



# Recent Progress on Multi-bubble Blow-ups and Multi-solitons to (Stochastic) Focusing Nonlinear Schrödinger Equations

Viorel Barbu<sup>1</sup> · Michael Röckner<sup>2</sup> · Deng Zhang<sup>3</sup>

Received: 30 October 2022 / Accepted: 19 February 2023

© Vietnam Academy of Science and Technology (VAST) and Springer Nature Singapore Pte Ltd. 2023

## Abstract

We review the recent progress on the long-time behavior for a general class of focusing  $L^2$ -critical nonlinear Schrödinger equations (NLS) with lower order perturbations. Two canonical models are the stochastic NLS driven by linear multiplicative noise and the classical deterministic NLS. We show the construction and uniqueness of the corresponding blow-up solutions and solitons, including the multi-bubble Bourgain–Wang type blow-up solutions and non-pure multi-solitons, which provide new examples for the mass quantization conjecture and the soliton resolution conjecture. The refined uniqueness of pure multi-bubble blow-ups and pure multi-solitons to NLS under very low asymptotical rate is also reviewed. Finally, as a new result, we prove the qualitative properties of stochastic blow-up solutions, including the concentration of mass, universality of critical mass blow-up profiles, as well as the vanishing of the virial at the blow-up time.

**Keywords** Multi-bubble blow-ups · Multi-solitons · Uniqueness · Nonlinear Schrödinger equation

**Mathematics Subject Classification (2010)** 35B44 · 35C08 · 35Q55 · 60H15

---

Dedicated to Professor Carlos Kenig on the occasion of his 70th birthday.

---

✉ Deng Zhang  
dzhang@sjtu.edu.cn

Viorel Barbu  
vb41@uaic.ro

Michael Röckner  
roeckner@math.uni-bielefeld.de

<sup>1</sup> Octav Mayer Institute of Mathematics (Romanian Academy) and A.I.I. Cuza University, 700506 Iași, Romania

<sup>2</sup> Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

<sup>3</sup> School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai, China

# 1 Introduction

We are concerned with a general class of focusing  $L^2$ -critical nonlinear Schrödinger equations with lower order perturbations

$$i \partial_t v + \Delta v + a_1 \cdot \nabla v + a_0 v + |v|^{\frac{4}{d}} v = 0 \tag{gNLS}$$

on  $\mathbb{R}^d$ , the coefficients of lower order perturbations are of form

$$a_1(t, x) = 2i \sum_{l=1}^N \nabla \phi_l(x) h_l(t), \tag{1.1}$$

$$a_0(t, x) = - \sum_{j=1}^d \left( \sum_{l=1}^N \partial_j \phi_l(x) h_l(t) \right)^2 + i \sum_{l=1}^N \Delta \phi_l(x) h_l(t), \tag{1.2}$$

with spatial functions  $\phi_l \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ , and temporal functions  $h_l \in C(\mathbb{R}^+; \mathbb{R})$ ,  $1 \leq l \leq N < \infty$ .

Equation (gNLS) is mainly motivated by the following two canonical models:

• **Stochastic nonlinear Schrödinger equations.** One important model closely related to (gNLS) is the stochastic nonlinear Schrödinger equation (SNLS for short) driven by linear multiplicative noise

$$i dX + \Delta X dt + |X|^{\frac{4}{d}} X dt = -i \mu X dt + i X dW(t), \tag{SNLS}$$

where  $W$  is a Wiener process

$$W(t, x) = \sum_{l=1}^N i \phi_l(x) B_l(t), \quad x \in \mathbb{R}^d, t \geq 0,$$

$\{\phi_l\} \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$ ,  $\{B_l\}$  are independent standard  $N$ -dimensional real valued Brownian motions on a normal stochastic basis  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ , and  $\mu = \frac{1}{2} \sum_{l=1}^N \phi_l^2$ . The last term  $X dW(t)$  in (SNLS) is taken in the sense of Gubinelli’s controlled rough path, which coincides with the usual Itô stochastic integration if the corresponding processes are  $\{\mathcal{F}_t\}$ -adapted (cf. [37, 38]). We take  $N < \infty$  for simplicity only.

The relationship between (gNLS) and (SNLS) can be seen from the Doss–Sussman type transformation

$$v := e^{-W} X,$$

which transforms the stochastic equation (SNLS) to a random equation (gNLS) with the temporal functions  $\{h_l\}$  being exactly the Brownian motions  $\{B_l\}$ .

The physical significance of SNLS is well known. One significant model arises from molecular aggregates with thermal fluctuations, and the multiplicative noise corresponds to scattering of exciton by phonons, due to thermal vibrations of the molecules. Many physical and numerical experiments have been made to study the noise effects on blow-up and solitons, see, e.g., [1, 2] for noise effect on the coherence of the ground state solitary solution in dimension two, [70] for the case of the critical quintic nonlinearity in dimension one. We also refer to [9] for applications to open quantum systems, where the martingale feature of mass plays an important role.

• **Nonlinear Schrödinger equations.** When  $a_1, a_0 = 0$ , i.e., the lower order perturbations vanish, (gNLS) reduces to the canonical focusing  $L^2$ -critical nonlinear Schrödinger equation (NLS for short)

$$i \partial_t v + \Delta v + |v|^{\frac{4}{d}} v = 0. \tag{NLS}$$

NLS is one of fundamental dispersive PDEs and is important in continuum mechanics, plasma physics and optics ([33]). In particular, for the cubic nonlinearity in the critical dimension two, the phenomenon of mass concentration near collapse gives a rigorous basis to the physical concept of “strong collapse” ([75]). (NLS) has the conservation laws for the mass

$$M(v) := \|v\|_{L^2},$$

and the Hamiltonian

$$H(v) := \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{d}{2d+4} \|v\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}. \tag{1.3}$$

The  $H^1$  local well-posedness of (NLS) is well-known, see, e.g., [14, 76]. The SNLS has been studied in [12, 20, 22], by using stochastic Strichartz estimates. Regarding the pathwise solvability of (SNLS) and the related (gNLS,) we refer to [4–6, 39, 81, 83] and the related applications in control theory [8, 82]. In particular, we refer to [5] for the  $H^1$  local well-posedness of (gNLS), and one has the conservation law of mass

$$\|v(t)\|_{L^2}^2 = \|v(0)\|_{L^2}^2,$$

for all  $t \in [0, \tau^*)$  with  $\tau^*( > 0)$  being the maximal existing time.

Let us mention that, when treating the stochastic equation (SNLS), the Doss–Sussman transformation (or, rescaling transformation) enables us to perform sharp analysis in a pathwise way, which is in general difficult for the usual Itô calculus. This pathwise treatment, particularly, gives path-by-path uniqueness and hence the cocycle property, i.e., we obtain a random dynamical system (cf. [19]). This provides a convenient platform to analyze the long-time dynamical mechanism including the current blow-up and soliton dynamics. For the interested readers, we refer to [80] for a review of the rescaling approach.

The long-time behavior of solutions to (gNLS) is much more delicate. An important role here is played by the ground state, which is the unique positive radial solution to the nonlinear elliptic equation

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0.$$

In view of [27, 78], the mass of the ground state is the sharp threshold for the global well-posedness, scattering and blow-up. We also would like to mention that, for the case of energy-critical NLS, Kenig and Merle [44] developed the concentration-compactness and rigidity method, and proved a sharp condition, characterized by the Aubin–Talenti solution, for the global well-posedness, scattering and blow-up.

In the critical mass case where  $\|v\|_{L^2} = \|Q\|_{L^2}$ , the following two important dynamics arise:

- Pseudo-conformal blow-up solutions

$$S_T(t, x) = (w(T - t))^{-\frac{d}{2}} Q \left( \frac{x - x^*}{w(T - t)} \right) e^{-\frac{i}{4} \frac{|x-x^*|^2}{T-t} + \frac{i}{w^2(T-t)} + i\vartheta},$$

- Solitary wave

$$W(t, x) := w^{-\frac{d}{2}} Q \left( \frac{x - ct}{w} \right) e^{i(\frac{1}{2}c \cdot x - \frac{1}{4}|c|^2 t + w^{-2}t + \vartheta)},$$

where  $x^*, c \in \mathbb{R}^d$ ,  $w > 0$  and  $T, \vartheta \in \mathbb{R}$ . Both dynamics are related closely to each other through the *pseudo-conformal transform* in the pseudo-conformal space  $\Sigma := \{u \in H^1 : xu \in L^2\}$  defined by

$$C_T(u)(t, x) := \frac{1}{(T-t)^{\frac{d}{2}}} u \left( \frac{1}{T-t}, \frac{x}{T-t} \right) e^{-i \frac{|x|^2}{4(T-t)}}, \quad t \neq T.$$

Note that  $S_T = C_T(W)$ ,  $x^*$  is the singularity corresponding to the velocity  $c$  of  $W$ , and  $S_T$  blows up at time  $T$ . In the seminal paper [55], Merle proved that the pseudo-conformal blow-up solution is the unique critical mass blow-up solution to the  $L^2$ -critical NLS, up to the symmetries of the equation. It was conjectured that the only non-scattering solutions to (NLS), up to symmetries, are the soliton or the pseudo-conformal transformation of the soliton. This conjecture has been recently solved by Dodson [28, 29].

Moreover, in the small supercritical mass case where  $\|Q\|_{L^2} < \|v_0\|_{L^2} < \|Q\|_{L^2} + \varepsilon$ , with small  $\varepsilon (> 0)$ , one stable dynamics is the log-log blow-up solution, which was first constructed by Perelman [67], and has been extensively studied in a series of works by Merle and Raphaël [57–59, 61].

Another important dynamics is the Bourgain–Wang blow-up solution, which was first constructed by Bourgain and Wang [11], and behaves asymptotically as a sum of a singular profile  $S_T$  and a regular profile  $z$ , i.e.,

$$v(t) - S_T(t) - z(t) \rightarrow 0, \quad \text{as } t \rightarrow T.$$

Unlike the log-log blow-up solutions, Bourgain–Wang solutions are unstable under  $H^1$  perturbation and lie on the boundary of two  $H^1$  open sets of global scattering solutions and log-log blow-up solutions ([62]).

In the even large mass regime, the complete characterization of general blow-up solutions to  $L^2$ -critical NLS remains open. In [10], Bourgain raised an open problem on the quantization property of blow-up solutions, namely, whether the concentration of mass is of the form  $k\|Q\|_{L^2}^2$ ,  $k \in \mathbb{Z}_+$ , see also [11]. Merle and Raphaël [60] formulated precisely the *mass quantization conjecture*: blow-up solutions are expected to decompose into a singular part and an  $L^2$  residual, and the singular part expands asymptotically as multiple bubbles, each of which concentrates a mass of no less than  $\|Q\|_{L^2}^2$  at the blow-up point. In particular, the Bourgain–Wang solutions provide examples of the mass quantization conjecture in the single-bubble case. Multi-bubble blow-up solutions without regular profiles to (NLS) were first constructed by Merle [54]. Thus, a natural question is whether there exist multi-bubble Bourgain–Wang type blow-up solutions.

Furthermore, according to the famous *soliton resolution conjecture*, global solutions to a nonlinear dispersive equation are expected to decompose at large time as a sum of solitons plus a scattering remainder. Important progress has recently been made for the energy critical wave equation, we refer to [18, 30–32] and the references therein. There are also many papers on the construction of multi-solitons behaving like a sum of multi-solitons. However, the existence of *non-pure multi-solitons* (including dispersive part) predicted by the soliton resolution conjecture seemed not available, see the lecture notes by Cazenave [15]. An interesting question is then to construct this kind of non-pure multi-solitons.

Let us also mention that the *uniqueness* of multi-solitons to  $L^2$ -(sub)critical NLS remains open (see [50]). The complete study of the uniqueness problem of multi-solitons was done for the  $L^2$ -(sub)critical gKdV in the remarkable paper by Martel [49]. Multi-solitons to  $L^2$ -supercritical gKdV were classified by Combet [16]. We also would like to refer to [41] for the classification of the strongly interacting kink-antikink pair. For the NLS, recent progress has

been made by Côte and Friederich [17] on the smoothness and uniqueness in the (sub)critical case when the asymptotic rate is large enough. A natural question is thus to prove the uniqueness of solitons particularly in the low asymptotic regime.

Turning to the stochastic case, there are several physical and numerical papers on the study of blow-up and solitons. It was observed in [24–26] that multiplicative noise has the effect to delay blow-up, while white noise even can prevent blow-up. Moreover, de Bouard and Debussche [23] proved that, in the  $L^2$ -supercritical case, the conservative noise accelerates blow-up with positive probability. Similar noise effect was also proved for additive noise ([21]). In [7], the authors proved that the non-conservative noise can provide damping effects, and so prevent explosion with high probability. Recently, in [65, 66], quantitative behavior of blow-up solutions, including the blow-up rate and blow-up profiles, was studied by numerical experiments. The main challenges in the stochastic case are the absence of pseudo-conformal symmetry and even of a conservation law for the Hamiltonian, due to the presence of lower order perturbations (or noise).

In this note, we review the recent progress [35, 71–74] on the construction and uniqueness of blow-up solutions and of solitons for the nonlinear Schrödinger equations with lower order perturbations (gNLS). In particular, the construction of multi-bubble Bourgain–Wang type blow-up solutions and non-pure multi-solitons provide new examples for the mass quantization conjecture and the soliton resolution conjecture. Furthermore, we also review the refined uniqueness results in [13] for pure multi-solitons to the  $L^2$ -critical (NLS) in the low asymptotic rate regime. Finally, inspired by [3, 40, 55, 56], as a new result, in Section 3 we prove qualitative properties of stochastic blow-up solutions, including the concentration of mass, the universality of blow-up profiles with critical mass and the vanishing of the virial at the blow-up time.

## 2 Multi-bubble Blow-up Solutions and Multi-solitons

Let us first review the stochastic single-bubble blow-up solutions in both the critical and small supercritical mass cases. Then, we show the existence and uniqueness results for the multi-bubble blow-up solutions and multi-solitons in Section 2.2 and Section 2.3, respectively. In Section 2.4, we show the refined uniqueness results in the low asymptotical rate regime.

