SDEs with singular drifts and multiplicative noise on general space-time domains

Chengcheng Ling*[a,c], Michael Röckner[c], and Xiangchan Zhu[b,c]

[a]School of science, Beijing Jiaotong University, Beijing 100044, China
[b]Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[c]Department of Mathematics, University of Bielefeld, D-33615, Germany

Abstract

In this paper, we prove the existence and uniqueness of maximally defined strong solutions to SDEs driven by multiplicative noise on general space-time domains $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$, which have continuous paths on the one-point compactification $Q \cup \partial$ of Q where $\partial \notin Q$ and $Q \cup \partial$ is equipped with the Alexandrov topology. If the SDE is of gradient type (see (2.5) below) we prove that under suitable Lyapunov type conditions the life time of the solution is infinite and its distribution has sub-Gaussian tails. This generalizes earlier work [7] by Krylov and one of the authors to the case where the noise is multiplicative.

Key words

Krylov's estimate, stochastic differential equation, well-posedness, non-explosion of the solution, maximally defined local solution to SDE, singular drift, multiplicative noise

1 Introduction

Consider the following stochastic differential equation (abbreviated as SDE):

$$X_t = x + \int_0^t b(s+r, X_r)dr + \int_0^t \sigma(s+r, X_r)dW_r, \quad t \geq 0, \tag{1.1}$$

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Email: cling@math.uni-bielefeld.de (C. Ling), roeckner@math.uni-bielefeld.de (M. Röckner), zhuxiangchan@126.com (X. Zhu)

in an open set $Q \subset [0, \infty) \times \mathbb{R}^d$ with measurable coefficients $b = (b_i)_{1 \le i \le d} : Q \to \mathbb{R}^d$ and $\sigma = (\sigma_{ij})_{1 \le i,j \le d} : Q \to L(\mathbb{R}^d)$ ($:= d \times d$ real valued matrices). Here $(s, x) \in Q$ is the initial point, and $(W_t)_{t \ge 0}$ is a d-dimensional $(\mathcal{F}_t)$-Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, P)$. Define

$$\xi := \inf \{t \ge 0 : (t + s, X_t) \notin Q\}. \tag{1.2}$$

ξ is called the explosion time (or life time) of the process $(t + s, X_t)_{t \ge 0}$ in the domain Q.

There are many known results on studying existence and uniqueness of strong solutions to the SDE (1.1). In the seminal paper [16], Veretennikov proved that for $Q = \mathbb{R}_+ \times \mathbb{R}^d$, if the coefficient σ is Lipschitz continuous in the space variable x uniformly with respect to the time variable t, $\sigma\sigma^*$ is uniformly elliptic, and b is bounded and measurable, then the SDE (1.1) admits a unique global strong solution (i.e. $\xi = \infty$ *a.s.* where ξ is defined in (1.2)). In [7] under the assumptions that $\sigma = \mathbb{I}_{d\times d}$ (i.e. additive noise, $\mathbb{I}_{d\times d}$ denotes the unit matrix in $\mathbb{R}^d$) and $|bI_{Q^n}| \in L^{q(n)}(\mathbb{R}; L^{p(n)}(\mathbb{R}^d))$ for $p(n), q(n) \in (2, \infty)$ and $d/p(n) + 2/q(n) < 1$, where Q^n are open bounded subsets of Q with $\overline{Q^n} \subset Q^{n+1}$ and $Q = \cup_n Q^n$, Krylov and Röckner proved the existence of a unique maximal local strong solution to (1.1) when Q is a subset of $\mathbb{R}^{d+1}$, in the sense that there exists a unique strong solution $(s + t, X_t)_{t \ge 0}$ solving (1.1) on $[0, \xi)$ such that $[0, \infty) \ni t \to (s + t, X_t) \in Q' := Q \cup \partial$ (=Alexandrov compactification of Q) is continuous and this process is defined to be in ∂ if $t \ge \xi$. To this end they applied the Girsanov transformation to get existence of a weak solution firstly and then proved pathwise uniqueness of (1.1) by Zvonkin's transformation invented in [24]. Then, the well-known Yamada-Watanabe theorem [21] yields existence and uniqueness of a maximal local strong solution. Fedrizzi and Flandoli [4] introduced a new method to prove existence and uniqueness of a global strong solution to the SDE (1.1) by using regularizing properties of solutions to the Kolmogrov equation corresponding to (1.1), assuming that $\sigma = \mathbb{I}_{d\times d}$, $|b| \in L^q_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^d))$ with $p, q \in (2, \infty)$ and $d/p + 2/q < 1$. This method was extended by Von der Lühe to the multiplicative noise case in her work [17]. Zhang in [23] proved existence and uniqueness of a strong solution to the SDE (1.1) on $Q = \mathbb{R}_+ \times \mathbb{R}^d$ for $t < \tau$ *a.s.*, where τ is some stopping time, under the assumptions that σ is bounded and uniformly continuous in x locally uniformly with respect to t, $\sigma\sigma^*$ is uniformly elliptic, and $|b|, |\nabla\sigma| \in L^{q(n)}_{loc}(\mathbb{R}_+; L^{p(n)}(B_n))$ (where $\nabla\sigma$ denotes the weak gradient of σ with respect to x) with $p(n), q(n)$ satisfying $p(n), q(n) \in (2, \infty)$ and $d/p(n) + 2/q(n) < 1$, where B_n is the ball in $\mathbb{R}^d$ with radius $n \in \mathbb{N}$ centered at zero. Zvonkin's transformation plays a crucial role in Zhang's proof (see also [16], [22], [20] for further interesting results on this topic, which however do not cover our results in this paper). The above results include the case where the coefficients of the SDE (1.1) are time dependent. For the time independent case, Wang [18] and Trutnau [9] used generalized Dirichlet forms to get existence and uniqueness and also non-explosion results for the SDE (1.1) on $Q = \mathbb{R}^d$.

As mentioned in [7], there are several interesting situations arising from applications, say diffusions in random media and particle systems, where the domain Q of (1.1) is not the full space $\mathbb{R} \times \mathbb{R}^d$ but a subdomain (e.g. $Q = \mathbb{R} \times (\mathbb{R}^d \backslash \gamma^\rho)$, where $\gamma^\rho = \{x \in \mathbb{R}^d | dist(x, \gamma) \le \rho\}$, $\rho > 0$, and γ is a locally finite subset of $\mathbb{R}^d$), where none of the above results mentioned can be applied to get global solutions, except for the one in [7]. Moreover, Krylov and Röckner in [7] not only proved the existence and uniqueness of a maximal local strong solution of the equation on Q, but also they obtained that if $b = -\nabla\phi$, i.e., b is minus the gradient in space of a nonnegative function ϕ and if there exist a constant $K \in [0, \infty)$ and an integrable function h on Q defined as above such that the following Lyapunov conditions hold in the distributional sense

$$2D_t\phi \leq K\phi, \quad 2D_t\phi + \Delta\phi \leq he^{\epsilon\phi}, \quad \epsilon \in [0,2), \tag{1.3}$$

the strong solution does not blow up, which means $\xi = \infty$ $a.s.$. Here $D_t\phi$ denotes the derivative of ϕ with respect to t. This result can be applied to diffusions in random environment and also finite interacting particle systems to show that if the above Lyapunov conditions hold, the process does not exit from Q or go to infinity in finite time. However, [7] is restricted to the case where the equation (1.1) is driven by additive noise, that is, the diffusion term is a Brownian motion.

Our aim in this paper is to extend these results on existence and uniqueness of maximally defined local solutions and also the non-explosion results in [7] to the multiplicative noise case on general space-time domains Q. In order to prove the maximal local well-posedness result, we use a localization technique and the well-posedeness result in [23]. We want to point out that as Krylov and Röckner did in [7], we also prove the continuity of the paths of the solution not only in the domain Q but also on $Q' = Q \cup \partial$, which essentially follows from Lemmas 3.5 and 3.6 below. As far as the non-explosion result is concerned, we have to take into account that having non-constant σ instead of $\mathbb{I}_{d\times d}$ in front of the Brownian motion in (1.1) means that we have to consider a different geometry on $\mathbb{R}^d$, and that this effects the Lyapunov function type conditions which are to replace (1.3) and also the form of the equation. In Remark 2.5 by comparing the underlying Kolmogrov operators, we explain why the SDE (2.5) should be considered and why (2.3) states the right Lyapunov type conditions which are analog to the ones in (1.3). This leads to some substantial changes in the proof of our non-explosion result in comparison with the one in [7]. In addition, we give some examples to show our well-posedness and non-explosion results in Sections 6.1 and 6.2. We also give two applications to diffusions in random media and particle systems. Both are generalizations of the examples in [7, Section 9] to the case of multiplicative noise.

The organization of this paper is as follows: We state our notions and main results in Section 2 . In Section 3 we prove that there exists a pathwise unique maximal strong solution $(s+t, X_t)_{t\geq 0}$ solving the SDE (1.1) on $[0, \xi)$, and that the paths of $(s+t, X_t)_{t\geq 0}$ are continuous in $Q' = Q \cup \partial$. Section 4 is devoted to the preparation of the proof of our non-explosion result, which is subsequently proved in Section 5. We discuss examples and applications of our results in Section 6 . The Appendix contains technical lemmas used in the proofs of our main results.

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2 Main results

Let Q be an open subset of $\mathbb{R}_+ \times \mathbb{R}^d$ and Q^n, $n \geq 1$, be bounded open subsets of Q such that $\overline{Q^n} \subset Q^{n+1}$ and $\cup_n Q^n = Q$. We add an object $\partial \notin Q$ to Q and define the neighborhoods of ∂ as the complements in Q of closed bounded subsets. Then $Q' = Q \cup \partial$ becomes a compact topological space, which is just the Alexandrov compactification of Q. For p, $q \in [1, \infty)$ and

$0 \le S < T < \infty$, let $\mathbb{L}_p^q(S,T)$ denote the space of all real Borel measurable functions on $[S,T] \times \mathbb{R}^d$ with the norm

$$\|f\|_{\mathbb{L}_p^q(S,T)} := \left(\int_S^T \left(\int_{\mathbb{R}^d} |f(t,x)|^p dx \right)^{q/p} dt \right)^{1/q} < +\infty.$$

For simplicity, we write

$$\mathbb{L}_p^q = \mathbb{L}_p^q(0,\infty), \quad \mathbb{L}_p^q(T) = \mathbb{L}_p^q(0,T), \quad \mathbb{L}_p^{q,loc} = L_q^{loc}(\mathbb{R}_+, L_p(\mathbb{R}^d)).$$

Let $\mathcal{C}([0,\infty),\mathbb{R}^d)$ denote the space of all continuous $\mathbb{R}^d$-valued functions defined on $[0,\infty)$, by $\mathcal{C}([0,\infty),Q')$ we denote all continuous Q'-valued paths, $\mathcal{C}_b^n(\mathbb{R}^d)$ denotes the set of all bounded n times continuously differentiable functions on $\mathbb{R}^d$ with bounded derivatives of all orders. Set $(a_{ij})_{1\le i,j\le d} := \sigma\sigma^*$, where σ^* denotes the transpose of σ. For $f \in L^1_{loc}(\mathbb{R}^d)$ we define $\partial_j f(x) := \frac{\partial f}{\partial x_j}(x)$ and $\nabla f := (\partial_i f)_{1\le i\le d}$ denotes the gradient of f. Here the derivatives are meant in the sense of distributions. For a real valued function $g \in \mathcal{C}^1([0,\infty))$, $D_t g$ denotes the derivative of g with respect to t. $L(\mathbb{R}^d)$ denotes all $d\times d$ real valued matrices in $\mathbb{R}^d$.

We first state the result about maximally local well-posedness of the SDE (1.1) on a domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$.

Theorem 2.1. *Let $(W_t)_{t\ge 0}$ be an d-dimensional Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$, let $(\mathcal{F}_t)_{t\ge0} = (\mathcal{F}_t^W)_{t\ge0}$ be the normal filtration generated by $(W_t)_{t\ge0}$. Assume that for any $n\in\mathbb{N}$ and some p_n, $q_n \in (2,\infty)$, satisfying $d/p_n + 2/q_n < 1$,*
(i) $|bI_{Q^n}|$, $|I_{Q^n}\nabla\sigma| \in \mathbb{L}_{p_n}^{q_n}$,
(ii) for all $1\le i,j\le d$, $Q \ni (t,x) \to \sigma_{ij}(t,x) \in \mathbb{R}$ is continuous in x uniformly with respect to t on Q^n, and there exists a positive constant δ_n such that for all $(t,x)\in Q^n$,

$$|\sigma^*(t,x)\lambda|^2 \ge \delta_n |\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d.$$

Then for any $(s,x)\in Q$, there exists an $(\mathcal{F}_t)$-stopping time $\xi := \inf\{t\ge 0 : z_t \notin Q\}$ and an $(\mathcal{F}_t)$-adapted, pathwise unique and Q'-valued process $(z_t)_{t\ge0} := (s+t, X_t)_{t\ge0}$ which is continuous in Q' such that

$$X_t = x + \int_0^t b(s+r,X_r)dr + \int_0^t \sigma(s+r,X_r)dW_r, \quad \forall t\in[0,\xi), a.s. \tag{2.1}$$

and for any $t\ge0$, $z_t = \partial$ on the set $\{\omega : t \ge \xi(\omega)\}$ a.s..

Remark 2.2. *In above theorem the condition $p,q\in(2,\infty)$ is automatically fulfilled when $d\ge 2$ since we also assume $d/p+2/q<1$. When $d=1$, we can refer to the result from Engelbert and Schmidt [2] to obtain the existence and uniqueness of a strong solution to homogeneous SDE on $\mathbb{R}^d$. They proved that if $\sigma(x)\ne 0$ for all $x\in\mathbb{R}$ and $b/\sigma^2 \in L^1_{loc}(\mathbb{R})$, and there exists a constant $C>0$ such that*

$$|\sigma(x)-\sigma(y)| \le C\sqrt{|x-y|}, \quad x,y\in\mathbb{R},$$

$$|b(x)| + |\sigma(x)| \le C(1+|x|),$$

then there exists a pathwise unique and $(\mathcal{F}_t)$-adapted process $(X_t)_{t\ge0}$ such that the SDE $X_t = x + \int_0^t b(X_t)dt + \int_0^t \sigma(X_t)dW_t$ holds a.s..

Below we give the non-explosion result for the solution to an SDE which is in a special form of (2.1) on a domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$ under the following assumptions.

