The invariance principle for nonlinear Fokker–Planck equations

Viorel Barbu* **Michael Röckner†‡**

Abstract

One studies here, via the La Salle invariance principle for nonlinear semigroups in Banach spaces, the properties of the ω-limit set $\omega(u_0)$ corresponding to the orbit $\gamma(u_0) = \{u(t,u_0);\ t \geq 0\}$, where $u = u(t,u_0)$ is the solution to the nonlinear Fokker–Planck equation

$$\begin{aligned} &u_t - \Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \ \text{ in } (0,\infty)\times\mathbb{R}^d,\\ &u(0,x) = u_0(x),\quad x\in\mathbb{R}^d,\quad u_0\in L^1(\mathbb{R}^d),\ d\geq 3.\end{aligned}$$

Here, $\beta\in C^1(\mathbb{R})$ and $\beta'(r)>0$, $\forall r\neq 0$. Moreover, β is a sublinear function, possibly degenerate in the origin, $b\in C^1(\mathbb{R})$, b bounded, $b\geq b_0\in(0,\infty)$, D is bounded such that $D=-\nabla\Phi$, where $\Phi\in C(\mathbb{R}^d)$ is such that $\Phi\geq 1$, $\Phi(x)\to\infty$ as $|x|\to\infty$ and satisfies a condition of the form $\Delta\Phi-\alpha|\nabla\Phi|^2\leq 0$, a.e. on $\mathbb{R}^d$. The main conclusion is that the equation has an equilibrium state and the set $\omega(u_0)$ is a nonempty, compact subset of $L^1(\mathbb{R}^d)$ while, for each $t\geq 0$, the operator $u_0\to u(t,u_0)$ is an isometry on $\omega(u_0)$. In the nondegenerate case $0<\gamma_0\leq\beta'\leq\gamma_1$ studied in [2], it follows that $\lim\limits_{t\to\infty} S(t)u_0 = u_\infty$ in $L^1(\mathbb{R}^d)$, where u_∞ is the unique bounded stationary solution to the equation.

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*Octav Mayer Institute of Mathematics of the Romanian Academy and Al.I. Cuza University, Iaşi, Romania

†Fakultät für Mathematik, Universität Bielefeld, D-33501, Bielefeld, Germany

‡Academy of Mathematics and Systems Science, CAS, Beijing, China

1 Introduction

Consider here the nonlinear Fokker–Planck equation (NFPE)

$$\begin{aligned} & u_t - \Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \ \text{ in } (0,\infty)\times\mathbb{R}^d, \\ & u(0,x) = u_0(x), \quad x\in\mathbb{R}^d,\ d\ge 3,\ u_0\in L^1(\mathbb{R}^d), \end{aligned} \tag{1.1}$$

under the following assumptions on $\beta : \mathbb{R}\to\mathbb{R}$ and $D : \mathbb{R}^d\to\mathbb{R}^d$

(i) $\beta\in C^1(\mathbb{R}),\ \beta'(r)>0,\ \forall\, r\in\mathbb{R}\setminus\{0\}$, $\beta(0)=0$, and

$$\mu_1\min\{|r|^\nu,|r|\}\le|\beta(r)|\le\mu_2|r|,\ \forall\, r\in\mathbb{R}, \tag{1.2}$$

for $\mu_1,\mu_2>0$ and $\nu>\frac{d-1}{d},\ d\ge 3$.

(ii) $D\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} D\in(L^1(\mathbb{R}^d)+L^\infty(\mathbb{R}^d))\cap(L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d))$ and $D=-\nabla\Phi$, where $\Phi\in C(\mathbb{R}^d)\cap W^{1,1}_{\rm loc}(\mathbb{R}^d)$, satisfies the conditions

$$\begin{gathered} \Phi(x)\ge 1,\ \forall\, x\in\mathbb{R}^d,\quad \lim_{|x|\to\infty}\Phi(x)=+\infty, \\ \Phi^{-m}\in L^1(\mathbb{R}^d) \text{ for some } m\ge 2, \end{gathered} \tag{1.3}$$

$$\mu_2\Delta\Phi(x)-b_0|\nabla\Phi(x)|^2\le 0,\ \text{a.e. } x\in\mathbb{R}^d\setminus\{0\}. \tag{1.4}$$

(iii) $b\in C^1(\mathbb{R})$, b bounded, $b(r)\ge b_0>0$ for all $r\in[0,\infty)$.

We note that (1.4) implies that $(\operatorname{div} D)^-\in L^\infty(\mathbb{R}^d)$. It should also be noted that assumption (i) does not preclude the degeneracy of the nonlinear diffusion function β in the origin. For instance, any continuous, increasing function $\beta:\mathbb{R}\to\mathbb{R}$ of the form

$$\beta(r)=\begin{cases} \mu_1 r|r|^{d-1} & \text{for } |r|\le r_0, \\ \mu_2 h(r) & \text{for } |r|>r_0, \end{cases}$$

where $r_0>0,\ \mu_1,\mu_2>0,\ |h(r)|\le L|r|,\ \forall\, r\in\mathbb{R},\ L>0$, satisfies (1.2) for a suitable γ_1. As regards Hypothesis (ii), an example of such a function Φ is

$$\Phi(x)=\begin{cases} |x|^2\log|x|+\mu & \text{for } |x|\le\delta=\exp\left(-\frac{d+2}{2d}\right), \\ \varphi(|x|)+\eta|x|+\mu & \text{for } |x|>\delta, \end{cases} \tag{1.5}$$

where the constants $\mu, \eta > 0$ are sufficiently large, and $\varphi : [\delta, \infty) \to \mathbb{R}$ is given by

$$\varphi(r) = \delta^2 \log \delta - \eta\delta - \int_\delta^r \frac{ds}{\frac{s}{\delta}\left(\frac{d}{2\delta+\eta d} + \frac{\delta}{\gamma_1(d-2)}\left(\left(\frac{r}{\delta}\right)^{d-2} - 1\right)\right)}, \quad \forall r \geq \delta.$$

(See [2], Appendix.)

Such an equation arises in statistical physics (see, e.g., [8], [9], [11], [14]) and is relevant in nonequilibrium statistical mechanics where it describes the dynamics of particle densities $\rho = \rho(t, x)$ in disordered media subject to anomalous diffusion (see, e.g., [8], [9], [11], [14]). The condition $D = -\nabla\Phi$ in Hypothesis (ii) means that the force field $D = D(x)$ *is conservative* and this property is related to the reversibility of stationary states (see [13]). The classical Einstein and Smoluchowski equations $u_t - \Delta u + \mathrm{div}(Du) = 0$ are associated with the Boltzmann-Gibbs distributions, while NFPE (1.1) corresponds to a generalized entropy (for instance, the Tsallis entropy in the case where $\beta(r) \equiv ar^q$). This equation can be derived also from the standard master equation associated with the particle transport model [9]. It should be emphasized that in all physical models governed by NFPE (1.1), $u(t, x)$ is either the density of particles at time t or a probability density associated with the corresponding McKean-Vlasov equation

$$\begin{aligned} &dX(t) = b(u(t, X(t)))D(X(t))dt + \frac{1}{\sqrt{2}}\left(\frac{\beta(u(t, X(t)))}{u(t, X(t))}\right)^{\frac{1}{2}} dW(t), \\ &X(0) = X_0. \end{aligned} \tag{1.6}$$

More exactly, if $u : [0, \infty) \to L^1(\mathbb{R}^d)$ is a distributional solution to (1.1), which is weakly t-continuous, then there is a weak solution X to (1.6) on some probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ such that $u(t, x)dx = \mathbb{P} \circ (X(t))^{-1}(dx)$, $u_0(x)dx = \mathbb{P} \circ (X_0)^{-1}(dx)$. (See [3], [4].) In this sense, all our results in this paper have a probabilistic interpretation and, in particular, we thus prove (see Theorem 3.2) the existence of an invariant measure for (1.6) for the class of degenerate cases, where β is as in Hypothesis (i).

An efficient functional way to treat NFPE (1.1) in $L^1(\mathbb{R}^d)$, which is the natural state space for the well-posedness of this equation, is to represent it as an infinite dimensional Cauchy problem in $L^1(\mathbb{R}^d)$.