### 2.1 Single Bubble Blow-up Solutions

For the spatial functions  $\{\phi_l\}$  we assume:

(A0) Asymptotical flatness: For every  $1 \leq l \leq N$  and multi-index  $\nu \neq 0$ ,

$$\lim_{|x| \rightarrow \infty} \langle x \rangle^2 |\partial_x^\nu \phi_l(x)| = 0.$$

(A1) Flatness at the origin: For every  $1 \leq l \leq N$  and multi-index  $0 \leq |\nu| \leq 5$ ,

$$\partial_x^\nu \phi_l(0) = 0.$$

The large spatial assumption in (A0) ensures the local well-posedness of (gNLS) (cf. [4, 5]), while the local spatial assumption in (A1) is mainly used for the blow-up analysis.

**Theorem 2.1** (Critical mass blow-up solution to SNLS, [73]) *Let  $d = 1, 2$ . Assume Assumptions (A0) and (A1) to hold. Then, there exists  $\tau^* \in (0, \infty)$  such that for any  $T \in (0, \tau^*]$ ,*

there exists  $v_0 \in H^1$  satisfying  $\|v_0\|_{L^2} = \|Q\|_{L^2}$  and the corresponding solution  $v$  to (gNLS) blows up at time  $T$ . Moreover, there exist  $\delta, C(T) > 0$  such that for  $t$  close to  $T$ ,

$$\|v(t) - S_T(t)\|_{H^1} \leq C(T)(T - t)^\delta.$$

**Remark 2.2**

- (i) A direct application of Theorem 2.1 gives the existence of critical mass blow-up solutions to (SNLS). It was known that solutions to (SNLS) exist globally in the  $H^1$ -subcritical case or for initial data below the ground state, see [5, 12, 22, 36, 64]. Hence, the mass of the ground state still serves as a threshold for the global well-posedness and blow-up in the stochastic case.
- (ii) The proof of Theorem 2.1 is based on the modulation method developed in [69], including the geometric decomposition, a bootstrap device and the backward propagation from the singularity. Delicate analysis has also been made to control the variation of the Hamiltonian, related estimates of geometric parameters and the polynomial type of perturbation orders when linearizing (gNLS) around the ground state.
- (iii) We also mention that the control of the first order term in (gNLS) is non-trivial and is based on the local smoothing estimates (see, e.g., [53, 83]). The local smoothing estimates also play the key role in the well-posedness of quasi-linear Schrödinger equations, see [45, 46] and the references therein.

The next result is concerned with the construction of log-log blow-up solutions to (gNLS) in the small supercritical mass regime. We refer to [34, 57–59, 61, 67] for (NLS) in the deterministic case. For simplicity, we focus on the case where  $\{\phi_l\}$  are Schwartz functions.

**Theorem 2.3** (Log-log blow-up solution to SNLS, [35]) *Consider (gNLS) with  $d = 1, 2$  and  $\{\phi_l\}$  being Schwartz functions. Then, there exists an initial datum  $v_0 \in H^1$  such that the corresponding solution  $v$  to (gNLS) blows up in finite time  $T$  according to a log-log law in the sense that there exist parameters  $(x, \gamma, \lambda) \in C^1((0, T); \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+)$ , such that*

$$v(t, x) = \frac{1}{\lambda^{d/2}(t)} (Q + \epsilon) \left( t, \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)}, \quad t \in (0, T),$$

with

$$\lambda(t)^{-1} \sim \sqrt{\frac{\ln |\ln T - t|}{T - t}} \quad \text{and} \quad \int |\nabla \epsilon|^2 + |\epsilon|^2 e^{-|y|} dy \xrightarrow{t \rightarrow T} 0,$$

and the blow up point  $x(t)$  converges as  $t \rightarrow T$ .

## 2.2 Multi-bubble Bourgain–Wang Type Blow-up Solutions

This subsection is concerned with the multi-bubble blow-up solutions to (gNLS) at the given distinct singularities  $\{x_k\}$ . Similarly to (A0) and (A1), we assume the following conditions for the spatial functions  $\{\phi_l\}$  and the regular profiles  $z^*$  (see, e.g., [60, 62]).

( $H1_{\nu_*}$ ) Asymptotical flatness: For any multi-index  $\nu \neq 0$  and  $1 \leq l \leq N$ ,

$$\lim_{|x| \rightarrow \infty} \langle x \rangle^2 |\partial_x^\nu \phi_l(x)| = 0.$$

Flatness at singularities: There exists  $\nu_* \in \mathbb{N}^+$  such that for every  $1 \leq l \leq N$  and every multi-index  $\nu$  with  $|\nu| \leq \nu_*$ ,

$$\partial_x^\nu \phi_l(x_k) = 0, \quad 1 \leq l \leq N, \quad 1 \leq k \leq K.$$

( $H2_{m,\alpha^*}$ ) Smallness: There exist  $m \in \mathbb{N}$ ,  $\alpha^* \in (0, \infty)$  such that for every regular profile  $z^*$  satisfying

$$\|z^*\|_{H^{2m+2+d}} \leq \alpha^*, \quad \|\langle x \rangle z^*\|_{H^1} \leq \alpha^*, \tag{2.1}$$

we have: Flatness at singularities: For any multi-index  $\nu$  with  $|\nu| \leq 2m$ ,

$$\partial_x^\nu z^*(x_k) = 0, \quad 1 \leq k \leq K.$$

We first show the existence and conditional uniqueness of pure multi-bubble blow-up solutions without the regular profile.

**Theorem 2.4** (Pure multi-bubble blow-up solutions to gNLS, [74]) *Consider (gNLS) with  $d = 1, 2$ . Assume that  $(H1_{\nu_*})$  holds with  $\nu_* \geq 5$ . Let  $K \in \mathbb{N}^+$ ,  $T \in \mathbb{R}^+$ ,  $\{\vartheta_j\}_{j=1}^K \subseteq \mathbb{R}$ ,  $\zeta \in (0, 1)$ . Then:*

- (a) *For any distinct points  $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$ ,  $w \in (0, \infty)$ , there exists  $\varepsilon > 0$  such that for any  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$  with  $\max_{1 \leq k \leq K} |w_k - w| \leq \varepsilon$ , assertions (i) and (ii) below hold.*
- (b) *For any  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$  there exists  $\varepsilon > 0$  such that for any distinct points  $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$  with  $\min_{1 \leq k \leq K} |x_j - x_k| \geq \varepsilon^{-1}$  assertions (i) and (ii) below hold, where:*

- (i) *Existence: There exists  $\tau^* > 0$  small enough such that for any  $T \in (0, \tau^*)$ , there exist  $v_0 \in \Sigma$  and a corresponding blow-up solution  $v$  to (gNLS) satisfying*

$$\left\| v(t) - \sum_{k=1}^K S_k(t) \right\|_{\Sigma} \leq C(T-t)^{\frac{1}{2}(\nu_*-5)+\tilde{\zeta}}, \quad t \in [0, T),$$

where  $C > 0$ ,  $\tilde{\zeta} \in (0, 1)$ , and  $\{S_k\}$  are the pseudo-conformal blow-up solutions

$$S_k(t, x) = (w_k(T-t))^{-\frac{d}{2}} Q \left( \frac{x-x_k}{w_k(T-t)} \right) e^{-\frac{i}{4} \frac{|x-x_k|^2}{T-t} + \frac{i}{w_k^2(T-t)} + i\vartheta_k}. \tag{2.2}$$

- (ii) *Conditional uniqueness: There exists a unique blow-up solution  $v$  to (gNLS) satisfying*

$$\left\| v(t) - \sum_{k=1}^K S_k(t) \right\|_{H^1} \leq C(T-t)^{3+\zeta}, \quad t \in [0, T),$$

where  $C > 0$ .

**Remark 2.5**

- (i) The above uniqueness result shows that multi-bubble blow-up solutions are unique in the energy class with the asymptotic rate  $(T-t)^{3+}$ . Later in Theorem 2.12, we can enlarge the unique energy class with a much lower convergence rate  $(T-t)^{0+}$  in the case of (NLS).
- (ii) The proof of Theorem 2.4 is based on a localization procedure and, in particular, the delicate decoupling between the interactions of different remainder profiles in the geometric decomposition. In order to respect the multi-bubble structure, a new generalized energy is constructed, which permits to decouple different bubbles and maintains the key monotonicity property.

Furthermore, for the multi-bubble Bourgain–Wang type blow-up solutions we have

**Theorem 2.6** (Multi-bubble Bourgain–Wang solutions to gNLS, [71]) *Consider (gNLS) with  $d = 1, 2$ . Let  $K \in \mathbb{N}^+$ ,  $T \in \mathbb{R}^+$ ,  $\{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}$ ,  $\zeta \in (0, 1)$ . Then, assertions (a) and (b) from Theorem 2.4 hold with (i), (ii) there, replaced by:*

- (i) *Existence: If  $(H1_{v_*})$  holds for  $v_* \geq 5$  and  $(H2_{m,\alpha^*})$  for some  $\alpha^* > 0$ ,  $m \geq 3$  if  $d = 2$ , and  $m \geq 4$  if  $d = 1$ , then there exists  $\varepsilon^* \in (0, \alpha^*]$  such that for every regular profile  $z^*$  satisfying (2.1) with  $\alpha^*$  replaced by  $\varepsilon^*$ , there exists a solution  $v$  to (gNLS) satisfying that for  $t$  close to  $T$ ,*

$$\left\| v(t) - \sum_{k=1}^K S_k(t) - z(t) \right\|_{L^2} + (T - t) \left\| v(t) - \sum_{k=1}^K S_k(t) - z(t) \right\|_{\Sigma} \leq C(T - t)^{\frac{1}{2}(\kappa - 1)},$$

where  $\kappa := (m + \frac{d}{2} - 1) \wedge (v_* - 2)$ ,  $C > 0$ ,  $\{S_k\}$  are as in (2.2), and  $z$  is the unique solution of the equation

$$\begin{aligned} i \partial_t z + \Delta z + a_1 \cdot \nabla z + a_0 z + |z|^{\frac{4}{d}} z &= 0, \\ z(T) &= z^*, \end{aligned}$$

where the coefficients  $a_1, a_0$  are given by (1.1) and (1.2), respectively.

- (ii) *Conditional uniqueness: If  $(H1_{v_*})$  holds with  $v_* \geq 12$  and  $(H2_{m,\alpha^*})$  holds for some  $\alpha^* > 0$  and  $m \geq 10$ , then there exists a unique solution  $v$  to (gNLS) satisfying that for  $t$  close to  $T$ ,*

$$\left\| v(t) - \sum_{k=1}^K S_k(t) - z(t) \right\|_{L^2} + (T - t) \left\| \nabla v(t) - \nabla \sum_{k=1}^K S_k(t) - \nabla z(t) \right\|_{L^2} \leq C(T - t)^{4+\zeta}.$$

**Remark 2.7** (i) To the best of our knowledge, Theorem 2.6 provides the first examples of multi-bubble blow-ups with a regular profile. As a direct application, it provides new examples for the mass quantization conjecture, the constructed solution satisfies that as  $t \rightarrow T$ ,

$$\begin{aligned} |v(t)|^2 &\rightharpoonup \sum_{k=1}^K \|Q\|_{L^2}^2 \delta_{x_k} + |z^*|^2, \\ v(t) &\rightarrow z^* \quad \text{in } L^2 \left( \mathbb{R}^d - \bigcup_{k=1}^K B(x_k, R) \right) \end{aligned}$$

for any  $R > 0$ . In particular, it concentrates the mass  $\|Q\|_{L^2}^2$  at each singularity and the remaining part converges to a regular profile  $z^*$ .

(ii) The conditional uniqueness result reveals the rigidity of the flow around multi-bubble pseudo-conformal blow-up solutions and the regular profile.

(iii) The main effort of the proof is dedicated to decoupling the interactions between three types of profiles: the main blow-up profile, the regular profile and the remainder profile. Unlike in [62], because multi-bubble blow-up solutions are in general not radial, two new geometric parameters are introduced, which in turn require coercivity type control of the energy. Let us also mention that additional temporal regularity is gained by subtracting the energy of the regular profile from that of the approximate solutions, which is important to run the bootstrap arguments.

### 2.3 (Non-pure) Multi-solitons

By virtue of the pseudo-conformal invariance of  $L^2$ -critical NLS, Theorem 2.6 also gives the corresponding results of non-pure multi-solitons to (NLS).

**Theorem 2.8** (Non-pure multi-solitons to NLS, [71]) *Consider (NLS) with  $d = 1, 2$ . Let  $K \in \mathbb{N}^+, \{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}, \zeta \in (0, 1)$ . Then, assertions (a) and (b) from Theorem 2.4 hold with  $x_k$  replaced by  $c_k, 1 \leq k \leq K$ , and (i) and (ii) there, replaced by:*

- (i) *Existence: If  $(H2_{m,\alpha^*})$  holds for some  $\alpha^* > 0$  with  $m \geq 6$ , then there exists  $\varepsilon^* \in (0, \alpha^*]$  such that for every regular profile  $z^*$  satisfying (2.1) with  $\alpha^*$  replaced by  $\varepsilon^*$ , there exists a solution  $u$  to (NLS) satisfying*

$$\left\| u(t) - \sum_{k=1}^K W_k(t) - \tilde{z}(t) \right\|_{\Sigma} \leq Ct^{-\frac{1}{2}\kappa + \frac{5}{2}}, \quad \text{for } t \text{ large enough,}$$

where  $\kappa = m + \frac{d}{2} - 1, C > 0, \{W_k\}$  are the solitary waves

$$W_k(t, x) = w_k^{-\frac{d}{2}} Q\left(\frac{x - c_k t}{w_k}\right) e^{i\left(\frac{1}{2}c_k x - \frac{1}{4}|c_k|^2 t + w_k^{-2} t + \vartheta_k\right)}, \quad (2.3)$$

and  $\tilde{z}$  corresponds to the regular part  $z$  in Theorem 2.6 through the inverse of the pseudo-conformal transform:

$$\tilde{z}(t, x) = C_T^{-1} z(t, x) = t^{-\frac{d}{2}} z\left(T - \frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}.$$

- (ii) *Conditional uniqueness: If  $(H2_{m,\alpha^*})$  holds for some  $\alpha^* > 0$  with  $m \geq 16$ , then there exists a unique non-pure multi-soliton  $u$  to (NLS) satisfying*

$$\left\| u(t) - \sum_{k=1}^K W_k(t) - \tilde{z}(t) \right\|_{\Sigma} \leq Ct^{-5-\zeta}, \quad \text{for } t \text{ large enough.}$$

**Remark 2.9** (i) To our best knowledge, Theorem 2.8 provides first examples of non-pure multi-solitons including a dispersive part to  $L^2$ -critical (NLS) predicted by the soliton resolution conjecture.

