. This implies (A_2) as explained in the proof of Lemma 5.1(2). □

We now conclude that all assertions in Theorems 2.1-Theorem 2.5 hold, except Theorem 2.2(4) where the condition (2.16) is to be verified for this model, for $d=n+2, d'=d''=n$ and

$$\begin{aligned}q_\alpha&:=\frac{2n+4}{3n+2-2\alpha}\le 2,\quad \alpha(d,d')=\tilde\alpha(n):=\frac12\sqrt{4+2n(n+2)}-1,\\ \gamma_{\alpha,p,q}&:=\frac n2+\frac{n+2}{2}(2-p^{-1}-q^{-1})-\alpha-1.\end{aligned}\tag{5.10}$$

Noting that when $n=1$ the condition $\alpha>\tilde\alpha(n)$ becomes $\alpha>\frac12\sqrt{10}-1$, which implies $1=n<2(1+\alpha)$ and $\frac{6}{5-2\alpha}=q_\alpha>\frac{n}{2\alpha}=\frac{1}{2\alpha}$, while $q_\alpha\le 2$ yields $\mathrm{i}(q)=1$ for $q\in[1,q_\alpha)$, we have the following result according to Theorems 2.1-Theorem 2.5.

Theorem 5.4. *Let $\hat L:=L_0+2\nabla\log h_0$, and $L=\hat L+Z$ for some C_b^1-vector field Z satisfying (2.2). The following assertions hold for $q_\alpha,\tilde a(n)$ and, $\gamma_{\alpha,p,q}$ in (5.10), and a constant $\kappa\ge1$ with $\kappa=1$ when ∂M is convex.*

(1) *When $n=1$ and $\alpha\in(\frac12\sqrt{10}-1,1]$,*

$$\lim_{t\to\infty}\sup_{\nu\in\mathscr P}\mathbb E^\nu\big[\big|\{t\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^++\{t\kappa\mathbb W_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^-\big|^q\big]=0,\ \ q\in\Big[1,\frac{6}{5-2\alpha}\Big).$$

(2) *If $n < 2(1+\alpha)$, then for any $q \in [1, q_\alpha)$ and $k \in (\frac{n+2}{2\alpha}, \infty] \cap [1, \infty]$,*

$$\lim_{t\to\infty} \sup_{\nu \in \mathscr{P}_{k,R}} \mathbb{E}^\nu \Big[\Big| \{t\mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t)\}^+ + \{t\kappa \mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t)\}^- \Big|^q \Big] = 0, \quad R \in (0, \infty).$$

(3) *If $n < 2(1+\alpha)$, then*

$$\kappa^{-1}\eta_Z^B \le \liminf_{t\to\infty} \inf_{\nu\in\mathscr{P}} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le \limsup_{t\to\infty} \sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le \eta_Z^B < \infty.$$

(4) *Let $n = 2(1+\alpha)$, i.e. $(n, \alpha) \in \{(3, \frac{1}{2}), (4, 1)\}$. Then there exist constants $c, t_0 > 0$ such that*

$$\sup_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le ct^{-1}\log t, \quad t \ge t_0.$$

If ∂M is convex or $(n, \alpha) = (3, \frac{1}{2})$, then there exists a constant $c' > 0$ such that

$$\inf_{\nu\in\mathscr{P}} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \ge c't^{-1}\log t, \quad t \ge t_0.$$

(5) *If $n > 2(1+\alpha)$, then there exist constants $c, t_0 > 1$ such that*

$$c^{-1}t^{-\frac{2}{n-2\alpha}} \le \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B, \mu)]\big)^2 \le \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] \le ct^{-\frac{2}{n-2\alpha}}, \quad t \ge t_0, \nu \in \mathscr{P}.$$

(6) *If $p, q \in [1, \infty)$ with $\gamma_{\alpha,p,q} < 0$, then there exist constants $c, t_0 > 1$ such that*

$$c^{-1}t^{-1} \le \big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B, \mu)]\big)^2 \le \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}]\big)^{\frac{1}{q}} \le ct^{-1}, \quad t \ge t_0, \nu \in \mathscr{P}.$$