In fact, it was shown in [3] (see, Lemmas 3.1, 3.2 therein) that the operator $A_0 : D(A_0) \subset L^1 \to L^1$ defined by

$$\begin{aligned} A_0u &= -\Delta\beta(u)+\mathrm{div}(Db(u)u),\ \forall\, u\in D(A_0),\\ D(A_0) &= \{u\in L^1;\ -\Delta\beta(u)+\mathrm{div}(Db(u)u)\in L^1\}, \end{aligned} \tag{1.7}$$

satisfies

$$R(I+\lambda A_0)=L^1,\ \forall\lambda>0, \tag{1.8}$$

and, for each $\lambda>0$, there is $J_\lambda : L^1\to D(A_0)$ such that

$$J_\lambda(u)\in(I+\lambda A_0)^{-1}\{u\},\ \forall u\in L^1(\mathbb{R}^d), \tag{1.9}$$

$$|J_\lambda(u)-J_\lambda(v)|_1\le|u-v|_1,\ \forall\, u,v\in L^1,\ \lambda>0, \tag{1.10}$$

where $L^1=L^1(\mathbb{R}^d)$ with its norm denoted by $|\cdot|_1$. We also set $L^1_{\rm loc}=L^1_{\rm loc}(\mathbb{R}^d)$ and

$$\mathcal{P}=\left\{u\in L^1;\ \int_{\mathbb{R}^d}u(x)dx=1,\ u\ge 0,\ \text{a.e. on } \mathbb{R}^d\right\}.$$

Then the operator $A : D(A)\subset L^1\to L^1$, defined by

$$\begin{aligned} Au &= A_0u,\ \forall\, u\in D(A),\\ D(A) &= J_\lambda(L^1), \end{aligned} \tag{1.11}$$

is m-accretive in L^1 (that is, satisfies (1.8)–(1.10)), $(I+\lambda A)^{-1}=J_\lambda$, and one has also

$$(I+\lambda A)^{-1}0=0,\ (I+\lambda A)^{-1}\mathcal{P}\subset\mathcal{P},\ \ \forall\lambda>0, \tag{1.12}$$

$$(I+\lambda A)^{-1}(L^1\cap L^\infty)\subset L^1\cap L^\infty,\ \ \forall\,\lambda\in(0,\lambda_0), \tag{1.13}$$

for some $\lambda_0>0$.

We denote by $C=\overline{D(A)}$ the closure of $D(A)$ in L^1. We note that, by [3, Theorem 2.2], if $\beta\in C^2(\mathbb{R})$, then $C=L^1$. Then (see, e.g, [1], p. 139), by the Crandall & Liggett generation theorem for each $u_0\in C$, the Cauchy problem

$$\begin{aligned} &\frac{du}{dt}+Au=0,\ \ t\ge 0,\\ &u(0)=u_0, \end{aligned} \tag{1.14}$$

has a unique mild solution $u\in C([0,\infty);L^1)$ and $S(t)u_0=u(t),\ t\ge 0$, is a semigroup of nonlinear contractions in L^1. In addition (see [3, Theorem 2.2]), it leaves $\mathcal{P}\cap C$ invariant for all $t\ge 0$, that is,

$$|S(t)u_0 - S(t)v_0|_1 \le |u_0 - v_0|_1, \quad \forall u_0, v_0 \in C,\ t \ge 0, \tag{1.15}$$

$$S(t)C \subset C,\ S(t)(\mathcal{P} \cap C) \subset \mathcal{P} \cap C, \quad \forall\, t \ge 0, \tag{1.16}$$

$$S(t)(C \cap L^\infty) \subset L^\infty \cap L^1, \quad \forall\, t > 0, \tag{1.17}$$

$$S(t+s) = S(t)S(s), \quad \forall\, t, s \ge 0. \tag{1.18}$$

$S(t)$ is called the contraction semigroup in L^1 generated by A on C.

We have

$$S(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\ A\right)^{-n} u_0 \text{ in } L^1,\ \forall u_0 \in C, \tag{1.19}$$

uniformly in t on compact intervals. This means that, for each $h > 0$ and $0 < T < \infty$,

$$S(t)u_0 = \lim_{h\to 0} u_h(t) \text{ in } L^1 \text{ uniformly on } [0,T],$$

where

$$\begin{aligned} &u_h(t) = u_h^{i+1}, \quad t \in [ih, (i+1)h),\ i = 0, 1, ..., N-1,\ N = \left[\tfrac{T}{h}\right], \\ &u_h^{i+1} + hAu_h^{i+1} = u_h^i,\ i = 0, 1, ..., N-1. \end{aligned} \tag{1.20}$$

We shall call $S : [0,\infty) \to \mathrm{Lip}_1(C)$ (= the space of all Lipschitz mappings on C with Lipschitz constant less than 1) the nonlinear semigroup (semiflow) corresponding to NFPE (1.1), (*the nonlinear Fokker–Planck semiflow*) and $u(t) = S(t)u_0$, $t \ge 0$, *the generalized* (or *mild*) *solution to NFPE* (1.1). We note that this generalized solution u is also a Schwartz distributional solution to (1.1) on $[0,\infty) \times \mathbb{R}^d$, but the uniqueness in the class of Schwartz distributional solutions requires some more regularity on β (see [5]).

As a matter of fact, since the solution $J_\lambda(f)$ of (1.9) in general might not be unique, it should be emphasized that equation (1.14) with A of the form (1.11) *is only one realization of a solution to the Fokker–Planck equation* (1.1) *as a Cauchy problem in the space L^1, which depends on the choice of the resolvent $\{J_\lambda\}_{\lambda>0}$.* More precisely, J_λ for $\lambda > 0$ is constructed as a limit of an approximation J_λ^ε as $\varepsilon \to 0$, and in general it depends on the choice of this approximation. In Section 2 below we shall explain which approximation we choose in this paper.

Now, assume that u_0 is a probability density. Equation (1.1) describes the evolution of an open system *far from equilibrium* and the transition of a state $u(t)$ to an equilibrium state, that is, the convergence of the solution $u(t)$ for $t \to \infty$ to a stationary probability density u_∞ is our objective here. To analyze this problem with the orbit $\gamma(u_0) = \{S(t)u_0,\ t \geq 0\}$ of $S(t)u_0$, where $u_0 \in C$, we associate the ω-limit set

$$\begin{aligned} \omega(u_0) &= \{u_\infty = \lim S(t_n)u_0 \text{ in } L^1 \text{ for some } \{t_n\} \to \infty\} \\ &= \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)u_0}. \end{aligned} \tag{1.21}$$

The properties and structure of the set $\omega(u_0) \subset C$ are important for the long time dynamics of the semiflow $S(t)$, $t \geq 0$, and play a central role in the theory of general dynamical systems in finite or infinite dimension (see, e.g., [6], [7], [10], [12]). In particular, if $\omega(u_0) \neq \emptyset$ and consists of one element u_∞ only, this means that

$$\lim_{t\to\infty} S(t)u_0 = u_\infty \text{ in } L^1.$$

In [2], it was proved that under additional assumptions on β and, more exactly, if β is not degenerate in the origin, that is,

$$0 < \gamma_0 \leq \beta'(r) \leq \gamma_1, \quad \forall\, r \in \mathbb{R}, \tag{1.22}$$

(which again implies that $C = L^1$), then, for each $u_0 \in \mathcal{P}$, such that

$$u_0 \ln(u_0) \in L^1(\mathbb{R}^d),\ \|u_0\| = \int_{\mathbb{R}^d} u_0(x)\Phi(x)dx < \infty, \tag{1.23}$$

one has $\omega(u_0) = \{u_\infty\}$, where u_∞ is an equilibrium solution to (1.1), and is the unique solution in $(L^1 \cap L^\infty)(\mathbb{R}^d)$ to the stationary equation

$$-\Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{1.24}$$

Moreover, u_∞ is an equilibrium solution, that is, it minimizes the free energy of the system. (See Remark 3.5 below.)

This result, which links the initial nonequilibrium state u_0 with the final equilibrium state u_∞, can be viewed as an H-theorem type for NFPE (1.1). It also has some deep implications for *the McKean–Vlasov stochastic differential equation* (1.6) associated with NFPE (1.1).

