A note on the ergodicity of Fokker–Planck flows in $L^1(\mathbb{R}^d)$

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Abstract

One proves that the nonlinear semigroups in $L^1(\mathbb{R}^d)$, $d \geq 3$, associated with the nonlinear Fokker–Planck equation $u_t - \Delta\beta(u) + \mathrm{div}(Db(u)u) = 0$ in $(0, \infty) \times \mathbb{R}^d$ is ergodic under suitable conditions on the coefficients $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$. In particular, this implies the ergodicity of probabilistically weak solutions to the corresponding McKean–Vlasov stochastic differential equations. Such a result completes the asymptotic results established in [7] in the case where the Fokker–Planck flow $S(t)$ in $L^1(\mathbb{R}^d)$ has not a fixed point.

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1 Introduction

Consider the nonlinear Fokker–Planck equation

$$\begin{aligned} &u_t - \Delta\beta(u) + \mathrm{div}(Db(u)u) = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^d, \\ &u(0, x) = u_0(x), \ x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 3$, and $b : \mathbb{R} \to \mathbb{R}$ are given functions to be made precise in the following.

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This equation describes, in statistical physics and mean field theory, the dynamics of a set of interacting particles or of many body systems in disordered media (the so-called anomalous diffusion). (See, e.g., [13].) In such a situation, for each $t \geq 0$, $u = u(t, \cdot)$ is a probability density for each probability density u_0. Another source for equation (1.1) is the description of the dynamics of Itô stochastic processes $X(t)$ in terms of their probability densities $u = u(t, x)$. Namely, if $u \in L^1_{\rm loc}((0, \infty) \times \mathbb{R}^d)$ is a Schwartz distributional solution to (1.1) such that $t \to u(t, \cdot)dx$ is weakly continuous and $u(t, \cdot)$ is a probability density, then there is a probabilistically weak solution X to the McKean–Vlasov stochastic differential equation in $\mathbb{R}^d$

$$\begin{aligned}
&dX(t) = D(X(t))b\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)dt + \sqrt{\frac{2\beta\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)}{\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))}}\,dW(t), \\
&\mathcal{L}_{X(t)}(dx) = u(t, x)dx,\ \ t \geq 0,
\end{aligned} \tag{1.2}$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$, where W is a d-dimensional $(\mathcal{F}_t)$-Brownian motion. Here, $\mathcal{L}_{X(t)}$ is the law of the process $X(t)$ under $\mathbb{P}$ and $\mathcal{L}_{X_0} = u_0 dx$ (see [3] for details).

A function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is called a *mild solution* to (1.1) if $u \in C([0, \infty); L^1(\mathbb{R}^d))$ and

$$u(t) = \lim_{h\to 0} u_h(t) \text{ strongy in } L^1(\mathbb{R}^d),\ \forall t \geq 0, \tag{1.3}$$

where $u_h : [0, \infty) \to L^1(\mathbb{R}^d)$ is the solution to the equation

$$\begin{aligned}
&\frac{1}{h}\left(u_h(t) - u_h(t-h)\right) + A_0 u_h(t) = 0 && \text{for } t \geq 0, \\
&u_h(t) = u_0 && \text{for } t < 0,
\end{aligned} \tag{1.4}$$

and A_0 is the operator in $L^1(\mathbb{R}^d)$ defined by

$$A_0(y) = -\Delta\beta(y) + \operatorname{div}(Db(y)y) \text{ in } \mathcal{D}'(\mathbb{R}^d),\ \forall y \in D(A_0), \tag{1.5}$$

$$\begin{aligned}
D(A_0) = \{&y \in L^1(\mathbb{R}^d);\ \beta(y) \in L^1_{\rm loc}(\mathbb{R}^d),\ Db(y)y \in L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d); \\
&-\Delta\beta(y) + \operatorname{div}(Db(y)y) \in L^1(\mathbb{R}^d)\}.
\end{aligned} \tag{1.6}$$

The existence of a mild solution u to (1.1) was studied under various hypotheses on β, d and b in the works [3]–[7]. The idea, previously used by

