# Stochastic Generalized Porous Media Equations over $\sigma$-finite Measure Spaces with Non-continuous Nonlinearity *

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**Abstract.** In this paper, we prove that stochastic porous media equations over $\sigma$-finite measure spaces $(E, \mathcal{B}, \mu)$, driven by time-dependent multiplicative noise, with the Laplacian replaced by a self-adjoint transient Dirichlet operator $L$ and the nonlinearity given by a maximal monotone multi-valued function $\Psi$ of polynomial growth, have a unique solution. This generalizes previous results in that we work on general measurable state spaces, allow non-continuous (nonlinear) monotone functions $\Psi$, for which, no further coercivity assumptions are needed, but only that their multi-valued extensions are maximal monotone and of at most polynomial growth. The result in particular applies to cases where $E$ is a manifold or a fractal, and to non-local operators $L$, as e.g. $L = -(-\Delta)^\alpha$, $\alpha \in (0, \frac{d}{2}) \cap (0, 1]$.

**Keywords:** Wiener process; Porous media equation; Dirichlet form; Maximal monotone graph; Yosida approximation; $L^p$-Itô formula in expectation.

## 1 Introduction

The purpose of this paper is to solve multi-valued stochastic porous media equations (SPMEs) on $(E, \mathcal{B}, \mu)$ of the following type:

$$\begin{cases} dX(t) - L\Psi(X(t))dt \ni B(t, X(t))dW(t), \text{ in } [0, T] \times E, \\ X(0) = x \text{ on } E \ (x \in \mathscr{F}_e^*), \end{cases} \tag{1.1}$$

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where $(E, \mathcal{B})$ is a standard measurable space (see [31]) with a $\sigma$-finite measure $\mu$. $(L, D(L))$ is the generator of a symmetric strongly continuous contraction sub-Markovian semigroup on $L^2(\mu)$, which additionally is assumed to be the generator of transient Dirichlet form (cf. Section 2.1 below). $\Psi(\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ denotes a maximal monotone graph with polynomial growth (cf. (H1) in Section 3 below). $B$ is a Hilbert-Schmidt operator-valued map fulfilling certain Lipschitz and growth conditions (cf. (H2) and (H3) in Section 3 below). $W$ is an $L^2(\mu)$-valued cylindrical $\mathscr{F}_t$-adapted Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal filtration $(\mathscr{F}_t)_{t\geq 0}$. Explicit assumptions and more explanations will be given in Section 3.

One motivation for studying this equation is that an important problem from physics, i.e., the following self-organized criticality (SOC) model, is of type (1.1):

$$dX(t) = \Delta H(X(t) - x_c)dt + (X(t) - x_c)dW(t), \tag{1.2}$$

where $H$ is the Heaviside function and $x_c$ is the critical state (see [9, 19]). Eq (1.2) is a continuum version of the original sand pile model or the Bak-Tang-Wiesenfeld (BTW) model [3, 2] via the cellular automaton algorithm. SOC systems have the properties of a critical point as attractor and to reach spontaneously a critical state. Finite time extinction for fast diffusions, which are also special cases of (1.1), will be done in future work. Apart from the SOC phenomenon mentioned above, Eq (1.1) models the dynamics of flows in porous media, the phase transitions (including melting and solidification processes), diffusion processes in kinetic gas theory, heat transfer in plasmas and population dynamics.

At least since [14], SPMEs with maximal monotone (possibly multi-valued) functions $\Psi$, have been studied in a variety of papers, see e.g. [8, 6, 7, 5, 10, 15, 24] and the references there in. (For the deterministic case, we refer to [38] including its references.) In the special case with $E$ being $\mathbb{R}^d$, $d \geq 3$, $L$ is equal to the Laplace operator $\Delta$ and $B$ is time-independent linear multiplicative, in [10, Section 4] the existence and uniqueness of solutions in $\mathcal{H}^{-1}$ for (1.1) were proved. Here $\mathcal{H}^{-1}$ is the dual space of $\mathcal{H}$ with

$$\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d); \xi \mapsto |\xi|\mathcal{F}(\varphi)(\xi) \in L^2(\mathbb{R}^d)\},$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of all tempered distributions on $\mathbb{R}^d$ and $\mathcal{F}(\varphi)$ is the Fourier transform of $\varphi$. The intention of this paper is to obtain analogous results as in [10, Section4] on more general spaces and more general operators $L$.

A natural approach to get the existence of solutions for (1.1) is to consider approximating equations of the following form with initial value $X_\lambda(0) \in \mathscr{F}_e^*$ (:=dual of the extended transient Dirichlet space with generator $L$; see Section 2.1):

$$dX_\lambda - L(\Psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = B(t, X_\lambda)dW(t), \quad t \in (0, T). \tag{1.3}$$

Here $\lambda > 0$ and

$$\Psi_\lambda(x) = \frac{1}{\lambda}\big(x - (1 + \lambda\Psi)^{-1}(x)\big) \in \Psi\big((1 + \lambda\Psi)^{-1}(x)\big)$$

is the Yosida approximation of $\Psi$. Then passing to the limit $\lambda \to 0$ we solve (1.1). In [35] the authors construct a suitable Gelfand triple with $\mathscr{F}_e^*$ as pivôt space and prove existence and uniqueness of solutions for the following stochastic generalized porous media equation in the state space $\mathscr{F}_e^*$:

$$dX(t) = (L\Psi(t, X(t)) + \Phi(t, X(t)))dt + B(t, X(t))dW(t),$$

where $L$ is as above, but $\Psi$ is only a single-valued map (as is $\Phi$) and $\Psi$ satisfies a certain coercivity condition, which is not assumed in this paper. So, our result is more general in the case $\Phi = 0$. In [36] the results in [35] are improved to initial conditions in the dual space of the Dirichlet space generated by $L$ which is larger than $\mathscr{F}_e^*$, but to achieve this, the condition that the Dirichlet form is local and satisfies Nash's inequality has to be imposed. However, all these results are restricted to single-valued continuous functions $\Psi$, not including the case of noncontinuous functions $\Psi$, which is covered in this paper. As said before, another main point of this paper is that we can drop the coercivity assumption on $\Psi$ made in [35, 36], only assuming its maximal monotonicity and its polynomial growth. In this paper we analyze (1.1) also in $L^{m+1}(\mu)$ (i.e., with initial condition $x \in L^{m+1}(\mu)$), where $m \in (1,\infty)$ being the exponential in the polynomial growth condition for $\Psi$ (see Hypothesis (H1) in Section 3). A crucial ingredient in our proofs is, therefore, an $L^p(\mu)$-Itô formula, which in the case $E = \mathbb{R}^d$ was proved in [26]. In the latter paper approximations by convolution with smooth functions were crucial. Since our space $E$ has no further structure, we could not use this approach in our case. As a substitute we use an $L^p$-Itô formula in expectation to get some crucial a priori $L^p(\mu)$-estimates for our approximating solutions. We include a complete proof for this type of $L^p(\mu)$-Itô formula in Appendix 7 of this paper. In comparison with [10], i.e., the special case $E = \mathbb{R}^d$, $L = \Delta$, we use the same strategy of proof, i.e., we use a similar "triple approximation" to solve equation (1.1). But due to our much more general situation, our proofs are much more involved with a substantial number of obstacles to be overcome, which do not occur in [10].

This paper is organized as follows. In Section 2, we introduce some notations and recall some known results for preparation. In addition, we prove some necessary technical auxiliary results, which will be used to construct the solutions to (1.1) in $\mathscr{F}_e^*$. In Section 3, we will present our assumptions and the two main results for (1.1) and (1.3). Detailed proofs of the existence and uniqueness results for (1.3) will be given in Section 4, while the ones for (1.1) will be given in Section 5. Some examples that are covered under our framework will be presented in Section 6, including nonlocal operators $L$. In order to make the main structure of the proofs more transparent, we shift the proofs of some estimates to Appendix 7.1. In addition, we include a detailed proof of an $L^p(\mu)$-Itô formula in expectation in Appendix 7.2, which is crucial for the proof of our main result. In Appendix 7.3, some explanations are included to justify the application of Itô's formula on Gelfand triples (see e.g. [29]) in our cases.

# 2 Notations and preliminaries

## 2.1 Dirichlet spaces and auxiliary results

Let $(E, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space, which we fix in the entire paper. We assume that $(E, \mathcal{B})$ is a standard measurable space (i.e., $\sigma$-isomorphic to a Polish space, see [31]). This assumption is used in the proof of the $L^p(\mu)$-Itô formula in expectation, but also in the proof of Lemma 4.1 below, where we apply [36, Lemma 5.1], in which this assumption on $(E, \mathcal{B})$ was crucially used. Let $(P_t)_{t\geq 0}$ be a strongly continuous, symmetric, sub-Markovian contraction semigroup on $L^2(\mu)$. Let $(L, D(L))$ be its infinitesimal generator (see e.g. [18, 30]), which is a negative definite self-adjoint operator on $L^2(\mu)$. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|_2$ for the inner product and the norm in $L^2(\mu)$ respectively. More generally, we set $\langle f, g\rangle := \mu(fg) := \int fg d\mu$ for any two measurable functions $f$, $g$ such that $fg \in L^1(\mu)$. For the rest of this paper we fix $(P_t)_{t\geq 0}$ with generator $(L, D(L))$ on $L^2(\mu)$ with $(E, \mathcal{B}, \mu)$ as above.

Consider the $\Gamma$-transform $V_r(r>0)$ of $(P_t)_{t\geq 0}$

$$V_r u=\Gamma(\frac{r}{2})^{-1}\int_0^\infty s^{\frac{r}{2}-1}e^{-s}P_s u ds,\ r>0,\ u\in L^2(\mu).$$

From [17] and [23], we can define the Bessel-potential space $(F_{1,2},\|\cdot\|_{F^*_{1,2}})$ by

$$F_{1,2}:=V_1(L^2(\mu)),\text{ with norm } \|u\|_{F_{1,2}}=|f|_2,\quad\text{for}\ \ u=V_1 f,\quad f\in L^2(\mu),$$

where the norm $|\cdot|_2$ is defined as $|f|_2=\langle f,f\rangle_2:=(\int_E|f|^2d\mu)^{\frac{1}{2}}$. From [17], we know that

$$V_1=(1-L)^{-\frac{1}{2}},\quad\text{so that}\quad F_{1,2}=D\big((1-L)^{\frac{1}{2}}\big)\quad\text{and}\quad\|u\|_{F_{1,2}}=|(1-L)^{\frac{1}{2}}u|_2.$$

The dual space of $F_{1,2}$ is denoted by $F^*_{1,2}$ and $F^*_{1,2}=D((1-L)^{-\frac{1}{2}})$, it is equipped with the norms

$$\|\eta\|_{F^*_{1,2,\nu}}:=\langle\eta,(\nu-L)^{-1}\eta\rangle_2^{\frac{1}{2}},\quad\eta\in F^*_{1,2},\quad 0<\nu<\infty.$$

Denote the duality between $F^*_{1,2}$ and $F_{1,2}$ by ${}_{F^*_{1,2}}\langle\cdot,\cdot\rangle_{F_{1,2}}$.

Consider the Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(\mu)$ associated with $(L,D(L))$, i.e.,

$$\begin{aligned}&D(\mathscr{E}):=F_{1,2},\quad\text{and}\\&\mathscr{E}(u,v):=\mu(\sqrt{-L}u\sqrt{-L}v),\quad u,v\in F_{1,2}.\end{aligned}$$

Let $D(\mathscr{E})$ be equipped with the inner product $\mathscr{E}_1:=\mathscr{E}+\langle\cdot,\cdot\rangle_2$.

If $(\mathscr{E},D(\mathscr{E}))$ is a transient Dirichlet space, that is, there exists $g\in L^1(\mu)\cap L^\infty(\mu)$, $g>0$, such that $\mathscr{F}_e\subset L^1(g\cdot\mu)$ continuously, let $(\mathscr{E},\mathscr{F}_e)$ be the corresponding extended Dirichlet space (see [18]), which is the completion of $F_{1,2}$, with respect to the norm

$$\|\cdot\|_{\mathscr{F}_e}:=\mathscr{E}(\cdot,\cdot)^{\frac{1}{2}}.$$

Then $F_{1,2}=\mathscr{F}_e\cap L^2(\mu)$. Let $\mathscr{F}^*_e$ be its dual space with inner product $\langle\cdot,\cdot\rangle_{\mathscr{F}^*_e}$ and corresponding norm $\|\cdot\|_{\mathscr{F}^*_e}$, which is induced by the Riesz map $\mathscr{F}_e\ni u\mapsto\mathscr{E}(\cdot,u)\in\mathscr{F}^*_e$. Denote the duality between $\mathscr{F}^*_e$ and $\mathscr{F}_e$ by ${}_{\mathscr{F}^*_e}\langle\cdot,\cdot\rangle_{\mathscr{F}_e}$. Both $\mathscr{F}_e$ and $\mathscr{F}^*_e$ are Hilbert spaces. For more background knowledge on Dirichlet forms, we refer to [18, 30]. From now on we assume:

**(L.1)** The symmetric Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ associated with $(L,D(L))$ is transient.

Consider the inner product $\mathscr{E}_\nu:=\mathscr{E}+\nu\langle\cdot,\cdot\rangle_2$, $\nu\in(0,\infty)$, on $F_{1,2}$, i.e.,

$$\|v\|^2_{F_{1,2,\nu}}:=\mathscr{E}(v,v)+\nu\int|v|^2d\mu=\|v\|^2_{\mathscr{F}_e}+\nu\int|v|^2d\mu,\ \text{for}\ v\in F_{1,2},\tag{2.1}$$

and

$$\|l\|_{F^*_{1,2,\nu}}:={}_{F^*_{1,2}}\langle l,(\nu-L)^{-1}l\rangle^{\frac{1}{2}}_{F_{1,2}}:=\sup_{\substack{v\in F_{1,2}\\ \|v\|_{F_{1,2,\nu}}\leq 1}}l(v),\ l\in F^*_{1,2},\tag{2.2}$$

$$\|l\|_{\mathcal{F}^*_e}:=\sup_{\substack{v\in\mathcal{F}_e\\ \|v\|_{\mathcal{F}_e}\leq 1}}l(v),\ l\in\mathcal{F}^*_e.\tag{2.3}$$

Since $F_{1,2}\subset\mathcal{F}_e$ continuously and densely, we have

$$\mathcal{F}^*_e\subset F^*_{1,2}\ \text{continuously and densely.}\tag{2.4}$$

**Proposition 2.1** *Let $l \in \mathscr{F}_e^*$. Then $\nu \mapsto \|l\|_{F^*_{1,2,\nu}}$ is decreasing,*

$$\lim_{\nu\to 0}\|l\|_{F^*_{1,2,\nu}} = \sup_{\nu>0}\|l\|_{F^*_{1,2,\nu}} = \|l\|_{\mathscr{F}_e^*}, \tag{2.5}$$

$$\|l\|_{F^*_{1,2}} \le \|l\|_{F^*_{1,2,\nu}} \le \frac{1}{\sqrt{\nu}}\|l\|_{F^*_{1,2}}, \quad \forall\ 0<\nu<1. \tag{2.6}$$

*Proof* Firstly, note that for all $l \in F^*_{1,2}$ and $0<\nu'\le\nu<\infty$, we have

$$\|l\|_{F^*_{1,2,\nu}} =: \sup_{\substack{v\in F_{1,2}\\ \|v\|_{F_{1,2,\nu}}\le 1}} l(v) \le \sup_{\substack{v\in F_{1,2}\\ \|v\|_{F_{1,2,\nu'}}\le 1}} l(v) = \|l\|_{F^*_{1,2,\nu'}},$$

i.e., $\forall\ l\in F^*_{1,2}$, $\|l\|_{F^*_{1,2,\nu}}$ is decreasing in $\nu$. In particular, the first equality in (2.5) and the first inequality in (2.6) hold.

Let $l \in \mathscr{F}_e^*$. Since $\mathscr{F}_e^* \subset F^*_{1,2}$ continuously and densely, we have $l\in F^*_{1,2}$ and

$$\|l\|_{F^*_{1,2,\nu}} = \sup_{\substack{v\in F_{1,2}\\ \|v\|_{F_{1,2,\nu}}\le 1}} l(v) \le \sup_{\substack{v\in \mathscr{F}_e\\ \|v\|_{\mathscr{F}_e}\le 1}} l(v) = \|l\|_{\mathscr{F}_e^*}.$$

Hence $\forall\ l\in\mathscr{F}_e^*$,

$$\lim_{\nu\to 0}\|l\|_{F^*_{1,2,\nu}} = \sup_{\nu>0}\|l\|_{F^*_{1,2,\nu}} \le \|l\|_{\mathscr{F}_e^*}. \tag{2.7}$$

To prove the converse inequality of (2.7), fix $l\in\mathscr{F}_e^*$ and let $\varepsilon,\ \delta\in(0,1)$. Then there exists $v_\varepsilon \in F_{1,2}$ with $\|v_\varepsilon\|_{\mathscr{F}_e}=1$ and

$$l(v_\varepsilon) \ge \|l\|_{\mathscr{F}_e^*} - \varepsilon.$$

Let $\nu_0 := \frac{\delta^2}{1+|v_\varepsilon|_2^2}$. From (2.1), we see that

$$\|v_\varepsilon\|_{F_{1,2,\nu_0}} = \sqrt{\|v_\varepsilon\|^2_{\mathscr{F}_e} + \nu_0|v_\varepsilon|_2^2} \le \sqrt{1+\delta^2} \le 1+\delta,$$

so for $\bar{v_\varepsilon} := \frac{v_\varepsilon}{1+\delta}$, we have

$$\|\bar{v_\varepsilon}\|_{F_{1,2,\nu_0}} \le 1.$$

Consequently,

$$\begin{aligned}\lim_{\nu\to 0}\|l\|_{F^*_{1,2,\nu}} &= \sup_{\nu>0}\|l\|_{F^*_{1,2,\nu}}\\ &\ge \|l\|_{F^*_{1,2,\nu_0}} \ge l(\bar{v_\varepsilon}) = \frac{1}{1+\delta}l(v_\varepsilon) \ge \frac{1}{1+\delta}(\|l\|_{\mathscr{F}_e^*}-\varepsilon),\end{aligned}$$

letting $\delta\to 0$, $\varepsilon\to 0$, yields the desired converse inequality. Hence (2.5) is proved.

