Existence, Uniqueness and Smoothness for Incompressible Navier-Stokes Equations *

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Abstract

For any divergence free initial datum u_0 with $\|u_0\|_\infty + \|\nabla u_0\|_{L^p} + \|\nabla^2 u_0\|_{L^p} < \infty$ for some $p > d$ ($d \geq 2$), the well-posedness and smoothness are proved for incompressible Navier-Stokes equations on $\mathbb{R}^d$ or $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, up to a time given by the initial datum and two constants coming from the upper bounds of the heat kernel and the Riesz transform. A mild well-posedness is also proved for L^p-bounded initial data. The blow-up is proved for both type solutions with finite maximal time.

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1 Introduction and main results

Consider the following incompressible Navier-Stokes equation on $E := \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$ ($d \geq 2$):

$$\begin{aligned} &\partial_t u_t = \kappa\Delta u_t - (u_t\cdot\nabla)u_t - \nabla\wp_t, \\ &\nabla\cdot u_t := \sum_{i=1}^d \partial_i u_t^i = 0, \quad t\in[0,T], \end{aligned} \tag{1.1}$$

where $T > 0$ is a fixed time, and

$$u := (u^1, \cdots, u^d) : [0,T]\times E \to \mathbb{R}^d, \quad \wp : [0,T]\times E \to \mathbb{R}.$$

This equation describes viscous incompressible fluids, where u is the velocity field of a fluid flow, $\wp$ is the pressure, and $\kappa > 0$ is the viscosity constant. The real-world model in physics is

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for $d = 3$, for which a challenging problem is to prove the well-posedness and characterize the regularity of solutions.

For any $p \in [1, \infty]$, let L^p be the space of (real or vector valued) functions f on E such that

$$\|f\|_{L^p} := \bigg(\int_E |f(x)|^p \mathrm{d}x \bigg)^{\frac{1}{p}} < \infty,$$

where we normalize the volume measure when $E = \mathbb{T}^d$. For $d = 3$, Leray [5] proved the weak existence for $u_0 \in L^3$ and studied the blow-up property. See [2, 6, 8] and references within for the blow-up in $L^p, p \geq 3$, and see [7, 9] and references within for the study using probabilistic approaches.

In this paper, we consider solutions of (1.1) in the class

$$\mathscr{U}_p := \big\{ f : [0,T] \times E \to \mathbb{R}^d : \|f\|_\infty + \|\nabla f\|_{p,\infty} + \|\nabla^2 f\|_{p,\infty} < \infty \big\}$$

for some $p > d$, where for a function f on $[0,T] \times E$,

$$\|f\|_\infty := \sup_{(t,x) \in [0,T] \times E} |f_t(x)|, \quad \|f\|_{p,\infty} := \sup_{t \in [0,T]} \|f_t\|_{L^p}.$$

By the Sobolev embedding theorem, $f \in \mathscr{U}_p$ for some $p > d$ implies

$$\|\nabla f\|_\infty \leq c\big(\|\nabla^2 f\|_{L^p} + \|\nabla f\|_{L^p} \big) < \infty$$

for some constant $c > 0$.

For any $n \in \mathbb{Z}^+$, we denote $f \in \mathscr{C}_b^n$ if it is a function on E with

$$\|f\|_{\mathscr{C}_b^n} := \sum_{i=0}^n \|\nabla^i f\|_\infty < \infty, \quad \nabla^0 f := f.$$

For any $\alpha \in (0,1)$, we write $f \in \mathscr{C}_b^{n+\alpha}$ if

$$\|f\|_{\mathscr{C}_b^{n+\alpha}} := \|f\|_{\mathscr{C}_b^n} + \sup_{x \neq y} \frac{|\nabla^n f(x) - \nabla^n f(y)|}{|x-y|^\alpha} < \infty.$$

Lemma 1.1. *Let $P_t = \mathrm{e}^{\kappa t \Delta}$ be the heat semigroup generated by $\kappa\Delta$, and let*

$$\mathscr{R}_t := (1 + \nabla(-\Delta)^{-1} \nabla \cdot) P_t, \quad t \geq 0.$$

For any $p \in (1, \infty)$, we have

$$\sup_{t \geq 0} \|\mathscr{R}_t\|_{L^p} \leq 1 + \|\nabla(-\Delta)^{-1} \nabla \cdot \|_{L^p} < \infty, \tag{1.2}$$

where $\|\cdot\|_{L^p}$ is the operator norm in L^p, and

$$\begin{aligned} \alpha_p &:= \sup_{t>0} t^{\frac{1}{2}} \|\nabla \mathscr{R}_t\|_{L^p} < \infty, \\ \beta_p &:= \sup_{t>0} t^{\frac{d}{2p}} \|\mathscr{R}_t\|_{L^p \to L^\infty} < \infty, \end{aligned} \tag{1.3}$$

where $\|\cdot\|_{L^p \to L^\infty}$ is the operator norm from L^p to L^∞.

Proof. We first observe that $\nabla(-\Delta)^{-1}\nabla\cdot$ is a bounded operator in L^p for functions $E\to\mathbb{R}^d$, so that (1.2) follows from the L^p-contraction of P_t. This is implied by the L^p-boundedness of the Riesz transform $\nabla(-\Delta)^{-\frac12}$ (for $E=\mathbb{T}^d$ it is restricted to functions $f\in L^p(E)$ with $\mu(f):=\int_E f(x)\mathrm{d}x=0$), see [1], and the fact that

$$\begin{aligned}\|\nabla(-\Delta)^{-1}\nabla\cdot f\|_{L^p}&=\Big\|\sum_{i=1}^d\nabla(-\Delta)^{-1}\partial_i\big\{f^i-\mu(f^i)1_{E=\mathbb{T}^d}\big\}\Big\|_{L^p}\\&\le\sum_{i=1}^d\big\|\nabla(-\Delta)^{-\frac12}\partial_i(-\Delta)^{-\frac12}\big\{f^i-\mu(f^i)1_{E=\mathbb{T}^d}\big\}\big\|_{L^p}\\&\le\sum_{i=1}^d\|\nabla(-\Delta)^{-\frac12}\|_{L^p}^2\big\|f^i-\mu(f^i)1_{E=\mathbb{T}^d}\big\|_{L^p}.\end{aligned}$$

