On nonlinear Markov processes in the sense of McKean

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Abstract

We study nonlinear Markov processes in the sense of McKean's seminal work [28] and present a large new class of examples. Our notion of nonlinear Markov property is in McKean's spirit, but, in fact, more general in order to include examples of such processes whose one-dimensional time marginals solve a nonlinear parabolic PDE (more precisely, a nonlinear Fokker–Planck–Kolmogorov equation) such as Burgers' equation, the porous media equation, or variants of the latter with transport-type drift. We show that the associated nonlinear Markov process is given by path laws of weak solutions to a corresponding distribution-dependent stochastic differential equation whose coefficients depend singularly (i.e. Nemytskii-type) on its one-dimensional time marginals. Moreover, we show that also for general nonlinear Markov processes, their path laws are uniquely determined by one-dimensional time marginals of suitable associated conditional path laws.

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1 Introduction

In this work we study nonlinear Markov processes in the sense of McKean's seminal work [28]. We suggest a modified definition of such processes (see Definition 2.1) and present as well as analyze several classes of examples which, to the best of our knowledge, have not been considered in the context of nonlinear Markov processes in the literature before. These include nonlinear Markov processes associated with the classical porous media equation, in all spatial dimensions $d \geq 1$, with the Barenblatt solutions as its one-dimensional time marginal densities, and variants thereof with more general diffusivity functions and additional drift part of transport type.

Linear situation. Let us briefly recall how the theory of classical linear Markov processes is linked to stochastic differential equations (SDEs) and linear parabolic Fokker–Planck–Kolmogorov equations (FPKEs). If

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq s, \quad X_s = x$$

has a unique probabilistic weak solution $X^{s,x}$ for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, then the corresponding path laws $\mathbb{P}_{s,x}$ form a Markov process, and, as a simple consequence of Itô's formula, its one-dimensional time marginals μ_t, $t \geq s$, solve the linear FPKE

$$\partial_t \mu_t = \partial^2_{ij}(a_{ij}(t, x)\mu_t) - \partial_i(b_i(t, x)\mu_t), \quad t \geq s, \quad \mu_s = \delta_x,$$

where $a = (a_{ij})_{i,j \leq d} = \frac{1}{2}\sigma\sigma^T$ and we use Einstein summation convention. In particular, $(\mu_t)_{t \geq s}$ is a curve in the space of all Borel probability measures $\mathcal{P}$ on $\mathbb{R}^d$ equipped with the weak topology.

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Vice versa, if for each (s,x) this linear FPKE has a unique nonnegative (distributional) solution with constant mass equal to one and if the coefficients satisfy a mild integrability condition with respect to these solutions, then by the Ambrosio-Figalli-Trevisan superposition principle [1, 17, 36] (see also the more recent paper [15]) the above SDE is weakly well-posed and the path laws $\mathbb{P}_{s,x}$ of its weak solutions constitute a Markov process. More precisely, for $A \in \mathcal{B}(\mathbb{R}^d)$ and the natural projections π_t^s on $C([s,\infty),\mathbb{R}^d)$, $\pi_t^s(w) := w(t)$, they satisfy the Markov property

$$\mathbb{P}_{s,\zeta}(\pi_t^s \in A | \sigma(\pi_\tau^s, s \le \tau \le r)) = \mathbb{P}_{r,\pi_r^s}(\pi_t^r \in A) \quad \mathbb{P}_{s,\zeta} - \text{a.s.} \tag{ℓMP}$$

for all $\zeta \in \mathcal{P}$ and $0 \le s \le r \le t$, where $\mathbb{P}_{s,\zeta} := \int_{\mathbb{R}^d} \mathbb{P}_{s,x}\, \zeta(dx)$. We make a precise comparison to this linear theory in Section 2.1 and show that our notion of nonlinear Markov processes is a proper generalization of it.

McKean's vision. McKean seems to have been the first who realized that the core Markovian feature, i.e. that the path law $\mathbb{P}_{s,\zeta}$ restricted to $\sigma(\pi_\tau^s, r \le \tau)$ conditioned on $\sigma(\pi_\tau^s, s \le \tau \le r)$ is a function of s, r and the current position π_r^s only, is more general than the specific formula (ℓMP) (see also the formulation in [35, p.145]). In [28], he suggested a generalization by replacing $\mathbb{P}_{r,\pi_r^s}$ by the one-dimensional marginal at time t of a regular conditional probability of $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\,\cdot\,|\sigma(\pi_r^r)]$, where $\mu_r^{s,\zeta}$ denotes the one-dimensional time marginal of $\mathbb{P}_{s,\zeta}$ at r. Hence the right-hand side of (ℓMP) remains a function of s, r and π_r^s. McKean's envisioned program was to connect this new Markov property to nonlinear parabolic PDEs, more precisely to nonlinear FPKEs of *Nemytskii-type*, i.e., to equations of the form

$$\partial_t u = \partial_{ij}^2(a_{ij}(t,u,x)u) - \partial_i(b_i(t,u,x)u) \tag{1}$$

by constructing nonlinear Markovian families $(\mathbb{P}_{s,\zeta})_{(s,\zeta)}$ whose one-dimensional time marginals have densities which solve such a PDE and, thereby, to provide a probabilistic representation for solutions to (1). In a later paper, McKean [27] also studied the connection to distribution-dependent SDEs (DDSDEs, also called *McKean–Vlasov equations of Nemytskii-type*) of the form

$$dX_t = b(t,u(t,X_t),X_t)dt + \sigma(t,u(t,X_t),X_t)dB_t, \quad u(t,\cdot)dx = \mathcal{L}_{X_t}, \tag{2}$$

($\mathcal{L}_{X_t}$ = distribution of X_t). If (2) is (probabilistically) weakly well-posed, then so is the associated nonlinear FPKE (1) (in the distributional sense), and the path laws of the unique weak solutions to (2) constitute a nonlinear Markov process in the sense of McKean. However, at the time of [28, 27], the connection between DDSDEs and nonlinear FPKEs of type (1) had not been studied extensively, and well-posedness results for either equation were rare, in particular for arbitrary probability measures as initial data.

In addition to special examples, such as discrete state space models from kinetic gas theory and DDSDEs with unit σ and $b(t,u(t,x),x) = \int_{\mathbb{R}}(y-x)u(t,y)dy$, he proposed to study very interesting models of type (1), such as Burgers' equation and the classical porous media equation in dimension $d = 1$. But at the time it was unclear whether these equations fit his notion of nonlinear Markov process. Though to the best of our knowledge he did not go beyond such suggestions in [28] or in subsequent works, he was clearly envisioning a much larger theory.

Goals of our paper. The aim of this paper is to realize and further develop McKean's suggestions on a notion of nonlinear Markov processes and the underlying theory, oriented by large new classes of examples associated with DDSDEs and nonlinear FPKEs, and to pave the way for their analysis. In particular, we contribute to McKean's idea of interpreting solutions to nonlinear FPKEs as one-dimensional time marginals of solution processes to DDSDEs. Our final goal is to develop an as rich theory as in the linear case, with the (linear) Markov process given by Brownian motion and its connection to the heat equation as its most prominent example. We would like to point out here that in the nonlinear case a corresponding example is given by the porous media equation and a nonlinear analogue of Brownian motion, explained in part (iv) of Section 4.2.

Our definition of a nonlinear Markov process is in the spirit of [28], but indeed more general since we allow a possibly restricted class $\mathcal{P}_0 \subseteq \mathcal{P}$ of initial data as well as time-inhomogeneous cases, see

Definition 2.1 for details. This is of high relevance since then the theory can be applied to many more examples in which equation (1) can only be solved for initial measures from a restricted class $\mathcal{P}_0$. Furthermore, we show that with our definition the path laws $\mathbb{P}_{s,\zeta}$ of nonlinear Markov processes are still uniquely determined by a family of one-dimensional time marginals, more precisely by the one-dimensional time marginals of the regular conditional probabilities of $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\,\cdot\,|\sigma(\pi_r^r)]$, $s \leq r$, cf. (4) and Proposition 2.3.

In our approach a key role in the entire theory is played by the *flow-property* for the one-dimensional time-marginals $\mu_t^{s,\zeta}$ of $\mathbb{P}_{s,\zeta}$, i.e.

$$\mu_t^{s,\zeta} \in \mathcal{P}_0, \quad \mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall\, 0 \leq s \leq r \leq t,\, \zeta \in \mathcal{P}_0. \tag{3}$$

This is one main difference from the classical linear theory, where (3) is valid as well, but much more attention is usually paid to the (a priori more restrictive) *Chapman–Kolmogorov equations*, i.e.

$$\mu_t^{s,\delta_x} = \int_{\mathbb{R}^d} \mu_t^{r,\delta_y}\, d\mu_r^{s,\delta_x}(y), \quad \forall\, 0 \leq s \leq r \leq t, x \in \mathbb{R}^d.$$

Unsurprisingly, the latter usually does not hold in our nonlinear case. In the linear theory the Chapman-Kolmogorov equations follow from the convexity of the class of all solutions to the linear FPKE. This convexity is clearly lost for nonlinear FPKEs. McKean, of course, also realized that (3) is satisfied for his notion of nonlinear Markov processes, but since for his specific examples mentioned above the nonlinear Markov property could be read off from explicitly given path laws, he did not put the notion of flow in the center of his theory.

For us it was extremely helpful to realize that as long as the flow property holds for solutions $u^{s,\zeta}$ to a nonlinear FPKE as in (1), one can construct a nonlinear Markov process with one-dimensional time marginals $u^{s,\zeta}(t,\cdot)dx$ without requiring that $u^{s,\zeta}$ are unique solutions to (1) in any sense. Indeed, it suffices to have a very mild restricted uniqueness property on the level of the *linearized* FPKE corresponding to (1), see Theorem 3.3 for details. The resulting nonlinear Markov process consists of path laws of weak solutions to the associated DDSDE. Moreover, if one additionally knows that the $u^{s,\zeta}$ are unique (in a subclass of solutions), then the associated Markov process is given by path laws of the (in a corresponding subclass) unique weak solutions to the DDSDE.