The situation is different in the degenerate case considered here, that is, where condition (1.22) is weakened to (i). As seen below (see Theorem 3.1), in this case $\omega(u_0)$ *is a nonempty*, compact subset of L^1 and, for every fix point of $S(t)$, $t > 0$, it is contained in some sphere centered at this fix point. This means that there is a compact set $\omega(u_0)$ of probability densities which attracts the trajectory which starts from a nonequilibrium state u_0. This behaviour is specific to open systems far from thermodynamic equilibrium ([8], [9]). It should be mentioned that Hypothesis (ii) part (1.3) excludes the special case of the porous media equation

$$u_t - \Delta\beta(u) = 0, \quad t \geq 0, \ x \in \mathbb{R}^d.$$

Notations. We denote by L^p, $1 \leq p \leq \infty$, the space of Lebesgue p-integrable functions on $\mathbb{R}^d$ and by $L^p_{\rm loc}$ the space $L^p_{\rm loc}(\mathbb{R}^d)$. The norm in L^p is denoted by $|\cdot|_p$ and the scalar product in L^2 is denoted by $\langle\cdot,\cdot\rangle_2$. Let $C^k(\mathbb{R}^\ell)$, $k = 1, 2$, $\ell \geq 1$, denote the space of k-differentiable functions on $\mathbb{R}^d$ and $C_b(\mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d$. Let $W^{k,p}(\mathbb{R}^d)$, $k = 1$, $1 \leq p \leq \infty$, denote the classical Sobolev spaces on $\mathbb{R}^d$ and $\Delta, \nabla, {\rm div}$ the standard differential operators on $\mathbb{R}^d$ taken in the sense of Schwartz distributions, i.e. on $\mathcal{D}'(\mathbb{R}^d)$. We set $H^k = W^{k,2}(\mathbb{R}^d)$ and denote by H^{-k} the dual space of H^k. The norm of $\mathbb{R}^d$ will be denoted by $|\cdot|$. We shall also use the notation

$$\|u\| = \int_{\mathbb{R}^d} \Phi(x)|u(x)|dx, \quad \forall\, u \in L^1, \tag{1.25}$$

and denote by $\mathcal{M}$ the subspace of L^1 with the norm (1.25) finite. For each $\eta > 0$, we set

$$\mathcal{M}_\eta = \{u \in \mathcal{M};\ \|u\| \leq \eta\}. \tag{1.26}$$

We also set

$$\mathcal{M}_+ = \{u \in \mathcal{M};\ u \geq 0, \ \text{a.e. in } \mathbb{R}^d\}. \tag{1.27}$$

Furthermore, $\mathcal{P}$ denotes the set of all probability densities on $\mathbb{R}^d$.

2 Construction of a solution semigroup to (1.1) with stationary point

Though, as explained in the introduction, the existence of a solution semigroup to (1.1) follows from [3, Theorem 2.2], in this section we shall present a construction of a solution semigroup $S(t)$, $t \geq 0$, for which we can prove that it has a stationary point, i.e., there exists $a \in L^1$ such that $S(t)a = a$, $\forall t \geq 0$. So, let us fix $M \in [1, \infty]$, $\varepsilon \in (0, 1]$ and assume that Hypotheses (i), (ii), (iii) hold. Consider the following approximating operator on L^1

$$(A_0)_{\varepsilon,M}u = -\Delta\beta_{\varepsilon,M}(u) + \mathrm{div}(Db(u)u), \tag{2.1}$$

$$D((A_0)_{\varepsilon,M}) = \{u \in L^1;\ -\Delta\beta_{\varepsilon,M}(u) + \mathrm{div}(Db(u)u) \in L^1\}, \tag{2.2}$$

where

$$\beta_{\varepsilon,M}(r) := \begin{cases} \beta(r) + \varepsilon r, & \text{if } |r| \le M, \\ \beta(M) + \beta'(M)(r - M) + \varepsilon r, & \text{if } r > M, \\ \beta(-M) + \beta'(M)(r + M) + \varepsilon r, & \text{if } r < -M, \end{cases} \tag{2.3}$$

so that

$$\beta_\varepsilon := \beta_{\varepsilon,\infty} = \beta + \varepsilon I. \tag{2.4}$$

Then, by [3, Lemmas 3.1 and 3.2] for every $\lambda > 0$ there is a right inverse $J_\lambda^{\varepsilon,M} : L^1 \to D((A_0)_{\varepsilon,M})$ of the operator $I + \lambda(A_0)_{\varepsilon,M} : D((A_0)_{\varepsilon,M}) \to L^1$, i.e.

$$R(I + \lambda(A_0)_{\varepsilon,M}) = L^1 \ \text{ and } \ (I + \lambda(A_0)_{\varepsilon,M})J_\lambda^{\varepsilon,M} = I$$

such that $A_{\varepsilon,M} := (A_0)_{\varepsilon,M \upharpoonright J_\lambda^{\varepsilon,M}(L^1)}$ is m-accretive on L^1 and $J_\lambda^{\varepsilon,M}(L^1)$ is independent of $\lambda > 0$ and $J_\lambda^{\varepsilon,M}(L^1 \cap L^\infty) \subset L^1 \cap L^\infty$. Applying the Crandall & Liggett generation theorem (see above) to the operator $A_{\varepsilon,M}$, we obtain the corresponding mild solution given by a nonlinear semigroup $S_{\varepsilon,M}(t)$, $t \ge 0$, on L^1 (see [3, Theorem 2.2] and note that $D(A_{\varepsilon,M}) := J_\lambda^{\varepsilon,M}(L^1)$ is dense in L^1, since $\varepsilon > 0$).

Furthermore, by Theorem 2.1 in [5], if $u_0 \in L^1 \cap L^\infty$, then $S_{\varepsilon,M}(t)u_0$, $t \in [0,T]$, is the unique narrowly continuous (in $t \ge 0$) weak solution in $(L^1 \cap L^\infty)((0,T) \times \mathbb{R}^d)$ of $(1.1)_{\varepsilon,M}$, where $(1.1)_{\varepsilon,M}$ denotes the NFPE (1.1) with $A_{\varepsilon,M}$ replacing A. In addition, for $M < \infty$, by Theorem 6.1 in [2] there exists $a_{\varepsilon,M} \in \mathcal{P} \cap \mathcal{M} \cap L^\infty(\mathbb{R}^d)$ which is given by

$$a_{\varepsilon,M}(x) = g_{\varepsilon,M}^{-1}(\mu_{\varepsilon,M} - \Phi(x)),\ x \in \mathbb{R}^d, \tag{2.5}$$

where

$$g_{\varepsilon,M}(r) := \int_1^r \frac{\beta'_{\varepsilon,M}(s)}{sb(s)}\, ds, \quad r > 0, \tag{2.6}$$

and $\mu_{\varepsilon,M} \in \mathbb{R}$ is the unique number such that

$$\int_{\mathbb{R}^d} g_{\varepsilon,M}^{-1}(\mu_{\varepsilon,M} - \Phi(x))dx = 1, \tag{2.7}$$

such that $S_{\varepsilon,M}(t)a_{\varepsilon,M} = a_{\varepsilon,M}$ for all $t \geq 0$, $\lim\limits_{t\to\infty} S_{\varepsilon,M}(t)\widetilde{u}_0 = a_{\varepsilon,M}$ for all $\widetilde{u}_0 \in \mathcal{P} \cap \mathcal{M}$ with $\widetilde{u}_0 \ln \widetilde{u}_0 \in L^1$ and (see [2, Corollary 6.3])

$$|a_{\varepsilon,M}|_\infty \leq \max\left(1, e^{\frac{|b|_\infty}{\varepsilon}\ (\mu_{\varepsilon,M}-1)}\right). \tag{2.8}$$

Furthermore, by [2, Theorem 6.4],

$$-\Delta\beta_\varepsilon(a_{\varepsilon,M} + \operatorname{div}(Db(a_{\varepsilon,M})a_{\varepsilon,M}) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{2.9}$$

We note that, obviously, $g_{\varepsilon,M} : [1,\infty) \to [0,\infty)$ and $g_{\varepsilon,M} : (0,1) \to (-\infty, 0)$. Therefore, by (2.7),

$$1 \geq \int_{\{\Phi \leq \mu_{\varepsilon,M}\}} 1\, dx,$$

hence

$$\sup\{\mu_{\varepsilon,M} \mid M \in [1,\infty),\ \varepsilon \in (0,1]\} < \infty. \tag{2.10}$$

Defining

$$\mu_\varepsilon := \sup_{M\in[1,\infty)} \mu_{\varepsilon,M}$$

we deduce from (2.8) that, for all $M \in [1,\infty)$,

$$|a_{\varepsilon,M}|_\infty \leq \max\left(1, e^{\frac{|b|_\infty}{\varepsilon}\ (\mu_\varepsilon-1)}\right) =: M_0. \tag{2.11}$$

It follows by the weak uniqueness result from [5] mentioned above that

$$a_\varepsilon := a_{\varepsilon,M_0} = a_{\varepsilon,M}, \quad \forall M \in [M_0, \infty). \tag{2.12}$$