Next, it is classical that for some constant $c>0$ we have

$$\|\nabla P_t\|_{L^p}\le ct^{-\frac12},\quad \|P_t-\mu(\cdot)1_{E=\mathbb{T}^d}\|_{L^p\to L^\infty}\le ct^{-\frac{d}{2p}},\quad t>0.$$

Combining this with (1.2) we prove (1.3). □

For $p>d$, let

$$\Theta_{p,T}:=\bigg\{(\theta_1,\theta_2):\ 0<\theta_1<\frac{1}{2\alpha_p\sqrt{T}},\ \ 0<\theta_2<\frac{(2p-d)T^{\frac{d-2p}{2p}}}{2p\beta_p}\bigg\}.\tag{1.4}$$

For any initial datum u_0 and $(\theta_1,\theta_2)\in\Theta_{p,T}$, we solve (1.1) in the class

$$\Gamma_{\theta_1,\theta_2}(u_0):=\big\{\gamma:[0,T]\times E\to\mathbb{R}^d:\ \gamma_0=u_0,\ \|\gamma\|_\infty\le\theta_1,\ \|\nabla\gamma\|_{p,\infty}\le\theta_2\big\}.$$

The main result of the paper is the following.

Theorem 1.2. *Let $p\in(d,\infty),(\theta_1,\theta_2)\in\Theta_{p,T}$, and let $u_0:[0,T]\times E\to\mathbb{R}^d$ satisfy*

$$\nabla\cdot u_0=0,\quad \|u_0\|_\infty+\|\nabla u_0\|_\infty+\|\nabla^2u_0\|_{L^p}<\infty$$

and

$$\begin{aligned}\|u_0\|_\infty&\le\theta_1\bigg(1-\frac{2p\theta_2\beta_pT^{\frac{2p-d}{2p}}}{2p-d}\bigg),\\\|\nabla u_0\|_{L^p}&\le\theta_2\Big(1-2\theta_1\alpha_p\sqrt{T}\Big).\end{aligned}\tag{1.5}$$

Then the following assertions hold.

(1) (1.1) *has a unique solution $(u,\nabla\wp)$ with*

$$u\in\mathscr{U}_p,\quad \|\nabla\wp\|_\infty+\|\nabla^2\wp\|_{p,\infty}<\infty.\tag{1.6}$$

Moreover, the solution satisfies $u\in\Gamma_{\theta_1,\theta_2}(u_0)$ and

$$\nabla\wp_t=\nabla(-\Delta)^{-1}\nabla\cdot\big\{(u_t\cdot\nabla)u_t\big\},\quad t\in[0,T].\tag{1.7}$$

(2) *If* $\|u_0\|_{L^p}+\|\nabla^2 u_0\|_\infty<\infty$ *then* $\|u\|_{p,\infty}+\|\nabla^2 u\|_\infty<\infty$. *If moreover* $\sum_{i=1}^n \|\nabla^i u_0\|_{L^p}<\infty$ *for some* $n\geq 3$, *then*

$$\sum_{i=1}^n \left\{\|\nabla^i u\|_{p,\infty}+\|\nabla^i \wp\|_{p,\infty}\right\}<\infty. \tag{1.8}$$

Consequently,

$$\sup_{t\in[0,T]}\left\{\|u_t\|_{\mathscr{C}_b^{n-\frac{d}{p}}}+\|\nabla \wp_t\|_{\mathscr{C}^{n-1-\frac{d}{p}}}\right\}<\infty. \tag{1.9}$$

If furthermore $\|\nabla^n u_0\|_\infty<\infty$, *then*

$$\sup_{t\in[0,T]}\left\{\|u_t\|_{\mathscr{C}_b^n}+\|\nabla \wp_t\|_{\mathscr{C}_b^{n-1}}\right\}<\infty. \tag{1.10}$$

By taking for instance

$$\theta_1(T)=\frac{1}{4\alpha_p\sqrt{T}},\quad \theta_2(T)=\frac{2p-d}{4p\beta_p T^{\frac{2p-d}{2p}}},$$

(1.5) holds for $(\theta_1,\theta_2)=(\theta_1(T),\theta_2(T))\in\Theta_{p,T}$ if

$$\|u_0\|_\infty\leq\frac{\theta_1(T)}{2},\quad \|\nabla u_0\|_{L^p}\leq\frac{\theta_2(T)}{2}.$$

By Theorem 1.2, (1.1) has a unique solution $(u_t,\nabla\wp_t)$ satisfying (1.6) for

$$T=T_0^*(u_0):=\min\left\{\Big(\frac{1}{8\alpha_p\|u_0\|_\infty}\Big)^2,\ \Big(\frac{2p-d}{8p\beta_p\|\nabla u_0\|_{L^p}}\Big)^{\frac{2p}{2p-d}}\right\}>0. \tag{1.11}$$

We may apply this assertion to (1.1) starting from time $T_0^*(u_0)$ with initial datum $u_{T_0^*(u_0)}$, such that (1.1) has a unique solution satisfying (1.6) for

$$T=T_1^*(u_0):=T_0^*(u_0)+T_0^*(u_{T_0^*(u_0)}).$$

Repeating this procedure, we have the well-posedness of (1.1) up to the maximal time

$$\zeta(u_0):=\sum_{n=0}^\infty T_n^*(u_0), \tag{1.12}$$

where $T_n^*(u_0):=T_0^*(u_{\tau_{n-1}(u_0)})$ with

$$\tau_{n-1}(u_0):=\sum_{i=0}^{n-1}T_i^*(u_0),\quad n\geq 1.$$

Moreover, (1.6) holds for any $T\in(0,\zeta(u_0))$. We have the following blow-up result at time $\zeta(u_0)$.

Theorem 1.3. *Let $\nabla\cdot u_0=0$ and $\|u_0\|_\infty+\|\nabla u_0\|_{L^p}+\|\nabla^2 u_0\|_{L^p}<\infty$ for some $p\in(d,\infty)$. Then (1.1) has a unique solution up to time $\zeta(u_0)$ such that (1.6) holds for all $T\in(0,\zeta(u_0))$. When $\zeta(u_0)<\infty$, for any continuous increasing function $\phi:[0,\infty)\to[0,\infty)$ with $\int_1^\infty \frac{\phi^{-1}(r)}{r^2}\mathrm{d}r<\infty$,*

$$\limsup_{t\to\zeta(u_0)}\phi\Big(\|u_t\|_\infty^2+\|\nabla u_t\|_{L^p}^{\frac{2p}{2p-d}}\Big)\big(\zeta(u_0)-t\big)=\infty. \tag{1.13}$$

In the following three sections, we prove Theorem 1.2(1), Theorem 1.2(2) and Theorem 1.3 respectively. Finally, in Section 5 investigate the mild well-posedness and blow up for (1.1) with $\|u_0\|_{L^p}<\infty$.