(ii) It is worth noting that the uniqueness of non-pure multi-solitons holds in the energy class with the decay rate  $t^{-5-}$ , which is larger than the usual class of fast exponential decay rate, which multi-solitons naturally lie in (see, e.g., [47, 48]). For multi-solitons without scattering part, we obtain the uniqueness with even lower asymptotic rate, see Theorems 2.13 and 2.14 below.

Unlike the  $L^2$ -critical NLS, because of the absence of pseudo-conformal invariance, stochastic multi-solitons to (SNLS) cannot be obtained from multi-bubble blow-ups in Section 2.2. Hence, the construction of stochastic multi-solitons requires a direct method at the level of solitons. This is the content of Theorem 2.10 below.

Below we consider the  $L^2$ -(sub)critical SNLS of similar structure as that of (SNLS):

$$dX(t) = i\Delta X(t)dt + i|X(t)|^{p-1}X(t)dt - \mu(t)X(t)dt + \sum_{k=1}^N X(t)G_k(t)dB_k(t), \quad (2.4)$$

where  $1 < p \leq 1 + \frac{4}{d}, d \geq 1, T_0 \geq 0, \{B_k\}$  and  $\mu$  as in (SNLS),  $G_k(t, x) = i\phi_k(x)g_k(t), x \in \mathbb{R}^d, t \geq 0, \{\phi_k\} \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R}), \{g_k\}$  are  $\{\mathcal{F}_t\}$ -adapted processes with paths in  $C^\alpha(\mathbb{R}^+, \mathbb{R})$ ,

$\alpha \in (\frac{1}{3}, \frac{1}{2})$ , that are controlled by  $\{B_k\}$  and  $X(t)G_k(t)dB_k(t)$  is taken in the sense of controlled rough paths (see, e.g., [37, 38]).

The basic conditions here are as follows:

(H1)' For every  $1 \leq l \leq N$ ,

$$\lim_{|x| \rightarrow \infty} |x|^2 |\partial_x^\nu \phi_l(x)| = 0, \quad \nu \neq 0.$$

(H2)' For every  $1 \leq l \leq N$ ,  $\{g_l\}$  are  $\{\mathcal{F}_t\}$ -adapted continuous processes and controlled by the Brownian motions  $\{B_l\}$ , i.e.,  $\{g_l\} \subseteq \mathcal{D}_B^{2\alpha}(\mathbb{R}^+; \mathbb{R}^N)$  with Gubinelli derivative  $\{g'_{lj}\}_{j,l=1}^N$ . In addition, for  $1 \leq l \leq N$ ,  $\phi_l$  and  $g_l$  satisfy one of the following two cases:

Case (I)':  $g_l \in L^2(\mathbb{R}^+)$ ,  $\mathbb{P}$ -a.s., and there exists  $c_l > 0$  such that

$$\sum_{|\nu| \leq 4} |\partial^\nu \phi_l(x)| \leq C e^{-c_l |x|}.$$

Case (II)':  $\mathbb{P}$ -a.s.,  $g_l \in L^2(\mathbb{R}^+)$  and there exists  $c^* > 0$  such that for  $t$  large enough,

$$\int_t^\infty g_l^2 ds \log \left( \int_t^\infty g_l^2 ds \right)^{-1} \leq \frac{c^*}{t^2}. \tag{2.5}$$

In addition, let  $\nu_* \in \mathbb{N}$  be such that  $\phi_l$  satisfies

$$\sum_{|\nu| \leq 4} |\partial^\nu \phi_l(x)| \leq C |x|^{-\nu_*}.$$

We see that the asymptotic behavior (2.5) is closely related to the Levy Hölder continuity of Brownian motion. For simplicity, we focus on the case where  $c_l = 1, 1 \leq l \leq N$ , and set

$$\phi(x) := \begin{cases} e^{-|x|} & \text{in Case (I)'}, \\ |x|^{-\nu_*} & \text{in Case (II)'}. \end{cases}$$

**Theorem 2.10** (Multi-solitons to SNLS, [72]) *Consider (2.4) with  $1 < p \leq 1 + \frac{4}{d}, d \geq 1$ . Let  $w_k^0 > 0, \theta_k^0 \in \mathbb{R}, x_k^0 \in \mathbb{R}^d, v_k \in \mathbb{R}^d \setminus \{0\}, 1 \leq k \leq K$ , such that  $v_j \neq v_k$  for any  $j \neq k$ . Assume (H1)' and (H2)' with  $\nu_*$  sufficiently large in Case (II)'.*

*Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , there exist  $T_0 = T_0(\omega)$  sufficiently large and  $X_*(\omega) \in H^1$ , such that there exists an  $H^1$  solution  $X(\omega)$  to (2.4) on  $[T_0, \infty)$  satisfying  $X(\omega, T_0) = X_*(\omega)$  and*

$$\left\| e^{-W_*(t)} X(t) - \sum_{k=1}^K R_k(t) \right\|_{H^1} \leq C \int_t^\infty s \phi^{\frac{1}{2}}(\delta s) ds, \quad t \geq T_0. \tag{2.6}$$

Here,  $\{R_k\}$  are the solitary waves

$$R_k(t, x) := Q_{w_k^0} \left( x - v_k t - x_k^0 \right) e^{i \left( \frac{1}{2} v_k \cdot x - \frac{1}{4} |v_k|^2 t + (w_k^0)^{-2} t + \theta_k^0 \right)},$$

with  $Q_w(x) := w^{-\frac{2}{p-1}} Q \left( \frac{x}{w} \right)$ , and

$$W_*(t, x) = - \sum_{l=1}^N \int_t^\infty i \phi_l(x) g_l(s) dB_l(s).$$

Moreover, in the  $L^2$ -subcritical case  $1 < p < 1 + \frac{4}{d}$ , there exists a solution  $X$  to (2.4) on the whole time interval  $[0, \infty)$ , satisfying the asymptotic behavior (2.6).

**Remark 2.11** (i) We note that the temporal asymptotic rate in (2.6) can be of either exponential and polynomial type, respectively, in Case (I)' and Case (II)', which is closely related to the spatial decay rate of the noise. This reflects the effect of noise on the soliton dynamics.

(ii) The proof of Theorem 2.10 is based on the rescaling approach, involving two types of Doss–Sussman transforms as in the stochastic scattering context [39], and on the modulation method in [17, 51, 52]. One crucial ingredient is the monotonicity of the Lyapunov type functional constructed in [52]. Let us also mention that the geometric decomposition in the proof is chosen in a quite unified manner in both the  $L^2$ -subcritical and critical cases.

### 2.4 Refined Uniqueness

In this subsection, we review the refined uniqueness results on the multi-bubble blow-up solutions and multi-solitons to (NLS), particularly, in the low asymptotic regime.

**Theorem 2.12** (Refined uniqueness of multi-bubble blow-ups to NLS, [13]) *Consider (NLS) in dimensions  $d = 1, 2$ . Let  $T \in \mathbb{R}$ ,  $K \in \mathbb{N} \setminus \{0\}$ ,  $\{\vartheta_k\} \subseteq \mathbb{R}$ . Then, for any  $\zeta \in (0, 1)$ , for any distinct points  $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$ ,  $w > 0$  (resp.  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$ ), there exists  $\varepsilon > 0$  such that for any  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$  with  $\max_{1 \leq k \leq K} |w_k - w| \leq \varepsilon$  (resp.  $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$  with  $\min_{1 \leq k \leq K} |x_j - x_k| \geq \varepsilon^{-1}$ ), there exists a unique multi-bubble blow-up solution  $v$  to (NLS) satisfying*

$$\left\| v(t) - \sum_{k=1}^K S_k(t) \right\|_{L^2} + (T - t) \left\| \nabla v(t) - \sum_{k=1}^K \nabla S_k(t) \right\|_{L^2} = o(1), \quad \text{as } t \text{ close to } T,$$

and additionally

$$\frac{1}{T - t} \int_t^T \frac{1}{T - s} \int_s^T \left\| v(r) - \sum_{k=1}^K S_k(r) \right\|_{H^1}^2 dr ds = \mathcal{O}((T - t)^\zeta),$$

where  $S_k$ ,  $1 \leq k \leq K$ , are the pseudo-conformal blow-up solutions given by (2.2). Moreover, the asymptotic rate of the unique solution  $v$  can be enhanced to the exponentially fast rate in the pseudo-conformal space. Namely, there exists  $\delta > 0$  such that

$$\left\| v(t) - \sum_{k=1}^K S_k(t) \right\|_{\Sigma} = \mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right), \quad \text{for } t \text{ close to } T.$$

In particular, the above results hold for the multi-bubble blow-up solutions  $v$  to (NLS) such that

$$\left\| v(t) - \sum_{k=1}^K S_k(t) \right\|_{H^1} = \mathcal{O}((T - t)^\zeta), \quad \text{for } t \text{ close to } T.$$

Regarding the pure multi-solitons to (NLS), we have the following uniqueness result in the energy class.

**Theorem 2.13** (Refined uniqueness in  $H^1$  of multi-solitons to NLS, [13]) *Consider the situation as in Theorem 2.12. Then, for any  $\zeta \in (0, 1)$ , for any distinct speeds  $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d$ ,  $w > 0$  (resp.  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$ ), there exists  $\varepsilon > 0$  such that for any  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$  with*

$\max_{1 \leq k \leq K} |w_k - w| \leq \varepsilon$  (resp.  $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d$  with  $\min_{1 \leq k \leq K} |c_j - c_k| \geq \varepsilon^{-1}$ ), there exists a unique multi-soliton  $u$  to (NLS) satisfying

$$\left\| u(t) - \sum_{k=1}^K W_k(t) \right\|_{H^1} = \mathcal{O} \left( \frac{1}{t^{2+\zeta}} \right), \quad \text{for } t \text{ large enough,}$$

where  $\{W_k\}$  are the solitons given by (2.3). Moreover, the unique multi-soliton  $u$  converges exponentially fast in the pseudo-conformal space, i.e., for some  $\delta > 0$ ,

$$\left\| u(t) - \sum_{k=1}^K W_k(t) \right\|_{\Sigma} = \mathcal{O} (e^{-\delta t}), \quad \text{for } t \text{ large enough.}$$

In the case of the pseudo-conformal space, the uniqueness class can be further enlarged in the even lower asymptotic regime.

**Theorem 2.14** (Refined uniqueness in  $\Sigma$  of multi-solitons to NLS, [13]) *Consider the situation as in Theorem 2.13. Then, for any  $\zeta \in (0, 1)$ , for any distinct non-zero speeds  $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d \setminus \{0\}$ ,  $w > 0$  (resp.  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$ ), there exists  $\varepsilon > 0$  such that for any  $\{w_k\}_{k=1}^K \subseteq \mathbb{R}^+$  with  $\max_{1 \leq k \leq K} |w_k - w| \leq \varepsilon$  (resp.  $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d \setminus \{0\}$  with  $\min_{1 \leq k \leq K} |c_j - c_k| \geq \varepsilon^{-1}$ ), there exists a unique multi-soliton  $u$  to (NLS) satisfying*

$$\left\| u(t) - \sum_{k=1}^K W_k(t) \right\|_{\Sigma} = \mathcal{O} \left( \frac{1}{t^{\frac{1}{2}+\zeta}} \right), \quad \text{for } t \text{ large enough.}$$

Moreover, the unique multi-soliton  $u$  converges exponentially fast to  $\sum_{k=1}^K W_k$  in  $\Sigma$ .

**Remark 2.15** (i) The uniqueness of multi-solitons to (NLS) was first obtained by Côte and Friederich [17] in the  $L^2$ -(sub)critical and critical cases, provided that the convergence rate is  $(1/t)^N$  for  $N$  large enough. Theorem 2.13 shows that the uniqueness class of multi-solitons to  $L^2$ -critical (NLS) can be enlarged in the low convergence regime with rate  $(1/t)^{2+}$ .

(ii) The proof proceeds in several upgradation steps, each step requires the monotonicity of different functionals appropriately constructed with respect to the multi-bubble structure. More delicately, the analysis of the functionals relies on suitable estimates of the remainder and geometric parameters in the previous step.

Another main difficulty is the presence of the localized mass, which is absent in the single-bubble case. The key idea is to upgrade the localized mass and the remainder together in the upgradation procedure.

The a priori low asymptotic rate also gives rise to a challenging problem to identify the energy. The important fact here is that the remainder exhibits dispersion in the energy space along a sequence, which leads to the energy quantization phenomenon that is the key towards the derivation of the refined energy estimate.

### 3 Mass Concentration and Universality

This section is devoted to the qualitative descriptions of the dynamical properties of blow-up solutions to (SNLS), including the concentration of  $L^2$ -norm, the universality of critical mass blow-up solutions, as well as the vanishing of the virial at the blow-up time.

We consider the  $H^1$  blow-up solution  $X$  to (SNLS) on  $[0, \tau^*)$ , where  $d \geq 1$ , the stochastic integration is taken in the sense of Itô and  $\tau^*( > 0)$  denotes the maximal existing time. We

note that by the construction in [5],  $\tau^*$  is an  $\{\mathcal{F}_t\}$ -stopping time and  $\tau^* > 0$ ,  $\mathbb{P}$ -a.s.. Hence, below we focus on the case  $\mathbb{P}(\tau^* < \infty) > 0$ .

The first result is concerned with the mass concentration property along a time sequence.