Assumption 1. ***(i)*** *ϕ is a nonnegative continuous function defined on Q.*
(ii) *For each n there exist $p = p(n)$, $q = q(n)$ satisfying*

$$p, q \in (2, \infty], \quad \frac{d}{p} + \frac{2}{q} < 1 \tag{2.2}$$

such that $|I_{Q^n} \nabla \phi|$, $|I_{Q^n} \nabla \sigma| \in \mathbb{L}_p^q$.
(iii) *For each $1 \le i, j \le d$, $Q \ni (t, x) \to \sigma_{ij}(t, x) \in \mathbb{R}$ is uniformly continuous in x locally uniformly with respect to t, and there exists a positive constant K such that for all $(t, x) \in Q$,*

$$\frac{1}{K}|\lambda|^2 \le |\sigma^*(t, x)\lambda|^2 \le K|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d.$$

(iv) *For some constants $K_1 \in [0, \infty)$ and $\epsilon \in [0, 2)$, in the sense of distributions on Q we have*

$$2D_t\phi \le K_1\phi, \quad 2D_t\phi + \sum_{i,j=1}^{d} \partial_j(a_{ij}\partial_i\phi) \le he^{\epsilon\phi}, \tag{2.3}$$

where h is a continuous nonnegative function defined on Q satisfying the following condition:
(H) For any $a > 0$ and $T \in (0, \infty)$ there is an $r = r(T, a) \in (1, \infty)$ such that

$$H(T, a, r) := H_Q(T, a, r) := \int_Q h^r(t, x) I_{(0,T)}(t) e^{-a|x|^2} dt dx < \infty.$$

(v) *For all $1 \le i, j \le d$, for all (t, x), $(s, y) \in Q$,*

$$|a_{ij}(t, x) - a_{ij}(s, y)| \le K(|x - y| \vee |t - s|^{1/2}), \tag{2.4}$$

and for all $n \in \mathbb{N}$, and (t, x), $(s, y) \in Q^n$, there exists $C_n \in [0, \infty)$ such that

$$|\partial_j a_{ij}(t, x) - \partial_j a_{ij}(s, y)| \le C_n(|x - y| \vee |t - s|^{1/2}).$$

(vi) *The function ϕ blows up near the parabolic boundary of Q, that is for any $(s, x) \in Q$, $\tau \in (0, \infty)$, and any continuous bounded $\mathbb{R}^d$-valued function x_t defined on $[0, \tau)$ and such that $(s + t, x_t) \in Q$ for all $t \in [0, \tau)$ and*

$$\liminf_{t \uparrow \tau} dist((s + t, x_t), \partial Q) = 0,$$

we have

$$\limsup_{t \uparrow \tau} \phi(s + t, x_t) = \infty.$$

Remark 2.3. *Observe that $H(T, a, r) < \infty$ if h is just a constant. Furthermore, Assumption 1 (iii) shows that σ is bounded on Q, invertible for every $(t, x) \in Q$, and the inverse σ^{-1} is also bounded on Q.*

Theorem 2.4. *Let Assumption 1 be satisfied. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and $(W_t)_{t\geq 0}$ be as in Theorem 2.1. Then for any $(s,x) \in Q$ there exists a continuous $\mathbb{R}^d$-valued and $(\mathcal{F}_t)$-adapted random process $(X_t)_{t\geq 0}$ such that almost surely for all $t \geq 0$, $(s+t, X_t) \in Q$,*

$$X_t = x + \int_0^t (-\sigma\sigma^* \nabla\phi)(s+r, X_r)dr + \frac{1}{2}(\sum_{j=1}^d \int_0^t \partial_j a_{ij}(s+r, X_r)dr)_{1\leq i\leq d} + \int_0^t \sigma(s+r, X_r)dW_r. \tag{2.5}$$

Furthermore, for each $T \in (0,\infty)$ and $m \geq 1$ there exists a constant N, depending only on K, K_1, d, $p(m+1)$, $q(m+1)$, ϵ, T, $\|I_{Q^{m+1}}\nabla\phi\|_{\mathbb{L}^{q(m+1)}_{p(m+1)}}$, $dist(\partial Q^m, \partial Q^{m+1})$, $\sup_{Q^{m+1}}\{\phi + h\}$, and the function H, such that for $(s,x) \in Q^m$, $t \leq T$, we have

$$E \sup_{t\leq T} \exp(\mu\phi(s+t, X_t) + \mu\nu|X_t|^2) \leq N, \tag{2.6}$$

where

$$\mu = (\delta/2)e^{-TK_1/(2\delta)}, \quad \delta = 1/2 - \epsilon/4, \quad \nu = \mu/(12KT). \tag{2.7}$$

Remark 2.5. *Obviously, the Kolmogrov operator $\mathcal{L}$ corresponding to (2.5) is given by*

$$\mathcal{L} = div(\sigma\sigma^*\nabla) - \langle \sigma^*\nabla\phi, \sigma^*\nabla\rangle, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^d$. Recalling that $div\circ\sigma$ is the adjoint of the 'geometric' gradient $\sigma^*\nabla$ (i.e. taking into account the geometry given to $\mathbb{R}^d$ through σ), we see that (2.5) is the geometrically correct analogue of the SDE

$$dX_t = -\nabla\phi(X_t)dt + dW_t, \quad t \geq 0$$

studied in [7]. So, the Laplacian Δ in [7] is replaced by the Laplace-Beltrami operator $div(\sigma\sigma^*\nabla)(= \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i))$ and the Euclidean gradient ∇ in [7] is replaced by the 'geometric' gradient $\sigma^*\nabla$. Also condition (2.3) is then the exact analogue of condition (1.3) above, which was assumed in [7].

3 Existence and uniqueness of a maximal local strong solution to the SDE (1.1) on an arbitrary domain in $\mathbb{R}_+ \times \mathbb{R}^d$

Theorem 2.1 says that there exists a unique maximally local strong solution to the SDE (1.1). Before going to its proof we give some results as preparation.

3.1 Preparation

Consider the SDE (1.1) in $[0,\infty) \times \mathbb{R}^d$. First we recall two results from [23].

Lemma 3.1. *([23, Theorem 1.1]) Assume that p, $q \in (2,\infty)$ satisfying $d/p + 2/q < 1$ and the following conditions hold.*
(i) $|b|$, $|\nabla\sigma| \in \mathbb{L}^{q,loc}_p$.

(ii) For all $1 \leq i, j \leq d$, $[0,\infty) \times \mathbb{R}^d \ni (t,x) \to \sigma_{ij}(t,x) \in \mathbb{R}$ is uniformly continuous in x locally uniformly with respect to $t \in [0,\infty)$, and there exist positive constants K and δ such that for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$

$$\delta|\lambda|^2 \leq |\sigma^*(t,x)\lambda|^2 \leq K|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d. \tag{3.1}$$

Then for any $(\mathcal{F}_t)$-stopping time τ and $x \in \mathbb{R}^d$, there exists a unique $(\mathcal{F}_t)$-adapted continuous $\mathbb{R}^d$-velued process $(X_t)_{t\geq 0}$ such that

$$P\left\{\omega : \int_0^T |b(r, X_r(\omega))|dr + \int_0^T |\sigma(r, X_r(\omega))|^2 dr < \infty, \forall T \in [0, \tau(\omega))\right\} = 1, \tag{3.2}$$

and

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \quad \forall t \in [0,\tau) \quad a.s, \tag{3.3}$$

which means that if there is another $(\mathcal{F}_t)$-adapted continuous stochastic process $(Y_t)_{t\geq 0}$ also satisfying (3.2) and (3.3), then

$$P\{\omega : X_t(\omega) = Y_t(\omega), \forall t \in [0, \tau(\omega))\} = 1.$$

Moreover, for almost all ω and all $t \geq 0$, $x \to X_t(\omega, x)$ is a homeomorphism on $\mathbb{R}^d$ and there exists a function $t \to C_t \in (0,\infty)$ such that $C_t \to \infty$ as $t \to \infty$ and for all $t > 0$ and all bounded measurable function ψ, for x, $y \in \mathbb{R}^d$,

$$|E\psi(X_t(x)) - E\psi(X_t(y))| \leq C_t \|\psi\|_\infty |x - y|.$$

Below we shall make essential use of Krylov's estimate. Therefore, we recall them here for readers' convenience.

Lemma 3.2. *([23, Theorem 2.1, Theorem 2.2]) Suppose σ satisfies the conditions in Lemma 3.1 and let $(X_t)_{t\geq 0}$ be continuous and $(\mathcal{F}_t)$-adapted $\mathbb{R}^d$-valued process satisfying (3.2) and (3.3). Fix an $(\mathcal{F}_t)$-stopping time τ and let $T_0 > 0$.*
(1) If b is Borel measurable and bounded, then for p, $q \in (1,\infty)$ with

$$\frac{d}{p} + \frac{2}{q} < 2,$$

there exists a positive constant $N = N(K, d, p, q, T_0, \|b\|_\infty)$ such that for all $f \in \mathbb{L}_p^q(T_0)$ and $0 \leq S < T \leq T_0$,

$$E\left(\int_{S\wedge\tau}^{T\wedge\tau} |f(s, X_s)|ds \middle| \mathcal{F}_S\right) \leq N\|f\|_{\mathbb{L}_p^q(S,T)}. \tag{3.4}$$

(2) If $b \in \mathbb{L}_p^q$ provided with

$$\frac{d}{p} + \frac{2}{q} < 1, \quad p, q \in (1,\infty), \tag{3.5}$$

then there exists a positive constant $N = N(K, d, p, q, T_0, \|b\|_{\mathbb{L}_p^q(T_0)})$ such that for all $f \in \mathbb{L}_p^q(T_0)$ and $0 \leq S < T \leq T_0$,

$$E\left(\int_{S\wedge\tau}^{T\wedge\tau} |f(s, X_s)|ds \middle| \mathcal{F}_S\right) \leq N\|f\|_{\mathbb{L}_p^q(S,T)}.$$

We note that actually condition $f \in \mathbb{L}_p^q(T_0)$ with $p, q \in (1, \infty)$ and $\frac{d}{p} + \frac{2}{q} < 1$ in the above Lemma 3.2 can be improved to $f \in \mathbb{L}_{p'}^{q'}(T_0)$ with $p', q' \in (1, \infty)$ and $\frac{d}{p'} + \frac{2}{q'} < 2$ without assuming that b is bounded, which we shall prove in the following lemma. Let K_0 and T_0 be some positive constants and we give the following assumption.

Assumption 2. *(i) For all $1 \leq i, j \leq d$, $[0, \infty) \times \mathbb{R}^d \ni (t, x) \to \sigma_{ij}(t, x) \in \mathbb{R}$ is uniformly continuous in x locally uniformly with respect to $t \in [0, \infty)$, and there exist positive constants K and δ such that for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$*

$$\delta|\lambda|^2 \leq |\sigma^*(t, x)\lambda|^2 \leq K|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d. \tag{3.6}$$

And $|\nabla\sigma| \in \mathbb{L}_p^{q,loc}$ with p, $q \in (2, \infty)$ satisfying $d/p + 2/q < 1$.
(ii) $b(t, x)$ is Borel measurable with $\|b\|_{\mathbb{L}_p^q} \leq K_0$ and $b(t, x) = 0$ for $t > T_0$.

Lemma 3.3. *Let Assumption 2 hold. Let $(X_t)_{t \geq 0}$ be a continuous $(\mathcal{F}_t)$-adapted process such that (3.2) and (3.3) are satisfied. Then for any Borel function $f \in \mathbb{L}_{p'}^{q'}(S, T)$ with $p', q' \in (1, \infty)$ and $d/p' + 2/q' < 2$, and for $0 \leq S < T \leq T_0$, we have*

$$E \int_S^T |f(t, X_t)| dt \leq N(d, p', q', K, \|b\|_{\mathbb{L}_p^q(T_0)}) \|f\|_{\mathbb{L}_{p'}^{q'}(S,T)}. \tag{3.7}$$

Furthermore, for any constant $\kappa \geq 0$ and $g \in \mathbb{L}_p^q(T_0)$,

$$E \exp(\kappa \int_0^{T_0} |g(t, X_t)|^2 dt) < \infty. \tag{3.8}$$

Proof. By Lemma 3.1 we obtain that there exists a unique $(\mathcal{F}_t)$-adapted $\mathbb{R}^d$-valued process $(M_t)_{t \geq 0}$ such that $M_t = x + \int_0^t \sigma(s, M_s) dW_s$, $t \geq 0$. For any p_1, $q_1 \in (1, \infty)$ satisfying

$$\frac{d}{p_1} + \frac{2}{q_1} < 2,$$

Lemma 3.2 implies that for $0 < S < T \leq T_0$, and $f \in \mathbb{L}_{p_1}^{q_1}(S, T)$

$$E \left(\int_S^T |f(t, M_t)| dt \middle| \mathcal{F}_S \right) \leq N \|f\|_{\mathbb{L}_{p_1}^{q_1}(S,T)}, \tag{3.9}$$

where N depends only on d, K, p_1, q_1, T_0. Applying (3.9) to $f = |g|^2$ we get

$$E \left(\int_S^T |g(t, M_t)|^2 dt \middle| \mathcal{F}_S \right) \leq N \|g^2\|_{\mathbb{L}_{p/2}^{q/2}(S,T)} = N \|g\|^2_{\mathbb{L}_p^q(S,T)}.$$

By Lemma A.1, for any $\kappa \in [0, \infty)$ we have

$$E \exp(\kappa \int_0^{T_0} |g(t, M_t)|^2 dt) \leq N(\kappa, K, d, p, q, T_0, \|g\|_{\mathbb{L}_p^q(T_0)}). \tag{3.10}$$

And also

$$E \exp(\kappa \int_0^{T_0} |b(t, M_t)|^2 dt) \leq N(\kappa, K, K_0, d, p, q, T_0). \tag{3.11}$$

The integral over $(0,T_0)$ in (3.11) can be replaced with the one over $(0,\infty)$ since $b(t,x)=0$ for $t>T_0$. Thus for any $\kappa\in[0,\infty)$

$$E\exp(\kappa\int_0^\infty |b(t,M_t)|^2dt)<\infty, \tag{3.12}$$

which and (3.6) implies that for any $c\in[0,\infty)$

$$E\exp(c\int_0^\infty (b^*(\sigma\sigma^*)^{-1}b)(t,M_t)dt)\le E\exp(\frac{c}{\delta}\int_0^\infty |b(t,M_t)|^2dt)<\infty. \tag{3.13}$$