2 Proof of Theorem 1.2(1)

We first introduce an equivalent equation of (1.1) where $\nabla\wp_t$ is formulated with u_t.

Lemma 2.1. *For a solution $(u,\nabla\wp)$ of (1.1) satisfying (1.6), the formula (1.7) holds so that (1.1) becomes*

$$\partial_t u_t=\kappa\Delta u_t-\big\{1+\nabla(-\Delta)^{-1}\nabla\cdot\big\}\big[(u_t\cdot\nabla)u_t\big],\quad t\in[0,T],\ \nabla\cdot u_0=0. \tag{2.1}$$

On the other hand, if $u\in\mathscr{U}_p$ solves (2.1), then it solves (1.1) with $\wp$ given by (1.7) such that (1.6) holds.

Proof. Let $(u,\nabla\wp)$ solve (1.1) such that (1.6) holds. By Duhamel's formula, we have

$$u_t=P_t u_0-\int_0^t P_{t-s}\big\{(u_s\cdot\nabla)u_s+\nabla\wp_s\big\}\mathrm{d}s,\quad t\in[0,T].$$

Taking divergence both sides and using $\nabla\cdot u_t=0$, we derive

$$\int_0^t P_{t-s}\big\{\nabla\cdot[(u_s\cdot\nabla)u_s]+\Delta\wp_s\big\}\mathrm{d}s=0,\quad t\in[0,T].$$

Therefore, (1.7) holds.

On the other hand, if $u\in\mathscr{U}_p$ solves (2.1), then it solves (1.1) with $\wp$ given by (1.7). By (1.2), we see that (1.6) holds. □

To solve (2.1), we present the following lemma.

Lemma 2.2. *For $p\in(\frac{d}{2},\infty)$, let $(\theta_1,\theta_2)\in\Theta_{p,T}$. If u_0 satisfies (1.5), then*

$$u_t^\gamma:=P_t u_0-\int_0^t\mathscr{R}_{t-s}\Big\{(\gamma_s\cdot\nabla)\gamma_s\Big\}\mathrm{d}s,\quad t\in[0,T] \tag{2.2}$$

defines a map $u:\Gamma_{\theta_1,\theta_2}(u_0)\to\Gamma_{\theta_1,\theta_2}(u_0);\ \gamma\mapsto u^\gamma$.

Proof. Let $\gamma \in \Gamma_{\theta_1,\theta_2}(u_0)$. By (1.3), $\nabla P_t = P_t\nabla$ and the L^p-contraction of P_t, for any $\gamma \in \Gamma_{\theta_1,\theta_2}(u_0)$ we have

$$\begin{aligned}\|u_t^\gamma\|_\infty &\le \|u_0\|_\infty + \beta_p \int_0^t (t-s)^{-\frac{d}{2p}} \big\|(\gamma_s\cdot\nabla)\gamma_s\big\|_{L^p}\mathrm{d}s \\ &\le \|u_0\|_\infty + \beta_p\|\gamma\|_\infty\|\nabla\gamma\|_{p,\infty}\int_0^t (t-s)^{-\frac{d}{2p}}\mathrm{d}s \\ &\le \|u_0\|_\infty + \frac{2p\beta_p T^{\frac{2p-d}{2p}}}{2p-d}\theta_1\theta_2,\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_t^\gamma\|_{L^p} &\le \|u_0\|_{L^p} + \alpha_p \int_0^t (t-s)^{-\frac{1}{2}} \big\|(\gamma_s\cdot\nabla)\gamma_s\big\|_{L^p}\mathrm{d}s \\ &\le \|\nabla u_0\|_{L^p} + 2\alpha_p\sqrt{T}\|\gamma\|_\infty\|\nabla\gamma\|_{p,\infty} \\ &\le \|\nabla u_0\|_{L^p} + 2\alpha_p\sqrt{T}\theta_1\theta_2.\end{aligned}$$

Combining these with (1.5) we obtain

$$\|u^\gamma\|_\infty \le \theta_1, \quad \|\nabla u^\gamma\|_{p,\infty} \le \theta_2.$$

Therefore, $u^\gamma \in \Gamma_{\theta_1,\theta_2}(u_0)$.

□

Finally, we introduce a result concerning the regularity of Kolmogorov equations. For any $p, q > 1$, a (real or vector valued) function f on $[0,T]\times E$ is said in the class L_q^p, if

$$\|f\|_{L_q^p} := \bigg(\int_0^T \|f_t\|_{L^p}^q \mathrm{d}t\bigg)^{\frac{1}{q}} < \infty.$$

Lemma 2.3. *Let $p, q \in (1,\infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$. For any $f : [0,T]\times E \to \mathbb{R}^d$ with $f \in L_q^p$,*

$$(\partial_t + \kappa\Delta)u_t = f_t, \quad t \in [0,T], u_T = 0 \tag{2.3}$$

has a unique solution in

$$H_q^{2,p} := \big\{f : [0,T]\times E \to \mathbb{R}^d;\ \|f\|_{L_q^p} + \|\nabla f\|_{L_q^p} + \|\nabla^2 f\|_{L_q^p} < \infty\big\},$$

and the unique solution satisfies

$$\|u\|_\infty + \|\nabla u\|_\infty + \|\nabla^2 u\|_{L_q^p} \le c\|f\|_{L_q^p} \tag{2.4}$$

for some constant $c > 0$ independent of f.