Furthermore, for $M = \infty$, we obtain that for $S_\varepsilon(t) := S_{\varepsilon,\infty}(t)$, $t \geq 0$, we have

$$S_\varepsilon(t)a_\varepsilon = a_\varepsilon \ \text{ for all } t \geq 0. \tag{2.13}$$

Lemma 2.1. *Let* $\varepsilon \in (0,1]$ *and* $(A_0)_\varepsilon := (A_0)_{\varepsilon,\infty}$, $J^\varepsilon_\lambda := J^{\varepsilon,\infty}_\lambda$ *and*

$$A_\varepsilon := (A_0)_{\varepsilon \upharpoonright J^\varepsilon_\lambda(L^1)}, \quad D(A_\varepsilon) := J^\varepsilon_\lambda(L^1),$$

which (*as seen above*) *is m-accretive on* L^1. *Then:*

(i) *Let* $f \in L^1 \cap L^\infty$. *Then there exists* $\lambda_0 \in (0,\infty)$ *such that, for every* $\lambda \in (0, \lambda_0]$, $J^\varepsilon_\lambda(f)$ *is the unique solution* $u_\lambda \in L^1 \cap L^\infty$ *of the equation*

$$u_\lambda + \lambda(A_0)_\varepsilon u_\lambda = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{2.14}$$

(ii) $J^\varepsilon_\lambda a_\varepsilon = a_\varepsilon$, $\forall \lambda \in (0, \lambda_0]$ *and* a_ε *as above.*

(iii) *For every* $\lambda \in (0, \lambda_0]$, $\lim\limits_{\varepsilon\to 0} J^\varepsilon_\lambda f = J_\lambda f$ *in* L^1, *where* J_λ *is as in* (1.8), (1.9) *in the introduction.*

(iv) *Let* $K \subset \mathbb{R}^d$, K *compact,* $q \in \left(1, \frac{d}{d-1}\right)$ *and* $\lambda \in (0, \lambda_0]$. *Then there exists* $C \in (0, \infty)$ *only depending on* $K, q, \lambda, |D|_\infty$ *and* $|b|_\infty$ *such that, for all* $f \in L^1$,

$$\|\beta(J_\lambda f)\|_{L^q(K)} \le C|f|_1, \tag{2.15}$$

and, for all $\nu \in \mathbb{R}^d$,

$$\|\beta(J_\lambda f)^\nu\|_{L^q(K)} \le C \left(\|E^\nu\|_{M^{\frac{d}{d-2}}} + \|\nabla E^\nu\|_{M^{\frac{d}{d-1}}} \right) |f|_1 \xrightarrow[\nu\to 0]{} 0\,, \tag{2.16}$$

where for a function $v : \mathbb{R}^d \to \mathbb{R}$ *we set* $v^\nu(x) := v(x+\nu) - v(x)$, $x \in \mathbb{R}^d$, $E(x) = \omega_d |x|^{2-d}$, $x \in \mathbb{R}^d$, *with* $\omega_d =$ *the volume of the unit ball and* $\|\cdot\|_{M^p}$, $p > 1$, *the norm of the Marcinkievicz space* (*see* [3] *and the references therein*).

Proof. See the Appendix. □

Theorem 2.2. *Suppose* $a := \lim\limits_{\varepsilon\to\infty} a_\varepsilon$ *exists in* L^1. *Then* $J_\lambda(a) = a$, *for all* $\lambda \in (0, \lambda_0]$ *with* λ_0 *as in Lemma* 2.1 (i). *In particular,* $a \in D(A)$ *and* $S(t)a = a$, *for all* $t \ge 0$.

Proof. We have by Lemma 2.1 (ii), for $\lambda \in (0, \lambda_0]$,

$$|J_\lambda a - a|_1 \le |J_\lambda a - J^\varepsilon_\lambda a|_1 + |J^\varepsilon_\lambda a - J^\varepsilon_\lambda a_\varepsilon|_1 + |a_\varepsilon - a|_1.$$

Now, the assertion follows by Lemma 2.1 (iii), since J^ε_λ is a Lipschitz contraction for all $\varepsilon > 0$, $\lambda > 0$. The last part of the assertion now follows by (1.11) and (1.19). □

In Theorem 3.2 below, we shall show that under a mild condition in addition to Hypotheses (i)-(iii) we indeed have that $a := \lim\limits_{\varepsilon\to 0} a_\varepsilon$ exists in L^1.

3 The main results

Theorem 3.1 below is the main result of this work which will be proved in Section 4.

Theorem 3.1. *Assume that Hypotheses* (i)-(iii) *hold and let* $\eta > 0$ *be arbitrary but fixed. Let* $u_0 \in \mathcal{M}_\eta \cap \mathcal{P} \cap C$*. Then,* $\omega(u_0) \subset \mathcal{M}_\eta \cap \mathcal{P} \cap C$ *is nonempty and for all* $t \geq 0$*,* $\omega(u_0)$ *is compact in* L^1*,* $\omega(u_0) = \overline{\{S(t)u_0 \mid t \geq 0\}}^{L^1}$*, and invariant under* $S(t)$*. Moreover,* $S(t)$ *is, for every* $t \geq 0$*, an isometry on* $\omega(u_0)$ *and it is a homeomorphism from* $\omega(u_0)$ *onto itself for each* $t \geq 0$*. If* $a \in \mathcal{M}_\eta \cap \mathcal{P} \cap C$ *is such that*

$$S(t)a = a, \quad \forall\, t \geq 0, \tag{3.1}$$

then $\omega(u_0) \subset \{y \in \mathcal{M}_\eta \cap \mathcal{P} \cap C;\ |y - a|_1 = r\}$*, for some* $0 \leq r \leq |u_0 - a|_1$.

In particular, it follows by Theorem 3.1 that $S(t),\ t \geq 0$, is *a continuous group* on $\omega(u_0)$. Moreover, *the function* $t \to S(t)v$ *is equi-almost periodic* in L^1 for each $v \in \omega(u_0)$, i.e., for every $\varepsilon > 0$ there exists $\ell_\varepsilon > 0$ such that for every interval I in $\mathbb{R}$ of length ℓ_ε there exists $\tau \in I$ such that

$$|S(t+\tau)y - S(t)y|_1 \leq \varepsilon,\ \forall t \in \mathbb{R},\ y \in \omega(u_0).$$

Furthermore,

$$\mathrm{dist}_H(\omega(u_0), \omega(\bar{u}_0)) \leq |u_0 - \bar{u}_0|_1,\ \forall\, u_0, \bar{u}_0 \in \mathcal{M}_\eta \cap \mathcal{P} \cap C,$$

where dist_H is the Hausdorff distance.

Since the main interest is to get solutions $u(t) = S(t)u_0$ to (1.1) in the class $\mathcal{P}$, the initial data u_0 was taken in the same set $\mathcal{P}$ which, by virtue of (1.16), implies that $u(t) \in \mathcal{P}$, $\forall\, t \geq 0$. As seen below, the set $\mathcal{M}_\eta$ is still invariant under the semigroup $S(t)$, but we note that this choice for initial data (that is, $u_0 \in \mathcal{M}_\eta$) was taken for technical reasons which will become clear in the proof of Theorem 3.1.

In order to apply Theorem 3.1, a nontrivial problem is the existence of a fixed point a for the semigroup $S(t)$.

This problem is quite delicate and will be treated in Theorem 3.2 below. To this purpose, consider the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(r) = \int_1^r \frac{\beta'(s)}{sb(s)}\, ds, \quad \forall\, r > 0. \tag{3.2}$$

We have

Theorem 3.2. *Assume that, besides Hypotheses* (i), (ii), (iii)*, the following conditions hold:*

$$\lim_{r\to+\infty} g(r) = +\infty, \quad if \quad \nu \in \left(1-\tfrac{1}{d}, 1\right]; \tag{3.3}$$

$$\lim_{r\to 0} g(r) = -\infty, \quad if \quad \nu \in (1,\infty). \tag{3.4}$$

Let $a_\varepsilon \in \mathcal{P}\cap\mathcal{M}\cap L^\infty$ be as in the previous section (see (2.12), (2.13)). Then $a := \lim\limits_{\varepsilon\to 0} a_\varepsilon$ exists in L^1 and

$$a(x) = g^{-1}(\mu - \Phi(x)), \quad x \in \mathbb{R}^d,$$

where $\mu \in \mathbb{R}$ is the unique number such that

$$\int_{\mathbb{R}^d} a^{-1}(\mu - \Phi(x))dx = 1.$$

Furthermore,

$$S(t)a = a, \quad \forall t \geq 0. \tag{3.5}$$

Proof of Theorem 3.2. The last part of the assertion follows by Theorem 2.2. So, it remains to prove its first part. To this end, we first note that by (3.3), (3.4), Lemma A.2 implies that $g : (0,\infty) \to \mathbb{R}$ is bijective, since g is strictly increasing.