(7) *Let $p, q \in [1, \infty)$ with $\gamma_{\alpha,p,q} \ge 0$. Then for any $\gamma > \gamma_{\alpha,p,q}$, there exists a constant $c > 0$ such that*

$$\sup_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}]\big)^{\frac{1}{q}} \le ct^{-\frac{1}{1+\gamma}}, \quad t \ge 1.$$

If (2.16) holds, then there exists a constant $c > 0$ such that

$$\sup_{\nu\in\mathscr{P}} \big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}]\big)^{\frac{1}{q}} \le ct^{-\frac{1}{1+\gamma_{\alpha,p,q}}} + ct^{-1}\log(1+t)1_{\{\gamma_{\alpha,p,q}=0\}}.$$

5.3 Subordinated Wright-Fisher diffusion process

Let $a, b > \frac{1}{4}$ be two constants, and let

$$\mu := 1_{[0,1]}(x)\frac{\Gamma(2a+2b)}{\Gamma(2a)\Gamma(2b)}x^{2a-1}(1-x)^{2b-1}\mathrm{d}x$$

be the Beta distribution on $M = [0, 1]$. The Fisher-Wright diffusion process $\hat{X}_t$ is generated by

$$\hat{L} := \frac{1}{2}x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \{a - (a+b)x\}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Under the Riemannian metric $\langle \partial_x, \partial_x \rangle = 2\{x(1-x)\}^{-1}$, we have

$$\Gamma(f,g) = \langle \nabla f, \nabla g \rangle := \frac{1}{2} x(1-x) f'(x) g'(x), \quad x \in M,$$

$$\rho(x,y) = \sqrt{2} \int_x^y \{s(1-s)\}^{-\frac{1}{2}} \mathrm{d}s, \quad 0 < x \le y < 1.$$

Since $\mathrm{div}_\mu Z = 0$ implies $Z = 0$, we have $L = \hat{L}$.

Lemma 5.5. *For the present model with $L = \hat{L}$, (A_1) with $d = 4(a \vee b)$ and (A_2) hold, (2.22) holds for $d' = 2$, (B) holds with $h(r) = \mathrm{e}^{Kr}$ for some constant $K > 0$, and (2.16) holds.*

Proof. Firstly, the condition (2.5) with $d = 4(a \vee b)$ is implied by [15, Corollary 2.3]. By [27, (2.4)], the Bakry-Emery curvature of $\hat{L}$ is bounded below by $-K < 0$ for some constant $K \ge 0$, so that

$$|\nabla P_t f| \le \mathrm{e}^{Kt} P_t |\nabla f|,$$

(2.7) and (B) hold for $\theta = m = 1$, $k(p) = \mathrm{e}^K$ and $h(r) = \mathrm{e}^{rK}$.

Next, for any $p \ge 1$ there exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\hat{L}\rho(X_0,\cdot)^{2p}(X_t) = 2p\rho(X_0,X_t)^{2(p-1)} L\rho(X_0,\cdot)(X_t)^2 + p(p-1)\rho(X_0,X_t)^{2(p-1)} \\ &\le c_1 \rho(X_0,X_t)^{2(p-1)}, \end{aligned}$$

so that

$$\mathbb{E}^\mu[\rho(X_0,X_t)^{2p}] \le c_1 \int_0^t \mathbb{E}^\mu[\rho(X_0,X_s)^{2(p-1)}] \mathrm{d}s. \tag{5.11}$$

In particular, for $p = 1$ we obtain (2.8), and for general $p \in \mathbb{N}$ it follows from (5.11) by the induction argument.

Moreover, we have $\lambda_i = (a+b)i$ so that (2.22) holds for $d' = 2$. Indeed, according to the proof of [14, Theorem 1.1], all eigenfunctions are polynomials. The trivial eigenvalue is $\lambda_0 = 0$ with $\phi_0 = 1$. For any $i \in \mathbb{N}$, let

$$\phi_i(x) := \sum_{j=0}^i \alpha_j x^j$$

with $\alpha_i > 0$ be the unitary eigenfunction for λ_i. We have

$$-\lambda_i \phi_i(x) = \hat{L}\phi_i(x).$$

Since the coefficients of x^i in left hand and right hand sides are $-\lambda_i \alpha_i$ and $-i(a+b)\alpha_i$ respectively, these two constants have to be equal each other, so that $\lambda_i = i(a+b)$.