**Theorem 3.1** (Mass concentration) *Assume the asymptotic flat condition (A0) to hold. Assume additionally that  $\mathbb{P}(\tau^* < \infty) > 0$ . Then, there exists a measurable set  $\Omega_0 \subseteq \Omega$  with full probability such that for every  $\omega \in \{\tau^* < \infty\} \cap \Omega_0$ , there exist  $t_n(\omega) \in (0, \tau^*)$  and  $y_n(\omega) \in \mathbb{R}^d$ ,  $n \in \mathbb{N}$ , such that  $t_n(\omega) \rightarrow \tau^*(\omega)$  and for any  $R > 0$ ,*

$$\liminf_{n \rightarrow \infty} \int_{|x - y_n(\omega)| \leq R} |X(t_n(\omega), x)(\omega)|^2 dx \geq \|Q\|_{L^2}^2. \tag{3.1}$$

**Remark 3.2** For the deterministic (NLS), the concentration of the  $L^2$ -norm was proved in [40, 63, 77] along all sequence of times converging to the blow-up time. The reason that Theorem 3.1 holds for a time sequence is mainly due to the failure of the conservation of the Hamiltonian in the stochastic case. The Itô evolution of the Hamiltonian actually contains an extra stochastic integration (see (3.10) below), which fluctuates in time and in general is not uniformly bounded.

The key point here is to make use of the specific structure of the stochastic integration, which can be controlled by using the Banica type inequality together with the Burkholder–Davis–Gundy inequality. Moreover, the quantitative large time behavior, i.e., the law of the iterated logarithm of a martingale, is also used in the control of the asymptotic behavior of the Hamiltonian of rescaled solutions.

In the critical mass case where  $\|X_0\|_{L^2} = \|Q\|_{L^2}$ , we have the universality of stochastic critical mass blow-up profiles, which extends the result in [79] to the stochastic case.

**Theorem 3.3** (Universality of critical mass blow-up profiles) *Assume the situation as in Theorem 3.1 to hold. Assume additionally that  $\|X(0)\|_{L^2} = \|Q\|_{L^2}$ . Then, there exists a measurable set  $\Omega_0 \subseteq \Omega$  with full probability such that for any  $\omega \in \{\tau^* < \infty\} \cap \Omega_0$ , there exist  $t_n(\omega) \in (0, \tau^*)$ ,  $y_n(\omega) \in \mathbb{R}^d$  and  $\theta(t_n(\omega)) \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , such that*

$$(\lambda_n(\omega))^{\frac{d}{2}} X(t_n(\omega), \lambda_n \cdot + y_n(\omega)) e^{i\theta(t_n(\omega))} \rightarrow Q \text{ in } H^1 \text{ as } n \rightarrow \infty, \tag{3.2}$$

where  $\lambda_n(\omega) := \frac{\|\nabla Q\|_{L^2}}{\|\nabla X(t_n(\omega))\|_{L^2}}$ ,  $n \geq 1$ .

A direct application of Theorem 3.3 yields that the critical mass blow-up solutions concentrate the whole mass at the singularity.

**Corollary 3.4** *Assume the conditions of Theorem 3.3 to hold and let  $\Omega_0$ ,  $y_n$  and  $t_n$  be as in Theorem 3.3. Then, we have for every  $\omega \in \{\tau^* < \infty\} \cap \Omega_0$ ,*

$$\int |X(t_n(\omega), x)|^2 dx - \|Q\|_{L^2}^2 \delta_{y_n(\omega)} \rightarrow 0 \text{ in distribution, as } n \rightarrow \infty. \tag{3.3}$$

**Remark 3.5** (i) In [55], Merle proved the rigidity of the critical mass blow-up solutions to (NLS), which are unique up to the symmetries of the equation. See also [40, 68] for simplified proofs. In the stochastic case, this strong rigidity is still unclear. The obstructions are due to the lack of pseudo-conformal symmetry, and the failure of the energy conservation and the virial identity. In the energy class with asymptotic rate  $(T - t)^{3+}$ , in view of Theorem 2.4 the uniqueness of stochastic critical mass blow-up solutions holds.

(ii) A sufficient condition for the finite time blow-up, i.e.,  $\mathbb{P}(\tau^* < \infty) > 0$ , is that  $H(X_0) < 0$  and  $\sum_{k=1}^N \|\nabla \phi_k\|_{L^\infty} \leq \varepsilon$  with  $\varepsilon$  small enough, see [7, Proposition 3.1]. In

particular, if  $H(X_0) < 0$  and  $\{\phi_l\}$  are constants (i.e., the noise  $W$  is spatially independent), then (SNLS) reduces to the classical NLS

$$idy = \Delta ydt + |y|^{\frac{4}{d}} ydt$$

with  $y(0) = X_0$ . Thus, the explosion time  $\tau^*$  is exactly the same as in the deterministic case.

In the general case, the numerical results in [24–26] suggest that, though the smooth multiplicative noise cannot prevent blow-up, i.e.,  $\tau^* < \infty$ ,  $\mathbb{P}$ -a.s., it has the effect to delay blow-up. The rigorous proof remains unclear. In the following we show that along certain sequence  $t_n \rightarrow \tau^*$ , the virial will tend to zero, which might be of use to understand the blow-up time.

**Theorem 3.6** (Vanishing of the virial at blow-up time) *Consider (SNLS) with  $X_0 \in H^1$ . Assume (A0) to hold. Assume additionally that  $\|X_0\|_{L^2} = \|Q\|_{L^2}$  and  $\mathbb{P}(\tau^* < \infty) = 1$ . Let  $t_n$  and  $y_n$ ,  $n \in \mathbb{N}$ , be as in Theorem 3.3. Then, there exists a measurable set  $\Omega_1$  with full probability such that for any  $\omega \in \Omega_1$ , there exists  $y^*(\omega) \in \mathbb{R}^d$  such that  $y_n(\omega) \rightarrow y^*(\omega)$  and*

$$\lim_{n \rightarrow \infty} \int |x - y^*(\omega)|^2 |X(t_n, x)(\omega)|^2 dx = 0. \tag{3.4}$$

Moreover, we have the following lower bound of the blow-up rate along the sequence  $\{t_n(\omega)\}$ ,

$$\|\nabla X(t_n)(\omega)\|_{L^2}^2 \geq C(\tau^*(\omega))(\tau^*(\omega) - t_n(\omega))^{-2}. \tag{3.5}$$

The proofs of Theorems 3.1, 3.3 and 3.6 are inspired by [3, 40, 55, 68] and are contained in the subsequent Sections 3.1, 3.2 and 3.3, respectively.

### 3.1 Concentration of $L^2$ -norm

Let us start with the compactness result in [40].

**Theorem 3.7** (Theorem 1.1, [40]) *Let  $\{v_n\}_{n=1}^\infty$  be a bounded family of  $H^1$  such that*

$$\limsup_{n \rightarrow \infty} \|\nabla v_n\|_{L^2} \leq M, \quad \limsup_{n \rightarrow \infty} \|v_n\|_{L^{2+\frac{4}{d}}} \geq m.$$

Then, there exists  $\{x_n\}_{n=1}^\infty \subseteq \mathbb{R}^d$  such that, up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly,}$$

with

$$\|V\|_{L^2} \geq \left(\frac{d}{d+2}\right)^{\frac{d}{4}} \left(\frac{m^{\frac{d}{2}+1}}{M^{\frac{d}{2}}}\right) \|Q\|_{L^2}.$$

**Lemma 3.8** (Sharp Gagliardo–Nirenberg inequality, [78]) *For any  $v \in H^1$ , we have*

$$\|v\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \leq \left(\frac{d+2}{d}\right) \left(\frac{\|v\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{d}} \|\nabla v\|_{L^2}^2.$$

In particular, for the Hamiltonian given by (1.3), we have

$$H(v) \geq \frac{1}{2} \left(1 - \left(\frac{\|v\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{d}}\right) \|\nabla v\|_{L^2}^2, \tag{3.6}$$

and  $H(v) \geq 0$  if  $\|v\|_{L^2} \leq \|Q\|_{L^2}$ .

The following result extends Banica’s inequality in [3] to the stochastic case.

**Lemma 3.9** (Banica type estimate) *Let  $X$  be the solution to (SNLS) on  $[0, \tau^*)$ . Suppose that  $\|X_0\|_{L^2} \leq \|Q\|_{L^2}$ . Then,  $\mathbb{P}$ -a.s. for any function  $\phi \in W^{1,\infty}$  and for any  $t \in [0, \tau^*)$  we have*

$$\left| \operatorname{Im} \int X(t) \nabla \overline{X(t)} \cdot \nabla \phi dx \right| \leq \left( 2H(X(t)) \int |X(t) \nabla \phi|^2 dx \right)^{\frac{1}{2}}. \tag{3.7}$$

**Proof** Thanks to the conservation law of mass, we adapt the arguments as in the proof of [3, Lemma 2.1]. On one hand, by (3.6) and the conservation of mass, for any  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned} H(e^{i\alpha\phi} X(t)) &\geq \frac{1}{2} \left( 1 - \left( \frac{\|e^{i\alpha\phi} X(t)\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{4}{d}} \right) \|\nabla(e^{i\alpha\phi} X(t))\|_{L^2}^2 \\ &\geq \frac{1}{2} \left( 1 - \left( \frac{\|X_0\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{4}{d}} \right) \|\nabla(e^{i\alpha\phi} X(t))\|_{L^2}^2 \geq 0. \end{aligned}$$

On the other hand, a straightforward expansion shows that

$$H(e^{i\alpha\phi} X(t)) = \frac{1}{2} \|\nabla \phi X(t)\|_{L^2}^2 \alpha^2 - \alpha \left( \operatorname{Im} \int X(t) \nabla \overline{X(t)} \cdot \nabla \phi dx \right) + H(X(t)).$$

Thus, we arrive at

$$\frac{1}{2} \|\nabla \phi X(t)\|_{L^2}^2 \alpha^2 - \alpha \left( \operatorname{Im} \int X(t) \nabla \overline{X(t)} \cdot \nabla \phi dx \right) + H(X(t)) \geq 0,$$

which yields (3.7) due to the arbitrariness of  $\alpha$ . □

**Lemma 3.10** (Control of Hamiltonian) *Let  $X$  be the solution to (SNLS) on  $[0, \tau^*)$ . Suppose that  $\|X_0\|_{L^2} \leq \|Q\|_{L^2}$ . Then, for any  $T > 0$ ,*

$$\mathbb{E} \sup_{t \in [0, \tau^* \wedge T)} H(X(t)) \leq C(T) < \infty. \tag{3.8}$$

*In particular,*

$$\sup_{t \in [0, \tau^* \wedge T)} H(X(t)) < \infty, \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

We remark that unlike in the NLS case, the Hamiltonian fails to be conserved in the stochastic case. Lemma 3.10 shows that the finiteness of the Hamiltonian can be derived in every bounded time regime, which is sufficient for the blow-up analysis.

**Proof of Lemma 3.10** As in [5, Theorem 3.1] we have the evolution formula of the Hamiltonian

$$\begin{aligned} H(X(t)) &= H(X_0) + \sum_{l=1}^N \int_0^t \frac{1}{2} \|\nabla \phi_l X(s)\|_{L^2}^2 ds - \sum_{l=1}^N \int_0^t \operatorname{Im} \langle \nabla \phi_l X(s), \nabla X(s) \rangle_2 dB_l(s) \\ &=: H(X_0) + H_1(t) + H_2(t). \end{aligned} \tag{3.10}$$

Note that, by the conservation law of mass,

$$\mathbb{E} \sup_{s \in [0, \tau^* \wedge t)} H_1(s) \leq \frac{1}{2} \left( \sum_{l=1}^N \|\nabla \phi_l\|_{L^\infty}^2 \right) \|X_0\|_{L^2}^2 t.$$

Moreover, by the Burkholder–Davis–Gundy inequality and the Banica type inequality (3.7),

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, \tau^* \wedge t]} |H_2(s)| &\leq C \mathbb{E} \left( \int_0^{\tau^* \wedge t} \sum_{l=1}^N |\operatorname{Im} \langle \nabla \phi_l X(s), \nabla X(s) \rangle_2|^2 ds \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} C \mathbb{E} \left( \int_0^{\tau^* \wedge t} \sum_{l=1}^N H(X(s)) \|X(s)\| \nabla \phi_l \|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} C \|X_0\|_{L^2} \left( \sum_{l=1}^N \|\nabla \phi_l\|_{L^\infty}^2 \right)^{\frac{1}{2}} \mathbb{E} \left( \int_0^{\tau^* \wedge t} H(X(s)) ds \right)^{\frac{1}{2}}. \end{aligned}$$

Note that  $H(X(s)) \geq 0$  for any  $s < \tau^*$ , due to Lemma 3.8 and the fact that  $\|X(s)\|_{L^2} = \|X_0\|_{L^2} \leq \|Q\|_{L^2}$ .