We consider this result, that no uniqueness is required on the level of the nonlinear PDE, as one of the main contributions of this paper, since it allows us to cover large classes of examples of nonlinear FPKEs whose solutions thus turn out to be the one-dimensional time marginals of an associated nonlinear Markov process. Among these are, apart from Burgers' equation, generalized porous media equations with drift (in particular including *nonlinear distorted Brownian motion*, see [5, 32, 9], but also equations where in the latter the Laplacian is replaced by a fractional Laplacian so that the corresponding DDSDEs have Lévy noise, see Section 4.2 (i)-(iii), as well as [10] and [33]. As already mentioned, we also prove that the classical Barenblatt solutions to the porous media equation on $\mathbb{R}^d$, $d \geq 1$, are one-dimensional time marginal densities of a nonlinear Markov process, cf. Section 4.2 (iv).

In the spirit of McKean's program, our focus is on nonlinear FPKEs of Nemytskii-type (1) and the associated DDSDEs (2), but our main result and our examples also cover nonlinear Markov processes with probability measures as one-dimensional time marginals, arising as solutions to more general nonlinear FPKEs than (1), namely to equations of type (FPE), as defined in Section 3.1. In these cases, the coefficients of the associated DDSDE do not depend pointwise on the one-dimensional time marginal density (which in general need not exist), but on the one-dimensional time marginal as a measure, see (DDSDE).

For such cases, it is usually assumed that the coefficients are weakly continuous in their measure variable or even Lipschitz continuous, e.g. with respect to a Wasserstein-distance, in order to obtain well-posedness for the nonlinear FPKE and its linearized version. In such well-posed situations, the nonlinear Markov property follows as a consequence of this uniqueness. More precisely, in this case the corresponding nonlinear and linearized martingale problems are well-posed (see [35]), which implies the linear Markov property for the unique solutions to each linearized DDSDE, from which also the nonlinear Markov property immediately follows, see Section 4.1. However, we stress that for our main examples of Nemytskii-type, i.e. for equations as in (1), the coefficients are not continuous in their measure variable with respect to the weak topology on $\mathcal{P}$, but only measurable with respect to the Borel σ-algebra of the latter (see the beginning of Section 4.2). In these cases, the nonlinear FPKE is usually not (uniquely) solvable for all initial probability measures.

Concerning the related literature, we would like to mention the very elaborate book by Kolokoltsov [25], in which nonlinear FPKEs with coefficients with Lipschitz continuous dependence in their measure variable (e.g. with respect to a Wasserstein distance) and their relation to distribution-dependent stochastic equations are studied. The book contains several well-posedness results for nonlinear FPKEs and its associated linearized equations (i.e. equations of type (FPE) and (ℓFPE), respectively, see Section (3.1) below), possibly including an additional jump operator. For such cases, the author associates with these well-posed equations a family of (linear) Markov processes in the following way: If for each initial datum $\zeta \in \mathcal{P}$ the linearized FPKE, obtained by fixing in the measure component of the coefficients the unique solution curve μ^ζ to the nonlinear FPKE with initial datum ζ, is well-posed, then as mentioned above, there is a unique (linear) Markov process related to each such linearized equation. More precisely, it is given by the path laws of the unique weak solutions to the stochastic equation associated with this linearized FPKE. The author defines this family of linear Markov processes as a nonlinear Markov process.

As mentioned before, all our main examples in the present work are of Nemytskii-type (1), and hence fail to satisfy the assumptions in [25]. Our notion of nonlinear Markov processes, however, is sufficiently general to also cover these Nemytskii cases and, hence, applies to a much larger class of examples than those treated in [25].

This paper was strongly motivated by the results in [32], where (among other results) it was shown that uniqueness (in a restricted class) of distributional solutions to nonlinear FPKEs and their linearizations implies weak uniqueness (in a restricted class) of the associated DDSDEs, and as a consequence, the nonlinear Markov property of the path laws of the weak solutions to the latter was shown (see [32, Cor.4.6]). In comparison with this, our main result, Theorem 3.3, does not require uniqueness of distributional solutions to the nonlinear FPKE, but only the existence of a solution flow. This enhances the realm of applications substantially (see Chapter 4 below) and, in particular, covers the so-called *nonlinear distorted Brownian motion* (NLDBM), which was already studied as a main example in [32], thus identifying NLDBM as a nonlinear Markov process in the sense of our Definition 2.1 (see Section 4.2 (i)).

We are not aware of any other paper on nonlinear Markov processes realizing McKean's program after [27, 28] in the Nemytskii-case, i.e. for PDEs as in (1). However, if one restricts oneself to the part of finding a probabilistic representation for the solutions of (1) as one-dimensional time marginal laws of a stochastic process without proving that it is a nonlinear Markov process, there are several papers on this in the literature, namely e.g. [12, 4, 14, 3, 24, 26, 16]. Here we would like to draw special attention to the pioneering work [13], which treats the classical porous media equation on $\mathbb{R}^1$ on the basis of very nice original ideas and without using the much later discovered technique from [7, 8] (see also [6]), based on the application of the already mentioned superposition principle to the associated linearized porous media equation. In the present paper we include an application of our main result to the case of $\mathbb{R}^d$ for all $d \geq 1$, thus, in particular, generalizing the results in [13] to all dimensions, see Section 4.2 (iv).

The structure of this paper is as follows. Section 2 contains our definition of nonlinear Markov processes, Proposition 2.3, which shows how Markovian path laws are uniquely determined by suitable one-dimensional time marginals, as well as a comparison to classical (linear) Markov processes in Subsection 2.1. In Section 3 we formulate and prove our main result, which is Theorem 3.3. Section 4 contains several applications to our main result, i.e. we associate nonlinear Markov processes to several important nonlinear FPKEs and McKean–Vlasov equations.

Notation. The Euclidean norm and inner product on $\mathbb{R}^d$ are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, and we set $\mathbb{R}_+ := [0, \infty)$. For topological spaces X and Y, $C(X, Y)$ denotes the space of continuous functions

$f : X \to Y$, $\mathcal{B}(X)$ the Borel σ-algebra on X, and $\mathcal{P}(X)$ the space of all probability measures on $\mathcal{B}(X)$, equipped with the weak topology, i.e. the initial topology of the maps $\mathcal{P}(X) \ni \mu \mapsto \int f d\mu$, for all bounded $f \in C(X, \mathbb{R})$. For $X = \mathbb{R}^d$, we write $\mathcal{P}$ instead of $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}_a$ for the space of measures $\mu \in \mathcal{P}$, which are absolutely continuous with respect to Lebesgue measure dx, i.e. $\mu \ll dx$. If $\mu \ll \nu$ and $\nu \ll \mu$, we write $\mu \sim \nu$. We let δ_x denote the Dirac measure in $x \in \mathbb{R}^d$.

Let $\mathcal{B}_b^{(+)}(\mathbb{R}^d)$ denote the space of bounded (non-negative) Borel functions $f : \mathbb{R}^d \to \mathbb{R}$, and $C_b(\mathbb{R}^d)$ its subspace of continuous functions. For $m \in \mathbb{N}$, $C^m_{(c)}(\mathbb{R}^d)$ are the spaces of (compactly supported) continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with continuous partial derivatives up to order m. We use the standard notation $L^p(\mathbb{R}^d; \mathbb{R}^m) = \{g : \mathbb{R}^d \to \mathbb{R}^m \text{ Borel}, |g|_{L^p} < \infty\}$ with the usual L^p-norms $|\cdot|_p$ for $p \in [1, \infty]$. The corresponding local spaces are denoted $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$. For $m = 1$, we write $L^p(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d; \mathbb{R})$.

The path space $C([s, \infty), \mathbb{R})$ with the topology of locally uniform convergence is denoted by Ω_s with Borel σ-algebra $\mathcal{B}(\Omega_s) = \sigma(\pi^s_\tau, \tau \geq s)$, where $\pi^s_t, t \geq s$, are the usual projections on Ω_s. We also use the notation $\mathcal{F}_{s,r} := \sigma(\pi^s_\tau, s \leq \tau \leq r)$ and $\Pi^s_r : \Omega_s \to \Omega_r$, $\Pi^s_r : w \mapsto w_{|[r,\infty)}$ for $s \leq r$.

2 Nonlinear Markov processes: definition and comparison to linear case

Definition 2.1. Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A *nonlinear Markov process* is a family $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ of probability measures $\mathbb{P}_{s,\zeta}$ on $\mathcal{B}(\Omega_s)$ such that for all $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$:

(i) The marginals $\mathbb{P}_{s,\zeta} \circ (\pi^s_t)^{-1} =: \mu^{s,\zeta}_t$ belong to $\mathcal{P}_0$.

(ii) The *nonlinear Markov property*

$$\mathbb{P}_{s,\zeta}(\pi^s_t \in A | \mathcal{F}_{s,r})(\cdot) = p_{(s,\zeta),(r,\pi^s_r(\cdot))}(\pi^r_t \in A) \quad \mathbb{P}_{s,\zeta} - \text{a.s.} \tag{MP}$$

holds, where $\{p_{(s,\zeta),(r,y)}\}_{y \in \mathbb{R}^d}$ is a regular conditional probability kernel of $\mathbb{P}_{r,\mu^{s,\zeta}_r}[\,\cdot\,|\sigma(\pi^r_r)]$ (i.e. in particular $p_{(s,\zeta),(r,y)} \in \mathcal{P}(\Omega_r)$ and $p_{(s,\zeta),(r,y)}(\pi^r_r = y) = 1$).

Remark 2.2. *(i) By a repeated monotone class-argument,* (MP) *is equivalent to: For any bounded $G : \Omega_s \to \mathbb{R}$, which is $\sigma(\pi^s_\tau, \tau \geq r)$-measurable*

$$\mathbb{E}_{s,\zeta}\big[G | \mathcal{F}_{s,r}\big](\cdot) = p_{(s,\zeta),(r,\pi^s_r(\cdot))}\big(G\big) \quad \mathbb{P}_{s,\zeta} - a.s.$$

(ii) The one-dimensional time marginals $\mu^{s,\zeta}_t = \mathbb{P}_{s,\zeta} \circ (\pi^s_t)^{-1}$ of a nonlinear Markov process satisfy the flow property*, i.e.*

$$\mu^{s,\zeta}_t = \mu^{r,\mu^{s,\zeta}_r}_t, \quad \forall\, 0 \leq s \leq r \leq t, \zeta \in \mathcal{P}_0.$$

Indeed, for all $A \in \mathcal{B}(\mathbb{R}^d)$:

$$\mu^{s,\zeta}_t(A) = \mathbb{E}_{s,\zeta}\big[\mathbb{P}_{s,\zeta}(\pi^s_t \in A | \mathcal{F}_{s,r})\big] = \mathbb{E}_{s,\zeta}\big[p_{(s,\zeta),(r,\pi^s_r)}(\pi^r_t \in A)\big] = \mathbb{P}_{r,\mu^{s,\zeta}_r}\big(\pi^r_t \in A\big) = \mu^{r,\mu^{s,\zeta}_r}_t(A).$$

The following proposition shows that the finite-dimensional distributions of the path measures $\mathbb{P}_{s,\zeta}$ of a nonlinear Markov process (and hence the path measures itself) are uniquely determined by the family of one-dimensional time marginals $p^{s,\zeta}_{r,t}(x, dz)$, $s \leq r, x \in \mathbb{R}^d$, introduced in (4).

