MULTI-BUBBLE BOURGAIN-WANG SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

MICHAEL RÖCKNER, YIMING SU, AND DENG ZHANG

ABSTRACT. We consider a general class of focusing L^2-critical nonlinear Schrödinger equations with lower order perturbations, for which the pseudo-conformal symmetry and the conservation law of energy are absent. In dimensions one and two, we construct Bourgain-Wang type solutions concentrating at K distinct singularities, $1 \leq K < \infty$, and prove that they are unique if the asymptotic behavior is within the order $(T-t)^{4+}$, for t close to the blow-up time T. These results apply to the canonical nonlinear Schrödinger equations and, through the pseudo-conformal transform, in particular yield the existence and conditional uniqueness of non-pure multi-solitons. Furthermore, through a Doss-Sussman type transform, these results also apply to stochastic nonlinear Schrödinger equations, where the driving noise is taken in the sense of controlled rough path.

CONTENTS

2010 *Mathematics Subject Classification.* 35B44, 35B40, 35Q55.
Key words and phrases. Bourgain-Wang solutions, L^2-critical, multi-bubbles, nonlinear Schrödinger equation, non-pure multi-solitons.

1. Introduction and main results

1.1. **Introduction.** We consider a general class of focusing L^2-critical nonlinear Schrödinger equations with lower order perturbations

$$i\partial_t v + \Delta v + a_1 \cdot \nabla v + a_0 v + |v|^{\frac{4}{d}} v = 0 \tag{1.1}$$

on $\mathbb{R}^d$, where $d = 1, 2$, the coefficients of lower order perturbations are of form

$$a_1(t,x) = 2i \sum_{l=1}^{N} \nabla\phi_l(x) h_l(t), \tag{1.2}$$

$$a_0(t,x) = -\sum_{j=1}^{d} \left(\sum_{l=1}^{N} \partial_j \phi_l(x) h_l(t) \right)^2 + i \sum_{l=1}^{N} \Delta\phi_l(x) h_l(t), \tag{1.3}$$

and $\phi_l \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$, $h_l \in C(\mathbb{R}^+; \mathbb{R})$, $1 \le l \le N$.

Equation (1.1) is mainly motivated by two canonical models. The first is the nonlinear Schrödinger equation (NLS), corresponding to the case without lower order perturbations, i.e.,

$$i\partial_t v + \Delta v + |v|^{\frac{4}{d}} v = 0. \tag{1.4}$$

NLS is a canonical equation of major importance in continuum mechanics, plasma physics and optics ([31]). In particular, for the cubic nonlinearity in the critical dimension two, the phenomenon of mass concentration near collapse gives a rigorous basis to the physical concept of "strong collapse" ([65]). For more physical interpretations we refer to [31, 36, 65].

Another important model is the stochastic nonlinear Schrödinger equation (SNLS)

$$idX + \Delta X dt + |X|^{\frac{4}{d}} X dt = -i\mu X dt + iX dW(t), \tag{1.5}$$

where W is a Wiener process of form

$$W(t,x) = \sum_{l=1}^{N} i\phi_l(x) B_l(t), \quad x \in \mathbb{R}^d, \ t \ge 0,$$

$\{\phi_l\} \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$, $\{B_l\}$ are standard N-dimensional real valued Brownian motions on a normal stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$, and $\mu = \frac{1}{2}\sum_{l=1}^N \phi_l^2$. The last term $XdW(t)$ in (1.5) is taken in the sense of controlled rough path (see Definition 1.8 below). The key relationship is that, through the Doss-Sussman type transformation $v := e^{-W}X$, v satisfies equation (1.1) with the functions $\{h_l\}$ being exactly the Brownian motions $\{B_l\}$.

The physical significance of SNLS is well known. One significant model arises from molecular aggregates with thermal fluctuations, where the multiplicative noise corresponds to scattering of exciton by phonons, due to thermal vibrations of the molecules. In particular, for the cubic nonlinearity in dimension two, the noise effect on the coherence of the ground state solitary solution was studied in [1, 2]. The case of quintic nonlinearity in the critical one dimensional case was studied in [61]. We also refer to [7] for applications to open quantum systems.

It is known that, equation (1.1) is locally well-posed in the space H^1, see, e.g., [12] for the NLS, and [22, 10, 4] for the SNLS.

The long time behavior of solutions is, however, more delicate. An important role here is played by the ground state, which is a positive radial solution to the elliptic equation

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0. \tag{1.6}$$

It is known that (see [12, Theorem 8.1.1]) Q is smooth and decays at infinity exponentially fast, i.e., there exist $C, \delta > 0$ such that for any multi-index $|\upsilon| \leq 3$,

$$|\partial_x^\upsilon Q(x)| \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^d. \tag{1.7}$$

More importantly, the mass of the ground state is the threshold of global well-posedness and blow-up. As a matter of fact, in the NLS case, solutions with subcritical mass (i.e., $\|v\|_{L^2} < \|Q\|_{L^2}$) exist globally and even scatter at infinity, [27, 67]. In contrast to that, in the critical mass regime, two important dynamics are exhibited: the non dispersive solitary wave

$$W(t,x) := w^{-\frac{d}{2}} Q\left(\frac{x-ct}{w}\right) e^{i(\frac{1}{2}c\cdot x - \frac{1}{4}|c|^2 t + w^{-2}t + \vartheta)}, \tag{1.8}$$

and the pseudo-conformal blow-up solutions

$$S_T(t,x) = (w(T-t))^{-\frac{d}{2}} Q\left(\frac{x-x^*}{w(T-t)}\right) e^{-\frac{i}{4}\frac{|x-x^*|^2}{T-t} + \frac{i}{w^2(T-t)} + i\vartheta}, \tag{1.9}$$

where $w > 0$, $c, x^* \in \mathbb{R}^d$ and $\vartheta \in \mathbb{R}$. Both dynamics are closely related to each other in the pseudo-conformal space $\Sigma := \{u \in H^1 : xu \in L^2\}$, through the *pseudo-conformal transform*

$$S_T(t,x) = \mathcal{C}_T(W)(t,x) := \frac{1}{(T-t)^{\frac{d}{2}}} W\left(\frac{1}{T-t}, \frac{x}{T-t}\right) e^{-i\frac{|x|^2}{4(T-t)}}, \quad t \neq T, \ x^* = c. \tag{1.10}$$

Note that, S_T blows up at time T, and x^* is the singularity corresponding to the velocity c of W. A remarkable result in the seminal paper by Merle [50] is that, the pseudo-conformal blow-up solution is the unique critical mass blow-up solution to L^2-critical NLS, up to symmetries of the equation.

In the small supercritical mass regime, two different kinds of blow-up solutions to NLS are exhibited. The first one is the Bourgain-Wang solution behaving asymptotically as a sum of a singular profile S_T and a regular profile z, i.e.,

$$v(t) - S_T(t) - z(t) \to 0, \quad as\ t \to T. \tag{1.11}$$

Note that, v blows up at time T with the pseudo-conformal speed

$$\|\nabla v(t)\|_{L^2} \sim (T-t)^{-1}.$$

This kind of solutions was first constructed in the pioneering work by Bourgain and Wang [9] in dimensions $d = 1, 2$. It was then extended by Krieger and Schlag [42] to prove the existence of a large set of initial data close to the ground state resulting in pseudo-conformal speed blow-up solutions in dimension $d = 1$, this set is a codimension one stable manifold in the measurable category. Moreover, the instability of such solutions was proved in the work by Merle, Raphaël and Szeftel [56], which shows that Bourgain-Wang solutions lie on the boundary of two H^1 open sets of global scattering solutions and loglog blow-up solutions. We also would like to refer to [41, 62] for the stable manifolds for the supercritical NLS, and [8] for the center-stable manifold for the $\dot{H}^{\frac{1}{2}}$-critical cubic NLS in dimension three.

Another important kind of blow-up solutions is of loglog blow-up rate

$$\|\nabla v(t)\|_{L^2} \sim \left((T-t)^{-1} \log|\log(T-t)|\right)^{\frac{1}{2}}.$$

Unlike Bourgain-Wang solutions, these solutions are stable under H^1 perturbations. In this respect, we refer to the pioneering work by Perelman [59] and a series of works of Merle and Raphaël [51, 52, 53, 55].

In the even larger mass regime, the construction of multi-bubble blow-up solutions was initiated by Merle [49], which behave like a sum of K pseudo-conformal blow-up solutions, $1 \leq K < \infty$. Through the pseudo-conformal transform, this also yields the existence of multi-solitons, [49]. Multi-bubble blow-up solutions with loglog speed have recently been constructed by Fan [32].

For general blow-up solutions to L^2-critical NLS, it is conjectured that the mass of blow-up solutions is quantized at each singularity and the remaining part of solutions converges strongly to a residue away from the singularities, see the *mass quantization conjecture* in [54], see also [9].

Let us also mention that, according to the famous *soliton resolution conjecture*, global solutions to a nonlinear dispersive equation are expected to decompose at large time as a sum of solitons plus a scattering remainder. We refer to [18, 28, 29, 30] and references therein for the important progress for the energy critical wave equation. For the NLS, except for the integrable one dimensional case, this conjecture is still open. A series of (pure) multi-solitons (i.e., solutions behaving as a sum of solitons without dispersive part) have been constructed for the NLS, see e.g. [17, 19, 20, 43, 44, 46, 49]. See [40] for the construction of two soliton solutions for the subcritical Hartree equation. For the gKdV equations, we also refer to [45, 14] for the existence and classification of multi-solitions, and [15, 16] for the construction of solutions behaving as a sum of solitons and of a linear term.

Hence, a natural question to ask is whether non-pure multi-solitons (including a dispersive part) can be constructed for the NLS, which, to the best of our knowledge, seems not to have been done in literature. See, e.g., the recent lecture notes of Cazenave [13].

The two conjectured long time dynamics are indeed the main motivations of the present work.

Furthermore, in the stochastic case, a remarkable result proved by de Bouard and Debussche [21, 23] is that, stochastic solutions can blow up at any short time with positive probability in the L^2-supercritical case. Several numerical experiments have been also made to investigate the dynamics of stochastic blow-up solutions, see, e.g., [24, 25, 26, 57, 58].

One major challenge in the stochastic case is that, in contrast to NLS, the classical pseudo-conformal symmetry is lost due to the input of noise. Moreover, the energy of solutions is no longer conserved, which makes it more difficult to understand the global behavior in the stochastic L^2-supercritical case, see [57, 58] for the numerical tracking of energy.

Recently, the quantitative construction of critical mass stochastic blow-up solutions to (1.5) is obtained in [63], the proof there relies mainly on the modulation method developed in the work by Raphaël and Szeftel [60] and also on the rescaling approach in [4, 5, 6, 37, 69, 70]. This also yields the threshold of the mass of the ground state for the global well-posedness and blow-up in the stochastic case. Later, stochastic blow-up solutions with loglog speed have been constructed in [33]. Furthermore, multi-bubble blow-up solutions to (1.5), behaving as a sum of pseudo-conformal blow-up solutions, were constructed and proved to be unique if the asymptotic behavior is of the order $(T-t)^{3+}$, [64]. The conditional uniqueness result has been further used in the very recent work [11] to enlarge the energy class for the uniqueness of both multi-bubble solutions and multi-solitons, particularly in the low asymptotical regime with the orders $O(T-t)^{0+}$ and s^{-2-}, respectively, where t is close to T and s is large.

In the present work, we study the Bourgain-Wang type solutions, concentrating at multiple points, in the large mass regime for both equations (1.4) and (1.5) in a uniform manner.

More precisely, in both dimensions one and two, we construct multi-bubble Bourgain-Wang solutions to (1.1), which behave asymptotically as a sum of pseudo-conformal blow-up solutions and a regular profile, i.e., for t close to T,

$$\|v(t) - \sum_{k=1}^{K} S_k(t) - z(t)\|_{L^2} + (T-t)\|\nabla v(t) - \nabla \sum_{k=1}^{K} S_k(t) - \nabla z(t)\|_{L^2} \leq C(T-t)^{\frac{1}{2}(\kappa-1)}, \tag{1.12}$$

where z is the regular profile propagating along the flow generated by equation (1.1) with $z(T) = z^*$, $\{S_k\}$ are the pseudo-conformal blow-up solutions as in (1.9) with distinct singularities, and the exponent κ (≥ 3) is closely related to the flatness at singularities of both the spatial functions $\{\phi_l\}$ and the residue z^*. Moreover, we prove that the multi-bubble Bourgain-Wang solutions are unique if their asymptotical behavior is within the order $(T-t)^{4+}$.

This provides examples of the conjectured mass quantization phenomena for both the L^2-critical NLS and SNLS. Furthermore, in the NLS case, through the pseudo-conformal transform, the existence and conditional uniqueness of non-pure multi-solutions are also obtained, which behave asymptotically as a sum of solitons with distinct velocities plus a dispersive part. To the best of our knowledge, this provides the first examples of non-pure multi-solitons to the L^2-critical NLS, predicted by the soliton resolution conjecture. Let us also mention that, the uniqueness holds in the energy class of solutions with decay rate t^{-5-}, where t is large enough, which is larger than the class of exponential convergence in which (pure) multi-solitons naturally lie.

Notations. For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and any multi-index $\upsilon = (\upsilon_1, \cdots, \upsilon_d)$, let $|\upsilon| = \sum_{j=1}^d \upsilon_j$, $\langle x\rangle = (1+|x|^2)^{1/2}$, $\partial_x^\upsilon = \partial_{x_1}^{\upsilon_1}\cdots\partial_{x_d}^{\upsilon_d}$, and $\langle\nabla\rangle = (I-\Delta)^{1/2}$.

We use the standard Sobolev spaces $H^{s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}, 1 \leq p \leq \infty$. In particular, $L^p := H^{0,p}(\mathbb{R}^d)$ is the space of p-integrable (complex-valued) functions, L^2 denotes the Hilbert space with the inner product $\langle v, w\rangle = \int_{\mathbb{R}^d} v(x)\overline{w}(x)dx$, and $H^s := H^{s,2}$. Let Σ denote the pseudo-conformal space, i.e., $\Sigma := \{u \in H^1, xu \in L^2\}$. The local smoothing space is defined by $L^2(I; H^\alpha_\beta) = \{u \in \mathscr{S}' : \int_I\int \langle x\rangle^{2\beta}|\langle\nabla\rangle^\alpha u(t,x)|^2 dxdt < \infty\}$, $\alpha,\beta \in \mathbb{R}$. Let C_c^∞ be the space of all compactly supported smooth functions on $\mathbb{R}^d$.

We also use the notation $\dot{g} = \frac{d}{dt}g$ for any C^1 function g on $\mathbb{R}$. For any Hölder continuous function $g \in C^\alpha(I)$, $\alpha > 0$, $I \subseteq \mathbb{R}^+$, let $\delta g_{st} := g(t) - g(s)$, $s,t \in I$, and $\|g\|_{\alpha,I} := \sup_{s,t\in I, s\neq t}\frac{|\delta g_{st}|}{|s-t|^\alpha}$. As $t \to T$ or $t \to \infty$, $f(t) = \mathcal{O}(g(t))$ means that $|f(t)/g(t)|$ stays bounded, and $f(t) = o(g(t))$ means that $|f(t)/g(t)|$ converges to zero.

Throughout this paper, the positive constants C and δ may change from line to line.

1.2. **Formulation of main results.** Let $K \in \mathbb{N}^+$ and $\{x_k\}_{k=1}^K$ denote the distinct blow-up points in $\mathbb{R}^d$. We assume that the spatial functions $\{\phi_l\}$ in the noise and the residue z^* satisfy the following hypotheses:

(H1) *Asymptotical flatness*: For any multi-index $\upsilon \neq 0$ and $1 \leq l \leq N$,

$$\lim_{|x|\to\infty} \langle x\rangle^2 |\partial_x^\upsilon \phi_l(x)| = 0. \tag{1.13}$$

Flatness at singularities: Let $\upsilon_* \in \mathbb{N}^+$. For every $1 \leq l \leq N$ and multi-index $|\upsilon| \leq \upsilon_*$,

$$\partial_x^\upsilon \phi_l(x_k) = 0, \quad 1 \leq l \leq N, \ 1 \leq k \leq K. \tag{1.14}$$

(H2) *Smallness*: Let α^* be a positive (small) constant, $m \in \mathbb{N}^+$. Let z^* satisfy

$$\|z^*\|_{H^{2m+2+d}} \leq \alpha^*, \tag{1.15}$$

$$\|\langle x\rangle z^*\|_{H^1} \leq \alpha^*. \tag{1.16}$$

Flatness at singularities: For any multi-index $|\upsilon| \leq 2m$,

$$\partial_x^\upsilon z^*(x_k) = 0, \quad 1 \leq k \leq K. \tag{1.17}$$

Remark 1.1. *On one hand, the asymptotical flatness condition* (1.13) *ensures the Strichartz and local smoothing estimates for the Laplacian with lower order perturbations, which guarantees the* *local solvability of equation* (1.1)*, see* [4, 48, 69]. *On the other hand, the flatness at singularities* (1.14) *and* (1.17) *permit to construct blow-up solutions, which reflect the local nature of the singularities.*

The main result of this paper is formulated in the following theorem.

Theorem 1.2. *Consider equation* (1.1) *with* $d = 1, 2$. *Let* $K \in \mathbb{N}^+$, $T \in \mathbb{R}_{>0}$, $\{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}$. *Assume that* $\{\phi_l\}_{l=1}^N$ *and* z^* *satisfy Hypotheses* ($H1$) *and* ($H2$)*, respectively, with* $\upsilon_* \geq 5$, $m \geq 3$ *if* $d = 2$ *and* $m \geq 4$ *if* $d = 1$.

Then, for any distinct points $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$, $w > 0$ *(resp. any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$*), there exists* $\varepsilon^* > 0$ *small enough such that for any* $\alpha^*, \varepsilon \in (0, \varepsilon^*)$ *and for any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$ *with* $|w_k - w| \leq \varepsilon$, $1 \leq k \leq K$ *(resp. any* $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$ *with* $|x_k - x_j| \geq \varepsilon^{-1}$, $j \neq k$*), the following holds:*

(i) *Existence. There exists a solution* v *to* (1.1) *satisfying that for* t *close to* T,

$$\|v(t) - \sum_{k=1}^K S_k(t) - z(t)\|_{L^2} \leq C(T-t)^{\frac{1}{2}(\kappa-1)}, \tag{1.18}$$

and

$$\|v(t) - \sum_{k=1}^K S_k(t) - z(t)\|_{\Sigma} \leq C(T-t)^{\frac{1}{2}(\kappa-3)}, \tag{1.19}$$

where $\kappa := (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$, $C > 0$, $\{S_k\}$ *are the pseudo-conformal blow-up solutions of the form*

$$S_k(t,x) = (w_k(T-t))^{-\frac{d}{2}} Q\left(\frac{x - x_k}{w_k(T-t)}\right) e^{-\frac{i}{4}\frac{|x-x_k|^2}{T-t} + \frac{i}{w_k^2(T-t)} + i\vartheta_k}. \tag{1.20}$$

and z *is the unique solution of the equation*

$$\begin{cases} i\partial_t z + \Delta z + a_1 \cdot \nabla z + a_0 z + |z|^{\frac{4}{d}} z = 0, \\ z(T) = z^*, \end{cases} \tag{1.21}$$

where the coefficients a_1, a_0 *are given by* (1.2) *and* (1.3)*, respectively.*

(ii) *Conditional uniqueness. Assume in addition that* $m \geq 10, \upsilon_* \geq 12$. *Then, for any small* $\zeta > 0$, *there exists a unique solution* v *to* (1.1) *satisfying that for* t *close to* T,

$$\|v(t) - \sum_{k=1}^K S_k(t) - z(t)\|_{L^2} + (T-t)\|\nabla v(t) - \nabla \sum_{k=1}^K S_k(t) - \nabla z(t)\|_{L^2} \leq C(T-t)^{4+\zeta}. \tag{1.22}$$

Remark 1.3. (i). *Theorem 1.2 mainly treats two cases of singularities* $\{x_k\}$ *and frequencies* $\{w_k\}$*:*

Case (I)*:* $\{x_k\}_{k=1}^K$ *are arbitrary distinct points in* $\mathbb{R}^d$, *and* $\{w_k\}_{k=1}^K (\subseteq \mathbb{R}_{>0})$ *satisfy that for some* $w > 0$, $|w_k - w| \leq \varepsilon$ *for every* $1 \leq k \leq K$.

Case (II)*:* $\{w_k\}_{k=1}^K$ *are arbitrary points in* $\mathbb{R}_{>0}$, *and* $\{x_k\}_{k=1}^K (\subseteq \mathbb{R}^d)$ *satisfy that* $|x_j - x_k| \geq \varepsilon^{-1}$ *for any* $1 \leq j \neq k \leq K$.

Cases (I) *and* (II) *roughly mean certain decoupling between the profiles. In particular,* Case (I) *allows the arbitrariness of singularities when the frequencies are the same. In the special single bubble case, both the singularity and the frequency can be arbitrary. Unlike in* Case (II)*, the arbitrariness of singularities in* Case (I) *is mainly due to the conservation law of mass, which gives a rapid exponential decay of the sum of the localized masses.*

(ii). *The decay order in* (1.18) *and* (1.19) *is closely related to the flatness of* $\{\phi_l\}$ *and* z^* *at the singularities. For* $\kappa \geq 4$, *the asymptotics hold in the more regular* $H^{\frac{3}{2}}$ *space.*

(iii). *It is important that the regular profile z propagates along the flow generated by equation* (1.1). *This fact permits to control the energy, particularly in the absence of the conservation law of energy, and to gain one more smallness of the remainder to fulfill the bootstrap arguments in the construction. The solvability of equation* (1.21) *can be guaranteed by the smallness of z^* in the Sobolev space and the Strichartz and local smoothing estimates for the Laplacian with lower order perturbations (see, e.g.,* [3, 4, 48, 69]*).*

(iv). *The conditional uniqueness reflects certain rigidity of the flow around multi-bubble pseudo-conformal blow-up solutions and the regular profile. It was first proved by Merle, Raphaël and Szeftel* [56] *in the single bubble case (i.e., $K = 1$) to ensure the continuity of the one-parameter curve in the instability result,* [56]. *See also* [39] *for Chern-Simons-Schrödinger equations and* [64] *for the SNLS case. It would be very interesting to prove the uniqueness in the low asymptotic regime, e.g. $(T-t)^{0+}$, as in the very recent work* [11]. *The main challenge here lies in the linear terms of the remainder in the control of localized mass and energy, which destroy the upgradation procedure in* [11]. *Let us also mention that, the proof of the conditional uniqueness in Theorem 1.2 is mainly due to the monotonicity of the generalized energy, which, fortunately, is stable under the effect of the regular profile.*

The applications to the NLS and SNLS cases are presented below.

Application 1: The NLS case. One main outcome of Theorem 1.2 is the following theorem concerning multi-bubble Bourgain-Wang solutions to L^2-critical NLS.

Theorem 1.4. *(Multi-bubble Bourgain-Wang solutions to NLS) Consider equation* (1.4) *with $d = 1, 2$. Let $K \in \mathbb{N}^+$, $T \in \mathbb{R}_{>0}$, $\{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}$. Assume that $z^* \in H^{2m+2+d}$ satisfying Hypothesis* ($H2$) *with $m \geq 3$ if $d = 2$ and $m \geq 4$ if $d = 1$.*

Then, for any distinct points $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$, $w > 0$ (resp. any $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$), there exists $\varepsilon^* > 0$ small enough such that for any $\alpha^*, \varepsilon \in (0, \varepsilon^*)$ and for any $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$ with $|w_k - w| \leq \varepsilon$, $1 \leq k \leq K$ (resp. any $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$ with $|x_k - x_j| \geq \varepsilon^{-1}$, $j \neq k$), there exists a solution v to (1.4) *satisfying the asymptotics* (1.18) *and* (1.19)*, where the regular profile z is the unique solution of equation*

$$\begin{cases} i\partial_t z + \Delta z + |z|^{\frac{4}{d}} z = 0, \\ z(T) = z^*. \end{cases} \tag{1.23}$$

Moreover, if in addition $m \geq 10$, then for any arbitrarily small $\zeta > 0$, there exists a unique solution to (1.4) *satisfying the asymptotic* (1.22).

Remark 1.5. *Theorem 1.4 gives examples for the conjectured mass quantization in* [54]. *Actually, by the asymptotical behavior* (1.18)*, for any $R > 0$,*

$$v(t) \to z^* \ \ in\ L^2\left(\mathbb{R}^d - \bigcup_{k=1}^K B(x_k, R)\right),$$

and

$$|v(t)|^2 \rightharpoonup \sum_{k=1}^K \|Q\|_{L^2}^2 \delta_{x_k} + |z^*|^2, \ \ as\ t \to T.$$

Hence, the solutions concentrate the mass $\|Q\|_{L^2}^2$ at each singularity and the remaining part converges to a regular residue z^*.

The next result is concerned with the non-pure multi-solitons to L^2-critical NLS, thanks to the pseudo-conformal transform which connects blow-up solutions and solitons.

Theorem 1.6. *(Non-pure multi-solitons to NLS) Consider equation* (1.4) *with* $d = 1, 2$. *Let* $K \in \mathbb{N}^+$, $\{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}$. *Assume that* $z^* \in H^{2m+2+d}$ *satisfying Hypothesis* (H2) *with* $m \geq 6$.

Then, for any distinct speeds $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d$, $w > 0$ *(resp. any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$*), there exists* $\varepsilon^* > 0$ *small enough such that for any* $\alpha^*, \varepsilon \in (0, \varepsilon^*)$ *and for any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$ *with* $|w_k - w| \leq \varepsilon$, $1 \leq k \leq K$ *(resp. any* $\{c_k\}_{k=1}^K \subseteq \mathbb{R}^d$ *with* $|c_j - c_k| \geq \varepsilon^{-1}$, $j \neq k$*), the following holds:*

(i) *Existence. There exists a solution u to* (1.4) *satisfying*

$$\|u(t) - \sum_{k=1}^{K} W_k(t) - \widetilde{z}(t)\|_\Sigma \leq Ct^{-\frac{1}{2}\kappa+\frac{5}{2}}, \quad for\ t\ large\ enough, \tag{1.24}$$

where $\kappa = m + \frac{d}{2} - 1$, $C > 0$, $\{W_k\}$ *are the solitary waves to* (1.4) *of form*

$$W_k(t, x) = w_k^{-\frac{d}{2}} Q\left(\frac{x - c_k t}{w_k}\right) e^{i(\frac{1}{2}c_k\cdot x - \frac{1}{4}|c_k|^2 t + w_k^{-2} t + \vartheta_k)}, \tag{1.25}$$

and $\widetilde{z}$ *corresponds to the regular part z for* (1.23) *through the inverse of the pseudo-conformal transform:*

$$\widetilde{z}(t, x) = C_T^{-1} z(t, x) = t^{-\frac{d}{2}} z\left(T - \frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}. \tag{1.26}$$

(ii) *Conditional uniqueness. If in addition* $m \geq 16$, *then for any arbitrarily small* $\zeta > 0$, *there exists a unique non-pure multi-soliton u to* (1.4) *satisfying*

$$\|u(t) - \sum_{k=1}^{K} W_k(t) - \widetilde{z}(t)\|_\Sigma \leq Ct^{-5-\zeta}, \quad for\ t\ large\ enough. \tag{1.27}$$

Remark 1.7. (i). *It is known (*[27]*) that in the subcritical mass regime* $\|z^*\|_{L^2} < \|Q\|_{L^2}$, *the solution z to* (1.23) *scatters both forward and backward in time, i.e.,* $\|z\|_{L^{2+\frac{4}{d}}(\mathbb{R}\times\mathbb{R}^d)} < \infty$. *Since the pseudo-conformal transform leaves the* L^2*-critical NLS and the* $L^{2+\frac{4}{d}}(\mathbb{R}\times\mathbb{R}^d)$*-norm invariant,* $\widetilde{z}$ *also scatters both forward and backward in time with small data* $\|z^*\|_{L^2} \leq \alpha^* << 1$. *Hence, by the asymptotics* (1.24)*, the constructed solution behaves as a sum of solitons plus a dispersive part. In particular, Theorem 1.6 provides new examples of non-pure multi-solitons to* L^2*-critical NLS, predicted by the soliton resolution conjecture.*

(ii). *It is also interesting to see that, the uniqueness of non-pure multi-solitons holds in the energy class of solutions with decay rate* t^{-5-}, *which is much larger than the class of exponential convergence in which multi-solitons naturally lie (see, e.g,* [43, 44]*). We also refer to* [17, 11] *for this kind of uniqueness in the case of pure multi-solitons to the* L^2*-critical NLS. It remains still open to prove the uniqueness or classification of even pure multi-solitons for the NLS, as done for the gKdV equations in* [45, 14].

(iii). *The relationship between the exponent m and the decay orders in Theorems 1.4 and 1.6 can be seen from the following estimates: for* $v := C_T u$,

$$\|u(t)\|_\Sigma \leq Ct\|v(T - \frac{1}{t})\|_\Sigma, \tag{1.28}$$

$$\|v(t)\|_\Sigma \leq \frac{C}{T - t}\|u(\frac{1}{T - t})\|_\Sigma. \tag{1.29}$$

Application 2: The SNLS case. Another important outcome of Theorem 1.2 is in stochastic case. Let us present the precise definition of solutions to equation (1.5) in the controlled rough path sense. For more details of the theory of (controlled) rough paths, we refer the interested readers to the monograph [34] and [35].

Definition 1.8. *We say that X is a solution to* (1.5) *on* $[0,\tau^*)$, *where* $\tau^* \in (0,\infty]$ *is a random variable, if* $\mathbb{P}$*-a.s. for any* $\varphi \in C_c^\infty$, $t \mapsto \langle X(t),\varphi\rangle$ *is continuous on* $[0,\tau^*)$ *and for any* $0 < s < t < \tau^*$

$$\langle X(t)-X(s),\varphi\rangle - \int_s^t \langle iX,\Delta\varphi\rangle + \langle i|X|^{\frac{4}{d}}X,\varphi\rangle - \langle \mu X,\varphi\rangle dr = \sum_{k=1}^N \int_s^t \langle i\phi_k X,\varphi\rangle dB_k(r).$$

Here, the integral $\int_s^t \langle i\phi_k X,\varphi\rangle dB_k(r)$ *is taken in the sense of controlled rough paths with respect to the rough paths* $(B,\mathbb{B})$, *where* $\mathbb{B} = (\mathbb{B}_{jk})$, $\mathbb{B}_{jk,st} := \int_s^t \delta B_{j,sr} dB_k(r)$ *with the integration taken in the sense of Itô and* $\delta B_{j,st} = B_j(t) - B_j(s)$. *That is,* $\langle i\phi_k X,\varphi\rangle \in C^\alpha([s,t])$,

$$\delta(\langle i\phi_k X,\varphi\rangle)_{st} = -\sum_{j=1}^N \langle \phi_j\phi_k X(s),\varphi\rangle \delta B_{j,st} + \delta R_{k,st}, \tag{1.30}$$

and $\|\langle\phi_j\phi_k X,\varphi\rangle\|_{\alpha,[s,t]} < \infty$, $\|R_k\|_{2\alpha,[s,t]} < \infty$, *where* $\frac{1}{3} < \alpha < \frac{1}{2}$.

The important fact is that, via the Doss-Sussman type transform

$$v = e^{-W}X, \tag{1.31}$$

the H^1 solvability of equations (1.1) and (1.5) is equivalent, see Theorem 2.10 in [63]. Thus, by virtue of Theorem 1.2, we have the following result for the L^2-critical SNLS.

Theorem 1.9. *(Multi-bubble Bourgain-Wang solutions to SNLS) Consider* (1.5) *with* $d = 1,2$. *Let* $K \in \mathbb{N}^+$, $\{\vartheta_k\}_{k=1}^K \subseteq \mathbb{R}$. *Assume that* $\{\phi_l\}_{l=1}^N$ *and* z^* *satisfy Hypotheses* $(H1)$ *and* $(H2)$, *respectively, with* $\upsilon_* \ge 5$, $m \ge 3$ *if* $d = 2$ *and* $m \ge 4$ *if* $d = 1$.

Then, for $\mathbb{P}$*-a.e* $\omega \in \Omega$ *and for any distinct points* $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$, $w > 0$ *(resp. any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$*), there exists* $\varepsilon^*(\omega) > 0$ *small enough such that for any* $\alpha^*, \varepsilon \in (0,\varepsilon^*)$ *and any* $\{w_k\}_{k=1}^K \subseteq \mathbb{R}_{>0}$ *with* $|w_k - w| \le \varepsilon$, $1 \le k \le K$ *(resp. any* $\{x_k\}_{k=1}^K \subseteq \mathbb{R}^d$ *with* $|x_k - x_j| \ge \varepsilon^{-1}$, $j \ne k$*), the following holds:*

There exists $\tau^*(\omega)$ *small enough such that for any* $T \in (0,\tau^*(\omega))$, *there exists a solution* X *to* (1.5) *satisfying for* t *close to* T,

$$\|e^{-W(t,\omega)}X(t,\omega) - \sum_{k=1}^K S_k(t) - z(t)\|_{L^2} \le C(T-t)^{\frac{1}{2}(\kappa-1)}, \tag{1.32}$$

and

$$\|e^{-W(t,\omega)}X(t,\omega) - \sum_{k=1}^K S_k(t) - z(t)\|_{\Sigma} \le C(T-t)^{\frac{1}{2}(\kappa-3)}, \tag{1.33}$$

where $\kappa := (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$, $C > 0$, $\{S_k\}$ *are the pseudo-conformal blow-up solutions as in* (1.20), *and* z *solves equation* (1.21).

Moreover, if in addition $m \ge 10$ *and* $\upsilon_* \ge 12$, *then for any arbitrarily small* $\zeta > 0$ *there exists a unique solution* X *to* (1.5) *such that*

$$\|e^{-W(t,\omega)}X(t,\omega) - \sum_{k=1}^K S_k(t) - z(t)\|_{\Sigma} \le C(T-t)^{4+\zeta}, \quad \text{for } t \text{ close to } T. \tag{1.34}$$

Remark 1.10. *The blow-up time $T \in (0, \tau^*)$ is chosen to be sufficiently small in Theorem 1.9 because the Brownian motions start moving at time zero.*

Sketch of Proof. The strategy of proof relies mainly on the modulation method developed in the works [60, 56] and on the multi-bubble analysis in [49, 11, 64].

The modulation method in [60] is very robust to handle the critical mass blow-up even in the absence of pseudo-conformal symmetry. It in particular enables us to treat equation (1.1) with lower order perturbations (or, the stochastic equation (1.5)). Moreover, as exhibited in [56], it also permits to construct Bourgain-Wang solutions as the limit of both the scattering and loglog blow-up solutions, rather than by the fixed point arguments in [9]. This inspires us to construct multi-bubble Bourgain-Wang solutions by using compactness arguments in the modulation framework, involving the backward integration from the singularity.

More precisely, we first decompose the approximating solution into three profiles

$$v(t,x) = U(t,x) + z(t,x) + R(t,x), \tag{1.35}$$

where U, z, R are the blow-up profile, the regular profile and the remainder, respectively, which satisfy suitable orthogonality conditions corresponding to the generalized null space of the linearized operators around the ground state. See Theorem 2.1 for the detailed statements.

Then, the localization analysis in [11, 64] and flatness conditions permit to reduce the analysis to an almost critical mass regime, in which more dynamical tools developed by [60] can be employed. One crucial ingredient here is the monotonicity of the generalized energy adapted to the multi-bubble case, which enables us to derive a uniform backwards control of the remainder. Hence, the desired blow-up solutions can be constructed by using compactness arguments as in [60, 64].