Finally, as explained in step (3) in the proof of Lemma 5.1, for (2.16) it suffices to verify (5.3) and (5.4). Since the curvature is bounded from below as indicated in the beginning of the proof, and since a one-dimensional ball is convex, (5.4) follows from [29, Theorem 1.4]. So, it remains to (5.3). With the transform $x \to 1-x$, we only need to prove this condition for $x \in [0, \frac{1}{2}]$. Let $x \in [0, \frac{1}{2}]$ and $B(x, r) := \{y \in [0, 1] : \rho(x, y) \le r\}$. Take, for instance, $r_0 = \frac{1}{8}\rho(\frac{1}{2}, 1)$ such that

$$x_0 := \sup B(1/2, 2r_0) \in (1/2, 1).$$

We have

$$c_0 := \inf_{x \in [0, \frac{1}{2}]} \mu(B(x, r_0)) > 0,$$

so that

$$\mu(B(x, 2r)) \le 1 \le c_0^{-1}\mu(B(x, r)), \quad r \ge r_0.$$

Hence, we only need to consider $r \in (0, r_0)$. On the other hand, we find constants $c_2 > c_1 > 0$ such that

$$c_1^{-1}\big|\sqrt{x} - \sqrt{y}\big| \ge \rho(x, y) \ge c_2^{-1}\big|\sqrt{x} - \sqrt{y}\big|, \quad x \in [0, 1/2], r \in (0, 2r_0),$$

so that for some constants $c_3, c_4 > 0$,

$$\big[\big\{(\sqrt{x} - c_1 r)^+\big\}^2, \big\{\sqrt{x} + c_1 r\big\}^2 \wedge x_0\big] \subset B(x, r) \subset \big[\big\{(\sqrt{x} - c_2 r)^+\big\}^2, \big\{\sqrt{x} + c_2 r\big\}^2 \wedge x_0\big], \ \ r \in (0, 2r_0).$$

Noting that $0 < 1 - x_0 \le 1 - s \le 1$ for $s \in B(x, 2r_0)$, we find constants $c_4 > c_3 > 0$ such that

$$\begin{aligned} &c_3\big\{(x + c_3 r)^{2a} - x^{2a}\big\} \le \mu(B(x, r)) \\ &\le \mu(B(x, 2r)) \le c_4\big\{(x + c_4 r)^{2a} - [(x - c_4 r)^+]^{2a}\big\}, \quad x \in [0, 1/2], r \in (0, r_0). \end{aligned}$$

since $\frac{(x+c_4 r)^{2a} - \{(x - c_4 r)^+\}^{2a}}{(x + c_3 r)^{2a} - x^{2a}}$ is a continuous function of $(x, r) \in [0, \frac{1}{2}] \times [0, r_0]$, where when $r = 0$ the function is understood as the limit $\frac{c_4}{c_3}$ as $r \to 0$, we obtain

$$\sup_{x \in [0,1/2], r \in (0, r_0)} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \le \sup_{x \in [0, \frac{1}{2}], r \in [0, r_0]} \frac{(x + c_4 r)^{2a} - \{(x - c_4 r)^+\}^{2a}}{(x + c_3 r)^{2a} - x^{2a}} < \infty.$$

Therefore, (5.3) holds. □

In conclusion, all assertions in Theorems 2.1-2.6(1) hold for $d = 4(a \vee b)$ and $d' = 2$ so that

(5.12)
$$\begin{aligned} q_\alpha &:= \frac{4(a \vee b)}{4(a \vee b) - \alpha}, \quad \alpha(d, d') = \tilde{\alpha} := (a \vee b)\big(\sqrt{5} - 1\big), \\ \gamma_{\alpha,p,q} &:= 2(a \vee b)(2 - p^{-1} - q^{-1}) - \alpha. \end{aligned}$$

Noting that $\alpha > \frac{4}{3}(a \vee b)$ implies $q_\alpha > \frac{4(a \vee b)}{2\alpha}$, $2(1 + \alpha) > d' = 2$ and $\alpha > \tilde{\alpha}$, Theorems 2.1-2.6(1) imply the following result.