M.G. Crandall in the existence theory of entropy solutions to a nonlinear conservation law equation [11], is to represent (1.1) as a Cauchy problem in $L^1(\mathbb{R}^d)$

$$\frac{du}{dt} + Au = 0,\ \forall\, t \geq 0; \quad u(0) = u_0, \tag{1.7}$$

where A is an m-accretive operator in $L^1(\mathbb{R}^d)$ such that $(I+\lambda A)^{-1}f \in (I+\lambda A_0)^{-1}f$, $\forall f \in^1 (\mathbb{R}^d)$, $\lambda > 0$. Then, by the Crandall & Liggett generation theorem (see, e.g., [1], [2]) there exists

$$S(t)u_0 = u(t,u_0) = \lim_{n\to\infty} \left(I + \frac{t}{n}\,A\right)^{-n} u_0, \quad \forall t \geq 0,\ u_0 \in \overline{D(A)}, \tag{1.8}$$

strongly in $L^1(\mathbb{R}^d)$, uniformly in t on bounded intervals. The function $u = u(t,u_0)$ is a mild solution to equation (1.1) in the sense of (1.3)–(1.4) and the mapping $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, is a continuous semigroup of contractions in $L^1(\mathbb{R}^d)$ on $\overline{D(A)}$ – the closure of the domain $D(A)$ of A in $L^1(\mathbb{R}^d)$. We call such a semigroup of contraction a *nonlinear Fokker–Planck flow.* It should be emphasized that, in general, such a semigroup $S(t)$ is not unique because its generator A is constructed from A_0 by

$$A(y) = A_0(J_\lambda(y)),\ \forall y \in D(A) = \{u = J_\lambda(f);\ f \in L^1(\mathbb{R}^d),\ \lambda > 0\}, \tag{1.9}$$

where $J_\lambda : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a family of contractions such that $J_\lambda(f) \in (I+\lambda A_0)^{-1}f$, $\forall f \in L^1(\mathbb{R}^d)$, $\lambda > 0$. Since $I + \lambda A_0$ is, in general, not one-to-one, hence $(I+\lambda A_0)^{-1}$ is multivalued, the family $\{J_\lambda\}_{\lambda>0}$ is not unique, hence so is the operator A. There is an alternative approach to existence theory for nonlinear Fokker–Planck equations developed by J.A. Carrillo [9], G.Q. Chen and B. Perthame [10] in the context of entropy and kinetic solutions, but we shall not pursue this approach in this paper. (In fact, a mild solution to (1.1) is a weaker concept of solution than that of entropy solutions as developed in [9], [10].) The semigroups $S(t)$ represents a section in the class of mild solutions.

Here, we shall consider equation (1.1) under the following hypotheses:

(H1) $\beta \in C^1(\mathbb{R}),\ \beta'(r) > 0,\ \forall r \in \mathbb{R} \setminus \{0\},\ \beta(0) = 0$, and

$$\mu_1 \min\{|r|^\nu, |r|\} \leq |\beta(r)| \leq \mu_2 |r|,\ \forall r \in \mathbb{R}, \tag{1.10}$$

for $\mu_1, \mu_2 > 0$ and $\nu > \frac{d-1}{d},\ d \geq 3$.

(H2) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} D \in (L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$, $D = -\nabla\Phi$, where $\Phi \in C(\mathbb{R}^d) \cap W^{1,1}_{\rm oc}(\mathbb{R}^d)$ and

$$\Phi(x) \ge 1,\ \forall x \in \mathbb{R}^d,\ \lim_{|x|\to\infty} \Phi(x) = +\infty,$$
$$\Phi^{-m} \in L^1(\mathbb{R}^d \ \text{ for some } m \ge 2,$$
$$\mu_2\Delta\Phi(x) - b_0|\nabla\Phi(x)|^2 \le 0,\ \text{ a.e. } x \in \mathbb{R}^d.$$

(H3) $b \in C^1(\mathbb{R}) \cap C_b(\mathbb{R})$, $b(r) \ge b_0 > 0$ for all $r \in [0,\infty)$.