For $f\in\mathbb{L}_{p'}^{q'}(S,T)$ with p', $q'\in(1,\infty)$, we can choose $\beta>1$ sufficiently close to 1 such that

$$\frac{d}{p'}+\frac{2}{q'}<\frac{2}{\beta}.$$

By Lemma 3.1 we obtain the existence and uniqueness of $(\mathcal{F}_t)$-adapted process $(X_t)_{t\ge0}$ which satisfies (3.2) and (3.3). By Lemma A.3, we have

$$\begin{aligned} E\int_S^T |f(t,X_t)|dt = E\int_S^T \rho|f(t,M_t)|dt &\le (E\int_S^T \rho^\alpha dt)^{1/\alpha}(E\int_S^T |f(t,M_t)|^\beta dt)^{1/\beta}\\ &\le (E\int_0^{T_0} \rho^\alpha dt)^{1/\alpha}(E\int_S^T |f(t,M_t)|^\beta dt)^{1/\beta}, \end{aligned} \tag{3.14}$$

where α, $\beta>1$ satisfying $1/\alpha+1/\beta=1$, and

$$\rho:=\exp(-\int_0^\infty b^*(\sigma^*)^{-1}(s,M_s)dW_s-\frac12\int_0^\infty (b^*(\sigma\sigma^*)^{-1}b)(s,M_s)ds).$$

Since

$$\begin{aligned} E\rho^\alpha = E\Big[&\Big(\exp(-2\alpha\int_0^\infty b^*(\sigma^*)^{-1}(s,M_s)dW_s-2\alpha^2\int_0^\infty (b^*(\sigma\sigma^*)^{-1}b)(s,M_s)ds)\Big)^{1/2}\\ &\Big(\exp((2\alpha^2-\alpha)\int_0^\infty (b^*(\sigma\sigma^*)^{-1}b)(s,M_s)ds)\Big)^{1/2}\Big], \end{aligned} \tag{3.15}$$

by Hölder's inequality and the fact that exponential martingale is a supermartingale and (3.13), we get

$$E\rho^\alpha\le N. \tag{3.16}$$

Then

$$\begin{aligned} E\int_S^T |f(t,X_t)|dt &\le N(T_0)(E\int_S^T |f(t,M_t)|^\beta dt)^{1/\beta}\\ &\le N(d,p_1,q_1,K,\|b\|_{\mathbb{L}_p^q(T_0)})\|f^\beta\|^{1/\beta}_{\mathbb{L}_{p_1}^{q_1}(S,T)}\\ &= N(d,p_1,q_1,K,\|b\|_{\mathbb{L}_p^q(T_0)})\|f\|_{\mathbb{L}_{\beta p_1}^{\beta q_1}(S,T)} \end{aligned}$$

for $d/p_1+2/q_1<2$, where $p_1=p'/\beta$, $q_1=q'/\beta$. Thus the above estimate implies (3.7).

Furthermore, according to Lemma A.3 and (3.10),

$$E\exp(\kappa\int_0^{T_0}|g(t,X_t)|^2dt)=E(\rho\exp(\kappa\int_0^{T_0}|g(t,M_t)|^2dt))$$
$$\leq(E\rho^2)^{1/2}(E\exp(2\kappa\int_0^{T_0}|g(t,M_t)|^2dt))^{1/2}<\infty.$$

□

Lemma 3.4. *Let $b^{(i)}(t,x)$, $i=1,2$ and σ satisfy Assumption 2 and let $|b^{(1)}(t,x)-b^{(2)}(t,x)|\leq\bar{b}(t,x)$, where $\bar{b}$ also satisfies Assumption 2. Let $(X_t^{(i)},W_t^{(i)})_{t\geq0}$ satisfy*

$$X_t^{(i)}=x+\int_0^t b^{(i)}(s,X_s^{(i)})ds+\int_0^t\sigma(s,X_s^{(i)})dW_s^{(i)},\quad t\geq0.$$

Then for any bounded Borel functions $f^{(i)}$, $i=1,2$ given on $\mathcal{C}:=\mathcal{C}([0,\infty),\mathbb{R}^d)$ we have

$$|Ef^{(1)}(X_\cdot^{(1)})-Ef^{(2)}(X_\cdot^{(2)})|\leq N(E|f^{(1)}(M_\cdot)-f^{(2)}(M_\cdot)|^2)^{1/2}+N\sup_{\mathcal{C}}|f^{(1)}|\|\bar{b}\|_{\mathbb{L}_p^q}\tag{3.17}$$

where $M_t=\int_0^t\sigma(s,M_s)dW_s$, $t\geq0$, and N is a constant independent of f.

Proof. According to Lemma A.3, we know that

$$Ef^{(2)}(X_\cdot^{(2)})=Ef^{(2)}(X_\cdot^{(1)})\bar{\rho}_\infty,$$

where for $t\geq0$, $\Delta b(t,X_t^{(1)}):=b^{(2)}(t,X_t^{(1)})-b^{(1)}(t,X_t^{(1)})$ and

$$\bar{\rho}_\infty:=\exp(\int_0^\infty\Delta b^*(\sigma^*)^{-1}(s,X_s^{(1)})dW_s^{(1)}-\frac{1}{2}\int_0^\infty(\Delta b^*(\sigma\sigma^*)^{-1}\Delta b)(s,X_s^{(1)})ds),$$

also $E\bar{\rho}_\infty=1$ by applying (3.8) and the fact that $\Delta b(t,x)=0$ for $t>T_0$ and (3.6). Hence the left-hand side of (3.17) is less than

$$E|f^{(1)}-f^{(2)}|(X_\cdot^{(1)})\bar{\rho}_\infty+\sup_{\mathcal{C}}|f^{(1)}|E|\bar{\rho}_\infty-1|=:I_1+I_2\sup_{\mathcal{C}}|f^{(1)}|.$$

Also we have that all moments of the exponential martingale

$$\bar{\rho}_t=\exp(\int_0^t\Delta b^*(\sigma^*)^{-1}(s,X_s^{(1)})dW_s^{(1)}-\frac{1}{2}\int_0^t(\Delta b^*(\sigma\sigma^*)^{-1}\Delta b)(s,X_s^{(1)})ds)$$

are finite by the same argument as getting (3.16) in Lemma 3.3. Hence we get

$$I_1^{3/2}\leq NE|f^{(1)}-f^{(2)}|^{3/2}(X_\cdot^{(1)})\tag{3.18}$$

and right hand side of (3.18) is controled by the first term on the right hand side of (3.17) by a similar argument as dealing with (3.14) in Lemma 3.3. To estimate I_2, we use Itô's formula to get for any $T\in[0,\infty)$,

$$\bar{\rho}_T=1+\int_0^T(\Delta b^*(\sigma^*)^{-1})(s,X_s^{(1)})\bar{\rho}_sdW_s^{(1)}.$$

It follows that for any $\beta > 1$

$$\begin{aligned}
I_2^2 \leq & E|\overline{\rho}_{T_0} - 1|^2 \\
\leq & E \int_0^{T_0} (\Delta b^* (\sigma\sigma^*)^{-1} \Delta b)(s, X_s^{(1)}) \overline{\rho}_s^2 ds \\
\leq & N(\int_0^{T_0} E\overline{\rho}_s^{2\beta/(\beta-1)} ds)^{1-1/\beta} (E \int_0^{T_0} |\Delta b(s, X_s^{(1)})|^{2\beta} ds)^{1/\beta} \\
\leq & N(\int_0^{T_0} E\overline{\rho}_s^{2\beta/(\beta-1)} ds)^{1-1/\beta} (E \int_0^{T_0} |\overline{b}(s, X_s^{(1)})|^{2\beta} ds)^{1/\beta}.
\end{aligned} \tag{3.19}$$

To estimate the second factor of the the right hand of (3.19) we use Lemma 3.3 with $\beta > 1$ close to 1 such that $2/q + d/p < 1/\beta$. The first factor of the the right hand of (3.19) is controlled by means of $E\overline{\rho}_{T_0}^{2\beta/(\beta-1)}$. Thus the result follows. □

3.2 Proof of Theorem 2.1

Now we are going to prove the maximal local well-posedness result on an arbitrary domain $Q \subset \mathbb{R}_+ \times \mathbb{R}^d$ by applying the localization technique, which is a modification of the proof of Theorem 1.3 in [23]. Furthermore we will prove the continuity of the solution on the domain $Q' = Q \cup \partial$, especially around the boundary $\partial Q'$.

Proof of Theorem 2.1. By Lemma A.2, for each $n \in \mathbb{N}$, we can find a nonnegative smooth function $\chi_n(t, x) \in [0, 1]$ in $\mathbb{R}^{d+1}$ such that $\chi_n(t, x) = 1$ for all $(t, x) \in Q^n$ and $\chi_n(t, x) = 0$ for all $(t, x) \notin Q^{n+1}$. For any $s, x \in Q$, let

$$b_s^n(t, x) := \chi_n(t + s, x) b(t + s, x)$$

and

$$\sigma_s^n(t, x) := \chi_{n+1}(t + s, x)\sigma(t + s, x) + (1 - \chi_n(t + s, x))(1 + \sup_{(t+s,x)\in Q^{n+2}} |\sigma(t + s, x)|)\mathbb{I}_{d\times d}.$$

By Lemma 3.1 there exists a unique $(\mathcal{F}_t)$-adapted continuous solution $(X_t^n)_{t\geq 0}$ satisfying

$$X_t^n = x + \int_0^t b_s^n(r, X_r^n) dr + \int_0^t \sigma_s^n(r, X_r^n) dW_r, \quad \forall t \in [0, \infty), a.s. \tag{3.20}$$

More precisely, for condition (i) in Lemma 3.1, for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$,

$$|b_s^n(t, x)| \leq |(bI_{Q^{n+2}})(s + t, x)|,$$

$$\begin{aligned}
|\nabla\sigma_s^n(t, x)| &\leq |(\nabla\chi_{n+1}\sigma)(t + s, x)| + |(\chi_{n+1}\nabla\sigma)(t + s, x)| + c|\nabla\chi_n(t + s, x)| \\
&\leq |(\nabla\chi_{n+1}\sigma I_{Q^{n+2}})(t + s, x)| + |(\nabla\sigma I_{Q^{n+2}})(t + s, x)| + c|\nabla\chi_n(t + s, x)|,
\end{aligned}$$

with constant $c > 0$, which means that we can take $p := p_{n+2}$, $q := q_{n+2}$. The continuity condition in Lemma 3.1 (ii) obviously holds. Further there exist constants $K(n)$ and δ_{n+1} such that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$,

$$|(\sigma_s^n)^*(t, x)\lambda|^2 \leq |(\sigma^* I_{Q^{n+2}} + (1 + \sup_{(s+t,x)\in Q^{n+2}} |\sigma^*(s + t, x)|)\mathbb{I}_{d\times d})(s + t, x)\lambda|^2 \leq K(n)|\lambda|^2,$$

and

$$
\begin{aligned}
|(\sigma_s^n)^*(t,x)\lambda|^2 \geq& |(\sigma^* I_{Q^{n+1}} + I_{(Q^{n+1})^c \cap Q^{n+2}} \\
&+ I_{(Q^{n+2})^c}(1 + \sup_{(s+t,x)\in Q^{n+2}} |\sigma^*(s+t,x)|) I_{d\times d})(s+t,x)\lambda|^2 \\
\geq& (\delta_{n+1} \wedge 1)|\lambda|^2.
\end{aligned}
$$

Thus equation (3.20) satisfies conditions (i) and (ii) in Lemma 3.1. For $n \geq k$, define

$$
\tau_{n,k} := \inf\left\{t \geq 0 : z_t^n := (s+t, X_t^n) \notin Q^k\right\},
$$

then it is easy to see that X_t^n, X_t^k, $t \geq 0$, satisfy

$$
X_{t\wedge\tau_{n,k}} = x + \int_0^{t\wedge\tau_{n,k}} b_s^k(r, X_r)dr + \int_0^{t\wedge\tau_{n,k}} \sigma_s^k(r, X_r)dW_r, \quad a.s..
$$

By the local uniqueness of the solution in Lemma 3.1, we have

$$
P\left\{\omega : X_t^n(\omega) = X_t^k(\omega), \forall t \in [0, \tau_{n,k}(\omega))\right\} = 1,
$$

which implies $\tau_{k,k} \leq \tau_{n,k} \leq \tau_{n,n}$ $a.s.$. Thus if we take $\xi_k := \tau_{k,k}$, then ξ_k is an increasing sequence of stopping times, and

$$
P\left\{\omega : X_t^n(\omega) = X_t^k(\omega), \forall t \in [0, \xi_k(\omega))\right\} = 1.
$$

Now for each $k \in \mathbb{N}$, the definitions

$$
X_t(\omega) := X_t^k(\omega) \text{ for } t < \xi_k, \quad \xi := \lim_{k\to\infty} \xi_k,
$$

and

$$
z_t = (s+t, X_t), \quad t < \xi, \quad z_t = \partial, \quad \xi \leq t < \infty, \tag{3.21}
$$

make sense almost surely. We may throw the set of ω where the above definitions do not make sense and work only on the remaining part of Ω. Then $(X_t)_{t\geq 0}$ satisfies the SDE (2.1) and ξ is the related explosion time.

The last thing is to prove that $(z_t)_{t\geq 0}$ from (3.21) is continuous on Q'. Since z_t coincides with (t, X_t^n) before ξ_n, the continuity of z_t before ξ_n follows from the continuity of (t, X_t^n), which can be obtained by Lemma 3.1. So we only need to show that z_t is left continuous at ξ $a.s.$. The argument essentially follows from [7].

We first show that $(z_t)_{t\geq 0}$ has strong Markov property. We use $P_{s,x}^n$ to denote the distribution of process $(z_t^n)_{t\geq 0} = (z_t^n(s,x))_{t\geq 0} := (s+t, X_t^n(s,(0,x)))_{t\geq 0}$ on $\mathcal{C}([0,\infty), \mathbb{R}^{d+1})$, where $(X_t^n(s,(0,x)))_{t\geq 0}$ means the solution $(X_t^n)_{t\geq 0}$ to (3.20) defined above with initial point $(0,x) \in \mathbb{R}^{d+1}$. $E_{s,x}^n$ denotes the expectation corresponding to $P_{s,x}^n$. The following argument is based on Proposition 4.3.3 of [10].