Let us mention that, different types of interactions are exhibited in the multi-bubble case:

- (i) Interactions between different blow-up profiles U_j and U_k, $j \neq k$. This kind of interaction is of exponentially small order (i.e., $e^{-\frac{\delta}{T-t}}$), due to the rapid decay of the ground state and the distinction of singularities. It was treated in the pioneering work by Merle [49] in the construction of multi-bubble pseudo-conformal solutions to L^2-critical NLS. See also [64] for the recent treatments in the stochastic case.
- (ii) Interactions between different localized remainders R_j and R_k, $j \neq k$. Unlike the previous interactions, the remainders are of low polynomial type decay orders. There is only little knowledge about remainders in the geometrical decomposition. Extra cancellations and decays have to be explored from the related localization and cut-off functions, e.g., to derive the monotonicity of the generalized energy and the coercivity of energy, [64].
- (iii) Interactions between the blow-up profile U and the remainder R. The typical interaction of this kind is the localized mass M_k defined in (3.1) below. It creates no difficulty in the single bubble case, as it is of second order $O(\|R\|_{L^2}^2)$ thanks to the conservation law of mass. However, the conservation law of mass fails for each localized mass in the multi-bubble case. Hence, more delicate analysis has to be performed to gain enough temporal regularity, [11, 64].
- (iv) Interactions between the remainder R and the regular profile z. This kind of interactions are acceptable in the construction procedure, as it is at least of the order $O(\|R\|_{L^2})$, which suffices for the bootstrap arguments.
- (v) Interactions between the blow-up profile U and the regular profile z. These interactions are treated by using the flatness condition (1.17). It is not difficult in the NLS case, as one may use Taylor's expansion and differentiate equation (1.4) enough times to get high temporal and spatial regularity, [9, 56]. However, this argument is not applicable in the

SNLS case, since the coefficients a_1, a_0 contain the rough paths of Brownian motions of merely temporal regularity $C_t^{\frac{1}{2}-}$. The key observation here is that, when interacting with the blow-up profile U, the spatial size $|x - x_k|$ is comparable to the temporal size $T - t$. This comparability between space and time permits to gain high temporal regularity from the spatial regularity of the residue z^*, and leads to an inductive expansion of solutions for which the continuity of coefficients suffices.

Another major difficulty is the failure of the conservation law of energy for the solutions to (1.1).

Actually, unlike in the radial case in [56], two new modulation parameters α_k and β_k need to be introduced in the multi-bubble case, due to the distinct singularities and the non-radialness of solutions. The control of these new parameters requires certain coercivity type control of energy, which however is no longer conserved. It might be tempting to use the variation control of energy as in [63, 64]. However, in the Bourgain-Wang regime under consideration, extra terms such as $\|z\|_{H^1}$ appear in the evolution formula of energy, which, unfortunately, give no temporal regularity and thus is far from sufficient to close the bootstrap arguments in the construction.

The key point here is, that the temporal regularity can be gained after subtracting the energy evolutions of the solutions and the regular profile. This leads us to introduce the evolution equation (1.21) for the regular profile, rather than the usual NLS. Similar structural consideration will also be used in the controls of localized mass and of the remainder in the pseudo-conformal space.

The remainder of this paper is organized as follows. Section 2 contains the geometrical decomposition and preliminary estimates for the modulation equations and different profiles. Section 3 is devoted to the controls of the localized mass, energy, and to the curial monotonicity of the generalized energy adapted to the multi-bubble case. Then, in Section 4 we mainly construct the multi-bubble Bourgain-Wang solutions to (1.1). The conditional uniqueness result in Theorem 1.2 is then proved in Section 5. At last, preliminaries concerning the linearized operators, expansion of the nonlinearity and the technical proof for modulation equations are collected in Appendix, i.e., Section 6.

2. Geometrical decomposition

2.1. **Geometrical decomposition.**

For each $1 \leq k \leq K$, define the modulation parameters by $\mathcal{P}_k := (\lambda_k, \alpha_k, \beta_k, \gamma_k, \theta_k) \in \mathbb{Y} := \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, where $\lambda_k, \gamma_k, \theta_k \in \mathbb{R}$, $\alpha_k, \beta_k \in \mathbb{R}^d$. Set $\mathcal{P} := (\mathcal{P}_1, \cdots, \mathcal{P}_K) \in \mathbb{Y}^K$.

Given any K distinct blow-up points $\{x_k\}$, set $P_k := |\lambda_k| + |\alpha_k - x_k| + |\beta_k| + |\gamma_k|$, $1 \leq k \leq K$, and $P := \sum_{k=1}^K P_k$. Let $S(t, x) = \sum_{k=1}^K S_k(t, x)$, where $\{S_k\}$ are given by (1.20).

Theorem 2.1. *(Geometrical decomposition) Given $T \in \mathbb{R}_{>0}$. Assume that $v \in C([\widetilde{t}, T_*]; H^1)$ solves* (1.1) *and $v(T_*) = S(T_*) + z(T_*)$, where $T_* < T$. Then, for α^* sufficiently small and for T_* close to T, there exist $t^* < T_*$ and unique modulation parameters $\mathcal{P} \in C^1((t^*, T_*); \mathbb{Y}^K)$, such that u admits the geometrical decomposition*

$$v(t, x) = U(t, x) + z(t, x) + R(t, x), \quad t \in [t^*, T_*], \ x \in \mathbb{R}^d, \tag{2.1}$$

where the main blow-up profile

$$U(t, x) = \sum_{k=1}^K U_k(t, x), \tag{2.2}$$

with

$$U_k(t, x) = \lambda_k(t)^{-\frac{d}{2}} Q_k(t, \frac{x - \alpha_k(t)}{\lambda_k(t)}) e^{i\theta_k(t)}, \quad Q_k(t, y) = Q(y) e^{i\left(\beta_k(t)\cdot y - \frac{1}{4}\gamma_k(t)|y|^2\right)}, \tag{2.3}$$

the regular profile z solves equation (1.21)*,* $R(T_*) = 0$*, and the modulation parameters satisfy*

$$\mathcal{P}_k(T_*) = (w_k(T-T_*), x_k, 0, w_k^2(T-T_*), w_k^{-2}(T-T_*)^{-1} + \vartheta_k), \quad 1 \le k \le K. \tag{2.4}$$

Moreover, for each $1 \le k \le K$*, the following orthogonality conditions hold on* $[t^*, T_*]$*:*

$$\begin{aligned} &\mathrm{Re} \int (x-\alpha_k) U_k(t) \overline{R}(t) dx = 0, \quad \mathrm{Re} \int |x-\alpha_k|^2 U_k(t) \overline{R}(t) dx = 0, \\ &\mathrm{Im} \int \nabla U_k(t) \overline{R}(t) dx = 0, \quad \mathrm{Im} \int \Lambda_k U_k(t) \overline{R}(t) dx = 0, \quad \mathrm{Im} \int \varrho_k(t) \overline{R}(t) dx = 0, \end{aligned} \tag{2.5}$$

where $\Lambda_k = \frac{d}{2} I_d + (x-\alpha_k) \cdot \nabla$*, and*

$$\varrho_k(t,x) = \lambda(t)_k^{-\frac{d}{2}} \rho_k(t, \frac{x-\alpha_k(t)}{\lambda_k(t)}) e^{i\theta_k(t)} \ \ with \ \ \rho_k(t,y) := \rho(y)^{i(\beta_k(t)\cdot y - \frac{1}{4}\gamma_k(t)|y|^2)}, \tag{2.6}$$

and ρ *is given by* (6.2).

Theorem 2.1 is mainly based on the implicit function theorem. The case $z^* = 0$ is proved in [64]. Since the smallness condition of z still keeps the non-degeneracy of the determinant of Jacobian matrix, the arguments in [64] are also applicable here. For simplicity, the proof is omitted.

2.2. **Modulation equations.** Let $\dot{g} := \frac{d}{dt} g$ for any C^1 function g. For each $1 \le k \le K$, define *the vector of modulation equations* by

$$Mod_k := |\lambda_k \dot{\lambda}_k + \gamma_k| + |\lambda_k^2 \dot{\gamma}_k + \gamma_k^2| + |\lambda_k \dot{\alpha}_k - 2\beta_k| + |\lambda_k^2 \dot{\beta}_k + \gamma_k \beta_k| + |\lambda_k^2 \dot{\theta}_k - 1 - |\beta_k|^2|. \tag{2.7}$$

Set $Mod := \sum_{k=1}^K Mod_k$.

The modulation equations mainly characterize the dynamics of geometrical parameters. The main estimate is contained in Theorem 2.2 below.

Theorem 2.2. *(Control of modulation equations) Assume that u admits the geometrical decomposition* (2.1) *on* $[t^*, T_*] \subseteq [0,T)$ *with the modulation parameters* $\mathcal{P} = (\lambda, \alpha, \beta, \gamma, \theta) \in \mathbb{Y}^K$*. Assume additionally that for some* $C_1, C_2 > 0$*,*

$$C_1(T-t) \le \lambda_k \le C_2(T-t), \quad t \in [t^*, T_*], \ 1 \le k \le K. \tag{2.8}$$

Then, for t^* *close to* T*, there exists* $C > 0$ *such that for any* $t \in [t^*, T_*]$*,*

$$Mod \le C \left(\sum_{k=1}^K |M_k| + P^2 D + D^2 + \alpha^*(T-t)^{m+1+\frac{d}{2}} + P^{\upsilon_*+1} \right), \tag{2.9}$$

where υ_* *is the index of flatness in* (1.14)*,* M_k *is the localized mass*

$$M_k = 2\mathrm{Re}\langle R_k, U_k \rangle + \int |R|^2 \Phi_k dx, \tag{2.10}$$

and D *is the important quantity to measure the size of remainder, defined by*

$$D := \|R\|_{L^2} + (T-t)\|\nabla R\|_{L^2}. \tag{2.11}$$

Moreover, we have the improved estimate

$$|\lambda_k \dot{\lambda}_k + \gamma_k| \le C \left(P^2 D + D^2 + \alpha^*(T-t)^{m+1+\frac{d}{2}} + P^{\upsilon_*+1} \right). \tag{2.12}$$

Remark 2.3. *By Lemma 4.3 below, we shall see that estimate* (2.12) *gains one more fact* $T-t$ *than* (2.9)*, which is important in the derivation of the monotonicity of generalized energy* $\mathscr{I}$*.*

Theorem 2.2 can be proved analogously as in [63, 64], and hence is postponed to the Appendix for simplicity.

2.3. **Estimates of profiles.** We collect in this subsection the estimates of three profiles in the above geometrical decomposition (2.1), which will be frequently used in the squeal.

The blow-up profile U. Let us first see that, by the explicit formula (2.3), U_k satisfies the equation

$$i\partial_t U_k+\Delta U_k + |U_k|^{\frac{4}{d}} U_k = \psi_k = \frac{e^{i\theta}}{\lambda_k^{2+\frac{d}{2}}}\Psi_k(t, \frac{x-\alpha_k}{\lambda_k}), \tag{2.13}$$

where $1 \leq k \leq K$, and

$$\begin{aligned}\Psi_k =& -(\lambda_k^2\dot{\theta}_k - 1 - |\beta_k|^2)Q_k - (\lambda_k^2\dot{\beta}_k + \gamma_k\beta_k)\cdot yQ_k + \frac{1}{4}(\lambda_k^2\dot{\gamma}_k + \gamma_k^2)|y|^2 Q_k \\ &- i(\lambda_k\dot{\alpha}_k - 2\beta_k)\cdot\nabla Q_k - i(\lambda_k\dot{\lambda}_k + \gamma_k)\Lambda Q_k.\end{aligned} \tag{2.14}$$

Lemma 2.4. *Suppose that $P = O(1)$ and $\lambda_k \geq C(T-t)$, $C > 0$. Then, for any $p \geq 2$, there exists $C > 0$ such that for all $t \in [t^*, T_*]$, $1 \leq k \leq K$,*

$$\|U(t)\|_{L^p}^p \leq C(T-t)^{-d(\frac{p}{2}-1)}. \tag{2.15}$$

Proof. Estimate (2.15) follows from the Gagliardo-Nirenberg inequality that for any $2 \leq p < \infty$,

$$\|g\|_{L^p} \leq C\|g\|_{L^2}^{1-d(\frac{1}{2}-\frac{1}{p})}\|\nabla g\|_{L^2}^{d(\frac{1}{2}-\frac{1}{p})}, \quad \forall g \in H^1, \tag{2.16}$$

and the estimates

$$\|U_k(t)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla U_k(t)\|_{L^2} = \lambda_k^{-1}\|\nabla Q_k\|_{L^2} \leq C(T-t)^{-1}. \tag{2.17}$$

□

Because the blow-up profile U_k is almost localized around $\frac{x-\alpha_k}{\lambda_k}$ and the singularities are separated from each other, the interactions between different blow-up profiles are exponentially small. Lemma 2.5 below is a slight modification of [64, Lemma 3.1].

Lemma 2.5. *(Interactions between blow-up profiles) Let $0 < t^* < T_* < \infty$. For $1 \leq k \leq K$, set*

$$G_k(t,x) := \lambda_k^{-\frac{d}{2}} g_k(t, \frac{x-\alpha_k}{\lambda_k})e^{i\theta_k}, \quad with \quad g_k(t,y) := g(y)e^{i(\beta_k(t)\cdot y - \frac{1}{4}\gamma_k(t)|y|^2)}, \tag{2.18}$$

where $g \in C_b^2(\mathbb{R}^d)$ decays exponentially fast at infinity

$$|\partial^\upsilon g(y)| \leq Ce^{-\delta|y|}, \quad |\upsilon| \leq 2,$$

with $C, \delta > 0$, $\mathcal{P}_k := (\lambda_k, \alpha_k, \beta_k, \gamma_k, \theta_k) \in C([t^*, T_*]; \mathbb{Y})$ satisfies that for $t \in [t^*, T_*]$,

$$\frac{1}{2} \leq \frac{\lambda_k(t)}{w_k(T-t)} \leq 2, \quad |\alpha_k(t) - x_k| \leq \min_{j\neq k}\{\frac{1}{12}|x_j - x_k|\} \wedge \frac{1}{2}, \quad |\beta_k(t)| + |\gamma_k(t)| \leq 1, \tag{2.19}$$

and

$$(T-t^*)(1 + \max_{1\leq k\leq K}|x_k|) \leq 1. \tag{2.20}$$

Then, there exist $C, \delta > 0$ such that for any $1 \leq k \neq l \leq K$, $m \in \mathbb{N}$ and multi-index υ with $|\upsilon| \leq 2$,

$$\int_{\mathbb{R}^d} |x-\alpha_l|^n |\partial^\upsilon G_l(t)||x-\alpha_k|^m |G_k(t)| dx \leq Ce^{-\frac{\delta}{T-t}}, \quad t \in [t^*, T_*]. \tag{2.21}$$

Moreover, let Φ_k *be defined in* (2.63) *below. Then, for any* $h \in L^1$ *or* L^2, $1 \leq k \neq l \leq K$, $m, n \in \mathbb{N}$ *and multi-index* υ *with* $|\upsilon| \leq 2$,

$$\int_{\mathbb{R}^d} |x - \alpha_l|^n |\partial^\upsilon G_l(t)| |x - \alpha_k|^m |h| \Phi_k dx \leq C e^{-\frac{\delta}{T-t}} \min\{\|h\|_{L^1}, \|h\|_{L^2}\}, \quad t \in [t^*, T_*]. \tag{2.22}$$

In the sequel, we take t and T^* close to T such that (2.19) and (2.20) hold. In particular, λ_k is comparable with $T - t$:

$$\frac{1}{2} w_k(T - t) \leq \lambda_k(t) \leq 2 w_k(T - t). \tag{2.23}$$

Hence, Lemma 2.5 is applicable.

The regular profile z**.** The main estimates of regular profile are contained in the following lemma.

Lemma 2.6. *Let* z^* *satisfy Hypothesis* $(H2)$. *Let* z *be the corresponding solution to equation* (1.21). *For every* $1 \leq k \leq K$, *define the renormalized variables* $\varepsilon_{z,k}$ *by*

$$z(t, x) = \lambda_k^{-\frac{d}{2}} \varepsilon_{z,k}\left(t, \frac{x - \alpha_k}{\lambda_k}\right) e^{i\theta_k}, \tag{2.24}$$

Then, for $\alpha^* = \alpha^*(T, m)$ *sufficiently small, the following estimates hold:*

(i) *(Smallness.) For t close to* T,

$$\|z\|_{L^\infty(t,T;H^{2m+d+2})} \leq C_{T,m} \alpha^*, \tag{2.25}$$

$$\|\partial_t z\|_{C([t,T];L^2)} \leq C_T \alpha^*. \tag{2.26}$$

In particular,

$$\|\varepsilon_{z,k}\|_{L^2} \leq C\alpha^*, \quad \|\nabla \varepsilon_{z,k}\|_{L^2} \leq C_T \alpha^* (T - t). \tag{2.27}$$

If in addition $xz \in H^1$, *then for t close to* T,

$$\|xz\|_{L^\infty(t,T;H^1)} \leq C_T \alpha^*. \tag{2.28}$$

(ii) *(Interaction between the profiles U and z.) If in addition* $P(t) = \mathcal{O}(T - t)$ *and* $|\alpha_k - x_k| < \frac{1}{2}$ *for t close to* T, *then for any* $\delta > 0$, *there exists* $C_{T,m,\delta} > 0$ *such that*

$$\sum_{|\upsilon| \leq 2} \|e^{-\delta|y|} \partial_y^\upsilon \varepsilon_{z,k}\|_{L^\infty} \leq C_{T,m,\delta} \alpha^* (T - t)^{m+1+\frac{d}{2}}. \tag{2.29}$$

Remark 2.7. *Note that, by the exponential decay* (1.7) *of ground state,*

$$\int U_k(t, x) \overline{z(t, x)} dx = \int Q_k(y) \overline{\varepsilon_{z,k}(t, y)} dy \leq C \|e^{-\delta|y|} \varepsilon_{z,k}\|_{L^\infty}.$$

Hence, estimate (2.29) *controls the interactions between the blow-up profile and the regular profile. As explained in Introduction, this estimate in the NLS case follows from Taylor's expansion of z and differentiating equation* (1.4) *enough times to get high temporal orders. For more general equation* (1.1), *including particularly the SNLS* (1.5) *where the coefficients* a_1, a_0 *are only* $C_t^{\frac{1}{2}-}$*-regular in time, we shall use a different inductive expansion of solutions and the comparability between the spatial size* $|x - \alpha_k|$ *and the temporal size* $T - t$, *due to the well localization of blow-up profile* U_k.

Proof of Lemma 2.6. (i). For simplicity, we set $p = 2 + \frac{4}{d}$, $p' = \frac{2d+4}{d+4}$. Applying the derivative $\langle\nabla\rangle^{2m+d+2}$ to both sides of equation (1.21) we have

$$i\partial_t\langle\nabla\rangle^{2m+d+2}z + (\Delta + a_1\cdot\nabla + a_0)\langle\nabla\rangle^{2m+d+2}z + [\langle\nabla\rangle^{2m+d+2}, a_1\cdot\nabla + a_0]z + \langle\nabla\rangle^{2m+d+2}(|z|^{\frac{4}{d}}z) = 0, \tag{2.30}$$

with $\langle\nabla\rangle^{2m+d+2}z(T) = \langle\nabla\rangle^{2m+d+2}z^*$, where $[\langle\nabla\rangle^{2m+d+2}, a_1\cdot\nabla + a_0]$ is the commutator $\langle\nabla\rangle^{2m+d+2}(a_1\cdot\nabla + a_0) - (a_1\cdot\nabla + a_0)\langle\nabla\rangle^{2m+d+2}$. Then using the Strichartz and local smoothing estimates (see [69, Theorem 2.11]) we have

$$\begin{aligned}\|z\|_{L^\infty(t,T;H^{2m+d+2})} \leq& C_T\left(\|z_*\|_{H^{2m+d+2}} + \|[\langle\nabla\rangle^{2m+d+2}, a_1\cdot\nabla + a_0]z\|_{L^2(t,T;H_1^{-\frac{1}{2}})} + \|\langle\nabla\rangle^{2m+d+2}(|z|^{\frac{4}{d}}z)\|_{L^{p'}(t,T;L^{p'})}\right)\\ \leq& C_T\left(\alpha^* + \|z\|_{L^2(t,T;H_{-1}^{2m+d+\frac{3}{2}})} + \|z\|^{1+\frac{4}{d}}_{L^\infty(t,T;H^{2m+d+2})}\right).\end{aligned} \tag{2.31}$$

Then, using the interpolation (see [69, Lemma 3.6])

$$\|z\|_{H_{-1}^{2m+d+\frac{3}{2}}} \leq C\delta^{\frac{1}{2}}\|z\|_{H_{-1}^{2m+d+2}} + C\delta^{-(2m+d+\frac{3}{2})}\|z\|_{L^2} \leq C\delta^{\frac{1}{2}}\|z\|_{H^{2m+d+2}} + C\delta^{-(2m+d+\frac{3}{2})}\alpha^*, \tag{2.32}$$

where the last step is due to $\langle x\rangle^{-1} \leq 1$ and the mass conservation $\|z\|_{L^2} = \|z^*\|_{L^2} \leq \alpha^*$, we lead to

$$\|z\|_{L^\infty(t,T;H^{2m+d+2})} \leq C_T\left((1 + T^{\frac{1}{2}}\delta^{-(2m+d+\frac{3}{2})})\alpha^* + T^{\frac{1}{2}}\delta^{\frac{1}{2}}\|z\|_{L^\infty(t,T;H^{2m+d+2})} + \|z\|^{1+\frac{4}{d}}_{L^\infty(t,T;H^{2m+d+2})}\right). \tag{2.33}$$

Here and in the sequel, the constant C_T may change from line to line.

Taking δ small enough such that $C_T T^{\frac{1}{2}}\delta^{\frac{1}{2}} < 1/2$ we obtain

$$\|z\|_{L^\infty(t,T;H^{2m+d+2})} \leq C_{T,m}\left(\alpha^* + \|z\|^{1+\frac{4}{d}}_{L^\infty(t,T;H^{2m+d+2})}\right). \tag{2.34}$$

Hence, taking α^* small enough we obtain (2.25).

Estimate (2.26) then follows from (2.25) and equation (1.21), and estimates in (2.27) follow directly from the identities:

$$\varepsilon_{z,k}(t,y) = \lambda_k^{\frac{d}{2}} z(t, \lambda_k y + \alpha_k)e^{-i\theta_k}, \tag{2.35}$$

and

$$\nabla\varepsilon_{z,k}(t,y) = \lambda_k^{\frac{d}{2}+1}\nabla z(t, \lambda_k y + \alpha_k)e^{-i\theta_k}. \tag{2.36}$$

It remains to prove (2.28). For this purpose, we derive from (1.21) that, for every $1 \leq j \leq d$,

$$i\partial_t(x_j z) + \Delta(x_j z) + a_1\cdot\nabla(x_j z) + a_0(x_j z) - 2\partial_j z - a_{1,j}z + x_j f(z) = 0, \tag{2.37}$$

and $x_j z(T) = x_j z^*$, $a_{1,j}$ is the j-th component of the vector a_1. Then, applying Strichartz estimates and using (2.25) we get

$$\begin{aligned}\|x_j z\|_{L^\infty(t,T;L^2)} \leq& C_T\left(\|x_j z^*\|_{L^2} + \|2\partial_j z + a_{1,j}z - x_j f(z)\|_{L^1(t,T;L^2)}\right)\\ \leq& C_T\left(\|x_j z^*\|_{L^2} + (T-t)\|z\|_{L^\infty(t,T;H^1)} + (T-t)\|z\|^{\frac{4}{d}}_{L^\infty(t,T;L^\infty)}\|x_j z\|_{L^\infty(t,T;L^2)}\right)\\ \leq& C_T\left(\alpha^* + (T-t)\alpha^*\|x_j z\|_{L^\infty(t,T;L^2)}\right),\end{aligned} \tag{2.38}$$

which yields that for t close to T,

$$\|x_j z\|_{L^\infty(t,T;L^2)} \leq C_T\alpha^*. \tag{2.39}$$

Moreover, for every $1 \leq l \leq d$, $x_j\partial_l z$ satisfies

$$i\partial_t(x_j\partial_l z) + \Delta(x_j\partial_l z) + a_1 \cdot \nabla(x_j\partial_l z) + a_0(x_j\partial_l z) + \mathcal{N} = 0, \tag{2.40}$$

where $x_j\partial_l z(T) = x_j\partial_l z^*$, and

$$\mathcal{N} = -2\partial_{jl}z - a_{1,j}\partial_l z + x_j(\partial_l a_1) \cdot \nabla z + x_j(\partial_l a_0)z + x_j\partial_l f(z). \tag{2.41}$$

Hence, by Strichartz estimates, (1.13) and (2.25),

$$\begin{aligned}\|x_j\partial_l z\|_{L^\infty(t,T;L^2)} \leq & C_T\left(\|x_j\partial_l z^*\|_{L^2} + \|\mathcal{N}\|_{L^1(t,T;L^2)}\right)\\ \leq & C_T\left(\|x_j\partial_l z^*\|_{L^2} + (T-t)\|z\|_{L^\infty(t,T;H^2)} + (T-t)\|z\|^{\frac{4}{d}}_{L^\infty(t,T;L^\infty)}\|x_j\partial_l z\|_{L^\infty(t,T;L^2)}\right)\\ \leq & C_T\left(\alpha^* + (T-t)\alpha^*\|x_j\partial_l z\|_{L^\infty(t,T;L^2)}\right),\end{aligned}$$

which yields that for t close to T,

$$\|x_j\partial_l z\|_{L^\infty(t,T;L^2)} \leq C_T\alpha^*. \tag{2.42}$$

Thus, estimate (2.28) is proved.

(ii). Let us set $m^* := 2m + 1$ and define the operator $\mathcal{D}_t$ by

$$\mathcal{D}_t := -i\left(\Delta + a_1(t) \cdot \nabla + a_0(t) + |z(t)|^{\frac{4}{d}}\right).$$

Then, by equation (1.1) and the mean valued theorem,

$$z(t) = z^* + \int_t^T \mathcal{D}_r z(r)dr = z^* + (T-t)\mathcal{D}_{t_1}z(t_1), \tag{2.43}$$

where $t_1 \in (t, T)$. Further expansion of $z(t_1)$ by (1.1) yields

$$\begin{aligned}z(t) =& z^* + (T-t)\mathcal{D}_{t_1}\left(z^* + \int_{t_1}^T \mathcal{D}_r z(r)dr\right)\\ =& z^* + (T-t)\mathcal{D}_{t_1}z^* + (T-t)(T-t_1)\mathcal{D}_{t_1} \circ \mathcal{D}_{t_2}z(t_2),\end{aligned}$$

where $t_2 \in (t_1, T)$. Then, further expansion by (2.43) and inductive arguments lead to

$$z(t) = z^* + \sum_{j=1}^n \prod_{l=0}^{j-1}(T-t_l)\mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_j}z^* + \prod_{l=0}^{n}(T-t_l)\mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_{n+1}}z(t_{n+1}), \tag{2.44}$$

where $t_0 := t$, $t_l \in (t, T)$, $1 \leq l \leq n$. By (1.17),

$$|\mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_j}z^*(x)| \leq C_T|x - x_k|^{m^*-2j}, \tag{2.45}$$

and by the Sobolev embedding $H^{2m+2+d} \hookrightarrow C_b^{2(m+1)}$, for $2 \leq n \leq m$,

$$\|\mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_{n+1}}z(t_{n+1})\|_{L^\infty} \leq C_{T,m}\|z\|_{L^\infty(t,T;C_b^{2(m+1)})} \leq C_{T,m}\|z\|_{L^\infty(t,T;H^{2m+2+d})}. \tag{2.46}$$

Hence, we derive that for $2 \leq n \leq m$,

$$|z(t)| \leq C_{T,m}\left(|x - x_k|^{m^*} + \sum_{j=1}^n (T-t)^j|x - x_k|^{m^*-2j} + (T-t)^{n+1}\right), \quad for\ |x - x_k| < 1. \tag{2.47}$$

Note that,

$$|\lambda_k y + \alpha_k - x_k| \leq CP\langle y\rangle \leq C(T-t)\langle y\rangle. \tag{2.48}$$

Moreover, since $|\alpha_k - x_k| < \frac{1}{2}$, $|y| > \frac{1}{2\lambda_k}$ in the regime $\{y \in \mathbb{R}^d : |\lambda_k y + \alpha_k - x_k| \geq 1\}$, and so

$$\|e^{-\delta|y|} z(t, \lambda_k y + \alpha_k) I_{|\lambda_k y + \alpha_k - x_k| \geq 1}\|_{L^\infty} \leq C_{T,m} e^{-\frac{\delta}{T-t}}.$$

Taking into account (2.35), (2.47) and (2.48) we obtain

$$\begin{aligned} \|e^{-\delta|y|} \varepsilon_{z,k}(y)\|_{L^\infty} \leq & \lambda_k^{\frac{d}{2}} \|e^{-\delta|y|} z(\lambda_k y + \alpha_k)(I_{|\lambda_k y + \alpha_k - x_k| < 1} + I_{|\lambda_k y + \alpha_k - x_k| \geq 1})\|_{L^\infty} \\ \leq & C_{T,m}(T-t)^{\frac{d}{2}} \left((T-t)^{m^*-n} + (T-t)^{n+1} \right) + C_{T,m} e^{-\frac{\delta}{T-t}}. \end{aligned} \tag{2.49}$$

This yields that, for $n = m$ and t close to T,

$$\|e^{-\delta|y|} \varepsilon_{z,k}(y)\|_{L^\infty} \leq C_{T,m,\delta}(T-t)^{m+1+\frac{d}{2}}. \tag{2.50}$$

Similarly, by (2.44), for any multi-index $|\upsilon| \leq 2$,

$$\partial_x^\upsilon z(t) = \partial_x^\upsilon z^* + \sum_{j=1}^n \prod_{l=0}^{j-1} (T - t_l) \partial_x^\upsilon \circ \mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_j} z^* + \prod_{l=0}^n (T - t_l) \partial_x^\upsilon \circ \mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_{n+1}} z(t_{n+1}). \tag{2.51}$$

As in (2.45) and (2.46), we have

$$|\partial_x^\upsilon \circ \mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_j} z^*(x)| \leq C_T |x - x_k|^{m^*-2j-|\upsilon|}, \tag{2.52}$$

and for $n \leq m-1$,

$$\|\partial_x^\upsilon \circ \mathcal{D}_{t_1} \circ \cdots \circ \mathcal{D}_{t_{n+1}} z(t_{n+1})\|_{L^\infty} \leq C_{T,m} \|z\|_{L^\infty(t,T;H^{2m+2+d})}. \tag{2.53}$$

Thus, for any multi-index $|\upsilon| \leq 2$ and $n \leq m - |\upsilon|$,

$$|\partial_x^\upsilon z(t)| \leq C_{T,m} \left(|x - x_k|^{m^*-|\upsilon|} + \sum_{j=1}^n (T-t)^j |x - x_k|^{m^*-2j-|\upsilon|} + (T-t)^{n+1} \right), \quad for\ |x - x_k| < 1. \tag{2.54}$$

Taking into account

$$\partial_x^\upsilon \varepsilon_{z,k}(t,y) = \lambda_k^{\frac{d}{2}+|\upsilon|} \partial_y^\upsilon z(t, \lambda_k y + \alpha_k) e^{-i\theta_k} \tag{2.55}$$

and arguing as in the proof of (2.49) we get

$$\begin{aligned} \|e^{-\delta|y|} \partial_y^\upsilon \varepsilon_{z,k}(t)\|_{L^\infty} \leq & C_{T,m}(T-t)^{\frac{d}{2}+|\upsilon|} \left((T-t)^{m^*-|\upsilon|} + \sum_{j=1}^n (T-t)^{m^*-j-|\upsilon|} + (T-t)^{n+1} \right) + C_{T,m} e^{-\frac{\delta}{T-t}} \\ \leq & C_{T,m} \left((T-t)^{m^*-n+\frac{d}{2}} + (T-t)^{n+|\upsilon|+1+\frac{d}{2}} + e^{-\frac{\delta}{T-t}} \right) \\ \leq & C_{T,m,\delta}(T-t)^{m+1+\frac{d}{2}}, \end{aligned} \tag{2.56}$$

where in the last step we chose $n = m - |\upsilon|$ for $1 \leq |\upsilon| \leq 2$.

Therefore, the proof of Lemma 2.6 is complete. □

For the coefficients of lower order perturbations, we have the following estimates.

Lemma 2.8. *For any multi-index υ, $|\upsilon| \leq 2$, set $\widetilde{\partial^\upsilon \phi_{l,k}}(y) := (\partial^\upsilon \phi_l)(\lambda_k y + \alpha_k)$, $1 \leq l \leq N$. Then,*

$$|\widetilde{\partial^\upsilon \phi_{l,k}}(y)| \leq C P^{\upsilon_*+1-|\upsilon|} \langle y \rangle^{\upsilon_*+1}, \quad 0 \leq |\upsilon| \leq \upsilon_*, \tag{2.57}$$

where υ_* is the index of flatness in Hypothesis $(H1)$. In particular, for $\widetilde{a}_{1,k}(t,y) := a_1(t, \lambda_k y + \alpha_k)$, $\widetilde{a}_{0,k}(t,y) := a_0(t, \lambda_k y + \alpha_k)$, we have that for any multi-index υ, $|\upsilon| \leq 2$, there exists $C > 0$ such that

$$|\partial_y^\upsilon (\widetilde{a}_{1,k}(t,y))| \leq C \lambda_k^{|\upsilon|} P^{\upsilon_*-|\upsilon|} \langle y \rangle^{\upsilon_*+1}, \tag{2.58}$$

$$|\partial_y^\upsilon(\widetilde{a}_{0,k}(t,y))| \leq C\lambda_k^{|\upsilon|}P^{\upsilon_*-1-|\upsilon|}\langle y\rangle^{2\upsilon_*+2}. \tag{2.59}$$

Proof. By Taylor's expansion and (1.14),

$$|\widetilde{\partial^\upsilon\phi_{l,k}}(y)| \leq C(\lambda_k y+\alpha_k-x_k)^{\upsilon_*+1-|\upsilon|} \leq CP^{\upsilon_*+1-|\upsilon|}\langle y\rangle^{\upsilon_*+1},\ \ 0\leq|\upsilon|\leq\upsilon_*.$$

This yields (2.57). Estimates (2.58) and (2.59) then follow from (2.57), (1.2) and (1.3). □

The remainder profile R**.** Lemma 2.9 below permits to control the H^1 and L^p-norms of remainder.

Lemma 2.9. *(*[64, Lemma 2.7]*) There exists* $C>0$ *such that*

$$\|R\|_{H^1} \leq C(T-t)^{-1}D,\ \ \|R\|_{L^2}\|\nabla R\|_{L^2} \leq (T-t)^{-1}D^2, \tag{2.60}$$

$$\|R\|_{L^p}^p \leq C(T-t)^{-d(\frac{p}{2}-1)}D^p. \tag{2.61}$$

In order to deal with the multi-bubble case, it is useful to decompose the remainder R into K localized profiles concentrating at the singularities. As in [64], since equation (1.4) is invariant under orthogonal transforms, we may take an orthonormal basis $\{\mathbf{v_j}\}_{j=1}^d$ of $\mathbb{R}^d$, such that $(x_j-x_l)\cdot\mathbf{v_1}\neq 0$ for any $1\leq j\neq l\leq K$. Hence, without loss of generality, we assume that $x_1\cdot\mathbf{v_1}<x_2\cdot\mathbf{v_1}<\cdots<x_K\cdot\mathbf{v_1}$. Then, set

$$\sigma := \frac{1}{12}\min_{1\leq k\leq K-1}\{(x_{k+1}-x_k)\cdot\mathbf{v_1}\}>0. \tag{2.62}$$

Let $\Phi(x)$ be a smooth function on $\mathbb{R}^d$ such that $0\leq\Phi(x)\leq 1$, $|\nabla\Phi(x)|\leq C\sigma^{-1}$, $\Phi(x)=1$ for $x\cdot\mathbf{v_1}\leq 4\sigma$ and $\Phi(x)=0$ for $x\cdot\mathbf{v_1}\geq 8\sigma$. Define the localization functions $\{\Phi_k\}$ by

$$\begin{aligned}&\Phi_1(x):=\Phi(x-x_1),\ \ \Phi_K(x):=1-\Phi(x-x_{K-1}),\\&\Phi_k(x):=\Phi(x-x_k)-\Phi(x-x_{k-1}),\ \ 2\leq k\leq K-1.\end{aligned} \tag{2.63}$$

One has the partition of unity $1=\sum_{j=1}^K\Phi_k$. Then,

$$R=\sum_{k=1}^K R_k,\ \ with\ \ R_k:=R\Phi_k. \tag{2.64}$$

The corresponding renormalized remainders ε_k, $1\leq k\leq K$, are defined by

$$R_k(t,x)=\lambda_k^{-\frac{d}{2}}\varepsilon_k(t,\frac{x-\alpha_k}{\lambda_k})e^{i\theta_k}. \tag{2.65}$$

The following almost orthogonality between profiles $\{R_k\}$ and $\{U_k\}$ is a consequence of the orthogonality (2.5) and the decoupling Lemma 2.5.