It remains to prove the second inequality in (2.6). But

$$\|l\|_{F^*_{1,2}} = \sup_{\|v\|_{F_{1,2}}\le 1} l(v) = \sqrt{\nu}\sup_{\sqrt{\nu}\|v\|_{F_{1,2}}\le 1} l(v) \ge \sqrt{\nu}\sup_{\|v\|_{F_{1,2,\nu}}\le 1} l(v) = \sqrt{\nu}\|l\|_{F^*_{1,2,\nu}}.$$

□

## 2.2 Gelfand triples

Let $H$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and let $H^*$ be its dual space. Let $V$ be a reflexive Banach space, such that $V \subset H$ continuously and densely. Then for its dual space $V^*$ it follows that $H^* \subset V^*$ continuously and densely. Identifying $H$ and $H^*$ via the Riesz isomorphism we have that

$$V \subset H \subset V^*$$

continuously and densely. Let ${}_{V^*}\langle\cdot,\cdot\rangle_V$ denote the dualization between $V^*$ and $V$ (i.e. ${}_{V^*}\langle z,v\rangle_V := z(v)$ for $z \in V^*$, $v \in V$). Then it follows that

$${}_{V^*}\langle z,v\rangle_V = \langle z,v\rangle_H, \text{ for all } z \in H,\ v \in V. \tag{2.8}$$

$(V, H, V^*)$ is called a Gelfand triple.

In [37], we constructed a Gelfand triple with $V := L^2(\mu)$, $H := F^*_{1,2}$ and proved the following two lemmas.

**Lemma 2.1** *The map* $(1-L) : F_{1,2} \to F^*_{1,2}$ *is an isometric isomorphism. In particular,*

$$\big\langle (1-L)u,\ (1-L)v\big\rangle_{F^*_{1,2}} = \langle u,\ v\rangle_{F_{1,2}} \ \textit{for all}\ \ u,v \in F_{1,2}. \tag{2.9}$$

*Furthermore,* $(1-L)^{-1} : F^*_{1,2} \to F_{1,2}$ *is the Riesz isomorphism for* $F^*_{1,2}$*, i.e., for every* $u \in F^*_{1,2}$*,*

$$\langle u,\ \cdot\rangle_{F^*_{1,2}} = {}_{F_{1,2}}\langle (1-L)^{-1}u,\ \cdot\rangle_{F^*_{1,2}}. \tag{2.10}$$

**Lemma 2.2** *The map*

$$1-L : F_{1,2} \to F^*_{1,2}$$

*via the continuous embedding* $F^*_{1,2} \subset (L^2(\mu))^*$ *extends to a linear isometry*

$$1-L : L^2(\mu) \to (L^2(\mu))^*,$$

*and for all* $u,v \in L^2(\mu)$*,*

$${}_{(L^2(\mu))^*}\langle (1-L)u, v\rangle_{L^2(\mu)} = \int_E u\cdot v\ d\mu. \tag{2.11}$$

The following lemma was shown in [35, Lemma 3.3(i)].

**Lemma 2.3** *The map* $\bar{L} : \mathscr{F}_e \to \mathscr{F}^*_e$ *defined by*

$$\bar{L}v := -\mathscr{E}(v,\cdot),\ v \in \mathscr{F}_e \tag{2.12}$$

*(i.e. the Riesz isomorphism of* $\mathscr{F}_e$ *and* $\mathscr{F}^*_e$ *multiplied by (-1)) is the unique continuous linear extension of the map*

$$D(L) \ni v \mapsto \mu(Lv\cdot) \in \mathscr{F}^*_e. \tag{2.13}$$

For simplicity, we write $L$ instead of $\bar{L}$ and $u$ instead of $\bar{u}$ below. Throughout the paper, let $L^2([0,T]\times\Omega; L^2(\mu))$ denote the space of all $L^2(\mu)$-valued square-integrable functions on $[0,T]\times\Omega$, and $C([0,T];\mathscr{F}^*_e)$ the space of all continuous $\mathscr{F}^*_e$-valued functions on $[0,T]$. For two Hilbert spaces $H_1$ and $H_2$, the space of Hilbert-Schmidt operators from $H_1$ to $H_2$ is denoted by $L_2(H_1,H_2)$. For simplicity, the positive constants $c$, $C$, $C_1$, $C_2$, $C_3$, $C_4$, $C_5$ and $C_p$ used in this paper may change from line to line. We would like to refer to [29, 33] for more background information and results on SPDEs and [8] on SPMEs.

# 3 Assumptions and Main Results

In addition to condition **(L.1)** above, we study Eq.(1.1) under the following assumptions.

**(H1)** $\Psi(\cdot):\mathbb{R}\to\mathbb{R}$ is a maximal monotone graph (cf. Remark 3.1 (iii) below) such that $0\in\Psi(0)$ and there exist $C\in(0,\infty)$ and $m\in[1,\infty)$ such that

$$\sup\{|\eta|;\eta\in\Psi(r)\}\leq C|r|^m,\quad \forall\, r\in\mathbb{R}. \tag{3.1}$$

**(H2)** Let $K:=L^1(\mu)\cap L^\infty(\mu)\cap\mathscr{F}_e^*$. $B\colon[0,T]\times K\times\Omega\to L_2(L^2(\mu),L^2(\mu))$ is progressively measurable, i.e. for any $t\in[0,T]$, this mapping restricted to $[0,t]\times K\times\Omega$ is measurable w.r.t. $\mathscr{B}([0,t])\times\mathscr{B}(K)\times\mathscr{F}_t$, where $\mathscr{B}(\cdot)$ is the Borel $\sigma$-field for a topological space. For simplicity, below we will write $B(t,u)$ meaning the mapping $\omega\mapsto B(t,u,\omega)$, and $B(t,u)$ satisfies:

**(i)** There exists $C_1\in[0,\infty)$ such that for all $\nu\in(0,\infty)$,

$$\|B(\cdot,u)-B(\cdot,v)\|_{L_2(L^2(\mu),F^*_{1,2,\nu})}\leq C_1\|u-v\|_{F^*_{1,2,\nu}}\ \text{ for all } u,v\in K \text{ on } [0,T]\times\Omega.$$

**(ii)** There exists $C_2\in(0,\infty)$ such that for all $\nu\in(0,\infty)$,

$$\|B(\cdot,u)\|_{L_2(L^2(\mu),F^*_{1,2,\nu})}\leq C_2\|u\|_{F^*_{1,2,\nu}}\ \text{ for all } u\in K \text{ on } [0,T]\times\Omega.$$

**(H3)(i)** There exists $C_3\in(0,\infty)$ satisfying

$$\|B(\cdot,u)\|_{L_2(L^2(\mu),L^2(\mu))}\leq C_3|u|_2\ \text{ for all } u\in K \text{ on } [0,T]\times\Omega.$$

**(ii)** There exist an orthonormal basis $\{e_k\}_{k\geq1}$ of $L^2(\mu)$ and $C_4\in(0,\infty)$ satisfying

$$\int_E\Big(\sum_{k=1}^\infty|B(\cdot,u)e_k|^2\Big)^{\frac{p}{2}}d\mu\leq C_4|u|_p^p,\ \text{ for all } u\in K \text{ on } [0,T]\times\Omega.$$

**(H4)** There exists a symmetric, positive, bilinear mapping $\Gamma:F_{1,2}\times F_{1,2}\to L^1(\mu)$ satisfying:
**(i)**

$$\mathscr{E}(u,u)=\int\frac{1}{2}\Gamma(u,u)d\mu,\ \text{ for all }\ u\in F_{1,2};$$

**(ii)** There exists a constant $C_5\in(0,\infty)$ such that

$$\Gamma(\varphi(u),\varphi(u))\leq C_5\Gamma(u,\varphi(u)),\ \forall\, u\in F_{1,2},$$

for every non-decreasing Lipschitz function $\varphi:\mathbb{R}\to\mathbb{R}$ with $\varphi(0)=0$.

**Remark 3.1** **(i)** (2.5) and (H2)(i) imply that for all $u,v\in K$,

$$\|B(\cdot,u)-B(\cdot,v)\|^2_{L_2(L^2(\mu),\mathscr{F}_e^*)}\leq C_1\|u-v\|^2_{\mathscr{F}_e^*}\ \text{ on } [0,T]\times\Omega. \tag{3.2}$$

**(ii)** We emphasize that (H4)(ii) is automatically fulfilled, if $(\mathcal{E},D(\mathcal{E}))$ is a local Dirichlet form.

**(iii)** A multi-valued function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is called maximal monotone if it is monotone, i.e.

$$(u-v)(x-y) \geq 0, \quad \forall x \in \Psi(u),\ y \in \Psi(v),\ u, v \in \mathbb{R},$$

and $(I+\Psi)(\mathbb{R}) = \mathbb{R}$. If $\Psi$ is the sub-differential $\partial j : \mathbb{R} \to 2^{\mathbb{R}}$ of a lower semi-continuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$, i.e.,

$$\partial j(r) = \{\zeta \in \mathbb{R} : j(r) \leq \zeta(r-\overline{r}) + j(\overline{r}),\ \forall \overline{r} \in \mathbb{R}\},$$

then $\Psi$ is maximal monotone. Conversely, every maximal monotone function $\Psi$ is of the form $\partial j$, where $j$ is such a lower semicontinuous convex function on $\mathbb{R}$ (see [4, (2.51)] for its definition). This function $j$ is called the potential of $\Psi$. We note that by [4, (2.51)] and (H1), it follows that $|j(r)| \leq C|r|^{m+1}$, $r \in \mathbb{R}$.

**(iv)** If $\tilde{\Psi} : \mathbb{R} \to \mathbb{R}$ is an increasing function and if $\{r_i | i \in \mathbb{N}\} \subset \mathbb{R}$ is the set of all $r \in \mathbb{R}$ for which $\tilde{\Psi}$ is discontinuous in $r$, then one gets a maximal monotone multivalued function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ by filling the gaps, i.e., define

$$\Psi(r) := \begin{cases} \tilde{\Psi}(r), & \text{for } r \notin \{r_i | i \in \mathbb{N}\}, \\ [\tilde{\Psi}(r_j - 0), \tilde{\Psi}(r_j + 0)], & else. \end{cases}$$

This is a well-known fact (see [4, page:54]). Hence our result covers non-continuous nonlinearities $\Psi$, as is indicated in the title of the paper.

**(v)** By (L.1) there exists $g \in L^1(\mu) \cap L^\infty(\mu)$, $g > 0$, $\mu$-a.e., such that $\mathscr{F}_e \subset L^1(g \cdot \mu)$ continuously and it was proved in [35] (see the last part of the proof of Proposition 3.1 in [35]) that the linear space

$$\mathscr{G} := \{h \cdot g | h \in L^\infty(\mu)\}$$

is dense in $\mathscr{F}_e^*$. Furthermore, obviously $\mathscr{G} \subset L^1(\mu) \cap L^\infty(\mu)$. Hence it follows that $K$ (defined in (H2)) is dense in $\mathscr{F}_e^*$, and hence in $(F_{1,2}, F^*_{1,2,\nu_0})$ for every $\nu_0 > 0$. Therefore, by (H2)(i) the map

$$K \ni u \longrightarrow B(t,u) \in L_2(L^2(\mu), F^*_{1,2,\nu_0})$$

can be extended uniquely to a Lipschitz continuous map on all of $F^*_{1,2,\nu_0}$. Furthermore, (H2)(ii) trivially also holds for this extension, as well as (3.2). We shall use this extension below without further notice.

**Definition 3.1** *Let $x \in \mathscr{F}_e^*$. An $\mathscr{F}_e^*$-valued adapted process $X = X(t)$ is called strong solution to (1.1) if there exists $q \in [2,\infty)$ such that the following conditions hold:*

$$\begin{aligned} &X \text{ is } \mathscr{F}_e^* - \text{valued continuous on } [0,T],\ \mathbb{P}-a.s.; \\ &X \in L^q(\Omega \times (0,T) \times E); \end{aligned}$$

*there is $\eta \in L^{\frac{q}{m}}(\Omega \times (0,T) \times E)$ such that*

$$\eta \in \Psi(X),\ dt \otimes \mathbb{P} \otimes d\mu - a.e. \text{ on } \Omega \times (0,T) \times E;$$

*and $\mathbb{P}$-a.s.,*

$$\int_0^\cdot \eta(s)ds \in C([0,T];\mathscr{F}_e), \tag{3.3}$$

$$X(t) = x + L\int_0^t \eta(s)ds + \int_0^t B(s,X(s))dW(s) \quad \text{for all } t \in [0,T].$$

Theorem 3.1 below is the main existence and uniqueness result for Eq.(1.1).

**Theorem 3.1** *Assume that (L.1), (H1)-(H4) are satisfied and let* $m$ *be as in (3.1). Let* $p \in [2, \infty)$ *and* $x \in L^p(\mu) \cap L^2(\mu) \cap L^{2m}(\mu) \cap \mathscr{F}_e^*$*. Then there is a unique strong solution* $X$ *to (1.1) such that*

$$X \in L^2\big(\Omega; C([0,T]; \mathscr{F}_e^*)\big) \cap L^\infty\big([0,T]; (L^p \cap L^2 \cap L^{2m})(\Omega \times E)\big). \tag{3.4}$$

Theorem 3.1 will be proved in Section 5. The proof is based on an approximating equation of (1.1). More precisely, in Section 4 we shall establish the existence of solutions for the following Yosida approximating equation of (1.1)

$$\begin{cases} dX_\lambda - L(\Psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = B(t, X_\lambda)dW(t), \quad t \in [0,T], \\ X_\lambda(0) = x \text{ on } E. \end{cases} \tag{3.5}$$

Here $\lambda > 0$ and

$$\Psi_\lambda(x) = \frac{1}{\lambda}\big(x - (1+\lambda\Psi)^{-1}(x)\big) \in \Psi\big((1+\lambda\Psi)^{-1}(x)\big)$$

is the Yosida approximation of $\Psi$, which is monotone and $\frac{2}{\lambda}$-Lipschitz ([8, page:13]). We recall that (see [4, page:38, Proposition 2.2]) $\Psi_\lambda$ is single-valued and for all $r \in \mathbb{R}$

$$|\Psi_\lambda(r)| \leq \inf |\Psi(r)|, \tag{3.6}$$

$$\lim_{\lambda \to 0} \Psi_\lambda(r) = \Psi_0(r), \tag{3.7}$$

where $\Psi_0(r)$ is the unique element in $\Psi(r)$ with minimal absolute value. This element exists, since $\Psi(r)$ is convex and closed for all $r \in \mathbb{R}$ (see [4, page:29, Proposition 2.1]). Obviously, $\Psi_0$ is increasing. Note that $\Psi_\lambda = \partial j_\lambda$ with

$$j_\lambda(r) := \inf\Big\{\frac{|r-\overline{r}|^2}{2\lambda} + j(\overline{r}); \overline{r} \in \mathbb{R}\Big\}, \quad \forall r \in \mathbb{R}, \tag{3.8}$$

where $j$ is the potential of $\Psi$ (see Remark 3.1 (iii)).

We have the following result for Eq.(3.5).

**Theorem 3.2** *Assume that (L.1), (H1)-(H4) are satisfied and let* $m$ *be as in (3.1). Let* $\lambda \in (0,1)$*,* $p \in [2, \infty)$*,* $m$ *as in (3.1) and* $x \in \mathscr{F}_e^* \cap L^{2m}(\mu) \cap L^2(\mu) \cap L^p(\mu)$*. Then (3.5) has a unique strong solution*

$$X_\lambda \in L^2(\Omega; C([0,T]; \mathscr{F}_e^*)) \cap L^\infty\big([0,T]; (L^p \cap L^2 \cap L^{2m})(\Omega \times E)\big), \tag{3.9}$$

*satisfying*

$$\int_0^\cdot \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds \in C([0,T]; \mathscr{F}_e) \ \mathbb{P}\text{-a.s.}, \tag{3.10}$$

*and* $\mathbb{P}$*-a.s.,*

$$X_\lambda(t) = x + L\int_0^t \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds + \int_0^t B(s, X_\lambda(s)) dW(s), \ \forall t \in [0,T]. \tag{3.11}$$

*Moreover, there exists* $C \in (0,\infty)$ *such that for all* $\lambda,\ \lambda' \in (0,1),\ t \in [0,T]$,

$$\mathbb{E}|X_\lambda(t)|_p^p \le C|x|_p^p,; \tag{3.12}$$

$$\mathbb{E}\int_0^T \int_E |\Psi_\lambda(X_\lambda(t))|^{\frac{p}{m}} d\mu dt \le C|x|_p^p, \tag{3.13}$$

$$\mathbb{E}\Big[\sup_{0\le t\le T} \|X_\lambda(t)\|^2_{\mathscr{F}_e^*}\Big] \le C\big(\|x\|^2_{\mathscr{F}_e^*} + |x|_{2m}^{2m}\big), \tag{3.14}$$

$$\mathbb{E}\Big[\sup_{0\le t\le T} \|X_\lambda(t) - X_{\lambda'}(t)\|^2_{\mathscr{F}_e^*}\Big] \le C(\lambda+\lambda')(|x|_2^2 + |x|_{2m}^{2m}). \tag{3.15}$$

# 4 Proof of Theorem 3.2

As said in the introduction, the proof of Theorem 3.2 is based on the strategy in [10, Section 4], but with major modifications.