Define the space $\mathbb{W}_0 := \left\{w \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) | w(0) = 0\right\}$ equipped with the supremum norm and Borel σ-algebra $\mathcal{B}(\mathbb{W}_0)$, the class $\mathcal{E}$ collects all the maps $F : \mathbb{R}^d \times \mathbb{W}_0 \to \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ such that for every probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ there exists a $\overline{\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{W}_0)}^{\mu\times P^W}$ $/\mathcal{B}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d))$ measurable map $F_\mu : \mathbb{R}^d \times \mathbb{W}_0 \to \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ such that for μ-a.e. $x \in \mathbb{R}^d$ we have $F(x,w) = F_\mu(x,w)$ for P^W-a.e. $w \in \mathbb{W}_0$. Here $\overline{\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{W}_0)}^{\mu\times P^W}$ means the completion of $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{W}_0)$ with respect to $\mu \times P^W$, and P^W denotes the distribution of the standard d-dimensional Wiener process $(W_t)_{t\geq 0}$ on $(\mathbb{W}_0, \mathcal{B}(\mathbb{W}_0))$. For each $n \in \mathbb{N}$, since we already have the pathwise uniqueness and existence of strong solution $(X_t^n)_{t\geq 0}$ to (3.20), by applying Theorem E.8 in [10], we obtain that there exists a map $F \in \mathcal{E}$ such that for $u \leq t$ we have $X_t^n(s,(0,x))(\omega) = F_{P\circ(X_u^n(s,(0,x)))^{-1}}(X_u^n(s,(0,x))(\omega), (W_\cdot - W_u)(\omega))(t)$ for P-a.e. $\omega \in \Omega$. Then for every bounded measurable function h defined on $\mathbb{R}^d$ and all $u,t \in [0,\infty)$ with $u \leq t$ we have for P-a.e. $\omega \in \Omega$

$$
\begin{aligned}
E[h(X_t^n(s,(0,x)))|\mathcal{F}_u](\omega) &= E[h(F_{P\circ(X_u^n(s,(0,x)))^{-1}}(X_u^n(s,(0,x))(\omega), W_\cdot - W_u)(t))] \\
&= E[h(F_{\delta_{X_u^n(s,(0,x))(\omega)}}(X_u^n(s,(0,x))(\omega), W_\cdot - W_u)(t))] \\
&= E[h(X_t^n(s,(u,X_u^n(s,(0,x))))(\omega))], \qquad (3.22)
\end{aligned}
$$

which shows the Markov property of the process $(X_t^n)_{t\geq 0}$. Here $X_t^n(s,(u,X_u^n(s,(0,x))))$ means the solution $(X_t^n)_{t\geq 0}$ to (3.20) with starting point $(u, X_u^n(s,(0,x))) \in \mathbb{R}^{d+1}$. Combining with the Feller property of $(X_t^n)_{t\geq 0}$ yielding from the second statement of Lemma 3.1 and well known results about Markov processes (see e.g. [1, Theorem 16.21]), we get that $(X_t^n)_{t\geq 0}$ is a strong Markov process.

Now we are going to prove that $(z_t^n)_{t\geq 0}$ is a strong Markov process. Observing that for $u \geq 0$, $(\hat{W}_t)_{t\geq 0} := (W_{t+u} - W_u)_{t\geq 0}$ is still a Brownian motion. For any $(s,x) \in Q$, and for any Borel bounded function f on $\mathbb{R}^{d+1}$, by (3.22), we have for any $u,t \geq 0$, P-a.e.

$$
\begin{aligned}
&X_{t+u}^n(s,(u,X_u^n(s,(0,x)))) \\
&\quad = X_u^n(s,(0,x)) + \int_u^{u+t} \sigma_s^n(r, X_r^n(s,(0,x)))d(W_r - W_u) \\
&\qquad\qquad + \int_u^{u+t} b_s^n(r, X_r^n(s,(0,x)))dr \\
&\quad = X_u^n(s,(0,x)) + \int_0^t \sigma_s^n(r+u, X_{u+r}^n(s,(0,x)))d\hat{W}_r \\
&\qquad\qquad + \int_0^t b_s^n(r+u, X_{r+u}^n(s,(0,x)))dr,
\end{aligned}
$$

and

$$
\begin{aligned}
&X_t^n(s+u,(0,X_u^n(s,(0,x)))) \\
&\quad = X_u^n(s,(0,x)) + \int_0^t \sigma_s^n(u+r, X_r^n(u+s,(0,X_u^n(s,(0,x)))))d\hat{W}_r \\
&\qquad\qquad + \int_0^t b_s^n(u+r, X_r^n(u+s,(0,X_u^n(s,(0,x)))))dr \\
&\quad = X_u^n(s,(0,x)) + \int_0^t \sigma_{s+u}^n(r, X_r^n(u+s,(0,X_u^n(s,(0,x)))))d\hat{W}_r \\
&\qquad\qquad + \int_0^t b_{s+u}^n(r, X_r^n(u+s,(0,X_u^n(s,(0,x)))))dr.
\end{aligned}
$$

Since $\sigma_s^n(u+r,\cdot) = \sigma_{s+u}^n(r,\cdot)$, and $b_s^n(u+r,\cdot) = b_{s+u}^n(r,\cdot)$, by the pathwise uniqueness of the the following equation

$$
dX_t = \sigma_{s+u}^n(t,X_t)d\hat{W}_t + b_{s+u}^n(t,X_t)dt, \quad X_0 = X_u^n(s,(0,x)),
$$

we have for arbitrary Borel bounded function h on $\mathbb{R}^d$, $Eh(X^n_{t+u}(s,(u,X^n_u(s,(0,x))))) = Eh(X^n_t(s+u,(0,X^n_u(s,(0,x)))))$. Hence for arbitrary Borel bounded function f on $\mathbb{R}^{d+1}$ and for $P-a.e.\ \omega\in\Omega$,

$$\begin{aligned}E[f(z^n_{t+u}(s,x))|\mathcal{F}_u](\omega) &= E[f(s+t+u,X^n_{t+u}(s,(0,x)))|\mathcal{F}_u](\omega)\\ &= E[f(s+t+u,X^n_{t+u}(s,(u,X^n_u(s,(0,x))))(\omega))]\\ &= E[f(s+t+u,X^n_t(s+u,(0,X^n_u(s,(0,x)))))(\omega))]\\ &= E^n_{z^n_u(s,x)(\omega)}f(z^n_t).\end{aligned}$$

So $(z^n_t)_{t\geq 0}$ is a Markov process. Furthermore, for any $(s,x)\in Q$, by applying Ito's formula to process $X^n_r(s,(0,x))$, we get that $u^n_s(t,x)=Eh(X^n_t(s,(0,x)))$ is the solution to the following equation

$$\begin{cases} D_r u^n_s(r,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a^n_{s,ij}(r,x)\partial_i\partial_j u^n_s(r,x) + b^n_s(r,x)\cdot\nabla u^n_s(r,x) \text{ on } (0,\infty)\times\mathbb{R}^d,\\ u^n_s(0,x)=h(x),\end{cases}\tag{3.23}$$

with $(a^n_{s,ij})_{1\leq i,j\leq d}=\sigma^n_s\cdot(\sigma^n_s)^*$, and Borel bounded continuous function h defined on $\mathbb{R}^d$. Let $u^n(t,x)$ be the solution to the following equation

$$\begin{cases} D_r u^n(r,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a^n_{ij}(r,x)\partial_i\partial_j u^n(r,x) + b^n(r,x)\cdot\nabla u^n(r,x) \text{ on } (s,\infty)\times\mathbb{R}^d,\\ u^n(s,x)=h(x),\end{cases}\tag{3.24}$$

with $(a^n_{ij})_{1\leq i,j\leq d}=\sigma^n\cdot(\sigma^n)^*$, and σ^n and b^n are defined as following

$$b^n(r,x):=b^n_0(r,x),\quad \sigma^n(r,x):=\sigma^n_0(r,x).$$

Then it is easy to see that $u^n(s+t,x)$ also satisfies (3.23), which by using uniqueness of solution to (3.23) implies $u^n_s(t,x)=u^n(s+t,x)=Eh(X^n_t(s,(0,x)))$. By Remark 10.4 [7] (or see Theorem 3.1 [19]), we know that the unique solution $u^n(t,x)$ to the above equation (3.24) is continuous on $(t,x)\in[s,\infty)\times\mathbb{R}^d$, which yields the continuity of $Eh(X^n_t(s,(0,x)))$ with respect to $(s,x)\in[0,\infty)\times\mathbb{R}^d$ for any $t\in[0,\infty)$. Then the second statement of Lemma 3.1 and dominated convergence theorem imply that for any Borel bounded continuous function g on $\mathbb{R}^{d+1}$, and for any $(s,x)\in[0,\infty)\times\mathbb{R}^d$

$$\begin{aligned}\lim_{(u,y)\to(s,x)} E^n_{u,y}g(z^n_t) &= \lim_{(u,y)\to(s,x)} Eg(u+t,X^n_t(u,(0,y)))\\ &= \lim_{(u,y)\to(s,x)}\Big(Eg(u+t,X^n_t(u,(0,y)))-Eg(u+t,X^n_t(u,(0,x)))\Big)\\ &\quad+\lim_{(u,y)\to(s,x)}\Big(Eg(u+t,X^n_t(u,(0,x)))-Eg(s+t,X^n_t(u,(0,x)))\Big)\\ &\quad+\lim_{(u,y)\to(s,x)} Eg(s+t,X^n_t(u,(0,x)))\\ &\leq \lim_{(u,y)\to(s,x)} C_t\|g(u+t,\cdot)\|_\infty|x-y| + Eg(s+t,X^n_t(s,(0,x)))\\ &= Eg(t+s,X^n_t(s,(0,x))) = E^n_{s,x}g(z^n_t).\end{aligned}$$

It shows that $(z^n_t)_{t\geq 0}$ also has Feller property, hence $(z^n_t)_{t\geq 0}$ is a strong Markov process. Then for any $(s,x)\in Q$, for any $(\mathcal{F}_t)$-adapted stopping time η and for any Borel bounded function f on $\mathbb{R}^{d+1}$,

$$E_{s,x}f(z_{\eta+t}) = f(\partial) + E_{s,x}(f(z_{\eta+t}) - f(\partial))I_{\xi>\eta+t}. \tag{3.25}$$

Since

$$\begin{aligned} E_{s,x}f(z_{\eta+t})I_{\xi>\eta+t} &= \lim_{n\to\infty} E_{s,x}f(z_{\eta+t})I_{\xi_n\geq\eta+t} \\ &= \lim_{n\to\infty} E^n_{s,x}f(z^n_{\eta+t})I_{\xi_n\geq\eta+t}I_{\xi_n\geq\eta} \\ &= \lim_{n\to\infty} E^n_{s,x}f(\eta+t, X^n_{\eta+t})I_{\xi_n\geq\eta+t}I_{\xi_n\geq\eta}, \end{aligned}$$

and $\{\xi_n\geq\eta\}\subset\mathcal{F}_\eta$, by the strong Markov property of $(z^n_t)_{t\geq 0}$, we get

$$\begin{aligned} \lim_{n\to\infty} E^n_{s,x}I_{\xi_n\geq\eta}E^n_{(\eta,X^n_\eta)}f(t,X^n_t)I_{\xi_n\geq\eta} &= \lim_{n\to\infty} E^n_{s,x}I_{\xi_n\geq\eta}E^n_{z^n_\eta}f(z^n_t)I_{\xi_n\geq\eta} \\ &= E_{s,x}I_{\xi>\eta}E_{(\eta,X_\eta)}f(t,X_t)I_{\xi>\eta}. \\ &= E_{s,x}I_{\xi>\eta}E_{z_\eta}f(z_t)I_{\xi>\eta} \end{aligned}$$

Then (3.25) yields

$$E_{s,x}f(z_{\eta+t}) = E_{s,x}E_{z_\eta}f(z_t). \tag{3.26}$$

We can find that (3.26) also holds if we replace (s,x) with ∂. Hence we get the strong Markov property of the process $(z_t)_{t\geq 0}$.

In the following we will prove another two auxiliary lemmas in order to show that our solution does not bounce back deep into the interior of Q from near ∂Q too often on any finite interval of time, which is crucial for us to prove the desired continuity. By shifting the origin in $\mathbb{R}^{d+1}$, without losing generality, we assume $(s,x)=(0,0)$.

Lemma 3.5. *For arbitrary $n\geq 0$, define $\nu_0=0$,*

$$\mu_k = \inf\left\{t\geq\nu_k : (t,X_t)\notin Q^{n+1}\right\}, \quad \nu_{k+1} = \inf\left\{t\geq\mu_k : (t,X_t)\in\overline{Q^n}\right\}. \tag{3.27}$$

Then for any $S\in(0,\infty)$ there exists a constant N, depending only on d, p, q, S, $\|bI_{Q^{n+1}}\|_{\mathbb{L}^q_p}$, $\sup_{(t,x)\in Q^{n+1}}|\sigma(t,x)|$, and the diameter of Q^{n+1}, such that

$$\sum_{k=0}^{\infty}(E|X_{S\wedge\mu_k}-X_{S\wedge\nu_k}|^2)^2\leq N, \quad \sum_{k=0}^{\infty}(E|S\wedge\mu_k - S\wedge\nu_k|^2)^2\leq S^4.$$

Proof. We have $E|X_{S\wedge\mu_k}-X_{S\wedge\nu_k}|^2\leq 2I_k+2J_k$, where

$$I_k := E|\int_{S\wedge\nu_k}^{S\wedge\mu_k}\sigma(s,X_s)dW_s|^2, \quad J_k := E|\int_{S\wedge\nu_k}^{S\wedge\mu_k}b(s,X_s)ds|^2.$$

Observe that on the set $\{S\wedge\nu_k < S\wedge\mu_k\}$ we have $S\wedge\nu_k=\nu_k$ and $(\nu_k, X_{\nu_k})\in\overline{Q^n}\subset Q^{n+1}$. Furthermore, $(t,X_t)\in Q^{n+1}$ for $S\wedge\nu_k<t<S\wedge\mu_k$, and we have

$$E|\int_{S\wedge\nu_k}^{S\wedge\mu_k}\sigma(s,X_s)dW_s|^2\leq\sum_{i,j=1}^{d}E|\int_{S\wedge\nu_k}^{S\wedge\mu_k}\sigma^2_{ij}(s,X_s)ds|\leq Cd^2E|S\wedge\mu_k - S\wedge\nu_k|,$$

$$I_k^2 \le Cd^4 E|S\wedge\mu_k - S\wedge\nu_k|^2 =: Cd^4 \bar I_k \le Cd^4 SE|S\wedge\mu_k - S\wedge\nu_k|,$$

$$\sum_{k=0}^{\infty}(E|\int_{S\wedge\nu_k}^{S\wedge\mu_k}\sigma(s,X_s)dW_s|^2)^2 \le Cd^4S^2, \quad \sum_{k=0}^{\infty}(\bar I_k)^2 \le (\sum_{k=0}^{\infty}\bar I_k)^2 \le S^4.$$

Moreover, by Hölder's inequality we have

$$J_k \le E|S\wedge\mu_k - S\wedge\nu_k|\int_{S\wedge\nu_k}^{S\wedge\mu_k}|b(s,X_s)|^2ds, \quad J_k^2 \le \bar I_k \bar J_k,$$

where

$$\bar J_k := E(\int_{S\wedge\nu_k}^{S\wedge\mu_k}|b(s,X_s)|^2ds)^2.$$

Let $\tau_n := \inf\{t\ge 0 : z_t \notin Q^n\}$. By the strong Markov property of $(z_t)_{t\ge 0}$, it follows that

$$\bar J_k \le \sup_{(s,x)\in Q^{n+1}} E_{s,x}(\int_0^{S\wedge\tau_{n+1}}|b(s+t,X_t)|^2dt)^2 = \sup_{(s,x)\in Q^{n+1}} E_{s,x}(\int_0^{S}|bI_{Q^{n+1}}(s+t,X_t)|^2dt)^2, \tag{3.28}$$

Since for $t\le\tau_{n+1}$, $X_t = X_t^{n+1}$, we see that the second right part of (3.28) will not change if we change arbitrarily b outside of Q^{n+1} only preserving the property that new b belongs to $\mathbb{L}_p^q$. We choose to let b be zero outside of Q^{n+1} and then get the desired estimate from (3.8). The lemma is proved.