Lemma 2.10. *(Almost orthogonality* [64, Lemma 4.4]*) Let* t^* *be as in Theorem 2.2. Then, for* t^* *and* T_* *close to* T*, there exists* $\delta>0$ *such that for every* $1\leq k\leq K$ *and any* $t\in[t^*,T_*]$*,*

$$\begin{aligned}&|\mathrm{Re}\int(x-\alpha_k)U_k\overline{R_k}dx|+|\mathrm{Re}\int|x-\alpha_k|^2U_k\overline{R_k}dx|\leq Ce^{-\frac{\delta}{T-t}}\|R\|_{L^2},\\&|\mathrm{Im}\int\nabla U_k\overline{R_k}dx|+|\mathrm{Im}\int\Lambda_kU_k\overline{R_k}dx|+|\mathrm{Im}\int\varrho_k\overline{R_k}dx|\leq Ce^{-\frac{\delta}{T-t}}\|R\|_{L^2}.\end{aligned} \tag{2.66}$$

Furthermore, by (1.1), (1.21) and (2.1), the remainder R satisfies the equation

$$i\partial_tR+\Delta R+a_1\cdot\nabla R+a_0R+(f(v)-f(U+z))=-\eta, \tag{2.67}$$

where

$$\eta=i\partial_tU+\Delta U+a_1\cdot\nabla U+a_0U+f(U+z)-f(z), \tag{2.68}$$

and $f(z) := |z|^{\frac{4}{d}}z$, $f(U+z)$ is defined similarly.

The estimates of η are contained in Lemma 2.11 below.

Lemma 2.11. *Suppose that $P = \mathcal{O}(T-t)$ and $|\alpha_k - x_k| < \frac{1}{2}$ for any $t \in [t^*, T_*]$. Then,*

$$|\eta(t,x)| \leq C(T-t)^{-\frac{d}{2}-2} \sum_{k=1}^{K} \left(Mod + |\varepsilon_{z,k}(y)| + (T-t)^{\upsilon_*+1}\right) e^{-\delta|y|}\Big|_{y=\frac{x-\alpha_k}{\lambda_k}} + C\widetilde{\eta}, \tag{2.69}$$

where $\widetilde{\eta}$ satisfies $\|\widetilde{\eta}(t)\|_{L^2} \leq Ce^{-\frac{\delta}{T-t}}$, and for any multi-index υ with $|\upsilon| \leq 2$,

$$\|\partial_x^\upsilon \eta(t)\|_{L^2} \leq C(T-t)^{-2-|\upsilon|}\left(Mod + \alpha^*(T-t)^{m+1+\frac{d}{2}} + (T-t)^{\upsilon_*+1}\right). \tag{2.70}$$

Proof. Let Ψ_k be as in (2.14). We decompose η into four parts:

$$\eta = \eta_1 + \eta_2 + \eta_3 + \eta_4, \tag{2.71}$$

where

$$\eta_1 = \sum_{k=1}^{K} \frac{e^{i\theta_k(t)}}{\lambda_k(t)^{2+\frac{d}{2}}} \Psi_k(t, \frac{x-\alpha_k(t)}{\lambda_k(t)}), \tag{2.72}$$

$$\eta_2 = f(U+z) - f(U) - f(z), \tag{2.73}$$

$$\eta_3 = f(U) - \sum_{k=1}^{K} f(U_k), \tag{2.74}$$

$$\eta_4 = a_1 \cdot \nabla U + a_0 U. \tag{2.75}$$

By the exponential decay (1.7) of ground state and $\|\varepsilon_{z,k}\|_{L^\infty} \leq C$,

$$|\eta_1 + \eta_2| \leq C(T-t)^{-\frac{d}{2}-2} \sum_{k=1}^{K} \left(Mod_k + |\varepsilon_{z,k}(t,y)|\right) e^{-\delta|y|}\Big|_{y=\frac{x-\alpha_k}{\lambda_k}}, \tag{2.76}$$

and $\widetilde{\eta} := |\eta_3|$ contains different blow-up profiles, and thus, by Lemma 2.5,

$$\|\widetilde{\eta}(t)\|_{L^2} \leq Ce^{-\frac{\delta}{T-t}}. \tag{2.77}$$

Moreover, since

$$\eta_4(t,x) = \sum_{k=1}^{K} \lambda_k^{-\frac{d}{2}-1}(t)\widetilde{a}_{1,k}(t,y)\nabla Q_k(t,y)e^{i\theta_k} + \lambda_k^{-\frac{d}{2}}(t)\widetilde{a}_{0,k}(t,y)Q_k(t,y)e^{i\theta_k}\Big|_{y=\frac{x-\alpha_k}{\lambda_k}}, \tag{2.78}$$

where $\widetilde{a}_{1,k}, \widetilde{a}_{0,k}$ are as in Lemma 2.8, using Lemma 2.8, (1.7) and $P \leq C(T-t)$ we get

$$\begin{aligned}
|\eta_4(t,x)| \leq & C \sum_{k=1}^{K} \left((T-t)^{-\frac{d}{2}-1} P^{\upsilon_*} \langle y \rangle^{\upsilon_*+1} e^{-\delta|y|} + (T-t)^{-\frac{d}{2}} P^{\upsilon_*-1} \langle y \rangle^{2\upsilon_*} e^{-\delta|y|}\right)\Big|_{y=\frac{x-\alpha_k}{\lambda_k}} \\
\leq & C \sum_{k=1}^{K} (T-t)^{\upsilon_*-\frac{d}{2}-1} e^{-\frac{\delta}{2}|y|}\Big|_{y=\frac{x-\alpha_k}{\lambda_k}}.
\end{aligned} \tag{2.79}$$

Hence, (2.76) and (2.79) together yield (2.69).

Concerning (2.70), by (2.14), it is clear that

$$\|\partial_x^\upsilon \eta_1\|_{L^2} \leq C(T-t)^{-2-|\upsilon|} Mod. \tag{2.80}$$

Moreover, expanding f and then using the exponential decay (1.7) of ground state we have

$$\|\partial_x^\upsilon \eta_2\|_{L^2} \leq C(T-t)^{-2-|\upsilon|} \sum_{|\upsilon|\leq 2} \|e^{-\delta|y|}\partial_y^\upsilon \varepsilon_{z,k}\|_{L^\infty} + Ce^{-\frac{\delta}{T-t}}. \tag{2.81}$$

Since η_3 contains the interactions between different blow-up profiles, by Lemma 2.5,

$$\|\partial_x^\upsilon \eta_3\|_{L^2} \leq Ce^{-\frac{\delta}{T-t}}. \tag{2.82}$$

At last, applying Lemma 2.8 we also infer that

$$\|\partial_x^\upsilon \eta_4\|_{L^2} \leq \sum_{k=1}^{K} \lambda_k^{-|\upsilon|+\frac{d}{2}} \|\partial_y^\upsilon(\widetilde{a}_1 \lambda_k^{-\frac{d}{2}-1} \nabla Q_k + \widetilde{a}_0 \lambda_k^{-\frac{d}{2}} Q_k)\|_{L^2} \leq C(T-t)^{\upsilon_*-|\upsilon|-1}. \tag{2.83}$$

Therefore, putting the above estimates altogether we obtain (2.70). □

3. Localized mass and (generalized) energy

This section is devoted to the key estimates of localized mass, energy and the generalized energy.

3.1. **Control of localized mass.** Recall that the localized mass is defined by

$$M_k := 2\mathrm{Re}\langle R_k, U_k\rangle + \int |R|^2 \Phi_k dx, \tag{3.1}$$

where $R_k = R\Phi_k$ and $\{\Phi_k\}$ are the localization functions given by (2.63).

The main estimate is contained in Theorem 3.1 below.

Theorem 3.1. *(Control of localized mass) Suppose that $P = O(T-t)$. Then, there exists $C > 0$ such that for every $1 \leq k \leq K$,*

$$|M_k(t)| \leq C \int_t^{T_*} \left(\alpha^* D + \frac{D^2}{T-s}\right) ds + C\left(\alpha^* D + \alpha^*(T-t)^{m+1+\frac{d}{2}}\right), \quad t \in [t^*, T_*]. \tag{3.2}$$

Proof. On one hand, the geometrical decomposition (2.1) yields the expansion:

$$\begin{aligned}
\int |v(t)|^2 \Phi_k dx &= \int (|U|^2 + |z|^2 + |R|^2)\Phi_k dx + 2\mathrm{Re} \int (R\overline{U} + z\overline{R} + z\overline{U})\Phi_k dx \\
&= \int |U|\Phi_k^2 dx + \int |z|^2 \Phi_k dx + \int |R|^2 \Phi_k dx \\
&\quad + 2\mathrm{Re} \int R_k \overline{U_k} dx + 2\mathrm{Re} \int z\overline{U_k} dx + 2\mathrm{Re} \int z\overline{R_k} dx + O(e^{-\frac{\delta}{T-t}} \|R\|_{L^2}),
\end{aligned} \tag{3.3}$$

where the last step is due to Lemma 2.5 and $\delta > 0$.

On the other hand, since $v(T_*) = S(T_*) + z(T_*)$, we have

$$\int |v(T_*)|^2 \Phi_k dx = \int (|S(T_*)|^2 + |z(T_*)|^2)\Phi_k dx + 2\mathrm{Re} \int (z\overline{S})(T_*)\Phi_k dx. \tag{3.4}$$

Note that, the integrations $\int |z|^2 \Phi_k dx$ and $\int |z(T_*)|^2 \Phi_k dx$ only contribute a small constant $(\alpha^*)^2$, which, however, is insufficient to close the bootstrap arguments later. The key point is that one more factor D can be explored by subtracting (3.4) from (3.3) and then using both the dynamics generated by equations (1.1) and (1.21).

To be precise, we derive from (3.3) and (3.4) that

$$|M_k(t)| \leq \left| \int (|v(t)|^2 - |v(T_*)|^2)\Phi_k dx - \int (|z(t)|^2 - |z(T_*)|^2)\Phi_k dx \right|$$

$$
\begin{aligned}
&+\left|\int(|U(t)|^2-|S(T_*)|^2)\Phi_k dx\right| \\
&+2\left(\left|\int(z\overline{U_k})(t)dx\right|+\left|\int(z\overline{R_k})(t)dx\right|+\left|\int(z\overline{S})(T_*)\Phi_k dx\right|\right)+Ce^{-\frac{\delta}{T-t}}\|R\|_{L^2} \\
=:\ &K_1+K_2+K_3+Ce^{-\frac{\delta}{T-t}}\|R\|_{L^2}. \qquad (3.5)
\end{aligned}
$$

Let us first treat the easier two terms K_2, K_3. Actually, it holds that (see [64, (5.22),(5.23)])

$$
\int|U(t)|^2\Phi_k dx=\|Q\|_{L^2}^2+O(e^{-\frac{\delta}{T-t}}), \qquad (3.6)
$$

$$
\int|S(T_*)|^2\Phi_k dx=\|Q\|_2^2+O(e^{-\frac{\delta}{T-T_*}})=\|Q\|_2^2+O(e^{-\frac{\delta}{T-t}}), \qquad (3.7)
$$

which yields that

$$
K_2(t)\leq Ce^{-\frac{\delta}{T-t}}. \qquad (3.8)
$$

Moreover, by (2.25) and (2.29),

$$
\begin{aligned}
K_3(t)\leq &C\left(\|z\|_{L^2}\|R(t)\|_{L^2}+\|e^{-\delta|y|}(|\varepsilon_{z,k}(t)|+|\varepsilon_{z,k}(T_*)|)\|_{L^\infty}+e^{-\frac{\delta}{T-T_*}}\right) \\
\leq &C\left(\alpha^*D+\alpha^*(T-t)^{m+1+\frac{d}{2}}\right) \qquad (3.9)
\end{aligned}
$$

Hence, it remains to treat the first term K_1 on the R.H.S. of (3.5).

For this purpose, we derive from equation (1.1) that

$$
\frac{d}{dt}\int|v|^2\Phi_k dx=\mathrm{Im}\int(2\overline{v}\nabla v+a_1|v|^2)\cdot\nabla\Phi_k dx. \qquad (3.10)
$$

Similarly, by equation (1.21),

$$
\frac{d}{dt}\int|z|^2\Phi_k dx=\mathrm{Im}\int(2\overline{z}\nabla z+a_1|z|^2)\cdot\nabla\Phi_k dx. \qquad (3.11)
$$

Thus,

$$
\frac{d}{dt}\int|v|^2\Phi_k dx-\frac{d}{dt}\int|z|^2\Phi_k dx=\mathrm{Im}\int 2(\overline{v}\nabla v-\overline{z}\nabla z)\cdot\nabla\Phi_k+(|v|^2-|z|^2)a_1\cdot\nabla\Phi_k dx. \qquad (3.12)
$$

Note that, by (2.1),

$$
\overline{v}\nabla v-\overline{z}\nabla z=\overline{U}\nabla(U+R+z)+(\overline{R}+\overline{z})\nabla U+\overline{z}\nabla R+\overline{R}\nabla z+\overline{R}\nabla R. \qquad (3.13)
$$

Since $P=O(T-t)$, $|x_k-\alpha_k(t)|\leq\sigma$, $1\leq k\leq K$, $\mathrm{supp}\nabla\Phi_k\subseteq\cap_{k=1}^K\{x:|x-\alpha_k|\geq 3\sigma\}$. By (1.7),

$$
\left|\int\left(\overline{U}\cdot\nabla(U+R+z)+(\overline{R}+\overline{z})\nabla U\right)\cdot\nabla\Phi_k dx\right|\leq Ce^{-\frac{\delta}{T-t}}. \qquad (3.14)
$$

Moreover, the integration by parts formula yields

$$
\left|\int\overline{z}\nabla R\cdot\nabla\Phi_k dx\right|=\left|\int R\nabla\overline{z}\cdot\nabla\Phi_k+R\overline{z}\Delta\Phi_k dx\right|\leq C\|z\|_{H^1}\|R\|_{L^2}. \qquad (3.15)
$$

Thus, it follows from (3.13)-(3.15) that

$$
\left|\mathrm{Im}\int(\overline{v}\nabla v-\overline{z}\nabla z)\cdot\nabla\Phi_k dx\right|\leq C\left(\|z\|_{H^1}\|R\|_{L^2}+\|R\|_{L^2}\|\nabla R\|_{L^2}+e^{-\frac{\delta}{T-t}}\right). \qquad (3.16)
$$

Similarly, we have

$$\begin{aligned}\left|\mathrm{Im}\int(|v|^2-|z|^2)a_1\cdot\nabla\Phi_k dx\right| \leq& C\int|\nabla\Phi_k|\,\left||U|^2+|R|^2+2\mathrm{Re}(R\overline{U}+z\overline{U}+z\overline{R})\right|dx\\ \leq& C\left(\|z\|_{L^2}\|R\|_{L^2}+\|R\|_{L^2}^2+e^{-\frac{\delta}{T-t}}\right).\end{aligned} \tag{3.17}$$

Hence, we conclude from (2.25), (3.12), (3.16) and (3.17) that

$$\begin{aligned}\left|\frac{d}{dt}\int|v|^2\Phi_k dx-\frac{d}{dt}\int|z|^2\Phi_k dx\right| \leq& C\left(\|z\|_{H^1}\|R\|_{L^2}+\|R\|_{L^2}^2+\|R\|_{L^2}\|\nabla R\|_{L^2}+e^{-\frac{\delta}{T-t}}\right)\\ \leq& C\left(\alpha^*D+\frac{D^2}{T-t}+e^{-\frac{\delta}{T-t}}\right).\end{aligned}$$

where $\delta>0$. Integrating both sides we then obtain

$$K_1\leq C\int_t^{T_*}\alpha^*D+\frac{D^2}{T-s}\,ds+Ce^{-\frac{\delta}{T-t}}. \tag{3.18}$$

Therefore, plugging (3.8), (3.9) and (3.18) into (3.5) we obtain (3.2). The proof is complete. □

3.2. **Refined estimate of β.** In this subsection we shall derive the refined estimate of parameter $\beta=(\beta_k)$ from the energy functional, defined by

$$E(v):=\frac{1}{2}\int_{\mathbb{R}^d}|\nabla v|^2dx-\frac{d}{2d+4}\int_{\mathbb{R}^d}|v|^{2+\frac{4}{d}}dx. \tag{3.19}$$

Unlike in the NLS case, the energy of solutions to (1.1) is no longer conserved, it is thus important to first control the variation of energy. This is the content of Lemma 3.2 below.

Lemma 3.2. *(Variation of the energy) Suppose $P=O(T-t)$. Then, there exists $C>0$ such that*

$$\left|\frac{d}{dt}E(v)-\frac{d}{dt}E(z)\right|\leq C\left(\alpha^*D+\frac{D^2}{(T-t)^2}+(T-t)^{\upsilon_*-3}\right),\ \ \forall t\in[t^*,T_*]. \tag{3.20}$$

Remark 3.3. *As in the proof of Theorem 3.1, the key point here is that one more factor D can be gained from the difference between the energies of v and z.*

Proof. As in the previous case of localized mass, we consider the difference between two energies of u and z. Using (1.1) and (1.21) we compute, as in [64, (5.26)],

$$\begin{aligned}\frac{d}{dt}E(v)=&-2\sum_{l=1}^N h_l\mathrm{Re}\int\nabla^2\phi_l(\nabla v,\nabla\overline{v})dx+\frac{1}{2}\sum_{l=1}^N h_l\int\Delta^2\phi_l|v|^2dx\\ &+\frac{2}{d+2}\sum_{l=1}^N h_l\int\Delta\phi_l|v|^{2+\frac{4}{d}}dx-\sum_{j=1}^d\mathrm{Im}\int\nabla\left(\sum_{l=1}^N\partial_j\phi_l h_l\right)^2\cdot\nabla v\overline{v}dx,\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\frac{d}{dt}E(z)=&-2\sum_{l=1}^N h_l\mathrm{Re}\int\nabla^2\phi_l(\nabla z,\nabla\overline{z})dx+\frac{1}{2}\sum_{l=1}^N h_l\int\Delta^2\phi_l|z|^2dx\\ &+\frac{2}{d+2}\sum_{l=1}^N h_l\int\Delta\phi_l|z|^{2+\frac{4}{d}}dx-\sum_{j=1}^d\mathrm{Im}\int\nabla\left(\sum_{l=1}^N\partial_j\phi_l h_l\right)^2\cdot\nabla z\overline{z}dx.\end{aligned} \tag{3.22}$$

In order to control the difference $\frac{d}{dt}E(v) - \frac{d}{dt}E(z)$, we first see that, by (2.1), integration by parts formula and (2.29),

$$
\begin{aligned}
&\left|\mathrm{Re}\int \nabla^2\phi_l(\nabla v, \nabla \overline{v})dx - \mathrm{Re}\int \nabla^2\phi_l(\nabla z, \nabla\overline{z})dx\right| \\
=&\Big|\sum_{i,j=1}^{d}\mathrm{Re}\int \partial_{ij}\phi_l\partial_i(U+R)\partial_j(\overline{U+R}) - \partial_{ijj}\phi_l\partial_j z(\overline{U+R}) - \partial_{iij}\phi_l(U+R)\partial_j\overline{z} \\
&\qquad - \partial_{ij}\phi_l\partial_{ij}z(\overline{U+R}) - \partial_{ij}\phi_l(U+R)\partial_{ij}\overline{z}dx\Big| \\
\le& C\sum_{i,j=1}^{d}\Bigg(\|R\|_{H^1}^2 + \int(|\partial_j z| + |\partial_{ij}z|)|R|dx + \left|\int \partial_{ij}\phi_l(\partial_i U\partial_j\overline{U} + \partial_i U\partial_j\overline{R} + \partial_i R\partial_j\overline{U})dx\right| \\
&\qquad + \left|\int z\partial_i(\partial_{ijj}\phi_l\overline{U}) + \overline{z}\partial_j(\partial_{iij}\phi_l U)dx\right| + \left|\int z\partial_{ij}(\partial_{ij}\phi_l\overline{U})dx\right|\Bigg) \\
\le& C\left(\|z\|_{H^2}\|R\|_{L^2} + \|R\|_{H^1}^2 + (T-t)^{\upsilon_*-3}(1+\sum_{k=1}^{K}\|\nabla\varepsilon_k\|_{L^2}) + (T-t)^{\upsilon_*-3}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + e^{-\frac{\delta}{T-t}}\right).
\end{aligned} \tag{3.23}
$$

Similarly,

$$
\begin{aligned}
\left|\int \Delta^2\phi_l(|v|^2 - |z|^2)dx\right| \le& C\left(\|R\|_{L^2}^2 + \int |z||R|dx + \int |\Delta^2\phi_l|(|U|^2 + |RU| + |zU|)\right)dx \\
\le& C\Big(\|R\|_{L^2}^2 + \|z\|_{L^2}\|R\|_{L^2} + (T-t)^{\upsilon_*-3} + (T-t)^{\upsilon_*-3}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\Big),
\end{aligned} \tag{3.24}
$$

and

$$
\begin{aligned}
\left|\int \Delta\phi_l(|v|^{2+\frac{4}{d}} - |z|^{2+\frac{4}{d}})dx\right| \le& C\int |\Delta\phi_l|\left(|U+R|^{2+\frac{4}{d}} + |z|^{1+\frac{4}{d}}|U+R|\right)dx \\
\le& C\left(\|R\|_{H^1}^{2+\frac{4}{d}} + \|z\|_{H^1}^{1+\frac{4}{d}}\|R\|_{L^2} + (T-t)^{\upsilon_*-3} + (T-t)^{\upsilon_*-3}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}^{1+\frac{4}{d}}\right).
\end{aligned} \tag{3.25}
$$

Moreover, by (3.13),

$$
\begin{aligned}
&\mathrm{Im}\int \nabla\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2\cdot(\nabla v\overline{v} - \nabla z\overline{z})dx \\
=&\mathrm{Im}\int \nabla\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2\cdot\left(\nabla U(\overline{U} + \overline{R} + \overline{z}) + (\nabla R + \nabla z)\overline{U} + \nabla R\overline{R} + \nabla R\overline{z} + \nabla z\overline{R}\right)dx.
\end{aligned} \tag{3.26}
$$

Note that, by the integration by parts formula, (2.29) and (2.57),

$$
\left|\int \nabla\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2(\nabla z\overline{U} + \nabla R\overline{z})dx\right|
$$

$$
\begin{aligned}
=&\left|\int \Delta\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2(z\overline{U}+R\overline{z})dx+\int\nabla\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2\cdot(z\nabla\overline{U}+R\nabla\overline{z})dx\right|\\
\leq& C\left(\|z\|_{H^1}\|R\|_{L^2}+(T-t)^{2\upsilon_*-2}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\right). \qquad (3.27)
\end{aligned}
$$

Using (2.57) again we also have

$$
\begin{aligned}
&\left|\mathrm{Im}\int\nabla\left(\sum_{j=1}^{d}\partial_j\phi_l\right)^2\cdot\left(\nabla U(\overline{U}+\overline{R}+\overline{z})+\nabla R\overline{U}\right)dx\right|\\
\leq& C\left((T-t)^{2\upsilon_*-2}\left(1+\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}+D\right)+e^{-\frac{\delta}{T-t}}\right)\\
\leq& C(T-t)^{2\upsilon_*-2}. \qquad (3.28)
\end{aligned}
$$

Plugging these into (3.26) we get

$$
\begin{aligned}
&\left|\mathrm{Im}\int\nabla\left(\sum\partial_j\phi_h\right)^2\cdot(\nabla v\overline{v}-\nabla z\overline{z})dx\right|\\
\leq& C\left(\|z\|_{H^1}\|R\|_{L^2}+\|R\|_{L^2}\|\nabla R\|_{L^2}+(T-t)^{2\upsilon_*-2}+(T-t)^{2\upsilon_*-2}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\right). \qquad (3.29)
\end{aligned}
$$

Therefore, we conclude from the estimates (3.23), (3.24), (3.25) and (3.29) that

$$
\begin{aligned}
\left|\frac{d}{dt}E(v)-\frac{d}{dt}E(z)\right|\leq& C\left(\left(\|z\|_{H^2}+\|z\|_{H^1}^{1+\frac{4}{d}}\right)\|R\|_{L^2}+\|R\|_{H^1}^2+\|R\|_{H^1}^{2+\frac{4}{d}}+(T-t)^{\upsilon_*-3}\right)\\
\leq& C\left(\alpha^*D+\frac{D^2}{(T-t)^2}+(T-t)^{\upsilon_*-3}\right),
\end{aligned}
$$

which yields (3.20), thereby finishing the proof. □

We are now in position to derive the refined estimate of $\beta=(\beta_k)$, which is essentially a consequence of the coercivity of energy around the ground state.

Theorem 3.4. *(Improved estimate of β) Suppose that $P=\mathcal{O}(T-t)$ and $D=o(1)$. Then, for any $t\in[t^*,T_*]$,*

$$
\sum_{k=1}^{K}\frac{|\beta_k|^2}{2\lambda_k^2}\|Q\|_{L^2}^2\leq\frac{\|yQ\|_{L^2}^2}{8}\sum_{k=1}^{K}\left(w_k^2-\frac{\gamma_k^2}{\lambda_k^2}\right)+\mathcal{O}(Er), \qquad (3.30)
$$

where the error term

$$
Er:=\int_t^T\left(\alpha^*D+\frac{D^2}{(T-s)^2}\right)ds+\alpha^*D+\sum_{k=1}^{K}\frac{|M_k|}{(T-t)^2}+(T-t)^{\upsilon_*-2}+\alpha^*(T-t)^{m-1+\frac{d}{2}}. \qquad (3.31)
$$

Proof. Let $F(v):=\frac{d}{2d+4}|v|^{2+\frac{4}{d}}$, $F(U+z)$ and $F(z)$ are defined similarly. Set $f(v):=|v|^{\frac{4}{d}}v$. Rewrite

$$
E(v)=E(v)+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Re}\int\overline{U_k}R_k+\frac{1}{2}|R|^2\Phi_k dx-\sum_{k=1}^{K}\frac{1}{2\lambda_k^2}M_k. \qquad (3.32)
$$

Using (2.1) and (6.8) we expand

$$
F(v)=F(U+z)+F'(U+z)\cdot R+F''(U+z,R)\cdot R^2.
$$

Applying (6.8) again to $F(U+z)$ we get

$$F(v) = F(U) + F'(U)\cdot z + F''(U,z)\cdot z^2 + F'(U+z)\cdot R + F''(U+z,R)\cdot R^2. \tag{3.33}$$

Then, taking into account $F'(U)\cdot z = \mathrm{Re}(f(U)\overline{z})$, $F'(U+z)\cdot z = \mathrm{Re}(f(U+z)\overline{z})$ and the expansion

$$\frac{1}{2}\|\nabla v\|_{L^2}^2 = \frac{1}{2}\|\nabla U\|_{L^2}^2 + \frac{1}{2}\|\nabla z\|_{L^2}^2 + \frac{1}{2}\|\nabla R\|_{L^2}^2 - \mathrm{Re}\langle \Delta U + \Delta z, R\rangle - \mathrm{Re}\langle \Delta U, z\rangle,$$

we obtain

$$\begin{aligned}
E(v) =& E(U) + E(z) - \sum_{k=1}^K \frac{1}{2\lambda_k^2} M_k - \left(\mathrm{Re}\int (\Delta U + \Delta z + |U+z|^{\frac{4}{d}}(U+z))\overline{R}dx - \sum_{k=1}^K \frac{1}{\lambda_k^2}\mathrm{Re}\int \overline{U_k} R_k dx\right)\\
&+\left(\int \frac{1}{2}|\nabla R|^2 dx + \sum_{k=1}^K \frac{1}{2\lambda_k^2}\int |R|^2\Phi_k dx - \mathrm{Re}\int F''(U+z,R)\cdot R^2 dx\right)\\
&- \mathrm{Re}\int (\Delta U + |U|^{\frac{4}{d}}U)\overline{z}dx - \mathrm{Re}\int (F''(U,z)\cdot z^2 - F(z))dx\\
=:& E(U) + E(z) - \sum_{k=1}^K \frac{1}{2\lambda_k^2}M_k + \sum_{l=1}^4 E_l.
\end{aligned} \tag{3.34}$$

Note that, E_1 and E_2 are ordered by the homogeneity of R, and E_3 and E_4 contain the perturbations with the regular profile z.

Next we estimate E_l separately, $1 \le l \le 4$.

For the linear term E_1, by Lemma 2.5,

$$\begin{aligned}
E_1 =& -\sum_{k=1}^K \mathrm{Re}\int (\Delta U_k - \lambda_k^{-2}U_k + |U_k|^{\frac{4}{d}}U_k)\overline{R_k}dx\\
&- \mathrm{Re}\int (\Delta z + |U+z|^{\frac{4}{d}}(U+z) - |U|^{\frac{4}{d}}U)\overline{R}dx + \mathcal{O}(e^{-\frac{\delta}{T-t}}\|R\|_{L^2})\\
=:& E_{11} + E_{12} + \mathcal{O}(e^{-\frac{\delta}{T-t}}\|R\|_{L^2}).
\end{aligned} \tag{3.35}$$

Using the identity (6.39) and the almost orthogonality (2.66) we have (see [11, (3.38)]),

$$\begin{aligned}
E_{11} =& -\sum_{k=1}^K \frac{1}{\lambda_k^2}\mathrm{Im}\int (\gamma_k\Lambda Q_k - 2\beta_k\cdot\nabla Q_k)\overline{\varepsilon_k}dx - \sum_{k=1}^K \frac{1}{\lambda_k^2}\mathrm{Re}\int |\beta_k - \frac{\gamma_k}{2}y|^2 Q_k\overline{\varepsilon_k}dx\\
=& -\sum_{k=1}^K \frac{|\beta_k|^2}{\lambda_k^2}\mathrm{Re}\int U_k\overline{R}_k dx + \mathcal{O}(e^{-\frac{\delta}{T-t}}\|R\|_{L^2})\\
=& -\sum_{k=1}^K \frac{|\beta_k|^2}{2\lambda_k^2}M_k + \mathcal{O}\left(\|R\|_{L^2}^2 + e^{-\frac{\delta}{T-t}}\|R\|_{L^2}\right).
\end{aligned} \tag{3.36}$$

Moreover, by (2.25) and (2.29),

$$\begin{aligned}
|E_{12}| \le& C\int (|\Delta z| + |U|^{\frac{4}{d}}|z| + |z|^{1+\frac{4}{d}})|R|dx\\
\le& C\left(\|\Delta z\|_{L^2} + \sum_{k=1}^K \lambda_k^{-2}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + \|z\|_{H^1}^{1+\frac{4}{d}} + e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^2}\\
\le& C\left(\alpha^*\|R\|_{L^2} + e^{-\frac{\delta}{T-t}}\right).
\end{aligned} \tag{3.37}$$

Thus, we obtain

$$E_1 = -\sum_{k=1}^{K} \frac{|\beta_k|^2}{2\lambda_k^2} M_k + O(\alpha^* \|R\|_{L^2} + e^{-\frac{\delta}{T-t}}) + o\left(\frac{D^2}{(T-t)^2}\right). \tag{3.38}$$

Concerning the second term E_2, set

$$\widetilde{E}_2 := -\mathrm{Re} \int F''(U+z, R)\cdot R^2 - F''(U,R)\cdot R^2 dx. \tag{3.39}$$

We estimate

$$\begin{aligned} E_2 =& \frac{1}{2}\int |\nabla R|^2 dx + \sum_{k=1}^{K} \frac{1}{2\lambda_k^2}\int |R|^2 \Phi_k dx - \mathrm{Re}\int F''(U,R)\cdot R^2 dx + \widetilde{E}_2 \\ =& \frac{1}{2}\int |\nabla R|^2 dx + \sum_{k=1}^{K} \frac{1}{\lambda_k^2}|R|^2\Phi_k - (1+\frac{2}{d})|U|^{\frac{4}{d}}|R|^2 - \frac{2}{d}|U|^{\frac{4}{d}-2}U^2\overline{R}^2 dx + \widetilde{E}_2 + O\left(\frac{D^3}{(T-t)^2}\right) \\ \geq& \widetilde{C}\frac{D^2}{(T-t)^2} + \widetilde{E}_2 + O\left(\sum_{k=1}^{K} \frac{M_k^2}{(T-t)^2} + e^{-\frac{\delta}{T-t}}\right), \end{aligned} \tag{3.40}$$

where $\widetilde{C} > 0$, the error in the second step is caused by the remainders of orders higher than two (see [11, (3.34)]), and the last step is mainly due to the local coercivity of linearized operator in Lemma 6.1, see the proof of [11, (3.39)], and $\frac{D^3}{(T-t)^2} = o\left(\frac{D^2}{(T-t)^2}\right)$. The error $\widetilde{E}_2$ can be bounded by

$$\begin{aligned} |\widetilde{E}_2| \leq& C\int (|U|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1})|z||R|^2 dx \\ \leq& C\left((T-t)^{-2}\sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|R\|_{L^2}^2 + \|z\|_{L^\infty}\|R\|_{L^{1+\frac{4}{d}}}^{1+\frac{4}{d}} + \|z\|_{L^\infty}^{\frac{4}{d}}\|R\|_{L^2}^2 + e^{-\frac{\delta}{T-t}}\right) \\ \leq& C\left((T-t)^{-2}\sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + \alpha^*(T-t)^{-2+\frac{d}{2}} + \alpha^*\right)D^2 + Ce^{-\frac{\delta}{T-t}}, \end{aligned} \tag{3.41}$$

where we also used (2.61) and $D = O(1)$ in the last step.