Proof. When $E = \mathbb{R}^d$ the assertion follows from Theorem 10.3 and Remark 10.4 in [4]. When $E = \mathbb{T}^d$ we extend f_t from $\mathbb{T}^d$ to $\mathbb{R}^d$ by letting

$$f_t(x+k) = f_t(x), \quad x \in [0,1)^d,\ k \in \mathbb{Z}^d.$$

Then

$$\|f\|_{\tilde{L}_q^p} := \sup_{z\in\mathbb{R}^d} \bigg(\int_0^T \|1_{B(z,1)} f_t\|_{L^p}^q \mathrm{d}t \bigg)^{\frac{1}{q}} < \infty,$$

where $B(z,1)$ is the unit ball in $\mathbb{R}^d$. By [10, Theorem 3.1], (2.3) for $\mathbb{R}^d$ replacing $\mathbb{T}^d$ has a unique solution in the class

$$\tilde{H}_q^{2,p} := \big\{ u : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d;\ \|u\|_{\tilde{L}_q^p} + \|\nabla u\|_{\tilde{L}_q^p} + \|\nabla^2 u\|_{\tilde{L}_q^p} < \infty \big\},$$

and the solution satisfies (2.4) for $\tilde{L}_q^p$ replacing L_q^p. By the periodicity of f_t, $u_t(\cdot+k)$ for $k \in \mathbb{Z}^d$ also solve the equation, so that the uniqueness implies $u_t(\cdot+k) = u_t$. Therefore, restricting to $\mathbb{T}^d$, u is the unique solution of (2.3), and (2.4) holds. □

We are now ready to prove the first assertion in the main result.

Proof of Theorem 1.2(1). By Lemma 2.1, it suffices to prove that (2.1) has a unique solution satisfying $u \in \mathscr{U}_p$, and the solution satisfies $u \in \Gamma_{\theta_1,\theta_2}(u_0)$.

(a) We first prove that the map u defined in Lemma 2.2 has a unique fixed point in $\Gamma_{\theta_1,\theta_2}(u_0)$. By (1.3), for any $\gamma, \gamma_2 \in \Gamma_{\theta_1,\theta_2}(u_0)$ we have

$$\begin{aligned}
\|u_t^\gamma - u_t^{\tilde{\gamma}}\|_\infty &\le \beta_p \int_0^t (t-s)^{-\frac{d}{2p}} \big\|(\gamma_s\cdot\nabla)\gamma_s - (\tilde{\gamma}_s\cdot\nabla)\tilde{\gamma}_s\big\|_{L^p}\mathrm{d}s \\
&\le \beta_p \int_0^t (t-s)^{-\frac{d}{2p}} \big\{\|\gamma_s-\tilde{\gamma}_s\|_\infty \|\nabla\gamma\|_{p,\infty} + \|\tilde{\gamma}\|_\infty \|\nabla(\gamma_s-\tilde{\gamma}_s)\|_{L^p}\big\}\mathrm{d}s \\
&\le \beta_p(\theta_1\vee\theta_2) T^{\frac{p-d}{2p}} \int_0^t (t-s)^{-\frac{1}{2}} \big\{\|\gamma_s-\tilde{\gamma}_s\|_\infty + \|\nabla(\gamma_s-\tilde{\gamma}_s)\|_{L^p}\big\}\mathrm{d}s, \quad t\in[0,T],
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|\nabla(u_t^\gamma - u_t^{\tilde{\gamma}})\|_{L^p} &\le \alpha_p \int_0^t (t-s)^{-\frac{1}{2}} \big\|(\gamma_s\cdot\nabla)\gamma_s - (\tilde{\gamma}_s\cdot\nabla)\tilde{\gamma}_s\big\|_{L^p}\mathrm{d}s \\
&\le \alpha_p(\theta_1\vee\theta_2) \int_0^t (t-s)^{-\frac{1}{2}} \big\{\|\gamma_s-\tilde{\gamma}_s\|_\infty + \|\nabla(\gamma_s-\tilde{\gamma}_s)\|_{L^p}\big\}\mathrm{d}s, \quad t\in[0,T].
\end{aligned}$$

Letting $C := \big(\big\{\beta_p T^{\frac{p-d}{2p}}\big\} + \alpha_p\big)(\theta_1\vee\theta_2)$, we derive

$$\begin{aligned}
&\|u_t^\gamma - u_t^{\tilde{\gamma}}\|_\infty + \|\nabla(u_t^\gamma - u_t^{\tilde{\gamma}})\|_{L^p} \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}} \big\{\|\gamma_s-\tilde{\gamma}_s\|_\infty + \|\nabla(\gamma_s-\tilde{\gamma}_s)\|_{L^p}\big\}\mathrm{d}s, \quad t\in[0,T].
\end{aligned} \tag{2.5}$$

For any $\lambda > 0$, let

$$\rho_\lambda(\gamma, \tilde{\gamma}) := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\big\{\|\gamma_t - \tilde{\gamma}_t\|_\infty + \|\nabla(\gamma_t - \tilde{\gamma}_t)\|_{L^p}\big\}, \quad \gamma, \tilde{\gamma} \in \Gamma_{\theta_1,\theta_2}(u_0).$$

Then (2.5) yields

$$\rho_\lambda(u^\gamma, u^{\tilde{\gamma}}) \le \varepsilon_\lambda \rho_\lambda(\gamma, \tilde{\gamma}), \quad \gamma, \tilde{\gamma} \in \Gamma_{\theta_1,\theta_2}(u_0),$$

where

$$\varepsilon_\lambda := C \int_0^T t^{-\frac{1}{2}} \mathrm{e}^{-\lambda t} \mathrm{d}t \to 0 \text{ as } \lambda \to \infty.$$

So, when λ is large enough, the map $u : \Gamma_{\theta_1,\theta_2}(u_0) \to \Gamma_{\theta_1,\theta_2}(u_0)$ is contractive in the complete metric ρ_λ. By the fixed point theorem, u has a unique fixed point $\gamma \in \Gamma_{\theta_1,\theta_2}(u_0)$. So, the equation

$$u_t = P_t u_0 - \int_0^t \mathscr{R}_{t-s}\big\{(u_s \cdot \nabla)u_s\big\}\mathrm{d}s, \quad t \in [0, T] \tag{2.6}$$

has a unique solution in $\Gamma_{\theta_1,\theta_2}(u_0)$.