For instance, any continuous, increasing function $\beta : \mathbb{R} \to \mathbb{R}$ of the form

$$\beta(r) = \begin{cases} \mu_1 r|r|^{d-1} & \text{for } |r| \le r_0, \\ \mu_2 h(r) & \text{for } |r| > r_0, \end{cases}$$

where $r_0 > 0$, $\mu_1, \mu_2 > 0$, $|h(r)| \le L|r|$, $\forall r \in \mathbb{R}$, $L > 0$, satisfies (1.10). As regards Hypothesis (H2), an example of such a function Φ is (see [5])

$$\Phi(x) = \begin{cases} |x|^2 \log|x| + \mu & \text{for } |x| \le \delta = \exp\left(-\frac{d+2}{2d}\right), \\ \varphi(|x|) + \eta|x| + \mu & \text{for } |x| > \delta, \end{cases}$$

where $\mu, \eta > 0$ are sufficiently large and φ is as in [5, Appendix]. As a matter of fact, in this case (see [4], [7]) the family $\{J_\lambda\}_{\lambda>0}$ of resolvents which defines the operator A is given by the viscosity approximation scheme

$$J_\lambda(f) = \lim_{\varepsilon\to 0} y_\varepsilon \ \text{ in } L^1(\mathbb{R}^d), \tag{1.11}$$

where y_ε is the solution to the equation

$$y_\varepsilon - \lambda\Delta(\beta_\varepsilon(y_\varepsilon) + \varepsilon y_\varepsilon) + \lambda \operatorname{div}(D_\varepsilon b_\varepsilon(y_\varepsilon) y_\varepsilon) = f, \tag{1.12}$$

where $\beta_\varepsilon, D_\varepsilon$ and b_ε are smooth approximations of β, D and b.

It turns out, however (see [6]), that if one further assumes that, for some $\alpha > 0$,

$$|b(r)r - b(\bar r)\bar r| \le \alpha|\beta(r) - \beta(\bar r)|,\ \forall r, \bar r \in \mathbb{R}, \tag{1.13}$$

then $(I + \lambda A_0)^{-1}$ is single-valued and so A is uniquely defined, more precisely $A = A_0$. In the following, we shall consider the semigroup $S(t)$ generated by A, which is given by (1.9), and we shall call it the nonlinear Fokker–Planck *flow.* This semigroup leaves invariant the set $\mathcal{P}$ of all the probability densities ρ on $\mathbb{R}^d$, that is,

$$\mathcal{P} = \left\{ \rho \in L^1(\mathbb{R}^d);\ \rho \geq 0,\ \text{a.e. in } \mathbb{R}^d;\ \int_{\mathbb{R}^d} \rho\, dx = 1 \right\}.$$

Now, consider the orbit $\gamma(u_0) = \{S(t)u_0,\ t \geq 0\}$ where $u_0 \in C := \overline{D(A)} = L^1(\mathbb{R}^d)$ if $\beta \in C^2(\mathbb{R})$. We associate to u_0 the ω-limit set

$$\begin{aligned} \omega(u_0) &= \{u_\infty = \lim S(t_n)u_0 \text{ in } L^1 \text{ for some } \{t_n\} \to \infty\} \\ &= \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u_0}. \end{aligned}$$

In particular, if $\omega(u_0) \neq \emptyset$ and consists of one element u_∞ only, then we have

$$\lim_{t \to \infty} S(t)u_0 = u_\infty \text{ in } L^1.$$

In [5], it was proved that, if β is not degenerate in the origin, that is,

$$0 < \gamma_0 \leq \beta'(r) \leq \gamma_1, \quad \forall\, r \in \mathbb{R}, \tag{1.14}$$

(which also implies that $C = L^1$), then, for each $u_0 \in \mathcal{P}$, such that

$$u_0 \ln(u_0) \in L^1(\mathbb{R}^d),\ \|u_0\| = \int_{\mathbb{R}^d} u_0(x)\Phi(x)dx < \infty, \tag{1.15}$$

one has $\omega(u_0) = \{u_\infty\}$, where u_∞ is the unique solution in $(L^1 \cap L^\infty)(\mathbb{R}^d)$ to the stationary equation