Proposition 2.3. *Let $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ be a nonlinear Markov process. For $\zeta \in \mathcal{P}_0$ and $0 \leq s \leq r \leq t$, define $p^{s,\zeta}_{r,t}(x, dz) \in \mathcal{P}$ by*

$$p^{s,\zeta}_{r,t}(x, dz) := p_{(s,\zeta),(r,x)} \circ (\pi^r_t)^{-1}(dz)\big(= \mathbb{P}_{r,\mu^{s,\zeta}_r}[\,\cdot\,|\pi^r_r = x] \circ (\pi^r_t)^{-1}\big), \tag{4}$$

which is uniquely determined for $\mu_r^{s,\zeta}$-a.e. $x \in \mathbb{R}^d$. Then for $n \in \mathbb{N}_0$, $f \in \mathcal{B}_b((\mathbb{R}^d)^n)$ and $s \le t_0 < \cdots < t_n$:

$$
\begin{aligned}
&\mathbb{E}_{s,\zeta}[f(\pi^s_{t_0},\dots,\pi^s_{t_n})]\\
&\quad= \int_{\mathbb{R}^d}\Big(\cdots\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} f(x_0\dots,x_n)\,p^{s,\zeta}_{t_{n-1},t_n}(x_{n-1},dx_n)\Big)p^{s,\zeta}_{t_{n-2},t_{n-1}}(x_{n-2},dx_{n-1})\dots\Big)d\mu^{s,\zeta}_{t_0}(x_0).
\end{aligned}
$$

Proof. Let $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$. By a monotone class argument, it suffices to consider $f = f_0 \otimes \cdots \otimes f_n$ with $f_i \in \mathcal{B}_b(\mathbb{R}^d)$. For $n = 0$,

$$\mathbb{E}_{s,\zeta}[f_0(\pi^s_{t_0})] = \int_{\mathbb{R}^d} f_0(x_0)d\mu^{s,\zeta}_{t_0}(x_0)$$

holds by definition of $\mu^{s,\zeta}_{t_0}$. For $n = 1$ and $f = f_0 \otimes f_1$, we have

$$
\begin{aligned}
\mathbb{E}_{s,\zeta}[f_0(\pi^s_{t_0})f_1(\pi^s_{t_1})] &= \mathbb{E}_{s,\zeta}\left[f_0(\pi^s_{t_0})\mathbb{E}_{s,\zeta}[f_1(\pi^s_{t_1})|\mathcal{F}_{s,t_0}]\right]\\
&= \mathbb{E}_{s,\zeta}\left[f_0(\pi^s_{t_0})p_{(s,\zeta),(t_0,\pi^s_{t_0})}(f_1(\pi^{t_0}_{t_1}))\right]\\
&= \int_{\mathbb{R}^d} f_0(x_0)p_{(s,\zeta),(t_0,x_0)}\big(f_1(\pi^{t_0}_{t_1})\big)d\mu^{s,\zeta}_{t_0}(x_0)\\
&= \int_{\mathbb{R}^d} f_0(x_0)\Big(\int f_1(x_1)p^{s,\zeta}_{t_0,t_1}(x_0,dx_1)\Big)d\mu^{s,\zeta}_{t_0}(x_0),
\end{aligned}
$$

where we used the nonlinear Markov property (MP) and Remark (2.2) (i) for the second equality. Now the claim follows by induction over n. □

Remark 2.4. *Hence, similarly as in the linear case, the path measure $\mathbb{P}_{s,\zeta}$ is reconstructed from one-dimensional time marginals $p^{s,\zeta}_{r,t}(x,\cdot)$, $t \ge r \ge s \ge 0, x \in \mathbb{R}^d$. However, the nonlinear nature becomes evident via the fact that even in the case $\mathcal{P}_0 = \mathcal{P}$ it is usually* not *true that $p^{s,\zeta}_{r,t}(x,\cdot) = \mathbb{P}_{r,\delta_x} \circ (\pi^r_t)^{-1}$, see Remark 4.1.*

2.1 Comparison with linear Markov processes

Here we comment on the connection to classical "linear" Markov processes. More precisely, we show that our Definition 2.1 is a proper extension of the linear definition. Afterwards we shortly recall how classical Markov processes typically arise from linear diffusion operators.

Let $(\mathbb{P}_{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ be a (classical, i.e. linear) normal and possibly time-inhomogeneous *Markov process*, i.e. for each (s,x), $\mathbb{P}_{s,x}$ is a probability measure on $\mathcal{B}(\Omega_s)$ with $\mathbb{P}_{s,x} \circ (\pi^s_s)^{-1} = \delta_x$, such that

(i) $x \mapsto \mathbb{P}_{s,x}(B)$ is measurable for all $s \ge 0$ and $B \in \mathcal{B}(\Omega_s)$,

(ii) the (classical) Markov property holds, i.e. for all $s \le r \le t$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}_{s,x}(\pi^s_t \in A|\mathcal{F}_{s,r})(\cdot) = \mathbb{P}_{r,\pi^s_r(\cdot)}(\pi^r_t \in A) \quad \mathbb{P}_{s,x} - \text{a.s.}$$

It is natural to set $\mathbb{P}_{s,\zeta} := \int_{\mathbb{R}^d} \mathbb{P}_{s,y}\, d\zeta(y)$ for any $s \ge 0$ and non-Dirac $\zeta \in \mathcal{P}$. Then $\mathbb{P}_{s,\zeta} \circ (\pi^s_s)^{-1} = \zeta$.

Proposition 2.5. *$(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}}$ is a nonlinear Markov process with $\mathcal{P}_0 = \mathcal{P}$.*

Proof. We have $\mathbb{P}_{r,\mu^{s,\zeta}_r} = \int_{\mathbb{R}^d} \mathbb{P}_{r,y}\, d\mu^{s,\zeta}_r(y)$ and $\mathbb{P}_{r,y}$ is concentrated on $\{\pi^r_r = y\}$. Hence $\{\mathbb{P}_{r,y}\}_{y\in\mathbb{R}^d}$ is a regular conditional probability kernel of $\mathbb{P}_{r,\mu^{s,\zeta}_r}[\,\cdot\,|\sigma(\pi^r_r)]$, and thus (MP) holds with $p_{(s,\zeta),(r,\pi^s_r(\cdot))} = \mathbb{P}_{r,\pi^s_r(\cdot)}$. □

We stress again that for nonlinear Markov processes, $p_{(s,\zeta),(r,\pi^s_r(\cdot))} = \mathbb{P}_{r,\pi^s_r(\cdot)}$ is usually not true, even if $\mathcal{P}_0 = \mathcal{P}$.

Remark 2.6. *The transition probabilities* $\mu_t^{s,x} := \mu_t^{s,\delta_x} = \mathbb{P}_{s,x} \circ (\pi_t^s)^{-1}$ *of a linear Markov process satisfy the* Chapman-Kolmogorov equations

$$\mu_t^{s,x}(A) = \int_{\mathbb{R}^d} \mu_t^{r,y}(A)\, d\mu_r^{s,x}(y), \quad \forall A \in \mathcal{B}(\mathbb{R}^d). \tag{CK}$$

They also satisfy the flow property, since

$$\mu_t^{s,\zeta}(A) = \mathbb{E}_{s,\zeta}\left[\mathbb{P}_{s,\zeta}(\pi_t^s \in A|\mathcal{F}_{s,r})\right] = \mathbb{E}_{s,\zeta}\left[\mathbb{P}_{r,\pi_r^s}(\pi_t^r \in A)\right] = \int \mathbb{P}_{r,y}(\pi_t^r \in A)\, d\mu_r^{s,\zeta}(y) = \mu_t^{r,\mu_r^{s,\zeta}}(A).$$

(of course this also follows directly by Proposition (2.5) *and Remark 2.2 (ii)). To us it seems that the Chapman-Kolmogorov equations do not have a natural counterpart in the nonlinear case, since even in the rather special case* $\mathcal{P}_0 = \mathcal{P}$, (CK) *will usually not be satisfied. Indeed,* (CK) *is not satisfied when the curves* $(\mu_t^{s,x})_{t\geq s}$ *are the unique solutions to a nonlinear FPKE, which is the case for all our examples in Section 4. From our point of view, with regard to nonlinear Markov processes, the flow property is the key property on the level of the one-dimensional time marginals, see also our main result, Theorem 3.3.*

Linear Markov processes from linear diffusion operators. Before we continue with the nonlinear case, let us recall how linear Markov processes typically arise from linear diffusion operators L of type

$$L_t\varphi(x) = a_{ij}(t,x)\partial_{ij}^2\varphi(x) + b_i(t,x)\partial_i\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

where a_{ij} and b_i, $1 \leq i,j \leq d$, are Borel coefficients on $\mathbb{R}_+ \times \mathbb{R}^d$ such that $a := (a_{ij})_{i,j\leq d}$ is symmetric and non-negative definite. The stochastic differential equation (SDE) associated with L is

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad \mathcal{L}(X_s) = \delta_x, \qquad t \geq s,$$

where $\sigma = (\sigma_{ik})_{i\leq d, k\leq m}$, $m \geq 1$, is a square root of a so that $1/2\sigma\sigma^T = a$, and B is a Brownian motion in $\mathbb{R}^m$. Probabilistic weak solutions to this SDE are equivalent to solutions to the martingale problem for L (which we do not recall here, but see [35]). By Itô's formula and the Ambrosio-Figalli-Trevisan *superposition principle* (see for instance [36]), $X^{s,x}$ is a weak solution to the SDE if and only if its one-dimensional time marginal curve $t \mapsto \mu_t^{s,x}$ solves the Cauchy problem of the FPKE

$$\partial_t\mu_t = L_t^*\mu_t, \quad t \geq s, \quad \mu_s = \delta_x, \tag{5}$$

in the distributional sense (L^* denotes the formal dual operator of L) and satisfies a mild integrability condition with respect to a and b. It is well-known [35, 36] that if either the Cauchy problem for the martingale problem, the SDE or the FPKE is well-posed for every initial datum (s,x), then all three problems are well-posed. In this case, the path laws $\mathbb{P}_{s,x}$ of the unique weak solutions to the SDE (which are exactly the martingale problem solutions) constitute a linear Markov process.