□

Following the same argument in [7, Corollary 4.3] and use Lemma 3.5 we get the following result. In order to reduce duplicate, the proof is omitted.

Lemma 3.6. *We say that on the time interval $[\nu_k,\mu_k]$ the trajectory $(t,X_t)_{t\ge0}$ makes a run from $\overline{Q^n}$ to $(Q^{n+1})^{\mathsf{c}}$ provided that $\mu_k<\infty$. Denote by $\nu(S)$ the number of runs which $(t,X_t)_{t\ge0}$ makes from $\overline{Q^n}$ to $(Q^{n+1})^{\mathsf{c}}$ before time S. Then for any $\alpha\in[0,1/2)$, $E\nu^\alpha(S)$ is dominated by a constant N, which depends only on α, d, p, q, S, $\|bI_{Q^{n+1}}\|_{\mathbb{L}_p^q}$, $\sup_{(t,x)\in Q^{n+1}}|\sigma(t,x)|$, the diameter of Q^{n+1}, and the distance between the boundaries of Q^n and Q^{n+1}.*

Now we go back to prove that z_t is left continuous at ξ *a.s.*. We denote $\nu_k(S)$ the number of runs of z_t from $\overline{Q^k}$ to $(Q^{k+1})^{\mathsf{c}}$ before $S\wedge\xi$. For $n>k+1$ obviously, $\nu_k(S\wedge\xi_n)$ is also the number of runs that $(t,X_t^n)_{t\ge0}$ makes from $\overline{Q^k}$ to $(Q^{k+1})^{\mathsf{c}}$ before $S\wedge\xi_n$, since (t,X_t) coincides with (t,X_t^n) before ξ_n. $\nu_k(S\wedge\xi_n)$ increase if we increase the time interval to S. By Lemma 3.6 $E\nu_k^{1/4}(S\wedge\xi_n)$ is bounded by a constant independent of n. By Fatou's Lemma $E\nu_k^{1/4}(S\wedge\xi)$ is finite. In particular, on the set $\{\omega:\xi(\omega)<\infty\}$ *a.s.* we have $\nu_k(\xi)<\infty$. The latter also holds on the set $\{\omega:\xi(\omega)=\infty\}$ because z_t is continuous on $[0,\xi)$ and Q^k is bounded. Thus $\nu_k(\xi)<\infty$ *a.s.* for any k. Since $(\xi^n, X^n_{\xi^n})\in\partial Q^n$ we conclude that *a.s.* there can exist only finitely many n such that z_t visits $\overline{Q^k}$ after exiting from Q^n. This is the same as to say that $z_t\to\partial$ as $t\uparrow\xi$ *a.s.*.

About the uniqueness, if there is another continuous $(\mathcal{F}_t)$-adapted Q'-valued solution $(z'_t)_{t\ge0} = (s+t,X'_t)_{t\ge0}$ to the SDE (2.1) with explosion time ξ', and for $t<\xi'$ it is Q-valued. Then for any $n\ge1$

$$\tau^n(X'_\cdot) := \inf\{t\ge0 : (s+t,X'_t)\notin Q^n\} < \xi' \tag{3.29}$$

and

$$\bar{\xi} := \lim_{n\to\infty} \tau^n(X'_\cdot) = \xi' \quad a.s.. \tag{3.30}$$

Precisely $\bar{\xi} \le \xi'$ by (3.29). On the other hand, on the set where $\bar{\xi} < \xi'$, we have $z'_{\bar{\xi}} \in Q$ since $\bar{\xi} < \xi'$, we also have $z'_{\bar{\xi}} = \partial$ since $z'_{\bar{\xi}}$ is the limit of points getting outside of any Q^n. Observe that before $\tau^n(X'_\cdot)$, X'_t also satisfies the SDE (3.20), by the local strong uniqueness of equation (3.20) proved by Lemma 3.1, we get $X^n_t = X'_t$ for $t \le \tau^n(X'_\cdot)$, so $\tau^n(X'_\cdot) = \tau_{n,n}$. And by (3.30) we see that

$$\xi' = \bar{\xi} = \lim_{n\to\infty} \tau^n(X'_\cdot) = \lim_{n\to\infty} \tau_{n,n} = \xi \quad a.s.,$$

which implies that for $t \le \xi = \xi'$, and z'_t coincides with z_t from our above construction (3.21). □

4 Preparations of the proof of Theorem 2.4

4.1 Probabilistic representation of solutions to parabolic partial differential equations

In this subsection, we give a probabilistic representation of the solution to the following backward parabolic partial differential equation with a potential term $V(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$,

$$\begin{cases} D_t u(t,x) + \mathcal{L}u(t,x) + V(t,x)u(t,x) = 0, \quad 0 \le t \le T, \\ \qquad\qquad\qquad\qquad\qquad\qquad u(T,x) = f(x). \end{cases} \tag{4.1}$$

Here $T \in (0,\infty)$ and

$$\mathcal{L}u(t,x) := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + b(t,x)\cdot \nabla u(t,x), \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1}),$$

where $(a_{ij})_{1\le i,j\le d} = \sigma\sigma^*$. We first give the assumptions which make the representation formula hold.

Assumption 3. *(i) For all $1 \le i,j \le d$, $\sigma_{ij} \in \mathcal{C}([0,T]\times\mathbb{R}^d)$,*
(ii) There exist positive constants K and δ such that for all $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\delta|\lambda|^2 \le |\sigma^*(t,x)\lambda|^2 \le K|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d,$$

(iii) b, $V \in \mathcal{C}_b([0,T]\times\mathbb{R}^d)$,
(iv) For all (t,x), $(s,y) \in [0,T]\times\mathbb{R}^d$, there exist constants C_1, C_2 and C_3 such that

$$\begin{aligned} |a_{ij}(t,x) - a_{ij}(s,y)| &\le C_1(|x-y| \vee |t-s|^{1/2}), \\ |b(t,x) - b(s,y)| &\le C_2(|x-y| \vee |t-s|^{1/2}), \\ |V(t,x) - V(s,y)| &\le C_3(|x-y| \vee |t-s|^{1/2}). \end{aligned}$$

(v) $f \in \mathcal{C}_c^2(\mathbb{R}^d)$.

Theorem 4.1. *If Assumption 3 holds, then there exists a unique solution $u(t,x)$ to the equation* (4.1) *and it can be represented by the following formula*

$$u(t,x) = E\Big[f(X(T,t,x))e^{\int_t^T V(u,X(u,t,x))du}\Big], \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{4.2}$$

where $X(T,t,x)$ is the solution to the SDE (1.1) *with initial point $(t,x) \in [0,T] \times \mathbb{R}^d$. Furthermore, for $t \in [0,T)$ we have*

$$u(t,\cdot), \quad D_t u(t,\cdot), \quad \nabla u(t,\cdot), \quad \nabla^2 u(t,\cdot) \in L^1(\mathbb{R}^d). \tag{4.3}$$

Proof. On one hand by classical results of partial differential equation (see [8, Theorem 5.1]), we know that under our assumption there exists a unique solution $u(t,x) \in \mathcal{C}^{1,2}([0,T],\mathbb{R}^d)$ to the equation (4.1), which can be written in the form of a potential with kernel k (see [8, (14.2)]):

$$u(t,x) = \int_{\mathbb{R}^d} k(T,y;t,x)f(y)dy, \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

satisfying

$$\lim_{t\to T} u(t,x) = \lim_{t\to T} \int_{\mathbb{R}^d} k(T,y;t,x)f(y)dy = f(x),$$

and for $s = 0,1,2$ there exists a constant C such that for $0 \le t < T$ (see [8, (13.1)])

$$\partial_x^s k(T,y;t,x) \le C(T-t)^{-\frac{d+s}{2}} \exp\Big(-C\frac{|y-x|^2}{T-t}\Big).$$

Then for $s = 0,1,2$, for $t \in [0,T)$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\partial_x^s u(t,x)|dx \le& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)\partial_x^s k(T,y;t,x)|dydx \\ \overset{Fubini}{=}& \int_{\mathbb{R}^d} |f(y)| \int_{\mathbb{R}^d} |\partial_x^s k(T,y;t,x)|dxdy \\ \le& C(T-t)^{-\frac{s}{2}} \int_{\mathbb{R}^d} |f(y)|dy < \infty,\end{aligned}$$

which implies that for $t \in [0,T)$,

$$u(t,\cdot), \quad \nabla u(t,\cdot), \quad \nabla^2 u(t,\cdot) \in L^1(\mathbb{R}^d). \tag{4.4}$$

Since b is bounded, we get $D_t u(t,\cdot) \in L^1(\mathbb{R}^d)$ following from the equation (4.1) and (4.4). On another hand, from our assumption we know that $\sigma\sigma^*$ is uniformly elliptic, $b(t,x)$ and $\sigma_{ij}(t,x)$, $1 \le i,j \le d$ are bounded for $(t,x) \in [0,T) \times \mathbb{R}^d$ and continuous in t and Lipschitz continuous in x, by a known result (eg. see [6, IV Theorem 2.2]) we get the existence and uniqueness of the global solution $(X_t)_{t\ge 0}$ to the SDE (1.1). Then by [12, Theorem 8.2.1] we get that (4.2) solves the equation (4.1). Hence combining these two sides we get the desired result and also (4.3) holds. □

4.2 Some auxiliary proofs

In order to show that under certain conditions our solutions will not blow up, we need some auxiliary proofs which we collect in this subsection. We fix an $T \in (0, \infty)$, for $t \in [0, T]$ define

$$Q_T := (0, T) \times \mathbb{R}^d, \quad B_r := \{x \in \mathbb{R}^d : |x| < r\}, \quad Q^{t,r} := [0, t) \times B_r.$$

Assumption 4. *(i)* ψ *is a nonnegative function defined on* $\mathbb{R}^{d+1}$ *and* $\psi \in \mathcal{C}_b^\infty(\mathbb{R}^{d+1})$,
(ii) $|\nabla\psi| \in \mathbb{L}_p^{q,loc}$ *with* $p, q \in (2, \infty)$ *and* $d/p + 2/q < 1$,
(iii) σ *satisfies the conditions in Assumption 2 (i),*
(iv) For all (t, x), $(s, y) \in [0, T] \times \mathbb{R}^d$, *there exist constants* K_0, $K \in [0, \infty)$ *such that for all* $1 \leq i, j \leq d$,

$$|a_{ij}(t, x) - a_{ij}(s, y)| \leq K(|x - y| \vee |t - s|^{1/2}),$$
$$|\partial_j a_{ij}(t, x) - \partial_j a_{ij}(s, y)| \leq K_0(|x - y| \vee |t - s|^{1/2}).$$

Let $(W_t)_{t\geq 0}$ be a d-dimensional Wiener process on a given complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, denote $(a_{ij})_{1\leq i,j\leq d} = \sigma\sigma^*$. Let $(s, x) \in [0, \infty) \times \mathbb{R}^d$, we introduce the process $(Y(t, s, x))_{t\geq s}$, satisfying

$$Y(t, s, x) = x + \int_s^t \sigma(r, Y(r, s, x))dW_r + (\frac{1}{2}\sum_{j=1}^d \int_s^t \partial_j a_{ij}(r, Y(r, s, x))dr)_{1\leq i\leq d}, \tag{4.5}$$

and process $(X(t, s, x))_{t\geq s}$ satisfying

$$X(t, s, x) = x + \int_s^t \sigma(r, X(r, s, x))dW_r + (\frac{1}{2}\sum_{j=1}^d \int_s^t \partial_j a_{ij}(r, X(r, s, x))dr)_{1\leq i\leq d} - \int_s^t (\sigma\sigma^*\nabla\psi)(r, X(r, s, x))dr. \tag{4.6}$$

Since for $1 \leq i, j \leq d$, $\partial_j a_{ij} = \sum_{k=1}^d \sigma_{ik}(\partial_j\sigma_{jk}) + \sum_{k=1}^d (\partial_j\sigma_{ik})\sigma_{jk}$, and $|\nabla\sigma| \in \mathbb{L}_p^{q,loc}$, from Assumption 4 (iii), we get $\sum_{j=1}^d |\partial_j a_{ij}| \in \mathbb{L}_p^{q,loc}$. Then Lemma 3.1 can be applied here to guarantee the existence and uniqueness of global $(\mathcal{F}_t)$-adapted solutions $(Y(t, s, x))_{t\geq s}$ and $(X(t, s, x))_{t\geq s}$ corresponding to SDEs (4.5) and (4.6) if Assumption 4 holds.