(b) We intend to prove that u solves (2.1) with $\|\nabla^2 u\|_{p,\infty} < \infty$. Let

$$f_t := \big(1 + \nabla(-\Delta)^{-1}\nabla\cdot\big)\big\{(u_{T-t} \cdot \nabla)u_{T-t}\big\}, \quad t \in [0, T]. \tag{2.7}$$

By (1.2) and $u \in \Gamma_{\theta_1,\theta_2}(u_0)$, we have $\|f\|_{L^p_q} < \infty$ for any $q > 1$. Let $q \in (\frac{2p}{p-d}, \infty)$ such that $\frac{d}{p} + \frac{2}{q} < 1$. By Lemma 2.3, the PDE

$$(\partial_t + \kappa\Delta)\tilde{u}_t = f_t, \quad \tilde{u}_T = 0$$

has a unique solution satisfying

$$\|\tilde{u}\|_\infty + \|\nabla\tilde{u}\|_\infty < \infty, \quad \|\nabla^2\tilde{u}\|_{L^q_p} < \infty. \tag{2.8}$$

Since $\tilde{u}_{T-\cdot}$ solves the PDE

$$\partial_t \tilde{u}_{T-t} = \kappa\Delta\tilde{u}_{T-t} - f_{T-t}, \quad t \in [0, T],\ \tilde{u}_{T-0} = 0, \tag{2.9}$$

by Duhamel's formula and (2.7), we obtain

$$\begin{aligned}\tilde{u}_{T-t} &= -\int_0^t P_{t-s}\Big\{\big(1 + \nabla(-\Delta)^{-1}\nabla\cdot\big)\big[(u_s \cdot \nabla)u_s\big]\Big\}\mathrm{d}s \\ &= -\int_0^t \mathscr{R}_{t-s}\big[(u_s \cdot \nabla)u_s\big]\Big\}\mathrm{d}s, \quad t \in [0, T].\end{aligned}$$

Combining this with (2.6) we get

$$u_t = P_t u_0 + \tilde{u}_{T-t}, \quad t \in [0, T], \tag{2.10}$$

so that (2.8) and (2.9) yield that u solves (2.1) and by $\|\nabla^2 u_0\|_{L^p} < \infty$, we have $\|\nabla^2 u\|_{L^p_q} < \infty$.

By Sobolev embedding theorem, $\|\nabla^2 u_0\|_{L^p} < \infty$ and $\|\nabla u_0\|_{L^p} < \infty$ imply $\|\nabla u_0\|_\infty < \infty$, which together with (2.8) and (2.10) implies $\|\nabla u\|_\infty < \infty$. Combining this with (2.6), (1.3) and $\partial_i\partial_j \mathscr{R}_{t-s} = \partial_i \mathscr{R}_{t-s}\partial_j$ for $1 \le i, j \le d$, we obtain we find constants $c_2, c_3 > 0$ such that for $q \in (\frac{2p}{p-d}, \infty)$,

$$\begin{aligned}
\|\nabla^2 u_t\|_{L^p} &\le \|\nabla^2 u_0\|_{L^p} + \int_0^t \big\|\nabla \mathscr{R}_{t-s}\big\{\nabla[(u_s\cdot\nabla)u_s]\big\}\big\|_{L^p}\mathrm{d}s \\
&\le \alpha_p \int_0^t (t-s)^{-\frac{1}{2}}\big(\|u\|_\infty \|\nabla^2 u_s\|_{L^p} + \|\nabla u\|_\infty \|\nabla u\|_{p,\infty}\big)\mathrm{d}s \\
&\le 2\alpha_p\sqrt{T}\|\nabla u\|_\infty\|\nabla u\|_{p,\infty} + \alpha_p\|u\|_\infty \|\nabla^2 u\|_{L^p_q}\bigg(\int_0^T s^{-\frac{q}{2(q-1)}}\mathrm{d}s\bigg)^{\frac{q-1}{q}} < \infty, \quad t\in[0,T].
\end{aligned}$$

Therefore, $\|\nabla^2 u\|_{p,\infty} < \infty$. Hence, (1.6) holds.

(c) If (1.1) has another solution $(\tilde{u}_t, \nabla\tilde{\wp}_t)$ satisfying (1.6) and with $\tilde{u}_0 = u_0$, by Lemma 2.1 we have $\nabla\tilde{u}_t = \nabla(-\Delta)^{-1}\nabla\cdot\big\{(\tilde{u}_t\cdot\nabla)\tilde{u}_t\big\}$ and

$$\tilde{u}_t = P_t u_0 - \int_0^t \mathscr{R}_{t-s}\big\{(\tilde{u}_s\cdot\nabla)\tilde{u}_s\big\}\mathrm{d}s, \quad t\in[0,T].$$

Combining this with (2.6) and repeating the argument in step (a) with $u, \tilde{u} \in \mathscr{U}_p$, we prove $u_t = \tilde{u}_t$. □

3 Proof of Theorem 1.2(2)

We first introduce a simple lemma collecting two inequalities, where (3.1) is well known and (3.2) is included in [3, Lemma 2.10].

Lemma 3.1. *Let $p \in (1,\infty)$. For any $m \in \mathbb{N}$ and $\alpha \in [0, \frac{m}{2}]$, there exists a constant $c > 0$ such that for any function f,*

$$\|\nabla^m P_t f\|_{L^p} \le ct^{-\alpha}\|(1-\Delta)^{\frac{m}{2}-\alpha} f\|_{L^p}, \quad t > 0. \tag{3.1}$$

Moreover, for any $\alpha > 0$, there exists a constant $c > 0$ such that for any functions f, g,

$$\begin{aligned}
&\|(1-\Delta)^\alpha(fg) - f(1-\Delta)^\alpha g\|_{L^p} \\
&\le c\big(\|\nabla f\|_\infty\|(1-\Delta)^{\alpha+\frac{1}{2}}g\|_{L^p} + \|g\|_\infty\|(1-\Delta)^\alpha f\|_{L^p}\big).
\end{aligned} \tag{3.2}$$

Proof of Theorem 1.2. Let $(u, \nabla\wp)$ be the solution given in Theorem 1.2(1) with $\|\nabla^2 u_0\|_\infty < \infty$.