Thus, for t close to T such that $C\left(\sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + 2(T-t)^{\frac{d}{2}}\right) \leq \frac{1}{2}\widetilde{C}$, it follows that

$$E_2 \geq \frac{\widetilde{C}}{2}\frac{D^2}{(T-t)^2} + O\left(\sum_{k=1}^{K} \frac{M_k^2}{(T-t)^2} + e^{-\frac{\delta}{T-t}}\right), \tag{3.42}$$

The last two terms E_3 and E_4 can be estimated easily by using (1.7) and (2.29):

$$|E_3| \leq C(T-t)^{-2}\sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + Ce^{-\frac{\delta}{T-t}} \leq C\alpha^*(T-t)^{m-1+\frac{d}{2}}, \tag{3.43}$$

$$|E_4| \leq C\sum_{j=2}^{1+\frac{4}{d}} \int |U|^{2+\frac{4}{d}-j}|z|^j dx \leq C(T-t)^{-2}\sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + Ce^{-\frac{\delta}{T-t}} \leq C\alpha^*(T-t)^{m-1+\frac{d}{2}}. \tag{3.44}$$

Thus, combining (3.34), (3.38), (3.42), (3.43) and (3.44) we conclude that for some $C > 0$,

$$E(v) \geq E(U) + E(z) + C\frac{D^2}{(T-t)^2} - \sum_{k=1}^{K} \frac{1+|\beta_k|^2}{2\lambda_k^2}M_k$$

$$+ O\left(\sum_{k=1}^{K}\frac{M_k^2}{(T-t)^2} + \alpha^* D + \alpha^*(T-t)^{m-1+\frac{d}{2}}\right). \tag{3.45}$$

Furthermore, since $v(T_*) = S(T_*) + z(T_*)$ we derive that

$$\begin{aligned} E(v(T_*)) =& E(S(T_*)) + E(z(T_*)) + \mathrm{Re}\int \nabla S(T_*)\nabla \bar{z}(T_*)dx \\ &- \int F(v(T_*)) - F(S(T_*)) - F(z(T_*))dx. \end{aligned} \tag{3.46}$$

As in (3.43) and (3.44), by (2.29) and $T - T_* \leq T - t$,

$$\begin{aligned} \left|\mathrm{Re}\int \nabla S(T_*)\nabla \bar{z}(T_*)dx\right| \leq& C(T-t)^{-2}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}(T_*)\|_{L^\infty} + Ce^{-\frac{\delta}{T-T_*}} \\ \leq& C\alpha^*(T-t)^{m-1+\frac{d}{2}}, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \left|\int F(v(T_*)) - F(S(T_*)) - F(z(T_*))dx\right| \leq& C(T-t)^{-2}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}(T_*)\|_{L^\infty} + Ce^{-\frac{\delta}{T-T_*}} \\ \leq& C\alpha^*(T-t)^{m-1+\frac{d}{2}}. \end{aligned} \tag{3.48}$$

Thus, it follows from (3.46), (3.47) and (3.48) that

$$E(v(T_*)) = E(S(T_*)) + E(z(T_*)) + O\left(\alpha^*(T-t)^{m-1+\frac{d}{2}}\right). \tag{3.49}$$

Now, plugging (3.49) into (3.45) we derive

$$\begin{aligned} E(U(t)) + C\frac{D^2}{(T-t)^2} \leq& (E(v(t)) - E(v(T_*))) - (E(z(t)) - E(z(T_*))) + E(S(T_*)) \\ &+ \sum_{k=1}^{K}\frac{1+|\beta_k|^2}{2\lambda_k^2}M_k + O\left(\sum_{k=1}^{K}\frac{M_k^2}{(T-t)^2} + \alpha^* D + \alpha^*(T-t)^{m-1+\frac{d}{2}}\right). \end{aligned} \tag{3.50}$$

Thus, by the variation control (3.20),

$$E(U(t)) + C\frac{D^2}{(T-t)^2} \leq E(S(T_*)) + O(Er), \tag{3.51}$$

where Er is as in (3.31). Moreover, (2.3) and Lemma 2.5 yield

$$E(U(t)) = \sum_{k=1}^{K}\left(\frac{|\beta_k|^2}{2\lambda_k^2}\|Q\|_{L^2}^2 + \frac{\gamma_k^2}{8\lambda_k^2}\|yQ\|_{L^2}^2\right) + O(e^{-\frac{\delta}{T-t}}), \tag{3.52}$$

$$E(S(T_*)) = \sum_{k=1}^{K}\frac{w_k^2}{8}\|yQ\|_{L^2}^2 + O(e^{-\frac{\delta}{T-T_*}}), \tag{3.53}$$

Therefore, plugging (3.52) and (3.53) into (3.51) we obtain (3.30). The proof is complete. □

3.3. **Monotonicity of generalized energy.** This subsection is mainly devoted to the crucial monotonicity property of generalized energy.

Let $\chi(x) = \psi(|x|)$ be a smooth radial function on $\mathbb{R}^d$, where ψ satisfies $\psi'(r) = r$ if $r \le 1$, $\psi'(r) = 2 - e^{-r}$ if $r \ge 2$, and

$$\left|\frac{\psi'''(r)}{\psi''(r)}\right| \le C, \quad \frac{\psi'(r)}{r} - \psi''(r) \ge 0. \tag{3.54}$$

Let $\chi_A(x) := A^2\chi(\frac{x}{A})$, $A \ge 1$, $f(v) := |v|^{\frac{4}{d}}v$ and $F(v) := \frac{d}{2d+4}|v|^{2+\frac{4}{d}}$.

The generalized energy, adapted to the multi-bubble case, is defined by

$$\begin{aligned}\mathscr{I}(t) :=& \frac{1}{2}\int |\nabla R|^2 dx + \frac{1}{2}\sum_{k=1}^K \int \frac{1}{\lambda_k^2}|R|^2\Phi_k dx - \mathrm{Re}\int F(v) - F(U+z) - f(U+z)\overline{R}dx \\ &+ \sum_{k=1}^K \frac{\gamma_k}{2\lambda_k}\mathrm{Im}\int (\nabla\chi_A)\left(\frac{x-\alpha_k}{\lambda_k}\right)\cdot \nabla R\overline{R}\Phi_k dx =: \mathscr{I}^{(1)} + \mathscr{I}^{(2)}. \end{aligned}\tag{3.55}$$

where $\mathscr{I}^{(1)}$ mainly contains the quadratic terms of remainder (up to acceptable errors) and $\mathscr{I}^{(2)}$ is a Morawetz type functional. The key monotonicity property is formulated below.

Theorem 3.5. *(Monotonicity of generalized energy) Suppose that $P = O(T-t)$, $|\beta_k| + D(t) = O((T-t)^2)$. Then, there exist $C_1, C_2 > 0$ such that for A large enough and $t \in [t^*, T_*]$,*

$$\frac{d\mathscr{I}}{dt} \ge C_1 \sum_{k=1}^K \frac{\gamma_k}{\lambda_k^2}\int (|\nabla R_k|^2 + \frac{1}{\lambda_k^2}|R_k|^2)e^{-\frac{|x-\alpha_k|}{A\lambda_k}}dx - C_2 A\mathcal{E}_r. \tag{3.56}$$

where

$$\begin{aligned}\mathcal{E}_r =& \sum_{k=1}^K \frac{\lambda_k\dot{\lambda}_k + \gamma_k}{\lambda_k^4}M_k + \left(\frac{Mod}{(T-t)^3} + \alpha^*(T-t)^{m-3+\frac{d}{2}} + (T-t)^{\upsilon_*-3}\right)D \\ &+ \frac{D^2}{(T-t)^2} + \varepsilon\frac{D^2}{(T-t)^3} + \frac{M_k^2}{(T-t)^3} + e^{-\frac{\delta}{T-t}}. \end{aligned}\tag{3.57}$$

The functionals $\mathscr{I}^{(1)}$ and $\mathscr{I}^{(2)}$ will be treated in Lemmas 3.6 and 3.7 below, respectively.

Lemma 3.6. *(Control of $\mathscr{I}^{(1)}$) Consider the situations as in Theorem 3.5. Then, there exists $C > 0$ such that for any $t \in [t^*, T_*]$,*

$$\begin{aligned}\frac{d\mathscr{I}^{(1)}}{dt} \ge& \sum_{k=1}^K \frac{\gamma_k}{\lambda_k^4}\|\varepsilon_k\|_{L^2}^2 - \sum_{k=1}^K \frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int (1+\frac{2}{d})|Q_k|^{\frac{4}{d}}|\varepsilon_k|^2 + \frac{2}{d}|Q_k|^{\frac{4}{d}-2}\overline{Q_k}^2\varepsilon_k^2 dy \\ &- \sum_{k=1}^K \frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int y\cdot\nabla\overline{Q_k}\,(f''(Q_k)\cdot\varepsilon_k^2)dy - C\mathcal{E}_r, \end{aligned}\tag{3.58}$$

where $\mathcal{E}_r$ is the error as in (3.57) but without the term $\frac{M_k^2}{(T-t)^3}$.

Proof. Let η be as in (2.68). Using equation (2.67) and (6.8) we compute as in [64, (5.32)] that

$$\begin{aligned}\frac{d\mathscr{I}^{(1)}}{dt} =& -\sum_{k=1}^K \dot{\lambda}_k\lambda_k^{-3}\int |R|^2\Phi_k dx - \sum_{k=1}^K \lambda_k^{-2}\mathrm{Im}\langle f'(U+z)\cdot R, R_k\rangle \\ &- \mathrm{Re}\langle f''(U+z,R)\cdot R^2, \partial_t(U+z)\rangle - \sum_{k=1}^K \lambda_k^{-2}\mathrm{Im}\langle R\nabla\Phi_k, \nabla R\rangle \end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{K} \lambda_k^{-2} \mathrm{Im}\langle f''(U+z,R)\cdot R^2, R_k\rangle \\
&- \mathrm{Im}\langle \Delta R - \sum_{k=1}^{K} \lambda_k^{-2} R_k + f(v) - f(U+z), a_1\cdot\nabla R + a_0 R\rangle \\
&- \mathrm{Im}\langle \Delta R - \sum_{k=1}^{K} \lambda_k^{-2} R_k + f(v) - f(U+z), \eta\rangle \\
=: &\sum_{l=1}^{7} \mathscr{I}_{t,l}^{(1)}. \qquad (3.59)
\end{aligned}
$$

The terms $\{\mathscr{I}_{t,l}^{(1)}\}$ are estimated as follows:

(i) *Estimate of* $\mathscr{I}_{t,1}^{(1)}$. Since $P = O(T-t)$, $D = O((T-t)^2)$, by Theorem 2.2,

$$
Mod = O((T-t)^2). \qquad (3.60)
$$

Hence, we compute

$$
-\frac{\dot\lambda_k}{\lambda_k^3} = \frac{\gamma_k}{\lambda_k^4} - \frac{\lambda_k\dot\lambda_k + \gamma_k}{\lambda_k^4} = \frac{\gamma_k}{\lambda_k^4} + O\left(\frac{Mod}{\lambda_k^4}\right) = \frac{\gamma_k}{\lambda_k^4} + O\left(\frac{D^2}{(T-t)^2}\right),
$$

which yields that

$$
\mathscr{I}_{t,1}^{(1)} = \sum_{k=1}^{K} \frac{\gamma_k}{\lambda_k^4} \int |R|^2 \Phi_k dx + O\left(\frac{D^2}{(T-t)^2}\right). \qquad (3.61)
$$

(ii) *Estimates of* $\mathscr{I}_{t,2}^{(1)}$ *and* $\mathscr{I}_{t,3}^{(1)}$. Rewrite

$$
\mathscr{I}_{t,2}^{(1)} + \mathscr{I}_{t,3}^{(1)} = -\sum_{k=1}^{K} \lambda_k^{-2} \mathrm{Im}\langle f'(U)\cdot R, R_k\rangle - \mathrm{Re}\langle f''(U,R)\cdot R^2, \partial_t U\rangle + er,
$$

where er denotes the difference

$$
\begin{aligned}
er := &- \sum_{k=1}^{K} \lambda_k^{-2} \mathrm{Im}\langle f'(U+z)\cdot R - f'(U)\cdot R, R_k\rangle \\
&- \Big(\mathrm{Re}\langle f''(U+z,R)\cdot R^2, \partial_t(U+z)\rangle - \mathrm{Re}\langle f''(U,R)\cdot R^2, \partial_t U\rangle\Big) \\
=:& er_1 + er_2. \qquad (3.62)
\end{aligned}
$$

We claim that

$$
er = O\Big(\alpha^*(T-t)^{-2}D^2\Big). \qquad (3.63)
$$

To this end, by (6.14), the renormalized variable $\varepsilon_{z,k}$ in (2.24),

$$
\begin{aligned}
|er_1| \le & C(T-t)^{-2} \int (|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1})|z||R|^2 dx \\
\le & C\left((T-t)^{-4} \sum_{k=1}^{K} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|R\|_{L^2}^2 + (T-t)^{-2}\|z\|_{L^\infty}^{\frac{4}{d}}\|R\|_{L^2}^2 + e^{-\frac{\delta}{T-t}}\|R\|_{L^2}^2\right) \\
\le & C\left(\alpha^*(T-t)^{m-3+\frac{d}{2}} + \alpha^*(T-t)^{-2}\right) D^2. \qquad (3.64)
\end{aligned}
$$

Regarding the second term er_2, note that

$$\begin{aligned} er_2 =& \mathrm{Re}\langle f''(U+z,R)\cdot R^2, \partial_t z\rangle + \mathrm{Re}\langle f''(U+z,R)\cdot R^2 - f''(U,R)\cdot R^2, \partial_t U\rangle \\ =: & \, er_{21} + er_{22}. \end{aligned}$$

By (6.16) and estimates (2.15), (2.25), (2.26) and (2.61),

$$\begin{aligned} |er_{21}| \le& C\int (|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1})|R|^2|\partial_t z|dx \\ \le& C\|\partial_t z\|_{L^2}\left(\|U\|^{\frac{4}{d}-1}_{L^{4(\frac{4}{d}-1)}}\|R\|^2_{L^8} + \|z\|^{\frac{4}{d}-1}_{L^\infty}\|R\|^2_{H^1} + \|R\|^{\frac{4}{d}+1}_{L^{\frac{8}{d}+2}}\right) \\ \le& C\alpha^*(T-t)^{-2}D^2. \end{aligned} \tag{3.65}$$

Moreover, since by (2.13) and $Mod = \mathcal{O}(1)$,

$$\|\partial_t U_k(t)\|_{L^\infty} \le C(T-t)^{-\frac{d}{2}-2}. \tag{3.66}$$

Then, using (6.15), (2.29) and (2.61) we get

$$\begin{aligned} |er_{22}| \le& C\int (|U|^{\frac{4}{d}-2} + |R|^{\frac{4}{d}-2} + |z|^{\frac{4}{d}-2})|z||R|^2|\partial_t U|dx \\ \le& C(T-t)^{-4}\sum_{k=1}^K \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|R\|^2_{L^2} + C(T-t)^{-d-2}\sum_{k=1}^K \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|R\|^{\frac{4}{d}}_{L^{\frac{4}{d}}} + Ce^{-\frac{\delta}{T-t}}\|R\|^2_{L^2} \\ \le& C(T-t)^{-4}\sum_{k=1}^K \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}D^2 + Ce^{-\frac{\delta}{T-t}}D^2 \\ \le& C\alpha^*(T-t)^{m-3+\frac{d}{2}}D^2. \end{aligned} \tag{3.67}$$

Hence, plugging (3.64), (3.65) and (3.67) into (3.62) we prove (3.63), as claimed.

Thus, since

$$|\beta_k| + D(t) + Mod(t) = \mathcal{O}((T-t)^2),$$

computing as in the proof of [11, (4.18),(4.20)] we obtain

$$\begin{aligned} \mathscr{I}^{(1)}_{t,2} + \mathscr{I}^{(1)}_{t,3} =& -\sum_{k=1}^K \frac{\gamma_k}{\lambda_k^2}\mathrm{Re}\int (1+\frac{2}{d})|U_k|^{\frac{4}{d}}|R_k|^2 + \frac{2}{d}|U_k|^{\frac{4}{d}-2}\overline{U_k}^2 R_k^2 dx \\ & -\sum_{k=1}^K \frac{\gamma_k}{\lambda_k}\mathrm{Re}\int (\frac{x-\alpha_k}{\lambda_k})\cdot\nabla\overline{U_k}\, f''(U_k)\cdot R_k^2 dx + \mathcal{O}((T-t)^{-2}D^2(t)). \end{aligned} \tag{3.68}$$

(*iii*) *Estimate of* $\mathscr{I}^{(1)}_{t,4}$. We use the smallness in Case (I) or Case (II) to control this term and have that (see [11, (4.21),(4.22)])

$$|\mathscr{I}^{(1)}_{t,4}| \le C\varepsilon(T-t)^{-3}D^2. \tag{3.69}$$

(*iv*) *Estimate of* $\mathscr{I}^{(1)}_{t,5}$. Using (6.16), (2.15), (2.61) and $D \le C(T-t)^2$ we get

$$\begin{aligned} |\mathscr{I}^{(1)}_{t,5}| \le& C(T-t)^{-2}\left(\|U\|^{\frac{4}{d}-1}_{L^{2(\frac{4}{d}-1)}}\|R\|^3_{L^6} + \|z\|^{\frac{4}{d}-1}_{L^\infty}\|R\|^3_{L^3} + \|R\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}}\right) \\ \le& C(T-t)^{-2}\left((T-t)^{-2}D^3 + \alpha^*(T-t)^{-\frac{d}{2}}D^3 + (T-t)^{-2}D^{\frac{4}{d}+2}\right) \\ \le& C(T-t)^{-2}D^2. \end{aligned} \tag{3.70}$$

(v) *Estimate of* $\mathscr{I}^{(1)}_{t,6}$. Using the integration by parts formula and (6.13) we get (see also [64, (5.42)])

$$
\begin{aligned}
&|\mathrm{Im}\langle \Delta R - \sum_{k=1}^{K} \lambda_k^{-2} R_k + f(v) - f(U+z), a_1 \cdot \nabla R\rangle| \\
\leq& C\left(\|\nabla R\|_{L^2}^2 + (T-t)^{-2}\|R\|_{L^2}^2\right) + C\int (|U|^{\frac{4}{d}} + |z|^{\frac{4}{d}} + |R|^{\frac{4}{d}})|R||a_1 \nabla R| dx. \qquad (3.71)
\end{aligned}
$$

Then, by (1.2), the change of variables and (2.57),

$$
\begin{aligned}
&\int (|U|^{\frac{4}{d}} + |z|^{\frac{4}{d}} + |R|^{\frac{4}{d}})|R||a_1 \cdot \nabla R| dx \\
\leq& C(T-t)^{-2} \sum_{l=1}^{N}\sum_{k=1}^{K} \|e^{-\delta|y|}\nabla\phi_l(\lambda_k y + \alpha_k)\|_{L^\infty}\|R\|_{L^2}\|\nabla R\|_{L^2} \\
&+ C\left(\|z\|_{L^\infty}^{\frac{4}{d}}\|R\|_{L^2}\|\nabla R\|_{L^2} + \|R\|_{L^{\frac{8}{d}+2}}^{\frac{4}{d}+1}\|\nabla R\|_{L^2}\right) + Ce^{-\frac{\delta}{T-t}}\|R\|_{L^2}^2 \\
\leq& C\left((T-t)^{\upsilon_*-3} + \alpha^*(T-t)^{-1}\right)D^2 + C(T-t)^{-3}D^{2+\frac{4}{d}} + Ce^{-\frac{\delta}{T-t}}D^2. \qquad (3.72)
\end{aligned}
$$

Since $D \leq C(T-t)$, we come to

$$
|\mathrm{Im}\langle \Delta R - \sum_{k=1}^{K} \lambda_k^{-2} R_k + f(u) - f(U+z), a_1 \cdot \nabla R\rangle| \leq C(T-t)^{-2}D^2. \qquad (3.73)
$$

Similarly, we have

$$
\begin{aligned}
|\mathrm{Im}\langle \Delta R - \sum_{k=1}^{K} \lambda_k^{-2} R_k + f(v) - f(U+z), a_0 R\rangle| \leq& C\left(\|R\|_{H^1}^2 + \lambda_k^{-2}\|R\|_{L^2}^2 + \int (|U|^{\frac{4}{d}} + |z|^{\frac{4}{d}} + |R|^{\frac{4}{d}})|R|^2 dx\right) \\
\leq& C\left((T-t)^{-2}D^2 + \alpha^* D^2 + (T-t)^{-2}D^{2+\frac{4}{d}}\right) \\
\leq& C(T-t)^{-2}D^2. \qquad (3.74)
\end{aligned}
$$

Thus, we conclude from (3.73) and (3.74) that

$$
|\mathscr{I}^{(1)}_{t,6}| \leq C(T-t)^{-2}D^2. \qquad (3.75)
$$

(vi) *Estimate of* $\mathscr{I}^{(1)}_{t,7}$. It remains to treat the delicate inner product $\mathscr{I}^{(1)}_{t,7}$ involving the η term. First, we claim that

$$
\begin{aligned}
\mathscr{I}^{(1)}_{t,7} =& -\sum_{k=1}^{K} \mathrm{Im}\langle \Delta R_k - \lambda_k^{-2} R_k + f'(U_k)\cdot R_k, \eta\rangle \\
&+ O\left(\alpha^*(T-t)^{m-1+\frac{d}{2}}D + (T-t)^{-2}D^2 + e^{-\frac{\delta}{T-t}}\right). \qquad (3.76)
\end{aligned}
$$

This means that, the inner products involving R_k of orders higher than one are acceptable errors.

To this end, we use the expansion (6.8) to get

$$
\begin{aligned}
f(v) - f(U+z) =& f'(U+z)\cdot R + f''(U+z,R)\cdot R^2 \\
=& f'(U)\cdot R + (f'(U+z)\cdot R - f'(U)\cdot R) + f''(U+z,R)\cdot R^2. \qquad (3.77)
\end{aligned}
$$

Define the renormalized variable $\varepsilon_{R,k}$ by

$$
R(t,x) = \lambda_k^{-\frac{d}{2}} \varepsilon_{R,k}(t, \frac{x-\alpha_k}{\lambda_k})e^{i\theta_k}.
$$

Then, by (6.14) and (2.69),

$$\begin{aligned}
&\operatorname{Im}\langle f'(U+z)\cdot R - f'(U)\cdot R, \eta\rangle \\
\le& C\int (|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1})|z||R||\eta| dx \\
\le& C(T-t)^{-4}\sum_{k=1}^{K}\int \left(e^{-\delta|y|} + |\varepsilon_{z,k}|^{\frac{4}{d}-1}\right)|\varepsilon_{z,k}||\varepsilon_{R,k}|\left(Mod + |\varepsilon_{z,k}| + (T-t)^{\nu_*+1}\right)e^{-\delta|y|}dy + Ce^{-\frac{\delta}{T-t}} \\
\le& C(T-t)^{-4}\sum_{k=1}^{K}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|R\|_{L^2}\left(Mod + \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + (T-t)^{\nu_*+1}\right) + Ce^{-\frac{\delta}{T-t}} \\
\le& C\alpha^*(T-t)^{m-1+\frac{d}{2}}D + Ce^{-\frac{\delta}{T-t}},
\end{aligned} \tag{3.78}$$

Moreover, by (2.15), (2.61), (2.70), (3.60) and $Mod \le C(T-t)^2$,

$$\begin{aligned}
\operatorname{Im}\langle f''(U+z,R)\cdot R^2, \eta\rangle \le& C\|U+z\|^{\frac{4}{d}-1}_{L^{4(\frac{4}{d}-1)}}\|R\|^2_{L^8}\|\eta\|_{L^2} + C\|R\|^{\frac{4}{d}+1}_{H^1}\|\eta\|_{L^2} \\
\le& C(T-t)^{-2}\|\eta\|_{L^2}D^2 + C\|\eta\|_{L^2}\|R\|^2_{H^1} \\
\le& C(T-t)^{-2}D^2.
\end{aligned} \tag{3.79}$$

Thus, combining (3.77), (3.78) and (3.79) and using Lemma 2.5 we obtain (3.76), as claimed.

Next, in order to treat the remaining linear terms on the R.H.S. of (3.76), we decompose η into four parts $\eta = \sum_{l=1}^4 \eta_l$ as in (2.71). Note that, by Lemma 2.5, (2.73) and (2.74),

$$\begin{aligned}
&|\operatorname{Im}\langle \Delta R_k - \lambda_k^{-2}R_k + f'(U_k)\cdot R_k, \eta_2+\eta_3\rangle| \\
\le& C\lambda_k^{-4}\sum_{j=1}^{\frac{4}{d}}\int |\nabla\varepsilon_k|(|\varepsilon_{z,k}|^j + |\nabla\varepsilon_{z,k}||\varepsilon_{z,k}|^{j-1})e^{-\delta|y|}dy + C\sum_{j=1}^{\frac{4}{d}}\int |\varepsilon_k||\varepsilon_{z,k}^j|e^{-\delta|y|}dy \\
&+ C\lambda_k^{-4}\sum_{j=1}^{\frac{4}{d}}\int |f'(Q_k)\varepsilon_k||Q_k|^{1+\frac{4}{d}-j}|\varepsilon_{z,k}|^j dy + Ce^{-\frac{\delta}{T-t}} \\
\le& C\lambda_k^{-4}\|e^{-\delta|y|}(|\varepsilon_{z,k}| + |\nabla\varepsilon_{z,k}|)\|_{L^\infty}D + Ce^{-\frac{\delta}{T-t}} \\
\le& C\alpha^*(T-t)^{m-3+\frac{d}{2}}D + Ce^{-\frac{\delta}{T-t}},
\end{aligned} \tag{3.80}$$

where the last step is due to (2.29). Moreover, by (2.75) and Lemma 2.8,

$$\begin{aligned}
&|\operatorname{Im}\langle \Delta R_k - \lambda_k^{-2}R_k + f'(U_k)\cdot R_k, \eta_4\rangle| \\
\le& C\lambda_k^{-2}|\operatorname{Im}\langle \nabla\varepsilon_k, \nabla(\lambda_k^{-1}\widetilde{a}_{1,k}\cdot\nabla Q_k + \widetilde{a}_{0,k}Q_k)\rangle| \\
&+ C\lambda_k^{-2}\int (|\varepsilon_k| + |f'(Q_k)\varepsilon_k|)|\lambda_k^{-1}\widetilde{a}_{1,k}\cdot\nabla Q_k + \widetilde{a}_{0,k}Q_k|dy \\
\le& C\lambda_k^{\nu_*-3}(\|\varepsilon_k\|_{L^2} + \|\nabla\varepsilon_k\|_{L^2}) \\
\le& C(T-t)^{\nu_*-3}D.
\end{aligned} \tag{3.81}$$

Hence, taking into account (2.72) and using Lemma 2.5 again we obtain

$$\begin{aligned}
\operatorname{Im}\langle \Delta R_k - \lambda_k^{-2}R_k + f'(U_k)\cdot R_k, \eta\rangle =& \lambda_k^{-4}\operatorname{Im}\langle \Delta\varepsilon_k - \varepsilon_k + f'(Q_k)\cdot\varepsilon_k, \Psi_k\rangle \\
&+ O\left((\alpha^*(T-t)^{m-3+\frac{d}{2}} + (T-t)^{\nu_*-3})D + e^{-\frac{\delta}{T-t}}\right),
\end{aligned} \tag{3.82}$$

where Ψ_k is given by (2.14).

The analysis is now reduced to that of the inner product involving Ψ_k.

By the proximity

$$Q_k = Q + O(Pe^{-\delta|y|}) \tag{3.83}$$

and the definition of linearized operators in (6.1),

$$\begin{aligned}&\mathrm{Im}\langle \Delta\varepsilon_k - \varepsilon_k + f'(Q_k)\cdot\varepsilon_k, \Psi_k\rangle\\ =&\mathrm{Im}\langle \Delta\varepsilon_k - \varepsilon_k + f'(Q)\cdot\varepsilon_k, \widetilde{\Psi}_k\rangle + O(PMod\|R\|_{L^2})\\ =&\langle L_+\varepsilon_{k,1}, \widetilde{\Psi}_{k,2}\rangle - \langle L_-\varepsilon_{k,2}, \widetilde{\Psi}_{k,1}\rangle + O(PModD).\end{aligned} \tag{3.84}$$

where $\widetilde{\Psi}_k$ is defined as in (2.14) with Q replacing Q_k, $\varepsilon_{k,1} = \mathrm{Re}\varepsilon_k$, $\varepsilon_{k,2} = \mathrm{Im}\varepsilon_{k,2}$, and $\widetilde{\Psi}_{k,1}$, $\widetilde{\Psi}_{k,2}$ are defined similarly. Then, by (2.14) and the algebraic identities in (6.3),

$$\begin{aligned}\langle L_+\varepsilon_{k,1}, \widetilde{\Psi}_{k,2}\rangle =& -(\lambda_k\dot{\lambda}_k + \gamma_k)\langle \varepsilon_{k,1}, L_+\Lambda Q\rangle\\ =&2(\lambda_k\dot{\lambda}_k + \gamma_k)\mathrm{Re}\langle U_k, R_k\rangle + O(PMod\|R\|_{L^2}),\end{aligned} \tag{3.85}$$

and

$$\begin{aligned}\langle L_-\varepsilon_{k,2}, \widetilde{\Psi}_{k,1}\rangle =& -(\lambda_k^2\beta_k + \gamma_k\beta_k)\langle \varepsilon_{k,2}, L_-Q\rangle + \frac{1}{4}(\lambda_k^2\dot{\gamma}_k + \gamma_k^2)\langle \varepsilon_{k,2}, L_-|y|^2Q\rangle\\ =&2(\lambda_k^2\beta_k + \gamma_k\beta_k)\langle \varepsilon_{k,2}, \nabla Q\rangle - (\lambda_k^2\dot{\gamma}_k + \gamma_k^2)\langle \varepsilon_{k,2}, \Lambda Q\rangle\\ =&O\left(PMod\|R\|_{L^2} + e^{-\frac{\delta}{T-t}}\|R\|_{L^2}\right),\end{aligned} \tag{3.86}$$

where in the last step we also used the almost orthogonality (2.66). Hence, plugging (3.84), (3.85) and (3.86) into (3.82) we obtain

$$\begin{aligned}&\mathrm{Im}\langle \Delta R_k - \lambda_k^{-2}R_k + f'(U_k)\cdot R_k, \eta\rangle\\ =&2\lambda_k^{-4}(\lambda_k\dot{\lambda}_k + \gamma_k)\mathrm{Re}\langle U_k, R_k\rangle\\ &+ O\left(\left((T-t)^{-3}Mod + \alpha^*(T-t)^{m-3+\frac{d}{2}} + (T-t)^{\upsilon_*-3}\right)D + e^{-\frac{\delta}{T-t}}\right).\end{aligned} \tag{3.87}$$

Thus, combining (3.76) and (3.87) together we arrive at

$$\begin{aligned}\mathscr{I}^{(1)}_{t,7} =& -2\sum_{k=1}^{K}\lambda_k^{-4}(\lambda_k\dot{\lambda}_k + \gamma_k)\mathrm{Re}\langle U_k, R_k\rangle\\ &+ O\left(\left((T-t)^{-3}Mod + \alpha^*(T-t)^{m-3+\frac{d}{2}} + (T-t)^{\upsilon_*-3}\right)D + (T-t)^{-2}D^2 + e^{-\frac{\delta}{T-t}}\right).\end{aligned} \tag{3.88}$$

Finally, plugging estimates (3.61), (3.68), (3.69), (3.70), (3.75) and (3.88) into (3.59) we obtain (3.58) and finish the proof of Lemma 3.6. □

Lemma 3.7. *(Control of $\mathscr{I}^{(2)}$) Consider the situations as in Theorem 3.5. Then, there exists $C > 0$ such that for all $t \in [t^*, T_*]$,*

$$\begin{aligned}\frac{d\mathscr{I}^{(2)}}{dt} \geq& -\sum_{k=1}^{K}\frac{\gamma_k}{4\lambda_k^4}\int \Delta^2\chi_A(y)|\varepsilon_k|^2dy + \sum_{k=1}^{K}\frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int \nabla^2\chi_A(y)(\nabla\varepsilon_k, \nabla\overline{\varepsilon_k})dy\\ &+ \sum_{k=1}^{K}\frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int \nabla\chi_A(y)\cdot\nabla\overline{Q_k}\, f''(Q_k)\cdot\varepsilon_k dy - CA\mathcal{E}'_r,\end{aligned} \tag{3.89}$$

where

$$\mathcal{E}'_r = \left(\frac{Mod}{(T-t)^3} + \alpha^*(T-t)^{m-2+\frac{d}{2}} + (T-t)^{\upsilon_*-2}\right)D + \frac{D^2}{(T-t)^2} + e^{-\frac{\delta}{T-t}}. \tag{3.90}$$

Proof. We compute as in [64, (5.48)],

$$\begin{aligned}\frac{d\mathscr{I}^{(2)}}{dt} =& -\sum_{k=1}^{K}\frac{\dot{\lambda}_k\gamma_k-\lambda_k\dot{\gamma}_k}{2\lambda_k^2}\mathrm{Im}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot\nabla R, R_k\rangle \\ &+\sum_{k=1}^{K}\frac{\gamma_k}{2\lambda_k}\mathrm{Im}\langle\partial_t(\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k}))\cdot\nabla R, R_k\rangle+\sum_{k=1}^{K}\frac{\gamma_k}{2\lambda_k^2}\mathrm{Im}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, \partial_t R\rangle \\ &+\sum_{k=1}^{K}\frac{\gamma_k}{2\lambda_k}\mathrm{Im}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k), \partial_t R\rangle \\ =:& \sum_{k=1}^{K}\left(\mathscr{I}^{(2)}_{t,k1}+\mathscr{I}^{(2)}_{t,k2}+\mathscr{I}^{(2)}_{t,k3}+\mathscr{I}^{(2)}_{t,k4}\right).\end{aligned} \tag{3.91}$$

(i) *Estimate of* $\mathscr{I}^{(2)}_{t,k1}$ *and* $\mathscr{I}^{(2)}_{t,k2}$. Since $|\frac{\dot{\lambda}_k\gamma_k-\lambda_k\dot{\gamma}_k}{\lambda_k^2}| \leq C\lambda_k^{-3}Mod_k$ and $|\partial_t(\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k}))| \leq CA\lambda_k^{-2}(Mod+P)$, by (3.60),

$$\begin{aligned}|\mathscr{I}^{(2)}_{t,k1}+\mathscr{I}^{(2)}_{t,k2}| \leq& CA\lambda_k^{-3}(Mod+P^2)\|\nabla R\|_{L^2}\|R\|_{L^2} \\ \leq& CA(\lambda_k^{-4}ModD^2+\lambda_k^{-2}D^2) \leq CA(T-t)^{-2}D^2.\end{aligned} \tag{3.92}$$

(ii) *Estimate of* $\mathscr{I}^{(2)}_{t,k3}$. We claim that

$$\begin{aligned}\mathscr{I}^{(2)}_{t,k3} =& -\frac{\gamma_k}{4\lambda_k^4}\mathrm{Re}\int\Delta^2\chi_A(\frac{x-\alpha_k}{\lambda_k})|R_k|^2dx+\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\int\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})|\nabla R_k|^2dx \\ &-\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U_k)\cdot R_k\rangle \\ &+O\Big(A(T-t)^{-2}D^2+\Big((T-t)^{-3}Mod+\alpha^*(T-t)^{m-2+\frac{d}{2}}+(T-t)^{\upsilon_*-2}\Big)D+e^{-\frac{\delta}{T-t}}\Big).\end{aligned} \tag{3.93}$$

For this purpose, by (2.67) and (6.8),

$$\mathscr{I}^{(2)}_{t,k3} = -\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, \Delta R+f'(U+z)\cdot R+f''(U+z,R)\cdot R^2+a_1\cdot\nabla R+a_0R+\eta\rangle. \tag{3.94}$$

First, we have from [11, (3.67)] that

$$\begin{aligned}-\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, \Delta R\rangle =& -\frac{\gamma_k}{4\lambda_k^4}\mathrm{Re}\int\Delta^2\chi_A(\frac{x-\alpha_k}{\lambda_k})|R_k|^2dx \\ &+\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\int\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})|\nabla R_k|^2dx+O(A\|R\|_{H^1}^2).\end{aligned} \tag{3.95}$$

Let us mention that, an extra factor $T-t$ is gained here from the decay properties of the cut-off function, i.e., for $|y| \geq 2A$,

$$|\nabla\Delta\chi_A(y)| \leq CA|y|^{-2}, \ \ |\partial_{x_kx_l}\chi_A(y)| \leq CA|y|^{-1}, \ \ 1 \leq k,l \leq d. \tag{3.96}$$

Moreover, rewrite

$$\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U+z)\cdot R\rangle = \mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U)\cdot R\rangle + \widetilde{er},$$

where the difference

$$\widetilde{er} := \mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U+z)\cdot R - f'(U)\cdot R\rangle. \tag{3.97}$$

Then, by the bound $\|\Delta\chi_A\|_{L^\infty} \le C$, (2.25) and (6.14),

$$\begin{aligned}|\widetilde{er}| \le& C\int(|U|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1})|z||R|^2dx\\ \le& C\left((T-t)^{-\frac{d}{2}(\frac{4}{d}-1)}+\|z\|_{L^\infty}^{\frac{4}{d}-1}\right)\|z\|_{L^\infty}\|R\|_{L^2}^2\\ \le& C(T-t)^{-2}D^2.\end{aligned}$$

This along with Lemma 2.5 yields that

$$\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U+z)\cdot R\rangle = \mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f'(U_k)\cdot R_k\rangle + O\left((T-t)^{-2}D^2+e^{-\frac{\delta}{T-t}}\right). \tag{3.98}$$

It also follows from (6.16), (2.15), (2.61) and $D = O((T-t)^2)$ that

$$\begin{aligned}\left|\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, f''(U+z,R)\cdot R^2\rangle\right| \le& C\lambda_k^{-1}(\|R\|_{L^6}^3\|U+z\|_{L^{2(\frac{4}{d}-1)}}^{\frac{4}{d}-1}+\|R\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2})\\ \le& C(T-t)^{-1}\left((T-t)^{-2}D^3+(T-t)^{-2}D^{\frac{4}{d}+2}\right)\\ \le& C(T-t)^{-2}D^2.\end{aligned} \tag{3.99}$$

Furthermore, by (2.70),

$$\begin{aligned}&\left|\frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\langle\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})R_k, a_1\cdot\nabla R+a_0R+\eta\rangle\right|\\ \le& C\lambda_k^{-1}(\|R\|_{L^2}\|\nabla R\|_{L^2}+\|R\|_{L^2}^2)+C\lambda_k^{-1}\|R\|_{L^2}\|\eta\|_{L^2}\\ \le& C\left((T-t)^{-2}D^2+\left((T-t)^{-3}Mod+\alpha^*(T-t)^{m-2+\frac{d}{2}}+(T-t)^{\upsilon_*-2}\right)D\right)\end{aligned} \tag{3.100}$$

Hence, plugging (3.95), (3.98), (3.99) and (3.100) into (3.94) we obtain (3.93), as claimed.
(*iii*) *Estimate of* $\mathscr{I}_{t,k4}^{(2)}$. The estimate of $\mathscr{I}_{t,k4}^{(2)}$ is similar to that of $\mathscr{I}_{t,k3}^{(2)}$. We claim that

$$\begin{aligned}\mathscr{I}_{t,k4}^{(2)} =& \frac{\gamma_k}{\lambda_k^2}\mathrm{Re}\int\nabla^2\chi_A(\frac{x-\alpha_k}{\lambda_k})(\nabla R_k,\nabla\overline{R_k})dx - \frac{\gamma_k}{2\lambda_k^2}\mathrm{Re}\int\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})|\nabla R_k|^2dx\\ &-\frac{\gamma_k}{\lambda_k}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot\nabla R_k, f'(U_k)\cdot R_k\rangle\\ &+O\left(A(T-t)^{-2}D^2+A\left((T-t)^{-3}Mod+\alpha^*(T-t)^{m-2+\frac{d}{2}}+(T-t)^{\upsilon_*-2}\right)D\right).\end{aligned} \tag{3.101}$$

For this purpose, using (2.67) again and (6.8) we derive

$$\begin{aligned}\mathscr{I}_{t,k4}^{(2)} =& -\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),\\ &\Delta R+f'(U+z)\cdot R_k+f''(U+z,R)\cdot R^2+(a_1\cdot\nabla+a_0)R+\eta\rangle.\end{aligned} \tag{3.102}$$

Similarly to (3.95), we have (see [11, (3.73)])

$$
\begin{aligned}
&-\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),\Delta R\rangle\\
=&\frac{\gamma_k}{\lambda_k^2}\mathrm{Re}\int\nabla^2\chi_A(\frac{x-\alpha_k}{\lambda_k})(\nabla R_k,\nabla\overline{R_k})-\frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})|\nabla R_k|^2dx+O\Big(A(T-t)^{-2}D^2\Big).
\end{aligned}\tag{3.103}
$$

We note that, the second order terms $\partial_{x_kx_l}\overline{R}$ are cancelled by the integration by parts formula. For the detailed computations we refer to [64, (5.66)].