$$-\Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

This is an H-theorem type result for the Fokker–Planck equation (1.1) (see, e.g., [14] for physical significance and examples). In [7], the nondegeneracy condition was relaxed to (H1), which along with (H2)–(H3) leads to the conclusion that, if $u_0 \in \mathcal{P} \cap C$ and $\|u_0\| \leq \eta$ for some $\eta > 0$, then $\omega(u_0)$ is nonempty, invariant under $S(t)$, $t \geq 0$, and compact in $L^1(\mathbb{R}^d)$ which implies that it is an attractor for the trajectory $\gamma(u_0)$. If, in addition, there is $a \in \mathcal{P} \cap C$ such that $\|a\| \leq \eta$ and $S(t)a = a$ for $t > 0$, then $\omega(u_0)$ lies on a ball $\{y \in L^1(\mathbb{R}^d);\ |y - a|_{L^1(\mathbb{R}^d)} = r\}$. In [7], sufficient conditions on β and b for the existence of such a fixed point for $S(t)$ were given. For instance, this happens if

$$\lim_{r \to +\infty} \int_1^r \frac{\beta'(s)}{sb(s)}\, ds = +\infty \text{ if } \nu \in \left(1 - \tfrac{1}{d}, 1\right]$$

and

$$\lim_{r\to 0}\int_1^r \frac{\beta'(s)}{sb(s)}\,ds = -\infty \text{ if } \nu > 1.$$

Here, no such condition will be imposed and so it is not cledar whether the semigroup $S(t)$ has a fixed point a in $\mathcal{P}\cap C$, but the nature of the omega-limit set $\omega(u_0)$ will be made clear from the asymptotic properties of the semigroup $S(t)$. Namely, we shall prove that the flow $t \to S(t)$ is ergodic in $L^1(\mathbb{R}^d)$ (Theorem 2.1).

Notation. $L^p(\mathbb{R}^d) = L^p,\ 1 \le p \le \infty$, is the space of real-valued Lebesgue measurable, p-integrable functions on $\mathbb{R}^d$. The space $L^p(\mathbb{R}^d;\mathbb{R}^d)$ is analogously defined and $W^{1,1}(\mathbb{R}^d;\mathbb{R}^d)$ is the Sobolev space $\{u \in L^1(\mathbb{R}^d;\mathbb{R}^d);\ D_iu_j \in L^1(\mathbb{R}^d),\ i = 1,...,d; u = (u_j)_{j=1}^d\}$. By $W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ we denote the corresponding local space. Let $C_b(\mathbb{R})$ denote the space of continuous and bounded functions on $\mathbb{R}$ and $C^1(\mathbb{R})$ the space of continuously differentiable functions on $\mathbb{R}$.

An operator $A : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space, is called m-accretive if $R(I + \lambda A) = \mathcal{X}$, $\forall \lambda > 0$, and

$$\|u_1 - u_2 + \lambda(Au_1 - Au_2)\|_{\mathcal{X}} \ge \|u_1 - u_2\|_{\mathcal{X}},\ \forall\lambda > 0,\ u_1, u_2 \in D(A),$$

where $D(A)$ is the domain of A and $R(I + \lambda A)$ is the range of $I + \lambda A$. (See, e.g., [1], [2].) For each $\eta > 0$, we set

$$\mathcal{M}_\eta := \left\{u \in L^1; \|u\| = \int_{\mathbb{R}^d} |u(x)|\Phi(x)dx \le \eta\right\},$$

where Φ is the potential of D defined in (H2).

2 The main results

Let $S(t) : C \to C$, $C = \overline{D(A)}$, be the semigroup generated by the operator A given above by (1.9) and let

$$\mathcal{K} := \mathcal{M}_\eta \cap C \cap \mathcal{P}$$

for a given $\eta > 0$. We shall assume that hypotheses (H1)–(H3) hold. Then we have

Theorem 2.1. *Let $\mathcal{X}$ be a real Banach space and let $F : \mathcal{K} \to \mathcal{X}$ be a uniformly continuous mapping. Then, for each $u_0 \in \mathcal{K}$, the set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$ and*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(S(t)u_0)dt = \int_{\omega(u_0)} F(\xi)d\xi, \tag{2.1}$$

$\omega(u_0)$ is endowed with its natural commutative group structure (*recalled below in the proof of the theorem*) *and where $d\xi$ is the normalized Haar measure on it.*

We recall that a Haar measure μ on a locally compact topological commutative group G is a nonzero Borel measure μ which is invariant on G, that is, $\mu(gS) = \mu(Sg) = \mu(S)$ for any Borel subset $S \subset G$.