(a) By $\|\nabla^2 u_0\|_\infty + \|u\|_\infty + \|\nabla u\|_\infty + \|\nabla u\|_{p,\infty} + \|\nabla^2 u\|_{p,\infty} < \infty$, (1.3) and $\partial_i\partial_j\mathscr{R}_{t-s} = \partial_i\mathscr{R}_{t-s}\partial_j$ for $1 \le i, j \le d$, we find constants $c_1, c_2 > 0$ such that (2.6) implies

$$\begin{aligned}
\|\nabla^2 u_t\|_\infty &\le \|\nabla^2 u_0\|_\infty + c_1\int_0^t (t-s)^{-\frac{d+p}{2p}}\big\|\nabla[(u_s\cdot\nabla)u_s]\big\|_{L^p}\mathrm{d}s \\
&\le \|\nabla^2 u_0\|_\infty + c_1\int_0^t (t-s)^{-\frac{d+p}{2p}}\big\{\|u_s\|_\infty\|\nabla^2 u_s\|_{L^p} + \|\nabla u_s\|_\infty\|\nabla u_s\|_{L^p}\big\}\mathrm{d}s \le c_2.
\end{aligned}$$

Thus,

$$\sum_{i=0}^{2} \|\nabla^i u\|_\infty < \infty. \tag{3.3}$$

Similarly, by (1.2) and (2.6), we find a constant $c > 0$ such that $\|u_0\|_{L^p} < \infty$ and $u \in \Gamma_{\theta_1,\theta_2}(u_0)$ imply

$$\|u\|_{p,\infty} \le \|u_0\|_{L^p} + c \sup_{t\in[0,T]} \int_0^t \|u_s\|_\infty \|\nabla u_s\|_{L^p} \mathrm{d}s \le \|u_0\|_{L^p} + cT\theta_1\theta_2 < \infty.$$

(b) By the Sobolev embedding theorem, (1.8) implies (1.9). By (1.2) and (1.7), to get (1.8) it suffices to prove

$$\sum_{i=0}^{n} \|\nabla^i u\|_{p,\infty} < \infty. \tag{3.4}$$

Assume that for some $2 \le m \le n-1$ we have

$$\sum_{i=0}^{m} \|\nabla^i u\|_{p,\infty} < \infty, \tag{3.5}$$

it remains to prove $\|\nabla^{m+1} u\|_{p,\infty} < \infty$. By the Sobolev embedding theorem, (3.5) and $\|u\|_\infty < \infty$ imply

$$\sup_{t\in[0,T]} \|u_t\|_{\mathscr{C}_b^{m-1}} \le \sup_{t\in[0,T]} \|u_t\|_{\mathscr{C}_b^{m-\frac{d}{p}}} < \infty. \tag{3.6}$$

By (1.2), (2.6), (3.5), (3.6) and the fact that (3.1) yields

$$\|\nabla^{m+1}(1-\Delta)^{-\frac{1}{4}} P_t f\|_{L^p} \le ct^{-\frac{3}{4}} \|\nabla^{m-1} f\|_{L^p}$$

for some constant $c > 0$, we find constants $c_1, c_2 > 0$ such that for any $t \in [0,T]$,

$$\begin{aligned} &\|\nabla^{m+1}(1-\Delta)^{-\frac{1}{4}} u_t\|_{L^p} - \|\nabla^{m+1}(1-\Delta)^{-\frac{1}{4}} u_0\|_{L^p} \\ &\le c_1 \int_0^t (t-s)^{-\frac{3}{4}} \Big\| \nabla^{m-1}\big[(u_s\cdot\nabla)u_s\big] \Big\|_{L^p} \mathrm{d}s \le c_2, \quad t \in [0,T]. \end{aligned}$$

Combining this with the fact that

$$\begin{aligned} &\|(1-\Delta)^{\frac{2m+1}{4}} f\|_{L^p} + \|(1-\Delta)^{\frac{2m-1}{4}} f\|_{L^p} + \|(1-\Delta)^{\frac{2m-1}{4}} \nabla f\|_{L^p} \\ &\le c\big(\|f\|_{L^p} + \|\nabla^{m+1}(1-\Delta)^{-\frac{1}{4}} f\|_{L^p}\big) \end{aligned}$$

for some constant $c > 0$, we derive

$$\|(1-\Delta)^{\frac{2m+1}{4}} u\|_{p,\infty} + \|(1-\Delta)^{\frac{2m-1}{4}} u\|_{p,\infty} + \|(1-\Delta)^{\frac{2m-1}{4}} \nabla u\|_{p,\infty} < \infty. \tag{3.7}$$

Moreover, (3.1) with $\alpha = -\frac{3}{4}$ and (3.2) with $\alpha = \frac{2m-1}{4}$ imply

$$\|\nabla^{m+1} P_{t-s} f\|_{L^p} \le c(t-s)^{-\frac{3}{4}} \|(1-\Delta)^{\frac{2m-1}{4}} f\|_{L^p},$$

$$\begin{aligned}&\|(1-\Delta)^{\frac{2m-1}{4}}(fg) - f(1-\Delta)^{\frac{2m-1}{4}} g\|_{L^p} \\ &\le c\big(\|\nabla f\|_\infty \|(1-\Delta)^{\frac{2m+1}{4}} g\|_{L^p} + \|(1-\Delta)^{\frac{2m-1}{4}} f\|_{L^p} \|g\|_\infty\big)\end{aligned}$$

for some constant $c > 0$. By combining these with (1.2), we find constants $c_3, c_4, c_5 > 0$ such that

$$\begin{aligned}&\|\nabla^{m+1} u_t\|_{L^p} - \|\nabla^{m+1} u_0\|_{L^p} \\ &\le c_3 \int_0^t (t-s)^{-\frac{3}{4}} \big\|(1-\Delta)^{\frac{2m-1}{4}} \big(1 + \nabla(-\Delta)^{-1}\nabla\cdot\big)\big[(u_s\cdot\nabla)u_s\big]\big\|_{L^p} \mathrm{d}s \\ &\le c_4 \int_0^t (t-s)^{-\frac{3}{4}} \big\|(1-\Delta)^{\frac{2m-1}{4}} [(u_s\cdot\nabla)u_s]\big\|_{L^p} \mathrm{d}s \\ &\le c_5 \int_0^t (t-s)^{-\frac{3}{4}} \Big\{ \|\nabla u_s\|_\infty \|(1-\Delta)^{\frac{2m-1}{4}} u_s|\big\|_{L^p} + \|\nabla^2 u_s\|_\infty \|(1-\Delta)^{\frac{2m+1}{4}} u_s\|_{L^p} \\ &\qquad + \|(1-\Delta)^{\frac{2m-1}{4}} \nabla u_s\|_{L^p} \|u_s\|_\infty \Big\} \mathrm{d}s.\end{aligned}$$

This together with (3.3), (3.5) and (3.7) implies $\|\nabla^{m+1} u\|_{p,\infty} < \infty$.