Moreover, as in (3.97), rewrite

$$
\begin{aligned}
&\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),f'(U+z)\cdot R\rangle\\
=&\frac{\gamma_k}{\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot\nabla R_k,f'(U_k)\cdot R_k\rangle+\widehat{er}+O(Ae^{-\frac{\delta}{T-t}}),
\end{aligned}\tag{3.104}
$$

where the last step is due to Lemma 2.5 and the error term is of form

$$
\widehat{er}:=\frac{\gamma_k}{\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),f'(U+z)\cdot R-f'(U)\cdot R\rangle.
$$

We use the bound $\|\nabla\chi_A\|_{L^\infty}\leq CA$, (2.25), (2.29) and (6.14) to bound

$$
\begin{aligned}
|\widehat{er}|\leq&CA\int(|\nabla R|+|R|)(|U|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1})|z||R|dx\\
\leq&CA\left(\sum_{k=1}^K(T-t)^{-2}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}+\|z\|_{L^\infty}^{\frac{4}{d}}\right)\int(|\nabla R|+|R|)|R|dx+CAe^{-\frac{\delta}{T-t}}\\
\leq&CA\left(\alpha^*(T-t)^{m-1+\frac{d}{2}}+\alpha^*\right)(T-t)^{-1}D^2+CAe^{-\frac{\delta}{T-t}}\\
\leq&CA\left((T-t)^{-1}D^2+e^{-\frac{\delta}{T-t}}\right).
\end{aligned}\tag{3.105}
$$

Plugging this into (3.104) yields that

$$
\begin{aligned}
&-\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),f'(U+z)\cdot R\rangle\\
=&-\frac{\gamma_k}{\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot\nabla R_k,f'(U_k)\cdot R_k\rangle+O\Big(A(T-t)^{-1}D^2+Ae^{-\frac{\delta}{T-t}}\Big).
\end{aligned}\tag{3.106}
$$

For the remaining inner products in (3.102), by (2.15) and (2.61),

$$
\begin{aligned}
&\left|\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),f''(U+z,R)\cdot R^2+a_1\cdot\nabla R+a_0R\rangle\right|\\
\leq&CA\int(|\nabla R|+|R|)(|U+z|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1})|R|^2dx+CA\int(|\nabla R|+|R|)^2dx\\
\leq&CA\left((T-t)^{-\frac{d}{2}(\frac{4}{d}-1)}+\|z\|_{L^\infty}^{\frac{4}{d}-1}\right)\int(|\nabla R|+|R|)|R|^2dx+CA\int(|\nabla R|+|R|)|R|^{\frac{4}{d}+1}dx+CA\|R\|_{H^1}^2\\
\leq&CA(T-t)^{-2+\frac{d}{2}}\left(\|\nabla R\|_{L^2}\|R\|_{L^4}^2+\|R\|_{L^3}^3\right)+CA\left(\|\nabla R\|_{L^2}\|R\|^{\frac{4}{d}+1}_{L^{2(\frac{4}{d}+1)}}+\|R\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}}\right)+CA\|R\|_{H^1}^2.
\end{aligned}\tag{3.107}
$$

Then, by (2.61), the R.H.S. above can be bounded by, up to a universal constant CA,

$$
\begin{aligned}
&(T-t)^{-2+\frac{d}{2}}\left((T-t)^{-\frac{d}{2}-1}+(T-t)^{-\frac{d}{2}}\right)D^3+\left((T-t)^{-3}+(T-t)^{-2}\right)D^{\frac{4}{d}+2}+(T-t)^{-2}D^2\\
\leq&(T-t)^{-3}D^3+(T-t)^{-3}D^{2+\frac{4}{d}}+(T-t)^{-2}D^2
\end{aligned}
$$

$$\leq (T-t)^{-2} D^2. \tag{3.108}$$

Finally, the last inner product involving η can be bounded easier than the previous $\mathscr{I}^{(1)}_{t,7}$ in $\mathscr{I}^{(1)}$. By (2.70),

$$\begin{aligned}
&\left|\frac{\gamma_k}{2\lambda_k}\mathrm{Re}\langle\nabla\chi_A(\frac{x-\alpha_k}{\lambda_k})\cdot(\nabla R_k+\nabla R\Phi_k),\eta\rangle\right| \\
\leq& CA\|R\|_{H^1}\|\eta\|_{L^2} \\
\leq& CA\left((T-t)^{-3}Mod+\alpha^*(T-t)^{m-2+\frac{d}{2}}+(T-t)^{\upsilon_*-2}\right)D.
\end{aligned} \tag{3.109}$$

Hence, we conclude from (3.103), (3.106), (3.108) and (3.109) that (3.101) holds.

Now, putting the estimates (3.92), (3.93) and (3.101) altogether and using the renormalized variable ε_k in (2.65) we arrive at

$$\begin{aligned}
\frac{d\mathscr{I}^{(2)}}{dt} =& -\sum_{k=1}^K\frac{\gamma_k}{4\lambda_k^4}\int\Delta^2\chi_A(y)|\varepsilon_k|^2dy+\sum_{k=1}^K\frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int\nabla^2\chi_A(y)(\nabla\varepsilon_k,\nabla\overline{\varepsilon_k})dy \\
&-\sum_{k=1}^K\mathrm{Re}\langle\frac{\gamma_k}{2\lambda_k^4}\Delta\chi_A(y)\varepsilon_k+\frac{\gamma_k}{\lambda_k^4}\nabla\chi_A(y)\cdot\nabla\varepsilon_k, f'(Q_k)\cdot\varepsilon_k\rangle+O(A\mathcal{E}_r'),
\end{aligned}$$

where $\mathcal{E}_r'$ is given by (3.90). Taking into account the identity

$$\begin{aligned}
&-\sum_{k=1}^K\mathrm{Re}\langle\frac{\gamma_k}{2\lambda_k^4}\Delta\chi_A(\frac{x-\alpha_k}{\lambda_k})\varepsilon_k+\frac{\gamma_k}{\lambda_k^4}\nabla\chi_A(\frac{x-\alpha_j}{\lambda_k})\cdot\nabla\varepsilon_k, f'(Q_k)\cdot\varepsilon_k\rangle \\
=&\sum_{k=1}^K\frac{\gamma_k}{\lambda_k^4}\mathrm{Re}\int\nabla\chi_A(y)\cdot\nabla\overline{Q_k}\,f''(Q_k)\cdot\varepsilon_k^2dy,
\end{aligned} \tag{3.110}$$

we thus obtain (3.89), thereby finishing the proof of Lemma 3.7. □

We are now in position to prove Theorem 3.5.

Proof of Theorem 3.5. Combining (3.58) and (3.89) altogether and then using the renormalized variable ε_k in (2.65) we obtain

$$\begin{aligned}
\frac{d\mathscr{I}}{dt}\geq&\sum_{k=1}^K\frac{\gamma_k}{\lambda_k^4}\Big(\int\nabla^2\chi_A(y)(\nabla\varepsilon_k,\nabla\overline{\varepsilon_k})dy+\int|\varepsilon_k|^2dy \\
&-\int(1+\frac{4}{d})|Q_k|^{\frac{4}{d}}\varepsilon_{k,1}^2+|Q_k|^{\frac{4}{d}-2}\overline{Q_k}^2\varepsilon_{k,2}^2dy-\frac{1}{4}\int\Delta^2\chi_A(y)|\varepsilon_k|^2dy \\
&+\mathrm{Re}\int(\nabla\chi_A(y)-y)\cdot\nabla\overline{Q_k}\,(f''(Q_k)\cdot\varepsilon_k^2)dy\Big)-CA\mathcal{E}_r,
\end{aligned} \tag{3.111}$$

where $\varepsilon_{k,1}=\mathrm{Re}\varepsilon_k$ and $\varepsilon_{k,2}=\mathrm{Im}\varepsilon_k$, $1\leq k\leq K$.

Then, arguing as in the proof of [11, (3.83)] we obtain that for A large enough,

$$\frac{d\mathscr{I}}{dt}\geq C\sum_{k=1}^K\frac{\gamma_k}{\lambda_k^4}\int|\nabla\varepsilon_k|^2e^{-\frac{|y|}{A}}+|\varepsilon_k|^2dy+O\left(\sum_{k=1}^K\frac{\gamma_k}{\lambda_k^4}Scal(\varepsilon_k)+\mathcal{E}_r\right). \tag{3.112}$$

Thus, using the inequality (see [11, (3.87)])

$$\sum_{k=1}^K Scal(\varepsilon_k)\leq C\sum_{k=1}^K\left(M_k^2+\|R\|_{L^2}^4+P^2\|R\|_{L^2}^2+e^{-\frac{\delta}{T-t}}\right) \tag{3.113}$$

we arrive at (3.56). Therefore, the proof is complete. □

Below we will fix a large constant A such that Theorem 3.5 is valid.

4. Construction of multi-bubble Bourgain-Wang solutions

In this section we construct the multi-bubble Bourgain-Wang solutions to (1.1) and derive several properties which will be used in the conditional uniqueness part in Section 5 later.

Throughout this section, we will take ε, α^* sufficiently small and t close to T such that

$$C\left((\varepsilon+\alpha^*)^{\frac{1}{2}} + (1+\max_{1\leq k\leq K}|x_k|)(T-t)^{\frac{d}{8+4d}}\right) \leq \frac{1}{2}, \tag{4.1}$$

where C is a universal constant, independent of ε, α^* and larger than the constants in the estimates in this section. Let us mention that, the exponent $d/(8+4d)$ is used in the derivation of (4.64) below. For the construction of blow-up solutions, it will be sufficient to take the exponent $1/4$.

Let us start with the bootstrap estimates of the remainder and geometrical parameters, which are the key towards the derivation of uniform estimates of solutions.

4.1. Bootstrap estimates.

Given any $\upsilon_* \geq 5$, $m \geq 3$ if $d = 2$ and $m \geq 4$ if $d = 1$, set

$$\kappa := (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2). \tag{4.2}$$

Note that $\kappa \geq 3$.

Proposition 4.1. *(Bootstrap estimates) Suppose that there exists $t^* \in (0, T_*)$ such that u admits the unique geometrical decomposition* (2.1) *on $[t^*, T_*]$ and the following estimates hold:*

(i) *For the remainder,*

$$\|R(t)\|_{L^2} \leq (T-t)^{\kappa+1}, \ \ \|\nabla R(t)\|_{L^2} \leq (T-t)^{\kappa}. \tag{4.3}$$

(ii) *For the modulation parameters, $1 \leq k \leq K$,*

$$|\lambda_k(t) - w_k(T-t)| + |\gamma_k(t) - w_k^2(T-t)| \leq (T-t)^{\kappa}, \tag{4.4}$$

$$|\alpha_k(t) - x_k| + |\beta_k(t)| \leq (T-t)^{\frac{\kappa}{2}+\frac{1}{2}}, \tag{4.5}$$

$$|\theta_k(t) - (w_k^{-2}(T-t)^{-1} + \vartheta_k)| \leq (T-t)^{\kappa-2}. \tag{4.6}$$

Then, there exists $t_* \in [0, t^*)$ such that the decomposition (2.1) *and the following improved estimates hold on the larger interval $[t_*, T_*]$: for $1 \leq k \leq K$,*

$$\|R(t)\|_{L^2} \leq \frac{1}{2}(T-t)^{\kappa+1}, \quad \|\nabla R(t)\|_{L^2} \leq \frac{1}{2}(T-t)^{\kappa}, \tag{4.7}$$

$$|\lambda_k(t) - w_k(T-t)| + |\gamma_k(t) - w_k^2(T-t)| \leq \frac{1}{2}(T-t)^{\kappa}, \tag{4.8}$$

$$|\alpha_k(t) - x_k| + |\beta_k(t)| \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+\frac{1}{2}}, \tag{4.9}$$

$$|\theta_k(t) - (w_k^{-2}(T-t)^{-1} + \vartheta_k)| \leq \frac{1}{2}(T-t)^{\kappa-2}. \tag{4.10}$$

Remark 4.2. *Since $\kappa \geq 3$,*

$$\lambda_k, \gamma_k, P \approx (T-t), \ \ |\beta_k| + |\alpha_k - x_k| + D = O((T-t)^2), \tag{4.11}$$

where the implicit constants are independent of ε, α^*. Hence, the results in the previous Sections 2 and 3 are all valid. Moreover, since $\kappa = (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$, we have

$$(T-t)^{m+\frac{d}{2}} + (T-t)^{\upsilon_*-1} \leq C(T-t)^{\kappa+1}. \tag{4.12}$$

In order to prove Proposition 4.1, by the continuity of Jacobian matrix, the local well-posedness theory of (1.1) and C^1-regularity of modulation parameters, we may take $t_*(< t^*)$ sufficiently close to t^*, such that the geometrical decomposition (2.1) and the following estimates hold on the larger interval $[t_*, T_*]$:

$$\|R(t)\|_{L^2} \leq 2(T-t)^{\kappa+1}, \quad \|\nabla R(t)\|_{L^2} \leq 2(T-t)^{\kappa}, \tag{4.13}$$

$$|\lambda_k(t) - w_k(T-t)| + |\gamma_k(t) - w_k^2(T-t)| \leq 2(T-t)^{\kappa}, \tag{4.14}$$

$$|\alpha_k(t) - x_k| + |\beta_k(t)| \leq 2(T-t)^{\frac{\kappa}{2}+\frac{1}{2}}, \tag{4.15}$$

$$|\theta_k(t) - (w_k^{-2}(T-t)^{-1} + \vartheta_k)| \leq 2(T-t)^{\kappa-2}. \tag{4.16}$$

By virtue of Theorem 2.2, 3.1, 3.4 and 3.5 we obtain

Lemma 4.3. *There exists $C > 0$ such that for any $t \in [t_*, T_*]$,*

$$M_k \leq C\alpha^*(T-t)^{\kappa+1}, \tag{4.17}$$

$$Mod \leq C\alpha^*(T-t)^{\kappa+1}, \tag{4.18}$$

$$|\lambda_k\dot{\lambda}_k + \gamma_k| \leq C(T-t)^{\kappa+2}, \tag{4.19}$$

and for the errors Er and $\mathcal{E}_r$ in (3.31) *and* (3.57)*, respectively,*

$$|Er| \leq C\alpha^*(T-t)^{\kappa-1}, \tag{4.20}$$

$$|\mathcal{E}_r| \leq C(\varepsilon + \alpha^*)(T-t)^{2\kappa-1} + C(T-t)^{2\kappa}. \tag{4.21}$$

Remark 4.4. *In comparison with* (4.18)*, one more factor* $(T-t)$ *is gained in* (4.19) *for the particular modulation equation $\lambda_k\dot{\lambda}_k + \gamma_k$. This fact is important to derive* (4.21) *and to close the bootstrap estimates of remainder.*

We are now in position to prove the bootstrap estimates in Proposition 4.1.

Proof of Proposition 4.1. *(i) Estimate of R.* On one hand, similarly to (6.15),

$$|F''(U+z,R)\cdot R^2 - F''(U,R)\cdot R^2| \leq C\left(|U|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1}\right)|z||R|^2, \tag{4.22}$$

we see that

$$\begin{aligned}
&\left|\int F''(U+z,R)\cdot R^2 dx - \int F''(U,R)\cdot R^2 dx\right| \\
\leq & C\int \left(|U|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1}\right)|z||R|^2 dx \\
\leq & C\left((T-t)^{-\frac{d}{2}(\frac{4}{d}-1)}\|z\|_{L^\infty}\|R\|_{L^2}^2 + \|z\|_{L^\infty}^{\frac{4}{d}}\|R\|_{L^2}^2 + \|z\|_{L^\infty}\|R\|_{L^{1+\frac{4}{d}}}^{1+\frac{4}{d}}\right) \\
\leq & C\left(\alpha^*(T-t)^{-2+\frac{d}{2}}D^2 + \alpha^*(T-t)^{-d(\frac{2}{d}-\frac{1}{2})}D^{1+\frac{4}{d}}\right) \\
= & o\left((T-t)^{-2}D^2\right).
\end{aligned} \tag{4.23}$$

Taking into account $F''(U,R)\cdot R^2 = F(U+R) - F(U) - \mathrm{Re}(f(U)\overline{R})$ we thus get

$$\begin{aligned}
\mathscr{I} = & \frac{1}{2}\int |\nabla R|^2 + \frac{1}{2}\sum_{k=1}^{K}\int \frac{1}{\lambda_k^2}|R|^2\Phi_k dx - \mathrm{Re}\int F(U+R) - F(U) - f(U)\overline{R}dx \\
& + \sum_{k=1}^{K}\frac{\gamma_k}{2\lambda_k}\mathrm{Im}\int \nabla\chi_A\left(\frac{x-\alpha_k}{\lambda_k}\right)\cdot\nabla R\overline{R}\Phi_k dx + o\left((T-t)^{-2}D^2\right).
\end{aligned} \tag{4.24}$$

Then, we use the expansion

$$F(U+R)-F(U)-\mathrm{Re}(f(U)\overline{R})=\frac{1}{2}(1+\frac{2}{d})|U|^{\frac{4}{d}}|R|^2+\frac{1}{d}|U|^{\frac{4}{d}-2}\mathrm{Re}(U^2\overline{R}^2)+O\left(\sum_{j=3}^{2+\frac{4}{d}}|U|^{2+\frac{4}{d}-j}|R|^j\right)$$

to derive

$$\begin{aligned}\mathscr{I}=&\frac{1}{2}\mathrm{Re}\int|\nabla R|^2+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}|R|^2\Phi_k-(1+\frac{2}{d})|U|^{\frac{4}{d}}|R|^2-\frac{2}{d}|U|^{\frac{4}{d}-2}U^2\overline{R}^2dx\\&+O\left(\sum_{j=3}^{2+\frac{4}{d}}\int|U|^{\frac{4}{d}+2-j}|R|^jdx+\|R\|_{L^2}\|\nabla R\|_{L^2}\right)+o\left((T-t)^{-2}D^2\right).\end{aligned}\tag{4.25}$$

Note that, the last second line on the R.H.S. above is of order $o((T-t)^{-2}D^2)$, see [11, (4.29)], while for the quadratic terms the following coercivity type estimate holds (see [11, (3.39)]):

$$\begin{aligned}&\frac{1}{2}\mathrm{Re}\int|\nabla R|^2+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}|R|^2\Phi_k-(1+\frac{2}{d})|U|^{\frac{4}{d}}|R|^2-\frac{2}{d}|U|^{\frac{4}{d}-2}U^2\overline{R}^2dx\\\geq&C\frac{D^2(t)}{(T-t)^2}+O\left(\sum_{k=1}^{K}\frac{M_k^2}{(T-t)^2}+e^{-\frac{\delta}{T-t}}\right),\end{aligned}\tag{4.26}$$

where $C>0$. Thus, by (4.25) and (4.26), for t close to T,

$$\mathscr{I}\geq\frac{C}{2}\frac{D^2}{(T-t)^2}-C\left(\sum_{k=1}^{K}\frac{M_k^2}{(T-t)^2}+e^{-\frac{\delta}{T-t}}\right).\tag{4.27}$$

On the other hand, Theorem 3.5 yields that for any $t\in[t_*,T_*]$,

$$\frac{d\mathscr{I}}{dt}\geq-C\mathcal{E}_r.\tag{4.28}$$

Thus, we infer from (4.27), (4.28) and the boundary condition $\mathscr{I}(T_*)=0$ that for any $t\in[t_*,T_*]$,

$$\frac{D^2}{(T-t)^2}\leq C\left(\int_t^{T_*}|\mathcal{E}_r|ds+\sum_{k=1}^{K}\frac{M_k^2}{(T-t)^2}+e^{-\frac{\delta}{T-t}}\right),$$

or, equivalently,

$$D\leq C\left((T-t)\left(\int_t^{T}|\mathcal{E}_r|ds\right)^{\frac{1}{2}}+\sum_{k=1}^{K}|M_k|+e^{-\frac{\delta}{T-t}}\right).\tag{4.29}$$

Taking into account (4.17) and (4.21) we then obtain

$$\begin{aligned}D\leq&C\left((\varepsilon+\alpha^*)^{\frac{1}{2}}(T-t)^{\kappa+1}+(T-t)^{\kappa+\frac{3}{2}}+\alpha^*(T-t)^{\kappa+1}\right)\\\leq&C\left(\varepsilon+\alpha^*+(T-t)\right)^{\frac{1}{2}}(T-t)^{\kappa+1},\end{aligned}\tag{4.30}$$

which along with (4.1) yields

$$D\leq\frac{1}{2}(T-t)^{\kappa+1}.\tag{4.31}$$

Thus, estimate (4.7) is verified.

(ii) *Estimates of λ_k and γ_k.* By (4.18),

$$\left|\frac{d}{dt}\left(\frac{\gamma_k}{\lambda_k}\right)\right| = \frac{|\lambda_k^2\dot{\gamma}_k - \lambda_k\dot{\lambda}_k\gamma_k|}{\lambda_k^3} \leq 2\frac{Mod}{\lambda_k^3} \leq C\alpha^*(T-t)^{\kappa-2}, \tag{4.32}$$

which along with the boundary condition $(\frac{\gamma_k}{\lambda_k})(T_*) = w_k$ yields that

$$\left|\frac{\gamma_k}{\lambda_k} - w_k\right| \leq \int_t^{T_*}\left|\frac{d}{dr}\left(\frac{\gamma_k}{\lambda_k}\right)\right|dr \leq C\alpha^*(T-t)^{\kappa-1}. \tag{4.33}$$

This in turn yields that

$$\left|\frac{d}{dt}(\lambda_k - w_k(T-t))\right| = \left|\dot{\lambda}_k + \frac{\gamma_k}{\lambda_k} + w_k - \frac{\gamma_k}{\lambda_k}\right| \leq \frac{Mod}{\lambda_k} + C\alpha^*(T-t)^{\kappa-1} \leq C\alpha^*(T-t)^{\kappa-1},$$

and thus, by (4.1),

$$\begin{aligned}|\lambda_k - w_k(T-t)| \leq& \int_t^{T_*}\left|\frac{d}{dr}(\lambda_k - w_k(T-r))\right|dr \\ \leq& C\alpha^*(T-t)^{\kappa} \leq \frac{1}{2}(T-t)^{\kappa}.\end{aligned} \tag{4.34}$$

Hence, we prove the estimate of λ_k in (4.8).

Regarding γ_k, by (4.18) and (4.33),

$$\left|\frac{d}{dt}(\gamma_k - w_k^2(T-t))\right| = \left|\dot{\gamma}_k + \frac{\gamma_k^2}{\lambda_k^2} + w_k^2 - \frac{\gamma_k^2}{\lambda_k^2}\right| \leq \frac{Mod}{\lambda_k^2} + C\left|w_k - \frac{\gamma_k}{\lambda_k}\right| \leq C\alpha^*(T-t)^{\kappa-1}.$$

Thus, taking into account $\gamma_k(T_*) = \omega_k^2(T-T_*)$ and (4.1) we get

$$|\gamma_k(t) - w_k^2(T-t)| \leq \int_t^{T_*}\left|\frac{d}{dr}(\gamma_k(r) - w_k^2(T-r))\right|dr \leq C\alpha^*(T-t)^{\kappa} \leq \frac{1}{2}(T-t)^{\kappa}. \tag{4.35}$$

This gives the estimate of γ_k in (4.8).

(iii) *Estimates of β_k and α_k.* By the improved estimate (3.30), (4.20) and (4.33),

$$\frac{|\beta_k|^2}{\lambda_k^2} \leq C\left(\sum_{k=1}^{K}\left|w_k - \frac{\gamma_k}{\lambda_k}\right| + Er\right) \leq C\alpha^*(T-t)^{\kappa-1}, \tag{4.36}$$

which along with (4.1) yields that

$$|\beta_k| \leq C(\alpha^*)^{\frac{1}{2}}(T-t)^{\frac{\kappa}{2}+\frac{1}{2}} \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+\frac{1}{2}}. \tag{4.37}$$

Moreover, by (4.18) and (4.37),

$$|\dot{\alpha}_k| = \left|\frac{\lambda_k\dot{\alpha}_k - 2\beta_k}{\lambda_k} + \frac{2\beta_k}{\lambda_k}\right| \leq \frac{Mod}{\lambda_k} + \frac{2|\beta_k|}{\lambda_k} \leq C\alpha^*(T-t)^{\frac{\kappa}{2}-\frac{1}{2}}. \tag{4.38}$$

Integrating both sides and using (4.1) and the boundary condition $\alpha_k(T_*) = x_k$ we get

$$|\alpha_k(t) - x_k| \leq \int_t^{T_*}|\dot{\alpha}_k(r)|dr \leq C\alpha^*(T-t)^{\frac{\kappa}{2}+\frac{1}{2}} \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+\frac{1}{2}}, \tag{4.39}$$

thereby proving the estimate of α_k in (4.9).

(iv) *Estimate of θ_k.* It remains to estimate θ_k. By (4.11), (4.18), (4.34) and (4.36),

$$\left|\frac{d}{dt}(\theta_k - w_k^{-2}(T-t)^{-1} + \vartheta_k)\right| = \left|\frac{\lambda_k^2\dot{\theta}_k - 1 - |\beta_k|^2}{\lambda_k^2} + \frac{|\beta_k|^2}{\lambda_k^2} + \frac{1}{\lambda_k^2} - \frac{1}{w_k^2(T-t)^2}\right|$$

$$\leq \frac{Mod}{\lambda_k^2} + \frac{|\beta_k|^2}{\lambda_k^2} + \frac{|\lambda_k - w_k(T-t)||\lambda_k + w_k(T-t)|}{w_k^2\lambda_k^2(T-t)^2}$$
$$\leq C\alpha^*(T-t)^{\kappa-3}, \tag{4.40}$$

which along with (4.1) and the boundary $\theta_k(T_*) = w_k^{-2}(T-T_*)^{-1} + \vartheta_k$ yields that

$$|\theta_k - (w_k^{-2}(T-t)^{-1} + \vartheta_k)| \leq \int_t^{T_*} \left|\frac{d}{dr}(\theta - w_k^{-2}(T-r)^{-1} + \vartheta_k)\right| dr$$
$$\leq C\alpha^*(T-t)^{\kappa-2} \leq \frac{1}{2}(T-t)^{\kappa-2}. \tag{4.41}$$

Hence, the estimate (4.10) is verified. Therefore, the proof of Proposition 4.1 is complete. □

4.2. **Proof of existence.** We are now in position to prove the existence part in Theorem 1.2. Consider the approximating solutions v_n satisfying the equation

$$\begin{cases} i\partial_t v_n + \Delta v_n + a_1 \cdot \nabla v_n + a_0 v_n + |v_n|^{\frac{4}{d}} v_n = 0, \\ v_n(t_n) = \sum_{k=1}^{K} S_k(t_n) + z(t_n), \end{cases} \tag{4.42}$$

where $\{t_n\}$ is any increasing sequence converging to T, the coefficients a_1, a_0 are given by (1.2) and (1.3), respectively, $\{S_k\}$ are the pseudo-conformal blow-up solutions defined in (1.20), and z solves equation (1.21).

As a consequence of bootstrap estimates, we have the key uniform estimates below.

Lemma 4.5. *(Uniform estimates) There exists $t_* \in [0, T)$ such that for n large enough, v_n admits the unique geometrical decomposition $v_n = U_n + z + R_n$ as in* (2.1)*, with the parameters $\mathcal{P}_{n,k} := (\lambda_{n,k}, \alpha_{n,k}, \beta_{n,k}, \gamma_{n,k}, \theta_{n,k})$, $1 \leq k \leq K$, and the estimates* (4.3)-(4.6) *hold on* $[t_*, t_n]$. *Moreover, there exists $C > 0$ such that*

$$\sup_n \|R_n(t)\|_\Sigma \leq C(T-t)^\kappa, \tag{4.43}$$

and

$$\sup_n \|xv_n\|_{C([t_*,t_n];L^2)} \leq C(1 + \max_{1\leq k\leq K} |x_k|)^2, \tag{4.44}$$

Proof. The proof of the existence of a universal time t_* and uniform estimates (4.3)-(4.6) is similar to that of [63, Theorem 5.1], mainly based on the bootstrap estimates in Proposition 4.1 and bootstrap arguments (see, e.g., [66, Proposition 1.21]). Thus, the details are omitted here for simplicity. Below let us mainly prove estimates (4.43) and (4.44).

Let $M := 1 + \max_{1\leq k\leq K} |x_k|$. Let $\varphi(x) \in C^1(\mathbb{R}^d, \mathbb{R})$ be a radial cutoff function such that $\varphi(x) = 0$ for $|x| \leq r$, and $\varphi(x) = (|x| - r)^2$ for $|x| > r$, where $r = 2\max_{1\leq k\leq K}\{|x_k|, 1\}$. Note that, $|\nabla\varphi| \leq C\varphi^{\frac{1}{2}}$ for a universal constant $C > 0$.