Furthermore, $g((0,1)) \subset (-\infty,0)$, $g([1,\infty)) \subset [0,\infty)$, $g \in C^1((0,\infty))$ and for its inverse

$$g^{-1} : \mathbb{R} \to (0,\infty),$$

we have $g^{-1} \in C^1(\mathbb{R})$, $(g^{-1})' > 0$, $g^{-1}([0,\infty)) \subset [1,\infty)$, $g^{-1}((-\infty,0)) \subset (0,1)$. Define (cf. (2.4) and (2.6))

$$g_\varepsilon(r) := \int_1^r \frac{\beta'_\varepsilon}{sb(s)}\, ds = \int_1^r \frac{\beta'(s)+\varepsilon}{sb(s)}\, ds, \quad r > 0. \tag{3.6}$$

Then g_ε and its inverse g_ε^{-1} have the same properties as g, g^{-1} above. Clearly,

$$g_\varepsilon \xrightarrow[\varepsilon\to 0]{} g \ \text{ locally uniformly on } \mathbb{R}, \tag{3.7}$$

hence

$$g_\varepsilon^{-1} \xrightarrow[\varepsilon\to 0]{} g^{-1} \ \text{ locally uniformly on } \mathbb{R}. \tag{3.8}$$

We recall that (see (2.5)-(2.7) and (2.12))

$$a_\varepsilon(x) = g_\varepsilon^{-1}(\mu_\varepsilon - \Phi(x)), \quad x \in \mathbb{R}^d, \tag{3.9}$$

where $\mu_\varepsilon \in \mathbb{R}$ is the unique number such that

$$\int_{\mathbb{R}^d} g_\varepsilon^{-1}(\mu_\varepsilon - \Phi(x))dx = 1. \tag{3.10}$$

Since $g_\varepsilon^{-1}([0,\infty)) \subset [1,\infty)$, by (3.10) we have

$$1 \geq \int_{\{\Phi \leq \mu_\varepsilon\}} 1 \, dx,$$

so

$$\sup_{\varepsilon \in (0,1]} \mu_\varepsilon < \infty.$$

Suppose there exist $\varepsilon_n \in [0,1]$, $n \in \mathbb{N}$, such that $\mu_{\varepsilon_n} \to -\infty$ as $n \to \infty$. Then, by (3.10),

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}^d} g_{\varepsilon_n}^{-1}(\mu_{\varepsilon_n} - \Phi(x))dx.$$

But, by Lemma A.2 applied to β_ε instead of β, it follows by (A.4), (A.8) and (1.3) that the limit on the r.h.s. is equal to zero. This contradiction implies that

$$\inf_{\varepsilon \in (0,1]} \mu_\varepsilon > -\infty,$$

so $\{\mu_\varepsilon \mid \varepsilon \in (0,1]\}$ is bounded, which implies that there exist $\varepsilon_n \in (0,1]$, $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ and $\mu := \lim\limits_{n\to\infty} \mu_{\varepsilon_n}$ exists in $\mathbb{R}$.

Furthermore, by (A.5) and (A.9) (again applied to β_ε) we have

$$\sup_n \int_{\mathbb{R}^d} g_{\varepsilon_n}^{-1}(\mu_{\varepsilon_n} - \Phi(x))\Phi(x)dx < \infty, \tag{3.11}$$

and by (3.8) we have

$$g_{\varepsilon_n}^{-1}(\mu_{\varepsilon_n} - \Phi) \xrightarrow[n\to\infty]{} g^{-1}(\mu - \Phi) \tag{3.12}$$

uniformly on compact subsets on $\mathbb{R}^d$. Hence the first part of the assertion follows by Lemma A.1 in the Appendix. □

Corollary 3.3. *Assume that Hypotheses* (i), (ii), (iii) *and* (3.3)–(3.4) *are satisfied. Then all the conclusions of Theorem* 3.1 *hold. In particular, for each* $u_0 \in \mathcal{M}_\eta \cap \mathcal{P} \cap C$, *the* ω*-limit set* $\omega(u_0)$ *lies in the set*

$$\{y \in \mathcal{M}_\eta \cap \mathcal{P} \cap C;\ |y - a|_1 = r\},$$

where $a \in \mathcal{M}_\eta \cap \mathcal{P} \cap C$ *satisfies* (3.5) *and* $0 \leq r \leq |u_0 - a_{\mu^*}|_1$.

Of course, if the set $\omega(y_0) \cap \{a \in C;\ S(t)a = a, \forall t > 0\}$ is nonempty, it follows by Corollary 3.3 that $\omega(y_0)$ contains only one element (the equilibrium state a) and so $\lim\limits_{t\to\infty} S(t)y_0 = a$ strongly in L^1.

It should be said that Theorem 3.1 and Corollary 3.3 provide a weak form of the H-theorem for NFPE (1.1) in the degenerate case we consider here. Roughly speaking, it amounts to saying that $S(t)u_0 \to \omega(u_0)$ in L^1 for $t \to \infty$.

Remark 3.4. One simple example of a function β which satisfies all conditions of Theorem 3.2 is the following:

$$\beta(r) = \int_0^r \theta(s)ds, \quad \forall\, r \geq 0,$$

where

$$\theta(s) = \begin{cases} -\dfrac{1}{\log s} & \text{for} \quad 0 < s \leq \delta < 1, \\ \zeta(s) & \text{for} \quad s > \delta, \end{cases}$$

and, for $r \in (-\infty, 0)$,

$$\beta(x) = -\beta(-r),$$

where $\delta > 0$, $\zeta \in C^1[\delta, +\infty)$, bounded, $\zeta \geq \zeta_0 \in (0, \infty)$, and ζ is such that $\theta \in C^1(0, \infty)$. Then it is elementary to check that β satisfies Hypothesis (i) with $\nu = 2$. Furthermore, obviously $g(r) = \text{const} - \log|\log r|$ for $r \in (0, 1]$. Hence (3.4) also holds and Theorem 3.2 applies.

Remark 3.5. Following [2], we can consider to associate with the dynamics $S(t)$ the Lyapunov function

$$V(u) = \int_{\mathbb{R}^d} \sigma(u(x))dx + \int_{\mathbb{R}^d} \Phi(x)u(x)dx = S[u] + E[u],\ \forall\, u \in \mathcal{M} \cap \mathcal{P}, \quad (3.13)$$

where

$$\sigma(r) = -\int_0^r d\tau \int_\tau^1 \frac{\beta'(s)}{sb(s)}\, ds, \quad \forall\, r \geq 0.$$

The function V is the free energy of the system, $S[u]$ is the generalized entropy of the system and $E[u]$ is the internal energy. (In the special case $\beta(r) \equiv r$, $S[u]$ is just the Boltzmann-Gibbs entropy.) Arguing as in [2], it follows that $t \to V(S(t)u_0)$ is nonincreasing on $[0, \infty)$ and, since V is lower-semicontinuous and positive on K, we have by Theorem 3.1 that

$$V(u_\infty) = \lim_{t\to\infty} V(S(t)u_0) = \inf V, \quad \forall u_\infty \in \omega(u_0).$$

Hence, u_∞ minimizes the free energy and, as mentioned earlier, this means that u_∞ is an equilibrium solution to (1.1). We note that, if $\nabla_x \beta(u(t,\cdot)) \in L^2$, then we have

$$\frac{d}{dt}\, S[u(t)] = -\int_{\mathbb{R}^d} J(u(t,x)) \cdot \nabla_x \beta(u(t,x))dx,\ t>0.$$

In particular, this implies that, if the current of probability J vanishes at the equilibrium solution u_∞, then the system is in a state of thermodynamic equilibrium with zero entropy production in u_∞, that is, it is reversible [13].

In the nondegenerate case (1.22), u_∞ is the unique equilibrium state of the system and coincides with the stationary solution to (1.1). It should be said, however, that in our case the equilibrium solution u_∞ might not be unique.

If $u_0 \in L^\infty \cap \mathcal{M} \cap \mathcal{P}$, then condition (1.2) in Hypothesis (i) can be relaxed to

$$\mu_1 r^\nu \le \beta(r), \quad \forall\, r>0, \tag{1.2$'$}$$

where $\mu_1 > 0$ and $\nu > \frac{d-1}{d},\ d \ge 3$.