Lemma 4.2. *Let Assumption 4 be satisfied. Take a nonnegative Borel function* f *on* $\mathbb{R}^{d+1}$. *For* $t \in [0, T]$ *introduce*

$$\beta_T(t, x) = \exp(- \int_t^T \nabla\psi^*\sigma(s, Y(s, t, x))dW_s - \frac{1}{2}\int_t^T |\nabla\psi^*\sigma\sigma^*\nabla\psi|(s, Y(s, t, x))ds - 2\int_t^T D_t\psi(s, Y(s, t, x)ds),$$

$$v_T(t, x) = E\beta_T(t, x)f(T, Y(T, t, x)), \quad c(t) = \int_{\mathbb{R}^d} e^{-2\psi(t,x)}v_T(t, x)dx.$$

Then $c(t)$ *is a constant for* $t \in [0, T]$.

Proof. Using a standard approximation argument it suffices to prove the result for $f \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$. First by Assumption 4 (i) and (iii), we have

$$E\exp(\frac{1}{2}\int_t^T |\nabla\psi^*\sigma\sigma^*\nabla\psi|(s,Y(s,t,x))ds) < \infty.$$

Girsanov transformation yields

$$v_T(t,x) = E\exp(-\int_t^T 2D_t\psi(s,X(s,t,x))ds)f(T,X(T,t,x)).$$

By Assumption 4 (i), (iii) and (iv), we get that $(\frac{1}{2}\sum_{j=1}^d \partial_j a_{ij})_{1\le i\le d}$ and $-\sigma\sigma^*\nabla\psi$ are bounded and also satisfy Assumption 3 (iv). By Theorem 4.1, $v_T(t,x)$ is the solution to the following Kolmogrov equation with a potential term $-2D_t\psi$:

$$\begin{cases} D_t v_T(t,x) + \frac{1}{2}\sum_{i,j=1}^d \partial_j(a_{ij}\partial_i v_T(t,x)) - ((\sigma\sigma^*\nabla\psi)^*\nabla v_T)(t,x) \\ \qquad\qquad - v_T(t,x)2D_t\psi(t,x) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \\ v_T(T,x) = f(T,x). \end{cases} \tag{4.7}$$

And Theorem 4.1 shows that for $t\in[0,T)$, $v_T(t,\cdot)$, $D_tv_T(t,\cdot)$, $\nabla v_T(t,\cdot)$, $\nabla^2 v_T(t,\cdot)\in L^1(\mathbb{R}^d)$, also there exists a kernel $k(T,y;t,x)$ such that

$$v_T(t,x) = \int_{\mathbb{R}^d} k(T,y;t,x)f(T,y)dy$$

and there exists a constant C such that ([8, (13.1)])

$$D_t k(T,y;t,x) \le C(T-t)^{-\frac{d+2}{2}}\exp\Big(-C\frac{|y-x|^2}{T-t}\Big).$$

Then by mean value theorem for $h\in\mathbb{R}$ with $t+h\in(0,T)$ there exists an $\theta\in(0,1)$ such that

$$\frac{|k(T,y;t+h,x)-k(T,y;t,x)|}{h} = D_tk(T,y;t+\theta h,x) \le C(T-t-\theta h)^{-\frac{d+2}{2}}\exp\Big(-C\frac{|y-x|^2}{T-t-\theta h}\Big),$$

then

$$\begin{aligned} \Big|\frac{v_T(t+h,x)-v_T(t,x)}{h}\Big| &\le \int_{\mathbb{R}^d}\Big|\frac{k(T,y;t+h,x)-k(T,y;t,x)}{h}\Big|f(T,y)dy \\ &\le C(T-t-\theta h)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}\exp\Big(-C\frac{|y-x|^2}{T-t-\theta h}\Big)f(T,y)dy \\ &\le C'(T-t-\theta h)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}\exp\Big(-C\frac{|y-x|^2}{T-t}\Big)f(T,y)dy. \end{aligned} \tag{4.8}$$

Denote $g(t,x) = e^{-2\psi(t,x)}v_T(t,x)$, we have for $t\in[0,T)$, $t+h\in(0,T)$,

$$\Big|\frac{g(t+h,x)-g(t,x)}{h}\Big| = \Big|\frac{e^{-\psi(t+h,x)}(v_T(t+h,x)-v_T(t,x))}{h} + \frac{v_T(t,x)(e^{-\psi(t+h,x)}-e^{-\psi(t,x)})}{h}\Big|$$

$$
\begin{aligned}
&\le \Big|\frac{v_T(t+h,x)-v_T(t,x)}{h}\Big| + \Big|\frac{v_T(t,x)(e^{-\psi(t+h,x)}-e^{-\psi(t,x)})}{h}\Big| \\
&\le C'(T-t-\theta h)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}\exp\Big(-C\frac{|y-x|^2}{T-t}\Big)f(T,y)dy + C''v_T(t,x) \\
&=: G_T(t,x),
\end{aligned}
$$

the last inequality holds because of (4.8) and mean value theorem. Since for for $t\in[0,T)$, $v_T(t,\cdot)\in L^1(\mathbb{R}^d)$ and

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(T-t-\theta h)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}\exp\Big(-C\frac{|y-x|^2}{T-t}\Big)f(T,y)dydx \\
&\quad=\int_{\mathbb{R}^d}\Big(\frac{T-t}{T-t-\theta h}\Big)^{\frac{d+2}{2}}(T-t)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}\exp\Big(-C\frac{|y-x|^2}{T-t}\Big)f(T,y)dydx \\
&\quad\le C(T-t)^{-1}\int_{\mathbb{R}^d}f(T,y)dy<\infty
\end{aligned}
$$

for $|h|\ll 1$, it yields that $G_T(t,\cdot)\in L^1(\mathbb{R}^d)$. Then by dominated convergence theorem, we have

$$
\lim_{h\to 0}\frac{\int_{\mathbb{R}^d}g(t+h,x)-g(t,x)dx}{h}=\lim_{h\to 0}\int_{\mathbb{R}^d}\frac{g(t+h,x)-g(t,x)}{h}dx=\int_{\mathbb{R}^d}D_tg(t,x)dx.
$$

That is to say

$$
D_t\int_{\mathbb{R}^d}e^{-2\psi(t,x)}v_T(t,x)dx=\int_{\mathbb{R}^d}D_t(e^{-2\psi}v_T)(t,x)dx. \tag{4.9}
$$

Besides, we can write the first equation in (4.7) in an equivalent form as

$$
D_t(e^{-2\psi}v_T)+\frac{1}{2}\sum_{i,j=1}^d\partial_i(e^{-2\psi}a_{ij}\partial_j v_T)=0. \tag{4.10}
$$

Now we are going to prove

$$
\int_{\mathbb{R}^d}div(F)(t,x)dx:=\int_{\mathbb{R}^d}\sum_{i,j=1}^d\partial_i(e^{-2\psi}a_{ij}\partial_j v_T)(t,x)dx=0,\quad t\in[0,T). \tag{4.11}
$$

Since ψ is nonnegative, $\partial_i\psi$ and a_{ij} are bounded on $[0,\infty)\times\mathbb{R}^d$ for $1\le i,j\le d$, then there exist constants C_1 and C_2 such that

$$
F_i=\sum_{j=1}^d e^{-2\psi}a_{ij}\partial_j v_T\le C_1\sum_{j=1}^d|\partial_j v_T|,
$$

and

$$
div(F)=\sum_{i,j=1}^d\partial_i(e^{-2\psi}a_{ij}\partial_j v_T)
$$

$$= \sum_{i,j=1}^{d} (-2\partial_i \psi e^{-2\psi} a_{ij} \partial_j v_T + \partial_i a_{ij} e^{-2\psi} \partial_j v_T + e^{-2\psi} a_{ij} \partial_i \partial_j v_T)$$
$$\leq C_2 \sum_{i,j=1}^{d} (|\partial_j v_T| + |\partial_i \partial_j v_T|).$$

According to (4.3) we know that $F(t, \cdot), divF(t, \cdot) \in L^1(\mathbb{R}^d)$ for any $t \in [0, T)$. For $n \in \mathbb{N}$, take smooth function χ_n on $\mathbb{R}^d$ such that $\chi_n(x) = 1$ when $|x| \leq n$ and $\chi_n(x) = 0$ when $|x| > n + 2$. Then by dominated convergence theorem and integration by parts formula for $t \in [0, T)$,

$$\int_{\mathbb{R}^d} div(F)(t, x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \chi_n(x) div(F)(t, x)dx = -\lim_{n\to\infty} \int_{\mathbb{R}^d} \nabla\chi_n(x) \cdot F(t, x)dx = 0.$$

Hence from (4.10), (4.9) and (4.11) we get

$$D_t \int_{\mathbb{R}^d} e^{-2\psi(t,x)} v_T(t, x)dx = 0.$$

This yields that $c(t)$ is a constant for $t \in [0, T)$. Since $c(t)$ is continuous for $t \in [0, T]$, it shows that $c(t)$ is a constant for $t \in [0, T]$. □

Remark 4.3. *As it was already pointed out in [7, Remark 7.3], the above lemma plays a very important role in later proof. Instead of the probabilistic way used in [7] to obtain the desired result, which is corresponding to [7, Lemma 7.1] and [7, Corollary 7.2], we applied analytic method in the proof of Lemma 4.2. The original idea of the analytic method that we used is inspired from [7, Remark 7.3] in which a rough outline of analytic proof was given. Due to the fact that $B_t =: W_{T-t} - W_T$ is still a Wiener process on $[0, T]$, the probabilistic way and analytic method both work in [7]. However if we replace the Wiener process W_t with a martingale Y_t, neither can we make the stochastic integral with respect to $B'_t =: Y_{T-t} - Y_T$ well-defined nor obtain a critical formula which is similar to [7, (7.1)]. This essentially decides the form of the studied SDEs and also the type of the Lyapunov function in what follows.*

Theorem 4.1 talks about Cauchy problem with terminal data for the equation (4.1) in the domain $[0, T] \times \mathbb{R}^d$. In the cylindrical domain $Q^{r^2,r}$ with surface $\partial Q^{r^2,r} := ((0, r^2) \times \partial B_r) \cup (\{r^2\} \times B_r)$ for $r \in (0, 1]$, we consider the first boundary value problem to the following parabolic equation on $\overline{Q^{r^2,r}}$ with assuming that f is a continuous function on $\partial Q^{r^2,r}$:

$$\begin{cases} \mathcal{L}u(t, x) = D_t u(t, x) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{d} \partial_i (a_{ij}(t, x) \partial_j u(t, x)) = 0 \qquad on \quad Q^{r^2,r}, \\ u(t, x) = f(t, x) \qquad on \quad \partial Q^{r^2,r}, \end{cases} \tag{4.12}$$

where $(a_{ij})_{1\leq i,j\leq d} = \sigma\sigma^*$. If Assumption 4 (iii) and (iv) hold, from [14, Theorem 3.1] and [14, Corollary 3.2] the solution $u(t, x)$ to (4.12) has a representation as following:

$$u(t, x) = \int_{\partial Q^{r^2,r}} f(s, y)p(s, y; t, x)dS(s, y),$$

where dS denotes the surface measure on $\partial Q^{r^2,r}$, and $p(s,y;t,x)$ is the Poisson kernel on $Q^{r^2,r}$ corresponding to (4.12), which has the following upper bound estimate on $Q^{r^2,r}$ ([14]) with a constant c independent of f

$$p(s,y;t,x) \leq c(s-t)^{-\frac{(d+1)}{2}} \exp(-c\frac{|y-x|^2}{s-t}) \tag{4.13}$$

for all $(t,x) \in Q^{r^2,r}$, $(s,y) \in \partial Q^{r^2,r}$, $0 \leq t < s$.

Besides, we also can represent the solution to the above equation (4.12) in a probabilistic way. For $(t,x) \in Q^{r^2,r}$, let

$$\tau_r := \inf\left\{s \geq 0 : (s, Y(s,t,x)) \notin Q^{r^2,r}\right\},$$

by applying Itô's formula to $u(s, Y(s,t,x))$ and taking expectation, we have for $(t,x) \in Q^{r^2,r}$,

$$u(t,x) = E^{(t,x)}[u(\tau_r, Y(\tau_r,t,x))] - E^{(t,x)}[\int_t^{\tau_r} \mathcal{L}u(s,Y(s,t,x))ds] = E^{(t,x)}[f(\tau_r, Y(\tau_r,t,x))].$$

Hence

$$E^{(t,x)}[f(\tau_r, Y(\tau_r,t,x))] = \int_{\partial Q^{r^2,r}} f(s,y)p(s,y;t,x)dS(s,y).$$

We take $(0,0)$ as the start point of the process $(s, Y(s,t,x))$, then denote $Y_s := Y(s,0,0)$ and

$$E[f(\tau_r, Y_{\tau_r})] = \int_{\partial Q^{r^2,r}} f(s,y)p(s,y;0,0)dS(s,y). \tag{4.14}$$

Lemma 4.4. *If Assumption 4 (iii) and (iv) hold, then on an extension of the probability space there is a stopping time γ such that the distribution of (γ, Y_γ) has a bounded density concentrated on $Q^{1,1}$.*

Proof. Let $n = d+3$. On an extension of our probability space there exists a random variable ρ with values in $[0,1]$ and density function $h(r) = nr^{n-1}$ such that ρ is independent of all $(\mathcal{F}_t)_{t\geq 0}$. Then ρ is also independent to $(t, Y_t)_{t\geq 0}$, since $(Y_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Let $\hat{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\rho), t \geq 0$, and define γ as the first exit time of $(t, Y_t)_{t\geq 0}$ from $Q^{\rho^2,\rho}$. Then γ is a bounded $(\hat{\mathcal{F}}_t)_{t\geq 0}$ stopping time. We claim that γ is a random variable of the type that we are looking for.