*Proof* For each fixed $\lambda$, firstly we consider the following approximating equation for (3.5)

$$\begin{cases} dX_\lambda^\nu(t) + (\nu - L)\big(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)\big)dt = B(t, X_\lambda^\nu(t))dW(t), \text{ in } (0,T)\times E, \\ X_\lambda^\nu(0) = x \in L^2(\mu)\cap L^p(\mu), \end{cases} \tag{4.1}$$

where $\nu \in (0,1)$. By [37, Lemma 3.1], (4.1) has a unique solution $X_\lambda^\nu \in L^2\big(\Omega; L^\infty([0,T]; L^2(\mu))\big) \cap L^2\big(\Omega\times[0,T]; F_{1,2}\big) \cap L^2(\Omega; C([0,T]; F_{1,2}^*))$.

To prove that (3.9)-(3.14) hold with $X_\lambda^\nu$ replacing $X_\lambda$, with a constant $C$ independent of $\nu$ and $\lambda$, we consider the following approximating equation for (4.1).

$$\begin{cases} dX_\lambda^{\nu,\varepsilon}(t) + A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(t))dt = B(t, X_\lambda^{\nu,\varepsilon}(t))dW(t), \text{ in } (0,T)\times E, \\ X_\lambda^{\nu,\varepsilon}(0) = x \in L^2(\mu)\cap L^p(\mu), \end{cases} \tag{4.2}$$

where $A_\lambda^{\nu,\varepsilon} : F_{1,2}^* \to F_{1,2}^*$, defined by

$$A_\lambda^{\nu,\varepsilon}(x) = \frac{1}{\varepsilon}\big(x - (I + \varepsilon A_\lambda^\nu)^{-1}(x)\big),\ x \in F_{1,2}^*,\ \varepsilon \in (0,1),$$

is the Yosida approximation of the operator $A_\lambda^\nu(x) := (\nu - L)(\Psi_\lambda(x) + \lambda I(x))$, $\forall\, x \in D(A_\lambda^\nu) := F_{1,2}$. Here and below $I$ denotes the identity map on the respective space. Clearly, $I + \varepsilon A_\lambda^\nu : F_{1,2} \to F_{1,2}^*$ is a bijection, since so is $\Psi_\lambda + \lambda I : F_{1,2} \to F_{1,2}$. Furthermore, since obviously $A_\lambda^\nu$ with domain $F_{1,2}$ is monotone on $F_{1,2}^*$, it follows that $A_\lambda^\nu$ is maximal monotone on $F_{1,2}^*$. Fix $x \in F_{1,2}^*$ and set $y := J_\varepsilon(x) := (I + \varepsilon A_\lambda^\nu)^{-1}x \in F_{1,2}$, i.e., $(I + \varepsilon A_\lambda^\nu)(y) = x$, equivalently,

$$y + \varepsilon(\nu - L)(\Psi_\lambda + \lambda I)(y) = x. \tag{4.3}$$

In particular, $(\Psi_\lambda + \lambda I)(y) \in D(L)$, if $x \in L^2(\mu)$.

Before giving the well-posedness result for (4.2), we need some preparations.

**Lemma 4.1** *For all* $0 < \varepsilon < 1$, *we have*

$$\|J_\varepsilon(x) - J_\varepsilon(\widetilde{x})\|_{F_{1,2,\nu}^*} \le \|x - \widetilde{x}\|_{F_{1,2,\nu}^*}, \forall\ x, \widetilde{x} \in F_{1,2}^*. \tag{4.4}$$

$$|J_\varepsilon(x) - J_\varepsilon(\widetilde{x})|_2 \le \frac{1}{\sqrt{\nu\varepsilon\lambda}}|x - \widetilde{x}|_2,\ \forall\ x, \widetilde{x} \in L^2(\mu). \tag{4.5}$$

$$|J_\varepsilon(x)|_p \le |x|_p,\ \ \forall\ x \in L^p(\mu)\cap L^2(\mu),\ 2 \le p < \infty. \tag{4.6}$$

*Proof* Firstly, let us prove (4.4). For $x,\ \tilde{x} \in F^*_{1,2}$, set $y := J_\varepsilon(x)$ and $\widetilde{y} := J_\varepsilon(\widetilde{x})$, we have

$$y - \tilde{y} + \varepsilon A^\nu_\lambda(y) - \varepsilon A^\nu_\lambda(\tilde{y}) = x - \tilde{x}. \tag{4.7}$$

Taking the scalar product of $y - \tilde{y}$ with both sides in $(F^*_{1,2}, \|\cdot\|_{F^*_{1,2,\nu}})$, we get

$$\langle y - \tilde{y}, y - \tilde{y}\rangle_{F^*_{1,2,\nu}} + \varepsilon\langle A^\nu_\lambda(y) - A^\nu_\lambda(\tilde{y}), y - \tilde{y}\rangle_{F^*_{1,2,\nu}} = \langle x - \tilde{x}, y - \tilde{y}\rangle_{F^*_{1,2,\nu}}. \tag{4.8}$$

For the second term in the left hand-side of (4.8), by (2.10), we know

$$\begin{aligned}
&\langle A^\nu_\lambda(y) - A^\nu_\lambda(\widetilde{y}), y - \widetilde{y}\rangle_{F^*_{1,2,\nu}} \\
&= \big\langle (\nu - L)((\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})), y - \widetilde{y}\big\rangle_{F^*_{1,2,\nu}} \\
&= {}_{F_{1,2}}\big\langle (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y}), y - \widetilde{y}\big\rangle_{F^*_{1,2}} \\
&= \big\langle (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y}), y - \widetilde{y}\big\rangle_2 \geq 0,
\end{aligned} \tag{4.9}$$

Since $y - \tilde{y} \in F_{1,2} \subset L^2(\mu)$.
(4.8) and (4.9) imply

$$\|y - \tilde{y}\|^2_{F^*_{1,2,\nu}} \leq \|x - \tilde{x}\|_{F^*_{1,2,\nu}} \cdot \|y - \tilde{y}\|_{F^*_{1,2,\nu}},$$

from which (4.4) follows.

Secondly, to prove the Lipschitz continuity of $J_\varepsilon$ in $L^2(\mu)$, we take $x, \tilde{x} \in L^2(\mu)$ and apply

$${}_{F^*_{1,2}}\big\langle\ \cdot,\ (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_{F_{1,2}}$$

to both sides of (4.7). Then

$$\begin{aligned}
&{}_{F^*_{1,2}}\big\langle y - \tilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_{F_{1,2}} \\
&+ {}_{F^*_{1,2}}\big\langle \varepsilon A^\nu_\lambda(y) - \varepsilon A^\nu_\lambda(\tilde{y}), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_{F_{1,2}} \\
&= {}_{F^*_{1,2}}\big\langle x - \tilde{x}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_{F_{1,2}}.
\end{aligned} \tag{4.10}$$

For the second term in the left hand-side of (4.10), by (2.8)-(2.10) (under the Gelfand triple $F_{1,2} \subset L^2(\mu) \subset F^*_{1,2}$), we obtain

$$\begin{aligned}
&{}_{F^*_{1,2}}\big\langle \varepsilon A^\nu_\lambda(y) - \varepsilon A^\nu_\lambda(\widetilde{y}), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big\rangle_{F_{1,2}} \\
&= {}_{F^*_{1,2}}\big\langle (I - L)(\varepsilon(\Psi_\lambda + \lambda I)(y) - \varepsilon(\Psi_\lambda + \lambda I)(\widetilde{y})), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big\rangle_{F_{1,2}} \\
&\quad + {}_{F^*_{1,2}}\big\langle \varepsilon(\nu - 1)((\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda)(\widetilde{y})), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big\rangle_{F_{1,2}} \\
&= \varepsilon\|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\|^2_{F_{1,2}} + \varepsilon(\nu - 1)|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})|^2_2 \\
&\geq \nu\varepsilon|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})|^2_2.
\end{aligned} \tag{4.11}$$

For the first term in the left hand-side of (4.10), since $\Psi_\lambda$ is monotone, by (2.8) (under the Gelfand triple $F_{1,2} \subset L^2(\mu) \subset F^*_{1,2}$), we know

$$\begin{aligned}
&{}_{F^*_{1,2}}\big\langle y - \tilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_{F_{1,2}} \\
&= \big\langle y - \tilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\big\rangle_2 \\
&\geq \lambda|y - \tilde{y}|^2_2.
\end{aligned} \tag{4.12}$$

Similarly, since $x, \widetilde{x} \in L^2(\mu)$, by (2.8), we have

$$\begin{aligned}
&{}_{F_{1,2}^*}\langle x-\widetilde{x}, (\Psi_\lambda+\lambda I)(y)-(\Psi_\lambda+\lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&= \langle x-\widetilde{x}, (\Psi_\lambda+\lambda I)(y)-(\Psi_\lambda+\lambda I)(\widetilde{y})\rangle_2. \qquad (4.13)
\end{aligned}$$

Taking (4.11), (4.12) and (4.13) into (4.10), by Young's inequality, we obtain

$$\begin{aligned}
&\lambda|y-\widetilde{y}|_2^2+\nu\varepsilon\big|(\Psi_\lambda+\lambda I)(y)-(\Psi_\lambda+\lambda I)(\widetilde{y})\big|_2^2 \\
&\le |x-\widetilde{x}|_2\cdot\big|(\Psi_\lambda+\lambda I)(y)-(\Psi_\lambda+\lambda I)(\widetilde{y})\big|_2 \\
&\le \frac{1}{\nu\varepsilon}|x-\widetilde{x}|_2^2+\nu\varepsilon\big|(\Psi_\lambda+\lambda I)(y)-(\Psi_\lambda+\lambda I)(\widetilde{y})\big|_2^2, \qquad (4.14)
\end{aligned}$$

and therefore

$$|y-\widetilde{y}|_2^2 \le \frac{1}{\nu\varepsilon\lambda}|x-\widetilde{x}|_2^2,$$

which yields (4.5) as claimed.

Now, let us prove (4.6). Let $x \in L^2(\mu)\cap L^p(\mu)$, $p \ge 2$. Since the function $h(r) := r|r|^{p-2}(1+k|r|^{p-2})^{-1}$ is Lipschitz, and $h(0)=0$, we have $h(y)\in F_{1,2}$, because $y \in F_{1,2}$. Hence applying ${}_{F_{1,2}^*}\big\langle\ \cdot,\ y|y|^{p-2}(1+k|y|^{p-2})^{-1}\big\rangle_{F_{1,2}}$, $k>0$, to both sides of (4.3), we obtain

$$\begin{aligned}
&{}_{F_{1,2}^*}\Big\langle y, \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_{F_{1,2}} + {}_{F_{1,2}^*}\Big\langle \varepsilon(\nu-L)(\Psi_\lambda(y)+\lambda y), \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_{F_{1,2}} \\
&= {}_{F_{1,2}^*}\Big\langle x, \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_{F_{1,2}}. \qquad (4.15)
\end{aligned}$$

Under the Gelfand triple $F_{1,2}\subset L^2(\mu)\subset F_{1,2}^*$, by (2.8), (4.15) yields

$$\begin{aligned}
&\Big\langle y, \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_2 + {}_{F_{1,2}^*}\Big\langle \varepsilon(\nu-L)(\Psi_\lambda(y)+\lambda y), \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_{F_{1,2}} \\
&= \Big\langle x, \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_2. \qquad (4.16)
\end{aligned}$$

For the second term in the left hand-side of (4.16), since $x \in L^2(\mu)$, $y \in F_{1,2}\subset L^2(\mu)$, from (4.3) we deduce that

$$(\nu-L)(\Psi_\lambda(y)+\lambda y)\in L^2(\mu).$$

Then by (2.8), we know

$${}_{F_{1,2}^*}\Big\langle \varepsilon(\nu-L)(\Psi_\lambda(y)+\lambda y), \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_{F_{1,2}} = \Big\langle \varepsilon(\nu-L)(\Psi_\lambda(y)+\lambda y), \frac{y|y|^{p-2}}{1+k|y|^{p-2}}\Big\rangle_2.$$

To estimate the term above, notice that for all Lipschitz and increasing function $g:\mathbb{R}\to\mathbb{R}$ with $g(0)=0$, we have

$$\int_E (\nu-L)\big(\Psi_\lambda(y)+\lambda y\big)\cdot g(y)d\mu \ge 0, \qquad (4.17)$$

because on one hand, $\Psi_\lambda$ is Lipschitz and monotone with $\Psi_\lambda(0)=0$, then obviously,

$$\int_E \nu\big(\Psi_\lambda(y)+\lambda y\big)\cdot g(y)d\mu \ge 0. \qquad (4.18)$$

On the other hand, we can prove the following term, i.e.,

$$\begin{aligned}&\big\langle(-L)(\Psi_\lambda(y)+\lambda y), g(y)\big\rangle\\ &=\mathscr{E}\big(\Psi_\lambda(y)+\lambda y, g(y)\big)\\ &=\lim_{\varepsilon\to 0}\mathscr{E}^{(\varepsilon)}\big(\Psi_\lambda(y)+\lambda y, g(y)\big),\end{aligned}\tag{4.19}$$

is non-negative. Indeed, by [36, Lemma 5.1], with $p$ being the kernel corresponding to $P := (I-\varepsilon L)^{-1}$, we know, setting $f := \Psi_\lambda + \lambda I$,

$$\begin{aligned}\mathscr{E}^{(\varepsilon)}\big(f(y), g(y)\big) &:= \frac{1}{\varepsilon}\big\langle f(y), \big(I-(I-\varepsilon L)^{-1}\big)g(y)\big\rangle_2\\ &=\frac{1}{2\varepsilon}\int_E\int_E \big((f(y(\widetilde{\xi}))) - f(y(\xi))\big)\cdot\big(g(y(\tilde{\xi}))-g(y(\xi))\big)p(\xi, d\tilde{\xi})\mu(d\xi)\\ &\quad+\frac{1}{\varepsilon}\int_E(1-P1(\xi))f(y(\xi))g(y(\xi))\mu(d\xi),\end{aligned}$$

since $f, g$ are monotone with $f(0)=g(0)=0$ and $P1\le 1$, we deduce that

$$\mathscr{E}^{(\varepsilon)}\big(f(y), g(y)\big)\ge 0,$$

which implies that (4.19) is non-negative. As a short remark, the assumption that $(E,\mathcal{B})$ is a standard measurable space is needed in [36, Lemma 5.1] to ensure the existence of the kernel $p$ above.

Thus,

$$\int_E\frac{|y|^p}{1+k|y|^{p-2}}d\mu\le\int_E\frac{xy|y|^{p-2}}{1+k|y|^{p-2}}d\mu.$$

Letting $k\to 0$ and by Hölder's inequality, we obtain

$$|y|_p^p\le\int_E xy|y|^{p-2}d\mu\le|x|_p|y|_p^{p-1}.$$

Hence, since $y=J_\varepsilon(x)$,

$$|J_\varepsilon(x)|_p\le|x|_p.$$

□

As shown in Lemma 4.1, $J_\varepsilon$ is Lipschitz in both $L^2(\mu)$ and $F^*_{1,2}$. Since $A^{\nu,\varepsilon}_\lambda=\frac{1}{\varepsilon}(I-J_\varepsilon)$, $A^{\nu,\varepsilon}_\lambda$ is also Lipschitz in $L^2(\mu)$ and $F^*_{1,2}$. If $x\in F^*_{1,2}$, (4.2) has a unique adapted solution $X^{\nu,\varepsilon}_\lambda\in L^2(\Omega; C([0,T];F^*_{1,2}))$ and by Itô's formula (see e.g. [29, Theorem 4.2.5]) we have

$$\begin{aligned}&\mathbb{E}\|X^{\nu,\varepsilon}_\lambda(t)\|^2_{F^*_{1,2,\nu}}+2\mathbb{E}\int_0^t\big\langle A^{\nu,\varepsilon}_\lambda(X^{\nu,\varepsilon}_\lambda(s)), X^{\nu,\varepsilon}_\lambda(s)\big\rangle_{F^*_{1,2,\nu}}ds\\ &=\|x\|^2_{F^*_{1,2,\nu}}+\mathbb{E}\int_0^t\|B(s,X^{\nu,\varepsilon}_\lambda(s))\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}ds\\ &\quad+2\mathbb{E}\int_0^t\langle X^{\nu,\varepsilon}_\lambda(s), B(s,X^{\nu,\varepsilon}_\lambda(s))dW(s)\rangle_{F^*_{1,2,\nu}},\quad t\in[0,T],\end{aligned}$$

which, by virtue of (H2)(ii) and the fact that the second term on the left hand-side is nonnegative by (4.4), yields

$$\mathbb{E}\|X^{\nu,\varepsilon}_\lambda(t)\|^2_{F^*_{1,2,\nu}}\le e^{C_2T}\|x\|^2_{F^*_{1,2,\nu}},\ \forall\varepsilon>0,\ t\in[0,T],\ x\in F^*_{1,2}.\tag{4.20}$$

Similarly, if $x \in L^2(\mu)$, we know that $X_\lambda^{\nu,\varepsilon} \in L^2(\Omega; C([0,T]; L^2(\mu)))$ and again by Itô's formula we obtain

$$\begin{aligned}
&\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\mathbb{E}\int_0^t \big\langle A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s)), X_\lambda^{\nu,\varepsilon}(s)\big\rangle_2 ds \\
&= |x|_2^2 + \mathbb{E}\int_0^t \|B(s, X_\lambda^{\nu,\varepsilon}(s))\|^2_{L_2(L^2(\mu),L^2(\mu))} ds \\
&\quad +2\mathbb{E}\int_0^t \langle X_\lambda^{\nu,\varepsilon}(s), B(s, X_\lambda^{\nu,\varepsilon}(s))dW(s)\rangle_2, \qquad (4.21)
\end{aligned}$$

which, by virtue of (H3)(i) and the fact that the second summand on the left hand-side is nonnegative by (4.6) applied to $p = 2$, yields

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 \le e^{C_3T}|x|_2^2,\ \forall \varepsilon > 0,\ t \in [0,T],\ x \in L^2(\mu). \qquad (4.22)$$