Remark 3.6. If, in addition to (i)–(iii), one assumes that $\beta'(r) > \gamma > 0$, $\forall r > 0$, then we have that $C = L^1$, hence $\mathcal{M}_\eta \cap \mathcal{P} \cap C = \mathcal{M}_\eta \cap \mathcal{P}$ and so Theorem 3.1 and Corollary 3.3 are true for all $u_0 \in \mathcal{M}_\eta \cap \mathcal{P}$.

Remark 3.7. Theorems 3.1, 3.2 can be rephrased in terms of the McKean-Vlasov equation (1.6), that is, in terms of convergence of the time marginal laws of the corresponding probabilistically weak solution $X = X(t)$ for $t \to \infty$.

4 Proof of Theorem 3.1

We shall prove Theorem 3.1 in three steps indicated by lemmas which follow. Let $\eta > 0$ and let $K = \mathcal{M}_\eta \cap \mathcal{P} \cap C$. Clearly, K is a closed and bounded set of L^1.

Lemma 4.1. *We have*

$$\|(I+\lambda A)^{-1} y\| \le \|y\|, \quad \forall\, y \in \mathcal{M} \cap \mathcal{P},\ \lambda > 0, \tag{4.1}$$

and

$$(I+\lambda A)^{-1}(K) \subset \mathcal{M}_\eta \cap \mathcal{P} \cap D(A) \subset D(A) \cap K,\ \forall \lambda > 0. \tag{4.2}$$

In particular,

$$\|S(t)u_0\| \le u_0, \quad \forall u_0 \in C,\ t \ge 0, \tag{4.3}$$

and $S(t)(K) \subset K, \quad \forall\, t \ge 0.$

Proof. By [3, Lemma 6.2], we have $\|J_\lambda^\varepsilon y\| \le \|y\|$ for all $y \in \mathcal{M} \cap \mathcal{P}$, $\lambda > 0$. Hence (4.1) follows by Lemma 2.1(iii) and Fatou's lemma. The last assertion then follows by (1.16) and (1.19). □

In the following, we shall denote by $\widetilde{A}$ the restriction of the operator A to K, that is,

$$\widetilde{A}u = Au, \quad \forall\, u \in D(\widetilde{A}) = D(A) \cap K. \tag{4.4}$$

By (4.2) it follows that, for every $\lambda > 0$,

$$\overline{D(\widetilde{A})} \subset K \subset (I + \lambda\widetilde{A})(D(\widetilde{A})) = R(I + \lambda\widetilde{A}) \tag{4.5}$$

and we have by definition that

$$(I + \lambda\widetilde{A})^{-1} = (I + \lambda A)^{-1} \ \text{ on } R(I + \lambda\widetilde{A}). \tag{4.6}$$

Furthermore, $(\widetilde{A}, D(\widetilde{A}))$ is accretive on L^1 and, since $\overline{D(\widetilde{A})} \subset K \subset \overline{D(A)}$, we conclude by (1.19) that $\forall u_0 \in \overline{D(\widetilde{A})}$, $t \ge 0$,

$$S(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\,\widetilde{A}\right)^{-1} u_0 \in \overline{D(\widetilde{A})}. \tag{4.7}$$

Therefore, $\widetilde{S}(t) := S(t)_{\uparrow\overline{D(\widetilde{A})}}$, $t \ge 0$, is the contraction semigroup generated by $\widetilde{A}$ on $\overline{D(\widetilde{A})}$. We are going to apply Theorem 3 in [7] to this semigroup $\widetilde{S}(t)$, $t \ge 0$, to prove Theorem 3.1. For this we need:

Lemma 4.2. *The operator* $(I + \lambda\widetilde{A})^{-1}$ *restricted to* K *is compact for* $\lambda \in (0, \lambda_0]$ *with* λ_0 *as in Lemma* 2.1.

Proof. Let $f_n \in K,\ n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} |f_n|_1 < \infty. \tag{4.8}$$

Then, since $\sup\limits_{n\in\mathbb{N}} \|f_n\| \le \eta$, by (4.1), (4.6) and Lemma A.1, it suffices to prove that (selecting a subsequence if necessary) for $\lambda \in (0, \lambda_0]$, $J_\lambda f_n = (I+\lambda A)^{-1} f_n$

converges in $L^1_{\mathrm{loc}}(K)$ as $n \to \infty$ for every compact set $K \subset \mathbb{R}^d$. Let $q \in \left(1, \frac{d}{d-1}\right)$. By (4.8), (2.15), (2.16) and the Riesz-Kolmogorov compactness theorem, it follows that (selecting a subsequence if necessary)

$$\beta(J_\lambda f_n) \xrightarrow[n\to\infty]{} \eta \ \text{ in } L^q(K) \tag{4.9}$$

and (since this holds for every such K)

$$\beta(J_\lambda f_n) \xrightarrow[n\to\infty]{} \eta\,, \ \text{ a.e. on } \mathbb{R}^d,$$

hence, since $\beta \in C^1$ and $\beta' > 0$,

$$J_\lambda f_n \xrightarrow[n\to\infty]{} \beta^{-1}(\eta) \ \text{ a.e. on } \mathbb{R}^d. \tag{4.10}$$

Furthermore, because $\nu > \frac{d-1}{d}$, we may choose q so close to $\frac{d}{d-1}$ that $\nu q > 1$, and hence by (1.2)

$$\mu_1^q \min(|r|^q, |r|^{\nu q}) \le |\beta(r)|^q,\ \forall r \in \mathbb{R}.$$

This implies due to (4.9) that $\{J_\lambda f_n \mid n \in \mathbb{N}\}$ is equi-integrable in $L^1(K)$, hence by (4.10)

$$J_\lambda\, f_n \xrightarrow[n\to\infty]{} \beta^{-1}(\eta) \quad \text{in } L^1(K).$$ □

Lemma 4.3. *For each $u_0 \in K$, the orbit $\gamma(u_0) = \{\widetilde{S}(t)u_0,\ t \ge 0\}$ is precompact in L^1.*

Proof. By Theorem 3 in [7], applied to the operator A with domain K, it suffices to show that $\overline{D(\widetilde{A})} \subset R(I + \lambda\widetilde{A})$, the operator $(I + \lambda\widetilde{A})^{-1}$ is compact on K for some $\lambda > 0$ and that the orbit $\gamma(u_0)$ is bounded in L^1. In fact, in [7] one assumes that $0 \in R(\widetilde{A})$ to have the latter, but in our case this follows, because $|\widetilde{S}(t)u_0|_1 = |S(t)u_0|_1 \le |u_0|_1$, since $S(t)$ is a Lipschitz contraction on L^1 with $S(t)(0) = 0$ (by (1.12) and (1.19)). The first condition in [7, Theorem 3] is just (4.5), the second is just Lemma 4.2. □

Proof of Theorem 3.1 (*continued*). Since, by Lemma 4.3, $\gamma(u_0)$ is precompact, it follows that $\omega(u_0) \ne \emptyset$ and that $\omega(u_0)$ is compact. Then, the conclusions of the theorem follow by Theorem 1 in [7]. □

5 Appendix

Proof of Lemma 2.1.

(i): We have seen above that $J_\lambda^\varepsilon f$ is such a solution of (2.14). So, let $u_\lambda, \widetilde{u}_\lambda \in L^1 \cap L^\infty$ be two solutions of (2.14). Then, obviously, $u_\lambda, \widetilde{u}_\lambda \in H^1$ and $\beta_\varepsilon(u_\lambda), \beta_\varepsilon(\widetilde{u}_\lambda) \in H^2$. Let $u := u_\lambda - \widetilde{u}_\lambda$. Then, by (2.14),

$$u - \lambda\Delta(\beta_\varepsilon(u_\lambda) - \beta_\varepsilon(\widetilde{u}_\lambda)) = -\lambda \operatorname{div}(D(b(u_\lambda)u_\lambda - b(\widetilde{u}_\lambda)\widetilde{u}_\lambda)).$$