Theorem 5.6. *For the above $L = \hat{L}$ and $\eta^B = \eta^B_Z$ for $Z = 0$, the following assertions hold for $q_\alpha, \tilde{a}$ and, $\gamma_{\alpha,p,q}$ in (5.12).*

(1) *If $\alpha > \frac{4}{3}(a \vee b)$, then*

$$\lim_{t \to \infty} \sup_{\nu \in \mathscr{P}} \mathbb{E}^\nu\big[\big|t\mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t)\big|^q\big] = 0, \quad q \in [1, q_\alpha).$$

(2) *For any* $q \in [1, q_\alpha)$ *and* $k \in (\frac{d}{2\alpha \mathrm{i}(q)}, \infty] \cap [1, \infty]$, *where we set* $(\frac{d}{2\alpha \mathrm{i}(q)}, \infty] = \{\infty\}$ *if* $\alpha = 0$,

$$\lim_{t\to\infty} \sup_{\nu \in \mathscr{P}_{k,R}} \mathbb{E}^\nu \left[\left| t\mathbb{W}_2(\mu_t^B, \mu)^2 - \Xi^B(t) \right|^q \right] = 0, \quad R \in (0, \infty).$$

(3) $\eta^B < \infty$ *and* $\limsup_{t\to\infty} \sup_{\nu \in \mathscr{P}} \left| t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B, \mu)^2] - \eta^B \right| = 0.$

(4) *Let* $p, q \in [1, \infty)$ *with* $\gamma_{\alpha,p,q} < 0$. *There exist constants* $c, t_0 > 1$ *such that*

$$c^{-1}t^{-1} \le \left(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B, \mu)] \right)^2 \le \left(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}] \right)^{\frac{1}{q}} \le ct^{-1}, \quad t \ge t_0, \nu \in \mathscr{P}.$$

(5) *Let* $p, q \in [1, \infty)$ *with* $\gamma_{\alpha,p,q} \ge 0$. *Then there exists a constant* $c > 0$ *such that*

$$\sup_{\nu \in \mathscr{P}} \left(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B, \mu)^{2q}] \right)^{\frac{1}{q}} \le ct^{-\frac{1}{1+\gamma_{\alpha,p,q}}} + c1_{\{\gamma_{\alpha,p,q}=0\}} t^{-1} \log(1+t), \quad t \ge 1.$$

5.4 Subordinated subelliptic diffusions on $\mathbf{SU}(2)$

Let $M = \mathbf{SU}(2)$ be the space of 2×2, complex, unitary matrices with determinant 1, which is a 3-dimensional compact Lie group, with Lie algebra $\mathbf{su}(2)$ and Riemannian metric $\langle \cdot, \cdot \rangle$ given by

$$\mathbf{su}(2) := \mathrm{span}\{U_1, U_2, U_3\}, \quad \langle U_i, U_j \rangle = 1_{\{i=j\}}, \quad 1 \le i, j \le 3,$$

where for $\mathbf{i} = \sqrt{-1}$,

$$U_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U_2 := \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad U_3 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}.$$

For each $1 \le i \le 3$, U_i is understood as a left-invariant vector field defined as

$$U_i f(x) := \lim_{\varepsilon \downarrow 0} \frac{f(\mathrm{e}^{\varepsilon U_i} x) - f(x)}{\varepsilon}, \quad f \in C^1(\mathbf{SU}(2)).$$

Then $[U_1, U_2] = 2U_3$, so that

$$\hat{L} := U_1^2 + U_2^2$$

satisfies Hörmader's condition. Moreover, $\hat{L}$ is symmetric in $L^2(\mu)$ where μ is the normalized Haar measure on $\mathbf{SU}(2)$, and the intrinsic distance ρ induced by

$$\Gamma(f, g) := (U_1 f)(U_1 g) + (U_2 f)(U_2 g)$$

is the Carnot-Carathéodory distance. By Chow's theorem, (M, ρ) is a compact geodesic space.