Thus, plugging the estimates above into (3.10) we get that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, \tau^* \wedge t]} H(X(s)) &\leq CT + C \mathbb{E} \left( \int_0^t \sup_{r \in [0, \tau^* \wedge s]} H(X(r)) ds \right)^{\frac{1}{2}} \\ &\leq C(1 + T) + C \int_0^t \mathbb{E} \sup_{r \in [0, \tau^* \wedge s]} H(X(r)) ds, \quad t \in [0, T], \end{aligned}$$

which along with the Gronwall inequality implies that

$$\mathbb{E} \sup_{t \in [0, \tau^* \wedge T]} H(X(t)) \leq C(1 + T)e^{CT} < \infty,$$

thereby yielding (3.8) and finishing the proof. □

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1** Set  $\lambda(t) := \frac{\|\nabla Q\|_{L^2}}{\|\nabla X(t)\|_{L^2}}$  and rescale  $X_\lambda(t, x) := \lambda(t)^{\frac{d}{2}} X(t, \lambda(t)x)$ ,  $t \in [0, \tau^*)$ ,  $x \in \mathbb{R}^d$ . Then, for any  $t \in [0, \tau^*)$ ,

$$\|X_\lambda(t)\|_{L^2}^2 = \|X(t)\|_{L^2}^2 = \|Q\|_{L^2}^2, \quad \|\nabla X_\lambda(t)\|_{L^2}^2 = \lambda(t)^2 \|\nabla X(t)\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2. \tag{3.11}$$

In view of (1.3) and (3.10), we deduce that for  $t \in [0, \tau^*)$ ,

$$\begin{aligned} H(X_\lambda(t)) &= \lambda^2(t) H(X(t)) \\ &= \|\nabla Q\|_{L^2}^2 \left( \frac{H(X_0)}{\|\nabla X(t)\|_{L^2}^2} + \frac{1}{2} \sum_{l=1}^N \frac{\int_0^t \|\nabla \phi_l X(s)\|_{L^2}^2 ds}{\|\nabla X(t)\|_{L^2}^2} - \frac{M(t)}{\|\nabla X(t)\|_{L^2}^2} \right), \tag{3.12} \end{aligned}$$

where

$$M(t) := \sum_{l=1}^N \int_0^t \operatorname{Im} \langle \nabla \phi_l X(s), \nabla X(s) \rangle_2 dB_l(s).$$

Note that since  $\int_0^t \|\nabla \phi_l X(s)\|_{L^2}^2 ds \leq \|\nabla \phi_l\|_{L^\infty}^2 \|X_0\|_{L^2}^2 t$  and  $\|\nabla X(t)\|_{L^2} \rightarrow \infty$  as  $t \rightarrow \tau^*$ ,

$$\|\nabla Q\|_{L^2}^2 \left( \frac{H(X_0)}{\|\nabla X(t)\|_{L^2}^2} + \frac{1}{2} \sum_{l=1}^N \frac{\int_0^t \|\nabla \phi_l X(s)\|_{L^2}^2 ds}{\|\nabla X(t)\|_{L^2}^2} \right) \rightarrow 0 \quad \text{as } t \rightarrow \tau^*. \tag{3.13}$$

Regarding the term  $M(t)$ , using the time-change for martingales (see e.g. [42, Theorem 16.4]) we deduce that there exists a Brownian motion  $\tilde{B}$  such that

$$M(t) = \tilde{B}(\langle M \rangle_t), \quad t \in (0, \tau^*), \quad \text{a.s.}, \tag{3.14}$$

where  $\langle M \rangle$  denotes the quadratic variation process of  $M$ , i.e.,

$$\langle M \rangle_t = \sum_{l=1}^N \int_0^t (\text{Im} \langle \nabla \phi_l X(s), \nabla X(s) \rangle_2)^2 ds, \quad t \in [0, \tau^*). \tag{3.15}$$

Moreover, by the law of iterated logarithm for Brownian motions (see [43, Theorem 11.18]),

$$\limsup_{t \rightarrow \tau^*} \frac{|M(t)|}{\sqrt{2 \langle M \rangle_t \ln \ln \langle M \rangle_t}} = 1, \quad \text{a.s.} \tag{3.16}$$

Hence, we can take a measurable set  $\Omega_0 \subseteq \Omega$  with full probability such that both (3.14) and (3.16) hold on  $\Omega_0$ . Below we consider  $\omega \in \{\tau^* < \infty\} \cap \Omega_0$ . For simplicity, the dependence on  $\omega$  is omitted in the notation.

Let  $t_n := \inf\{t \in (0, \tau^*) : \|\nabla X(t)\|_{L^2} \geq n\}$ ,  $n \geq N_0 := 2\|X_0\|_{H^1}$ . Then, using the continuity of  $X$  in  $H^1$  and the explosion of  $\|\nabla X(t)\|_{L^2}$  at  $\tau^*$  we have  $t_n \rightarrow \tau^*$  and  $\sup_{t \in [0, t_n]} \|\nabla X(t)\|_{L^2} = \|\nabla X(t_n)\|_{L^2}$ .

Then, in the case where  $\sup_{0 \leq t < \tau^*} \langle M \rangle_t < \infty$ , using the continuity of Brownian motion we have  $\sup_{0 \leq t < \tau^*} |\tilde{B}(\langle M \rangle_t)| < \infty$ , and thus

$$\frac{|M(t)|}{\|\nabla X(t)\|_{L^2}^2} \leq \frac{\sup_{0 \leq t < \tau^*} |\tilde{B}(\langle M \rangle_t)|}{\|\nabla X(t)\|_{L^2}^2} \rightarrow 0 \quad \text{as } t \rightarrow \tau^*. \tag{3.17}$$

Moreover, in the case where  $\langle M \rangle_t \rightarrow \infty$  as  $t \rightarrow \tau^*$ , by (3.15) and Hölder’s inequality,

$$\begin{aligned} \langle M \rangle_{t_n} &\leq \sum_{l=1}^N \|\nabla \phi_l\|_{L^\infty}^2 \|X_0\|_{L^2}^2 \int_0^{t_n} \|\nabla X(s)\|_{L^2}^2 ds \\ &\leq \sum_{l=1}^N t_n \|\nabla \phi_l\|_{L^\infty}^2 \|X_0\|_{L^2}^2 \|\nabla X(t_n)\|_{L^2}^2. \end{aligned}$$

which along with (3.16) yields that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{|M(t_n)|}{\|\nabla X(t_n)\|_{L^2}^2} &\leq \limsup_{n \rightarrow \infty} \frac{\sqrt{2 \langle M \rangle_{t_n} \ln \ln \langle M \rangle_{t_n}}}{\|\nabla X(t_n)\|_{L^2}^2} \\ &\leq C \limsup_{n \rightarrow \infty} \frac{\sqrt{\ln \ln \|\nabla X(t_n)\|_{L^2}}}{\|\nabla X(t_n)\|_{L^2}} = 0. \end{aligned} \tag{3.18}$$

Thus, we conclude from (3.17) and (3.18) that

$$\frac{M(t_n)}{\|\nabla X(t_n)\|_{L^2}^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Taking into account (3.12) and (3.13), we obtain that the Hamiltonian of the rescaled solution tends to zero along the sequence  $\{t_n\}$ , namely,

$$\lim_{n \rightarrow \infty} H(X_{\lambda_n}(t_n)) = 0, \tag{3.19}$$

where  $\lambda_n := \lambda(t_n) = \|\nabla Q\|_{L^2} / \|\nabla X(t_n)\|_{L^2}$ ,  $n \geq 1$ .

Now, it follows from (3.11) and (3.19) that

$$\lim_{n \rightarrow \infty} \frac{d}{4 + 2d} \|X_{\lambda_n}(t_n)\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} = \lim_{n \rightarrow \infty} \left( \frac{1}{2} \|\nabla X_{\lambda_n}(t_n)\|_{L^2}^2 + H(X_{\lambda_n}(t_n)) \right) = \frac{1}{2} \|\nabla Q\|_{L^2}^2,$$

which implies that

$$\lim_{n \rightarrow \infty} \|X_{\lambda_n}(t_n)\|_{L^{2+\frac{4}{d}}} = \left( \frac{2+d}{d} \right)^{\frac{d}{4+2d}} \|\nabla Q\|_{L^2}^{\frac{d}{2}}.$$

Thus, by virtue of Theorem 3.7, we obtain a profile  $V \in H^1$  and a sequence of concentration points  $\{y_n\} \subseteq \mathbb{R}^d$  such that up to a subsequence (still denoted by  $\{n\}$ )

$$\|Q\|_{L^2} \leq \|V\|_{L^2}, \tag{3.20}$$

$$X_{\lambda_n}(t_n, \cdot + y_n) \rightharpoonup V \quad \text{weakly in } H^1, \text{ as } n \rightarrow \infty. \tag{3.21}$$

In particular, for any  $A > 0$  and  $R > 0$ ,

$$\begin{aligned} \int_{|x| \leq A} |V(x)|^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{|x| \leq A} \lambda_n^d |X(t_n, \lambda_n x + y_n)|^2 dx \\ &= \liminf_{n \rightarrow \infty} \int_{|x - y_n| \leq \lambda_n A} |X(t_n, x)|^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{|x - y_n| \leq R} |X(t_n, x)|^2 dx, \end{aligned} \tag{3.22}$$

where the last step is due to the fact that  $\lambda_n A \leq R$  for  $n$  large enough.

Therefore, letting  $A \rightarrow \infty$  and using (3.20) we obtain (3.1) and finish the proof.  $\square$

### 3.2 Universality of Stochastic Critical Mass Blow-up Profiles

We first recall the variational characterization of the ground state.

**Lemma 3.11** (Variational characterization of ground state, [68]) *Let  $v \in H^1$  be such that*

$$\|v\|_{L^2} = \|Q\|_{L^2}, \quad H(v) = 0.$$

*Then, there exist parameters  $\lambda_0 \in \mathbb{R}^+$ ,  $x_0 \in \mathbb{R}^d$  and  $\theta_0 \in \mathbb{R}$  such that*

$$v(x) = \lambda_0^{\frac{d}{2}} Q(\lambda_0 x + x_0) e^{i\theta_0}, \quad x \in \mathbb{R}^d.$$

Now we are prepared to prove Theorem 3.3.

**Proof of Theorem 3.3** Let  $\Omega_0, t_n, y_n$  and the profile  $V$  be as in Theorem 3.1. Below we consider  $\omega \in \{\tau^* < \infty\} \cap \Omega_0$ . For simplicity, we suppress the  $\omega$  dependence in the notation.

We claim that

$$\|V\|_{L^2} = \|Q\|_{L^2}, \tag{3.23}$$

$$\|\nabla V\|_{L^2} = \|\nabla Q\|_{L^2}, \tag{3.24}$$

$$H(V) = 0. \tag{3.25}$$

For this purpose, using (3.20) and the conservation of mass we note that

$$\|X_{\lambda_n}(\cdot + y_n)\|_{L^2} = \|X_0\|_{L^2} = \|Q\|_{L^2} \leq \|V\|_{L^2}, \tag{3.26}$$

where  $\lambda_n := \lambda(t_n)$ . Moreover, (3.22) yields that,

$$\|V\|_{L^2}^2 \leq \liminf_{n \rightarrow \infty} \int_{|x-y_n| \leq R} |X(t_n, x)|^2 dx \leq \|X(t_n)\|_{L^2}^2 = \|Q\|_{L^2}^2. \tag{3.27}$$

Thus, (3.23) follows immediately from (3.26) and (3.27).

Regarding (3.24) and (3.25), in view of (3.21) and (3.26), we first have

$$X_{\lambda_n}(t_n, \cdot + y_n) \rightarrow V \quad \text{in } L^2, \text{ as } n \rightarrow \infty, \tag{3.28}$$

which along with the uniform boundedness of  $\|\nabla X_{\lambda_n}(t_n)\|_{L^2}$ , implied by (3.11), and interpolation yields the strong convergence in  $L^{2+4/d}$ , i.e.,

$$X_{\lambda_n}(t_n, \cdot + y_n) \rightarrow V \quad \text{in } L^{2+\frac{4}{d}}, \text{ as } n \rightarrow \infty. \tag{3.29}$$

In order to prove the strong convergence in  $H^1$ , i.e.,

$$X_{\lambda_n}(t_n, \cdot + x_n) \rightarrow V \quad \text{in } H^1, \text{ as } n \rightarrow \infty. \tag{3.30}$$

We note that, on one hand, by (3.11), (3.19) and (3.29),

$$\begin{aligned} \|\nabla Q\|_{L^2}^2 &= \lim_{n \rightarrow \infty} \|\nabla X_{\lambda_n}(\cdot + y_n)\|_{L^2}^2 \\ &= \lim_{n \rightarrow \infty} 2H(X_{\lambda_n}(\cdot + y_n)) + \frac{d}{d+2} \|X_{\lambda_n}(\cdot + y_n)\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \\ &= \frac{d}{d+2} \|V\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}, \end{aligned}$$

which along with Lemma 3.8 and (3.23) yields

$$\|\nabla Q\|_{L^2}^2 \leq \|\nabla V\|_{L^2}^2.$$

On the other hand, (3.11) and (3.21) yield that

$$\|\nabla V\|_{L^2}^2 \leq \liminf_{n \rightarrow \infty} \|\nabla X_{\lambda_n}(t_n, \cdot + y_n)\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2. \tag{3.31}$$

Thus, we obtain

$$\|\nabla V\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2 = \|\nabla X_{\lambda_n}(\cdot + y_n)\|_{L^2}^2,$$

which yields (3.24), and, via (3.21) and (3.28), also gives the strong convergence (3.30) in  $H^1$ .

Therefore, combining (3.29) and (3.30), we obtain (3.25), as claimed.

Now, by virtue of (3.23), (3.24), (3.25) and the variational characterization of the ground state  $Q$  in Lemma 3.11, we obtain that there exist  $x_0 \in \mathbb{R}^d$  and  $\theta_0 \in [0, 2\pi)$  such that

$$V(x) = e^{i\theta_0} Q(x + x_0),$$

which along with (3.30) implies (3.2). The proof is complete. □

**Proof of Corollary 3.4** In order to prove (3.3), it suffices to prove that for any  $\varphi \in C_b^\infty$ ,

$$\int \varphi(x) |X(t_n, x)|^2 dx - \|Q\|_{L^2}^2 \varphi(y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For this purpose, by a change of variables we obtain

$$\int \varphi(x) |X(t_n, x)|^2 dx = \int \varphi(\lambda_n x + y_n) |\lambda_n^{\frac{d}{2}} X(t_n, \lambda_n x + y_n)|^2 dx.$$

This yields

$$\begin{aligned} & \left| \int \varphi(x) |X(t_n, x)|^2 dx - \varphi(y_n) \|Q\|_{L^2}^2 \right| \\ & \leq \int |\varphi(\lambda_n x + y_n)| \left| |\lambda_n^{\frac{d}{2}} X(t_n, \lambda_n x + y_n)|^2 - |Q(x)|^2 \right| dx \\ & \quad + \int |\varphi(\lambda_n x + y_n) - \varphi(y_n)| |Q(x)|^2 dx \\ & \leq 2 \|Q\|_{L^2} \|\varphi\|_{L^\infty} \|\lambda_n^{\frac{d}{2}} X(t_n, \lambda_n x + y_n) e^{i\theta(t_n)} - Q(x)\|_{L^2} + \lambda_n \|\varphi'\|_{L^\infty} \int |x| |Q(x)|^2 dx. \end{aligned}$$

Thus, in view of (3.2), the right-hand side above converges to zero. The proof is complete.  $\square$

### 3.3 Vanishing of Virial

We take the full probability set  $\Omega_0$  such that Theorem 3.3, Corollary 3.4 and Lemma 3.10 hold on  $\Omega_0$ . For every  $\omega \in \Omega_0$ , let  $\{t_n(\omega)\}$  and  $\{y_n(\omega)\}$  be as in Theorem 3.3. Up to a subsequence (still denoted by  $n$ ), we may assume that  $y_n(\omega) \rightarrow y^*(\omega)$ ,  $y^*(\omega)$  may be the Alexandrov point of  $\mathbb{R}^d$ . In the case where  $|y^*(\omega)| < \infty$ , we set

$$V_m(t)(\omega) := \int \theta_m(x - y^*(\omega)) |X(t)(\omega)|^2 dx, \tag{3.32}$$

where  $\theta_m(x) := m^2 \theta(\frac{x}{m})$ ,  $\theta \in C_0^\infty$  satisfies  $\theta(x) = |x|^2$  for  $x \in B(0, 1)$ ,  $|\nabla \theta(x)|^2 \leq C \theta(x)$  for  $x \in \mathbb{R}^d$ ,  $C > 0$ .