The question whether a Markov process exists in ill-posed regimes is more delicate, but positive answers are known: if no uniqueness is known at all, at least for continuous bounded coefficients one can select one particular martingale solution $\mathbb{P}_{s,x}$ for each initial datum such that the selected family constitutes a Markov process [35, Ch.12]. If partial well-posedness holds in the sense that for suitably rich subclasses of initial conditions $\mathcal{R}_s \subseteq \mathcal{P}$, the Cauchy problem (5) from initial time s has a unique solution in a suitable subclass $\mathcal{R}_{[s,\infty)}$, then for all $\zeta \in \mathcal{R}_s$ the corresponding martingale problem has a unique solution $\mathbb{P}_{s,\zeta}$ with one-dimensional time marginals in $\mathcal{R}_{[s,\infty)}$, and the path measures $\mathbb{P}_{s,\zeta}$ induce a Markov process via disintegration [36].

3 Construction of nonlinear Markov processes: main result

In this section we present our main result on the construction of nonlinear Markov processes from prescribed one-dimensional time marginals given as solution curves to nonlinear FPKEs. First, let us introduce the analytic and stochastic equations associated with a nonlinear diffusion operator.

3.1 Equations and solutions

Let $a_{ij}, b_i : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \to \mathbb{R}$, $1 \leq i, j \leq d$, be Borel maps (with respect to the weak topology on $\mathcal{P}$), set $a := (a_{ij})_{i,j\leq d}, b := (b_i)_{i\leq d}$, and consider for $(t, \mu) \in \mathbb{R}_+ \times \mathcal{P}$ the diffusion operator

$$L_{t,\mu}\varphi(x) = a_{ij}(t, \mu, x)\partial^2_{ij}\varphi(x) + b_i(t, \mu, x)\partial_i\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d). \tag{6}$$

We assume $a = \frac{1}{2}\sigma\sigma^T$ for some Borel map $\sigma : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \to \mathbb{R}^{d\times m}, m \in \mathbb{N}$, with coefficients σ_{ik}, and we denote by B an $\mathbb{R}^m$-standard Brownian motion. For a random variable Z, let $\mathcal{L}_Z$ denote its distribution. In particular, for a process X with paths in Ω_s, $\mathcal{L}_X$ denotes its path law on Ω_s.

As in the linear case, three nonlinear problems are associated with L and an initial datum $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$. On the level of path measures the distribution-dependent stochastic differential equation (also *McKean-Vlasov equation*)

$$dX_t = b(t, \mathcal{L}_{X_t}, X_t)dt + \sigma(t, \mathcal{L}_{X_t}, X_t)dB_t, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta, \tag{DDSDE}$$

and the nonlinear martingale problem, which consists of finding $P \in \mathcal{P}(\Omega_s)$ such that

$$P \circ (\pi^s_s)^{-1} = \zeta \text{ and } \varphi(\pi^s_t) - \int_s^t L_{r,\mathcal{L}_{\pi^s_r}}\varphi(\pi^s_r)\, dr \text{ is a martingale } \forall \varphi \in C^2_c(\mathbb{R}^d) \tag{MGP}$$

on $[s, \infty)$ with respect to $(\mathcal{F}_{s,t})_{t\geq s}$. On state space level, one has the nonlinear FPKE

$$\partial_t\mu_t = L^*_{t,\mu_t}\mu_t, \quad t \geq s, \quad \mu_s = \zeta, \tag{FPE}$$

where $L^*_{t,\zeta}$ denotes the formal dual operator of $L_{t,\zeta}$. For an initial datum (s, η), from (FPE) one obtains a *linear* FPKE by fixing a weakly continuous curve $t \mapsto \mu_t \in \mathcal{P}$ in a and b:

$$\partial_t\nu_t = L^*_{t,\mu_t}\nu_t, \quad t \geq s, \quad \nu_s = \eta. \tag{ℓFPE}$$

Analogously, one obtains a (non-distribution dependent) SDE from (DDSDE):

$$dX_t = b(t, \mu_t, X_t)dt + \sigma(t, \mu_t, X_t)dB_t, \quad t \geq s, \quad \mathcal{L}_{X_s} = \eta, \tag{ℓDDSDE}$$

without any a priori relation between X_t and μ_t, and a linear martingale problem, namely to find $P \in \mathcal{P}(\Omega_s)$ such that

$$P \circ (\pi^s_s)^{-1} = \eta \text{ and } \varphi(\pi^s_t) - \int_s^t L_{r,\mu_r}\varphi(\pi^s_r)\, dr \text{ is a martingale } \forall \varphi \in C^2_c(\mathbb{R}^d) \tag{ℓMGP}$$

on $[s, \infty)$ with respect to $(\mathcal{F}_{s,t})_{t\geq s}$.

Definition 3.1. Let $a_{ij}, b_i, \sigma_{ik} : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \to \mathbb{R}$ be as above, $s \geq 0$ and $\zeta, \eta \in \mathcal{P}$.

(a) **Solutions to** (DDSDE)**,** (MGP)**,** (FPE) **and linear counterparts.**

(i) A stochastic process $(X_t)_{t\geq s}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq s}, \mathbb{P}, B)$ with an $\mathbb{R}^m$-valued $(\mathcal{F}_t)$-adapted Brownian motion B is a *(probabilistic weak) solution to* (DDSDE) from (s, ζ), if it is $(\mathcal{F}_t)$-adapted, $\mathcal{L}_{X_s} = \zeta$,

$$\mathbb{E}\left[\int_s^T |b(t, \mathcal{L}_{X_t}, X_t)| + |\sigma(t, \mathcal{L}_{X_t}, X_t)|^2 dt\right] < \infty, \quad \forall T > s, \tag{7}$$

and $\mathbb{P}$-a.s.

$$X_t - X_s = \int_s^t b(r, \mathcal{L}_{X_r}, X_r)dr + \int_s^t \sigma(r, \mathcal{L}_{X_r}, X_r)dB_r, \quad \forall t \geq s. \tag{8}$$

(ii) A process X as in (i) is a solution to (ℓDDSDE) from (s,η), if $\mathcal{L}_{X_s} = \eta$ and (7) and (8) hold with μ_t instead of $\mathcal{L}_{X_t}$.

(iii) $P \in \mathcal{P}(\Omega_s)$ is a *solution to* (MGP) from (s,ζ), if

$$\mathbb{E}_P\left[\int_s^T |b(t,\mathcal{L}_{\pi_t^s},\pi_t^s)| + |a(t,\mathcal{L}_{\pi_t^s},\pi_t^s)|\,dt\right] < \infty, \quad \forall T > s, \tag{9}$$

and P solves (MGP).

(iv) Similarly, P is a solution to (ℓMGP) from (s,η), if (9) holds with μ_t in place of $\mathcal{L}_{\pi_t^s}$ and P solves (ℓMGP).

(v) A weakly continuous curve $\mu : t \mapsto \mu_t$ in $\mathcal{P}$ is a *solution to* (FPE) from (s,ζ), if

$$\int_s^T \int_{\mathbb{R}^d} |a(t,\mu_t,x)| + |b(t,\mu_t,x)|\,d\mu_t(x)dt < \infty, \quad \forall T > s, \tag{10}$$

and for all $\varphi \in C_c^2$

$$\int_{\mathbb{R}^d} \varphi\,d\mu_t - \int_{\mathbb{R}^d} \varphi\,d\zeta = \int_s^t \int_{\mathbb{R}^d} L_{\tau,\mu_\tau}\varphi\,d\mu_\tau d\tau, \quad \forall t \geq s.$$

For $\mathcal{P}_0 \subseteq \mathcal{P}$, a family $(\mu_t^{s,\zeta})_{t\geq s}$, $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, of solutions to (FPE) with initial datum $\mu_s^{s,\zeta} = \zeta$ is a *solution flow*, if it satisfies the flow property, i.e. if $\mu_t^{s,\zeta} \in \mathcal{P}_0$ for all $s \leq t, \zeta \in \mathcal{P}_0$, and

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall s \leq r \leq t, \zeta \in \mathcal{P}_0.$$

(vi) A weakly continuous curve $\nu : t \mapsto \nu_t$ in $\mathcal{P}$ is a *solution to* (ℓFPE) from (s,η), if

$$\int_s^T \int_{\mathbb{R}^d} |a(t,\mu_t,x)| + |b(t,\mu_t,x)|\,d\nu_t(x)dt < \infty, \quad \forall T > s,$$

and for all $\varphi \in C_c^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi\,d\nu_t - \int_{\mathbb{R}^d} \varphi\,d\eta = \int_s^t \int_{\mathbb{R}^d} L_{\tau,\mu_\tau}\varphi\,d\nu_\tau d\tau, \quad \forall t \geq s.$$

(b) **Uniqueness of solutions.**

(i) Solutions to (DDSDE) [resp. (ℓDDSDE)] from (s,ζ) [resp. (s,η)] are (weakly) unique, if for any two solutions X^1, X^2, $\mathcal{L}_{X_s^i} = \zeta$ [resp. $\mathcal{L}_{X_s^i} = \eta$], $i \in \{1,2\}$, implies $\mathcal{L}_{X^1} = \mathcal{L}_{X^2}$.

(ii) Solutions to (MGP) [resp. (ℓMGP)] from (s,ζ) [resp. (s,η)] are unique, if for any two solutions P^1, P^2, $P^i \circ (\pi_s^s)^{-1} = \zeta$ [resp. $P^i \circ (\pi_s^s)^{-1} = \eta$], $i \in \{1,2\}$, implies $P^1 = P^2$.