Let $w_n := U_n + R_n$, $n \geq 1$. Then, $v_n = w_n + z$. By equations (4.42) and (1.21), w_n solves equation

$$\begin{cases} i\partial_t w_n + \Delta w_n + a_1 \cdot \nabla w_n + a_0 w_n + f(v_n) - f(z) = 0, \\ w_n(t_n) = \sum_{k=1}^{K} S_k(t_n) \ (=: S(t_n)). \end{cases} \tag{4.45}$$

Then, by the integration by parts formula and $\mathrm{Im}\, w_n\overline{f(w_n)} = 0$,

$$\begin{aligned}\frac{d}{dt}\int |w_n|^2\varphi dx &= \mathrm{Im}\int (2\overline{w_n}\nabla w_n + a_1|w_n|^2)\cdot\nabla\varphi + 2w_n(\overline{f(v_n)} - \overline{f(w_n)} - \overline{f(z)})\varphi dx \\ &= \mathrm{Im}\int (2\overline{w_n}\nabla w_n + a_1|w_n|^2)\cdot\nabla\varphi dx + O\left(\sum_{j=1}^{4/d}\int |w_n^{2+\frac{4}{d}-j}z^j\varphi| dx\right). \end{aligned} \tag{4.46}$$

In order to estimate the R.H.S. of (4.46), we note that

$$\begin{aligned}&\left|\mathrm{Im}\int (2\overline{w_n}\nabla w_n + a_1|w_n|^2)\cdot\nabla\varphi dx\right| \\ \leq& C\int_{|x-x_k|\geq 1, 1\leq k\leq K}\left(|w_n||\nabla w_n| + |w_n|^2\right)\varphi^{\frac12}dx \\ \leq& C\left(\left(\int_{|x-x_k|\geq 1,1\leq k\leq K}|\nabla w_n|^2dx\right)^{\frac12} + \left(\int_{|x-x_k|\geq 1,1\leq k\leq K}|w_n|^2dx\right)^{\frac12}\right)\left(\int |w_n|^2\varphi dx\right)^{\frac12}, \end{aligned} \tag{4.47}$$

where $C>0$ is independent of n. By (2.1), (4.3) and (1.7),

$$\left|\int_{|x-x_k|\geq 1,1\leq k\leq K}|w_n(t)|^2 + |\nabla w_n(t)|^2dx\right| \leq C(\|R_n(t)\|_{H^1}^2 + e^{-\frac{\delta}{T-t}}) \leq C(T-t)^{2\kappa}. \tag{4.48}$$

This yields that for a universal constant $C>0$,

$$\left|\mathrm{Im}\int (2\overline{w_n}\nabla w_n + a_1|w_n|^2)\cdot\nabla\varphi dx\right| \leq C(T-t)^{\kappa}\left(\int |w_n|^2\varphi dx\right)^{\frac12}. \tag{4.49}$$

Moreover, for $1\leq j\leq \frac{4}{d}$, since $\mathrm{supp}\varphi\subseteq\{x: |x-x_k|\geq 1, 1\leq k\leq K\}$,

$$\begin{aligned}\int |w_n^{2+\frac4d-j}z^j\varphi|dx \leq& \left(\int |w_n|^2\varphi dx\right)^{\frac12}\left(\int |w_n|^{2+\frac8d-2j}|z|^{2j}\varphi dx\right)^{\frac12} \\ \leq& C\left(\int |w_n|^2\varphi dx\right)^{\frac12}\left(\int \left(|U_n|^{2+\frac8d-2j} + |R_n|^{2+\frac8d-2j}\right)|z|^{2j}\varphi dx\right)^{\frac12} \\ \leq& C\left(\int |w_n|^2\varphi dx\right)^{\frac12}\left(\int |R_n|^{2+\frac8d-2j}|z|^{2j}\varphi dx + M^2e^{-\frac{\delta}{T-t}}\right)^{\frac12} \\ \leq& C\left(\int |w_n|^2\varphi dx\right)^{\frac12}\left(\|R_n\|_{L^{2(2+\frac8d-2j)}}^{2+\frac8d-2j}\|xz\|_{L^4}^2\|z\|_{L^\infty}^{2j-2} + M^2e^{-\frac{\delta}{T-t}}\right), \end{aligned}$$

which along with (2.25), (2.28), (4.1) and (4.7) yields that for a universal constant $C>0$,

$$\int |w_n^{2+\frac4d-j}z^j\varphi|dx \leq C(T-t)^{2\kappa}\left(\int |w_n|^2\varphi dx\right)^{\frac12}. \tag{4.50}$$

Hence, plugging (4.49) and (4.50) into (4.46) we get

$$\left|\frac{d}{dt}\int |w_n(t)|^2\varphi dx\right| \leq C(T-t)^{\kappa}\left(\int |w_n(t)|^2\varphi dx\right)^{\frac12}. \tag{4.51}$$

Thus, integrating (4.51) from t to t_n, using (4.1) and the boundary estimate

$$\int |w_n(t_n)|^2 \varphi dx = \int \Big| \sum_{k=1}^{K} S_k(t_n) \Big|^2 \varphi dx \leq CM^2 e^{-\frac{\delta}{T-t_n}} \leq CM^2 e^{-\frac{\delta}{T-t}} \tag{4.52}$$

we obtain for $t \in [0, t_n]$,

$$\int |w_n(t)|^2 \varphi dx \leq C(T-t)^{2\kappa+2}. \tag{4.53}$$

In particular, this yields that

$$\int |R_n(t)|^2 \varphi dx \leq C\left(\int |U_n(t)|^2 \varphi dx + \int |w_n(t)|^2 \varphi dx \right) \leq C(T-t)^{2\kappa+2}. \tag{4.54}$$

Since $\varphi(x) \geq \frac{1}{4}|x|^2$ for $|x| \geq 4M$, by (4.1), (4.7) and (4.54),

$$\int |xR_n(t)|^2 dx \leq C\left(\int |R_n(t)|^2 \varphi dx + M^2 \int |R_n(t)|^2 dx \right) \leq CM^2 (T-t)^{2\kappa+2} \leq C(T-t)^{2\kappa+1}, \tag{4.55}$$

where C is independent of n and M. This along with (4.3) yields (4.43).

Similarly, we derive that

$$\begin{aligned} \int |xv_n(t)|^2 dx \leq& C\left(\int |xw_n|^2 dx + \|xz\|_{L^2} \right)^2 \\ \leq& C\left(\int |w_n(t)|^2 \varphi dx + M^2 \|w_n\|_{L^2}^2 + \|xz\|_{L^2}^2 \right) \\ \leq& C\left((T-t)^{2\kappa+2} + KM^2 \|Q\|_{L^2}^2 + M^2 (T-t)^{2\kappa+2} + \|xz\|_{L^2}^2 \right) \leq CM^2, \end{aligned} \tag{4.56}$$

where the last step is due to (2.28), (4.3), (4.53) and the conservation law of mass, and C is independent of n. This yields (4.44). Therefore, the proof of Lemma 4.5 is complete. □

Proof of existence part in Theorem 1.2. Let α^*, ε be small enough such that (4.1) holds and t_n as in Lemma 4.5. Let $M := 1 + \max_{1 \leq k \leq K} |x_k|$. By Lemma 4.5, $\{v_n(t_*)\}$ are uniformly bounded in Σ, and thus up to a subsequence (still denoted by $\{n\}$), $v_n(t_*)$ converges weakly to some $v_* \in \Sigma$. The weak convergence indeed can be enhanced to the strong one in the space L^2, i.e.,

$$v_n(t_*) \to v_*, \ \ in\ L^2,\ as\ n \to \infty. \tag{4.57}$$

This is due to the uniform integrability of $\{v_n(t_*)\}$ implied by the uniform estimate (4.44):

$$\sup_{n\geq 1} \|v_n(t_*)\|_{L^2(|x|>A)} \leq \frac{1}{A} \sup_{n\geq 1} \|xv_n(t_*)\|_{L^2(|x|>A)} \leq \frac{CM^2}{A} \to 0, \ \ as\ A \to \infty. \tag{4.58}$$

Thus, the L^2 local well-posedness theory (see, e.g. [3]) yields a unique L^2-solution v_c to (1.1) on $[t_*, T)$, satisfying that $v_c(t_*) = v_*$ and

$$\lim_{n\to\infty} \|v_n - v_c\|_{C([t_*,t];L^2)} = 0, \ \ t \in [t_*, T). \tag{4.59}$$

Moreover, since $v_* \in \Sigma$, the local well-posedness result also yields $v_c \in C([t_*, t]; \Sigma)$ for $t \in (t_*, T)$.

Next, we show that v_c is the desired multi-bubble Bourgain-Wang solution to (1.1).

As a matter of fact, let

$$(\lambda_{0,k}, \alpha_{0,k}, \beta_{0,k}, \gamma_{0,k}, \theta_{0,k}) := (w_k(T-t), x_k, 0, w_k^2(T-t), w_k^{-2}(T-t)^{-1} + \vartheta_k)$$

and $\mathcal{P}_{n,k} := (\lambda_{n,k}, \alpha_{n,k}, \beta_{n,k}, \gamma_{n,k}, \theta_{n,k})$ be the parameters corresponding to the geometrical decomposition of v_n. Then, analogous computations as in [63] and estimates (4.3)-(4.6) yield, for $\kappa \geq 1$,

$$\begin{aligned}\|U_n - S\|_{L^2} \leq & C\sum_{j=1}^{K}\left(\left|\frac{\lambda_{0,k}}{\lambda_{n,k}} - 1\right| + \left|\frac{\alpha_{n,k} - \alpha_{0,k}}{\lambda_{n,k}}\right| + \left|\beta_{n,k} - \beta_{0,k}\right| + \left|\gamma_{n,k} - \gamma_{0,k}\right|\right. \\ & \left. + \left|\frac{\lambda_{n,k}^{\frac{d}{2}} - \lambda_{0,k}^{\frac{d}{2}}}{\lambda_{0,k}^{\frac{d}{2}}}\right| + \left|\theta_{n,k} - \theta_{0,k}\right|\right) \\ \leq & C\left((T-t)^{\frac{1}{2}\kappa - \frac{1}{2}} + (T-t)^{\kappa - 2}\right) \\ \leq & C(T-t)^{\frac{1}{2}(\kappa - 1)}, \end{aligned} \tag{4.60}$$

which along with (4.59) yields that

$$\|v_c(t) - S(t) - z(t)\|_{L^2} \leq \lim_{n\to\infty} (\|U_n(t) - S(t)\|_{L^2} + \|R_n(t)\|_{L^2}) \leq C(T-t)^{\frac{1}{2}(\kappa-1)}. \tag{4.61}$$

Moreover, as in [64],

$$\begin{aligned}\|U_n - S\|_{\Sigma} \leq & CM\sum_{k=1}^{K}\left(\frac{1}{\lambda_{0,k}}\left|\frac{\lambda_{0,k}}{\lambda_{n,k}} - 1\right| + \left|\frac{\alpha_{n,k} - \alpha_{0,k}}{\lambda_{0,k}\lambda_{n,k}}\right| + \left|\frac{\beta_{n,k} - \beta_{0,k}}{\lambda_{0,k}}\right| + \left|\frac{\gamma_{n,k} - \gamma_{0,k}}{\lambda_{0,k}}\right|\right. \\ & \left. + \left|\frac{\lambda_{n,k}^{1+\frac{d}{2}} - \lambda_{0,k}^{1+\frac{d}{2}}}{\lambda_{n,k}\lambda_{0,k}^{1+\frac{d}{2}}}\right| + \left|\frac{\theta_{n,k} - \theta_{0,k}}{\lambda_{0,k}}\right|\right) \\ \leq & CM(T-t)^{\frac{1}{2}(\kappa-3)}, \end{aligned} \tag{4.62}$$

which, via (4.43), yields that

$$\|v_n(t) - S(t) - z(t)\|_{\Sigma} \leq \|U_n(t) - S(t)\|_{\Sigma} + \|R_n(t)\|_{\Sigma} \leq CM(T-t)^{\frac{1}{2}(\kappa-3)}. \tag{4.63}$$

Hence, possibly selecting a further subsequence (still denoted by $\{n\}$) and using (4.59) we obtain

$$v_n(t) - S(t) - z(t) \rightharpoonup v(t) - S(t) - z(t), \ \ \textit{weakly in } \Sigma, \ \textit{as } n \to \infty,$$

which yields that

$$\|v_c(t) - S(t) - z(t)\|_{\Sigma} \leq \liminf_{n\to\infty} \|v_n(t) - S_T(t) - z(t)\|_{\Sigma} \leq CM(T-t)^{\frac{1}{2}(\kappa-3)}.$$

Therefore, the proof of existence part in Theorem 1.2 is complete. □

4.3. **Further properties.**

We close this section with further properties of the constructed multi-bubble Bourgain-Wang solutions in Theorem 1.2, which will be used in Section 5 later.

Proposition 4.6. *($H^{\frac{3}{2}}$ boundedness) Consider the situations as in Theorem 1.2. Then,*

$$\|R_n(t)\|_{H^{\frac{3}{2}}} \leq (T-t)^{\kappa-2}, \ \ t \in [t_*, t_n), \tag{4.64}$$

where $\kappa := (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$.

Proof. Set $M := 1 + \max_{1\leq j\leq K} |x_j|$. Rewrite equation (2.67):

$$i\partial_t R_n + \Delta R_n + (a_1 \cdot \nabla + a_0)R_n = -\eta_n - f(R_n) - (f(v_n) - f(U_n + z) - f(R_n)), \tag{4.65}$$

where $R_n(t_n) = 0$ and η_n is given by (2.68). Applying $\langle\nabla\rangle^{\frac{3}{2}}$ to both sides of (4.65) yields

$$i\partial_t(\langle\nabla\rangle^{\frac{3}{2}}R_n) + \Delta(\langle\nabla\rangle^{\frac{3}{2}}R_n) + (a_1 \cdot \nabla + a_0)(\langle\nabla\rangle^{\frac{3}{2}}R_n)$$

$$=[a_1\cdot\nabla+a_0,\langle\nabla\rangle^{\frac32}]R_n-\langle\nabla\rangle^{\frac32}\eta_n-\langle\nabla\rangle^{\frac32}f(R_n)-\langle\nabla\rangle^{\frac32}\left(f(v_n)-f(U_n+z)-f(R_n)\right), \tag{4.66}$$

where $[a_1\cdot\nabla+a_0,\langle\nabla\rangle^{\frac32}]$ is the commutator $(a_1\cdot\nabla+a_0)\langle\nabla\rangle^{\frac32}-\langle\nabla\rangle^{\frac32}(a_1\cdot\nabla+a_0)$. Then, the Strichartz and local smoothing estimates yield

$$\begin{aligned}\|R_n\|_{C([t,t_n];H^{\frac32})}\le& C\Big(\|[a_1\cdot\nabla+a_0,\langle\nabla\rangle^{\frac32}]R_n\|_{L^2(t,t_n;H_1^{-\frac12})}+\|\langle\nabla\rangle^{\frac32}(f(R_n))\|_{L^{\frac{4+2d}{4+d}}(t,t_n;L^{\frac{4+2d}{4+d}})}\\ &+\|\langle\nabla\rangle^{\frac32}\eta_n\|_{L^{\frac{4+2d}{4+d}}(t,t_n;L^{\frac{4+2d}{4+d}})}+\|\langle\nabla\rangle^{\frac32}(f(u_n)-f(U_n+z)-f(R_n))\|_{L^2(t,t_n;H_1^{-\frac12})}\Big)\\ =:&\sum_{l=1}^4 J_l.\end{aligned} \tag{4.67}$$

To estimate the R.H.S. above, by the calculus of pseudo-differential operators, (1.13) and (4.7),

$$J_1\le C\|R_n\|_{L^2(t,t_n;H^1_{-1})}\le C(T-t)^{\frac12}\|R_n\|_{C([t,t_n];H^1)}\le C(T-t)^{\kappa+\frac12}. \tag{4.68}$$

Moreover, similarly to [64, (7.8)], by the product rule, Sobolev's embedding and (4.3),

$$\|\langle\nabla\rangle^{\frac32}(f(R_n))\|_{L^{\frac{4+2d}{4+d}}}\le C\|R_n\|_{H^1}^{\frac4d}\|R_n\|_{H^{\frac32}}\le C(T-t)^{\frac4d\kappa}\|R_n\|_{H^{\frac32}}, \tag{4.69}$$

which yields that

$$J_2\le C(T-t)^{\frac4d\kappa+\frac{4+d}{4+2d}}\|R_n\|_{C([t,t_n];H^{\frac32})}. \tag{4.70}$$

Regarding J_3, we use the decomposition $\eta_n=\sum_{l=1}^4\eta_l$ as in (2.71) to derive that for $p:=\frac{4+2d}{4+d}$ and any multi-index $|\upsilon|\le 2$, by (2.72) and (4.18),

$$\|\partial_x^\upsilon\eta_1\|_{L^p(t,t_n;L^p)}\le C\sum_{k=1}^K\lambda_k^{\frac1p}\lambda_k^{-2-|\upsilon|+d(\frac1p-\frac12)}Mod\le C\alpha^*(T-t)^{\kappa-2+\frac{d}{4+2d}}. \tag{4.71}$$

Moreover, by (2.73),

$$\|\partial_x^\upsilon\eta_2\|_{L^p}\le C(T-t)^{-4-\frac d2+\frac dp}\sum_{|\upsilon|\le2}\sum_{k=1}^K\|e^{-\delta|y|}\partial_y^\upsilon\varepsilon_{z,k}\|_{L^\infty}+Ce^{-\frac{\delta}{T-t}}, \tag{4.72}$$

which along with (2.29) and (4.12) yields

$$\begin{aligned}\|\partial_x^\upsilon\eta_2\|_{L^p(t,t_n;L^p)}\le& C(T-t)^{\frac1p-4+\frac{d}{2+d}}\sum_{|\upsilon|\le2}\sum_{k=1}^K\|e^{-\delta|y|}\partial_y^\upsilon\varepsilon_{z,k}\|_{L^\infty}+Ce^{-\frac{\delta}{T-t}}\\ \le& C\alpha^*(T-t)^{m-2+\frac2d+\frac{d}{4+2d}}\le C(T-t)^{\kappa-1}.\end{aligned} \tag{4.73}$$

Note that, because η_3 contains the interactions between different blow-up profiles, by Lemma 2.5,

$$\|\partial_x^\upsilon\eta_3\|_{L^p(t,t_n;L^p)}\le Ce^{-\frac{\delta}{T-t}}. \tag{4.74}$$

At last, by (2.75),

$$\partial_x^\upsilon\eta_4=\sum_{k=1}^K\lambda_k^{-\frac d2-|\upsilon|}\partial_y^\upsilon\left(\widetilde{a}_{1,k}\lambda_k^{-1}\cdot\nabla Q_k+\widetilde{a}_{0,k}Q_k\right)(\frac{x-\alpha_k}{\lambda_k}),$$

which by (2.58), (2.59) and (4.12) yields that

$$\|\partial_x^\upsilon\eta_4\|_{L^p(t,t_n;L^p)}\le C(T-t)^{\upsilon_*-2+\frac{d}{4+2d}}\le C(T-t)^{\kappa+\frac{d}{4+2d}}. \tag{4.75}$$

Hence, we conclude that

$$J_3 \leq \|\eta_n\|_{L^{\frac{4+2d}{4+d}}(t,t_n;H^{2,\frac{4+2d}{4+d}})} \leq C(T-t)^{\kappa-2+\frac{d}{4+2d}}. \tag{4.76}$$

It remains to estimate the last term J_4. We estimate

$$\begin{aligned} J_4 \leq & C\|\langle x\rangle(f(v_n)-f(U_n+z)-f(R_n))\|_{L^2(t,t_n;H^1)} \\ \leq & C\sum_{j=1}^{4/d}\Big(\|\langle x\rangle(|U_n|^{1+\frac{4}{d}-j}+|z|^{1+\frac{4}{d}-j})|R_n|^j\|_{L^2(t,t_n;L^2)} + \|\langle x\rangle(|\nabla U_n|+|\nabla z|)(|U_n|^{\frac{4}{d}-j}+|z|^{\frac{4}{d}-j})|R_n|^j\|_{L^2(t,t_n;L^2)} \\ & + \|\langle x\rangle(|U_n|^{1+\frac{4}{d}-j}+|z|^{1+\frac{4}{d}-j})|\nabla R_n||R_n|^{j-1}\|_{L^2(t,t_n;L^2)}\Big). \end{aligned} \tag{4.77}$$

Note that, by (2.25) and (2.28),

$$\|\langle x\rangle|z|^{1+\frac{4}{d}-j}|R_n|^j\|_{L^2(t,t_n;L^2)} \leq (T-t)^{\frac12}\|\langle x\rangle z\|_{L^2}\|z\|_{L^\infty}^{\frac{4}{d}-j}\|R\|^j_{C([t,t_n];H^1)} \leq C(T-t)^{\kappa j+\frac12}.$$

Since $\|\langle x\rangle\nabla U_n\|_{L^\infty} \leq CM(T-t)^{-\frac{d}{2}-1}$ and $\|\nabla U_n\|_{L^\infty} \leq C(T-t)^{-\frac{d}{2}}$, by (2.28) and (4.3),

$$\begin{aligned} & \|\langle x\rangle\left(|\nabla U_n||z|^{\frac{4}{d}-j}+|\nabla z||U_n|^{\frac{4}{d}-j}+|\nabla z||z|^{\frac{4}{d}-j}\right)|R_n|^j\|_{L^2(t,t_n;L^2)} \\ \leq & C\Big(M(T-t)^{-\frac{d}{2}-\frac12}\|z\|^{\frac{4}{d}-j}_{L^\infty(t,t_n;L^\infty)}\|R\|^j_{C([t,t_n];H^1)} + (T-t)^{-\frac{d}{2}(\frac{4}{d}-j)+\frac12}\|\langle x\rangle\nabla z\|_{L^\infty(t,t_n;H^1)}\|R\|^j_{C([t,t_n];H^1)} \\ & + (T-t)^{\frac12}\|\langle x\rangle\nabla z\|_{L^\infty(t,t_n;H^1)}\|z\|^{\frac{4}{d}-j}_{L^\infty(t,t_n;L^\infty)}\|R\|^j_{C([t,t_n];H^1)}\Big) \\ \leq & C\left(M(T-t)^{-\frac{d}{2}-\frac12+\kappa j}+(T-t)^{-2+\frac{d}{2}j+\frac12+\kappa j}+(T-t)^{\frac12+\kappa j}\right) \\ \leq & CM(T-t)^{\kappa-\frac32}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\langle x\rangle|z|^{1+\frac{4}{d}-j}|\nabla R_n||R_n|^{j-1}\|_{L^2(t,t_n;L^2)} \leq & C(T-t)^{\frac12}\|\langle x\rangle z\|_{L^\infty(t,t_n;H^1)}\|z\|^{\frac{4}{d}-j}_{L^\infty(t,t_n;L^\infty)}\|R\|^j_{C([t,t_n];H^1)} \\ \leq & C(T-t)^{\frac12+\kappa}. \end{aligned}$$

The remaining terms in (4.77) only involves U_n and R_n and can be bounded by, as in [64],

$$CM\left((T-t)^{\kappa-\frac32}+(T-t)^{\kappa-\frac32}\|R\|_{C([t,t_n];H^{\frac32})}\right).$$

Thus, we conclude that

$$J_4 \leq CM\left((T-t)^{\kappa-\frac32}+(T-t)^{\kappa-\frac32}\|R_n\|_{C([t,t_n];H^{\frac32})}\right). \tag{4.78}$$

Therefore, estimates (4.67), (4.68), (4.70), (4.76) and (4.78) altogether yield that

$$\|R_n\|_{C([t,t_n];H^{\frac32})} \leq CM\left((T-t)^{\kappa-2+\frac{d}{4+2d}}+(T-t)^{\kappa-\frac32}\|R_n\|_{C([t,t_n];H^{\frac32})}\right), \tag{4.79}$$

which along with (4.1) yields (4.64). □

As a consequence of Proposition 4.6 and the uniform estimates (4.3)-(4.6), the asymptotic behavior (1.19) can be taken in the more regular space $\dot{H}^{\frac32}$. Since the proof is similar to that of [64, Proposition 7.2], it is omitted here for simplicity.

Corollary 4.7. *Consider the situation as in Proposition 4.6 with $\upsilon_* \geq 6$, $m \geq 4$ if $d = 2$ and $m \geq 5$ if $d = 1$. Then, we have*

$$\|v_n(t) - S(t) - z(t)\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\frac{\kappa}{2}-2}, \tag{4.80}$$

where $\kappa = (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$. In particular, for the blow-up solution v_c constructed in Theorem 1.2, we have

$$\|v_c(t) - S(t) - z(t)\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\frac{\kappa}{2}-2}, \tag{4.81}$$

and the strong H^1 convergence holds: for any $t \in (t_*, T)$,

$$\|v_n - v_c\|_{C([t_*,t];H^1)} \to 0, \quad as\ n \to \infty. \tag{4.82}$$

The constructed blow-up solution v_c actually admits the geometrical decomposition on the existing time interval $[t_*, T)$, namely,

$$v_c(t,x) = \sum_{k=1}^{K} \lambda_k^{-\frac{d}{2}} Q_k(t, \frac{x-\alpha_k}{\lambda_k}) e^{i\theta_k} + z(t,x) + R(t,x) \quad (:= U(t,x) + z(t,x) + R(t,x)) \tag{4.83}$$

with

$$Q_k(t,y) := Q(y) e^{i(\beta_k(t)\cdot y - \frac{1}{4}\gamma_k(t)|y|^2)}, \tag{4.84}$$

the parameters $\mathcal{P} := \{\lambda, \alpha, \beta, \gamma, \theta\}$ are C^1 functions and the following orthogonality conditions hold on $[t_*, T)$: for $1 \leq k \leq K$,

$$\begin{aligned} &\mathrm{Re} \int (x-\alpha_k) U_k \overline{R} dx = 0, \quad \mathrm{Re} \int |x-\alpha_k|^2 U_k \overline{R} dx = 0, \\ &\mathrm{Im} \int \nabla U_k \overline{R} dx = 0, \quad \mathrm{Im} \int \Lambda U_k \overline{R} dx = 0, \quad \mathrm{Im} \int \varrho_k \overline{R} dx = 0. \end{aligned} \tag{4.85}$$

This fact is mainly due to the uniform estimate (4.18) of modulation equation, which ensures the equicontinuity of geometrical parameters $\{\mathcal{P}_n\}$ on every $[t_*, t_n]$, $n \geq 1$, and thus permits to take the limit procedure via the Arzelà-Ascoli Theorem. We refer to [64] for more details.

Hence, taking the limit $n \to \infty$ in the uniform estimates (4.3)-(4.6) and (4.64) we get the following estimates on $[t_*, T)$: for $1 \leq k \leq K$,

$$\|R(t)\|_{L^2} \leq (T-t)^{\kappa+1}, \quad \|R(t)\|_{H^1} \leq (T-t)^{\kappa}, \quad \|R(t)\|_{H^{\frac{3}{2}}} \leq (T-t)^{\kappa-2}, \tag{4.86}$$

$$|\lambda_k(t) - w_k(T-t)| + \left|\gamma_k(t) - w_k^2(T-t)\right| \leq (T-t)^{\kappa}, \tag{4.87}$$

$$|\alpha_k(t) - x_k| + |\beta_k(t)| \leq (T-t)^{\frac{\kappa}{2}+\frac{1}{2}}, \tag{4.88}$$

$$|\theta_k(t) - (w_k^{-2}(T-t)^{-1} + \vartheta_k)| \leq (T-t)^{\kappa-2}. \tag{4.89}$$

As a consequence, for any $t \in [t_*, T)$, λ_k, γ_k, P are comparable to $T-t$:

$$\lambda_k(t), \gamma_k(t), P(t) \approx T-t, \tag{4.90}$$

and

$$Mod(t) \leq C\alpha^*(T-t)^{\kappa+1}, \tag{4.91}$$

$$\|\partial_x^{\upsilon}\eta\|_{L^2} \leq C\alpha^*(T-t)^{\kappa-1-|\upsilon|}, \quad t \in [t_*, T), \ |\upsilon| \leq 2. \tag{4.92}$$

where $C > 0$ is a universal constant independent of ε, α^* and t.

5. Conditional uniqueness of multi-bubble Bourgain-Wang solutions

5.1. **Control of the difference.** In this subsection we assume Hypothesis $(H1)$ with $m \geq 10$, $\upsilon_* \geq 12$. Set $\kappa := (m + \frac{d}{2} - 1) \wedge (\upsilon_* - 2)$. Note that, $\kappa \geq 9 + \frac{d}{2}$.

Let v_c be the constructed multi-bubble Bourgain-Wang solution in Theorem 1.2, with the corresponding parameters $\mathcal{P} = (\lambda, \alpha, \beta, \gamma, \theta)$. Let v be any blow-up solution to (1.1) satisfying

$$\|v(t) - \sum_{k=1}^{K} S_k(t) - z(t)\|_{L^2} + (T-t)\|v(t) - \sum_{k=1}^{K} S_k(t) - z(t)\|_{H^1} \leq C(T-t)^{4+\zeta}, \quad t \in [t_*, T), \tag{5.1}$$

where ζ is any positive constant close to 0. Set

$$w := v - v_c = \sum_{k=1}^{K} w_k, \quad w_k := w\Phi_k, \ 1 \leq k \leq K, \tag{5.2}$$

where $\{\Phi_k\}$ are given by (2.63). Define the renormalized variable ϵ_k by

$$w_k(t,x) := \lambda_k(t)^{-\frac{d}{2}} \epsilon_k\Big(t, \frac{x - \alpha_k(t)}{\lambda_k(t)}\Big) e^{i\theta_k(t)}, \quad 1 \leq k \leq K. \tag{5.3}$$

Note that, ϵ_k is different from ε_k defined in (2.65). Similarly to (2.11), set

$$\widetilde{D}(t) := \|w(t)\|_{L^2} + (T-t)\|\nabla w(t)\|_{L^2}, \tag{5.4}$$

Then, by (4.86) and (5.1),

$$\|R(t)\|_{L^2} \leq (T-t)^{\kappa+1}, \ \|R(t)\|_{H^1} \leq (T-t)^{\kappa}, \ \|R(t)\|_{H^{\frac{3}{2}}} \leq (T-t)^{\kappa-2} \ \ \textit{with } \kappa \geq 9, \tag{5.5}$$

$$\widetilde{D}(t) \leq C(T-t)^{4+\zeta}, \tag{5.6}$$

and

$$\|w(t)\|_{L^p}^p \leq C(T-t)^{-d(\frac{p}{2}-1)} \widetilde{D}^p. \tag{5.7}$$

Moreover, by equations (1.1) and (5.6), w satisfies the equation

$$\begin{cases} i\partial_t w + \Delta w + a_1 \cdot \nabla w + a_0 w + f(v_c + w) - f(v_c) = 0, \quad t \in (t_*, T), \\ \lim\limits_{t \to T} \|w(t)\|_{H^1} = 0. \end{cases} \tag{5.8}$$

The crucial ingredient in the uniqueness proof is the following Lyapunov type functional, which is similar to the generalized energy $\mathscr{I}$ in (3.55),

$$\begin{aligned} \widetilde{\mathscr{I}} :=& \frac{1}{2}\int |\nabla w|^2 dx + \frac{1}{2}\sum_{k=1}^{K} \frac{1}{\lambda_k^2} \int |w|^2 \Phi_k dx - \mathrm{Re}\int F(v_c + w) - F(v_c) - f(v_c)\overline{w} dx \\ &+ \sum_{k=1}^{K} \frac{\gamma_k}{2\lambda_k} \mathrm{Im} \int (\nabla\chi_A)\Big(\frac{x-\alpha_k}{\lambda_k}\Big) \cdot \nabla w \overline{w} \Phi_k dx. \end{aligned} \tag{5.9}$$

Lemma 5.1. *There exist $C_1, C_2, C_3 > 0$ such that for $t \in [t_*, T)$,*

$$C_1(T-t)^{-2}\widetilde{D}^2 - C_2 \sum_{k=1}^{K} \frac{Scal_k}{\lambda_k^2} \leq \widetilde{\mathscr{I}} \leq C_3 A (T-t)^{-2} \widetilde{D}^2, \tag{5.10}$$

where

$$Scal_k(t) := \langle \epsilon_{k,1}, Q\rangle^2 + \langle \epsilon_{k,1}, yQ\rangle^2 + \langle \epsilon_{k,1}, |y|^2 Q\rangle^2 + \langle \epsilon_{k,2}, \nabla Q\rangle^2 + \langle \epsilon_{k,2}, \Lambda Q\rangle^2 + \langle \epsilon_{k,2}, \rho\rangle^2, \tag{5.11}$$

and $\epsilon_{k,1}, \epsilon_{k,2}$ are the real and imaginary parts of ϵ_k, respectively.

Proof. We first show that, the constructed blow-up solution v_c in (5.9) can be replaced by the blow-up profile U given by (4.83), up to the error $\mathcal{O}((T-t)^{-1}\widetilde{D}^2)$, i.e.,

$$\begin{aligned}\mathrm{Re}\int F(v_c+w)-F(v_c)-f(v_c)\overline{w}dx =&\mathrm{Re}\int F(U+w)-F(U)-f(U)\overline{w}dx\\ &+o\left((T-t)^{-1}\widetilde{D}^2\right).\end{aligned} \tag{5.12}$$

To this end, we note that

$$|F''(v_c,w)\cdot w^2-F''(U,w)\cdot w^2|\leq C\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|z+R||w|^2. \tag{5.13}$$

By (2.29), (4.86) and (5.6),

$$\begin{aligned}\int |U|^{\frac{4}{d}-1}|z+R||w|^2dx &\leq C(T-t)^{-2}\sum_{k=1}^K \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}\|w\|_{L^2}^2+C(T-t)^{-\frac{d}{2}(\frac{4}{d}-1)}\|R\|_{L^2}\|w\|_{H^1}^2+Ce^{-\frac{\delta}{T-t}}\|w\|_{L^2}^2\\ &\leq C\left(\alpha^*(T-t)^{m-1+\frac{d}{2}}\widetilde{D}^2+(T-t)^{\kappa-3+\frac{d}{2}}\widetilde{D}^2+e^{-\frac{\delta}{T-t}}\widetilde{D}^2\right)\\ &=o\left((T-t)^{-1}\widetilde{D}^2\right).\end{aligned} \tag{5.14}$$

Moreover, by (2.25), (4.86) and (5.6),

$$\begin{aligned}\int(|w|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1})|z+R||w|^2dx \leq& C\Big(\|z\|_{L^\infty}\|w\|_{H^1}^{1+\frac{4}{d}}+\|R\|_{L^2}\|w\|_{H^1}^{1+\frac{4}{d}}+\|z\|_{L^\infty}^{\frac{4}{d}}\|w\|_{L^2}^2\\ &+\|z\|_{L^\infty}^{\frac{4}{d}-1}\|R\|_{H^1}\|w\|_{H^1}^2+\|z\|_{L^\infty}\|R\|_{H^1}^{\frac{4}{d}-1}\|w\|_{H^1}^2+\|R\|_{H^1}^{\frac{4}{d}}\|w\|_{H^1}^2\Big)\\ =&o\left((T-t)^{-1}\widetilde{D}^2\right).\end{aligned} \tag{5.15}$$

Hence, (5.12) follows from (5.14) and (5.15), as claimed.

Next, for the R.H.S. of (5.12), note that,

$$\begin{aligned}\mathrm{Re}(F(U+w)-F(U)-f(U)\overline{w}) =&\frac{1}{2}(1+\frac{2}{d})|U|^{\frac{4}{d}}|w|^2+\frac{1}{d}|U|^{\frac{4}{d}-2}\mathrm{Re}(U^2\overline{w}^2)\\ &+\mathcal{O}\left((|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1})|w|^3\right).\end{aligned} \tag{5.16}$$

The error term above can be bounded by, via (5.6),

$$\int(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1})|w|^3dx \leq C\left((T-t)^{-2+\frac{d}{2}}\|w\|_{H^1}^3+\|w\|_{H^1}^{2+\frac{4}{d}}\right)\leq C(T-t)^{-1}\widetilde{D}^2. \tag{5.17}$$

Moreover, for the Morawetz type functional in (5.9),

$$\left|\frac{\gamma_k}{2\lambda_k}\mathrm{Im}\int(\nabla\chi_A)\left(\frac{x-\alpha_k}{\lambda_k}\right)\cdot\nabla w\overline{w}\Phi_k dx\right|\leq CA\|w\|_{L^2}\|\nabla w\|_{L^2}\leq CA(T-t)^{-1}\widetilde{D}^2(t). \tag{5.18}$$

Thus, we conclude from (5.12), (5.16)-(5.18) that

$$\widetilde{\mathscr{I}}=\frac{1}{2}\mathrm{Re}\int|\nabla w|^2+\sum_{k=1}^K\frac{1}{\lambda_k^2}|w|^2\Phi_k-(1+\frac{2}{d})|U|^{\frac{4}{d}}|w|^2-\frac{2}{d}|U|^{\frac{4}{d}-2}U^2\overline{w}^2dx+\mathcal{O}(A(T-t)^{-1}\widetilde{D}^2). \tag{5.19}$$

Now, on one hand, by Hölder's inequality, (5.19) and (5.4),

$$|\widetilde{\mathscr{I}}|\leq C\left(\|w\|_{H^1}^2+(T-t)^{-2}\|w\|_{L^2}^2+(T-t)^{-1}\widetilde{D}^2\right)\leq CA(T-t)^{-2}\widetilde{D}^2, \tag{5.20}$$

which yields the second inequality in (5.10).

On the other hand, the first inequality in (5.10) mainly follows from the coercivity type estimate below, which is similar to (4.26) mainly due to the local coercivity of linearized operators,

$$\widetilde{\mathscr{I}} \geq C_1(T-t)^{-2}\widetilde{D}^2 - C_2\left(A(T-t)^{-1}\widetilde{D}^2 + \sum_{k=1}^{K}\lambda_k^{-2}Scal_k + e^{-\frac{\delta}{T-t}}\widetilde{D}^2\right).$$

Hence, for t close to T such that $C_2\left(A(T-t)+e^{-\frac{\delta}{T-t}}\right) \leq \frac{1}{2}C_1$, it leads to

$$\widetilde{\mathscr{I}} \geq \frac{1}{2}C_1(T-t)^{-2}\widetilde{D}^2 - C_2\sum_{k=1}^{K}\lambda_k^{-2}Scal_k.$$

This verifies the first inequality in (5.10). Therefore, the proof is complete. □

The following monotonicity property of $\widetilde{\mathscr{I}}$ is crucial in the derivation of uniqueness.