In order to analyze the virial functional (3.32), we shall derive an evolution formula on a full probability set in Lemma 3.13 below. The delicate fact is that, it is unclear whether  $y^*$  is measurable with respect to  $\omega$ , so the application of Itô’s formula as in [7] is not possible directly. Below we show that the evolution formula can be derived for  $V_m(t)$ , but with  $y^*(\omega)$  replaced by any fixed  $y \in \mathbb{R}^d$ . The point is that this formula holds for any  $y \in \mathbb{R}^d$  on a universal full probability set  $\Omega_1$ , but the null set  $\Omega \setminus \Omega_1$  is independent of  $y$ . Thus, one can eventually replace  $y$  by  $y^*(\omega)$  in the evolution formula.

The proof is based on a mollification procedure. Let  $\varphi \in C_c^\infty$  be a real-valued non-negative function with unit integral,  $\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(\frac{x}{\varepsilon})$ ,  $x \in \mathbb{R}^d$ . Set  $h^\varepsilon := h * \varphi_\varepsilon$  for any locally integrable function  $h$  mollified by  $\varphi_\varepsilon$ ,  $f^\varepsilon(X) := (|X|^{\frac{d}{2}} X)^\varepsilon$  and  $f(X) := |X|^{\frac{d}{2}} X$ . We have that  $\mathbb{P}$ -a.s. for any  $t \in (0, \tau^*)$  and any  $q \in (1, \infty)$ , as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} X_0^\varepsilon &\rightarrow X_0 \quad \text{in } L^2, & X^\varepsilon &\rightarrow X \in L^q(0, t; H^1) \cap L^{2+\frac{4}{d}}(0, t; L^{2+\frac{4}{d}}), \\ f^\varepsilon(X) &\rightarrow f(X) \quad \text{in } L^{\frac{2d+4}{d+4}}(0, t; L^{\frac{2d+4}{d+4}}). \end{aligned} \tag{3.33}$$

**Lemma 3.12** *Consider the situation as in Theorem 3.6. Then, for any sequence  $\varepsilon_n \rightarrow 0$  when  $n \rightarrow \infty$ , there exists a further subsequence  $\{\varepsilon_{n_j}\}$  and a full probability set  $\tilde{\Omega}_0$ , such that for any  $\omega \in \tilde{\Omega}_0$  and any  $y \in \mathbb{R}^d \cap Q^d$ ,*

$$\begin{aligned} & \sum_{l=1}^N \left| \int_0^t \int \theta_m(x - y) \bar{X}^{\varepsilon_{n_j}}(s, x) ((X \phi_l)^{\varepsilon_{n_j}} - X^{\varepsilon_{n_j}} \phi_l)(s, x) dx dB_l(s) \right| \rightarrow 0 \\ & \text{as } \varepsilon_{n_j} \rightarrow 0, t \in (0, \tau^*). \end{aligned} \tag{3.34}$$

**Proof** Fix  $1 \leq l \leq N$ . For any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$  and any  $M \geq 1$ , we use the Burkholder–Davis–Gundy inequality and the conservation of mass to derive

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, \tau^* \wedge M]} \left| \int_0^s \int \theta_m(x - y) \overline{X}^{\varepsilon_n}(s, x) ((X\phi_l)^{\varepsilon_n} - X^{\varepsilon_n} \phi_l)(s, x) dx dB_l(s) \right|^2 \\ &= \mathbb{E} \left( \int_0^{\tau^* \wedge M} \left| \int \theta_m(x - y) \overline{X}^{\varepsilon_n}(s, x) ((X\phi_l)^{\varepsilon_n} - X^{\varepsilon_n} \phi_l)(s, x) dx \right|^2 ds \right)^{\frac{1}{2}} \\ &\lesssim m^2 \|X_0\|_{L^2} \mathbb{E} \left( \int_0^{\tau^* \wedge M} \|(X\phi_l)^{\varepsilon_n}(s) - X^{\varepsilon_n}(s)\phi_l\|_{L^2}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $\mathbb{P}$ -a.s. for any  $s \in (0, \tau^* \wedge M)$ ,

$$\|(X\phi_l)^{\varepsilon_n}(s) - X^{\varepsilon_n}(s)\phi_l\|_{L^2}^2 \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0,$$

and

$$\|(X\phi_l)^{\varepsilon_n}(s) - X^{\varepsilon_n}(s)\phi_l\|_{L^2}^2 \lesssim \|\phi_l\|_{L^\infty}^2 \|X(s)\|_{L^2}^2 \lesssim \|\phi_l\|_{L^\infty}^2 \|X_0\|_{L^2}^2,$$

the dominated convergence theorem gives

$$\mathbb{E} \sup_{s \in [0, \tau^* \wedge M]} \left| \int_0^s \int \theta_m(x - y) \overline{X}^{\varepsilon_n}(s, x) ((X\phi_l)^{\varepsilon_n} - X^{\varepsilon_n} \phi_l)(s, x) dx dB_l(s) \right|^2 \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0,$$

which implies that

$$\sup_{s \in [0, \tau^* \wedge M]} \left| \int_0^s \int \theta_m(x - y) \overline{X}^{\varepsilon_n}(s, x) ((X\phi_l)^{\varepsilon_n} - X^{\varepsilon_n} \phi_l)(s, x) dx dB_l(s) \right|^2 \rightarrow 0$$

in probability, as  $\varepsilon_n \rightarrow 0$ .

It follows that for any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$  and any  $M \geq 1$ , there exists a subsequence  $\{\varepsilon_{n_j}\}$  such that  $\mathbb{P}$ -a.s.,

$$\sup_{s \in [0, \tau^* \wedge M]} \left| \int_0^s \int \theta_m(x - y) \overline{X}^{\varepsilon_{n_j}}(s, x) ((X\phi_l)^{\varepsilon_{n_j}} - X^{\varepsilon_{n_j}} \phi_l)(s, x) dx dB_l(s) \right|^2 \rightarrow 0 \quad \text{as } \varepsilon_{n_j} \rightarrow 0. \tag{3.35}$$

Using a diagonal argument we can select a further subsequence, still denoted by  $\{\varepsilon_{n_j}\}$ , and a full probability set  $\widetilde{\Omega}_0$ , such that for any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$  and any  $M \geq 1$ , the convergence (3.35) holds on  $\widetilde{\Omega}_0$ . Taking into account that

$$\{t < \tau^*\} = \bigcup_{M \geq 1} \{t < \tau^* \wedge M\},$$

we thus obtain (3.34) and finish the proof. □

The key evolution formula of the virial  $V_m$  defined in (3.32) is contained in Lemma 3.13 below.

**Lemma 3.13** (Evolution of virial) *Consider the situation as in Theorem 3.6. Suppose that  $|y^*| < \infty$ . Then, there exists a measurable set  $\Omega_1$  with full probability such that for every  $\omega \in \Omega_1$ ,*

$$V_m(t)(\omega) = V_m(0) - 2\text{Im} \int_0^t \langle \nabla \theta_m(\cdot - y^*(\omega)) X(s)(\omega), \nabla X(s)(\omega) \rangle ds, \quad 0 \leq t < \tau^*(\omega). \tag{3.36}$$

**Proof** Taking convolution of both sides of (SNLS) with the mollifier  $\varphi_\varepsilon$  we derive from (SNLS) that for every  $x \in \mathbb{R}^d$ ,

$$X^\varepsilon(t, x) = \int_0^t (i \Delta X^\varepsilon + i f^\varepsilon(X) - (\mu X)^\varepsilon)(s, x) ds + \sum_{l=1}^N \int_0^t i X \phi_l^\varepsilon(s, x) dB_l(s), \quad t \in (0, \tau^*).$$

This along with the product rule yields

$$\begin{aligned} |X^\varepsilon(t, x)|^2 &= |X^\varepsilon(0, x)|^2 + 2\text{Re} \int_0^t [\overline{X}^\varepsilon (i \Delta X^\varepsilon + i f^\varepsilon(X) - (\mu X)^\varepsilon)](s, x) ds \\ &\quad + \sum_{l=1}^N \int_0^t |(X \phi_l)^\varepsilon(s, x)|^2 ds \\ &\quad - 2\text{Im} \sum_{l=1}^N \int_0^t \overline{X}^\varepsilon(s, x) (X \phi_l)^\varepsilon(s, x) dB_l(s), \quad t \in (0, \tau^*), \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.37}$$

Since both sides are continuous in  $x$ , we can find a universal null set  $\tilde{N}$  such that for every  $\omega \in \Omega \setminus \tilde{N}$ , (3.37) holds for any  $x \in \mathbb{R}^d$ .

Thus, we can take a full probability set  $\Omega_1$  such that Theorem 3.3, Lemmas 3.10 and 3.12 and (3.37) hold on  $\Omega_1$ . Below we consider  $\omega \in \Omega_1$  and omit the argument  $\omega$  in the notation for simplicity.

For any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$ , set

$$V_{m,y}^\varepsilon(t) := \int \theta_m(x - y) |X^\varepsilon(t, x)|^2 dx, \quad t \in (0, \tau^*).$$

Using (3.37) we have the following identity on  $\Omega_1$ :

$$\begin{aligned} V_{m,y}^\varepsilon(t) &= \int \theta_m(x - y) |X^\varepsilon(0, x)|^2 dx + \sum_{l=1}^N \int \theta_m(x - y) \int_0^t |(X \phi_l)^\varepsilon(s, x)|^2 ds dx \\ &\quad + 2\text{Re} \int \theta_m(x - y) \int_0^t [\overline{X}^\varepsilon (i \Delta X^\varepsilon + i f(X)^\varepsilon - (\mu X)^\varepsilon)](s, x) ds dx \\ &\quad - 2 \sum_{l=1}^N \text{Im} \int \theta_m(x - y) \int_0^t \overline{X}^\varepsilon(s, x) (X \phi_l)^\varepsilon(s, x) dB_l(s) dx \\ &=: \int \theta_m(x - y) |X^\varepsilon(0, x)|^2 dx + I_{1,y}^\varepsilon(t) + I_{2,y}^\varepsilon(t) + I_{3,y}^\varepsilon(t), \quad t \in (0, \tau^*). \end{aligned} \tag{3.38}$$

Since  $\|\theta_m(\cdot - y)\|_{L^\infty} \lesssim m^2$ , we infer from (3.33) that

$$\int \theta_m(x - y) |X^\varepsilon(0, x)|^2 dx \rightarrow \int \theta_m(x - y) |X(0, x)|^2 dx = V_m(0) \quad \text{as } \varepsilon \rightarrow 0. \tag{3.39}$$

Below we pass to the limit  $\varepsilon \rightarrow 0$  for each term  $I_{j,y}^\varepsilon(t)$ ,  $1 \leq j \leq 3$ .

First, using (3.33) again we have

$$I_{1,y}^\varepsilon(t) \rightarrow 2 \int \theta_m(x - y) \int_0^t \mu(x) |X(s, x)|^2 ds dx = 2 \int_0^t \int \theta_m(x - y) \mu(x) |X(s, x)|^2 dx ds, \tag{3.40}$$

where  $t \in (0, \tau^*)$  and the last step is due to the Fubini theorem.

Regarding the second term  $I_{2,y}^\varepsilon(t)$ , using Fubini’s theorem and the integration-by-parts formula we derive

$$\begin{aligned} &2\operatorname{Re} \int \theta_m(x-y) \int_0^t \overline{X}^\varepsilon(s,x) i \Delta X^\varepsilon(s,x) ds dx \\ &= 2\operatorname{Im} \int_0^t \int \nabla X^\varepsilon(s,x) \cdot \nabla \theta_m(x-y) \overline{X}^\varepsilon(s,x) dx ds, \end{aligned}$$

which along with (3.33) and the fact that  $\|\nabla \theta_m(\cdot - y)\|_{L^\infty} \lesssim m$  yields that as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} &2\operatorname{Re} \int \theta_m(x-y) \int_0^t \overline{X}^\varepsilon(s,x) i \Delta X^\varepsilon(s,x) ds dx \\ &\rightarrow 2\operatorname{Im} \int_0^t \int \nabla X(s,x) \cdot \nabla \theta_m(x-y) \overline{X}(s,x) dx ds. \end{aligned} \tag{3.41}$$

Moreover, it follows from (3.33) again that

$$\begin{aligned} &2\operatorname{Re} \int \theta_m(x-y) \int_0^t \overline{X}^\varepsilon(s,x) i f(X)^\varepsilon(s,x) ds dx \\ &\rightarrow -2\operatorname{Im} \int \theta_m(x-y) \int_0^t \overline{X}(s,x) f(X)(s,x) ds dx = 0 \end{aligned} \tag{3.42}$$

and

$$2\operatorname{Re} \int \theta_m(x-y) \int_0^t \overline{X}^\varepsilon(s,x) (\mu X)^\varepsilon(s,x) ds dx \rightarrow 2\operatorname{Re} \int_0^t \int \theta_m(x-y) \mu(x) |X(s,x)|^2 dx ds. \tag{3.43}$$

Thus, it follows from (3.41), (3.42) and (3.43) that as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} I_{2,y}^\varepsilon(t) &\rightarrow 2\operatorname{Im} \int_0^t \int \nabla X(s,x) \cdot \nabla \theta_m(x-y) \overline{X}(s,x) dx ds \\ &\quad - 2\operatorname{Re} \int_0^t \int \theta_m(x-y) \mu(x) |X(s,x)|^2 dx ds, \quad t \in (0, \tau^*). \end{aligned} \tag{3.44}$$