Finally, when $\|\nabla^n u_0\|_\infty < \infty$, by (1.9), to prove (1.10) it suffices to show that

$$\sup_{t\in[0,T]} \|\nabla^n u_t\|_\infty < \infty.$$

This can be shown as in step (a) using ∇^n to replace ∇^2 and applying (1.9).

□

4 Proof of Theorem 1.3

If (1.13) does not hold, then there exists a constant $c > 0$ such that

$$\phi\big(\|u_t\|_\infty^2 + \|\nabla u_t\|_{L^p}^{\frac{2p}{2p-d}}\big)\big(\zeta(u_0) - t\big) \le c, \quad t \in [0, \zeta(u_0)).$$

So,

$$A_n := \|u_{\tau_{n-1}(u_0)}\|_\infty^2 + \|\nabla u_{\tau_{n-1}(u_0)}\|_{L^p}^{\frac{2p}{2p-d}}, \quad n \ge 1$$

satisfies

$$\sum_{k=n}^\infty T_k^*(u_0) = \zeta(u_0) - \tau_{n-1}(u_0) \le \frac{c}{\phi(A_n)}, \quad n \ge 1.$$

By (1.11) we find a constant $c_1 > 0$ such that

$$T_k^*(u_0) \ge c_1 \min\big\{\|u_{\tau_{k-1}(u_0)}\|_\infty^{-2},\ \|\nabla u_{\tau_{k-1}(u_0)}\|_{L^p}^{-\frac{2p}{2p-d}}\big\} \ge \frac{c_1}{A_k}, \quad k \ge n.$$

Then for some constant $c_2 > 0$ we have

$$B_n := \sum_{k=n}^{\infty} A_k^{-1} \le \frac{c_2}{\phi(A_n)}, \quad n \ge 1.$$

Therefore,

$$B_{n+1} - B_n = -A_n^{-1} \le -\frac{1}{\phi^{-1}(c_2 B_n^{-1})}, \quad n \ge 1.$$

Noting that ϕ^{-1} is increasing while B_n is decreasing to 0 as $n \to \infty$, the linear interpolation $(B_s)_{s\in[1,\infty]}$ of $(B_n)_{n\ge1}$ satisfies

$$B_s' = B_{n+1} - B_n \le -\frac{1}{\phi^{-1}(c_2 B_n^{-1})} \le -\frac{1}{\phi^{-1}(c_2 B_s^{-1})}, \quad s \in [n, n+1).$$

Since $\int_1^\infty \frac{\phi^{-1}(s)}{s^2}\mathrm{d}s < \infty$, this implies that

$$-\infty \ge \int_1^\infty \phi^{-1}(c_2 B_s^{-1})\mathrm{d}B_s = -c_2 \int_{c_2 B_1^{-1}}^\infty \frac{\phi^{-1}(r)}{r^2}\mathrm{d}r > -\infty,$$

which is impossible. Thus, (1.13) has to be true.

5 The mild well-posedness

By Lemma 2.1, in the regular case as in Theorem 1.2, the solution of (1.1) is given by (1.7) and (2.6). Since $\nabla \cdot u_t = 0$, we may reformulate (1.5) as

$$u_t = P_t u_0 - \sum_{i=1}^d \int_0^t \partial_i \mathscr{R}_{t-s}\big\{u_s^i u_s\big\}\mathrm{d}s, \quad t \in [0,T]. \tag{5.1}$$

This leads to the following notion of mild solution to (1.1). Let

Definition 5.1. Let $p \in (d,\infty)$. We denote $\nabla \cdot f = 0$ for a function $f : E \to \mathbb{R}^d$, if

$$\int_E \langle f, \nabla h\rangle(x)\mathrm{d}x = 0, \quad h \in C_0^\infty(E),$$

where C_0^∞ is the class of C^∞ real functions on E with compact support.

A function $u : [0,T] \times E \to \mathbb{R}^d$ is called a weak solution of (1.1), if $\|u\|_{p,\infty} < \infty, \nabla \cdot u_t = 0$ and (5.1) holds.

For any $p \in (d,\infty)$, by the $L^{\frac{p}{2}}$ boundedness of the Riesz transform and that

$$\|\nabla P_t\|_{L^{\frac{p}{2}} \to L^p} \le c t^{-\frac{p+d}{2p}}, \quad t > 0$$

for some constant $c > 0$, we have

$$K_p := \sup_{t>0} t^{\frac{p+d}{2p}} \|\nabla \mathscr{R}_t\|_{\mathscr{L}^{\frac{p}{2}} \to L^p} < \infty.$$

Let

$$\theta_{p,T,d} := \frac{p-d}{2pK_p d^{\frac{p-1}{p}}} T^{\frac{d-p}{2p}}.$$

Moreover, let

$$\Gamma_\theta(u_0) := \big\{\gamma : [0,T]\times E \to \mathbb{R}^d;\ \gamma_0 = u_0,\ \nabla\cdot\gamma_t = 0,\ \|\gamma\|_{p,\infty} \le \theta\big\}.$$

Theorem 5.1. *Let $p \in (d,\infty)$, $u_0 \in L^p(E\to\mathbb{R}^d)$ with $\nabla\cdot u_0 = 0$, and $\theta \in (0,\theta_{p,T,d})$. If*

$$\|u_0\|_{L^p} \le \theta\Big(1 - \frac{\theta}{\theta_{p,T,d}}\Big), \tag{5.2}$$

then (1.1) *has a unique mild solution, and the solution is in $\Gamma_\theta(u_0)$. Consequently,* (1.1) *has a unique solution for*

$$T = \tilde{T}_0^*(u_0) := \Big(\frac{p-d}{8pK_p d^{\frac{p-1}{p}}\|u_0\|_{L^p}}\Big)^{\frac{2p}{p-d}}, \tag{5.3}$$

and the solution is in $\Gamma_\theta(u_0)$ for $\theta = \frac{1}{2}\theta_{p,T,d}$.