To formulate the diffusion process $\hat{X}_t$ generated by $\hat{L}$, we use the cylindrical coordinates introduced in [10]:

$$\left[0, \frac{\pi}{2}\right) \times [0, 2\pi] \times [-\pi, \pi] \ni (r, \theta, z) \mapsto \mathrm{e}^{r(\cos\theta)U_1 + r(\cos\theta)U_2} \mathrm{e}^{zU_3} \in M := \mathbf{SU}(2).$$

Under these coordinates, the diffusion process $\hat{X}_t := (r_t, \theta_t, z_t)$ is constructed by solving the SDEs

$$
\begin{aligned}
&\mathrm{d}r_t = 2\cot(2r_t)\mathrm{d}t + \mathrm{d}B_t, \\
&\mathrm{d}(\theta_t, z_t) = \Big(\frac{2}{\sin\theta_t}, \tan r_t\Big)\mathrm{d}\tilde{B}_t,
\end{aligned}
\tag{5.13}
$$

where $(B_t, \tilde{B}_t)$ is a two-dimensional Brownian motion, see [5, Remark 2.2]. The following lemma shows that conditions $(A_1), (A_2)$, (2.16) and (2.22) hold. However, due to the degeneracy of the diffusion, assumption (B) may be invalid.

Lemma 5.7. *Conditions* $(A_1), (A_2)$, (2.16) *and* (2.22) *hold for* $d = 4$ *and* $d' = 3$.

Proof. By [5, Theorem 4.10], for any $p > 1$ there exists a constant $c(p) > 0$ such that

$$
|\nabla \hat{P}_t f| \le c(p)\mathrm{e}^{-2t}(P_t|\nabla f|^p)^{\frac{1}{p}}, \quad t \ge 0.
$$

So, (2.7) holds.

According to [7], the generalized curvature-dimension condition $CD(\rho_1, \frac{1}{2}, 1, 2)$ holds, so that (2.16) is implied by [6, Theorem 1.2].

Let $\hat{p}_t$ be the heat kernel of $\hat{P}_t$ with respect to μ. By [5, Proposition 3.1] and the spectral representation of heat kernel, see also [8], all eigenvalues with multiplicities of $-\hat{L}$ are given by

$$
\{\lambda_i\}_{i\ge 0} = \big\{4k(k + |n| + 1) + 2|n| : \quad n \in \mathbb{Z}, k \in \mathbb{Z}_+\big\}.
$$

In particular, $\lambda_1 = 2$. It is easy to see that for large $i \in \mathbb{N}$,

$$
\#\big\{4k(k + |n| + 1) + 2|n| \le i : \quad n \in \mathbb{Z}, k \in \mathbb{Z}_+\big\}
$$

has order $i^{\frac{3}{2}}$, so that (2.22) holds for $d' = 3$.

To verify (2.5), we use the cylindrical coordinates (r, θ, z), for which the identity matrix becomes $\mathbf{0} := (0, 0, 0)$. Let $\hat{p}_t$ be the heat kernel of $\hat{P}_t$ with respect to μ. By [5, Proposition 3.9], there exists a constant $c > 0$ such that

$$
\hat{p}_t(\mathbf{0}, \mathbf{0}) \le ct^{-2}, \quad t \in (0, 1].
$$

By the left invariant of the heat kernel which follows from the same property of the generator $\hat{L}$, we obtain

$$
\|\hat{P}_t\|_{1\to\infty} = \sup_{x\in M} \hat{p}_t(x, x) \le ct^{-2}, \quad t \in (0, 1].
$$

This together with $\lambda_1 = 2$ implies (2.5) for $\lambda = 2$.