Finally, for the last stochastic term  $I_{3,y}^\varepsilon(t)$ . Since

$$\operatorname{Im} \int \theta_m(x-y) \overline{X}^\varepsilon(s,x) X^\varepsilon(s,x) \phi_l(x) dx = 0,$$

using the stochastic Fubini theorem and Lemma 3.12 we obtain that there exists a sequence  $\{\varepsilon_n\}$  such that the following convergence holds on a full probability set  $\Omega_1$ : for any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$  and any  $t \in (0, \tau^*)$ ,

$$I_{3,y}^{\varepsilon_n}(t) = -2 \sum_{l=1}^N \operatorname{Im} \int_0^t \int \theta_m(x-y) \overline{X}^{\varepsilon_n}(s,x) ((X\phi_l)^{\varepsilon_n} - X^{\varepsilon_n} \phi_l)(s,x) dx dB_l(s) \rightarrow 0$$

as  $\varepsilon_n \rightarrow 0$ . (3.45)

Therefore, we conclude from (3.38), (3.39), (3.40), (3.44) and (3.45) that on the set  $\Omega_1$ , as  $\varepsilon_n \rightarrow 0$ , for any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$  and any  $t \in (0, \tau^*)$ ,

$$V_{m,y}^{\varepsilon_n}(t) \rightarrow \int \theta_m(x-y) |X(0,x)|^2 dx + 2\operatorname{Im} \int_0^t \int \nabla X(s,x) \cdot \nabla \theta_m(x-y) \overline{X}(s,x) dx ds.$$

Taking into account

$$V_{m,y}^{\varepsilon_n}(t) \rightarrow \int \theta_m(x - y)|X(t, x)|^2 dx, \quad t \in (0, \tau^*),$$

we obtain that for every  $\omega \in \Omega_1$ , it holds that for any  $y \in \mathbb{R}^d \cap \mathbb{Q}^d$ ,

$$\begin{aligned} \int \theta_m(x - y)|X(t, x)(\omega)|^2 dx &= \int \theta_m(x - y)|X(0, x)|^2 dx \\ &- 2\text{Im} \int_0^t \langle \nabla \theta_m(\cdot - y)X(s)(\omega), \nabla X(s)(\omega) \rangle ds, \quad t \in (0, \tau^*(\omega)). \end{aligned} \quad (3.46)$$

We see that both sides of (3.46) are continuous in  $y$ . Actually, for any  $y_n \rightarrow y, n \rightarrow \infty$ , since  $X(\omega) \in C([0, t]; H^1), t \in (0, \tau^*(\omega))$ , we have

$$|X(\omega)\nabla \theta_m(x - y) \cdot \nabla X(\omega)| \leq Cm|X(\omega)||\nabla X(\omega)| \in L^1((0, t) \times \mathbb{R}^d),$$

where the constant  $C$  is independent of  $\{y_n\}$ . Taking into account  $\nabla \theta_m(x - y_n) \rightarrow \nabla \theta_m(x - y)$  and using the dominated convergence theorem we get

$$\begin{aligned} &\int_0^t \langle \nabla \theta_m(\cdot - y_n)X(s)(\omega), \nabla X(s)(\omega) \rangle ds \\ &\rightarrow \int_0^t \langle \nabla \theta_m(\cdot - y)X(s)(\omega), \nabla X(s)(\omega) \rangle ds \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Similar arguments also apply to the other two terms in (3.46).

Thus, we obtain that, on the set  $\Omega_1$ , (3.46) holds for any  $y \in \mathbb{R}^d$ . In particular, the null set is independent of  $y \in \mathbb{R}^d$ . Therefore, we can replace  $y$  by  $y^*(\omega)$  in (3.46), where  $\omega \in \Omega_1$ , and prove (3.36).  $\square$

Next, we show the boundedness of the virial of solutions, based on the control of the Hamiltonian in Lemma 3.10.

**Lemma 3.14** (Control of virial) *Consider the situation as in Theorem 3.6. Set*

$$V(t) := \int |x|^2 |X(t, x)|^2 dx, \quad t \in [0, \tau^*].$$

Then, there exists a full probability set  $\Omega_1$  such that for every  $\omega \in \Omega_1$ ,

$$\sup_{t \in [0, \tau^*(\omega)]} V(t)(\omega) < \infty. \quad (3.47)$$

**Proof** We take the full probability sets  $\Omega_1$  as in the proof of Lemma 3.13, and so Theorem 3.3, Corollary 3.4 and Lemmas 3.10 and 3.13 hold on  $\Omega_1$ . Below we consider  $\omega \in \Omega_1$  and omit the argument  $\omega$ .

Recall that  $\{t_n\}$  and  $\{y_n\}$  are as in Theorem 3.3 and  $y_n \rightarrow y^*$ , where  $y^*$  may be the Alexandrov point of  $\mathbb{R}^d$ . In the case where  $|y^*| < \infty$ , applying Lemma 3.13 we have

$$V_m(X(t)) = V_m(X_0) - 2\text{Im} \int_0^t \langle \nabla \theta_m(\cdot - y^*)X(s), \nabla X(s) \rangle ds, \quad 0 \leq t < \tau^*.$$

By (3.7), (3.9) and  $|\nabla \theta_m|^2 \leq C\theta_m$ ,

$$\left| \frac{d}{dt} V_m(t) \right| \leq C(H(X(t))\|\nabla \theta_m(x - y^*)X(t)\|_{L^2}^2)^{\frac{1}{2}} \leq C(\tau^*)V_m^{\frac{1}{2}}(t),$$

where  $C(\tau^*)$  is independent of  $m$ . This yields that

$$\left| \sqrt{V_m(t)} - \sqrt{V_m(0)} \right| \leq \frac{1}{2} C(\tau^*) t, \quad t \in [0, \tau^*]. \tag{3.48}$$

In particular, we have the following uniform bound for all  $m, n \geq 1$ ,

$$\sqrt{V_m(0)} \leq \frac{1}{2} C(\tau^*) \tau^* + \sqrt{V_m(t_n)}.$$

Applying Corollary 3.4 we infer that  $V_m(t_n) \rightarrow 0$  as  $n \rightarrow \infty$ , and so the following uniform-in- $m$  boundedness holds

$$\sqrt{V_m(0)} \leq \frac{1}{2} C(\tau^*) \tau^* < \infty. \tag{3.49}$$

In particular, since  $\theta_m(x - y^*) \rightarrow |x - y^*|^2$  as  $m \rightarrow \infty$ , Fatou’s lemma yields

$$\int |x - y^*|^2 |X_0|^2 dx \leq \frac{1}{4} C(\tau^*)^2 (\tau^*)^2 < \infty.$$

Thus, using (3.48), (3.49), the conservation of mass and Fatou’s lemma again we obtain

$$\begin{aligned} V(t) &\leq |y^*|^2 \|X_0\|_{L^2}^2 + \int |x - y^*|^2 |X(t)|^2 dx \\ &\leq |y^*|^2 \|X_0\|_{L^2}^2 + \liminf_{m \rightarrow \infty} V_m(t) \\ &\leq |y^*|^2 \|X_0\|_{L^2}^2 + \liminf_{m \rightarrow \infty} \left( \frac{1}{2} C(\tau^*) \tau^* + \sqrt{V_m(0)} \right)^2 \\ &\leq |y^*|^2 \|X_0\|_{L^2}^2 + (C(\tau^*) \tau^*)^2 < \infty. \end{aligned}$$

thereby yielding (3.47) in the case where  $|y^*| < \infty$ .

The proof for the case where  $y^*$  is the Alexandrov point of  $\mathbb{R}^d$  is similar. In this case, we modify  $V_m(t)$  by  $\tilde{V}_m(t) := \int \theta_m(x) |X(t, x)|^2 dx$ . Then, estimate (3.48) still holds for  $\tilde{V}_m(t)$ , and by Corollary 3.4,  $\tilde{V}_m(t_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, using similar arguments as those below (3.48) we obtain (3.47) and finish the proof.  $\square$

**Proof of Theorem 3.6** Below we perform the analysis on the full probability set  $\Omega_1$  defined in the proof of Lemma 3.13. Recall that  $\{y_n\}$  are the concentration points as in Corollary 3.4 and  $y_n \rightarrow y^*$ .

We claim that

$$y^*(\omega) = \frac{\lim_{n \rightarrow \infty} \int x |X(t_n(\omega), x)(\omega)|^2 dx}{\|Q\|_{L^2}^2}. \tag{3.50}$$

To this end, suppose that  $\{y_n\}$  converges to infinity, then for  $R$  large enough and for  $n$  very large,  $\{x : |x - y_n| \leq 1\} \subseteq \{|x| \geq R\}$ , which, via the  $L^2$  concentration in Corollary 3.4, yields that

$$\|Q\|_{L^2}^2 \leq \int_{|x-y_n| \leq 1} |X(t_n, x)|^2 dx \leq \int_{|x| > R} |X(t_n, x)|^2 dx.$$

But on the other hand, by the finiteness of the virial (3.47),

$$\int_{|x| > R} |X(t_n, x)|^2 dx \leq \frac{V(t_n)}{R^2} \leq \frac{\sup_{t \in [0, \tau^*]} V(t)}{R^2} \leq \varepsilon$$

for  $R$  large enough, which leads to a contradiction. It follows that  $|y^*| < \infty$ .

Hence, we may take  $R$  large enough such that  $\{y_n\} \subseteq B_R(0) := \{x : |x| \leq R\}$ . Let  $\chi_R \in C_0^\infty$  be a cut-off function such that  $\chi_R(x) = 1$  for  $|x| \leq R$ , and  $\chi_R(x) = 0$  for  $|x| \geq R + 1$ . In particular,  $\chi_R(y_n)y_n = y_n$  for all  $n \geq 1$ . Then, we estimate

$$\begin{aligned} \left| \int |X(t_n)|^2 x dx - \|Q\|_{L^2}^2 y_n \right| &\leq \left| \int |X(t_n)|^2 (1 - \chi_R(x)) x dx \right| \\ &\quad + \left| \int |X(t_n)|^2 \chi_R(x) x dx - \chi_R(y_n) y_n \|Q\|_{L^2}^2 \right| \\ &\leq \frac{1}{R} \sup_{t \in [0, \tau^*]} V(t) + \left| \int |X(t_n)|^2 \chi_R(x) x dx - \chi_R(y_n) y_n \|Q\|_{L^2}^2 \right|. \end{aligned}$$

In view of Corollary 3.4, the second term above tends to zero, which yields that

$$\limsup_{n \rightarrow \infty} \left| \int |X(t_n)|^2 x dx - \|Q\|_{L^2}^2 x_n \right| \leq \frac{1}{R} \sup_{t \in [0, \tau^*]} V(t).$$

Then, using (3.47) and letting  $R \rightarrow \infty$  we obtain

$$\int |X(t_n)|^2 x dx - \|Q\|_{L^2}^2 y_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.51}$$

Moreover, since  $\{X(t)\}$  is a continuous and adapted process in  $H^1$ , arguing as in the proof of [7, Lemma 4.2] we have that for any  $n, m \geq 1$ ,

$$\begin{aligned} \left| \int |X(t_m)|^2 x dx - \int |X(t_n)|^2 x dx \right| &\leq 2 \int_{t_n}^{t_m} \left| \operatorname{Im} \int X(s) \nabla \overline{X}(s) ds \right| ds \\ &\leq C \int_{t_n}^{t_m} (H(X(s)) \|X(s)\|_{L^2}^2)^{\frac{1}{2}} ds \\ &\leq C(\tau^*) \|X_0\|_{L^2} |t_m - t_n| \rightarrow 0, \end{aligned}$$

where the last two steps follow from Lemmas 3.9 and 3.10. This yields that the limit  $\lim_{n \rightarrow \infty} \int |X(t_n)|^2 x dx$  exists. Thus, taking into account (3.51) we obtain (3.50), as claimed.

Now, let  $V_m(t) := \int \theta_m(x - y^*) |X(t, x)|^2 dx$ . Arguing as in the proof of (3.48) we obtain that for any  $t_n \neq t_l < \tau^*$ ,

$$\left| \sqrt{V_m(t_n)} - \sqrt{V_m(t_l)} \right| \leq \frac{1}{2} C(\tau^*) |t_n - t_l|.$$

Since by Corollary 3.4,  $V_m(t_l) \rightarrow 0$  as  $l \rightarrow \infty$ , we infer that

$$\sqrt{V_m(t_n)} \leq \frac{1}{2} C(\tau^*) (\tau^* - t_n)$$

with  $C(\tau^*)$  independent of  $m$  and  $n$ . Then, Fatou’s lemma yields that as  $n \rightarrow \infty$ ,

$$\int |x - y^*|^2 |X(t_n)|^2 dx \leq \liminf_{m \rightarrow \infty} V_m(X(t_n)) \leq \frac{1}{4} C(\tau^*)^2 (\tau^* - t_n)^2 \rightarrow 0, \tag{3.52}$$

thereby yielding (3.4).

Furthermore, by the uncertainty principle,

$$\int |X(t_n, x + y^*)|^2 dx \leq C \int |x|^2 |X(t_n, x + y^*)|^2 dx \int |\nabla X(t_n, x + y^*)|^2 dx,$$

which along with the conservation of mass and (3.52) implies that

$$\begin{aligned}\|Q\|_{L^2}^2 &= \int |X(t_n, x)|^2 dx \leq C \int |x - y^*|^2 |X(t_n, x)|^2 dx \int |\nabla X(t_n, x)|^2 dx \\ &\leq \frac{1}{4} C C^2 (\tau^*) (\tau^* - t_n)^2 \|\nabla X(t_n)\|_{L^2}^2,\end{aligned}$$

thereby yielding (3.5). Therefore, the proof of Theorem 3.6 is complete.  $\square$

**Acknowledgements** The authors thank for the financial support by the Deutsche Forschungsgemeinschaft (DFG, German Science Foundation) through SFB 1283/2 2021-317210226 at Bielefeld University. D. Zhang is also grateful for the support by NSFC (No. 12271352) and Shanghai Rising-Star Program 21QA1404500.