Actually, according to independence and (4.14), for a nonnegative continuous function $f(t,x)$ on $[0,\infty) \times \mathbb{R}^d$ we have

$$\begin{aligned} Ef(\gamma, Y_\gamma) =& E[Ef(\tau_r, Y_{\tau_r})\Big|_{\rho=r}] = E[\int_{\partial Q^{r^2,r}} f(s,y)p(s,y;0,0)dS(s,y)\Big|_{\rho=r}] \\ =& \int_0^1 h(r)dr \int_{\partial Q^{r^2,r}} f(s,y)p(s,y;0,0)dS(s,y) \\ =& \int_0^1 h(r)dr \int_{(0,r^2)\times\partial B_r} f(s,y)p(s,y;0,0)dS(s,y) \\ &+ \int_0^1 h(r)dr \int_{B_r} f(r^2,y)p(r^2,y;0,0)dy =: I_1 + I_2. \end{aligned}$$

Then (4.13) and the fact that $\exp(-c\frac{|y|^2}{s})s^{-(d+1)/2}$ is bounded by Nr^{-d-1} on $(0,r^2)\times\partial B_r$ yield

$$
\begin{aligned}
I_1 &\le k\int_0^1 h(r)dr\int_0^{r^2}\int_{\partial B_r} f(s,y)\frac{\exp(-c\frac{|y|^2}{s})}{s^{(d+1)/2}}dS(s,y)\\
&\le N\int_0^1 h(r)r^{-d-1}dr\int_0^{r^2}\int_{\partial B_r} f(s,y)dS(s,y)\\
&\le N\int_0^1\int_0^{r^2}\int_{\partial B_1} r^{-d-1}f(s,ry)h(r)r^{d-1}d(\partial B_1)dsdr\\
&\le N\int_0^1\int_0^{1}\int_{\partial B_1} f(s,ry)r^{d}d(\partial B_1)dsdr\\
&\le N\int_{Q^{1,1}} f(s,y)dsdy,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le k\int_0^1\int_{B_r} f(r^2,y)h(r)\frac{\exp(-c\frac{|y|^2}{r})}{r^{d+1}}dydr\\
&\le N\int_0^1\int_{B_r} f(r^2,y)h(r)r^{-d-1}dydr\\
&= N\int_0^1\int_{B_r} f(r^2,y)r^{n-2-d}dydr\\
&\le N\int_{Q^{1,1}} f(s,y)dsdy.
\end{aligned}
$$

Hence

$$
Ef(\gamma,Y_\gamma)\le N\int_{Q^{1,1}} f(t,x)dxdt
$$

and N is independent of f.

For arbitrary nonnegative function $|fI_{Q^{1,1}}|\in\mathbb{L}_1^1$, we can use a standard method to approximate f via continuous functions. The conclusion is proved. □

Lemma 4.5. *Let Assumption 4 hold. Let $K_2\in[0,\infty)$ be a constant. Assume that*

$$
\psi I_{Q^{1,1}}\le K_2,\quad \|\nabla\psi I_{Q^{1,1}}\|_{\mathbb{L}_p^q}\le K_2.
$$

Take an $r\in(1,\infty)$ and a nonnegative Borel function $f=f(t,x)$ on $(0,\infty)\times\mathbb{R}^d$ such that $f(t,x)=0$ for $t>T$. For $0\le s\le t\le T$ and $x\in\mathbb{R}^d$ introduce

$$
\begin{aligned}
&\rho_t(s,x)=\exp(-\int_s^t\nabla\psi^*\sigma(u,Y(u,s,x))dW_s-\frac{1}{2}\int_s^t|\nabla\psi^*\sigma\sigma^*\nabla\psi|(u,Y(u,s,x))du),\\
&\alpha_t(s,x)=\exp(-2\int_s^t(D_t\psi)_+(u,Y(u,s,x))du),\\
&u_t(s,x)=E\rho_t(s,x)\alpha_t(s,x)f(t,Y(t,s,x)).
\end{aligned}
$$

Then there is a constant N, depending only on K, r, p, q, K_2 and T, such that

$$
\int_0^T u_t(0,0)dt\le N(\int_{(0,\infty)\times\mathbb{R}^d} f^re^{-2\psi}dtdx)^{1/r}+N(\int_{Q^{1,1}} f^{d+3}dtdx)^{1/(d+3)}. \tag{4.15}
$$

Proof. By the strong Markov property of $(Y_t)_{t\geq 0}$, which can be obtained from the similar argument as in the proof of Theorem 2.1, for any stopping time τ we have

$$EI_{\tau\leq t}\rho_t(0,0)\alpha_t(0,0)f(t,Y_t) = EI_{\tau\leq t}\rho_\tau(0,0)\alpha_\tau(0,0)u_t(\tau,Y_\tau).$$

Therefore, upon assuming without losing generality that $T\geq 1$, for γ from Lemma 4.4,

$$\int_0^T u_t(0,0)dt = E\int_0^\gamma \rho_t(0,0)\alpha_t(0,0)f(t,Y_t)dt + E\rho_\gamma(0,0)\alpha_\gamma(0,0)\int_\gamma^T u_t(\gamma,Y_\gamma)dt =: I_1+I_2.$$

Observe that $\alpha_t\leq 1$ and for $t\leq\gamma$ we have $(t,Y_t)\in Q^{1,1}$ so that, in particular, in the formula defining $\rho_t(0,0)$ we can replace $\nabla\psi$ with $\nabla\psi I_{Q^{1,1}}$ and hence all moments of $\rho_t(0,0)I_{t\leq\gamma}$ and $\rho_\gamma(0,0)$ are finite and uniformly bounded in t. Since by (3.7) we have

$$E[\exp(\frac{1}{2}\int_0^t |\nabla\psi^*\sigma\sigma^*\nabla\psi|I_{Q^{1,1}}(u,Y(u,s,x))du)] < \infty$$

for all $t\in[0,T]$. For the moments of $\rho_t(0,0)I_{t\leq\gamma}$ and $\rho_\gamma(0,0)$, by using the same way of getting (3.15) and (3.16) we get the desired results. We also can replace $\frac{1}{2}\sum_{j=1}^d\int_s^t \partial_j a_{ij}(r,Y(r,s,x))dr$ by $\frac{1}{2}\sum_{j=1}^d\int_s^t I_{Q^{1,1}}\partial_j a_{ij}(r,Y(r,s,x))dr)$ in the SDE (4.5) for all $1\leq i,j\leq d$, it follows by Hölder's inequality and (3.8) that for any $v\in(1,\infty)$

$$I_1\leq N(E\int_0^T |f^v I_{Q^{1,1}}|(t,Y_t)dt)^{1/v} \leq N\|f^v I_{Q^{1,1}}\|^{1/v}_{\mathbb{L}_{d+5/2}}.$$

We can choose v so that $v(d+5/2)=d+3$, and get that I_1 is less than the second term on the right in (4.15).

In what concerns I_2 we again use $\alpha_\gamma(0,0)\leq 1$ and the finiteness of all moments of $\rho_\gamma(0,0)$. Then we find

$$I_2\leq N(\int_0^1\int_s^T(\int_{B_1} u_t^r(s,x)dx)dtds)^{1/r}. \tag{4.16}$$

To estimate the interior integral with respect to x we insert there $\exp(-2\psi(s,x))$ and again use Hölder's inequality and the fact that $E\rho_t(s,x)\leq 1$. This yields

$$I_2(s,t) := \int_{B_1} u_t^r(s,x)dx \leq e^{2K_2}\int_{\mathbb{R}^d} e^{-2\psi(s,x)}\hat{v}_t(s,x)dx$$

where

$$\hat{v}_t(s,x) = E\rho_t(s,x)\alpha_t(s,x)f^r(t,Y(t,s,x)) \leq E\beta_t(s,x)f^r(t,Y(t,s,x)).$$

Hence by Lemma 4.2,

$$I_2(s,t)\leq e^{2K_2}\int_{\mathbb{R}^d} e^{-2\psi(t,x)}f^r(t,x)dx,$$

which and (4.16) show that I_2 is less than the first term on the right in (4.15). The Lemma is proved. □

Lemma 4.6. *Let the assumptions of Lemma 4.5 be satisfied and let $\epsilon\in[0,2)$ be a constant and h a nonnegative Borel function on bounded domain $Q\subset[0,\infty)\times\mathbb{R}^d$ such that on Q,*

$$2D_t\psi + \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i\psi) \leq he^{\epsilon\psi}. \tag{4.17}$$

Then for any $\delta \in [0, 2-\epsilon)$, $r \in (1, 2/(\delta+\epsilon)]$, there exists a constant N, depending only on K, T, p, q, K_2, ϵ, δ and r (but not Q) such that for any stopping time $\tau \le \tau_Q(Y.)$ we have

$$E\Phi_\tau \le N + N(\int_Q h^r e^{-(2-r\eta)\psi} dtdx)^{1/r} + N \sup_{Q^{1,1}} h, \tag{4.18}$$

where $\eta = \delta + \epsilon$ so that $r\eta \le 2$ and

$$\begin{aligned}\Phi_t := \exp(&- \int_0^t (\nabla\psi^*\sigma)(s, Y_s) dW_s - \frac{1}{2}\int_0^t |\nabla\psi^*\sigma\sigma^*\nabla\psi|(s, Y_s) ds \\ &- 2\int_0^t (D_t\psi)_+(s, Y_s) ds + \delta\psi(t, Y_t)).\end{aligned}$$

Proof. By Itô's formula,

$$\begin{aligned}\Phi_\tau = &\Phi_0 + m_\tau + \int_0^\tau \Phi_t[\delta D_t\psi + \frac{\delta}{2}\sum_{i,j=1}^d \partial_j(a_{ij}\partial_i\psi) - 2(D_t\psi)_+ \\ &+ \frac{1}{2}(|\delta-1|^2 - 1)|\nabla\psi^*\sigma\sigma^*\nabla\psi|](t, Y_t) dt\end{aligned}$$

where m_t is a local martingale starting at zero. By using (4.17), and the inequality $|\delta-1| \le 1$ we obtain

$$\Phi_\tau \le \Phi_0 + \delta\int_0^\tau \Phi_t h(t, Y_t)\exp(\epsilon\psi(t, Y_t)) dt + m_\tau. \tag{4.19}$$

Since $\Phi_t \ge 0$ we take the expectations of both sides and drop Em_τ. More precisely, we introduce $\tau_n := \inf\{t \ge 0 : |m_t| \ge n\}$ and substitute $\tau \wedge \tau_n$ in place of τ in (4.19). After that we take expectations, use the fact that $Em_{\tau\wedge\tau_n} = 0$, let $n \to \infty$, and finally use Fatou's Lemma with monotone convergence theorem. Furthermore, we denote $f = I_Q h \exp(\eta\psi)$ and notice that $\tau \le T$. Then in the notation of Lemma 4.5, we find that

$$\begin{aligned}E\Phi_\tau &\le N + NE\int_0^\tau \rho_t(0,0)\alpha_t(0,0) f(t, Y_t) dt \\ &\le N + N\int_0^T E\rho_t(0,0)\alpha_t(0,0) f(t, Y_t) dt = N + N\int_0^T u_t(0,0) dt.\end{aligned}$$

It only remains to note that the first term in the right-hand side of (4.15) is just the second one on the right in (4.18) and the second integral on the right in (4.15) is less than $vol Q^{1,1} \sup_{Q^{1,1}} h^{d+3} \exp[\eta K_2(d+3)]$. The Lemma is proved. □

Theorem 4.7. *Let Assumption 4 hold. Let K_1, $K_2 \in [0, \infty)$ and $\epsilon \in [0, 2)$ be some constants and let Q be a bounded subdomain of Q_T and h be a nonnegative Borel function on Q. Assume that*

$$hI_{Q^{1,1}} \le K_2, \quad \psi I_{Q^{1,1}} \le K_2, \quad \|I_{Q^{1,1}}\nabla\psi\|_{\mathbb{L}_p^q} \le K_2.$$

Also assume that on Q

$$\psi \ge 0, \quad 2D_t\psi \le K_1\psi,$$

$$2D_t\psi + \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i\psi) \le he^{\epsilon\psi}.$$

Denote by X_t, $t \in [0,T]$, the solution of

$$X_t = \int_0^t \sigma(s,X_s)dW_s + \int_0^t (-\sigma\sigma^*\nabla\psi)(s,X_s)ds + (\frac{1}{2}\sum_{j=1}^d \int_0^t \partial_j a_{ij}(s,X_s)ds)_{1\le i\le d}.$$

Then for any $r \in (1, 4/(2+\epsilon)]$ there exists a constant N, depending only on K, K_1, K_2, r, d, T, p, q, and ϵ, such that

$$E \sup_{t\le\tau_Q(X.)} \exp[\mu(\psi(t,X_t) + \nu|X_t|^2)] \le N + NH_Q(T,a,r) \tag{4.20}$$

where H_Q is introduced in Assumption 1, $a = (2-r\eta)\nu$, $\eta = 2\delta+\epsilon$, μ, ν and δ are taken from (2.7)*. Here $\tau_Q(X.) := \inf\{t \ge 0 : (t,X_t) \notin Q\}$.*

Proof. Define $\hat\psi = \psi + \nu|x|^2$,

$$M_t = \exp(\delta\hat\psi(t,X_t) - \frac{K_1}{2}\int_0^t \hat\psi(s,X_s)ds), \quad M_* = \sup_{t\le\tau_Q(X.)} M_t.$$

Then for $t \le \tau_Q(X.)$,

$$\hat\psi(t,X_t) \le \ln M_*^{1/\delta} + \frac{K_1}{2\delta}\int_0^t \hat\psi(s,X_s)ds$$

and hence by Gronwall's inequality

$$\hat\psi(t,X_t) \le e^{tK_1/(2\delta)} \ln M_*^{1/\delta} \le e^{TK_1/(2\delta)} \ln M_*^{1/\delta}.$$

Take $\mu = \frac{\delta}{2}e^{-TK_1/(2\delta)}$, then

$$\exp(\mu\hat\psi(t,X_t)) \le \sqrt{M_*}. \tag{4.21}$$

Therefore, to prove (4.20), it suffices to prove that $E\sqrt{M_*} \le N$. It turns by a well known result on transformations of stochastic inequalities (see Lemma 3.2 in [5]), if $EM_\tau \le N_1$ for all stopping times $\tau \le \tau_Q(X.)$. Then $E\sqrt{M_*} \le 3N_1$. Thus, it suffices to estimate EM_τ.