**Lemma 4.2** *For $p \in (2,\infty)$ and $x \in L^p(\mu)\cap L^2(\mu)$, we have that $X_\lambda^{\nu,\varepsilon} \in L^\infty\big([0,T]; L^p(\Omega; L^p(\mu))\big)$.*

*Proof* For $\alpha, R > 0$, consider the set

$$\mathcal{K}_R = \big\{X \in L^2\big([0,T]; C([0,T]; L^2(\mu))\big), e^{-p\alpha t}\mathbb{E}|X(t)|_p^p \le R^p,\ t \in [0,T]\big\}.$$

Since, by (4.2), $X_\lambda^{\nu,\varepsilon}$ is a fixed point of the map

$$F : X \mapsto\ e^{-\frac{\bullet}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^\bullet e^{-\frac{\bullet-s}{\varepsilon}} J_\varepsilon(X(s))ds + \int_0^\bullet e^{-\frac{\bullet-s}{\varepsilon}} B(s, X(s))dW(s),$$

obtained by iteration in $L^2\big(\Omega; C([0,T]; L^2(\mu))\big)$, it suffices to show that $F$ leaves the set $\mathcal{K}_R$ invariant for $\alpha, R > 0$ large enough. By (4.6) we have that for $X \in \mathcal{K}_R$, $t \ge 0$

$$\begin{aligned}
&\Big[e^{-p\alpha t}\mathbb{E}\Big|e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}} J_\varepsilon(X(s))ds\Big|_p^p\Big]^{\frac{1}{p}} \\
&\le e^{-\alpha t}e^{-\frac{t}{\varepsilon}}|x|_p + e^{-\alpha t}\Big[\mathbb{E}\Big(\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}|X(s)|_p ds\Big)^p\Big]^{\frac{1}{p}} \\
&\le e^{-(\alpha+\frac{1}{\varepsilon})t}|x|_p + e^{-\alpha t}\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}\big(\mathbb{E}|X(s)|_p^p\big)^{\frac{1}{p}} ds \\
&\le e^{-(\alpha+\frac{1}{\varepsilon})t}|x|_p + \frac{R}{1+\alpha\varepsilon}. \qquad (4.23)
\end{aligned}$$

Set

$$Y(t) = \int_0^t e^{-\frac{t-s}{\varepsilon}} B(s, X(s))dW(s),\ t \ge 0.$$

Then $Y$ is a solution to the following SDE on $L^2(\mu)$:

$$\begin{cases} dY(t) + \frac{1}{\varepsilon}Y(t)dt = B(t, X(t))dW(t),\ t \ge 0, \\ Y(0) = 0, \end{cases}$$

equivalently,

$$d\big(e^{\frac{t}{\varepsilon}}Y(t)\big) = e^{\frac{t}{\varepsilon}}B(t, X(t))dW(t),\ t \ge 0,\ Y(0) = 0.$$

By Hypothesis (H3)(ii), we may apply Theorem 7.1 in the Appendix with $u(t)$ replaced by $e^{\frac{t}{\varepsilon}}Y(t)$. Then by Hölder's and Young's inequality and (H3)(ii), we obtain for $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\big|e^{\frac{t}{\varepsilon}}Y(t)\big|_p^p\\
&= \frac{1}{2}p(p-1)\mathbb{E}\int_0^t \int_E \big|e^{\frac{s}{\varepsilon}}Y(s)\big|^{p-2}\cdot\sum_{k=1}^{\infty}\big|e^{\frac{s}{\varepsilon}}B(s,X(s))e_k\big|^2 d\mu ds\\
&\le \frac{1}{2}p(p-1)\mathbb{E}\int_0^t \Big(\int_E |e^{\frac{s}{\varepsilon}}Y(s)|^{p-2\cdot\frac{p}{p-2}}d\mu\Big)^{\frac{p-2}{p}}\cdot\bigg(\int_E\Big(\sum_{k=1}^{\infty}|e^{\frac{s}{\varepsilon}}B(s,X(s))e_k|^2\Big)^{\frac{p}{2}}d\mu\bigg)^{\frac{2}{p}}ds\\
&= \frac{1}{2}p(p-1)\mathbb{E}\int_0^t |e^{\frac{s}{\varepsilon}}Y(s)|_p^{p-2}\cdot\bigg(\int_E\Big(\sum_{k=1}^{\infty}|e^{\frac{s}{\varepsilon}}B(s,X(s))e_k|^2\Big)^{\frac{p}{2}}d\mu\bigg)^{\frac{2}{p}}ds\\
&\le \frac{1}{2}p(p-1)\mathbb{E}\int_0^t \frac{\Big(|e^{\frac{s}{\varepsilon}}Y(s)|_p^{p-2}\Big)^{\frac{p}{p-2}}}{\frac{p}{p-2}}+\frac{\Big(\int_E\big(\sum_{k=1}^{\infty}|e^{\frac{s}{\varepsilon}}B(s,X(s))e_k|^2\big)^{\frac{p}{2}}d\mu\Big)^{\frac{2}{p}\cdot\frac{p}{2}}}{\frac{p}{2}}ds\\
&= \frac{1}{2}(p-1)(p-2)\mathbb{E}\int_0^t |e^{\frac{s}{\varepsilon}}Y(s)\big|_p^p ds+(p-1)\mathbb{E}\int_0^t\int_E\Big(\sum_{k=1}^{\infty}|e^{\frac{s}{\varepsilon}}B(s,X(s))e_k|^2\Big)^{\frac{p}{2}}ds\\
&\le \frac{1}{2}(p-1)(p-2)\mathbb{E}\int_0^t |e^{\frac{s}{\varepsilon}}Y(s)\big|_p^p ds+C_4(p-1)\mathbb{E}\int_0^t |e^{\frac{s}{\varepsilon}}X(s)|_p^p ds, \qquad (4.24)
\end{aligned}$$

and therefore, by Gronwall's lemma, we obtain

$$\begin{aligned}
\mathbb{E}\big|e^{\frac{t}{\varepsilon}}Y(t)\big|_p^p &\le C_4(p-1)e^{\frac{(p-1)(p-2)T}{2}}\int_0^t \mathbb{E}|e^{\frac{s}{\varepsilon}}X(s)|_p^p ds\\
&\le C_4(p-1)e^{\frac{(p-1)(p-2)T}{2}}\int_0^t R^p e^{(\frac{p}{\varepsilon}+p\alpha)s}ds\\
&\le \frac{C_{T,p}R^p\varepsilon}{(1+\varepsilon\alpha)p}e^{\frac{(1+\varepsilon\alpha)pt}{\varepsilon}}, \qquad (4.25)
\end{aligned}$$

which yields

$$e^{-p\alpha t}\mathbb{E}|Y(t)|_p^p \le \frac{C_{T,p}\varepsilon R^p}{(1+\varepsilon\alpha)},\ \forall t\in[0,T]. \qquad (4.26)$$

Then, by formulas (4.23), (4.26), we infer that for $\alpha$ large enough and $R \ge 2|x|_p$, the map $F$ leaves $\mathcal{K}_R$ invariant as claimed. □

**Lemma 4.3** *For all $p \in [2,\infty)$ and $x \in L^p(\mu)\cap L^2(\mu)$, there exists $C_p \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]}\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_p^p \le C_p|x|_p^p,\ \forall \varepsilon,\lambda,\nu\in(0,1). \qquad (4.27)$$

*Proof* Applying the Itô formula to $|X_\lambda^{\nu,\varepsilon}(t)|_p^p$ (see Theorem 7.1 in the Appendix), we obtain

$$\begin{aligned}
\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_p^p = |x|_p^p &- p\mathbb{E}\int_0^t\int_E A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s))X_\lambda^{\nu,\varepsilon}(s)|X_\lambda^{\nu,\varepsilon}(s)|^{p-2}d\mu ds\\
&+\frac{1}{2}p(p-1)\mathbb{E}\int_0^t\int_E|X_\lambda^{\nu,\varepsilon}(s)|^{p-2}\cdot\sum_{k=1}^{\infty}|B(s,X_\lambda^{\nu,\varepsilon}(s))e_k|^2 d\mu ds. \qquad (4.28)
\end{aligned}$$

Recall that $A^{\nu,\varepsilon}_\lambda(X^{\nu,\varepsilon}_\lambda(s)) = \frac{1}{\varepsilon}(X^{\nu,\varepsilon}_\lambda(s) - J_\varepsilon(X^{\nu,\varepsilon}_\lambda(s)))$, so we have

$$\begin{aligned}
&\int_E A^{\nu,\varepsilon}_\lambda(X^{\nu,\varepsilon}_\lambda(s))X^{\nu,\varepsilon}_\lambda(s)|X^{\nu,\varepsilon}_\lambda(s)|^{p-2}d\mu \\
&= \frac{1}{\varepsilon}\int_E |X^{\nu,\varepsilon}_\lambda(s)|^p d\mu - \frac{1}{\varepsilon}\int_E J_\varepsilon(X^{\nu,\varepsilon}_\lambda(s))X^{\nu,\varepsilon}_\lambda(s)|X^{\nu,\varepsilon}_\lambda(s)|^{p-2}d\mu. \qquad (4.29)
\end{aligned}$$

By Hölder's inequality and (4.6), we conclude

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_E |X^{\nu,\varepsilon}_\lambda(s)|^p d\mu - \frac{1}{\varepsilon}\int_E J_\varepsilon(X^{\nu,\varepsilon}_\lambda(s))X^{\nu,\varepsilon}_\lambda(s)|X^{\nu,\varepsilon}_\lambda(s)|^{p-2}d\mu \\
&\geq \frac{1}{\varepsilon}\int_E |X^{\nu,\varepsilon}_\lambda(s)|^p d\mu - \frac{1}{\varepsilon}\Big[|J_\varepsilon(X^{\nu,\varepsilon}_\lambda(s))|_p \cdot |X^{\nu,\varepsilon}_\lambda(s)|_p^{p-1}\Big] \\
&\geq \frac{1}{\varepsilon}\int_E |X^{\nu,\varepsilon}_\lambda(s)|^p d\mu - \frac{1}{\varepsilon}|X^{\nu,\varepsilon}_\lambda(s)|_p \cdot |X^{\nu,\varepsilon}_\lambda(s)|_p^{p-1} \\
&= 0. \qquad (4.30)
\end{aligned}$$

By (4.28)-(4.30) and using a similar argument as in (4.24), we get

$$\begin{aligned}
\mathbb{E}|X^{\nu,\varepsilon}_\lambda(t)|_p^p &\leq |x|_p^p + \frac{1}{2}(p-1)\mathbb{E}\int_0^t (p-2)|X^{\nu,\varepsilon}_\lambda(s)|_p^p + 2C_4|X^{\nu,\varepsilon}_\lambda(s)|_p^p ds \\
&= |x|_p^p + \frac{1}{2}(p-1)(p-2+2C_4)\mathbb{E}\int_0^t |X^{\nu,\varepsilon}_\lambda(s)|_p^p ds.
\end{aligned}$$

As a result, by Gronwall's lemma, we obtain,

$$\text{ess sup}_{t\in[0,T]}\mathbb{E}|X^{\nu,\varepsilon}_\lambda(t)|_p^p \leq C_p|x|_p^p, \ \forall\ \varepsilon,\lambda,\nu \in (0,1).$$

Since $t \mapsto |X^{\nu,\epsilon}_\lambda(t)|_p$ is lower semi-continuous and hence so is $t \mapsto \mathbb{E}|X^{\nu,\epsilon}_\lambda(t)|_p^p$, (4.27) follows. □

**Lemma 4.4** *Let $p \in [2,\infty)$, $x \in L^2(\mu) \cap L^p(\mu)$ and $X^{\nu,\varepsilon}_\lambda$ as above. Then as $\varepsilon \longrightarrow 0$, we have*

$$X^{\nu,\varepsilon}_\lambda \longrightarrow X^\nu_\lambda \text{ strongly in } L^2(\Omega; C([0,T]; F^*_{1,2})),$$

*where $X^\nu_\lambda$ is the solution to (4.1). Furthermore, there exists $C_p \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}|X^\nu_\lambda(t)|_p^p \leq C_p|x|_p^p, \quad \forall\ \lambda,\nu \in (0,1). \qquad (4.31)$$

*Proof* We prove the lemma in two steps, which are given as two claims.

**Claim 4.1** *For each $x \in L^2(\mu)$, the sequence $\{X^{\nu,\varepsilon}_\lambda\}$ is Cauchy in $L^2(\Omega; C([0,T]; F^*_{1,2}))$.*

*Proof* Let $\varepsilon, \eta > 0$. Applying the Itô formula ([29, Theorem 4.2.5] with $V := L^2(\mu)$, $H := F^*_{1,2,\nu}$, $\alpha = 2$, $X_0 = x$) to $\|X^{\nu,\varepsilon}_\lambda - X^{\nu,\eta}_\lambda\|^2_{F^*_{1,2,\nu}}$, we have

$$\begin{aligned}
&d\|X^{\nu,\varepsilon}_\lambda(t) - X^{\nu,\eta}_\lambda(t)\|^2_{F^*_{1,2,\nu}} \\
&+2\big\langle(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X^{\nu,\varepsilon}_\lambda(t))) - (\Psi_\lambda + \lambda)(J_\eta(X^{\nu,\eta}_\lambda(t)))\big), X^{\nu,\varepsilon}_\lambda(t) - X^{\nu,\eta}_\lambda(t)\big\rangle_{F^*_{1,2,\nu}} dt \\
&= 2\big\langle X^{\nu,\varepsilon}_\lambda(t) - X^{\nu,\eta}_\lambda(t), \big(B(t, X^{\nu,\varepsilon}_\lambda(t)) - B(t, X^{\nu,\eta}_\lambda(t))\big)dW(t)\big\rangle_{F^*_{1,2,\nu}} \\
&+\big\|B(t, X^{\nu,\varepsilon}_\lambda(t)) - B(t, X^{\nu,\eta}_\lambda(t))\big\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})} dt. \qquad (4.32)
\end{aligned}$$

The second term in the left hand-side of the above equality, by (4.3), (2.10) and (2.8), is equal to

$$\begin{aligned}
&2\big\langle(\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t)))-(\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(t))),J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))-J_\eta(X_\lambda^{\nu,\eta}(t))\big\rangle_2 dt\\
&+2\big\langle(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)-(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big),\\
&\quad\varepsilon(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)-\eta(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big)\big\rangle_{F^*_{1,2,\nu}}dt.
\end{aligned}\tag{4.33}$$

Taking (4.33) into (4.32), then taking expectation of both sides, we obtain for all $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\|X_\lambda^{\nu,\varepsilon}(r)-X_\lambda^{\nu,\eta}(r)\|^2_{F^*_{1,2,\nu}}\\
&-2\mathbb{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r\big\langle X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s),(B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,X_\lambda^{\nu,\eta}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu}}\Big|\Big]\\
&+2\mathbb{E}\int_0^t\big\langle(\Psi_\lambda+\lambda I)\big(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big)-(\Psi_\lambda+\lambda I)\big(J_\eta(X_\lambda^{\nu,\eta}(s))\big),J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))-J_\eta(X_\lambda^{\nu,\eta}(s))\big\rangle_2 ds\\
\le\;&2\mathbb{E}\int_0^T\big|\big\langle(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)-(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big),\\
&\qquad\varepsilon(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)-\eta(\nu-L)\big((\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big)\big\rangle_{F^*_{1,2,\nu}}\big|ds\\
&+\mathbb{E}\int_0^t\big\|B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,X_\lambda^{\nu,\eta}(s))\big\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}ds\\
\le\;&3(\varepsilon+\eta)\mathbb{E}\int_0^T\|(\nu-L)(\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\|^2_{F^*_{1,2,\nu}}+\|(\nu-L)(\Psi_\lambda+\lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\|^2_{F^*_{1,2,\nu}}ds\\
&+\mathbb{E}\int_0^t\big\|B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,X_\lambda^{\nu,\eta}(s))\big\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}ds\\
\le\;&3(\varepsilon+\eta)(\frac{2}{\lambda}+\lambda+C_5)e^{C_3T}|x|_2^2+\mathbb{E}\int_0^t C_1\|X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s)\|^2_{F^*_{1,2,\nu}}ds,
\end{aligned}\tag{4.34}$$

where we used Proposition 7.1 (see Appendix) and (H2)(i) in the last inequality. For the second term in the left hand-side of (4.34), by using the Burkholder-Davis-Gundy inequality for $p=1$, we obtain for all $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r\big\langle X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s),(B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,X_\lambda^{\nu,\eta}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu}}\Big|\Big]\\
\le\;&\mathbb{E}\Big[\int_0^t\|X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s)\|^2_{F^*_{1,2,\nu}}\cdot C_1\|X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s)\|^2_{F^*_{1,2,\nu}}ds\Big]^{\frac12}\\
\le\;&\mathbb{E}\Big[\sup_{r\in[0,t]}\|X_\lambda^{\nu,\varepsilon}(r)-X_\lambda^{\nu,\eta}(r)\|^2_{F^*_{1,2,\nu}}\cdot C_1\int_0^t\|X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s)\|^2_{F^*_{1,2,\nu}}ds\Big]^{\frac12}\\
\le\;&\frac14\mathbb{E}\sup_{r\in[0,t]}\|X_\lambda^{\nu,\varepsilon}(r)-X_\lambda^{\nu,\eta}(r)\|^2_{F^*_{1,2,\nu}}+C_1\mathbb{E}\int_0^t\|X_\lambda^{\nu,\varepsilon}(s)-X_\lambda^{\nu,\eta}(s)\|^2_{F^*_{1,2,\nu}}ds.
\end{aligned}\tag{4.35}$$