Applying $\langle u, \cdot\rangle_{-1}$ to both sides of this equation, where $\langle \cdot, \cdot\rangle_{-1}$ denotes an inner product in H^{-1}, we find, since $\beta_\varepsilon = \beta + \varepsilon I$,

$$\begin{aligned} |u|_{-1}^2 + \varepsilon\lambda|u|_2^2 &\leq \lambda \langle \beta_\varepsilon(u_\lambda) - \beta_\varepsilon(\widetilde{u}_\lambda), u\rangle_{-1} \\ &\quad -\lambda \langle \operatorname{div}(D(b(u_\lambda)u_\lambda - b(\widetilde{u}_\lambda)\widetilde{u}_\lambda)), u\rangle_{-1} \\ &\leq \lambda(\beta_M + C|D|_\infty b_M)|u|_2|u|_{-1}, \end{aligned} \tag{5.0}$$

where we used that $|\operatorname{div} \cdot|_{-1} \leq C|\cdot|_2$ for some $C \in (0,\infty)$ and where

$$\beta_M := \sup\left\{\frac{\beta(r_1) - \beta(r_2)}{r_1 - r_2};\ r_1, r_2 \leq M,\ r_1 \neq r_2\right\} + 1,$$
$$b_M := \sup\left\{\frac{b(r_1)r_1 - b(r_2)r_2}{r_1 - r_2};\ r_1, r_2 \leq M,\ r_1 \neq r_2\right\},$$

and $M := \Big(1 + |(\operatorname{div} D)^- + |D||_\infty^{1/2}\, |f|\Big)$. We recall that by the proof of Lemma 3.1 in [3] (see formula (3.36)) there exists $\widetilde{\lambda}_0 \in (0,\infty)$ such that for any solution u_λ of (2.14) we have $|u_\lambda|_\infty \leq M$, $\forall \lambda \in (0, \widetilde{\lambda}_0)$. Hence, $\beta_M, b_M < \infty$ if $\lambda \in (0, \widetilde{\lambda}_0)$. Hence, by Young's inequality there exists $\lambda_0 \in (0, \widetilde{\lambda}_0)$, such that for some $C_\varepsilon \in (0,\infty)$

$$|u|_{-1}^2 \leq \lambda C_\varepsilon |u|_{-1}^2,\ \forall \lambda \in (0, \lambda_0]$$

Hence, $u = 0$.

(ii): We know by (2.9) and (2.11) that for $\lambda \in (0,\infty)$

$$a_\varepsilon + \lambda(A_0)_\varepsilon a_\varepsilon = a_\varepsilon$$

and that by (i) for $\lambda \in (0, \lambda_0]$

$$J_\lambda^\varepsilon a_\varepsilon + \lambda(A_0)_\varepsilon J_\lambda^\varepsilon a_\varepsilon = a_\varepsilon.$$

Hence by the uniqueness part of (i) assertion (ii) follows since $a_\varepsilon \in L^1 \cap L^\infty$ by (2.10), (2.11).

To prove (iii), for each $f \in L^1$, we set $u_\varepsilon = (I + \lambda A_\varepsilon)^{-1} f$, that is

$$u_\varepsilon - \lambda \Delta \beta_\varepsilon(u_\varepsilon) + \lambda \operatorname{div}(Db(u_\varepsilon)u_\varepsilon) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{5.1}$$

To prove that, for $\varepsilon \to 0$ it follows that $u_\varepsilon \to u$ in L^1, where u is a solution to

$$u - \lambda \Delta \beta(u) + \lambda \operatorname{div}(Db(u)u) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{5.2}$$

where $\beta_\varepsilon(u) = \beta(u) + \varepsilon u$ (see (2.4)), we shall proceed as in the proof of Lemmas 3.1 and 3.3 in [3]. However, since the proof is the same, it will be sketched only. Namely, using the fact that $\beta' > 0$, it follows that $u_\varepsilon \to u \in (I + \lambda A_0)^{-1} f$ strongly in L^1_{loc} and $\beta(u_\varepsilon) \to \beta(u)$ in L^1_{loc}.

Now, let $\widetilde{\Phi} \in C^2(\mathbb{R}^d)$ be such that $\widetilde{\Phi}(x) \to \infty$ as $|x| \to \infty$, $\nabla \widetilde{\Phi} \in L^\infty$, $\Delta \widetilde{\Phi} \in L^\infty$.

If we multiply (5.1) by $\widetilde{\Phi} \exp(-\nu \widetilde{\Phi}) \mathcal{X}_\delta(\widetilde{\beta}_\varepsilon(u_\varepsilon))$, where $\nu > 0$, and integrate on $\mathbb{R}^d$, we get after some calculation identical with that in the proof of Lemma 3.3 in [3] that, for each $f \in \widetilde{\mathcal{M}}$, we have the estimate

$$\|u_\varepsilon\|_* \le \|f\|_* + C\lambda(|\Delta \widetilde{\Phi})|_\infty + |D|_\infty |\nabla \widetilde{\Phi}|_\infty)|f|_1,$$

where

$$\|u\|_* = \int_{\mathbb{R}^d} |u(x)| \widetilde{\Phi}(x) dx, \;\; \widetilde{\mathcal{M}} = \{f \in L^1;\ \|f\|_* < \infty\}.$$

By Lemma A.1 below, this implies that, for $\varepsilon \to 0$, $u_\varepsilon \to u$ in L^1 and, letting $\varepsilon \to 0$ in (5.1), we get that u is a solution to (5.2), as claimed. □

(iv): It follows from Lemma 3.1 in [3] and its proof that there exists $C \in (0, \infty)$ only depending on $K, q, \lambda, |D|_\infty$ and $|b|_\infty$ such that for all $f \in L^1 \cap L^\infty$, $\nu \in \mathbb{R}^d$, $\varepsilon \in (0, 1]$, $|J^\varepsilon_\lambda f|_\infty \le C|f|_\infty$,

$$\|\beta_\varepsilon(J^\varepsilon_\lambda f)\|_{L^q(K)} \le C|f|_1 \tag{5.3}$$

and

$$\|\beta_\varepsilon(J^\varepsilon_\lambda f)^\nu\|_{L^q(K)} \le C \left(\|E^\nu\|_{M^{\frac{d}{d-2}}} + \|\nabla E^\nu\|_{M^{\frac{d}{d-1}}} \right) |f|_1. \tag{5.4}$$

We have that $\lim\limits_{n\to\infty} \|E^\nu\|_{M^{\frac{d}{d-2}}} + \|\nabla E^\nu\|_{M^{\frac{d}{d-1}}} = 0$, hence, by (iii) and the Riesz-Kolmogorov compactness theorem, along a subsequence $\varepsilon \to 0$,

$$\begin{aligned} J^\varepsilon_\lambda f &\to J_\lambda f \quad \text{weakly in } L^q(K), \\ \beta(J^\varepsilon_\lambda f) &\to \eta \qquad \text{strongly in } L^q(K). \end{aligned}$$

Since $u \mapsto \beta(u)$ is maximal monotone in each dual pair $(L^q(K), L^{q'}(K))$, hence weakly-strongly closed, we conclude that

$$\eta = \beta(J_\lambda f).$$

Hence, by Fatou's lemma we may pass to the limit in (5.3), (5.4) to obtain (2.15) and (2.16) for all $f \in L^1 \cap L^\infty$. Since $L^1 \cap L :^\infty$ is dense in L^1, applying Fatou's lemma again we obtain (2.15), (2.16) for all $f \in L^1$. □

We use the following well known result in this paper. We include a proof for the reader's convenience.

Lemma A.1. *Let $u_n \in L^1(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that for some $u \in L^1_{\text{loc}}(\mathbb{R}^d)$*

$$\lim_{n\to\infty} u_n = u \quad \textit{in } L^1_{\text{loc}}(\mathbb{R}^d).$$

Furthermore, let $\Phi : \mathbb{R}^d \to [1, \infty)$ be Borel-measurable with $\{\Phi \le c\}$ relatively compact for all $c \in (0, \infty)$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |u_n|\Phi\, dx < \infty. \tag{A.1}$$

Then $\int_{\mathbb{R}^d} |u|\Phi\, dx < \infty$ and $\lim\limits_{n\to\infty} u_n = u$ in $L^1(\mathbb{R}^d)$.