An example covered by Theorem 2.1 is $\mathcal{X} = \mathbb{R}$ and

$$F(u) = \int_{\mathbb{R}^d} g(x)u(x)dx, \quad \forall u \in L^1(\mathbb{R}^d), \tag{2.2}$$

where $g \in L^\infty(\mathbb{R}^d)$. Then, by Theorem 2.1, we obviously have

Corollary 2.2. *Under hypotheses* (H1)–(H3)*, for each $u_0 \in \mathcal{K}$ and $g \in L^\infty(\mathbb{R}^d)$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x)dx = \int_{\omega(u_0)} \int_{\mathbb{R}^d} g(x)\xi(x)\,dx\,d\xi. \tag{2.3}$$

Furthermore, the semigroup $S(t)$ is mean-ergodic, that is,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(t)u_0 dt = \int_{\omega(u_0)} \xi\, d\xi \ \text{ in } L^1(\mathbb{R}^d), \tag{2.4}$$

where $d\xi$ is, as above, the normalized Haar measure on $\omega(u_0)$.

By Corollary 2.2 it follows in particular that, under hypotheses (H1)–(H3) it is satisfied for the nonlinear Fokker–Planck flow $t \to S(t)u_0$ where $u_0 \in \mathcal{K}$ the classical *Boltzmann hypothesis* (see, e.g., [16], p. 389) is satisfied with the time average $\int_{\omega(u_0)} \xi\, d\xi$ which is the mean of the Haar measure $d\xi$.

Now, coming back to the McKean–Vlasov equation (1.2), we get by Corollary 2.2 the following ergodic result for solutions $X(t)$ to (1.2).

Corollary 2.3. *Let $u_0 \in \mathcal{K}$. Then, under hypotheses* (H1)–(H3) *there is a probabilistically weak solution X to* (1.2)*, where $\mathcal{L}_{X_0} = u_0 dx$, such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}[g(X(t))]dt = \int_{\omega(u_0)} \int g(x)\xi(x)dx\, d\xi,\ \forall g \in L^\infty. \tag{2.5}$$

Furthermore, we have

$$\frac{1}{T} \int_0^T \mathcal{L}_{X(t)}(dx)dt \to \int_{\omega(u_0)} \xi(x)dx\, d\xi$$

in the weak topology on $\mathcal{P}$ a.s. $T \to \infty$.

Remark 2.4. The case $d = 2$ is singular for the semigroup approach of equation (1.1), namely for the existence of an m-accretive realization A of the operator A_0 and this is the principal motivation to avoid it (see [4], [7]). However, the case $d = 1$ could be treated in a similar way following [4], but we omit the details.

3 Proofs

As shown in [7], Lemma 4.3, under hypotheses (H1)–(H3) for each $u_0 \in \mathcal{K}$, the orbit $\gamma(u_0)$ of $S(t)u_0$ is precompact in L^1. This implies that the ω-limit set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$. We recall that, for each $t \geq 0$, $\omega(u_0)$ is invariant under $S(t)$ which is an homeomorphism of $\omega(u_0)$ onto $\omega(u_0)$, that is, $S(t)$ is a group on $\omega(u_0)$. Then, $\omega(u_0)$ can be endowed with a topological commutative group structure with the product $g_1 \circ g_2 = S(t_1 + t_2)u_0$, $g_1, g_2 \in \omega(u_0)$, where $g_1 = S(t_1)u_0$, $g_2 = S(t_2)u_0$. Then, by the classical A. Weil theorem (see [15]), there is a unique normalized Haar measure on $\omega(u_0)$ and so, by Birkhoff's ergodic theorem (see [8] and Theorem 1 in [12]) it follows that (2.1) holds for each uniformly continuous mapping $F : \mathcal{K} \to \mathcal{X}$ and so Theorem 2.1 follows.

As regards Corollary 2.3, the existence of a weak solution to (1.2) follows from [3]. Furthermore, formula (2.5) follows by (2.3) taking into account that

$$\mathbb{E}[g(X(t))] = \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x)dx,\ \forall t \geq 0.$$

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