Substituting (4.35) into (4.34), we obtain

$$\begin{aligned}
&\frac12\mathbb{E}\sup_{r\in[0,t]}\|X_\lambda^{\nu,\varepsilon}(r)-X_\lambda^{\nu,\eta}(r)\|^2_{F^*_{1,2,\nu}}\\
&+2\mathbb{E}\int_0^t\big\langle(\Psi_\lambda+\lambda I)\big(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big)-(\Psi_\lambda+\lambda I)\big(J_\eta(X_\lambda^{\nu,\eta}(s))\big),J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))-J_\eta(X_\lambda^{\nu,\eta}(s))\big\rangle_2 ds\\
\le\;&3(\varepsilon+\eta)(\frac{2}{\lambda}+\lambda+C_5)e^{C_3T}|x|_2^2+3C_1\mathbb{E}\int_0^t\sup_{r\in[0,s]}\|X_\lambda^{\nu,\varepsilon}(r)-X_\lambda^{\nu,\eta}(r)\|^2_{F^*_{1,2,\nu}}ds.
\end{aligned}$$

By Gronwall's lemma, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|X_\lambda^{\nu,\varepsilon}(t)-X_\lambda^{\nu,\eta}(t)\|^2_{F^*_{1,2,\nu}}\\
&+4\mathbb{E}\int_0^t\big\langle(\Psi_\lambda+\lambda I)\big(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big)-(\Psi_\lambda+\lambda I)\big(J_\eta(X_\lambda^{\nu,\eta}(s))\big),J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))-J_\eta(X_\lambda^{\nu,\eta}(s))\big\rangle_2 ds\\
&\leq 6(\varepsilon+\eta)(\frac{2}{\lambda}+\lambda+C)e^{(6C_1+C_3)T}|x|_2^2. \qquad (4.36)
\end{aligned}$$

Since by the monotonicity of $\Psi_\lambda$ the second term on the left hand-side of inequality (4.36) is nonnegative, letting $\varepsilon,\eta\to 0$, we see that $\{X_\lambda^{\nu,\varepsilon}\}$ is Cauchy in $L^2(\Omega;C([0,T];F^*_{1,2}))$. $\square$

From Claim 4.1, we know there exists $\widetilde{X}\in L^2(\Omega;C([0,T];F^*_{1,2}))$ such that

$$\lim_{\varepsilon\to 0}X_\lambda^{\nu,\varepsilon}=\widetilde{X}\ \text{ in } L^2(\Omega;C([0,T];F^*_{1,2})), \qquad (4.37)$$

**Claim 4.2** $\widetilde{X}=X_\lambda^\nu$.

*Proof* We have

$$\lim_{\varepsilon\to 0}\int_0^\cdot B(s,X_\lambda^{\nu,\varepsilon}(s))dW(s)=\int_0^\cdot B(s,\widetilde{X}(s))dW(s)\ \text{ in } L^2(\Omega;C([0,T];F^*_{1,2})), \qquad (4.38)$$

since by the BDG inequality for $p=1$ and (H2)(i), we have

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,T]}\Big\|\int_0^r(B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,\widetilde{X}(s)))dW(s)\Big\|^2_{F^*_{1,2,\nu}}\\
&\leq C\mathbb{E}\int_0^T\|B(s,X_\lambda^{\nu,\varepsilon}(s))-B(s,\widetilde{X}(s))\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}ds\\
&\leq CT\mathbb{E}\sup_{s\in[0,T]}\|X_\lambda^{\nu,\varepsilon}(s)-\widetilde{X}(s)\|^2_{F^*_{1,2,\nu}}.
\end{aligned}$$

Next we show that $(\Psi_\lambda+\lambda)(\widetilde{X})\in L^2((0,T);L^2(\Omega;F_{1,2}))$ and that (4.1) is satisfied. From Lemma 4.3 we know that $\{X_\lambda^{\nu,\varepsilon}\}$ is bounded in $L^2((0,T)\times\Omega\times E)$ and therefore along a subsequence, again denoted by $\{\varepsilon\}$, we have

$$\lim_{\varepsilon\to 0}X_\lambda^{\nu,\varepsilon}=\widetilde{X}\ \text{ weakly in } L^2((0,T)\times\Omega\times E). \qquad (4.39)$$

From (4.3) and (7.4), we know

$$\begin{aligned}
&\mathbb{E}\int_0^T\|X_\lambda^{\nu,\varepsilon}(s)-J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\|^2_{F^*_{1,2,\nu}}ds\\
&=\varepsilon^2\mathbb{E}\int_0^T\|(\nu-L)(\Psi_\lambda(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))+\lambda J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\|^2_{F^*_{1,2,\nu}}ds\\
&\leq\frac{\varepsilon}{2}(\frac{2}{\lambda}+\lambda+C_5)e^{C_3T}|x|_2^2, \qquad (4.40)
\end{aligned}$$

which yields,

$$\lim_{\varepsilon\to 0}J_\varepsilon(X_\lambda^{\nu,\varepsilon})=\widetilde{X}\ \text{ in } L^2((0,T);L^2(\Omega;F^*_{1,2})). \qquad (4.41)$$

Recall from (4.6) that

$$|J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))|_2 \le |X_\lambda^{\nu,\varepsilon}(t)|_2, \ \ \forall\, t \in [0,T]. \tag{4.42}$$

Therefore, we infer by (4.39) and (4.40) that

$$\lim_{\varepsilon\to 0} J_\varepsilon(X_\lambda^{\nu,\varepsilon}) = \widetilde{X}, \text{ weakly in } L^2((0,T)\times\Omega\times E). \tag{4.43}$$

By the monotonicity of $\Psi_\lambda$, it follows from (4.36) that $J_\varepsilon(X_\lambda^{\nu,\varepsilon})$, $\varepsilon \in (0,1)$, is Cauchy in $L^2((0,T)\times\Omega\times E)$, so the convergence in (4.43) is strong and thus

$$\lim_{\varepsilon\to 0}(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon})) = (\Psi_\lambda+\lambda I)(\widetilde{X}) \ \text{ in } L^2((0,T)\times\Omega\times E), \tag{4.44}$$

since $\Psi_\lambda + \lambda I$ is Lipschitz.

From (7.4), we know that $(\nu-L)(\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}))$, $\varepsilon\in(0,1)$, is bounded in $L^2([0,T]\times\Omega;F_{1,2}^*)$, so $(\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}))$ is bounded in $L^2([0,T];L^2(\Omega,F_{1,2}))$. Hence there exists a subsequence, again denoted by $\{\varepsilon\}$ such that

$$\lim_{\varepsilon\to 0}(\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon})) = (\Psi_\lambda+\lambda I)(\widetilde{X}) \text{ weakly in } L^2([0,T]\times\Omega;F_{1,2}). \tag{4.45}$$

It is then easy to see that also

$$\lim_{\varepsilon\to 0}\int_0^\cdot (\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))ds = \int_0^\cdot (\Psi_\lambda+\lambda I)(\widetilde{X}(s))ds$$

weakly in $L^2([0,T]\times\Omega;F_{1,2})$, and thus

$$\lim_{\varepsilon\to 0}(\nu-L)\int_0^\cdot (\Psi_\lambda+\lambda I)J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))ds = (\nu-L)\int_0^\cdot (\Psi_\lambda+\lambda I)(\widetilde{X}(s))ds$$

weakly in $L^2([0,T]\times\Omega;F_{1,2}^*)$.

Consequently, taking into account (4.37), (4.38), as $\varepsilon\to 0$, we can pass to the weak limit in $L^2([0,T]\times\Omega;F_{1,2}^*)$ in the equation

$$X_\lambda^{\nu,\varepsilon}(t) = x + (\nu-L)\int_0^t (\Psi_\lambda+\lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))ds + \int_0^t B(s,X_\lambda^{\nu,\varepsilon}(s))dW(s),$$

and since each term is a $\mathbb{P}$-a.s. continuous path in $F_{1,2}^*$, we conclude that $\widetilde{X}$ is a strong solution to (4.1) in the sense of Definition 3.1 in [37]. Furthermore, by the uniqueness part of [37, Lemma 3.1], it follows that $X_\lambda^\nu = \widetilde{X}$ a.e. in $(0,T)\times\Omega\times E$. □

By (4.39) and Lemma 4.3, Claim 4.1 implies (4.31). This completes the proof of Lemma 4.4. □

**Remark 4.1** By Lemma 4.4 we know that

$$X_\lambda^\nu(t) = x + (\nu-L)\int_0^t \big(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\big) + \int_0^t B(s,X_\lambda^\nu(s))dW(s), \ t\in[0,T].$$

But, since $X_\lambda^\nu = \widetilde{X}$, by (4.45) we may interchange $(\nu-L)$ with the integral w.r.t. $ds$.

Let us now continue to prove Theorem 3.2. Choose $0 < \nu \le \nu_0 \le 1$, rewrite (4.1) as

$$\begin{aligned} &dX_\lambda^\nu(t) + (\nu_0 - L)(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t))dt \\ &= (\nu_0 - \nu)(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t))dt + B(t, X_\lambda^\nu(t))dW(t). \end{aligned} \tag{4.46}$$

Now by Remark 4.1 we may apply Itô's formula ([29, Theorem 4.2.5]) to $\|X_\lambda^\nu - X_\lambda^{\nu'}\|^2_{F^*_{1,2,\nu_0}}$, $\nu,\ \nu' \in (0, \nu_0]$, to obtain for all $t \in [0,T]$, $\lambda \in (0,1)$,

$$\begin{aligned} &\|X_\lambda^\nu(t) - X_\lambda^{\nu'}(t)\|^2_{F^*_{1,2,\nu_0}} \\ &+2\int_0^t \int_E \big(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_\lambda(X_\lambda^{\nu'}(s)) - \lambda X_\lambda^{\nu'}(s)\big) \cdot (X_\lambda^\nu(s) - X_\lambda^{\nu'}(s))d\mu ds \\ &= 2(\nu' - \nu)\int_0^t \big\langle \Psi_\lambda(X_\lambda^\nu(s)) - \Psi_\lambda(X_\lambda^{\nu'}(s)), X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\big\rangle_{F^*_{1,2,\nu_0}} ds \\ &+2(\nu' - \nu)\int_0^t \big\langle \lambda X_\lambda^\nu(s) - \lambda X_\lambda^{\nu'}(s), X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\big\rangle_{F^*_{1,2,\nu_0}} ds \\ &+\int_0^t \|B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s))\|^2_{L_2(L^2(\mu), F^*_{1,2,\nu_0})} ds \\ &+2\int_0^t \big\langle X_\lambda^\nu(s) - X_\lambda^{\nu'}(s), \big(B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s))\big)dW(s)\big\rangle_{F^*_{1,2,\nu_0}}. \end{aligned} \tag{4.47}$$

Since $(\Psi_\lambda(r) - \Psi_\lambda(r'))(r - r') \ge 0$ for $r, r' \in \mathbb{R}$, $L^2(\mu) \subset F^*_{1,2}$ continuously, by Minkowski inequality, Young's inequality and (H2)(i), (4.47) yields for all $t \in [0,T]$,

$$\begin{aligned} &\|X_\lambda^\nu(t) - X_\lambda^{\nu'}(t)\|^2_{F^*_{1,2,\nu_0}} + \int_0^t \int_E 2\lambda |X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)|^2 d\mu ds \\ &\le \frac{2(\nu' - \nu)}{\sqrt{\nu_0}} \int_0^t (|\Psi_\lambda(X_\lambda^\nu(s))|_2 + |\Psi_\lambda(X_\lambda^{\nu'}(s))|_2) \cdot (\|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|_{F^*_{1,2,\nu_0}}) ds \\ &+2\lambda|\nu' - \nu| \int_0^t \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|^2_{F^*_{1,2,\nu_0}} ds \\ &+2\int_0^t \big\langle X_\lambda^\nu - X_\lambda^{\nu'}, (B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu_0}} \\ &+\int_0^t C_1 \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|^2_{F^*_{1,2,\nu_0}} ds \\ &\le \frac{2(\nu' + \nu)^2}{\nu_0} \int_0^t |\Psi_\lambda(X_\lambda^\nu(s))|_2^2 + |\Psi_\lambda(X_\lambda^{\nu'}(s))|_2^2 ds \\ &+2\int_0^t \big\langle X_\lambda^\nu(s) - X_\lambda^{\nu'}(s), (B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu_0}} \\ &+(C_1 + 2\lambda + 1)\int_0^t \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|^2_{F^*_{1,2,\nu_0}} ds. \end{aligned} \tag{4.48}$$

Taking expectation to both sides of (4.48), by the BDG inequality for $p = 1$, and by the fact that, by (H1), $|\Psi_\lambda(r)| \le C|r|^m$, $\forall r \in \mathbb{R}$, with $C$ independent of $\lambda$, taking (H2)(i) and (4.27) into account, we obtain for all $t \in [0,T]$,

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\Big[\sup_{s\in[0,t]} \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|^2_{F^*_{1,2,\nu_0}}\Big] + 2\lambda\mathbb{E}\int_0^t |X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)|_2^2 ds \\ &\le \frac{C_T(\nu + \nu')^2}{\nu_0}|x|_{2m}^{2m} + C\mathbb{E}\int_0^t \sup_{r\in[0,s]} \|X_\lambda^\nu(r) - X_\lambda^{\nu'}(r)\|^2_{F^*_{1,2,\nu_0}} ds. \end{aligned} \tag{4.49}$$

Hence by Gronwall's lemma, we have for some $C_T \in (0,\infty)$

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[0,T]} \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|^2_{F^*_{1,2,\nu_0}}\Big] + 2\lambda\mathbb{E}\int_0^T |X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)|_2^2 ds\\ &\le \frac{C_T(\nu+\nu')^2}{\nu_0}|x|_{2m}^{2m}, \quad \forall\ \lambda \in (0,1),\ \nu,\nu' \in (0,\nu_0]. \qquad (4.50)\end{aligned}$$

Hence there exists an $(\mathscr{F}_t)_{t\ge0}$-adapted process $X_\lambda \in L^2(\Omega; C([0,T]; F^*_{1,2})) \cap L^2((0,T)\times\Omega\times E)$ such that

$$\lim_{\nu\to0}\Big\{\mathbb{E}\Big[\sup_{s\in[0,T]} \|X_\lambda^\nu(s) - X_\lambda(s)\|^2_{F^*_{1,2,\nu_0}}\Big] + 2\lambda\mathbb{E}\int_0^T |X_\lambda^\nu(s) - X_\lambda(s)|_2^2 ds\Big\} = 0. \qquad (4.51)$$

Consequently, by (H2)(i) we can pass to the limit with $\nu \to 0$ in (4.1) to obtain

$$\begin{aligned}X_\lambda(t) = x - \lim_{\nu\to0}(\nu - L)&\int_0^t \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) ds\\ &+ \int_0^t B(s, X_\lambda(s)) dW(s), \quad t \in [0,T], \qquad (4.52)\end{aligned}$$

where the limit exists in $L^2(\Omega; C([0,T]; F^*_{1,2}))$. Furthermore, it follows by (4.51), since $\Psi_\lambda$ is Lipschitz, that

$$\lim_{\nu\to0}\int_0^\cdot \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) ds = \int_0^\cdot \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds, \qquad (4.53)$$

in $L^2(\Omega; C([0,T]; L^2(\mu)))$, hence in $L^2(\Omega; C([0,T]; F^*_{1,2}))$. Writing $\nu - L = (1-L) + (\nu-1)I$, (4.52) and (4.53) imply that the convergence in (4.53) holds even in $L^2(\Omega; C([0,T]; F_{1,2}))$ and that the second term on the right hand-side of (4.52) is equal to

$$L\int_0^t \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds,$$

which shows that $X_\lambda$ is a solution of (3.11) in the sense of Definition 3.1 in [37] with state space $F^*_{1,2}$.

Now let us prove that, since $x \in \mathscr{F}^*_e (\subset F^*_{1,2})$, which so far we have not used, that $X_\lambda$ is indeed a solution of (3.11) on the smaller state space $\mathscr{F}^*_e$ and that (3.12)-(3.15) hold. Note that (3.10) trivially holds, since the convergence in (4.53) is in $L^2(\Omega; C([0,T]; F_{1,2}))$ and since $F_{1,2} \subset \mathscr{F}_e$ continuously.

To prove (3.12) we observe that by (4.51) it follows that as $\nu \to 0$, $X_\lambda^\nu \to X_\lambda$ in $dt \otimes \mathbb{P}$-measure. Hence we have by Fatou's lemma and (4.31) for all $\varphi \in L^1([0,T];\mathbb{R})$

$$\begin{aligned}\int_0^T |\varphi(t)|\mathbb{E}|X_\lambda(t)|_p^p dt &\le \liminf_{\nu\to0}\int_0^T |\varphi(t)|\mathbb{E}|X_\lambda^\nu(t)|_p^p dt\\ &\le |\varphi|_{L^1([0,T];\mathbb{R})} C_p |x|_p^p,\end{aligned}$$

which implies (3.12). Now (3.13) follows by (H1).

To prove (3.14) we note that by exactly the same arguments as in the proof of (4.50), except for using (H2)(ii) instead of (H2)(i), we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[0,T]} \|X_\lambda^\nu(s)\|^2_{F^*_{1,2,\nu_0}}\Big] + \lambda\mathbb{E}\int_0^T |X_\lambda^\nu(s)|_2^2 ds\\ &\le C_T\big(\|x\|^2_{F^*_{1,2,\nu_0}} + \nu_0|x|_{2m}^{2m}\big), \quad \forall\ \lambda \in (0,1),\ \nu \in (0,\nu_0]. \qquad (4.54)\end{aligned}$$

Hence we get by Fatou's lemma

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_\lambda(t)\|^2_{F^*_{1,2,\nu_0}}\Big]+\lambda\mathbb{E}\int_0^T|X_\lambda(s)|_2^2ds\\ &\le C_T\big(\|x\|^2_{F^*_{1,2,\nu_0}}+|x|^{2m}_{2m}\big),\quad\forall\ \lambda\in(0,1).\end{aligned}\tag{4.55}$$

Letting $\nu_0\to 0$ and taking (2.5) into account, we get

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_\lambda(t)\|^2_{\mathscr{F}^*_e}\Big]+\lambda\mathbb{E}\int_0^T|X_\lambda(s)|_2^2ds\le C_T\big(\|x\|^2_{\mathscr{F}^*_e}+|x|^{2m}_{2m}\big),\ \forall\ \lambda\in(0,1),\tag{4.56}$$

hence (3.14) follows.