## References

- Bang, O., Christiansen, P.L., If, F., Rasmussen, K.Ø., Gaididei, Y.B.: Temperature effects in a nonlinear model of monolayer Scheibe aggregates. *Phys. Rev. E* **49**, 4627–4636 (1994)
- Bang, O., Christiansen, P.L., If, F., Rasmussen, K.Ø., Gaididei, Y.B.: White noise in the two-dimensional nonlinear Schrödinger equation. *Appl. Anal.* **57**, 3–15 (1995)
- Banica, V.: Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3**, 139–170 (2004)
- Barbu, V., Röckner, M., Zhang, D.: The stochastic nonlinear Schrödinger equations with linear multiplicative noise: the rescaling approach. *J. Nonlinear Sci.* **24**, 383–409 (2014)
- Barbu, V., Röckner, M., Zhang, D.: Stochastic nonlinear Schrödinger equations. *Nonlinear Anal.* **136**, 168–194 (2016)
- Barbu, V., Röckner, M., Zhang, D.: The stochastic logarithmic Schrödinger equation. *J. Math. Pures Appl. (9)* **107**, 123–149 (2017)
- Barbu, V., Röckner, M., Zhang, D.: Stochastic nonlinear Schrödinger equations: no blow-up in the non-conservative case. *J. Differ. Equ.* **263**, 7919–7940 (2017)
- Barbu, V., Röckner, M., Zhang, D.: Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise. *Ann. Probab.* **46**, 1957–1999 (2018)
- Barchielli, A., Gregoratti, M.: Quantum Trajectories and Measurements in Continuous Time. The Diffusive Case. *Lecture Notes Physics*, vol. 782. Springer-Verlag, Berlin (2009)
- Bourgain, J.: Problems in Hamiltonian PDE's. In: Alon, N., Bourgain, J., Connes, A., Gromov, M., Milman, V. (eds.) *Visions in Mathematics. Modern Birkhäuser Classics*, pp. 32–56. Birkhäuser, Basel (2000)
- Bourgain, J., Wang, W.: Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25**, 197–215 (1997)
- Brzeźniak, Z., Millet, A.: On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Anal.* **41**, 269–315 (2014)
- Cao, D., Su, Y., Zhang, D.: On uniqueness of multi-bubble blow-up solutions and multi-solitons to  $L^2$ -critical nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.* **247**, 4 (2023)
- Cazenave, T.: *Semilinear Schrödinger Equations*. Courant Lecture Notes, vol. 10. American Mathematical Society and Courant Institute of Mathematical Sciences, Providence, RI (2003)
- Cazenave, T.: An overview of the nonlinear Schrödinger equation. Lecture notes (2020). <https://www.ljll.math.upmc.fr/cazenave/>
- Combet, V.: Multi-soliton solutions for the supercritical gKdV equations. *Commun. Partial Differ. Equ.* **36**, 380–419 (2010)
- Côte, R., Friederich, X.: On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations. *Commun. Partial Differ. Equ.* **46**, 2325–2385 (2021)
- Côte, R., Kenig, C., Lawrie, A., Schlag, W.: Profiles for the radial focusing  $4d$  energy-critical wave equation. *Commun. Math. Phys.* **357**, 943–1008 (2018)
- Crauel, H., Flandoli, F.: Attractors for random dynamical systems. *Probab. Theory Relat. Fields* **100**, 365–393 (1994)
- de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. *Commun. Math. Phys.* **205**, 161–181 (1999)
- de Bouard, A., Debussche, A.: On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probab. Theory Relat. Fields* **123**, 76–96 (2002)

22. de Bouard, A., Debussche, A.: The stochastic nonlinear Schrödinger equation in  $H^1$ . *Stoch. Anal. Appl.* **21**, 97–126 (2003)
23. de Bouard, A., Debussche, A.: Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise. *Ann. Probab.* **33**, 1078–1110 (2005)
24. de Bouard, A., Debussche, A., Di Menza, L.: Theoretical and numerical aspects of stochastic nonlinear Schrödinger equations. *Journées Équations aux Dérivées Partielles (Plestin-les-Grèves, 2001)*, Exp. No. III, 13 pp., Univ. Nantes, Nantes (2001)
25. Debussche, A., Di Menza, L.: Numerical simulation of focusing stochastic nonlinear Schrödinger equations. *Phys. D.* **162**, 131–154 (2002)
26. Debussche, A., Di Menza, L.: Numerical resolution of stochastic focusing NLS equations. *Appl. Math. Lett.* **15**, 661–669 (2002)
27. Dodson, B.: Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. *Adv. Math.* **285**, 1589–1618 (2015)
28. Dodson, B.: A determination of the blowup solutions to the focusing, quintic NLS with mass equal to the mass of the soliton. [arXiv:2104.11690](https://arxiv.org/abs/2104.11690) (2021)
29. Dodson, B.: A determination of the blowup solutions to the focusing NLS with mass equal to the mass of the soliton. *Ann. PDE* **9**, 3 (2023)
30. Duyckaerts, T., Jia, H., Kenig, C., Merle, F.: Soliton resolution along a sequence of times for the focusing energy critical wave equation. *Geom. Funct. Anal.* **27**, 798–862 (2017)
31. Duyckaerts, T., Kenig, C., Merle, F.: Classification of the radial solutions of the focusing, energy-critical wave equation. *Camb. J. Math.* **1**, 75–144 (2013)
32. Duyckaerts, T., Kenig, C., Merle, F.: Soliton resolution for the radial critical wave equation in all odd space dimensions. [arXiv:1912.07664v1](https://arxiv.org/abs/1912.07664v1) (2019)
33. Dyachenko, S., Newell, A.C., Pushkarev, A., Zakharov, V.E.: Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schrödinger equation. *Phys. D* **57**, 96–160 (1992)
34. Fan, C.: log-log blow up solutions blow up at exactly  $m$  points. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34**, 1429–1482 (2017)
35. Fan, C., Su, Y., Zhang, D.: A note on log-log blow up solutions for stochastic nonlinear Schrödinger equations. *Stoch. Partial Differ. Equ. Anal. Comput.* **10**, 1500–1514 (2022)
36. Fan, C., Xu, W.: Global well-posedness for the defocusing mass-critical stochastic nonlinear Schrödinger equation on  $\mathbb{R}$  at  $L^2$  regularity. *Anal. PDE* **14**, 2561–2594 (2021)
37. Friz, P., Hairer, M.: *A Course on Rough Paths. With an Introduction to Regularity Structures*. Universitext. Springer, Cham (2014)
38. Gubinelli, M.: Controlling rough paths. *J. Funct. Anal.* **216**, 86–140 (2004)
39. Herr, S., Röckner, M., Zhang, D.: Scattering for stochastic nonlinear Schrödinger equations. *Commun. Math. Phys.* **368**, 843–884 (2019)
40. Hmidi, T., Keraani, S.: Blowup theory for the critical nonlinear Schrödinger equations revisited. *Int. Math. Res. Not.* **2005**, 2815–2828 (2005)
41. Jendrej, J., Kowalczyk, M., Lawrie, A.: Dynamics of strongly interacting kink-antikink pairs for scalar fields on a line. *Duke Math. J.* **171**, 3643–3705 (2022)
42. Kallenberg, O.: *Foundations of Modern Probability. Probability and its Applications (New York)*. Springer-Verlag, New York (1997)
43. Karatzas, I., Shreve, S.E.: *Brownian Motion and Stochastic Calculus*, 2nd edn. Graduate Texts in Mathematics, vol. 113. Springer-Verlag, New York (1991)
44. Kenig, C.E., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.* **166**, 645–675 (2006)
45. Kenig, C.E., Ponce, G., Rolvung, C., Vega, L.: The general quasilinear ultrahyperbolic Schrödinger equation. *Adv. Math.* **206**, 402–433 (2006)
46. Kenig, C.E., Ponce, G., Vega, L.: The Cauchy problem for quasi-linear Schrödinger equations. *Invent. Math.* **158**, 343–388 (2004)
47. Le Coz, S., Li, D., Tsai, T.-P.: Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proc. R. Soc. Edinb. Sect. A* **145**, 1251–1282 (2015)
48. Le Coz, S., Tsai, T.-P.: Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity* **27**, 2689–2709 (2014)
49. Martel, Y.: Asymptotic  $N$ -soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.* **127**, 1103–1140 (2005)
50. Martel, Y.: Interaction of solitons from the PDE point of view. *Proceedings of the International Congress of Mathematicians-Rio de Janeiro 2018. Vol. III. Invited lectures*, pp. 2439–2466. World Scientific, Hackensack, NJ (2018)

51. Martel, Y., Merle, F.: Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**, 849–864 (2006)
52. Martel, Y., Merle, F., Tsai, T.-P.: Stability in  $H^1$  of the sum of  $K$  solitary waves for some nonlinear Schrödinger equations. *Duke Math. J.* **133**, 405–466 (2006)
53. Marzuola, J., Metcalfe, J., Tataru, D.: Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.* **255**, 1479–1553 (2008)
54. Merle, F.: Construction of solutions with exactly  $k$  blow-up points for the Schrödinger equation with critical nonlinearity. *Commun. Math. Phys.* **129**, 223–240 (1990)
55. Merle, F.: Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.* **69**, 427–454 (1993)
56. Merle, F.: Blow-up phenomena for critical nonlinear Schrödinger and Zakharov equations. In: *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*. Doc. Math., Extra Vol. III, pp. 57–66 (1998)
57. Merle, F., Raphaël, P.: Sharp upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Geom. Funct. Anal.* **13**, 591–642 (2003)
58. Merle, F., Raphaël, P.: On universality of blow-up profile for  $L^2$  critical nonlinear Schrödinger equation. *Invent. Math.* **156**, 565–672 (2004)
59. Merle, F., Raphaël, P.: The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. Math. (2)* **161**, 157–222 (2005)
60. Merle, F., Raphaël, P.: Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation. *Commun. Math. Phys.* **253**, 675–704 (2005)
61. Merle, F., Raphaël, P.: Sharp lower bound on the blow up rate for critical nonlinear Schrödinger equation. *J. Amer. Math. Soc.* **19**, 37–90 (2006)
62. Merle, F., Raphaël, P., Szeftel, J.: The instability of Bourgain-Wang solutions for the  $L^2$  critical NLS. *Amer. J. Math.* **135**, 967–1017 (2013)
63. Merle, F., Tsutsumi, Y.:  $L^2$  concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity. *J. Differ. Equ.* **84**, 205–214 (1990)
64. Millet, A.: Critical and supercritical stochastic NLS: additive or multiplicative noise. Lecture at the workshop “Touch down of stochastic analysis in Bielefeld”. Universität Bielefeld, September 25–26 (2019)
65. Millet, A., Rodriguez, A.D., Roudenko, S., Yang, K.: Behavior of solutions to the 1D focusing stochastic nonlinear Schrödinger equation with spatially correlated noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **9**, 1031–1080 (2021)
66. Millet, A., Roudenko, S., Yang, K.: Behaviour of solutions to the 1D focusing stochastic  $L^2$ -critical and supercritical nonlinear Schrödinger equation with space-time white noise. *IMA J. Appl. Math.* **86**, 1349–1396 (2021)
67. Perelman, G.: On the blow up phenomenon for the critical nonlinear Schrödinger equation in 1D. *Ann. Henri. Poincaré* **2**, 605–673 (2001)
68. Raphaël, P.: On the singularity formation for the nonlinear Schrödinger equation. In: Ellwood, D., Rodnianski, I., Staffilani, G., Wunsch, J. (eds.) *Evolution Equations*. Clay Mathematics Proceedings, vol. 17, pp. 269–323. Amer. Math. Soc, Providence, RI (2013)
69. Raphaël, P., Szeftel, J.: Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS. *J. Amer. Math. Soc.* **24**, 471–546 (2011)
70. Rasmussen, K.Ø., Gaididei, Yu.B., Bang, O., Christiansen, P.L.: The influence of noise on critical collapse in the nonlinear Schrödinger equation. *Phys. Lett. A.* **204**, 121–127 (1995)
71. Röckner, M., Su, Y., Zhang, D.: Multi-bubble Bourgain-Wang solutions to nonlinear Schrödinger equation. [arXiv:2110.04107](https://arxiv.org/abs/2110.04107) (2021). Accepted by *Trans. Amer. Math. Soc*
72. Röckner, M., Su, Y., Zhang, D.: Multi solitary waves to stochastic nonlinear Schrödinger equations. *Probab. Theory Relat. Fields* **186**, 813–876 (2023). [arXiv:2112.07483](https://arxiv.org/abs/2112.07483)
73. Su, Y., Zhang, D.: Construction of minimal mass blow-up solutions to rough nonlinear Schrödinger equations. *J. Funct. Anal.* **284**, 109796 (2023)
74. Su, Y., Zhang, D.: On the multi-bubble blow-up solutions to rough nonlinear Schrödinger equations. [arXiv:2012.14037v1](https://arxiv.org/abs/2012.14037v1) (2020)
75. Sulem, C., Sulem, P.-L.: *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*. Applied Mathematical Sciences, vol. 139. Springer, New York (1999)
76. Tao, T.: *Nonlinear Dispersive Equations. Local and Global Analysis*. CBMS Regional Conference Series in Mathematics, vol. 106. American Mathematical Society, Providence, RI (2006)
77. Tsutsumi, Y.: Rate of  $L^2$  concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power. *Nonlinear Anal.* **15**, 719–724 (1990)

78. Weinstein, M.: Nonlinear Schrödinger equations and sharp interpolation estimates. *Commun. Math. Phys.* **87**, 567–576 (1983)
79. Weinstein, M.: On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations. *Commun. Partial Differ. Equ.* **11**, 545–565 (1986)
80. Zhang, D.: Recent progress on stochastic nonlinear Schrödinger equations. In: Eberle, A., et al. (eds.) *Stochastic Partial Differential Equations and Related Fields*. Springer Proceedings in Mathematics & Statistics, vol. 229, pp. 279–289. Springer, Cham (2018)
81. Zhang, D.: Stochastic nonlinear Schrödinger equations in the defocusing mass and energy critical cases. [arXiv:1811.00167v2](https://arxiv.org/abs/1811.00167v2) (2018). Accepted by *Ann. Appl. Probab*
82. Zhang, D.: Optimal bilinear control of stochastic nonlinear Schrödinger equations: mass-(sub)critical case. *Probab. Theory Relat. Fields* **178**, 69–120 (2020)
83. Zhang, D.: Strichartz and local smoothing estimates for stochastic dispersive equations with linear multiplicative noise. *SIAM J. Math. Anal.* **54**, 5981–6017 (2022)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.