(iii) Solutions to (FPE) [resp. (ℓFPE)] from (s,ζ) [resp. (s,η)] are unique, if for any two solutions μ^1, μ^2 [resp. ν^1, ν^2], $\mu_s^i = \zeta$ [resp. $\nu_s^i = \eta$], $i \in \{1,2\}$, implies $\mu_t^1 = \mu_t^2$ [resp. $\nu_t^1 = \nu_t^2$] for all $t \geq s$.

For each of these problems, *uniqueness in a subclass of solutions* means any two solutions from a specified subclass with identical initial datum coincide.

Remark 3.2. *The following relations between solutions to the above problems are well-known. The path law P of a solution X to* (DDSDE)*, resp.* (ℓDDSDE)*, solves* (MGP)*, resp.* (ℓMGP)*, and vice versa, for a solution P to* (MGP)*, resp.* (ℓMGP)*, there exists a solution process X to* (DDSDE)*, resp.* (ℓDDSDE) *such that $P = \mathcal{L}_X$. These equivalences are in one-to-one correspondence, i.e. existence* *and uniqueness of solutions at the levels of the [linear] stochastic equation and the [linear] martingale problem are equivalent.*

By Itô's formula, a solution P to (MGP)*, resp.* (ℓMGP)*, induces a solution to the respective nonlinear or linear FPKE via its curve of one-dimensional time marginals. Conversely, by the Ambrosio-Figalli-Trevisan-superposition principle, to any solution $(\nu_t)_{t\geq s}$ to* (ℓFPE) *there is a solution to* (ℓMGP) *with marginals $(\nu_t)_{t\geq s}$, cf. [1, 17, 36]. An analogous version of this result for nonlinear equations is due to Barbu and the second author of this paper [7, 8]. Hence uniqueness for the [linear] martingale problem yields uniqueness for the [linear] FPKE. Conversely, uniqueness of* (ℓFPE) *for sufficiently many initial data implies uniqueness for* (ℓMGP)*. To obtain uniqueness of solutions to* (MGP)*, one needs not only uniqueness for* (FPE)*, but also for the associated linearized FPKE, see [11, 32].*

3.2 Main result

We want to construct nonlinear Markov processes with prescribed one-dimensional time marginals, which are given as solutions to nonlinear FPKEs. For an as mild as possible uniqueness assumption in Theorem 3.3 below, we introduce the following classes of measure-valued curves. For a $\mathcal{P}$-valued curve $\mu = (\mu_t)_{t\geq s}$ and $C > 0$ set

$$\mathcal{A}_{s,\leq}(\mu, C) := \big\{(\eta_t)_{t\geq s} \in C([s,\infty),\mathcal{P}) : \eta_t \leq C\mu_t,\ \forall t \geq s\big\} \text{ and } \mathcal{A}_{s,\leq}(\mu) := \bigcup_{C>0} \mathcal{A}_{s,\leq}(\mu, C).$$

Let $\mathcal{P}_0 \subseteq \mathcal{P}$ such that

$$\zeta \in \mathcal{P}_0, \eta \in \mathcal{P} \text{ and } \eta \leq C\zeta \text{ for some } C > 0 \implies \eta \in \mathcal{P}_0. \tag{P}$$

Theorem 3.3. *Let $\mathcal{P}_0 \subseteq \mathcal{P}$ have property* (P)*, and assume the following: $(\mu^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ is a solution flow to* (FPE)*, and for each $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, $\mu^{s,\zeta}$ is the unique solution to* (ℓFPE) *with $\mu_t^{s,\zeta}$ in place of μ_t in $\mathcal{A}_{s,\leq}(\mu^{s,\zeta})$ from (s,ζ).*
Let $\mathbb{P}_{s,\zeta}$ denote the path law of the unique weak solution to (DDSDE) *from (s,ζ) such that its one-dimensional time marginals are given by $(\mu_t^{s,\zeta})_{t\geq s}$. Then $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ is a nonlinear Markov process.*

Proof of Theorem 3.3. Since each $\mu^{s,\zeta}$ satisfies (10) and is obviously a solution to (ℓFPE) with $\mu_t^{s,\zeta}$ in place of μ_t, by [36] there exists a family $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ such that

(i) $\mathbb{P}_{s,\zeta} \in \mathcal{P}(\Omega_s)$ and $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = \mu_t^{s,\zeta}$,

(ii) $\mathbb{P}_{s,\zeta}$ solves (ℓMGP) with $\mu_t^{s,\zeta}$ in place of μ_t from (s,ζ).

Hence $\mathbb{P}_{s,\zeta}$ also solves (MGP), so there exists a weak solution $X^{s,\zeta}$ to (DDSDE) with path law $\mathbb{P}_{s,\zeta}$ and initial distribution ζ. Concerning the nonlinear Markov property, let $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$. Disintegrating $\mathbb{P}_{r,\mu_r^{s,\zeta}}$ with respect to $\mu_r^{s,\zeta}$ yields

$$\mathbb{P}_{r,\mu_r^{s,\zeta}}(\cdot) = \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)}(\cdot)d\mu_r^{s,\zeta}(y),$$

where the $\mu_r^{s,\zeta}$-almost surely determined family $p_{(s,\zeta),(r,y)}$, $y \in \mathbb{R}^d$, of Borel probability measures on Ω_r is as in Definition 2.1. By [36, Prop.2.8] and the flow property of $(\mu^{s,\zeta})_{(s,\zeta)}$, for $\mu_r^{s,\zeta}$-a.e. $y \in \mathbb{R}^d$ $p_{(s,\zeta),(r,y)}$ solves (ℓMGP) with $\mu_t^{s,\zeta}$ in place of μ_t from (r,δ_y). Hence, for any $\varrho \in \mathcal{B}_b^+(\mathbb{R}^d)$ with $\int \varrho\, d\mu_r^{s,\zeta} = 1$,

$$\mathbb{P}_\varrho := \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)}\, \varrho(y)d\mu_r^{s,\zeta}(y) \tag{11}$$

solves the same linearized martingale problem with initial datum $(r, \varrho d\mu_r^{s,\zeta})$. Let $n \in \mathbb{N}$, $s \le t_1 < \dots < t_n \le r$, $h \in \mathcal{B}_b^+((\mathbb{R}^d)^n)$ such that $a^{-1} \le h \le a$ for some $a > 1$, and let $\tilde{g} : \mathbb{R}^d \to \mathbb{R}_+$ be the $\mu_r^{s,\zeta}$-a.s. uniquely determined map such that

$$\mathbb{E}_{s,\zeta}\big[h(\pi^s_{t_1}, \dots, \pi^s_{t_n})|\sigma(\pi^s_r)\big] = \tilde{g}(\pi^s_r) \quad \mathbb{P}_{s,\zeta} - a.s.$$

Let $g := c_0\tilde{g}$, where $c_0 > 0$ is such that $\int g\, d\mu_r^{s,\zeta} = 1$, and let $\mathbb{P}_g$ be the martingale solution with g in place of ϱ in (11), with initial condition $(r, gd\mu_r^{s,\zeta})$.

Also, consider $\theta : \Omega_s \to \mathbb{R}$, $\theta := c_0 h(\pi^s_{t_1}, \dots, \pi^s_{t_n})$, i.e. $\mathbb{E}_{s,\zeta}[\theta] = 1$. Set

$$\mathbb{P}^\theta := (\theta \cdot \mathbb{P}_{s,\zeta}) \circ (\Pi^s_r)^{-1}.$$

Note that $a^{-1}c_0 \le g, \theta \le ac_0$ $\mu_r^{s,\nu}$-a.s., so in particular $gd\mu_r^{s,\zeta} \sim \mu_r^{s,\zeta}$. It is easy to see that also $\mathbb{P}^\theta$ solves (ℓMGP) with $\mu_t^{s,\zeta}$ in place of μ_t from $(r, gd\mu_r^{s,\zeta})$. In particular both one-dimensional time marginal curves $(\mathbb{P}_g \circ (\pi^r_t)^{-1})_{t\ge r}$ and $(\mathbb{P}^\theta \circ (\pi^r_t)^{-1})_{t\ge r}$ solve (ℓFPE) with $\mu_t^{s,\zeta}$ in place of μ_t from $(r, gd\mu_r^{s,\zeta})$. It is straightforward to check that

$$\mathbb{P}^\theta \circ (\pi^r_t)^{-1}, \mathbb{P}_g \circ (\pi^r_t)^{-1} \le ac_0\mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall t \ge r,$$

i.e. both these one-dimensional time marginal curves belong to $\mathcal{A}_{r,\le}(\mu^{r,\mu_r^{s,\zeta}})$. Hence by the assumption and Lemma 3.6 (ii)

$$\mathbb{P}_g \circ (\pi^r_t)^{-1} = \mathbb{P}^\theta \circ (\pi^r_t)^{-1}, \quad \forall t \ge r,$$

and therefore for $t \ge r$ and $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}\mathbb{E}_{s,\zeta}\big[h(\pi^s_{t_1}, \dots, \pi^s_{t_n})\mathbb{1}_{\pi^s_t \in A}\big] &= c_0^{-1}\mathbb{P}^\theta \circ (\pi^r_t)^{-1}(A) = c_0^{-1}\mathbb{P}_g \circ (\pi^r_t)^{-1}(A) \\ &= c_0^{-1}\int_{\mathbb{R}^d} p_{(s,\zeta),(r,\pi^s_r(\omega))}(\pi^r_t \in A)g(\pi^s_r(\omega))\, d\mathbb{P}_{s,\zeta}(\omega) \\ &= \int_{\mathbb{R}^d} p_{(s,\zeta),(r,\pi^s_r(\omega))}(\pi^r_t \in A)h(\pi^s_{t_1}(\omega), \dots, \pi^s_{t_n}(\omega))\, d\mathbb{P}_{s,\zeta}(\omega).\end{aligned}$$

Here we used the $\sigma(\pi^s_r)$-measurability of $\Omega_s \ni \omega \mapsto p_{(s,\zeta),(r,\pi^s_r(\omega))}(\pi^r_t \in A)$ for the final equality. By a monotone class-argument, the nonlinear Markov property follows.

It remains to prove the uniqueness claim about $\mathbb{P}_{s,\zeta}$. But this follows from Remark 3.5 below and the equivalence of solutions to (MGP) and (DDSDE), cf. Remark 3.2. □

Since the nonlinear Markov property is obviously always fulfilled for $s = r$, the corollary below follows from the above proof. We will make use of it in the examples in Section 4 to allow singular initial data (e.g., Dirac measures) for nonlinear Markov processes.