Now let us prove that $X_\lambda$ is a solution to (3.5) with state space $\mathscr{F}^*_e$. By (3.10) and Lemma 2.3, we have

$$L\int_0^\cdot\Psi_\lambda(X_\lambda(s))+X_\lambda(s)ds\in L^2(\Omega;C([0,T];\mathscr{F}^*_e)).$$

Furthermore, letting $\nu\to 0$ in (H2)(ii), we conclude from (4.55) that the stochastic integral in (3.11) is in $L^2(\Omega;C([0,T];\mathscr{F}^*_e))$ as well. Since $x\in\mathscr{F}^*_e$, (3.11) (which holds in $F^*_{1,2}$) implies that $X_\lambda\in L^2(\Omega;C([0,T];\mathscr{F}^*_e))$. So, altogether this implies that $X_\lambda$ is a strong solution of (3.5) with state space $\mathscr{F}^*_e$ in the sense of (3.9)-(3.11).

Now finally we prove (3.15). Firstly, we have

$$\begin{aligned}&d(X^\nu_\lambda(t)-X^\nu_{\lambda'}(t))+(\nu_0-L)\big(\Psi_\lambda(X^\nu_\lambda(t))-\Psi_{\lambda'}(X^\nu_{\lambda'}(t))+\lambda X^\nu_\lambda(t)-\lambda' X^\nu_{\lambda'}(t))\big)dt\\ &+(\nu-\nu_0)\big(\Psi_\lambda(X^\nu_\lambda(t))-\Psi_{\lambda'}(X^\nu_{\lambda'}(t))+\lambda X^\nu_\lambda(t)-\lambda' X^\nu_{\lambda'}(t))\big)dt\\ =&\ \big(B(t,X^\nu_\lambda(t))-B(t,X^\nu_{\lambda'}(t))\big)dW(t)\end{aligned}$$

By Remark 4.1 we may apply Itô's formula ([29, Theorem 4.2.5]) to $\frac12\|X^\nu_\lambda-X^\nu_{\lambda'}\|^2_{F^*_{1,2,\nu_0}}$, to obtain for $\nu\in(0,\nu_0]$, $t\in[0,T]$,

$$\begin{aligned}&\frac12\|X^\nu_\lambda(t)-X^\nu_{\lambda'}(t)\|^2_{F^*_{1,2,\nu_0}}\\ &+\int_0^t\int_E\big(\Psi_\lambda(X^\nu_\lambda(s))+\lambda X^\nu_\lambda(s)-\Psi_{\lambda'}(X^\nu_{\lambda'}(s))-\lambda' X^\nu_{\lambda'}(s)\big)\cdot(X^\nu_\lambda(s)-X^\nu_{\lambda'}(s))d\mu ds\\ &+(\nu-\nu_0)\int_0^t\big\langle\Psi_\lambda(X^\nu_\lambda(s))+\lambda X^\nu_\lambda(s)-\Psi_{\lambda'}(X^\nu_{\lambda'}(s))-\lambda' X^\nu_{\lambda'}(s),X^\nu_\lambda(s)-X^\nu_{\lambda'}(s)\big\rangle_{F^*_{1,2,\nu_0}}ds\\ =&\ \frac12\int_0^t\big\|B(s,X^\nu_\lambda(s))-B(s,X^\nu_{\lambda'}(s))\big\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu_0})}ds\\ &+\int_0^t\big\langle X^\nu_\lambda(s)-X^\nu_{\lambda'}(s),(B(s,X^\nu_\lambda(s))-B(s,X^\nu_{\lambda'}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu_0}}.\end{aligned}\tag{4.57}$$

Since $r=\lambda\Psi_\lambda(r)+(I+\lambda\Psi)^{-1}(r)$, for all $r\in\mathbb{R}$, we have for all $r'\in\mathbb{R}$

$$\begin{aligned}(\Psi_\lambda(r)-\Psi_{\lambda'}(r'))(r-r')=&\ \big[\Psi_\lambda(r)-\Psi_{\lambda'}(r')\big]\cdot\big[(I+\lambda\Psi)^{-1}(r)-(I+\lambda'\Psi)^{-1}(r')\big]\\ &+\big[\Psi_\lambda(r)-\Psi_{\lambda'}(r')\big]\cdot\big[\lambda\Psi_\lambda(r)-\lambda'\Psi_{\lambda'}(r')\big].\end{aligned}\tag{4.58}$$

Note that the first summand in the right hand-side is nonnegative since $\Psi$ is maximal monotone and since $\Psi_\lambda(r)\in\Psi((I+\lambda\Psi)^{-1}(r))$(see [4, page:61]). Plugging (4.58) into (4.57), and using that $\|\cdot\|_{F^*_{1,2,\nu_0}} \le \frac{1}{\sqrt{\nu_0}}|\cdot|_2$ and (H2)(i), we obtain for $\nu \in (0,\nu_0]$, $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}\|X^\nu_\lambda(t) - X^\nu_{\lambda'}(t)\|^2_{F^*_{1,2,\nu_0}}\\
&+\int_0^t \int_E \big(\Psi_\lambda(X^\nu_\lambda(s)) - \Psi_{\lambda'}(X^\nu_{\lambda'}(s))\big)\cdot\big(\lambda\Psi_\lambda(X^\nu_\lambda(s)) - \lambda'\Psi_{\lambda'}(X^\nu_{\lambda'}(s))\big)d\mu ds\\
&+\int_0^t \int_E \big(\lambda X^\nu_\lambda(s) - \lambda' X^\nu_{\lambda'}(s)\big)\cdot\big(X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)\big)d\mu ds\\
\le{}& \frac{(\nu_0-\nu)}{\sqrt{\nu_0}}\int_0^t \big|\Psi_\lambda(X^\nu_\lambda(s)) + \lambda X^\nu_\lambda(s) - \Psi_{\lambda'}(X^\nu_{\lambda'}(s)) - \lambda' X^\nu_{\lambda'}(s)\big|_2 \cdot \big\|X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)\big\|_{F^*_{1,2,\nu_0}} ds\\
&+\frac{C_1}{2}\int_0^t \|X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)\|^2_{F^*_{1,2,\nu_0}} ds\\
&+\int_0^t \big\langle X^\nu_\lambda(s) - X^\nu_{\lambda'}(s), (B(s,X^\nu_\lambda(s)) - B(s,X^\nu_{\lambda'}(s)))dW(s)\big\rangle_{F^*_{1,2,\nu_0}}.
\end{aligned} \tag{4.59}$$

By the Burkholder-Davis-Gundy inequality (for $p=1$) we get for all $\lambda,\ \lambda' > 0$, $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\Big[\sup_{s\in[0,t]}\|X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)\|^2_{F^*_{1,2,\nu_0}}\Big]\\
\le{}& C(\lambda+\lambda'+\nu_0)\mathbb{E}\int_0^t \big(|\Psi_\lambda(X^\nu_\lambda(s))|_2^2 + |\Psi_{\lambda'}(X^\nu_{\lambda'}(s))|_2^2 + |X^\nu_\lambda(s)|_2^2 + |X^\nu_{\lambda'}(s)|_2^2\big)ds\\
&+C\mathbb{E}\int_0^t \sup_{r\in[0,s]}\|X^\nu_\lambda(r) - X^\nu_{\lambda'}(r)\|^2_{F^*_{1,2,\nu_0}} ds.
\end{aligned} \tag{4.60}$$

Hence by (H1), (4.31) and Gronwall's lemma, there exists $C_T \in (0,\infty)$ independent of $\nu_0$, such that for all $\nu \in (0,\nu_0]$, $\lambda, \lambda' \in (0,1)$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|X^\nu_\lambda(t) - X^\nu_{\lambda'}(t)\|^2_{F^*_{1,2,\nu_0}}\Big] \le C_T(\lambda+\lambda'+\nu_0)(|x|_2^2 + |x|_{2m}^{2m}). \tag{4.61}$$

Then letting $\nu \to 0$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_\lambda(t) - X_{\lambda'}(t)\|^2_{F^*_{1,2,\nu_0}}\Big] \le C_T(\lambda+\lambda'+\nu_0)(|x|_2^2 + |x|_{2m}^{2m}), \tag{4.62}$$

so by letting $\nu_0 \to 0$ in (4.62) and taking into account (2.5) we obtain (3.15). Consequently, Theorem 3.2 is proved. □

# 5 Proof of Theorem 3.1

After all our preparations, to deduce that the solution $X_\lambda$, $\lambda \in (0,1)$, of equation (3.5) as $\lambda \to 0$ converges to the unique solution of equation (1.1) is now in principle quite standard (at least for experts on stochastic porous media equations), maybe except for proving (3.3). Since, however, there is no proof in the literature that covers our general case, we give a complete presentation of the arguments in this section.

*Proof* Let $X_\lambda$ be as in Theorem 3.2. Then it follows by Theorem 3.2 that there exists a process $X \in L^2(\Omega; C([0,T]; \mathscr{F}_e^*))$ such that, as $\lambda \to 0$,

$$\begin{aligned} X_\lambda \to X & \quad \text{strongly in } L^2(\Omega; C([0,T]; \mathscr{F}_e^*)), \\ X_\lambda \to X & \quad \text{weak-star in } L^\infty([0,T]; (L^p \cap L^2 \cap L^{2m})(\Omega \times E)), \\ \lambda X_\lambda \to 0 & \quad \text{strongly in } L^2([0,T] \times \Omega \times E), \\ \Psi_\lambda(X_\lambda) \to \eta & \quad \text{weakly in } L^{\frac{m+1}{m}}([0,T] \times \Omega \times E) \cap L^2([0,T] \times \Omega \times E). \end{aligned} \tag{5.1}$$

By (5.1) and (H2)(i) we may take the limit $\lambda \to 0$ in (3.11) in $L^2(\Omega; C([0,T]; \mathscr{F}_e^*))$ to obtain that

$$\begin{aligned} X = x + \lim_{\lambda \to 0} L \int_0^{\cdot} \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds \\ + \int_0^{\cdot} B(s, X(s)) dW(s), \end{aligned} \tag{5.2}$$

where we have used that by (2.5) we may take the limit $\nu_0 \to 0$ in (H2)(i). By Lemma 2.3 we conclude that

$$\lim_{\lambda \to 0} \int_0^{\cdot} \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds \tag{5.3}$$

exists in $L^2(\Omega; C([0,T]; \mathscr{F}_e))$, hence by (L.1) in $L^2(\Omega; C([0,T]; L^1(g \cdot \mu)))$ for some $g \in L^1(\mu) \cap L^\infty(\mu)$, $g > 0$. Hence the limit in (5.3) coincides with the limit in $dt \otimes \mathbb{P} \otimes d\mu$-measure. Therefore, $\int_0^{\cdot} \eta(s) ds \in L^2(\Omega; C([0,T]; \mathscr{F}_e))$ and (5.2) implies

$$X(t) = x + L \int_0^t \eta(s) ds + \int_0^t B(s, X(s)) dW(s), \quad t \in [0,T]. \tag{5.4}$$

Hence $X(t)$, $t \in [0,T]$, is a solution to (1.1) in the sense of Definition 3.1 if we can show that

$$\eta \in \Psi(X), \quad dt \otimes \mathbb{P} \otimes \mu - a.e.. \tag{5.5}$$

For this it suffices to show that

$$\limsup_{\lambda \to 0} \mathbb{E} \int_0^T \int_E \Psi_\lambda(X_\lambda) X_\lambda d\mu dt \leq \mathbb{E} \int_0^T \int_E \eta X d\mu dt. \tag{5.6}$$

Indeed, since $\Psi_\lambda = \partial j_\lambda$, where $j_\lambda$ is as in (3.8), we have for all $\lambda \in (0,1)$

$$\mathbb{E} \int_0^T \int_E \Psi_\lambda(X_\lambda)(X_\lambda - Z) d\mu dt \geq \mathbb{E} \int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(Z) d\mu dt, \tag{5.7}$$

for all $Z \in L^{m+1}((0,T) \times \Omega \times E)$, since $X_\lambda, |\Psi_\lambda(X_\lambda)| \in (L^{\frac{2}{m}} \cap L^2)((0,T) \times \Omega \times E) \subset L^{\frac{m+1}{m}}((0,T) \times \Omega \times E)$.
Let $\Psi^0$ be as defined in (3.7) and define the integral (see [4, page:54])

$$j(r) := \int_0^r \Psi^0(s) ds, \ r \in \mathbb{R}.$$

Then $j$ is a continuous and convex function on $\mathbb{R}$ satisfying

$$0 \leq j(r) \leq r \Psi^0(r), \forall r \in \mathbb{R}, \tag{5.8}$$

because $\Psi(0)=0$. Recall that by [4, page:48, Theorem 2.9]

$$j_\lambda \geq 0, \tag{5.9}$$

$$\lim_{\lambda\to 0} j_\lambda(r) = j(r), \forall r \in \mathbb{R}, \tag{5.10}$$

$$j_\lambda(r) \leq j(r), \forall r \in \mathbb{R}. \tag{5.11}$$

Consequently, for all $Z \in L^{m+1}((0,T)\times\Omega\times E)$

$$\limsup_{\lambda\to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(Z) d\mu dt \geq \mathbb{E}\int_0^T \int_E j(X) - j(Z) d\mu dt. \tag{5.12}$$

Indeed, by (5.8), (5.9) and (H1)

$$|j_\lambda(z)| \leq C|z|^{m+1}$$

and hence by Lebesgue's dominated convergence theorem,

$$\lim_{\lambda\to 0} \mathbb{E}\int_0^T \int_E j_\lambda(Z) d\mu dt = \mathbb{E}\int_0^T \int_E j(Z) d\mu dt.$$

Furthermore, by (3.6), (3.7), (H1) and because $X \in L^{2m}((0,T)\times\Omega\times E)$ we have as $\lambda\to 0$, $\Psi_\lambda(X) \to \Psi_0(X)$ in $L^2((0,T)\times\Omega\times E)$, hence

$$\begin{aligned} &\limsup_{\lambda\to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j(X) d\mu dt \\ &= \limsup_{\lambda\to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(X) d\mu dt \\ &\geq \limsup_{\lambda\to 0} \mathbb{E}\int_0^T \int_E \Psi_\lambda(X)(X_\lambda - X) d\mu dt \\ &= 0. \end{aligned}$$

Hence by (5.6), (5.7) and (5.12), we have $\forall Z \in L^{m+1}((0,T)\times\Omega\times E)$,

$$\mathbb{E}\int_0^T \int_E \eta(X-Z) d\mu dt \geq \mathbb{E}\int_0^T \int_E \big(j(X) - j(Z)\big) d\mu dt.$$

This yields

$$\mathbb{E}\int_0^T \int_E \eta(X-Z) d\mu dt \geq \mathbb{E}\int_0^T \int_E \zeta(X-Z) d\mu dt, \tag{5.13}$$

for all $Z \in L^{m+1}((0,T)\times\Omega\times E)$ and $\zeta \in L^{\frac{m+1}{m}}((0,T)\times\Omega\times E)$ such that $\zeta \in \Psi(Z)$ a.e. on $(0,T)\times\Omega\times E$.

By virtue of assumption (H1), $\Psi$ is maximal monotone in $L^{m+1}((0,T)\times\Omega\times E)\times L^{\frac{m+1}{m}}((0,T)\times\Omega\times E)$, so by [4, page:34, Theorem 2.2] one knows that the equation

$$J(Z) + \Psi(Z) \ni J(X) + \eta, \tag{5.14}$$

where $J(Z) = |Z|^{p-2}Z$, has a unique solution $Z \in L^{m+1}((0,T)\times\Omega\times E)$.

Now if in (5.13), we take $Z$ to be the solution of (5.14) and $\zeta := J(X) - J(Z) + \eta$, we obtain

$$\mathbb{E}\int_0^T \int_E (J(X) - J(Z))(X - Z) d\mu dt \leq 0,$$

i.e.,

$$\mathbb{E}\int_0^T\int_E(|X|^{p-2}X-|Z|^{p-2}Z)(X-Z)d\mu dt\le 0.$$

Since $J:r\to|r|^{p-2}r$ is strictly increasing, it follows that

$$\mathbb{E}\int_0^T\int_E((|X|^{p-2}X-|Z|^{p-2}Z)(X-Z)d\mu dt=0.$$

Hence $X=Z$ a.e. on $(0,T)\times\Omega\times E$, and thus by (5.14), we have $\eta\in\Psi(X)$, $\mathbb{P}\otimes dt\otimes\mu$, *a.e.*.

It remains to prove (5.6). By Appendix 7.3 we may apply Itô's formula from [29, Theorem 4.2.5] to the process in (3.5) to obtain

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\|X_\lambda(t)\|^2_{\mathscr{F}_e^*}-\mathbb{E}\int_0^t {}_{V^*}\big\langle\widetilde{L}\big(\Psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s)\big),X_\lambda(s)\big\rangle_V ds\\&=\frac{1}{2}\|x\|^2_{\mathscr{F}_e^*}+\frac{1}{2}\mathbb{E}\int_0^t\|B(s,X_\lambda(s))\|^2_{L_2(L^2(\mu),\mathscr{F}_e^*)}ds,\end{aligned}\tag{5.15}$$

By (7.16), where $\widetilde{L}$ and $V$ are as in Appendix 7.3, we know that

$$-{}_{V^*}\big\langle\widetilde{L}\big(\Psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s)\big),X_\lambda(s)\big\rangle_V=\int_E\big(\Psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s)\big)\cdot X_\lambda(s)d\mu.\tag{5.16}$$

By Appendix 7.3 we may also apply Itô's formula from [29, Theorem 4.2.5] to the process in (5.4) to obtain by (7.16)

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\|X(t)\|^2_{\mathscr{F}_e^*}+\mathbb{E}\int_0^t\int_E\eta\cdot X(s)ds\\&=\frac{1}{2}\|x\|^2_{\mathscr{F}_e^*}+\frac{1}{2}\mathbb{E}\int_0^t\|B(s,X(s))\|^2_{L_2(L^2(\mu),\mathscr{F}_e^*)}ds.\end{aligned}\tag{5.17}$$

Letting $\lambda\to 0$ in (5.15) after plugging in (5.16), using (5.2) and comparing with (5.17), we obtain (5.6) (even with " $=$ " replacing " $\le$ ").