Proof. We will use the fixed point theorem for the map

$$\gamma \mapsto u_t^\gamma := P_t u_0 - \sum_{i=1}^d \int_0^t \partial_i \mathscr{R}_{t-s}(\gamma_s^i\gamma_s)\mathrm{d}s,\quad t\in[0,T].$$

(a) For any $\gamma\in\Gamma_\theta(u_0)$, we intend to prove that $u^\gamma \in \Gamma_\theta(u_0)$. Indeed, by definition we have

$$\begin{aligned}\|u_t^\gamma\|_{L^p} &\le \|u_0\|_{L^p} + K_p\sum_{i=1}^d\int_0^t (t-s)^{-\frac{p+d}{2p}}\|\gamma_s^i\gamma_s\|_{L^{\frac{p}{2}}}\mathrm{d}s\\ &\le \|u_0\|_{L^p} + K_p\sum_{i=1}^d\int_0^t (t-s)^{-\frac{p+d}{2p}}\|\gamma_s^i\|_{L^p}\|\gamma_s\|_{L^p}\mathrm{d}s.\end{aligned}$$

Noting that $p>d\ge 2$ implies $\sum_{i=1}^d|\gamma_s^i|^p \le |\gamma_s|^p$, by Hölder's inequality,

$$\sum_{i=1}^d\|\gamma_s^i\|_{L^p} \le d^{\frac{p-1}{p}}\Big(\sum_{i=1}^d\int_E|\gamma_s^i(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}} \le d^{\frac{p-1}{p}}\|\gamma_s\|_{L^p}.$$

Hence,

$$\|u_t^\gamma\|_{L^p} \le \|u_0\|_{L^p} + \frac{2pK_p d^{\frac{p-1}{p}}T^{\frac{p-d}{2p}}}{p-d}\|\gamma\|_{p,\infty}^2 \le \|u_0\|_{L^p} + \frac{\theta^2}{\theta_{p,T,d}},\quad \gamma\in\Gamma_\theta(u_0).$$

So, (5.2) yields $\|u^\gamma\|_{p,\infty}\le\theta$. Moreover, by $\nabla\cdot u_0 = 0$ and $\nabla\cdot\mathscr{R}_{t-s}=0$, we have $\nabla\cdot u_t^\gamma = 0$. Therefore, $u_t^\gamma\in\Gamma_\theta(u_0)$.

(b) We find a constant $c>0$ such that

$$\begin{aligned}\|u_t^\gamma - u_t^{\tilde\gamma}\|_{L^p} &\le K_p \sum_{i=1}^d \int_0^t (t-s)^{-\frac{p+d}{2p}} \big\|(\gamma_s^i - \tilde\gamma_s^i)\gamma_s + \tilde\gamma_s^i(\gamma_s - \tilde\gamma_s)\big\|_{L^{\frac{p}{2}}} \\ &\le c\int_0^t (t-s)^{-\frac{p+d}{2p}} \|\gamma_s - \tilde\gamma_s\|_{L^p}\mathrm{d}s, \quad \gamma, \tilde\gamma \in \Gamma_\theta(u_0).\end{aligned}$$

So, when $\lambda>0$ is large enough, u is contractive on $\Gamma_\theta(u_0)$ under the complete metric

$$\tilde\rho_\lambda(\gamma,\tilde\gamma) := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\|\gamma_t - \tilde\gamma_t\|_{L^p}.$$

Hence, u has a unique fixed point in $\Gamma_\theta(u_0)$. This is the unique mild solution in the class $\Gamma_\theta(u_0)$.

Finally, if (1.1) has another mild solution $\tilde u_t$ with $\tilde u_0 = u_0$, by the same technique we prove $\tilde u_t = u_t$. □

As explained after Theorem 1.2, for any u_0 with $\nabla\cdot u_0 = 0$ and $\|u_0\|_{L^p}<\infty$, we have the mild well-posedness of (1.1) up to the maximal time

$$\tilde\zeta(u_0) := \sum_{n=0}^\infty \tilde T_n^*(u_0),$$

where $\tilde T_n^*(u_0) := \tilde T_0^*(u_{\tilde\tau_{n-1}(u_0)})$ with

$$\tilde\tau_{n-1}(u_0) := \sum_{i=0}^{n-1} \tilde T_i^*(u_0), \quad n\ge 1.$$

By (5.3), the same argument in the proof of Theorem 1.3 implies the following result.

Theorem 5.2. *Let $u_0\in L^p$ for some $p\in(d,\infty)$ and $\nabla\cdot u_0=0$. Then for any $T\in(0,\tilde\zeta(u_0))$, (1.1) has a unique mild solution with $\|u\|_{p,\infty}<\infty$. Moreover, when $\tilde\zeta(u_0)<\infty$, for any increasing continuous function $\phi:[0,\infty)\to[1,\infty)$ with $\int_1^\infty \frac{\phi^{-1}(r)}{r^2}\mathrm{d}r<\infty$, we have*

$$\limsup_{t\to\tilde\zeta(u_0)} \phi\Big(\|u_t\|_{L^p}^{\frac{2p}{p-d}}\Big)(\tilde\zeta(u_0)-t) = \infty.$$

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References

[1] D. Bakry, *Etude des transformations de Riesz dans les variétés riemanniennes á courbure de Ricci minorée,* Séminaire de Probabilités XXI, Lecture Notes in Math. 1247, 137-172, 1987, Springer.

[2] W. Feng, J. He, W. Wang, *Quantitative bounds for critically bounded solutions to the three-dimensional Navier-Stokes equations in Lorentz spaces,* arXiv:2201.04656v1.

[3] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation,* J. Amer. Math. Soc. 4(1991),323–347.

[4] N.V. Krylov, M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift,* Probab. Theory Related Fields 131(2005), 154–196.

[5] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace,* Acta Math. 63(1934), 193–248.

[6] S. Palasek, *Improved quantitative regularity for the Navier-Stokes equations in a scale of critical spaces,* Arch. Ration. Mech. Anal. 242(2021), 1479–1531.

[7] Z. Qian, E. Süli, Y. Zhang, *Random vortex dynamics via functional stochastic differential equations,* arXiv:2201.00448v1.

[8] T. Tao, *Quantitative bounds for critically bounded solutions to the Navier-Stokes equations,* arXiv:1908.04958.

[9] F.-Y. Wang, *A probabilistic characterization on Navier-Stokes equations,* arXiv:2201.06861.

[10] P. Xia, L. Xie, X. Zhang, G. Zhao, *$L^q(L^p)$-theory of stochastic differential equations,* Stoch. Proc. Appl. 130(2020), 5188–5211.