Corollary 3.4. *Let $\mathcal{P}_0 \subseteq \mathcal{P}$ have property* (P) *and let $(\mu^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ be a solution flow such that for each $r > s$ and $\zeta \in \mathcal{P}_0$, $\mu^{s,\zeta}_{|[r,\infty)}$ is the unique solution to* (ℓFPE) *with $\mu_t^{s,\zeta}$ in place of μ_t with initial datum $(r, \mu_r^{s,\zeta})$ in $\mathcal{A}_{r,\le}(\mu^{s,\zeta}_{|[r,\infty)})$.*
Then there exists a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ such that $\mathbb{P}_{s,\zeta} \circ (\pi^s_t)^{-1} = \mu_t^{s,\zeta}$, and $\mathbb{P}_{s,\zeta}$ is the path law of a weak solution to (DDSDE) *from (s,ζ). If $\zeta = \mu_s^{\tilde{s},\eta}$ for some $0 \le \tilde{s} < s$ and $\eta \in \mathcal{P}_0$, then $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to* (DDSDE) *with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t\ge s}$.*

Remark 3.5. *In the proof of Theorem 3.3 we used Lemma 3.6 (ii) to obtain uniqueness of solutions to* (ℓFPE) *with $\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}$ in place of μ_t in the class $\mathcal{A}_{r,\le}(\mu^{r,\mu_r^{s,\zeta}})$ for any initial datum $(r, gd\mu_r^{s,\zeta})$ with bounded probability densities g bounded away from* 0*. In fact, one can even prove that for any (s,ζ) and $r \ge s$, solutions to the martingale problem* (ℓMGP) *with $\mu_t^{r,\mu_r^{s,\zeta}}$ in place of μ_t with one-dimensional* *time marginals in* $\mathcal{A}_{r,\leq}(\mu^{r,\mu_r^{s,\zeta}})$ *are unique from time* r *for any initial datum* $gd\mu_r^{s,\zeta}$ *with as* g *as above. For this, it suffices to inspect and optimize the proof of [36, Lem.2.12] and to apply Lemma 3.6 (ii). In particular, under the assumption of the above theorem,* $\mathbb{P}_{s,\zeta}$ *is the unique solution to* (MGP) *from* (s,ζ) *with one-dimensional time marginals* $\mu_t^{s,\zeta}$, $t \geq s$.

As far as we know, a transfer-of-uniqueness result as in Lemma 3.6 (ii) below has not been considered in the literature before. Hence we also state part (i), which is not used in this paper, but might be of independent interest.

Lemma 3.6. *Consider a linear FPKE, i.e.* (FPE) *with coefficients independent of the measure variable. For this equation, the following holds.*

(i) If solutions (in the sense of Definition 3.1 (v) with a, b *independent of* $\mu \in \mathcal{P}$*) are unique from* (s,ζ)*, then solutions to the same equation are also unique from* (s,η) *for any* $\eta \in \mathcal{P}$ *such that* $\eta \sim \zeta$.

(ii) If $(\nu_t^{s,\zeta})_{t\geq s}$ *is the unique solution in* $\mathcal{A}_{s,\leq}(\nu^{s,\zeta})$ *from* (s,ζ)*, then in this class solutions are also unique from any* $(s, gd\zeta)$ *with* $g \in \mathcal{B}_b^+(\mathbb{R}^d)$, $\int g\,d\zeta = 1$ *and* $\delta \leq g$ *for some* $\delta > 0$.

Proof. (i) Assume the claim is wrong, i.e. there is $\nu \sim \zeta$, $\nu \in \mathcal{P}$, such that there are two solutions $(\eta_t^i)_{t\geq s}$ with $\eta_s^i = \nu$, $i \in \{1,2\}$, and $\eta_r^1 \neq \eta_r^2$ for some $r > s$. By the superposition principle, there are probability measures P^i on $\mathcal{B}(\Omega_s)$ with one-dimensional time marginal curves $P_t^i = \eta_t^i$ which solve the corresponding linear martingale problem. Disintegrating each P^i with respect to π_s^s yields two ν-a.s. (hence also ζ-a.s.) uniquely determined families $\{\theta_y^i\}_{y\in\mathbb{R}^d}$ such that $y \mapsto \theta_y^i$ is measurable in the sense that $y \mapsto \theta_y^i(A)$ is a Borel map for any $A \in \mathcal{B}(\Omega_s)$, and ν-a.e. (hence ζ-a.e.) θ_y^i solves the linear martingale problem with initial datum (s,δ_y), see [36, Prop.2.8]. Since

$$\int_{\mathbb{R}^d} \theta_y^1 \circ (\pi_r^s)^{-1}\, d\nu(y) = P_r^1 = \eta_r^1 \neq \eta_r^2 = P_r^2 = \int_{\mathbb{R}^d} \theta_y^2 \circ (\pi_r^s)^{-1}\, d\nu(y),$$

there is $A \in \mathcal{B}(\mathbb{R}^d)$ such that the Borel set $E := \{y : \theta_y^1(\pi_r^s \in A) > \theta_y^2(\pi_r^s \in A)\}$ has strictly positive ν- (hence strictly positive ζ-) measure. For $i \in \{1,2\}$, set

$$\tilde{\theta}_y^i := \begin{cases} \theta_y^i, & y \in E, \\ \theta_y^1, & y \in E^c. \end{cases}$$

Clearly, $y \mapsto \tilde{\theta}_y^i$ is measurable and the probability measures on $\mathcal{B}(\Omega_s)$ given by $\int \tilde{\theta}_y^i\, d\zeta(y)$ both solve the linear martingale problem with initial datum (s,ζ). Hence the curves $(\nu_t^i)_{t\geq s}$,

$$\nu_t^i := \int_{\mathbb{R}^d} \tilde{\theta}_y^i \circ (\pi_t^s)^{-1}\, d\zeta(y),$$

solve the FPKE from the assertion with initial datum (s,ζ), and for A and r as above, we have

$$\begin{aligned} \nu_r^1(A) &= \int_E \theta_y^1 \circ (\pi_r^s)^{-1}(A)\, d\zeta + \int_{E^c} \theta_y^1 \circ (\pi_r^s)^{-1}(A)\, \zeta \\ &> \int_E \theta_y^2 \circ (\pi_r^s)^{-1}(A)\, d\zeta + \int_{E^c} \theta_y^1 \circ (\pi_r^s)^{-1}(A)\, d\zeta = \nu_r^2(A), \end{aligned}$$

i.e. $\nu^1 \neq \nu^2$, which contradicts the assumption.

(ii) Proceed as in (i), but assume in addition that $\nu = gd\zeta$ with g as in the assertion and $\eta^i \in \mathcal{A}_{s,\leq}(\nu^{s,\zeta})$ for $i \in \{1,2\}$. Then, denoting by $|g^{-1}|_\infty$ the L^∞-norm with respect to either of the equivalent measures ζ and $gd\zeta$ (note that we even assume $\delta \leq g$ pointwise), we have

$$\begin{aligned} \nu_r^i &\leq |g^{-1}|_\infty \left(\int_{\mathbb{R}^d} \theta_y^i \circ (\pi_r^s)^{-1}\, g(y) d\zeta(y) + \int_{\mathbb{R}^d} \theta_y^1 \circ (\pi_r^s)^{-1}\, g(y) d\zeta(y) \right) \\ &= |g^{-1}|_\infty (\eta_r^i + \eta_r^1), \end{aligned}$$

and since $\eta^i \in \mathcal{A}_{s,\leq}(\nu^{s,\zeta})$, also $\nu^i \in \mathcal{A}_{s,\leq}(\nu^{s,\zeta})$, which contradicts the assumption in (ii). □

Remark 3.7. *Due to Lemma 3.6 (ii), the assumption of Corollary 3.3 can slightly be generalized as follows:*

Let $\mathcal{P}_0 \subseteq \mathcal{P}$ have property (P)*, and assume the following: $(\mu^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ is a solution flow to* (FPE)*, and for each $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ and $r > s$, there is $\eta \in \mathcal{P}$, $\eta \sim \mu^{s,\zeta}_r$, with bounded density with respect to $\mu^{s,\zeta}_r$ bounded away from* 0*, such that there is at most one solution to* (ℓFPE) *with $\mu^{s,\zeta}_t$ in place of μ_t with initial datum (r,η) in $\mathcal{A}_{r,\leq}(\mu^{s,\zeta}_{|[r,\infty)})$.*

4 Applications and examples

4.1 Well-posed nonlinear FPKEs

Theorem 3.3 particularly applies, if for any $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ the nonlinear FPKE (FPE) has a unique solution $(\mu^{s,\zeta}_t)_{t\geq s}$ from (s,ζ) (such that $\mu^{s,\zeta}_t \in \mathcal{P}_0$ for all $t \geq s$), which is also the unique solution to (ℓFPE) with $\mu^{s,\zeta}$ in place of μ_t from (s,ζ). The first part is particularly true when the associated McKean-Vlasov equation has a unique weak solution from (s,ζ) with one-dimensional time marginals in $\mathcal{P}_0$. This kind of uniqueness was obtained in [18] for the case of Lipschitz-continuous coefficients on $\mathbb{R}^d \times \mathcal{P}_p$, where for $p \geq 1$

$$\mathcal{P}_p := \left\{\zeta \in \mathcal{P} : \int_{\mathbb{R}^d} |x|^p d\zeta(x) < \infty\right\}$$

is equipped with the p-Wasserstein distance $d_{\mathbb{W}_p}$. The following generalization to one-sided Lipschitz drift b was proven in [38]: for $p \geq 2$, (DDSDE) has a unique weak solution $X^{s,\zeta}$ from (s,ζ), $\zeta \in \mathcal{P}_p$, with $\mu^{s,\zeta}_t := \mathcal{L}_{X^{s,\zeta}_t} \in \mathcal{P}_p$, if there is a strictly positive, increasing continuous function K on $\mathbb{R}_+$ such that for all $t \geq 0, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_p$

(A1) $|\sigma(t,\mu,x) - \sigma(t,\nu,y)|^2 \leq K(t)\big(|x-y|^2 + d_{\mathbb{W}_p}(\mu,\nu)^2\big)$,

(A2) $2\langle b(t,\mu,x) - b(t,\nu,y), x-y\rangle \leq K(t)\big(|x-y|^2 + d_{\mathbb{W}_p}(\mu,\nu)\big)$,

(A3) b is bounded on bounded sets in $\mathbb{R}_+ \times \mathcal{P}_p \times \mathbb{R}^d$, $b(t,\cdot,\cdot)$ is continuous on $\mathcal{P}_p \times \mathbb{R}^d$ and

$$|b(t,0,\mu)|^p \leq K(t)\bigg(1 + \int |z|^p \, d\mu(z)\bigg).$$

Under these assumptions $\mu^{s,\zeta}$ is also the unique solution to (ℓFPE) from (s,ζ) with $\mu^{s,\zeta}_t$ in place of μ_t. Hence by Theorem 3.3, the path laws $\mathbb{P}_{s,\zeta} := \mathcal{L}_{X^{s,\zeta}}$, $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_p$, form a nonlinear Markov process with one-dimensional time marginals $\mu^{s,\zeta}_t$.