Theorem 5.2. *(Monotonicity of $\widetilde{\mathscr{I}}$) There exist $C_1, C_2 > 0$ such that for A large enough and t close to T,*

$$\frac{d\widetilde{\mathscr{I}}}{dt} \geq C_1\sum_{k=1}^{K}\int\left(\frac{1}{\lambda_k}|\nabla w_k|^2 + \frac{1}{\lambda_k^3}|w_k|^2\right)e^{-\frac{|x-\alpha_k|}{A\lambda_k}}dx - C_2A\widetilde{\mathcal{E}}_r \tag{5.21}$$

where

$$\widetilde{\mathcal{E}}_r = \frac{\widetilde{D}^2}{(T-t)^2} + \varepsilon\frac{\widetilde{D}^2}{(T-t)^3} + \sum_{k=1}^{K}\frac{Scal_k(t)}{\lambda_k^3(t)}. \tag{5.22}$$

Remark 5.3. *Comparing with the error $\mathcal{E}_r$ in (3.57), we see that $\widetilde{\mathcal{E}}_r$ in (5.22) only contains the orders of D higher than one. This fact is important in the derivation of uniqueness.*

Proof. Using equation (5.8) we compute

$$\begin{aligned}
\frac{d\widetilde{\mathscr{I}}}{dt} =& -\sum_{k=1}^{K}\frac{\dot{\lambda}_k}{\lambda_k^3}\mathrm{Im}\int|w|^2\Phi_k dx - \sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f'(v_c)\cdot w, w_k\rangle - \mathrm{Re}\langle f''(v_c,w)\cdot w^2, \partial_t v_c\rangle \\
& - \sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle w\nabla\Phi_k, \nabla w\rangle - \sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f''(v_c,w)\cdot w^2, w_k\rangle \\
& - \mathrm{Im}\langle \Delta w - \sum_{k=1}^{K}\frac{1}{\lambda_k^2}w_k + f(v_c+w) - f(v_c), a_1\cdot\nabla w + a_0 w\rangle \\
& - \sum_{k=1}^{K}\frac{\dot{\lambda}_k\gamma_k - \lambda_k\dot{\gamma}_k}{2\lambda_k^2}\mathrm{Im}\langle\nabla\chi_A\left(\frac{x-\alpha_k}{\lambda_k}\right)\cdot\nabla w, w_k\rangle + \sum_{k=1}^{K}\frac{\gamma_k}{2\lambda_k}\mathrm{Im}\langle\partial_t\left(\nabla\chi_A\left(\frac{x-\alpha_k}{\lambda_k}\right)\right)\cdot\nabla w, w_k\rangle \\
& + \sum_{k=1}^{K}\mathrm{Im}\langle\frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A\left(\frac{x-\alpha_k}{\lambda_k}\right)w_k + \frac{\gamma_k}{2\lambda_k}\nabla\chi_A\left(\frac{x-\alpha_k}{\lambda_k}\right)\cdot(\nabla w_k + \nabla w\Phi_k), \partial_t w\rangle \\
=:& \sum_{l=1}^{9}\widetilde{\mathscr{I}_{t,l}}.
\end{aligned} \tag{5.23}$$

In order to reduce the analysis of (5.23) to the previous case in (3.59) and (3.91), we show that the reference solution v_c in $\widetilde{\mathscr{I}_{t,2}}$, $\widetilde{\mathscr{I}_{t,3}}$, $\widetilde{\mathscr{I}_{t,5}}$, $\widetilde{\mathscr{I}_{t,6}}$ and $\widetilde{\mathscr{I}_{t,9}}$ can be replaced by $U+z$, up to the acceptable error $(T-t)^{-2}\widetilde{D}^2$.

(i) *Estimate of* $\widetilde{\mathscr{I}_{t,2}}$. By (6.14),

$$\begin{aligned}&\Big|\widetilde{\mathscr{I}_{t,2}}+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f'(U+z)\cdot w, w_k\rangle\Big|\\ \le& C\sum_{k=1}^{K}\frac{1}{\lambda_k^2}\int(|U|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1})|R||w|^2dx\\ \le& C\Big((T-t)^{-4+\frac{d}{2}}\|R\|_{L^2}\|w\|_{L^4}^2+(T-t)^{-2}\|z\|_{L^\infty}^{\frac{4}{d}-1}\|R\|_{L^2}\|w\|_{L^4}^2+(T-t)^{-2}\|R\|_{H^1}^{\frac{4}{d}}\|w\|_{L^4}^2\Big).\end{aligned}$$

Then, by (2.25), (4.86) and (5.7),

$$\begin{aligned}\Big|\widetilde{\mathscr{I}_{t,2}}+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f'(U+z)\cdot w, w_k\rangle\Big|\le& C\left((T-t)^{\kappa-3}+(T-t)^{\frac{4}{d}\kappa-\frac{d}{2}-2}\right)\widetilde{D}^2\\ \le& C(T-t)^{-2}\widetilde{D}^2. \qquad (5.24)\end{aligned}$$

(ii) *Estimate of* $\widetilde{\mathscr{I}_{t,3}}$. By the decomposition (4.83),

$$\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t v_c\rangle=\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t(U+z)\rangle+\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t R\rangle.$$

Let us treat the two terms on the R.H.S. above separately.

First by (6.15),

$$\begin{aligned}&|\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t(U+z)\rangle-\mathrm{Re}\langle f''(U+z,w)\cdot w^2,\partial_t(U+z)\rangle|\\ \le& C\|\partial_t(U+z)\|_{L^\infty}\int\left(|U|^{\frac{4}{d}-2}+|z|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R||w|^2dx. \qquad (5.25)\end{aligned}$$

Since by (2.29) and (3.66), $\|\partial_t(U+z)\|_{L^\infty}\le C(T-t)^{-2-\frac{d}{2}}$. Then, by (4.86) and (5.6), the R.H.S. above can be bounded by, up to a universal constant,

$$\begin{aligned}&(T-t)^{-2-\frac{d}{2}}\left(\left((T-t)^{-2+d}+\|z\|_{L^\infty}^{\frac{4}{d}-2}\right)\|R\|_{L^2}\|w\|_{L^4}^2+\|R\|_{H^1}^{\frac{4}{d}-1}\|w\|_{L^4}^2+\|R\|_{L^2}\|w\|_{L^{\frac{8}{d}}}^{\frac{4}{d}}\right)\\ \le&(T-t)^{\kappa-3}\widetilde{D}^2\le(T-t)^{-2}\widetilde{D}^2. \qquad (5.26)\end{aligned}$$

Next we show that

$$\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t R\rangle=\mathcal{O}((T-t)^{-2}\widetilde{D}^2). \qquad (5.27)$$

To this end, by equation (2.67),

$$|\mathrm{Re}\langle f''(v_c,w)\cdot w^2,\partial_t R\rangle|=|\mathrm{Im}\langle f''(v_c,w)\cdot w^2,\Delta R+f(v_c)-f(U+z)+(a_1\cdot\nabla+a_0)R+\eta\rangle|.$$

Note that, by (4.64),

$$|\mathrm{Im}\langle f''(v_c,w)\cdot w^2,\Delta R\rangle|\le C\|R\|_{\dot H^{\frac{3}{2}}}\|f''(u,w)\cdot w^2\|_{\dot H^{\frac{1}{2}}}\le C(T-t)^{\kappa-2}\|f''(u,w)\cdot w^2\|_{\dot H^{\frac{1}{2}}}. \qquad (5.28)$$

Then, by (2.25), (4.83), (4.86), (5.6) and $\|U(t)\|_{H^1}\le C(T-t)^{-1}$,

$$\begin{aligned}\|f''(v_c,w)\cdot w^2\|_{\dot H^{\frac{1}{2}}}\le& C\sum_{j=2}^{1+\frac{4}{d}}\|v_c\|_{H^1}^{1+\frac{4}{d}-j}\|w\|_{H^1}^{j}\\ \le& C\sum_{j=2}^{1+\frac{4}{d}}(\|U\|_{H^1}^{1+\frac{4}{d}-j}+\|z\|_{H^1}^{1+\frac{4}{d}-j}+\|R\|_{H^1}^{1+\frac{4}{d}-j})\|w\|_{H^1}^{j}\end{aligned}$$

$$\leq C\sum_{j=2}^{1+\frac{4}{d}}(T-t)^{-(1+\frac{4}{d}-j)+(3+\zeta)(j-2)}\|w\|_{H^1}^2 \leq C(T-t)^{-3}\|w\|_{H^1}^2. \tag{5.29}$$

Plugging this into (5.28) and using $\kappa \geq 5$ we obtain

$$|\mathrm{Im}\langle f''(v_c,w)\cdot w^2, \Delta R\rangle| \leq C(T-t)^{\kappa-5}\|w\|_{H^1}^2 \leq C(T-t)^{-2}\widetilde{D}^2(t). \tag{5.30}$$

Moreover, since by (6.16),

$$\begin{aligned}|f''(v_c,w)\cdot w^2| \leq & C(|U|^{\frac{4}{d}-1}+|z|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1})|w|^2 \\ \leq & C\left((T-t)^{-2+\frac{d}{2}}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^2,\end{aligned} \tag{5.31}$$

and

$$|f(v_c)-f(U+z)| \leq C(|U|^{\frac{4}{d}}+|R|^{\frac{4}{d}}+|z|^{\frac{4}{d}})|R| \leq C\left((T-t)^{-2}+|R|^{\frac{4}{d}}\right)|R|, \tag{5.32}$$

taking into account (2.25), (4.86) and (5.6) we get

$$\begin{aligned}&\left|\mathrm{Im}\langle f''(v_c,w)\cdot w^2, f(v_c)-f(U+z)+(a_1\cdot\nabla+a_0)R\rangle\right| \\ \leq & C\int\left((T-t)^{-2+\frac{d}{2}}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^2\left((T-t)^{-2}|R|+|R|^{1+\frac{4}{d}}+|\nabla R|+|R|\right)dx \\ \leq & C\left((T-t)^{-2+\frac{d}{2}}+\|R\|_{H^1}^{\frac{4}{d}-1}+\|w\|_{H^1}^{\frac{4}{d}-1}\right)\|w\|_{H^1}^2\left((T-t)^{-2}\|R\|_{L^2}+\|R\|_{H^1}^{1+\frac{4}{d}}+\|R\|_{H^1}\right) \\ \leq & C(T-t)^{\kappa-5+\frac{d}{2}}\widetilde{D}^2 \leq C(T-t)^{-2}\widetilde{D}^2.\end{aligned} \tag{5.33}$$

Furthermore, by (4.92) and (5.31),

$$\begin{aligned}|\mathrm{Im}\langle f''(v_c,w)\cdot w^2,\eta\rangle| \leq & C(T-t)^{-2+\frac{d}{2}}\|\eta\|_{L^2}\|w\|_{H^1}^2 \\ \leq & C(T-t)^{\kappa-3}\|w\|_{H^1}^2 \\ \leq & C(T-t)^{-2}\widetilde{D}^2.\end{aligned} \tag{5.34}$$

Thus, estimates (5.30), (5.33) and (5.34) together yield (5.27), as claimed.
Therefore, we conclude from (5.25), (5.26) and (5.27) that

$$\widetilde{\mathscr{I}_{t,3}} = \mathrm{Re}\langle f''(U+z,w)\cdot w^2, \partial_t(U+z)\rangle + O\left((T-t)^{-2}\widetilde{D}^2\right). \tag{5.35}$$

(iii) *Estimate of* $\widetilde{\mathscr{I}_{t,5}}$. By (5.5), (5.6) and (6.15),

$$\begin{aligned}&\left|\widetilde{\mathscr{I}_{t,5}}+\sum_{k=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f''(U+z,w)\cdot w^2, w_k\rangle\right| \\ \leq & C(T-t)^{-2}\int\left(|U|^{\frac{4}{d}-2}+|z|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R||w|^3dx \\ \leq & C(T-t)^{-4+d}\|R\|_{H^1}\|w\|_{H^1}^3 \leq C(T-t)^{-2}\widetilde{D}^2.\end{aligned}$$

This yields that

$$\widetilde{\mathscr{I}_{t,5}} = -\sum_{j=1}^{K}\frac{1}{\lambda_k^2}\mathrm{Im}\langle f''(U,w)\cdot w^2, w_k\rangle + O\left((T-t)^{-2}\widetilde{D}^2\right). \tag{5.36}$$

(*iv*) *Estimate of* $\widetilde{\mathscr{I}_{t,6}}$. Since by (6.7),

$$
\begin{aligned}
&|f(v_c + w) - f(v_c) - (f(U + z + w) - f(U + z))|\\
=&|f'(v_c, w) \cdot w - f'(U + z, w) \cdot w|\\
\leq& C(|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1} + |w|^{\frac{4}{d}-1})|R||w|,
\end{aligned} \tag{5.37}
$$

we infer from (4.86) that

$$
\begin{aligned}
&\Big|\mathrm{Im}\langle f(v_c + w) - f(v_c) - (f(U + z + w) - f(U + z)), a_1 \cdot \nabla w + a_0 w\rangle\Big|\\
\leq& C(T - t)^{-\frac{d}{2}(\frac{4}{d}-1)}\|R\|_{H^1}\|w\|_{H^1}^2\\
\leq& C(T - t)^{-2}\widetilde{D}^2.
\end{aligned}
$$

This yields that

$$
\widetilde{\mathscr{I}_{t,6}} = -\mathrm{Im}\langle \Delta w - \sum_{k=1}^{K} \frac{1}{\lambda_k^2} w_k + f(U + z + w) - f(U + z), a_1 \cdot \nabla w + a_0 w\rangle + O((T - t)^{-2}\widetilde{D}^2). \tag{5.38}
$$

(*v*) *Estimate of* $\widetilde{\mathscr{I}_{t,9}}$. By equation (5.8),

$$
\begin{aligned}
\widetilde{\mathscr{I}_{t,9}} =& \sum_{k=1}^{K} \mathrm{Im}\langle \frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A(\frac{x - \alpha_k}{\lambda_k})w_k + \frac{\gamma_k}{2\lambda_k}\nabla\chi_A(\frac{x - \alpha_k}{\lambda_k}) \cdot (\nabla w_k + \nabla w\Phi_k),\\
& i\Delta w + i(a_1 \cdot \nabla w + a_0 w) + i(f(v_c + w) - f(v_c))\rangle.
\end{aligned} \tag{5.39}
$$

Note that, unlike in (2.67), we have $\eta = 0$ here.

Then, in view of (5.37), we see that

$$
\begin{aligned}
&\Big|\langle \frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A(\frac{x - \alpha_k}{\lambda_k})w_k + \frac{\gamma_k}{2\lambda_k}\nabla\chi_A(\frac{x - \alpha_k}{\lambda_k}) \cdot (\nabla w_k + \nabla w\Phi_k), i(f(v_c + w) - f(v_c))\rangle\\
&- \langle \frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A(\frac{x - \alpha_k}{\lambda_k})w_k + \frac{\gamma_k}{2\lambda_k}\nabla\chi_A(\frac{x - \alpha_k}{\lambda_k}) \cdot (\nabla w_k + \nabla w\Phi_k), i(f(U + z + w) - f(U + z))\rangle\Big|\\
\leq& CA\int \Big((T - t)^{-1}|w| + |\nabla w|\Big)\Big(|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1} + |w|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1}\Big)|R||w|dx\\
\leq& CA(T - t)^{-3+\frac{d}{2}}\|R\|_{H^1}\|w\|_{H^1}^2 \leq CA(T - t)^{-2}\widetilde{D}^2.
\end{aligned}
$$

This yields that

$$
\begin{aligned}
\widetilde{\mathscr{I}_{t,9}} =& \sum_{k=1}^{K} \mathrm{Im}\langle \frac{\gamma_k}{2\lambda_k^2}\Delta\chi_A(\frac{x - \alpha_k}{\lambda_j})w_k + \frac{\gamma_k}{2\lambda_k}\nabla\chi_A(\frac{x - \alpha_k}{\lambda_k}) \cdot (\nabla w_k + \nabla w\Phi_k),\\
& i\Delta w + i(f(U + z + w) - f(U + z)) + i(a_1 \cdot \nabla + a_0)w\rangle + O\Big(A(T - t)^{-2}\widetilde{D}^2\Big).
\end{aligned} \tag{5.40}
$$

Now, the reference solution v_c in (5.23) has been replaced by $U+z$ up to the order $O((T-t)^{-2}\widetilde{D}^2)$. Note that, by (4.86)-(4.89) and (5.6), the conditions in Theorem 3.5 are verified. Hence, arguing as in the proof of Theorem 3.5 with w replacing R and using (5.6) we obtain (5.21).

As mentioned below (5.39), because for the difference w we have $\eta = 0$, the errors involving M_k and the linear terms of D in (3.57) do not appear here, only the higher order terms of D remain.

Therefore, the proof is complete. □

As a consequence of Lemma 5.1 and Theorem 5.2 we have

Corollary 5.4. *For t close to T, set*

$$\widetilde{N}(t) := \sup_{t\leq s<T} \frac{\widetilde{D}^2(s)}{(T-s)^2}. \tag{5.41}$$

Then, there exists $C>0$ such that

$$\widetilde{N}(t) \leq C\left(\sum_{k=1}^{K} \sup_{t\leq s<T} \frac{Scal_k(s)}{\lambda_k^2(s)} + \int_t^T \sum_{k=1}^{K} \frac{Scal_k(s)}{\lambda_k^3(s)} + \varepsilon \frac{\widetilde{N}(s)}{T-s} ds\right). \tag{5.42}$$

Proof. By Lemma 5.1 and Theorem 5.2, for $t<\tilde{t}<T$,

$$\begin{aligned}
C_1 \frac{\widetilde{D}^2(t)}{(T-t)^2} &\leq \widetilde{\mathscr{I}}(t) + C_2 \sum_{k=1}^{K} \frac{Scal_k(t)}{\lambda_k^2(t)} \\
&= \widetilde{\mathscr{I}}(\tilde{t}) + C_2 \sum_{k=1}^{K} \frac{Scal_k(t)}{\lambda_k^2(t)} - \int_t^{\tilde{t}} \frac{d\widetilde{\mathscr{I}}}{ds}(s) ds \\
&\leq CA\left(\frac{\widetilde{D}^2(\tilde{t})}{(T-\tilde{t})^2} + \sum_{k=1}^{K} \frac{Scal_k(t)}{\lambda_k^2(t)} + \int_t^{\tilde{t}} \frac{\widetilde{D}^2(s)}{(T-s)^2} + \sum_{k=1}^{K} \frac{Scal_k(s)}{\lambda_k^3(s)} + \varepsilon \frac{\widetilde{D}^2(s)}{(T-s)^3} ds\right),
\end{aligned}$$

which yields that

$$\sup_{t\leq s\leq \tilde{t}} \frac{\widetilde{D}^2(s)}{(T-s)^2} \leq CA\left(\widetilde{N}(\tilde{t}) + \sum_{k=1}^{K} \sup_{t\leq s\leq \tilde{t}} \frac{Scal_k(s)}{\lambda_k^2(s)} + (\tilde{t}-t)\widetilde{N}(t) + \int_t^{\tilde{t}} \sum_{k=1}^{K} \frac{Scal_k(s)}{\lambda_k^3(s)} + \varepsilon \frac{\widetilde{N}(s)}{T-s} ds\right).$$

Since by (5.6), $\widetilde{N}(\tilde{t}) \to 0$ as $\tilde{t} \to T$, taking $\tilde{t} \to T$ and t close to T we obtain (5.42). □

5.2. **Control of the null space.** In this subsection we derive the control of scalar $Scal_k$. The main result is formulated in Theorem 5.8 below. The arguments follow the lines in the proof of [64, Theorem 7.7], mainly based on algebraic identities. For the reader's convenience, let us sketch the main arguments below.

For every $1 \leq k \leq K$, define the renormalized variables $\widetilde{e}_k$ and e_k by

$$w(t,x) = \lambda_k(t)^{-\frac{d}{2}} \widetilde{e}_k(t, \frac{x-\alpha_k(t)}{\lambda_k(t)}) e^{i\theta_k(t)}, \quad with\, \widetilde{e}_k(t,y) = e_k(t,y) e^{i(\beta_k(t)\cdot y - \frac{1}{4}\gamma_k(t)|y|^2)}. \tag{5.43}$$

Note that, the renormalized variable e_k is different from the previous one ϵ_k in (5.3).

We use (6.8) to expand

$$f(v_c+w) - f(v_c) = \partial_z f(v_c) w + \partial_{\bar{z}} f(v_c) \overline{w} + f''(v_c, w)\cdot w^2. \tag{5.44}$$

Then, using (6.8) again to further expand $\partial_z f(v_c)$ and $\partial_{\bar{z}} f(v_c)$ around the profile U we get

$$f(v_c+w) - f(v_c) = f'(U)\cdot w + G_1, \tag{5.45}$$

where

$$G_1 := w(\partial_z f)'(U, z+R)\cdot(z+R) + \overline{w}(\partial_{\bar{z}} f)'(U, z+R)\cdot(z+R) + f''(v_c, w)\cdot w^2. \tag{5.46}$$

Decompose $f'(U)\cdot w$ into three parts

$$\begin{aligned}
f'(U)\cdot w &= f'(U_k)\cdot w + \sum_{l\neq k} f'(U_l)\cdot w + [f'(U)\cdot w - \sum_{l=1}^{K} f'(U_l)\cdot w] \\
&=: f'(U_k)\cdot w + G_2 + G_3,
\end{aligned} \tag{5.47}$$

and set

$$G_4 := a_1 \cdot \nabla w + a_0 w, \tag{5.48}$$

where a_1, a_0 are given by (1.2) and (1.3), respectively.

Thus, by (5.45), (5.47) and (5.48), equation (5.8) can be reformulated:

$$i\partial_t w + \Delta w + f'(U_k)\cdot w = -\sum_{l=1}^{4} G_l. \tag{5.49}$$

Plugging (5.43) into (5.49) and using algebraic computations one has the equation of e_k below.

Lemma 5.5. *For every $1 \le k \le K$, e_k satisfies the equation*

$$i\lambda_k^2\partial_t e_k + \Delta e_k - e_k + (1+\frac{2}{d})Q^{\frac{4}{d}} e_k + \frac{2}{d}Q^{\frac{4}{d}}\overline{e_k} = -\sum_{l=1}^{4}\mathcal{H}_l + O\Big((\langle y\rangle^2|\widetilde{e}_k| + \langle y\rangle|\nabla\widetilde{e}_k|)Mod_k\Big), \tag{5.50}$$

where

$$\mathcal{H}_l(t,y) = \lambda_k^{2+\frac{d}{2}} e^{-i\theta_k} e^{-i(\beta_k\cdot y - \frac{1}{4}\gamma_k|y|^2)} G_l(t, \lambda_k y + \alpha_k), \ \ 1 \le l \le 4. \tag{5.51}$$

The error terms $\{\mathcal{H}_l\}$ in (5.51) can be controlled by Lemma 5.6 below.

Lemma 5.6. *Let $\mathcal{K}$ belong to the generalized kernels of the linearized operator L given by* (6.1) *below, i.e., $\mathcal{K} \in \{Q, yQ, |y|^2Q, \nabla Q, \Lambda Q, \rho\}$. Then, there exist $C, \delta > 0$ such that*

$$\int |\mathcal{H}_1(t,y)||\mathcal{K}(y)|dy \le C(T-t)^{4+\zeta}\widetilde{D}(t), \tag{5.52}$$

$$\int (|\mathcal{H}_2(t,y)| + |\mathcal{H}_3(t,y)|)|\mathcal{K}(y)|dy \le Ce^{-\frac{\delta}{T-t}}\|w\|_{L^2}, \tag{5.53}$$

$$\Big|\int \mathcal{H}_4(t,y)\mathcal{K}(y)dy\Big| \le C(T-t)^{\upsilon_*+1}\|w\|_{L^2}, \tag{5.54}$$

where υ_* is the flatness index of the spatial functions $\{\phi_l\}$ in Hypothesis ($H1$).

Proof. Estimates (5.53) and (5.54) were proved in [64, (7,95), (7.96)], hence we mainly focus on the estimate (5.52).

Define the renormalized variable $\varepsilon_{R,k}$ by

$$R(t,x) = \lambda_k^{-\frac{d}{2}}\varepsilon_{R,k}(t, \frac{x-\alpha_k}{\lambda_k})e^{i\theta_k}. \tag{5.55}$$

By (5.51),

$$\int |\mathcal{H}_1(t,y)||\mathcal{K}(y)|dy \le C(T-t)^{2-\frac{d}{2}}\int |G_1(t,x)\mathcal{K}(\frac{x-\alpha_k}{\lambda_k})|dx. \tag{5.56}$$

Since $\mathcal{K}(y) \le Ce^{-\delta|y|}$ and by (5.46),

$$|G_1| \le C\left(|U|^{\frac{4}{d}-1} + |z+R|^{\frac{4}{d}-1}\right)|z+R||w| + C\left(|U|^{\frac{4}{d}-1} + |z+R|^{\frac{4}{d}-1} + |w|^{\frac{4}{d}-1}\right)|w|^2,$$

taking into account Lemma 2.5 we derive

$$\begin{aligned}\int |\mathcal{H}_1(t,y)||\mathcal{K}(y)|dy \le & C\int e^{-\delta|y|}\Big(\left(e^{-\delta|y|} + |\varepsilon_{z,k}|^{\frac{4}{d}-1} + |\varepsilon_{R,k}|^{\frac{4}{d}-1}\right)|\varepsilon_{z,k} + \varepsilon_{R,k}||\widetilde{e}_k| \\ & + \left(e^{-\delta|y|} + |\varepsilon_{z,k}|^{\frac{4}{d}-1} + |\varepsilon_{R,k}|^{\frac{4}{d}-1} + |\widetilde{e}_k|^{\frac{4}{d}-1}\right)|\widetilde{e}_k|^2\Big)dy + Ce^{-\frac{\delta}{T-t}}. \end{aligned} \tag{5.57}$$

Then, by (2.25), (2.29), (4.86) and (5.6), the R.H.S. can be bounded by, up to a universal constant,

$$
\begin{aligned}
&\Big(\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + \|\varepsilon_{R,k}\|_{L^2} + \|e^{-\delta|y|}|\varepsilon_{z,k}|^{\frac{4}{d}-1}\|_{L^\infty}\|\varepsilon_{z,k}+\varepsilon_{R,k}\|_{L^2}\Big)\|\widetilde{e}_k\|_{L^2} + \|\varepsilon_{R,k}\|_{H^1}^{\frac{4}{d}-1}\|\varepsilon_{z,k}+\varepsilon_{R,k}\|_{H^1}\|\widetilde{e}_k\|_{H^1}\\
&\quad + (1+\|e^{-\delta|y|}|\varepsilon_{z,k}|^{\frac{4}{d}-1}\|_{L^\infty})\|\widetilde{e}_k\|_{L^2}^2 + \|\varepsilon_{R,k}\|_{H^1}^{\frac{4}{d}-1}\|\widetilde{e}_k\|_{H^1}^2 + \|\widetilde{e}_k\|_{H^1}^{\frac{4}{d}+1} + e^{-\frac{\delta}{T-t}}\\
\leq&\Big(\alpha^*(T-t)^{m+1+\frac{d}{2}} + (T-t)^{\kappa+1}\Big)\widetilde{D} + (T-t)^{\kappa(\frac{4}{d}-1)}\widetilde{D} + \widetilde{D}^2 + (T-t)^{\kappa(\frac{4}{d}-1)}\widetilde{D}^2 + \widetilde{D}^{\frac{4}{d}+1} + e^{-\frac{\delta}{T-t}}\\
\leq&(T-t)^{4+\zeta}\widetilde{D}.
\end{aligned}
$$

This yields (5.52) and finishes the proof. □

Applying Lemmas 5.5 and 5.6 and using algebraic identities in (6.3) one has the following ODE system of the renormalized variable e_k along the six directions in the null space.

Proposition 5.7. *Let e_k be as in* (5.43) *and $e_{k,1} := \mathrm{Re} e_k$, $e_{k,2} := \mathrm{Im} e_k$, Then, for every $1 \leq k \leq K$,*

$$\frac{d}{dt}\langle e_{k,1}, Q\rangle = \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.58}$$

$$\frac{d}{dt}\langle e_{k,2}, \Lambda Q\rangle = 2\lambda_k^{-2}\langle e_{k,1}, Q\rangle + \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.59}$$

$$\frac{d}{dt}\langle e_{k,1}, |y|^2Q\rangle = -4\lambda_k^{-2}\langle e_{j,2}, \Lambda Q\rangle + \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.60}$$

$$\frac{d}{dt}\langle e_{k,2}, \rho\rangle = \lambda_k^{-2}\langle e_{k,1}, |y|^2Q\rangle + \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.61}$$

$$\frac{d}{dt}\langle e_{k,2}, \nabla Q\rangle = \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.62}$$

$$\frac{d}{dt}\langle e_{k,1}, yQ\rangle = -2\lambda_k^{-2}\langle e_{k,2}, \nabla Q\rangle + \mathcal{O}((T-t)^{3+\zeta}\sqrt{\widetilde{N}}), \tag{5.63}$$

Proof. By (5.50),

$$
\begin{aligned}
\frac{d}{dt}\langle e_{k,1}, Q\rangle =& -\lambda_k^{-2}\mathrm{Im}\int Q\left(\left(\Delta e_k - e_k + (1+\frac{2}{d})Q^{\frac{4}{d}}e_k + \frac{2}{d}Q^{\frac{4}{d}}\overline{e_k}\right) + \sum_{l=1}^4 \mathcal{H}_l\right)dy\\
&+\mathcal{O}\left(\lambda_k^{-2}Mod_k\int Q(\langle y\rangle^2|\widetilde{e}_k| + \langle y\rangle|\nabla\widetilde{e}_k|)dy\right).
\end{aligned} \tag{5.64}
$$

Note that, by the definition of L_- and the identity $L_-Q = 0$ in (6.3),

$$\mathrm{Im}\int Q\left(\Delta e_k - e_k + (1+\frac{2}{d})Q^{\frac{4}{d}}e_k + \frac{2}{d}Q^{\frac{4}{d}}\overline{e_k}\right)dy = -\mathrm{Im}\int QL_-e_{k,2}dy = -\mathrm{Im}\int L_-Qe_{k,2}dy = 0.$$

Moreover, since $\widetilde{D} \leq C(T-t)\sqrt{\widetilde{N}}$, by Lemma 5.6,

$$\lambda_k^{-2}\left|\int Q\mathcal{H}_l dy\right| \leq C\lambda_k^{-2}(T-t)^{4+\zeta}\widetilde{D} \leq C(T-t)^{3+\zeta}\sqrt{\widetilde{N}}.$$

It also follows from (4.91) that for $\kappa \geq 4$,

$$
\begin{aligned}
\lambda_k^{-2}Mod_k\int Q\left(\langle y\rangle^2|\widetilde{e}_k| + \langle y\rangle|\nabla\widetilde{e}_k|\right)dy \leq& C\lambda_k^{-2}Mod\widetilde{D}\\
\leq& C\alpha^*(T-t)^{\kappa}\sqrt{\widetilde{N}} \leq C\alpha^*(T-t)^{3+\zeta}\sqrt{\widetilde{N}}.
\end{aligned}
$$

Hence, (5.58) follows from the above estimates. The proof of (5.59)-(5.63) is similar, see also the proof of [64, Proposition 7.12]. □

As a consequence, we have the control of scalar $Scal_k$ below. The proof is similar to that of [64, Theorem 7.7] and hence is omitted here.

Theorem 5.8. *(Control of $Scal_k$) There exists $C > 0$ such that for t close to T and $1 \le k \le K$,*

$$Scal_k(t) \le C(T-t)^{2+\zeta}\widetilde{N}(t). \tag{5.65}$$

5.3. **Proof of conditional uniqueness.** We are now in position to prove the conditional uniqueness part in Theorem 1.2.

Let ε be a sufficiently small constant to be specified later and let t close to T such that (4.1) holds. By Corollary 5.4, for any $t \in [t_*, T)$,

$$\widetilde{N}(t) \le C_1 \sum_{k=1}^{K} \sup_{t\le s<T} \frac{Scal_k(s)}{\lambda_k^2(s)} + C_1 \int_t^T \left(\sum_{k=1}^{K} \frac{Scal_k(s)}{\lambda_k^3(s)} + \varepsilon \frac{\widetilde{N}(s)}{T-s} \right) ds, \tag{5.66}$$

which along with Theorem 5.8 yields that for some $\zeta > 0$,

$$\widetilde{N}(t) \le C_2(T-t)^{\zeta}\widetilde{N}(t) + C_2\varepsilon \int_t^T \frac{\widetilde{N}(s)}{T-s} ds, \tag{5.67}$$

where C_2 is independent of ε and t. Then, taking t even closer to T such that $C_2(T-t)^{\zeta} \le \frac{1}{2}$ we obtain the Gronwall type inequality

$$\widetilde{N}(t) \le 2C_2\varepsilon \int_t^T \frac{\widetilde{N}(s)}{T-s} ds. \tag{5.68}$$

Moreover, by (5.6) and (5.41),

$$\widetilde{N}(t) \le C_3(T-t)^{6+\zeta}, \tag{5.69}$$

where $C_3 (\ge 1)$ is independent of ε and t.

We claim that for any t close to T and for any $l \ge 1$,

$$\widetilde{N}(t) \le \left(\frac{2C_2C_3\varepsilon}{6+\zeta}\right)^l (T-t)^{6+\zeta}. \tag{5.70}$$

To this end, plugging (5.69) into the Gronwall type inequality (5.68) we get

$$\widetilde{N}(t) \le 2C_2\varepsilon \int_t^T C_3(T-s)^{5+\zeta} ds \le \left(\frac{2C_2C_3\varepsilon}{6+\zeta}\right)(T-t)^{6+\zeta}, \tag{5.71}$$

which verifies (5.70) at the preliminary step $l = 1$. Moreover, plugging (5.70) into (5.68) we derive that (5.70) is still valid with $l+1$ replacing l. Thus, the induction arguments lead to (5.70).

Therefore, take ε small enough such that $\frac{2C_2C_3\varepsilon}{6+\zeta} < 1$. Then, it follows from (5.70) that

$$\widetilde{N}(t) \le \lim_{l\to\infty} \left(\frac{2C_2C_3\varepsilon}{6+\zeta}\right)^l (T-t)^{6+\zeta} = 0, \tag{5.72}$$

which yields $\widetilde{N}(t) = 0$ for t close to T, and so $w \equiv 0$. The proof of Theorem 1.2 is complete. □

6. Appendix

This Appendix mainly contains preliminaries of linearized operators around the ground state, the expansion of the nonlinearity and the proof of Theorem 2.2.

Coercivity of the linearized operators. Let $L = (L_+, L_-)$ be the linearized operator around the ground state, defined by

$$L_+ := -\Delta + I - (1+\frac{4}{d})Q^{\frac{4}{d}}, \quad L_- := -\Delta + I - Q^{\frac{4}{d}}. \tag{6.1}$$

The generalized null space of operator L is spanned by $\{Q, xQ, |x|^2Q, \nabla Q, \Lambda Q, \rho\}$, where $\Lambda := \frac{d}{2}I_d + x\cdot\nabla$, and ρ is the unique H^1 spherically symmetric solution to the equation

$$L_+\rho = -|x|^2Q, \tag{6.2}$$

which satisfies the exponential decay property (see, e.g., [40, 47]), i.e., for some $C, \delta > 0$,

$$|\rho(x)| + |\nabla\rho(x)| \leq Ce^{-\delta|x|}.$$

Moreover, it holds that (see, e.g., [68, (B.1), (B.10), (B.15)])

$$\begin{aligned} &L_+\nabla Q = 0, \quad L_+\Lambda Q = -2Q, \quad L_+\rho = -|x|^2Q, \\ &L_-Q = 0, \quad L_-xQ = -2\nabla Q, \quad L_-|x|^2Q = -4\Lambda Q. \end{aligned} \tag{6.3}$$

Lemma 6.1 below contains the key localized coercivity of the linearized operator.