**Uniqueness**

Suppose $X_1$, $X_2$ are two strong solutions to (1.1). We have with $\widetilde{L}$ as in Appendix 7.3

$$\begin{cases}d(X_1-X_2)-\widetilde{L}(\eta_1-\eta_2)dt=(B(t,X_1)-B(t,X_2))dW(t),\ \text{in }[0,T]\times E,\\X_1-X_2=0\text{ on }E,\end{cases}\tag{5.18}$$

where $\eta_i\in\Psi(X_i)$, $i=1,2$, a.e. on $\Omega\times(0,T)\times E$.

As above we may apply Itô's formula to get

$$\begin{aligned}&\frac{1}{2}d\|X_1-X_2\|^2_{\mathscr{F}_e^*}-{}_{V^*}\big\langle\widetilde{L}(\eta_1-\eta_2),X_1-X_2\big\rangle_V dt\\&=\frac{1}{2}\big\|B(t,X_1)-B(t,X_2)\big\|^2_{L_2(L^2(\mu),\mathscr{F}_e^*)}dt\\&\quad+\big\langle X_1-X_2,(B(s,X_1)-B(s,X_2))dW_t\big\rangle_{\mathscr{F}_e^*}.\end{aligned}\tag{5.19}$$

Since $\Psi$ is monotone, by (7.16) we have

$$\begin{aligned}&\mathbb{E}\int_0^T {}_{V^*}\big\langle -\widetilde{L}(\eta_1-\eta_2), X_1-X_2\big\rangle_V dt\\ =\ &\mathbb{E}\int_0^T\int_E(\eta_1-\eta_2)\cdot(X_1-X_2)d\mu dt\geq 0.\end{aligned}\tag{5.20}$$

Therefore, integrating (5.19) from 0 to $t$ and taking expectation, by (5.20) and Remark 3.1 (i), we obtain

$$\mathbb{E}\|X_1-X_2\|^2_{\mathscr{F}_e^*}\leq C_1\int_0^t\mathbb{E}\|X_1-X_2\|^2_{\mathscr{F}_e^*}ds,\quad \forall t\in[0,T],$$

and by Gronwall's inequality we get $X_1=X_2$, $\mathbb{P}-$a.s.. Thus Theorem 3.1 is proved. □

# 6 Applications

**Example 6.1 (Example for $B$, see [36, Remark 2.9])**

**(M)** Let $N\in\mathbb{N}\cup\{+\infty\}$ and $e_k\in L^2(\mu)\cap L^\infty(\mu)$, $1\leq k<N$, be an orthonormal system in $L^2(\mu)$ such that for every $1\leq k<N$ there exists $\xi_k\in(0,\infty)$ such that for all $\nu\in(0,\infty)$

$$|{}_{F^*_{1,2}}\langle x, e_k u\rangle_{F_{1,2}}|\leq\xi_k\|x\|_{F^*_{1,2,\nu}}\|u\|_{F_{1,2,\nu}},\quad\forall\ u\in F_{1,2},\ x\in F^*_{1,2}.$$

**(M)** means that each $e_k$ is a multiplier in $(F^*_{1,2},\|\cdot\|_{F^*_{1,2,\nu}})$ with norm independent of $\nu>0$. Choose $\mu_k\in(0,\infty)$ such that

$$\sum_{k=1}^{\infty}\mu_k^2(\xi_k^2+|e_k|^2_\infty)<\infty,\tag{6.1}$$

and define for $x\in F^*_{1,2}$, $B(x)\in L_2(L^2(\mu),F^*_{1,2})$ by

$$B(x)h:=\sum_{k=1}^{\infty}\mu_k\langle e_k,h\rangle x\cdot e_k,\quad h\in L^2(\mu).\tag{6.2}$$

Indeed, (extending $\{e_k|k\in\mathbb{N}\}$ to an orthonormal basis of $L^2(\mu)$ by **(M)**) we have for $x\in F^*_{1,2}$, $\nu\in(0,\infty)$

$$\|B(x)\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}=\sum_{k=1}^{\infty}\|B(x)e_k\|^2_{F^*_{1,2,\nu}}=\sum_{k=1}^{\infty}\mu_k^2\|xe_k\|^2_{F^*_{1,2,\nu}}\leq\sum_{k=1}^{\infty}\mu_k^2\xi_k^2\|x\|^2_{F^*_{1,2,\nu}},$$

which implies (H2)(ii), and since $x\to B(x)$ is linear, assumption (H2)(i) follows.

From (6.1) and (6.2), we see that for $x\in L^2(\mu)$, $B(x)\in L_2(L^2(\mu),L^2(\mu))$, since

$$\|B(x)\|^2_{L_2(L^2(\mu),L^2(\mu))}=\sum_{k=1}^{\infty}|B(x)e_k|_2^2=\sum_{k=1}^{\infty}\mu_k^2|xe_k|_2^2\leq\sum_{k=1}^{\infty}\mu_k^2|e_k|^2_\infty|x|_2^2,$$

which implies (H3)(i). Similarly, for $x\in L^2(\mu)\cap L^p(\mu)\subset L^p(\mu)$,

$$\Big(\int_E\big(\sum_{k=1}^{\infty}|B(x)e_k|^2\big)^{\frac{p}{2}}d\mu\Big)^{\frac{2}{p}}\leq\sum_{k=1}^{\infty}\Big(\int_E|B(x)e_k|^pd\mu\Big)^{\frac{2}{p}}=\sum_{k=1}^{\infty}\mu_k^2|xe_k|_p^2\leq\sum_{k=1}^{\infty}\mu_k^2|e_k|^2_\infty|x|_p^2,$$

which implies (H3)(ii).

**Example 6.2 (Example for local $\mathcal{E}$)**

Suppose $(\mathcal{E}, F_{1,2})$ is a local transient Dirichlet form with generator $L$ such that it admits a carré du champ $\Gamma$ ([12, Definition 4.1.2]), which is a unique positive symmetric and continuous bilinear map from $F_{1,2} \times F_{1,2}$ into $L^1(\mu)$ such that

$$\mathcal{E}(uw, v) + \mathcal{E}(vw, u) - \mathcal{E}(w, uv) = \int w\Gamma(u, v)d\mu, \ \forall u, v, w \in F_{1,2}. \tag{6.3}$$

From [12, Propostion 6.1.1], we know that then

$$\mathcal{E}(u, v) = \frac{1}{2}\int \Gamma(u, v)d\mu, \quad u, v \in F_{1,2},$$

which implies (H4)(i).

By [12, Corollary 7.1.2], we know that for every Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ which satisfies $\varphi(0) = 0$,

$$\Gamma(\varphi(u), v) = \varphi'(u)\Gamma(u, v), \quad \forall\ u, v \in F_{1,2}, \tag{6.4}$$

where $\varphi'$ is any version of the derivatives (defined Lebesgue-a.e.) of $\varphi$. Furthermore, if $\varphi$ is nondecreasing, then

$$\Gamma(u, \varphi(u)) = \varphi'(u)\Gamma(u, u) \geq 0, \ \forall\ u \in F_{1,2},$$

and

$$\Gamma(\varphi(u), \varphi(u)) = \varphi'(u)\Gamma(u, \varphi(u)) \leq \text{ ess sup}_{r\in\mathbb{R}}\ \varphi'(r)\Gamma(u, \varphi(u)), \ \forall\ u \in F_{1,2},$$

which implies (H4)(ii).

There is a abundance of examples of such Dirichlet forms on very general state spaces $E$, as e.g. finite or infinite dimensional manifolds. We refer e.g. to [12, 18, 30] and also [20].

**Example 6.3 (Example for nonlocal $\mathcal{E}$)**

As is well-known, under quite general assumptions according to the Beurling-Deny representation formula a Dirichlet can be written as the sum of a local Dirichlet form $\mathcal{E}^{(1)}$ (i.e. if it has a square field operator, it satisfies (6.4)) and a non-local part $\mathcal{E}^{(2)}$ (see [18, Section 3.2] or [21] for details). A typical form of $\mathcal{E}^{(2)}$ is as follows

$$\mathcal{E}^{(2)}(u, v) = \int_E\int_E (u(x) - u(y))(v(x) - v(y))J(x, dy)m(dx), \quad u, v \in D(\mathcal{E}^{(2)}),$$

where $J$ is a kernel from $E$ to $E$ and $m$ is a $\sigma$-finite measure on $(E, \mathcal{B})$. Therefore,

$$\mathcal{E}^{(2)}(u, v) = \int_E \Gamma(u, v)dm, \quad u, v \in D(\mathcal{E}^{(2)}),$$

where for $x \in E$

$$\Gamma(u, v)(x) = \int (u(x) - u(y))(v(x) - v(y))J(x, dy).$$

Clearly, $\Gamma$ does not satisfy (6.4), but it satisfies our condition (H4)(ii). Indeed, for every non-decreasing Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$ and $u \in D(\mathcal{E}^{(2)})$ we have

$$\begin{aligned}\Gamma(\varphi(u), \varphi(u)) &= \int_E \Big(\varphi(u(x)) - \varphi(u(y))\Big)^2 J(x, dy) \\ &\leq Lip\varphi\Big( \int_E 1_{\{u(x)\geq u\}}(u(x) - u(y))\big(\varphi(u(x)) - \varphi(u(y))\big) J(x, dy) \\ &\qquad + \int_E 1_{\{u(x)<u\}}(u(y) - u(x))\big(\varphi(u(y)) - \varphi(u(x))\big) J(x, dy)\Big) \\ &= Lip\varphi\Gamma(u, \varphi(u)).\end{aligned}$$

A concrete example of this is the following very classical case.

Let $E = \mathbb{R}^d$, $\mu = dx$ and let ” $\hat{}$ ” resp. ” $\check{}$ ” denote Fourier transform, i.e.,

$$\hat{f}(x) = (2\pi)^{-\frac{d}{2}} \int \exp[i\langle x, y\rangle_{\mathbb{R}^d}] f(y) dy,$$

resp. its inverse. Define for $\alpha > 0$

$$(-\Delta)^\alpha u := \big(|x|^{2\alpha}\hat{u}\big)^{\check{}} \quad (\in L^2(\mathbb{R}^d; dx)), \quad u \in C_0^\infty(\mathbb{R}^d).$$

Then $(-\Delta)^\alpha$ is a symmetric linear operator on $L^2(\mathbb{R}^d; dx)$ with dense domain $C_0^\infty(\mathbb{R}^d)$. Hence the form

$$\mathbb{D}^{(\alpha)}(u, v) := \frac{1}{2} \int \hat{u}\bar{\hat{v}}|x|^{2\alpha} dx, \quad u, v \in C_0^\infty(\mathbb{R}^d),$$

is closable, where ” $\bar{}$ ” means complex conjugation. Its closure $(\mathbb{D}^{(\alpha)}, H^{\alpha,2}(\mathbb{R}^d))$ is hence a symmetric closed form on $L^2(\mathbb{R}^d; dx)$. If $\alpha \in (0, \frac{d}{2}) \cap (0, 1]$, it is a transient Dirichlet form and for some constant $c_{\alpha,d} > 0$

$$\mathcal{E}(u, v) = c_{\alpha,d} \int\int \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{2\alpha+d}} dxdy, \quad u, v \in H^{\alpha,2}(\mathbb{R}^d).$$

For more details we refer to [30, page:43] and [35].

**Remark 6.1** Theorem 3.1 also applies to transient Dirichlet forms, where the corresponding state space E is a fractal, (see e.g. [28]).

# 7 Appendix

## 7.1 Auxiliary results

In this part we aim to prove (7.4), which has been used in the proof of Claim 4.3.

**Lemma 7.1** *For all $x \in F_{1,2}^*$ and all $\varepsilon > 0$, we have*

$$\begin{aligned}&\big\langle(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), x\big\rangle_{F^*_{1,2,\nu}} \\ &= \big\langle(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\big\rangle_2 + \varepsilon\|(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x)))\|^2_{F^*_{1,2,\nu}}. \qquad (7.1)\end{aligned}$$

*For all $x \in L^2(\mu)$,*

$$\begin{aligned}&\big\langle(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), x\big\rangle_2 \\ &= \big\langle(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\big\rangle_2 + \varepsilon\big|(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big|_2^2. \qquad (7.2)\end{aligned}$$

*Proof* Recall that

$$A_\lambda^{\nu,\varepsilon} = \frac{1}{\varepsilon}(I - J_\varepsilon) = (\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon),$$

For $x \in F_{1,2}^*$, to prove (7.1), we rewrite

$$\begin{aligned}
&\big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), x\big\rangle_{F^*_{1,2,\nu}} \\
&= \big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), J_\varepsilon(x)\big\rangle_{F^*_{1,2,\nu}} \\
&\quad + \big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), \varepsilon(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big)\big\rangle_{F^*_{1,2,\nu}} \\
&= \big\langle (\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x\big\rangle_2 + \varepsilon\|(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x)))\|^2_{F^*_{1,2,\nu}}.
\end{aligned}$$

The proof of (7.2) is analogous due to the fact that $J_\varepsilon$ is $\frac{1}{\sqrt{\nu\varepsilon\lambda}}$-Lipschitz in $L^2(\mu)$, so $A_\lambda^{\nu,\varepsilon} \in L^2(\mu)$ if $x \in L^2(\mu)$. □

**Lemma 7.2** *For each $x \in L^2(\mu)$, $T > 0$, and $0 < \varepsilon < 1$, there exists $C > 0$ such that $\forall\ \nu \in (0,1)$, $\lambda \in (0,+\infty)$,*

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(s)|_2^2 + 2\mathbb{E}\int_0^t \big\langle (\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big\rangle_2 ds \le e^{C_3 T}|x|_2^2, \ \forall t \in [0,T]. \tag{7.3}$$

*Proof* Applying Itô formula to $|X_\lambda^{\nu,\varepsilon}|_2^2$, we obtain

$$\begin{aligned}
&d|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t)))), (X_\lambda^{\nu,\varepsilon}(t)\big\rangle_2 dt \\
&= \|B(t, X_\lambda^{\nu,\varepsilon}(t))\|^2_{L_2(L^2(\mu),L^2(\mu))} dt + 2\big\langle X_\lambda^{\nu,\varepsilon}, B(t, X_\lambda^{\nu,\varepsilon}(t))dW(t)\big\rangle_2
\end{aligned}$$

which by (7.2) yields,

$$\begin{aligned}
&d|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t)))), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))\big\rangle_2 dt \\
&\quad + 2\varepsilon\big|(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))))\big|_2^2 dt \\
&= \|B(t, X_\lambda^{\nu,\varepsilon}(t))\|^2_{L_2(L^2(\mu),L^2(\mu))} dt + 2\big\langle X_\lambda^{\nu,\varepsilon}, B(t, X_\lambda^{\nu,\varepsilon}(t))dW(t)\big\rangle_2.
\end{aligned}$$

Taking expectation of both sides, by (H3)(i) we get

$$\begin{aligned}
&\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\mathbb{E}\int_0^t \big\langle (\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big\rangle_2 ds \\
&\quad + 2\varepsilon\mathbb{E}\int_0^t \big|(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))))\big|_2^2 ds \\
&\le |x|_2^2 + C_3\mathbb{E}\int_0^t |X_\lambda^{\nu,\varepsilon}(s)|_2^2 ds.
\end{aligned}$$

Then by (4.17) and Gronwall's lemma we get (7.3) as claimed. □

**Proposition 7.1** *For $x \in L^2(\mu)$, $t \in [0,T]$ and $0 < \varepsilon < 1$, $\nu \in (0,1)$, we have*

$$\mathbb{E}\int_0^t \big\|(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)\big\|^2_{F^*_{1,2,\nu}} ds \le \frac{1}{2}(\frac{2}{\lambda} + \lambda + C_5)e^{C_3 T}|x|_2^2. \tag{7.4}$$

*Proof* Let $x \in L^2(\mu)$. Then

$$\begin{aligned}
&\big\|(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big)\big\|^2_{F^*_{1,2,\nu}} \\
&= \|(\Psi_\lambda + \lambda I)(J_\varepsilon(x))\|^2_{F_{1,2,\nu}} \\
&= \int \frac{1}{2}\Gamma\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x)), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big) d\mu \\
&\quad + \nu\big\langle(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big\rangle_2 \\
&\leq C_5 \int \frac{1}{2}\Gamma\big(J_\varepsilon(x), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big) d\mu + \nu(\frac{2}{\lambda} + \lambda)\langle(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\rangle_2 \\
&\leq (\frac{2}{\lambda} + \lambda + C_5)\langle J_\varepsilon(x), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\rangle_{F_{1,2,\nu}} \\
&= (\frac{2}{\lambda} + \lambda + C_5)\langle(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\rangle_2,
\end{aligned}$$

where in the first inequality we used (H4) and the fact that $r(\Psi_\lambda(r) + \lambda r) \geq 0$ for all $r \in \mathbb{R}$, and the last equality comes from the fact that $(\Psi_\lambda + \lambda I)(J_\varepsilon(x)) \in D(L)$. Now from (7.3), we get the assertion. □

## 7.2 The $L^p$-Itô formula in expectation

The purpose in this section is to prove Theorem 7.1 below, which has been used in Lemmas 4.2 and 4.3.