It remains to verify (A_2). For any $(r, z) \in [0, \frac{\pi}{2}) \times [-\pi, \pi]$, let $\theta(r, z) \in [-\pi, \pi]$ be the unique solution to the equation

$$
\theta(r, z) - z = \frac{(\cos r)(\sin\theta(r, z)) \arccos[(\cos\theta(r, z))\cos r]}{\sqrt{1 - (\cos^2 r)\cos^2\theta(r, z)}}.
$$

By [5, Remark 3.12], the distance of $x := (r,\theta,z)$ to $\mathbf{0}$ depends only on (r,z), and there exists a constant $c_1>0$ such that

$$\rho(\mathbf{0},x)^2 = \frac{(\theta(r,z)-z)^2\tan^2 r}{\sin^2\theta(r,z)} \le c_1\sin^2 r \le c_1 r^2. \tag{5.14}$$

On the other hand, letting $\hat X_0=\mathbf{0}$, by (5.13) and Itô's formula, for any $p\ge 1$ we find a constant $c_1(p)>0$ such that

$$\mathrm{d}r_t^{2p} \le c(p) r_t^{2(p-1)}\mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t. So, we find a constant $c_2(p)>0$ such that

$$\mathbb{E}[r_t^{2p}] \le k(p)t^p,\quad t\ge 0.$$

Combining this with (5.14) and using the left invariance of the heat kernel, we obtain (A_2). □

By the above lemma and that $M=\mathbf{SU}(2)$ is a Polish space, we conclude that all assertions in Theorems 2.1-Theorem 2.3 and Theorem 2.6(1) hold for $d=4, d'=3$ and

$$q_\alpha := \frac{8}{9-2\alpha},\qquad \gamma_{\alpha,p,q} := \frac{1}{2}-\alpha+2(2-p^{-1}-q^{-1}). \tag{5.15}$$

Noting that $1<q_\alpha<2$ for $\alpha\in(\frac12,1]$, so that $\mathrm{i}(q)=1$ for $q\in[1,q_\alpha)$, we have the following result.

Theorem 5.8. *Let $\hat L := U_1^2+U_2^2$, and $L=\hat L+Z$ for some C_b^1-vector field Z satisfying* (2.2)*. The following assertions hold for q_α and $\gamma_{\alpha,p,q}$ in* (5.15)*.*

(1) *If $\alpha\in(\frac12,1]$, then for any $q\in[1,q_\alpha)$ and $k\in(\frac{2}{\alpha},\infty]\cap[1,\infty]$,*

$$\lim_{t\to\infty}\sup_{\nu\in\mathscr{P}_{k,R}}\mathbb{E}^\nu\Big[\big|\{t\mathbb{W}_2(\mu_t^B,\mu)^2-\Xi^B(t)\}^+\big|^q\Big]=0,\quad R\in(0,\infty).$$

(2) *If $\alpha\in(\frac12,1]$, then*

$$\limsup_{t\to\infty}\sup_{\nu\in\mathscr{P}} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2]\le \eta_Z^B<\infty.$$

(3) *If $\alpha\in(0,\frac12]$, then there exist constants $c,t_0>0$ such that*

$$\sup_{\nu\in\mathscr{P}}\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^B,\mu)^2]\le ct^{-\frac{2}{3-2\alpha}}+c1_{\{\alpha=\frac12\}}t^{-1}\log t,\quad t\ge t_0.$$

(4) *If $p,q\in[1,\infty)$ with $\gamma_{\alpha,p,q}<0$, then there exist constants $c,t_0>1$ such that*

$$c^{-1}t^{-1}\le\big(\mathbb{E}^\nu[\mathbb{W}_1(\mu_t^B,\mu)]\big)^2\le\big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac1q}\le ct^{-1},\quad t\ge t_0,\nu\in\mathscr{P}.$$

(5) *Let $p,q\in[1,\infty)$ with $\gamma_{\alpha,p,q}\ge 0$. Then for any $\gamma>\gamma_{\alpha,p,q}$, there exists a constant $c>0$ such that for any $t\ge 1$,*

$$\sup_{\nu\in\mathscr{P}}\big(\mathbb{E}^\nu[\mathbb{W}_{2p}(\mu_t^B,\mu)^{2q}]\big)^{\frac1q}\le ct^{-\frac{1}{1+\gamma_{\alpha,p,q}}}+c1_{\{\gamma_{\alpha,p,q}=0\}}t^{-1}\log(1+t).$$

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