Proof. By Fatou's Lemma and (A.1)

$$\int_{\mathbb{R}^d} |u|\Phi\, dx \le \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |u_n|\Phi\, dx < \infty.$$

Hence, $u \in L^1(\mathbb{R}^d)$ and, for all $c \in (0, \infty)$,

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u - u_n| dx &= \limsup_{n\to\infty} \int_{\{\Phi > c\}} |u - u_n| dx + \limsup_{n\to\infty} \int_{\overline{\{\Phi \le c\}}} |u - u_n| dx \\ &\le \frac{1}{c} \sup_n \int_{\mathbb{R}^d} (|u| + |u_n|)\Phi\, dx \underset{c\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

□

Lemma A.2. *Assume that Hypotheses* (i), (ii), (iii) *hold. Let g be as in* (3.2)*, i.e.,*

$$g(r) = \int_1^r \frac{\beta'(s)}{sb(s)}\, ds, \quad r \in (0, \infty).$$

(j) *Let* $\nu \in \left(1-\frac{1}{d}, 1\right]$. *Then*

$$\lim_{r\to 0} g(r) = -\infty. \tag{A.2}$$

If, in addition, $\lim_{r\to\infty} g(r) = \infty$, *then*

$$g^{-1}(r) \le e^{\frac{1}{\mu_2}(b_0 r+\beta(1)-\mu_1)},\ \forall r \in \left(-\infty, \tfrac{1}{b_0}(\mu_1-\beta(1))\right], \tag{A.3}$$

hence, for all $\mu \in \mathbb{R}$ *on all of* $\mathbb{R}^d$,

$$\begin{aligned} g^{-1}(\mu-\Phi) \le\ & 1_{\{\Phi\ge\mu+b_0^{-1}(\beta(1)-\mu_1)\}} e^{\frac{1}{\mu_2}(b_0\mu+\beta(1)-\mu_1)} e^{-\frac{b_0}{\mu_2}\Phi} \\ & + 1_{\{\Phi<\mu+b_0^{-1}(\beta(1)-\mu_1)\}} g^{-1}(\mu-\Phi), \end{aligned} \tag{A.4}$$

which in turn implies that, for all $\mu \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}^d} g^{-1}(\mu-\Phi(x))\Phi(x)dx \le\ & e^{\frac{1}{\mu_2}(b_0\mu+\beta(1)-\mu_1)} \int_{\mathbb{R}^d} e^{-\frac{b_0}{\mu_2}\Phi(x)}\Phi(x)dx \\ & + g^{-1}(\mu-1)\int_{\{\Phi<\mu+b_0^{-1}(\beta(1)-\mu_1)\}} \Phi(x)dx < \infty. \end{aligned} \tag{A.5}$$

(jj) *Let* $\nu \in (1,\infty)$. *Then,*

$$\lim_{r\to\infty} g(r) = \infty. \tag{A.6}$$

If, in addition, $\lim_{r\to 0} g(r) = -\infty$, *then*

$$g^{-1}(r) \le e^{\frac{1}{\mu_2}(b_0 r+\beta(1))},\ \forall r \in \left(-\infty, -\frac{\beta(1)}{b_0}\right], \tag{A.7}$$

hence, for all $\mu \in \mathbb{R}$ *on all of* $\mathbb{R}^d$,

$$\begin{aligned} g^{-1}(\mu-\Phi) \le\ & 1_{\left\{\Phi\ge\mu+\frac{\beta(1)}{b_0}\right\}} e^{\frac{1}{\mu_2}(b_0\mu+\beta(1))} e^{-\frac{b_0}{\mu_2}\Phi} \\ & + 1_{\left\{\Phi<\mu+\frac{\beta(1)}{b_0}\right\}} g^{-1}(\mu-\Phi), \end{aligned} \tag{A.8}$$

which in turn implies that, for all $\mu \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}^d} g^{-1}(\mu-\Phi(x))\Phi(x)dx \le\ & e^{\frac{1}{\mu_2}(b_0\mu+\beta(1))} \int_{\mathbb{R}^d} e^{-\frac{b_0}{\mu_2}\Phi(x)}\Phi(x)dx \\ & + g^{-1}(\mu-1)\int_{\left\{\Phi<\mu+\frac{\beta(1)}{b_0}\right\}} \Phi(x)dx < \infty. \end{aligned} \tag{A.9}$$

Proof. First, we note that by Hypothesis (iii) and integrating by parts we find

$$1_{(0,1]}\,\frac{\widetilde{g}}{b_0} + 1_{(1,\infty)}\,\frac{\widetilde{g}}{|b|_\infty} \le g \le 1_{(0,1]}\,\frac{\widetilde{g}}{|b|_\infty} + 1_{(1,\infty)}\,\frac{\widetilde{g}}{b_0}, \tag{A.10}$$

where

$$\widetilde{g}(r) := \frac{\beta(r)}{r} - \beta(1) + \int_1^r \frac{\beta(s)}{s^2}\,ds,\ r \in (0,\infty).$$

(j): Let $\nu \in \left(1-\frac{1}{d},1\right]$. Then, by (1.2), for $r \in (0,1]$,

$$\widetilde{g}(r) \le \mu_2 - \beta(1) - \int_r^1 \frac{\mu_1}{s}\,ds.$$

So, (A.10) implies (A.2). Now, additionally assume that $\lim\limits_{r\to\infty} g(r) = \infty$, so that $g : (0,\infty) \to \mathbb{R}$ is bijective. Again by (A.10) and (1.2) we obtain, for all $r \in (0,\infty)$,

$$g(r) \ge b_0^{-1} 1_{(0,1]}(r)(\mu_1 - \beta(1) + \mu_2 \ln r) + 1_{(1,\infty)}(r) g(r).$$

Replacing $r \in (0,\infty)$ by $e^{\frac{1}{\mu_2}(b_0 r + \beta(1) - \mu_1)} \in (0,1]$, for $r \in (-\infty, b_0^{-1}(\mu_1 - \beta(1))]$, we obtain, since g is increasing,

$$g\left(e^{\frac{1}{\mu_2}(b_0 r + \beta(1) - \mu_1)}\right) \ge r,$$

and thus

$$g^{-1}(r) \le e^{\frac{1}{\mu_2}(b_0 r + \beta(1) - \mu_1)},\ \forall r \in (-\infty, b_0^{-1}(\mu_1 - \beta(1))],$$

which, for all $\mu \in \mathbb{R}$, implies that on all of $\mathbb{R}^d$

$$\begin{aligned} g^{-1}(\mu - \Phi) \le\ & 1_{\{\Phi \ge \mu + b_0^{-1}(\beta(1)-\mu_1)\}} e^{\frac{1}{\mu_2}(b_0\mu + \beta(1) - \mu_1)} e^{-\frac{b_0}{\mu_2}\Phi} \\ & + 1_{\{\Phi < \mu + b_0^{-1}(\beta(1)-\mu_1)\}} g^{-1}(\mu - \Phi), \end{aligned}$$

which is (A.4). (A.5) is now obvious by (1.3) (which, in particular, implies that $\{\Phi < c\}$ is relatively compact $\forall c \in \mathbb{R}$) and because g^{-1} is increasing.

(jj): Let $\nu \in (1,\infty)$. Then, by (1.2), for $r \in (1,\infty)$,

$$\widetilde{g}(r) \ge \mu_1 - \beta(1) + \mu_1 \int_1^r \frac{1}{s}\,ds.$$

So, (A.10) implies (A.6). Now, additionally assume that $\lim\limits_{r\to 0} g(r) = -\infty$, so that $g : (0,\infty) \to \mathbb{R}$ is bijective. Again by (A.10) and (1.2) we obtain, for all $r \in (0,\infty)$,

$$\begin{aligned} g(r) &\geq 1_{(0,1]}(r) b_0^{-1} \left(\mu_1 r^{\nu-1} - \beta(1) - \int_r^1 \frac{\mu_2}{s}\, ds \right) + 1_{(1,\infty)}(r) g(r) \\ &\geq 1_{(0,1]} b_0^{-1} (\mu_2 \ln r - \beta(1)) + 1_{(1,\infty)}(r) g(r). \end{aligned}$$

Replacing $r \in (0,\infty)$ by $e^{\frac{1}{\mu_2}(b_0 r + \beta(1))} \in (0,1]$ for $r \in \left(-\infty, -\frac{\beta(1)}{b_0}\right]$, we obtain

$$g\left(e^{\frac{1}{\mu_0}(b_0 r + \beta(1))}\right) \geq r,$$

and thus

$$g^{-1}(r) \leq e^{\frac{1}{\mu_2}(b_0 r + \beta(1))},\ \forall r \in \left(-\infty, -\frac{\beta(1)}{b_0}\right],$$

which, for all $\mu \in \mathbb{R}$, implies that on all of $\mathbb{R}^d$

$$g^{-1}(\mu - \Phi) \leq 1_{\left\{\Phi \geq \mu + \frac{\beta(1)}{b_0}\right\}} e^{\frac{1}{\mu_2}(b_0 \mu + \beta(1))} e^{-\frac{b_0}{\mu_2}\Phi} + 1_{\left\{\Phi < \mu + \frac{\beta(1)}{b_0}\right\}} g^{-1}(\mu - \Phi).$$

Hence, (A.7), (A.8) are proved and (A.9) is then again an obvious consequence. □

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