Let $\ell_2$ be the space of all square-summable sequences in $\mathbb{R}$ and $p \in [1, \infty)$. In addition, to the real-valued $L^p$-space, $L^p(\mu) := L^p(E, \mu)$ we consider the $\ell_2$-valued $L^p$-space $L^p(\mu; \ell_2) := L^p(E, \mu; \ell_2)$. We set

$$|g|_p^p := |g|^p_{L^p(\mu;\ell_2)} = \int_E \|g(x)\|^p_{\ell_2} \mu(dx) = \int_E \Big(\sum_{k=1}^\infty |g_k(x)|^2\Big)^{\frac{p}{2}} \mu(dx).$$

Let $\mathscr{P}$ denote the predictable $\sigma$-algebra on $[0, T] \times \Omega$ corresponding to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$. For $p \in [1, \infty)$ we set

$$\mathbb{L}^p(T) := L^p([0, T] \times \Omega, \mathscr{P}; L^p(\mu))$$

and

$$\mathbb{L}^p(T; \ell_2) := L^p([0, T] \times \Omega, \mathscr{P}; L^p(\mu; \ell_2)),$$

equipped with its standard $L^p$-norms. Since $(E, \mathcal{B})$ is a standard measurable space, by definition there exists a complete metric $d$ on $E$, such that $(E, d)$ is separable, i.e., a Polish space, whose Borel $\sigma$-algebra coincides with $\mathcal{B}$. Below we fix this metric $d$ and denote the corresponding set of all bounded continuous functions by $C_b(E)$.

Let $\mathcal{E}$ be all $g = (g_k)_{k\in\mathbb{N}} \in L^\infty([0, T] \times \Omega; L^\infty(\mu; \ell_2) \cap L^1(\mu; \ell_2))$ such that there exists $j \in \mathbb{N}$ and bounded stopping times $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_j \leq T$ such that

$$g_k = \begin{cases} \sum_{i=1}^j g_k^i 1_{(\tau_{i-1}, \tau_i]}, & \text{if } k \leq j; \\ 0, & \text{if } k > j, \end{cases} \tag{7.5}$$

where $g_k^i \in C_b(E) \cap L^1(\mu)$, $1 \leq i \leq j$.

**Claim 7.1** *$\mathcal{E}$ is dense in $\mathbb{L}^p(T;\ell_2)$ for all $p\in[1,\infty)$.*

*Proof* Let $f=(f_k)_{k\in\mathbb{N}}\in L^q(T;\ell_2)$, with $q:=\frac{p}{p-1}$, be such that

$$_{\mathbb{L}^q(T;\ell_2)}\langle f,g\rangle_{\mathbb{L}^p(T;\ell_2)}=\mathbb{E}\int_0^T\int_E\sum_{k=1}^\infty f_kg_kd\mu ds=0\ \ \forall g\in\mathcal{E}.$$

Now let $\sigma\le\tau$ be two stopping times and $k\in\mathbb{N}$. Define $g\in\mathbb{L}^p(T;\ell_2)$ by $g=(g_k\delta_{ik})_{i\in\mathbb{N}}$, where

$$g_k:=g_k^kI_{(\sigma,\tau]}$$

and $g_k^k\in C_b(E)\cap L^1(\mu)$. Then $g\in\mathcal{E}$, hence

$$\begin{aligned}0&={}_{\mathbb{L}^q(T;\ell_2)}\langle f,g\rangle_{\mathbb{L}^p(T;\ell_2)}\\&=\mathbb{E}\int_0^T\int_E f_kg_k^kd\mu\ I_{(\sigma,\tau]}(t)dt,\end{aligned}$$

which implies that

$$\int_E f_kg_k^kd\mu=0\ \ dt\otimes\mathbb{P}-a.s.,$$

since all sets of the type $(\sigma,\tau]$ generate the $\sigma$-algebra $\mathscr{P}$ and since $f_k$ is $\mathscr{P}$-measurable. Therefore, since $C_b(E)\cap L^1(\mu)$ is dense in $L^p(\mu)$,

$$f_k=0\ \text{ in }\ L^q(\mu)\ \ dt\otimes\mathbb{P}-a.s.,\quad\text{for all }\ k\in\mathbb{N}.$$

Now the assertion follows by the Hahn-Banach theorem. □

**Remark 7.1** Let $\mathcal{S}$ be the set of all functions $f\in L^\infty([0,T]\otimes\Omega;L^\infty(\mu)\cap L^1(\mu))$ such that there exist $l\in\mathbb{N}$ and bounded stopping times $\tau_0'\le\tau_1'\le\cdots\le\tau_l'\le T$ such that $f=\sum_{i=1}^l f^i1_{(\tau_{i-1}',\tau_i']}$, where $f^i\in C_b(E)\cap L^1(\mu)$, $1\le i\le l$. Similarly to Claim 7.1, one can prove that $\mathcal{S}$ is dense in $\mathbb{L}^p(T)$ for all $p\in[1,\infty)$.

Define $\mathbf{M}:\mathcal{E}\longmapsto\bigcap_{p\ge1}L^p(\Omega;C([0,T];L^p(\mu)))$ as follows:

$$\begin{aligned}\mathbf{M}(g)(t)=\int_0^t gdW(s)\ &:=\sum_{k=1}^\infty\int_0^t g_kdW_k(s)\\&=\sum_{i,k=1}^j g_k^i\big(W_k(t\wedge\tau_i)-W_k(t\wedge\tau_{i-1})\big),\ t\in[0,T],\ g\in\mathcal{E}.\end{aligned}\tag{7.6}$$

Let us note that the right hand-side of (7.6) is $\mathbb{P}$-a.s. for every $t\in[0,T]$ a continuous $\mu$-version of $\mathbf{M}(g)(t)\in L^p(E,\mu)$, which for every $x\in E$ is a continuous real-valued martingale and is equal to

$$\sum_{k=1}^\infty\int_0^t g_k(s,x)dW_k(s),\ x\in E,\ t\in[0,T].\tag{7.7}$$

**Claim 7.2** *Let $p\in[2,\infty)$. Then $\mathbf{M}$ extends to a linear continuous map $\overline{\mathbf{M}}$ from $\mathbb{L}^p(T;\ell_2)$ to $L^p(\Omega;C([0,T];L^p(\mu)))$, such that $\overline{\mathbf{M}}(g)$ is a continuous martingale in $L^p(\mu)$ for all $g\in\mathbb{L}^p(T;\ell_2)$.*

*Proof* We have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\int_E \Big|\int_0^t g\,dW(s)\Big|^p d\mu\Big]\\
&= \mathbb{E}\Big[\sup_{t\in[0,T]}\int_E \Big|\sum_{k=1}^{\infty}\int_0^t g_k(s,x)dW_k(s)\Big|^p d\mu\Big]\\
&\le \int_E \Big[\mathbb{E}\sup_{t\in[0,T]}\Big|\sum_{k=1}^{\infty}\int_0^t g_k(s,x)dW_k(s)\Big|^p\Big] d\mu\\
&\le c_p\int_E \Big[\mathbb{E}\Big\langle\sum_{k=1}^{\infty}\int_0^{\cdot} g_k(s,x)dW_k(s)\Big\rangle_T^{\frac{p}{2}}\Big] d\mu\\
&= c_p\int_E \mathbb{E}\Big(\sum_{k=1}^{\infty}\int_0^{T} g_k^2(s,x)ds\Big)^{\frac{p}{2}} d\mu\\
&= c_p\mathbb{E}\Bigg[\int_E \Big(\int_0^{T} |g(s,x)|_{\ell_2}^2 ds\Big)^{\frac{p}{2}} d\mu\Bigg]^{\frac{2}{p}\cdot\frac{p}{2}}\\
&\le c_p\mathbb{E}\Bigg[\int_0^T \Big(\int_E |g(s,x)|_{\ell_2}^p d\mu\Big)^{\frac{2}{p}} ds\Bigg]^{\frac{p}{2}}\\
&\le c_p T^{\frac{p}{2}-1}\mathbb{E}\int_0^T \big|g(s,\cdot)\big|_{L^p(\mu;\ell_2)}^p ds, \qquad (7.8)
\end{aligned}
$$

where we have used the BDG inequality applied to the real-valued martingale in (7.7) in the third step, the assumption that $p \ge 2$ and Minkowski's inequality in the sixth step and Hölder's inequality in the last step. Hence the first part of the assertion follows.

To prove the second let $g \in \mathbb{L}^p(T;\ell_2)$. It suffices to prove that for all $f \in L^q(\mu)$ with $q := \frac{p}{p-1}$,

$$\int_E f\ \overline{\mathbf{M}}(g)(t)d\mu,\ t \in [0,T],$$

is a real-valued martingale (see e.g. [29, Remark 2.2.5]). But since for some $g_n \in \mathcal{E}$, $n \in \mathbb{N}$, we have $\forall\ t \in [0,T]$ that

$$\mathbf{M}(g_n)(t)\ \ \overrightarrow{n\to\infty}\ \ \overline{\mathbf{M}}(g)(t)\ \text{ in }\ L^p(\Omega;L^p(\mu)),$$

it follows that

$$\int_E f\ \mathbf{M}(g_n)(t)d\mu\ \ \overrightarrow{n\to\infty}\ \ \int_E f\ \overline{\mathbf{M}}(g)(t)d\mu\ \text{ in }\ L^1(\Omega).$$

So, we may assume that $g \in \mathcal{E}$. But in this case by (7.6) it follows immediately that $\int_E f\ \mathbf{M}(g)(t)d\mu$, $t \in [0,T]$, is a real-valued martingale. □

Below we define for $g \in \mathbb{L}^p(T;\ell_2)$, $p \in [2,\infty)$,

$$\int_0^t g(s)dW(s) := \overline{\mathbf{M}}(g)(t),\quad t \in [0,T],$$

where $\mathbf{M}$ is as in Claim 7.2.

Now we fix $p \in [2,\infty)$ and consider the following process

$$u : \Omega \times [0,T] \to L^p(\mu),$$

defined by

$$u(t) := u(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s), \tag{7.9}$$

where $u(0) \in L^p(\Omega, \mathcal{F}_0; L^p(\mu))$, $f \in \mathbb{L}^p(T)$ and $g \in \mathbb{L}^p(T;\ell_2)$.

**Theorem 7.1** ***"Itô-formula in expectation"*** *Let $p \in (2,\infty)$, $f \in \mathbb{L}^p(T)$, $g \in \mathbb{L}^p(T;\ell_2)$. Let $u$ be as in (7.9). Then for all $t \in [0,T]$,*

$$\begin{aligned}\mathbb{E}|u(t,x)|_p^p = \mathbb{E}|u(0)|^p &+ \mathbb{E}\int_0^t \int_E p|u(s,x)|^{p-2}u(s,x)f(s,x)\mu(dx)ds \\ &+\frac{1}{2}p(p-1)\mathbb{E}\int_0^t \int_E |u(s,x)|^{p-2}|g(s,x)|_{\ell_2}^2\mu(dx)ds.\end{aligned} \tag{7.10}$$

**Remark 7.2** In the case $E = \mathbb{R}^d$, $\mu$ =Lebesgue measure, N. Krylov proved Itô's formula for the $L^p$-norm of a large class of $W^{1,p}$-valued stochastic processes in his fundamental paper [26]. In particular, Lemma 5.1 in that paper gives a pathwise Itô formula for processes $u$ as in (7.9), which immediately implies (7.10). The proof, however, uses a smoothing technique by convoluting the process $u$ in $x$ with Dirac-sequence of smooth functions, which is not available in our more general case, where $(E,\mathcal{B})$ is just a standard measurable space with a $\sigma$-finite measure $\mu$, without further structural assumptions that we wanted to avoid to cover applications e.g. to underlying spaces $E$ which are fractals. Fortunately, the above Itô formula in expectation is enough to prove all main results in this paper without any further assumptions. After the preparations above, its proof is quite simple.

We recall the following well-known result (see e.g. Theorem 21.7 in [11]):

**Lemma 7.3** *Let $p \in [1,\infty)$, $v_n, v \in L^p(\mu)$ such that $v_n \to v$ in $\mu$-measure as $n \to \infty$ and*

$$\lim_{n\to\infty} |v_n|_p = |v|_p.$$

*Then*

$$\lim_{n\to\infty} v_n = v \quad in \quad L^p(\mu).$$

**Proof of Theorem 7.1** By Claim 7.1 and Remark 7.1, we can find $f_n \in \mathcal{S}$, $n \in \mathbb{N}$, and $g_n \in \mathbb{L}^p(T;\ell_2)$, $n \in \mathbb{N}$, such that as $n \to \infty$

$$f_n \to f \text{ in } \mathbb{L}^p(T), \tag{7.11}$$

and

$$g_n \to g \text{ in } \mathbb{L}^p(T;\ell_2). \tag{7.12}$$

For $n \in \mathbb{N}$, define

$$u_n(t) := u(0) + \int_0^t f_n(s)ds + \int_0^t g_n(s)dW(s). \tag{7.13}$$

By (7.9), (7.11), (7.12) and Claim 7.2, it follows that as $n \to \infty$,

$$\begin{aligned} \int_0^{\cdot} f_n(s)ds &\to \int_0^{\cdot} f(s)ds, \\ \int_0^{\cdot} g_n(s)dW(s) &\to \int_0^{\cdot} g(s)dW(s), \\ u_n &\to u, \end{aligned} \tag{7.14}$$

in $L^p(\Omega; C([0,T]; L^p(\mu)))$.

Applying the Itô formula to the real-valued semi-martingale $|u_n(t,x)|_p^p$ for each $x \in E$, and integrating w.r.t. $x \in E$ and $\omega \in \Omega$, we obtain

$$\begin{aligned} \mathbb{E}\int_E |u_n(t,x)|^p \mu(dx) = \mathbb{E}|u(0)|^p &+ \mathbb{E}\int_E \int_0^t p|u_n(s,x)|^{p-2} u_n(s,x) \cdot f_n(s,x) ds \mu(dx) \\ &+ \frac{1}{2} p(p-1)\mathbb{E}\int_E \int_0^t |u_n(s,x)|^{p-2} \cdot |g_n(s,x)|_{\ell_2}^2 ds \mu(dx). \end{aligned} \tag{7.15}$$

Note that by Lemma 7.3 and (7.14)

$$\begin{aligned} |u_n(s)|^{p-2} u_n(s) &\to |u(s)|^{p-2} u(s) \ \text{ in } \ L^{\frac{p}{p-1}}(\mu), \\ |u_n(s)|^{p-2} &\to |u(s)|^{p-2} \ \text{ in } \ L^{\frac{p}{p-2}}(\mu), \end{aligned}$$

as $n \to \infty$. Hence by (7.11) and (7.12) we may pass to the limit $n \to \infty$ in (7.15) to get (7.10). ☐

## 7.3 Justification for applying Itô's formula to the processes in (3.5) and (5.4)

To apply the Itô formula from [29, Theorem 4.2.5] we have to consider the equations (3.5) and (5.4) in an appropriate Gelfand triple. We need the following two lemmas whose assertions are special cases of [35, Proposition 3.1].

**Lemma 7.4** *$\mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)$ is dense both in $\mathscr{F}_e$ and $L^{\frac{m+1}{m}}(\mu)$.*

Define

$$V := \{u \in L^{m+1}(\mu) | \exists\, C \in (0,\infty) \text{ such that } |\mu(uv)| \le C\|v\|_{\mathscr{F}_e},\ \forall\, v \in \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)\}.$$

By Lemma 7.4, $V$ is a subspace of $\mathscr{F}_e^*$ and can be symbolically written as $V = L^{m+1}(\mu) \cap \mathscr{F}_e^*$. We note that $V$ is reflexive, since $L^{m+1}(\mu)$ and $\mathscr{F}_e^*$ is reflexive, hence so is $L^{m+1}(\mu) \times \mathscr{F}_e^*$. But

$$V \ni u \mapsto (u, \mu(u\ \cdot)) \in L^{m+1} \times \mathscr{F}_e^*$$

is a homeomorphic isomorphism, mapping $V$ onto a closed subspace of $L^{m+1} \times \mathscr{F}_e^*$, which is reflexive.

**Lemma 7.5** *(i) $V$ is dense both in $\mathscr{F}_e^*$ and $L^{m+1}(\mu)$.*
*(ii) For the map $L := \overline{L} : \mathscr{F}_e \to \mathscr{F}_e^*$ defined in Lemma 2.3 we have for all $v \in \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)$, $u \in V$,*

$$\langle Lv, u\rangle_{\mathscr{F}_e^*} = -\mu(vu). \tag{7.16}$$

Now we set $H := \mathscr{F}_e^*$ and consider the Gelfand triple

$$V \subset H \subset V^*.$$

Consider the operator

$$L : \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu) \to \mathscr{F}_e^* \subset V^*,$$

as $V^*$-valued, i.e., $Lv = -\mu(v\ \cdot) \in V^*$. Then by Lemma 7.5, $L$ is continuous w.r..t the norm $|\cdot|_{\frac{m+1}{m}}$ on $\mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)$, hence by Lemma 7.4 has a unique continuous linear extension

$$\widetilde{L} : L^{\frac{m+1}{m}}(\mu) \to V^*,$$

such that

$$_{V^*}\langle \widetilde{L}v, u\rangle_V = \mu(vu), \quad \forall v \in L^{\frac{m+1}{m}}(\mu), u \in V. \tag{7.17}$$

Now we consider equation (5.4) in the large space $V^*$. Then, since $\eta \in L^{\frac{m+1}{m}}([0,T] \times \Omega \times E)$,

$$L \int_0^t \eta(s)ds = \widetilde{L} \int_0^t \eta(s)ds = \int_0^t \widetilde{L}\eta(s)ds \in V^*,$$

so

$$X(t) = x + \int_0^t \widetilde{L}\eta(s)ds + \int_0^t B(s, X(s))dW(s), t \in [0,T],$$

and the Itô formula from [29, Theorem 4.2.5] applies. Likewise, it applies to the process in (3.5), since by the same argument we get for (3.5)

$$X(t) = x + \int_0^t \widetilde{L}(\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s))ds + \int_0^t B(s, X_\lambda(s))dW(s), t \in [0,T].$$

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