Further well-posedness results for equations of type (DDSDE) were obtained in [29, 22, 21, 34, 23]. While in these situations it has to be checked separately whether also the linear equations (ℓFPE) are well-posed, this is in general much easier than in the nonlinear case and thus often true in the above works. Then the path laws of these unique weak solutions to (DDSDE) form a nonlinear Markov process.

4.2 Restricted well-posed nonlinear FPKEs of Nemytskii-type

Without sufficient regularity in $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}$, e.g. Lipschitz-continuity with respect to a Wasserstein distance, or non-degeneracy for the diffusion coefficient, well-posedness for nonlinear (in fact, even for linear) FPKEs usually fails, but one might still be able to prove uniqueness in a subclass of solutions.

Here we present such examples, giving rise to nonlinear Markov processes in the case of singular *Nemytskii-type* coefficients, i.e. for $\mu \in \mathcal{P}_a$ with $\mu = \frac{d\mu}{dx}dx$

$$b(t,\mu,x) = \tilde{b}\bigg(t, \frac{d\mu}{dx}(x), x\bigg), \quad a(t,\mu,x) = \tilde{a}\bigg(t, \frac{d\mu}{dx}(x), x\bigg) \tag{12}$$

for Borel coefficients $\tilde{b}, \tilde{a}$ on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively. Without further mentioning, we always consider the version of $\frac{d\mu}{dx}$ which is 0 on those $x \in \mathbb{R}^d$ for which $\lim_{r\to 0} dx(B_r(0))^{-1}\mu(B_r(x))$ does not exist in $\mathbb{R}$ (here $B_r(0)$ denotes the Euclidean ball of radius $r > 0$ centered at x). By Lebesgue's differentiation theorem, the set of such x is a dx-zero set. Since $(\mu, y) \mapsto \frac{d\mu}{dx}(y)$ is $\mathcal{B}(\mathcal{P}_a)\otimes\mathcal{B}(\mathbb{R}^d)$-measurable by [20, Sect.4.2.], it follows that b and a are product Borel measurable on $\mathbb{R}_+ \times \mathcal{P}_a \times \mathbb{R}^d$ for the weak topology on $\mathcal{P}_a$.

(i) **Generalized porous media equation.** Let

$$\tilde{b}(t,z,x) = b(z)D(x), \quad \tilde{a}(t,z,x) = \frac{\beta(z)}{z}\,\mathrm{Id}, \tag{13}$$

where Id denotes the identity matrix. In this case, (FPE) is a generalized porous media equation for the densities $u(t,\cdot) = \frac{d\mu_t}{dx}(\cdot)$, namely

$$\partial_t u = \Delta\beta(u) - \mathrm{div}\,\big(Db(u)u\big). \tag{GPME}$$

A (distributional probability) solution to (GPME) from $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}$ is a map $u : (s,\infty) \to \mathcal{P}_a$ such that $t \mapsto u(t,\cdot)dx =: \mu_t$ is a weakly continuous solution to (FPE) in the sense of Definition 3.1 (v) with $\lim_{t\to s} u(t,\cdot)dx = \zeta$ in the weak topology on $\mathcal{P}$. Suppose the following assumptions hold.

(B1) $\beta(0) = 0, \beta \in C^2(\mathbb{R}), \beta' \geq 0$.

(B2) $b \in C^1(\mathbb{R}) \cap C_b(\mathbb{R}), b \geq 0$.

(B3) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d), \mathrm{div}\, D \in L^2_{\mathrm{loc}}(\mathbb{R}^d), (\mathrm{div}\, D)^- \in L^\infty(\mathbb{R}^d)$.

(B4) $\forall K \subset \mathbb{R}$ compact: $\exists \alpha_K > 0$ with $|b(r)r - b(s)s| \leq \alpha_K|\beta(r) - \beta(s)|\ \forall r, s \in K$.

Then by [2, Thm.2.2] for $\zeta \in \mathcal{P}_0 := \mathcal{P}_a \cap L^\infty$, there is a (distributional weakly continuous probability) solution $u^{s,\zeta}$ to (GPME) from (s,ζ) such that $u^{s,\zeta} \in \bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$, $u^{s,\zeta}_t \in \mathcal{P}_0$ for all $t \geq s$, and $(u^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ is a solution flow (we identify $u^{s,\zeta}$ with $(\mu^{s,\zeta}_t)_{t\geq s}$, $\mu^{s,\zeta}_t := u^{s,\zeta}(t,\cdot)dx$). Moreover, by [11, Cor.4.2], the linearized equation (ℓFPE) associated with (GPME) with initial datum (s,ζ) with $\mu^{s,\zeta}_t$ in place of μ_t has $\mu^{s,\zeta}$ as its unique (weakly continuous probability) solution in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$.

Hence Theorem 3.3 implies the existence of a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times(\mathcal{P}_a\cap L^\infty)}$ with one-dimensional time marginals $\mathbb{P}_{s,\zeta}\circ(\pi^s_t)^{-1} = u^{s,\zeta}(t,\cdot)dx$. More precisely, since by [11, Cor.3.4], $u^{s,\zeta}$ is the unique (distributional weakly continuous probability) solution to (GPME) from (s,ζ) in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$, it follows from the last part of Remark 3.5 that $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution $X^{s,\zeta}$ to the corresponding McKean-Vlasov equation

$$dX_t = b\big(u(t,X_t)\big)D(X_t))dt + \sqrt{\frac{2\beta(u(t,X_t))}{u(t,X_t)}}dB_t, \quad \mathcal{L}_{X_t} = u(t,\cdot)dx, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta \tag{14}$$

under the constraint $u \in \bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$. We stress that well-posedness for (GPME) or its linearized equations in a larger class then $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$ is not known.

The special case $\gamma_1 \leq \beta' \leq \gamma_2$ for $\gamma_i > 0$ (which already implies (B4)) and $D = -\nabla\Phi$ for

$$\Phi \in C^1(\mathbb{R}^d),\ \Phi \geq 1,\ \lim_{|x|\to\infty} \Phi(x) = \infty,\ \Phi^{-m} \in L^1(\mathbb{R}^d) \text{ for some } m \geq 2,$$

called *nonlinear distorted Brownian motion (NLDBM)*, was already studied in [32].

We continue with cases in which the set of admissible initial data $\mathcal{P}_0$ includes singular measures, e.g. the Dirac measures. For such initial data, (restricted) uniqueness of solutions to (FPE) and (ℓFPE) is difficult to obtain, in particular in the singular Nemytskii-case (although there are positive results for pure-diffusion equations, cf. [30]). However, if solutions started from singular measures are more regular after any strictly positive time, one can still obtain the nonlinear Markov property without any uniqueness for singular initial data. Let b, a be as in (12),(13), and assume the following:

(C1) Assumptions (B1)-(B4) hold. In addition assume:

(C2) $\beta'(r) \geq a|r|^{\alpha-1}$ and $|\beta(r)| \leq C|r|^\alpha$ for some $a, C > 0$ and $\alpha \geq 1$.

(C3) $D \in L^\infty \cap L^2(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} D \in L^2(\mathbb{R}^d)$, $\operatorname{div} D \geq 0$.

Then by [2, Thm.5.2], for each $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}$, there is a (weakly continuous probability) solution $u^{s,\zeta}$ to (GPME) from (s,ζ) such that $u^{s,\zeta} \in \bigcap_{r>s} L^\infty((r,\infty)\times\mathbb{R}^d)$. If $\zeta \in \mathcal{P}_a \cap L^\infty$, even $u^{s,\zeta} \in L^\infty((s,\infty)\times\mathbb{R}^d)$ and in this case, since (B1)-(B4) hold, it follows as in the first part of this example that $u^{s,\zeta}$ is the unique weakly continuous distributional probability solution from (s,ζ) in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$. Consequently, since for any $s < r$ and $\zeta \in \mathcal{P}$, both $(u_t^{s,\zeta})_{t\geq r}$ and $(u_t^{r,u_r^{s,\zeta}dx})_{t\geq r}$ solve (GPME) from $(r, u_r^{s,\zeta}dx)$ and belong to $\bigcap_{T>r} L^\infty((r,T)\times\mathbb{R}^d)$, it follows that $(\mu^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}}$, $\mu_t^{s,\zeta} := u^{s,\zeta}(t,\cdot)dx$, is a solution flow (in contrast to the previous example, this does not follow directly from the construction of $\mu^{s,\zeta}$ in [2]). Since for $s < r$ and $\zeta \in \mathcal{P}$ we have $(\mu_t^{s,\zeta})_{t\geq r} \in L^\infty((r,\infty)\times\mathbb{R}^d)$, it follows as in the previous example that the corresponding linearized equation with initial datum $(r, \mu_r^{s,\zeta})$ and with $\mu_t^{s,\zeta}$ in place of μ_t has $\mu^{s,\zeta} = \mu^{r,\mu_r^{s,\zeta}}$ as its unique solution in $L^\infty((r,\infty)\times\mathbb{R}^d)$.

Hence by Corollary 3.4 there is a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}}$ with one-dimensional time marginals $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = \mu_t^{s,\zeta}$, consisting of path laws of weak solutions to (14). For $\zeta \in \mathcal{P}_a \cap L^\infty$, $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (14) with one-dimensional time marginals in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$ (as in the previous example).