Lemma 6.1. *(Localized coercivity* [64, Corollary 3.4]*) Let ϕ be a positive smooth radial function on $\mathbb{R}^d$, such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = e^{-|x|}$ for $|x| \geq 2$, $0 < \phi \leq 1$, and $\left|\frac{\nabla\phi}{\phi}\right| \leq C$ for some $C > 0$. Set $\phi_A(x) := \phi\left(\frac{x}{A}\right)$, $A > 0$. Then, for A large enough we have*

$$\int (|f|^2 + |\nabla f|^2)\phi_A - (1+\frac{4}{d})Q^{\frac{4}{d}}f_1^2 - Q^{\frac{4}{d}}f_2^2 dx \geq C_1 \int (|\nabla f|^2 + |f|^2)\phi_A dx - C_2 Scal(f), \tag{6.4}$$

where $C_1, C_2 > 0$, f_1, f_2 are the real and imaginary parts of f, respectively, and $Scal(f)$ denotes the scalar products along the unstable directions in the null space

$$Scal(f) := \langle f_1, Q\rangle^2 + \langle f_1, xQ\rangle^2 + \langle f_1, |x|^2Q\rangle^2 + \langle f_2, \nabla Q\rangle^2 + \langle f_2, \Lambda Q\rangle^2 + \langle f_2, \rho\rangle^2, \tag{6.5}$$

Expansion of the nonlinearity. Let us recall the expansion that for any continuous differentiable function $g : \mathbb{C} \to \mathbb{C}$ and for any $v, w \in \mathbb{C}$, (see, e.g., [38, (3.10)])

$$g(v+w) = g(v) + g'(v,w)\cdot w \tag{6.6}$$

with

$$g'(v,w)\cdot w := w\int_0^1 \partial_z g(v+sw)ds + \overline{w}\int_0^1 \partial_{\overline{z}} g(v+sw)ds, \tag{6.7}$$

where $z = x + iy \in \mathbb{C}$, $\partial_z g$ and $\partial_{\overline{z}} g$ are the usual complex derivatives $\partial_z g = \frac{1}{2}(\partial_x g - i\partial_y g)$, $\partial_{\overline{z}} g = \frac{1}{2}(\partial_x g + i\partial_y g)$, respectively. Moreover, if $\partial_z g$ and $\partial_{\overline{z}} g$ are also continuously differentiable, we may expand g up to the second order

$$g(v+w) = g(v) + g'(v)\cdot w + g''(v,w)\cdot w^2, \tag{6.8}$$

where

$$\begin{aligned} g'(v)\cdot w &:= \partial_z g(v)w + \partial_{\overline{z}} g(v)\overline{w}, \\ g''(v,w)\cdot w^2 &:= w^2\int_0^1 t\int_0^1 \partial_{zz} g(v+stw)dsdt + 2|w|^2\int_0^1 t\int_0^1 \partial_{z\overline{z}} g(v+stw)dsdt \\ &\quad + \overline{w}^2\int_0^1 t\int_0^1 \partial_{\overline{z}\overline{z}} g(v+stw)dsdt, \end{aligned} \tag{6.9}$$

In particular, for $f(z) := |z|^{\frac{4}{d}}z$ with $d = 1, 2$, $z \in \mathbb{C}$, one has

$$f(v+w) = f(v) + f'(v)\cdot R + f''(v)\cdot w^2 + \mathcal{O}\left(\sum_{l=3}^{1+\frac{4}{d}} |v|^{1+\frac{4}{d}-l}|w|^l\right), \tag{6.10}$$

where

$$f'(v)\cdot w := \partial_z f(v)w + \partial_{\bar z} f(v)\overline{w} = (1+\frac{2}{d})|v|^{\frac{4}{d}}w + \frac{2}{d}|v|^{\frac{4}{d}-2}v^2\overline{w}, \tag{6.11}$$

$$\begin{aligned} f''(v)\cdot w^2 :=& \frac{1}{2}\partial_{zz} f(v)w^2 + \partial_{z\bar z} f(v)|w|^2 + \frac{1}{2}\partial_{\bar z\bar z} f(v)\overline{w}^2 \\ =& \frac{1}{d}(1+\frac{2}{d})|v|^{\frac{4}{d}-2}\overline{v}w^2 + \frac{2}{d}(1+\frac{2}{d})|v|^{\frac{4}{d}-2}v|w|^2 + \frac{1}{d}(\frac{2}{d}-1)|v|^{\frac{4}{d}-4}v^3\overline{w}^2. \end{aligned} \tag{6.12}$$

The following estimates are also useful:

$$|f(v_1) - f(v_2)| \le C(|v_1|^{\frac{4}{d}} + |v_2|^{\frac{4}{d}})|v_1 - v_2|, \tag{6.13}$$

$$|f'(v_1)\cdot w - f'(v_2)\cdot w| \le C(|v_1|^{\frac{4}{d}-1} + |v_2|^{\frac{4}{d}-1})|v_1 - v_2||w|, \tag{6.14}$$

$$|f''(v_1,w)\cdot w^2 - f''(v_2,w)\cdot w^2| \le C(|v_1|^{\frac{4}{d}-2} + |v_2|^{\frac{4}{d}-2} + |w|^{\frac{4}{d}-2})|v_1 - v_2||w|^2, \tag{6.15}$$

$$|f''(v,w)\cdot w^2| \le C(|v|^{\frac{4}{d}-1} + |w|^{\frac{4}{d}-1})|w|^2. \tag{6.16}$$

Proof of Theorem 2.2. We adapt the arguments as in [63, 64].

(i) *Reformulation of the equation of remainder.* By (2.1) and (6.8),

$$f(v) = f(U+z) + f'(U+z)\cdot R + f''(U+z,R)\cdot R^2. \tag{6.17}$$

Plugging this into (2.67) leads to the equation of R:

$$i\partial_t R + \sum_{k=1}^{K}\left(\Delta R_k + (1+\frac{2}{d})|U_k|^{\frac{4}{d}}R_k + \frac{2}{d}|U_k|^{\frac{4}{d}-2}U_k^2\overline{R_k} + i\partial_t U_k + \Delta U_k + |U_k|^{\frac{4}{d}}U_k\right) = -\sum_{l=1}^{5} H_l, \tag{6.18}$$

where H_1, H_2 contain the interactions between different blow-up profiles U_j and U_l, $j \neq l$,

$$H_1 := f'(U)\cdot R - \sum_{k=1}^{K} f'(U_k)\cdot R_k, \tag{6.19}$$

$$H_2 := f(U) - \sum_{k=1}^{K} f(U_k), \tag{6.20}$$

the terms H_3, H_4 contain the regular flow z, i.e.,

$$H_3 := f'(U+z)\cdot R - f'(U)\cdot R + f''(U+z,R)\cdot R^2, \tag{6.21}$$

$$H_4 := f(U+z) - f(U) - f(z), \tag{6.22}$$

and the lower order perturbations are contained in H_5:

$$H_5 := \sum_{l=1}^{K}\left(a_1\cdot\nabla(U_l + R_l) + a_0(U_l + R_l)\right), \tag{6.23}$$

where a_1, a_0 are the coefficients of lower order perturbations given by (1.2) and (1.3), respectively.

(ii) *Estimate of Modulation equations.* Let us take the modulation equation $\lambda_k^2\dot{\gamma}_k + \gamma_k^2$ to illustrate the main arguments below. As $R(T_*) = 0$, we may take t^* close to T such that $\|R\|_{C([t^*,T_*];H^1)} \le 1$.

Taking the inner product of (6.18) with $\Lambda_k U_k$ and then taking the real part we get

$$\begin{aligned}
&-\operatorname{Im}\langle \partial_t R, \Lambda U_k\rangle + \operatorname{Re}\langle \Delta R_k + (1+\frac{2}{d})|U_k|^{\frac{4}{d}} R_k + \frac{2}{d}|U_k|^{\frac{4}{d}-2} U_k^2 \overline{R_k}, \Lambda_k U_k\rangle \\
&+ \operatorname{Re}\langle i\partial_t U_k + \Delta U_k + |U_k|^{\frac{4}{d}} U_k, \Lambda_k U_k\rangle \\
=& -\operatorname{Re}\langle \sum_{j\neq k} (\Delta R_j + (1+\frac{2}{d})|U_j|^{\frac{4}{d}} R_j + \frac{2}{d}|U_j|^{\frac{4}{d}-2} U_j^2 \overline{R_j}), \Lambda_k U_k\rangle \\
&- \operatorname{Re}\langle \sum_{j\neq k} (i\partial_t U_j + \Delta U_j + |U_j|^{\frac{4}{d}} U_j), \Lambda_k U_k\rangle - \sum_{l=1}^{5} \operatorname{Re}\langle H_l, \Lambda_k U_k\rangle. \qquad (6.24)
\end{aligned}$$

First for the L.H.S. of (6.18), we have (see the proof of [64, (4.38)], [11, (6.43)])

$$\begin{aligned}
\lambda_k^2 \times (\text{L.H.S. of (6.24)}) =& -\frac{1}{4}\|yQ\|_2^2 (\lambda_k^2 \dot{\gamma}_k + \gamma_k^2) + M_k \\
&+ O\Big((P + \|R\|_{L^2} + e^{-\frac{\delta}{T-t}}) Mod + P^2 \|R\|_{L^2} + \|R\|_{L^2}^2 + e^{-\frac{\delta}{T-t}}\Big). \qquad (6.25)
\end{aligned}$$

Next we show that the R.H.S. of (6.24) contribute acceptable orders. This is mainly due to the exponentially small interactions between different blow-up profiles and to the flatness of both the regular profile z and lower order coefficients a_1, a_0 at the singularities.

To be precise, in view of Lemma 2.5 and (2.13), we have that for some $\delta > 0$,

$$\Big|\langle \sum_{j\neq k} \Big(\Delta R_j + (1+\frac{2}{d})|U_j|^{\frac{4}{d}} R_j + \frac{2}{d}|U_j|^{\frac{4}{d}-2} U_j^2 \overline{R_j}\Big) + H_1, \Lambda_k U_k\rangle\Big| \leq C\lambda_k^{-2} e^{-\frac{\delta}{T-t}} \|R\|_{L^2}, \qquad (6.26)$$

$$\Big|\langle \sum_{j\neq k} \Big(i\partial_t U_j + \Delta U_j + |U_j|^{\frac{4}{d}} U_j\Big) + H_2, \Lambda_k U_k\rangle\Big| \leq C\lambda_k^{-2} e^{-\frac{\delta}{T-t}} (1 + Mod), \qquad (6.27)$$

For the third term H_3, by (6.14) and Lemma 2.5,

$$\begin{aligned}
&\Big|\operatorname{Re}\langle f'(U+z)\cdot R - f'(U)\cdot R, \Lambda_k U_k\rangle\Big| \\
\leq& C\Big(\int (|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1})|z||R|\Lambda_k U_k| dx + e^{-\frac{\delta}{T-t}}\Big) \\
\leq& C\Big(\lambda_k^{-\frac{d}{2}(\frac{4}{d}+1)} \lambda_k^{\frac{d}{2}} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} \|R\|_{L^2} + \|z\|_{L^\infty}^{\frac{4}{d}-1} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} \|R\|_{L^2} + e^{-\frac{\delta}{T-t}}\Big) \\
\leq& C\Big(\lambda_k^{-2} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} D + e^{-\frac{\delta}{T-t}}\Big). \qquad (6.28)
\end{aligned}$$

Moreover, by (6.16) and (2.61),

$$\begin{aligned}
|\langle f''(U+z,R)\cdot R^2, \Lambda_k U_k\rangle| \leq& C\int \Big(|U|^{\frac{4}{d}-1} + |z|^{\frac{4}{d}-1} + |R|^{\frac{4}{d}-1}\Big)|R|^2 |\Lambda_k U_k| dx \\
\leq& C\Big(\lambda_k^{-2}\|R\|_{L^2}^2 + \lambda_k^{-2} \|\varepsilon_{z,k}\|_{L^\infty}^{\frac{4}{d}-2} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} \|R\|_{L^2}^2 + \lambda_k^{-\frac{d}{2}} \|R\|_{L^{1+\frac{4}{d}}}^{1+\frac{4}{d}} + e^{-\frac{\delta}{T-t}}\Big) \\
\leq& C\Big(\lambda_k^{-2} D^2 + e^{-\frac{\delta}{T-t}}\Big). \qquad (6.29)
\end{aligned}$$

Hence, we conclude from (6.28) and (6.29) that

$$\operatorname{Re}\langle H_3, \Lambda_k U_k\rangle = O\Big(\lambda_k^{-2} \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} D + \lambda_k^{-2} D^2 + e^{-\frac{\delta}{T-t}}\Big). \qquad (6.30)$$

We also see that

$$|\mathrm{Re}\langle H_4, \Lambda_k U_k\rangle| \leq C \sum_{j=1}^{4/d} \int |U|^{1+\frac{4}{d}-j}|z|^j|\Lambda_k U_k| dx \leq C\lambda_k^{-2}\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty}. \tag{6.31}$$

Regarding the H_5 term on the R.H.S. of (6.24), by Lemma 2.5, the change of variables and integrating by parts formula,

$$\begin{aligned}\mathrm{Re}\langle H_5, \Lambda_k U_k\rangle =&\mathrm{Re}\langle \lambda_k^{-1}\widetilde{a}_{1,k}\cdot\nabla(Q_k+\varepsilon_k)+\widetilde{a}_{0,k}(Q_k+\varepsilon_k), \Lambda Q_k\rangle + \mathcal{O}(e^{-\frac{\delta}{T-t}})\\ =&-\lambda_k^{-1}\mathrm{Re}\langle \mathrm{div}\widetilde{a}_{1,k}\,(Q_k+\varepsilon_k), \Lambda Q_k\rangle - \lambda_k^{-1}\mathrm{Re}\langle Q_k+\varepsilon_k, \overline{\widetilde{a}_{1,k}}\cdot\nabla(\Lambda Q_k)\rangle\\ &+\mathrm{Re}\langle\widetilde{a}_{0,k}(Q_k+\varepsilon_k),\Lambda Q_k\rangle + \mathcal{O}(e^{-\frac{\delta}{T-t}})\end{aligned} \tag{6.32}$$

where $\widetilde{a}_{1,k}$ and $\widetilde{a}_{0,k}$ are defined as in Lemma 2.8. Then, applying Lemma 2.8 we obtain

$$|\mathrm{Re}\langle H_5, \Lambda_k U_k\rangle| \leq C\left(\lambda_k^{-2}P^{\upsilon_*+1}+e^{-\frac{\delta}{T-t}}\right). \tag{6.33}$$

Hence, it follows from estimates (6.26), (6.27), (6.30), (6.31) and (6.33) that

$$\text{R.H.S. of (6.24)} \leq C\lambda_k^{-2}\left(e^{-\frac{\delta}{T-t}}Mod + D^2 + \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + P^{\upsilon_*+1} + e^{-\frac{\delta}{T-t}}\right). \tag{6.34}$$

Now, combining (6.25) and (6.34) together we conclude that for each $1 \leq k \leq K$,

$$\begin{aligned}|\lambda_k^2\dot{\gamma}_k+\gamma_k^2| \leq& C\Big((P+\|R\|_{L^2}+e^{-\frac{\delta}{T-t}})Mod + |M_k| + P^2D + D^2\\ &+\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + P^{\upsilon_*+1} + e^{-\frac{\delta}{T-t}}\Big).\end{aligned} \tag{6.35}$$

Similar arguments apply to the remaining four modulation equations $|\lambda_k\dot{\alpha}_k - 2\beta_k|$, $|\lambda_k\dot{\lambda}_k+\gamma_k|$, $|\lambda_k^2\dot{\beta}_k+\beta_k\gamma_k|$ and $|\lambda_k^2\dot{\theta}_k - 1 - |\beta_k|^2|$, by taking the inner products of equation (6.18) with $i(x-\alpha_k)U_k$, $i|x-\alpha_k|^2U_k$, ∇U_k, ϱ_k, respectively, and then taking the real parts. This leads to

$$Mod_k(t) \leq C\Big((P+\|R\|_{L^2}+e^{-\frac{\delta}{T-t}})Mod + |M_k| + P^2D + D^2 + \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + P^{\upsilon_*+1} + e^{-\frac{\delta}{T-t}}\Big). \tag{6.36}$$

Therefore, taking t^* even closer to T such that

$$(1+C)(P(t)+\|R(t)\|_{C([t^*,T_*];H^1)}+e^{-\frac{\delta}{T-t}}) \leq \frac{1}{2}$$

and then summing over k and using (2.29) we obtain (2.9).

(*iii*) *Improved estimate of* $\lambda_k\dot{\lambda}_k+\gamma_k$. Taking the inner product of equation (6.18) with $|x-\alpha_k|^2U_k$, then taking the imaginary part and arguing as in the proof of (6.34) we have that, similarly to (6.24),

$$\begin{aligned}&\mathrm{Re}\langle\partial_t R, |x-\alpha_k|^2U_k\rangle + \mathrm{Im}\langle\Delta R_k + (1+\frac{2}{d})|U_k|^{\frac{4}{d}}R_k + \frac{2}{d}|U_k|^{\frac{4}{d}-2}U_k^2\overline{R_k}, |x-\alpha_k|^2U_k\rangle\\ &+\mathrm{Im}\langle i\partial_t U_k + \Delta U_k + |U_k|^{\frac{4}{d}}U_k, |x-\alpha_k|^2U_k\rangle\\ =&\mathcal{O}\left(D^2+\|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + P^{\upsilon_*+1} + e^{-\frac{\delta}{T-t}}\right).\end{aligned} \tag{6.37}$$

Note that, the bound on the R.H.S. above equals to (6.34) multiplied by λ_k^2, which essentially relies on the exponential decay of ground state. We also used the fact that, by (2.9), $Mod = \mathcal{O}(1)$, and thus $e^{-\frac{\delta}{T-t}}Mod = \mathcal{O}(e^{-\frac{\delta}{T-t}})$.

Regarding the L.H.S. of (6.37), by the orthogonality condition (2.5), (2.13) and Lemma 2.5,

$$\begin{aligned}\mathrm{Re}\langle\partial_t R, |x-\alpha_k|^2U_k\rangle &= 2\dot{\alpha}_k\cdot\mathrm{Re}\langle R, (x-\alpha_k)U_k\rangle - \mathrm{Re}\langle R, |x-\alpha_k|^2\partial_t U_k\rangle\\ &= -\mathrm{Re}\langle R_k, |x-\alpha_k|^2\partial_t U_k\rangle + \mathcal{O}(e^{-\frac{\delta}{T-t}}\|R\|_{L^2}).\end{aligned} \tag{6.38}$$

Then, using (2.3), (2.13), (2.65) and the algebraic identity

$$\Delta Q_k - Q_k + |Q_k|^{\frac{4}{d}} Q_k = |\beta_k - \frac{\gamma_k}{2} y|^2 Q_k - i\gamma_k \Lambda Q_k + 2i\beta_k \cdot \nabla Q_k \tag{6.39}$$

we get

$$\begin{aligned} -\mathrm{Re}\langle R_k, |x-\alpha_k|^2 \partial_t U_k\rangle =& -\mathrm{Im}\langle \varepsilon_k, |y|^2(\Delta Q_k + |Q_k|^{\frac{4}{d}} Q_k)\rangle + \mathcal{O}(Mod\|\varepsilon_k\|_{L^2}) \\ =& -\mathrm{Im}\langle \varepsilon_k, |y|^2 Q_k\rangle - \gamma_k \mathrm{Re}\langle \varepsilon_k, |y|^2 \Lambda Q_k\rangle + 2\beta_k \cdot \mathrm{Re}\langle \varepsilon_k, |y|^2 \nabla Q_k\rangle \\ & + \mathcal{O}\Big((Mod + P^2)D\Big), \end{aligned}$$

By the integration by parts formula and the almost orthogonality (2.66),

$$\begin{aligned} & -\gamma_k \mathrm{Re}\langle \varepsilon_k, |y|^2 \Lambda Q_k\rangle + 2\beta_k \mathrm{Re}\langle \varepsilon_k, |y|^2 \nabla Q_k\rangle \\ =& \gamma_k \mathrm{Re}\langle \Lambda\varepsilon_k, |y|^2 Q_k\rangle - 2\beta_k \mathrm{Re}\langle \nabla\varepsilon_k, |y|^2 Q_k\rangle + \mathcal{O}(e^{-\frac{\delta}{T-t}}\|R\|_{L^2}). \end{aligned} \tag{6.40}$$

Thus, we obtain

$$\begin{aligned} \mathrm{Re}\langle \partial_t R, |x-\alpha_k|^2 U_k\rangle =& -\mathrm{Im}\langle \varepsilon_k, |y|^2 Q_k\rangle + \gamma_k \mathrm{Re}\langle \Lambda\varepsilon_k, |y|^2 Q_k\rangle - 2\beta_k \cdot \mathrm{Re}\langle \nabla\varepsilon_k, |y|^2 Q_k\rangle \\ & + \mathcal{O}\Big((Mod + P^2 + e^{-\frac{\delta}{T-t}})D\Big). \end{aligned} \tag{6.41}$$

Furthermore, using (2.13), the identities

$$\Lambda Q_k = \left(\Lambda Q + i(\beta_k \cdot y - \frac{1}{2}\gamma_k |y|^2) Q\right) e^{i(\beta_k \cdot y - \frac{1}{4}\gamma_k |y|^2)}, \tag{6.42}$$

$$\nabla Q_k = \left(\nabla Q + i(\beta_k - \frac{1}{2}\gamma_k y) Q\right) e^{i(\beta_k \cdot y - \frac{1}{4}\gamma_k |y|^2)}, \tag{6.43}$$

and $\langle \Lambda Q, |y|^2 Q\rangle = -\|yQ\|_{L^2}^2$ we compute

$$\mathrm{Im}\langle i\partial_t U_k + \Delta U_k + |U_k|^{\frac{4}{d}} U_k, |x-\alpha_k|^2 U_k\rangle = (\lambda_k \dot{\lambda}_k + \gamma_k)\|yQ\|_{L^2}^2. \tag{6.44}$$

Therefore, plugging (6.41) and (6.44) into (6.37) and using the change of variables, the bound $|M_k| \leq CD$ and (2.9) we obtain the equation for the renormalized variable ε_k below

$$\begin{aligned} & \mathrm{Im}\langle \Delta\varepsilon_k - \varepsilon_k + (1+\frac{2}{d})|Q_k|^{\frac{4}{d}}\varepsilon_k + \frac{2}{d}|Q_k|^{\frac{4}{d}-2} Q_k^2 \overline{\varepsilon_k}, |y|^2 Q_k\rangle \\ & + \gamma_k \mathrm{Re}\langle \Lambda\varepsilon_k, |y|^2 Q_k\rangle - 2\beta_k \cdot \mathrm{Re}\langle \nabla\varepsilon_k, |y|^2 Q_k\rangle + (\lambda_k \dot{\lambda}_k + \gamma_k)\|yQ\|_{L^2}^2 \\ =& \mathcal{O}(P^2 D + D^2 + \|e^{-\delta|y|}\varepsilon_{z,k}\|_{L^\infty} + P^{\nu_*+1} + e^{-\frac{\delta}{T-t}}). \end{aligned} \tag{6.45}$$

Writing the L.H.S. of (6.45) in terms of real and imaginary parts we see that the first three terms are exactly the first line of [63, (4.27)] and hence are of order $\mathcal{O}(P^2\|R\|_{L^2})$, due to [63, (4.28)]. This yields that

$$\text{L.H.S. of (6.45)} = (\lambda_k \dot{\lambda}_k + \gamma_k)\|yQ\|_{L^2}^2 + \mathcal{O}(P^2 D). \tag{6.46}$$

Therefore, plugging this into (6.45) and using (2.29) we obtain the desired estimate (2.12). □

Acknowledgements

M. Röckner and D. Zhang thank for the financial support by the Deutsche Forschungsgemeinschaft (DFG, German Science Foundation) through SFB 1283/2 2021-317210226 at Bielefeld University. Y. Su is supported by NSFC (No. 11601482). D. Zhang is also grateful for the support by NSFC (No. 11871337) and Shanghai Rising-Star Program 21QA1404500.

References

[1] O. Bang, P.L. Christiansen, F. If, K.O. Rasmussen, Temperature effects in a nonlinear model of monolayer Scheibe aggregates. *Phys. Rev. E* **49** (1994), 4627–4636.

[2] O. Bang, P.L. Christiansen, F. If, K.O. Rasmussen, Y.B. Gaididei, White noise in the two-dimensional nonlinear Schrödinger equation, *Appl. Anal*. **57** (1995), no. 1-2, 3–15.

[3] V. Barbu, M. Röckner, D.Zhang, The stochastic nonlinear Schrödinger equation with multiplicative noise: the rescaling aproach, *J. Nonlinear Sciences*, **24** (2014), 383–409.

[4] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equations. *Nonlinear Anal*. **136** (2016), 168–194.

[5] V. Barbu, M. Röckner, D. Zhang, The stochastic logarithmic Schrödinger equation. *J. Math. Pures Appl.* **(9)** 107 (2017), no. 2, 123-149.

[6] V. Barbu, M. Röckner, D. Zhang, Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise. *Ann. Probab*. **46** (2018), no. 4, 1957–1999.

[7] A. Barchielli, M. Gregoratti, Quantum Trajectories and Measurements in Continuous Case. The Diffusive Case, *Lecture Notes Physics* **782**, Springer Verlag, Berlin, 2009.

[8] M. Beceanu, A critical center-stable manifold for Schrödinger's equation in three dimensions. *Comm. Pure Appl. Math.* **65** (2012), no. 4, 431–507.

[9] J. Bourgain, W. Wang, Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **(4)** **25** (1997), no. 1-2, 197–215.

[10] Z. Brzézniak, A. Millet, On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Anal.* **41** (2014), no. 2, 269–315.

[11] D. Cao, Y. Su, D. Zhang, On uniqueness of multi-bubble blow-up solutions and multi-solitons to L^2-critical nonlinear Schrödinger equations, `arXiv:2105.14554`.

[12] T. Cazenave, Semilinear Schrödinger equations. *Courant Lecture Notes in Mathematics*, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[13] T. Cazenave, An overview of the nonlinear Schrödinger equation. Lecture notes, 2020. https://www.ljll.math.upmc.fr/cazenave/.

[14] V. Cômbet, Multi-soliton solutions for the supercritical gKdV equations. *Comm. Partial Differential Equations*. **36** (2011), no. 3, 380–419.

[15] R. Côte, Construction of solutions to the subcritical gKdV equations with a given asymptotical behavior. *J. Funct. Anal.* **241** (2006), no. 1, 143–211.

[16] R. Côte, Construction of solutions to the L^2-critical KdV equation with a given asymptotic behaviour. *Duke Math. J.* **138** (2007), no. 3, 487–531.

[17] R. Côte, X. Friederich, On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations. 2020. hal-02873307v2, to appear in *Commun. Partial Differ. Equ.*.

[18] R. Côte, C. Kenig, A. Lawrie, W. Schlag, Profiles for the radial focusing 4d energy-critical wave equation. *Comm. Math. Phys.* **357** (2018), no. 3, 943–1008.

[19] R. Côte, S. Le Coz, High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. Pures Appl.* **(9)** 96 (2011), no. 2, 135–166.

[20] R. Côte, Y. Martel, F. Merle, Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.* **27** (2011), no. 1, 273–302.

[21] A. de Bouard, A. Debussche, On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probab. Theory Related Fields* **123** (2002), no. 1, 76–96.

[22] A. de Bouard, A. Debusche, The stochastic nonlinear Schrödinger equation in H^1, *Stoch. Anal. Appl.* **21** (2003), 97–126.

[23] A. de Bouard, A. Debussche, Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise. *Ann. Probab.* **33** (2005), no. 3, 1078–1110.

[24] A. de Bouard, A. Debussche, L.D. Menza, Theoretical and numerical aspects of stochastic nonlinear Schrödinger equations, *Journées "Équations aux Dérivées Partielles"* (Plestin-les-Grèves, 2001), Exp. No. III, 13 pp., Univ. Nantes, Nantes, 2001.

[25] A. Debussche, L.D. Menza, Numerical simulation of focusing stochastic nonlinear Schrödinger equations, *Phys. D* **162** (2002), no. 3-4, 131–154.

[26] A. Debussche, L.D. Menza, Numerical resolution of stochastic focusing NLS equations. *Appl. Math. Lett.* **15** (2002), no. 6, 661–669.

[27] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. *Adv. Math.* **285** (2015), 1589–1618.

[28] T. Duyckaerts, H. Jia, C. Kenig, F. Merle, Soliton resolution along a sequence of times for the focusing energy critical wave equation. *Geom. Funct. Anal.* **27** (2017), no. 4, 798–862.

[29] T. Duyckaerts, C. Kenig, F. Merle, Classification of radial solutions of the focusing, energy-critical wave equation. *Camb. J. Math.* **1** (2013), no. 1, 75–144.

[30] T. Duyckaerts, C. Kenig, F. Merle, Soliton resolutin for the radial critical wave equation in all odd space dimensions. `arXiv:1912.07664v1`.

[31] S. Dyachenko, A.C. Newell, A. Pushkarev, V.E. Zakharov, Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schrödinger equation. *Phys. D* **57** (1992), no. 1–2, 96–160.

[32] C.J. Fan, log-log blow up solutions blow up at exactly m points. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), no. 6, 1429–1482.

[33] C.J. Fan, Y. Su, D. Zhang, A note on log-log blow up solutions for stochastic nonlinear Schrödinger equations, `arXiv:2011.12171,` to appear in *Stoch. Partial Differ. Equ. Anal. Comput.*.

[34] P. Friz, M. Hairer, A Course on Rough Paths. With an Introduction to Regularity Structures. *Universitext*. Springer, Cham, 2014. xiv+251 pp.

[35] M. Gubinelli, Controlling rough paths. *J. Funct. Anal.* **216** (2004), no. 1, 86–140.

[36] S.J. Gustafson, I.M. Sigal, Mathematical Concepts of Quantum Mechanics. Second edition. *Universitext*. Springer, Heidelberg, 2011. xiv+382 pp.

[37] S. Herr, M. Röckner, D. Zhang, Scattering for stochastic nonlinear Schrödinger equations. *Comm. Math. Phys*. **368** (2019), no. 2, 843–884.

[38] R. Killip, M. Visan, Nonlinear Schrödinger equations at critical regularity. Evolution equations, 325-437, *Clay Math. Proc.,* **17**, Amer. Math. Soc., Providence, RI, 2013.

[39] K. Kim, S. Kwon, On pseudoconformal blow-up solutions to the self-dual Chern-Simons-Schrödinger equation: existence, uniqueness, and instability, `arXiv:1909.01055`, to appear in *Mem. Amer. Math. Soc*.

[40] J. Krieger, Y. Martel, P. Raphaël, Two-soliton solutions to the three-dimensional gravitational Hartree equation. *Comm. Pure Appl. Math.* **62** (2009), no. 11, 1501–1550.

[41] J. Krieger, W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *J. Amer. Math. Soc.* **19** (2006), no. 4, 815–920.

[42] J. Krieger, W. Schlag, Non-generic blow-up solutions for the critical focusing NLS in 1-d. *J. Eur. Math. Soc.* **11** (2009), no.1, 1–125.

[43] S. Le Coz, D. Li, T.P. Tsai, Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), no. 6, 1251–1282.

[44] S. Le Coz, T.P. Tsai, Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity* **27** (2014), no. 11, 2689–2709.

[45] Y. Martel, Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.* **127** (2005), no. 5, 1103–1140.

[46] Y. Martel, F. Merle, Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23** (2006), no. 6, 849–864.

[47] Y. Martel, P. Raphaël, Strongly interacting blow up bubbles for the mass critical nonlinear Schrödinger equation. *Ann. Sci. Éc. Norm. Supér.* **51** (4) (2018), no. 3, 701-737.

[48] J. Marzuola, J. Metcalfe, D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, *J. Funct. Anal.*, **255** (6) (2008), 1479–1553.

[49] F. Merle, Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.* **129** (1990), no. 2, 223–240.

[50] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.* **69** (1993), no. 2, 427–454.

[51] F. Merle, P. Raphaël, Sharp upper bound on the blow up rate for critical nonlinear Schrödinger equation. *Geom. Funct. Anal.* **13** (2003), 591–642.

[52] F. Merle, P. Raphaël, On universality of blow up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.* **156** (2004), 565–672.

[53] F. Merle, P. Raphaël, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math.* 161 (2005), no. 1, 157–222.

[54] F. Merle, P. Raphaël, Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation. *Comm. Math. Phys.* 253 (2005), no. 3, 675–704.

[55] F. Merle, P. Raphaël, Sharp lower bound on the blow up rate for critical nonlinear Schrödinger equation. *J. Amer. Math. Soc.* **19** (2006), no. 1, 37–90.

[56] F. Merle, P. Raphaël, J. Szeftel, The instability of Bourgain-Wang solutions for the L^2 critical NLS. *Amer. J. Math.* **135** (2013), no. 4, 967–1017.

[57] A. Millet, A.D. Rodriguez, S. Roudenko, K. Yang, Behavior of solutions to the 1d focusing stochastic nonlinear Schrödinger equation with spatially correlated noise, `arXiv:2006.10695v1.`

[58] A. Millet, S. Roudenko, K. Yang, Behavior of solutions to the 1d focusing stochastic L^2-critical and supercritical nonlinear Schrödinger equation with space-time white noise, `arXiv:2005.14266v1.`

[59] G. Perelman, On the blow up phenomenon for the critical nonlinear Schrödinger equation in 1D. *Ann. Henri. Poincaré* **2** (2001), 605-673.

[60] P. Raphaël, J. Szeftel, Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS. *J. Amer. Math. Soc.* **24** (2011), no. 2, 471–546.

[61] K.O. Rasmussen, Y.B. Gaididei, O. Bang, P.L. Chrisiansen, The influence of noise on critical collapse in the nonlinear Schrödinger equation. *Phys. Letters A* **204** (1995), 121–127.

[62] W. Schlag, Stable manifolds for an orbitally unstable nonlinear Schrödinger equation. *Ann. of Math.* **(2) 169** (2009), no. 1, 139–227.

[63] Y. Su, D. Zhang, Minimal mass blow-up solutions to rough nonlinear Schrödinger equations. `arXiv:2002.09659v1.`

[64] Y. Su, D. Zhang, On the multi-bubble blow-up solutions to rough nonlinear Schrödinger equations, `arXiv:2012.14037v1.`

[65] C. Sulem, P.L. Sulem, The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse. *Applied Mathematical Sciences* **139**, Springer, New York, 1999.

[66] T. Tao, Nonlinear Dispersive Equations. Local and Global Analysis. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.

[67] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* **87** (1982/83), no. 4, 567–576.

[68] M. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* **16** (1985), no. 3, 472–491.

[69] D. Zhang, Strichartz and local smoothing estimates for stochastic dispersive equations. `arXiv:1709.03812`

[70] D. Zhang, Optimal bilinear control of stochastic nonlinear Schrödinger equations: mass-(sub)critical case. *Probab. Theory Related Fields* **178** (2020), no. 1–2, 69–120.

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: `roeckner@math.uni-bielefeld.de`

Department of mathematics, Zhejiang University of Technology, 310014 Zhejiang, China
E-mail address: `yimingsu@zjut.edu.cn`

School of mathematical sciences, Shanghai Jiao Tong University, 200240 Shanghai, China
E-mail address: `dzhang@sjtu.edu.cn`