(ii) **Burgers' equation.** Consider Burgers' equation in $\mathbb{R}^1$, i.e.

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - u\frac{\partial u}{\partial x},$$

which is one of the key examples McKean hinted at in his seminal work [28]. Here we put forward his program by realizing classical solutions to Burgers' equation as one-dimensional time marginals of a nonlinear Markov process. Since we restrict attention to smooth pointwise solutions, we consider the equivalent equation

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - \frac{1}{2}\frac{\partial u^2}{\partial x} \tag{BE}$$

instead, which is a nonlinear FPKE of Nemytskii-type with coefficients

$$b(t,\mu,x) = \tilde{b}\left(\frac{d\mu}{dx}(x)\right) := \frac{1}{2}\frac{d\mu}{dx}(x) \text{ and } a(t,\mu,x) := 1. \tag{15}$$

For $(s,\zeta) \in \mathbb{R}_+ \times (\mathcal{P}_a \cap L^\infty)$, (BE) has a unique smooth pointwise (i.e. strong) solution $u^{s,\zeta}$ on (s,∞) such that $u^{s,\zeta}(t,\cdot) \to \zeta$ in $L^1(dx)$ as $t \to s$, see [19] Furthermore, $0 \leq u^{s,\zeta}(t,\cdot) \leq |\zeta|_\infty$ and

$$\int_{\mathbb{R}} u^{s,\zeta}(t,x)\,dx = \int_{\mathbb{R}} u_0(x)\,dx$$

for all $t \geq s$. Hence $u^{s,\zeta}$ is in particular a distributional probability solution with initial datum (s,ζ), and its uniqueness in the class of pointwise solutions implies that $(u^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times(\mathcal{P}_a\cap L^\infty)}$ is a solution flow for (BE). Since $u^{s,\zeta} \in L^\infty((s,\infty)\times\mathbb{R}) \cap L^1((s,T)\times\mathbb{R})$ for all $T > s$, the superposition principle [36] applies, i.e. there is a probabilistic weak solution $X^{s,\zeta}$ on (s,∞) to

$$dX_t = \frac{1}{2}u(t,X_t)dt + dB_t, \quad \mathcal{L}_{X_t} = u(t,\cdot)\,dx,, \quad \mathcal{L}_{X_s} = \zeta, \tag{16}$$

with $u = u^{s,\zeta}$. Since $u^{s,\zeta} \in L^\infty((s,\infty)\times\mathbb{R})$, by [11, Cor.4.2] the linearized equation

$$\partial_t v = \frac{\partial^2 v}{\partial^2 x} - \frac{1}{2}\frac{\partial(u^{s,\zeta}v)}{\partial x}, \quad t \geq s, \quad v(s,\cdot) = \zeta$$

has $v = u^{s,\zeta}$ as its unique weakly continuous probability solution in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R})$ for each $(s,\zeta) \in \mathbb{R}_+ \times (\mathcal{P}_a \cap L^\infty)$. More precisely, to see this one replaces $\tilde{b}$ from (15) by a bounded C^1 map which coincides with $\tilde{b}$ on $[0, \sup_{t,x} u^{s,\zeta}(t,x)]$.

Consequently, Theorem 3.3 applies and the path laws $\mathbb{P}_{s,\zeta}$ of the solutions $X^{s,\zeta}$, $(s,\zeta) \in \mathbb{R}_+ \times (\mathcal{P}_a \cap L^\infty)$, form a nonlinear Markov process, and $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (16) from (s,ζ) with time marginal densities in $\cap_{T>s} L^\infty((s,T)\times\mathbb{R})$.

(iii) **Distribution-dependent stochastic equations with Lévy noise.** For $\alpha \in (\frac{1}{2}, 1)$ and $\zeta \in \mathcal{P}_a \cap L^\infty$, consider the Cauchy problem for the fractional generalized porous media equation

$$\partial_t u = (-\Delta)^\alpha \beta(u) - \operatorname{div}\big(Db(u)u\big), \quad t \geq s, \quad u(s,\cdot) = \zeta. \tag{17}$$

Since the operator $(-\Delta)^\alpha$ is non-local, such equations are not related to diffusion operators (6). Associated with this non-local equation is the distribution-dependent SDE

$$dX_t = b\big(u(t,X_t)\big)D(X_t)dt + \left(\frac{\beta(u(t,X_{t-}))}{u(t,X_{t-})}\right)^{\frac{1}{2\alpha}} dL_t, \quad \mathcal{L}_{X_t} = u(t,\cdot)dx, \quad t \geq s, \tag{18}$$

where L is a d-dimensional isotropic $2s$-stable process with Lévy measure $dz/|z|^{d+2\alpha}$. A probabilistic weak solution X to (18) is defined as in Def.(3.1) (i) with $m = d$ and L instead of B, see [10] for details. Suppose the following assumptions hold.

(D1) $\beta \in C^\infty(\mathbb{R})$, $\beta' > 0$, $\beta(0) = 0$.

(D2) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} D \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $(\operatorname{div} D)^- \in L^\infty(\mathbb{R}^d)$.

(D3) $b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$, $b \geq 0$.

Then, by [10, Thm.2.4&Thm.3.1], for $(s,\zeta) \in \mathbb{R}_+ \times (\mathcal{P}_a \cap L^\infty)$ there is a weakly continuous distributional solution $t \mapsto u_t^{s,\zeta} \in \mathcal{P}_A \cap L^\infty$ to (17) from (s,ζ) with the following properties. $(u^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times(\mathcal{P}_a\cap L^\infty)}$ is a solution flow in $\mathcal{P}_0 = \mathcal{P}_a \cap L^\infty$, and $u^{s,\zeta}$ is the unique weakly continuous probability solution to (17) in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$. Moreover, by [10, Thm.3.2], it is also the unique solution to the linearized equation

$$\partial_t v = (-\Delta)^\alpha \left(\frac{\beta(u^{s,\zeta})}{u^{s,\zeta}} v\right) - \operatorname{div}\big(Db(u^{s,\zeta})v\big), \quad t \geq s, \quad v(s,\cdot) = \zeta, \tag{19}$$

in $\bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$. Consequently, by a non-local version of the superposition principle [33], Theorem 3.3 also applies in this non-local case and implies the existence of a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times(\mathcal{P}_a\cap L^\infty)}$, where $\mathbb{P}_{s,\zeta}$ is the path law of a càdlàg weak solution $X^{s,\zeta}$ to (18) with one-dimensional time marginals $u^{s,\zeta}(t,\cdot)dx$. By [10, Thm.4.2], $X^{s,\zeta}$ is unique under the constraint $u \in \bigcap_{T>s} L^\infty((s,T)\times\mathbb{R}^d)$.

(iv) **Barenblatt solutions to the classical PME.** For the case of the classical porous media equation

$$\partial_t u = \Delta\big(|u|^{m-1}u\big), \quad m \geq 1, \tag{20}$$

it was shown in [30] that for any initial datum $(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}$, there is a unique weakly continuous distributional probability solution $u^{s,\zeta}$ in $\bigcap_{\tau>s,T>\tau} L^\infty((\tau,T)\times\mathbb{R}^d)$. In fact, it is shown that $u^{s,\zeta}$ is even L^1-continuous on (s,∞). Clearly, the uniqueness implies the flow property of the curves $t \mapsto u^{s,\zeta}(t,\cdot)dx$. For $\zeta = \delta_{x_0}$, $u^{s,\zeta}$ is the *Barenblatt solution* (see [37]), given by

$$u^{s,\delta_{x_0}}(t,x) = (t-s)^{-\alpha}\left[\big(C - k|x-x_0|^2(t-s)^{-2\beta}\big)^+\right]^{\frac{1}{m-1}}, \quad t > s,$$

where $\alpha = \frac{d}{d(m-1)+2}, \beta = \frac{\alpha}{d}, k = \frac{\alpha(m-1)}{2md}$, $f^+ := f \vee 0$, and $C = C(m,d) > 0$ is chosen such that $\int_{\mathbb{R}^d} u^{s,\zeta}(t,\cdot)dx = 1$ for all $t > s$. The corresponding McKean-Vlasov equation is

$$dX_t = \sqrt{2u(t,X_t)^{m-1}}dB_t, \quad \mathcal{L}_{X_t} = u(t,\cdot)dx, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta. \tag{21}$$

By Corollary 3.4 there is a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}}$ consisting of path laws $\mathbb{P}_{s,\zeta}$ of weak solutions to (21) with one-dimensional time marginals given by $u^{s,\zeta}(t,\cdot)dx$. For $\zeta \in \mathcal{P}_a \cap L^\infty$, $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (21) with one-dimensional time marginals in $\bigcap_{\tau>s,T>\tau} L^\infty((\tau,T)\times\mathbb{R}^d)$. This way, the Barenblatt solutions have a probabilistic interpretation as time marginal densities of a nonlinear Markov process. In the special case $d=1$, the identification of the Barenblatt solutions as one-dimensional time marginals of a stochastic process solving (21) was already obtained in [13], but no connection to any kind of nonlinear Markov property was drawn.

Remark 4.1. *With regard to Remark 2.4 we stress that in all the above examples, even in those where $\mathcal{P}_0 = \mathcal{P}$, it is not true that $p_{(s,\zeta),(r,y)} = \mathbb{P}_{r,\delta_y}$, since for the nonlinear equation* (FPE) *the set of all solutions with a common initial condition is not convex. Hence, unlike for classical linear cases, for the nonlinear Markov processes from the previous examples one cannot expect to calculate their path laws $\mathbb{P}_{s,\zeta}$ via the one-dimensional time marginals $\mathbb{P}_{r,y}\circ(\pi^s_r)^{-1}$, but via the formula given in Proposition 2.3.*

4.3 Ill-posed nonlinear FPKEs

Finally we remark that Theorem 3.3 applies when no uniqueness for the nonlinear FPKE is known at all, but if one can select a solution flow $(\mu^{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ for (FPE), and if one can establish the required restricted uniqueness for the linearized equations with $\mu^{s,\zeta}_t$ in place of μ_t. Solution flows for ill-posed nonlinear FPKEs have been constructed in [31]. Since in principle, (restricted) uniqueness for nonlinear equations is much harder to obtain than the mild uniqueness for (ℓFPE) required in Theorem 3.3, it might be possible to construct nonlinear Markov processes in such nonlinear completely ill-posed cases. This will be a point of further study in the future.

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