On a probability space carrying a d-dimensional Wiener process $(\hat W_t)_{t\ge0}$ introduce $(\hat X_t)_{t\ge0}$ as the solution of the equation

$$\hat X_t = \int_0^t \sigma(s,\hat X_s)d\hat W_s - \int_0^{t\wedge\tau_Q(\hat X.)} \sigma\sigma^*\nabla\hat\psi(s,\hat X_s)ds + (\int_0^{t\wedge\tau_Q(\hat X.)} \frac{1}{2}\sum_{j=1}^d \partial_j a_{ij})(s,\hat X_s)ds)_{1\le i\le d}. \tag{4.22}$$

Also set

$$\hat M_t = \exp(2\delta\hat\psi(t,\hat X_t) - 2\int_0^t (D_t\hat\psi)_+(s,\hat X_s)ds), \quad t \ge 0.$$

Write $\hat E$ for the expectation sign on the new probability space and observe that on Q

$$2D_t\hat\psi + \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i\hat\psi) = 2D_t\psi + \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i\psi) + 2\nu\sum_{i,j=1}^d x_i\partial_j a_{ij} + 2\nu\sum_{i,j=1}^d \partial_j a_{ij}$$

$$\leq (h+C)e^{\epsilon\hat{\psi}}. \tag{4.23}$$

Here $2\nu\sum_{i,j=1}^{d} x_i\partial_j a_{ij} + 2\nu\sum_{i,j=1}^{d}\partial_j a_{ij} \leq (h+C)e^{\epsilon\hat{\psi}}$ holds because of Assumption 4, which means that $|\partial_j a_{ij}|$ is bounded. Then after an obvious change of measure (cf. Lemma A.3) inequality (4.18) with 2δ, $\hat{E}$, $\hat{\psi}$, and $\hat{W}_t$ in place of δ, E, ψ, and W_t, respectively, $\eta = 2\delta+\epsilon$, and $r\in(1,4/(2+\epsilon)]\subset(1,2/(2\delta+\epsilon)]$ is written as

$$\hat{E}\hat{M}_\tau \leq N + N(\int_Q h^r I_{(0,T)}e^{-(2-r\eta)\hat{\psi}}dtdx)^{1/r}$$

and since $\hat{\psi}\geq \nu|x|^2$ on Q, we obtain

$$\hat{E}\hat{M}_\tau \leq N + N(\int_Q h^r I_{(0,T)}e^{-(2-r\eta)\nu|x|^2}dtdx)^{1/r} = N + NH_Q^{1/r}(T,(2-r\eta)\nu,r) =: N_0$$

for all stopping times $\tau\leq\tau_Q(\hat{X}.)$, which yields

$$\hat{E}\sqrt{\hat{M}_*}\leq 3N_0.$$

Combining this with the inequality

$$\exp(2\delta\hat{\psi}(t,\hat{X}_t) - K_1\int_0^t\hat{\psi}(x,\hat{X}_s)ds) \leq \hat{M}_t, \qquad t\leq\tau_Q(\hat{X}.),$$

the left-hand side of which is quite similar to M_t but with $2\hat{\psi}$ in place of $\hat{\psi}$, the above argument deduce

$$\hat{E}\sup_{t\leq\tau_Q(\hat{X}.)}\exp(2\mu\nu|\hat{X}_t|^2) \leq \hat{E}\sup_{t\leq\tau_Q(\hat{X}.)}\exp(2\mu\hat{\psi}(t,\hat{X}_t)) \leq NN_0. \tag{4.24}$$

We now estimate EM_τ through $\hat{E}\hat{M}_\tau$ by using Girsanov's theorem and Hölder's inequality. We use a certain freedom in choosing $\hat{X}_t$ and $\hat{W}_t$ and on the probability space where W_t and X_t are given we introduce a new measure by the formula:

$$\hat{P}(d\omega) = \exp(-2\nu\int_0^\infty X_t^*\sigma(t,X_t)I_{t<\tau_Q(X.)}dW_t - 2\nu^2\int_0^\infty X_t^*(\sigma\sigma^*)(t,X_t)X_tI_{t<\tau_Q(X.)}dt)P(d\omega).$$

Since Q is a bounded domain, then we have

$$E\exp\Big(2\nu^2\int_0^\infty X_t^*(\sigma\sigma^*)(t,X_t)X_tI_{t<\tau_Q(X.)}dt\Big) \leq E\exp\Big(2\nu^2K\int_0^T X_t^*X_tI_{t<\tau_Q(X.)}dt\Big) < \infty,$$

which implies that $\hat{P}$ is a probability measure. Furthermore, as is easy to see, for $t\leq\tau_Q(X.)$

$$\hat{X}_t := X_tI_{t<\tau_Q(X.)} + (\int_0^t\sigma(s,X_s)dW_s - \int_0^{\tau_Q(X.)}\sigma(s,X_s)dW_s + X_{\tau_Q(X.)})I_{t\geq\tau_Q(X.)}$$

coincides with X_t and satisfies (4.22) for $t\leq\tau_Q(X.)$ with

$$\hat{W}_t = W_t + 2\nu\int_0^{t\wedge\tau_Q(X.)}\sigma^*(s,X_s)X_sds$$

which is a Wiener process with respect to $\hat{P}$. In this situation for $\tau \le \tau_Q(X.) = \tau_Q(\hat{X}.)$

$$EM_\tau \le \hat{E}\hat{M}_\tau^{1/2} \exp(2\nu \int_0^\infty \hat{X}_t^* \sigma(t, \hat{X}_t) I_{t<\tau_Q(\hat{X}.)} d\hat{W}_t - 2\nu^2 \int_0^\infty \hat{X}_t^* (\sigma\sigma^*)(t, \hat{X}_t)\hat{X}_t I_{t<\tau_Q(\hat{X}.)} dt)$$
$$\le (\hat{E}\hat{M}_\tau)^{1/2} (\hat{E}\rho^{1/2} \exp(12v^2 \int_0^\infty \hat{X}_t^* (\sigma\sigma^*)(t, \hat{X}_t)\hat{X}_t I_{t<\tau_Q(\hat{X}.)} dt))^{1/2}$$

where

$$\rho = \exp(8\nu \int_0^\infty \hat{X}_t^* \sigma(t, \hat{X}_t) I_{t<\tau_Q(\hat{X}.)} d\hat{W}_t - 32\nu^2 \int_0^\infty \hat{X}_t^* (\sigma\sigma^*)(t, \hat{X}_t)\hat{X}_t I_{t<\tau_Q(\hat{X}.)} dt).$$

Observe that $\hat{E}\rho = 1$ and $\hat{E}\hat{M}_\tau \le N_0$. Therefore,

$$EM_\tau \le N_0^{1/2} (\hat{E} \exp(24\nu^2 \int_0^{\tau_Q(X.)} (\hat{X}_t^* (\sigma\sigma^*)(t, \hat{X}_t)\hat{X}_t) dt))^{1/4}.$$

It only remains to refer to (4.24) after noticing that

$$24\nu^2 \int_0^{\tau_Q(X.)} (\hat{X}_t^* (\sigma\sigma^*)(t, \hat{X}_t)\hat{X}_t) dt \le 24\nu^2 KT \sup_{t\le\tau_Q(X.)} |X_t|^2 = 2\mu\nu \sup_{t\le\tau_Q(X.)} |X_t|^2$$

and use the inequality $\iota^\alpha \le 1 + \iota$ if $\iota \ge 0$, $0 \le \alpha \le 1$, where $\nu = \mu/(12KT)$. The theorem is proved. □

5 Proof of Theorem 2.4

By Theorem 2.1 the strong solution $(t, X_t)_{t\ge0}$ to (2.5) is defined at least until the time ξ when $(s + t, X_t)_{t\ge0}$ exits from all Q^n. We claim that in order to prove $\xi = \infty$ *a.s.* and also to prove the second assertions of the theorem, it suffices to prove that for each $T \in (0, \infty)$ and $m \ge 1$ there exists a constant N, depending only on K, K_1, d, $p(m + 1)$, $q(m + 1)$, ϵ, T, $\|I_{Q^{m+1}}\nabla\phi\|_{\mathbb{L}^{q(m+1)}_{p(m+1)}}$, $dist(\partial Q^m, \partial Q^{m+1})$, $\sup_{Q^{m+1}} \{\phi + h\}$, and the function H, such that for $(s, x) \in Q^m$ we have

$$E \sup_{t<\xi\wedge T} \exp(\mu\phi(s + t, X_t) + \mu\nu|X_t|^2) \le N. \tag{5.1}$$

To prove the claim notice that (5.1) implies

$$\sup_{t<\xi\wedge T} (\phi(s + t, X_t) + |X_t|^2) < \infty \quad a.s.. \tag{5.2}$$

It follows that *a.s.* there exists an $n \ge 1$ such that up to time $\xi \wedge T$ the trajectory $(Z_t)_{t\ge0} = (s + t, X_t)_{t\ge0}$ lies in Q^n. Indeed, on the set of all ω where this is wrong, for the exit time ξ^n of Z_t from Q^n we have $\xi^n < T$ for all n. However owing to (5.2), the sequence X_{ξ^n} should be bounded, then the sequence Z_{ξ^n} has limit points on the boundary ∂Q. According to the Assumption 1 (vi), it only happens with probability zero. Hence, *a.s.* there is $n \ge 1$ such that $T \le \xi^n$. Since this happens for any T and $\xi^n < \xi$ we conclude that $\xi = \infty$ *a.s.*, which proves our intermediate claim.

Since $dist(\partial Q^m, \partial Q^{m+1}) > 0$ we can find $\kappa \in (0, 1]$ sufficiently small so that $(s, x)+Q^{\kappa^2,\kappa} \subset Q^{m+1}$ for all $(s, x) \in Q^m$. Therefore, by translation and dilation, without losing generality, we may assume that $s=0$, $x=0$ and $Q^{1,1}\subset Q^m$.

Next we notice that obviously, to prove (5.1) it suffices to prove that with N of the same kind as in (5.1) for any $n\geq m+2$,

$$E\sup_{t<\xi^n\wedge T}\exp(\mu\phi(t,X_t)+\mu\nu|X_t|^2)\leq N. \tag{5.3}$$

Fix an $n\geq m+2$. By virtue of Theorem 2.1, notice that the left-hand side of (5.3) will not change if we change $-\sigma\sigma^*\nabla\phi+(\frac{1}{2}\sum_{j=1}^d\partial_j a_{ij})_{1\leq i\leq d}$ outside of Q^n. Therefore we may replace ϕ with $\phi\eta$ and replace $\frac{1}{2}\sum_{j=1}^d\partial_j a_{ij}$ with $\frac{1}{2}\sum_{j=1}^d\partial_j a_{ij}\eta$ for each $1\leq i\leq d$, where η is an infinitely differentiable function equal 1 on a neighborhood of Q^n and equals 0 outside of Q^{n+1}. To simplify the notation we just assume that ϕ and $\frac{1}{2}\sum_{j=1}^d\partial_j a_{ij}$ vanish outside of Q^{n+1} and (2.3) holds in a neighborhood of $\overline{Q^n}$. This is harmless as long as we prove that N depends appropriately on the data.

Now we mollify ϕ by convolving it with a δ-like nonnegative smooth function $\zeta^\gamma(t,x)=\gamma^{-d-1}\zeta(t/\gamma,x/\gamma)$, ζ has compact support in Q^1. Denote by $\phi^{(\gamma)}$ the result of the convolution and use an analogous notation for the convolution of $\zeta^\gamma(t,x)$ with other functions. Also denote by $(X_t^\gamma)_{t\geq 0}$ the solution of the following SDE

$$X_t^\gamma=\int_0^t\sigma(s,X_s^\gamma)dW_s+\int_0^t(-\sigma\sigma^*\nabla\phi^{(\gamma)})(s,X_s^\gamma)ds+(\frac{1}{2}\sum_{j=1}^d\int_0^t\partial_j a_{ij}(s,X_s^\gamma)ds)_{1\leq i\leq d}.$$

For $x_\cdot\in\mathcal{C}([0,\infty),\mathbb{R}^d)$ we define $\xi_n(x_\cdot):=\inf\{t\geq 0:(t,x_t)\notin Q^n\}$. Consider the bounded function f on $\mathcal{C}([0,\infty),\mathbb{R}^d)$ given by the formula

$$f(x_\cdot)=\sup_{t\leq\xi^n(x_\cdot)\wedge T}\exp(\mu\phi(t,x_t)+\mu\nu|x_t|^2),$$

and let f^γ be defined by the same formula with $\phi^{(\gamma)}$ in place of ϕ. Since $\sigma\sigma^*$ is bounded, by using Lemma 3.4 we conclude that the left-hand side of (5.3) is equal to the limit as $\gamma\downarrow 0$ of

$$Ef^\gamma(X_\cdot^\gamma)=E\sup_{t<\xi^n(X_\cdot^\gamma)\wedge T}\exp(\mu\phi^{(\gamma)}(t,X_t^\gamma)+\mu\nu|X_t^\gamma|^2). \tag{5.4}$$

In fact, if we denote $M_t=\int_0^t\sigma(s,M_s)dW_s$, $t\geq 0$, according to Lemma 3.4

$$\begin{aligned}|Ef(X_\cdot)-Ef^\gamma(X_\cdot^\gamma)|&\leq N'(E|f(M_\cdot)-f^\gamma(M_\cdot)|^2)^{1/2}+N'\|f\|_\infty\|\sigma\sigma^*(\nabla\phi-\nabla\phi^{(\gamma)})I_{Q^n}\|_{L_p^q}\\&\leq N'(E|f(M_\cdot)-f^\gamma(M_\cdot)|^2)^{1/2}+KN'\|(\nabla\phi-\nabla\phi^{(\gamma)})I_{Q^n}\|_{L_p^q},\end{aligned}$$

which of course tends to 0 when $\gamma\to 0$, since ϕ is continuous and bounded on Q^n, $|I_{Q^n}\nabla\phi|\in\mathbb{L}_p^q$, then $f^\gamma\to f$ and $I_{Q^n}\nabla\phi^{(\gamma)}\to I_{Q^n}\nabla\phi$ in $\mathbb{L}_p^q$ as $\gamma\to 0$.

In the light of the fact that (2.3) holds in a neighborhood of Q^n we have that on Q^n for sufficiently small γ

$$\begin{aligned}2D_t\phi^{(\gamma)}+\sum_{i,j=1}^d\partial_j(a_{ij}\partial_i\phi^{(\gamma)})&\leq((he^{\epsilon\phi})^{(\gamma)}e^{-\epsilon\phi^{(\gamma)}}+\sum_{i,j=1}^d|\partial_j(a_{ij}\partial_i\phi^{(\gamma)})-(\partial_j(a_{ij}\partial_i\phi))^{(\gamma)}|)e^{\epsilon\phi^{(\gamma)}}\\&=:h^\gamma e^{\epsilon\phi^{(\gamma)}}.\end{aligned} \tag{5.5}$$

Since h is continuous, then $(he^{\epsilon\phi})^{(\gamma)}e^{-\epsilon\phi^{(\gamma)}} \to h$ uniformly on Q^n. Besides $\sum_{i,j=1}^d |\partial_j(a_{ij}\partial_i\phi^{(\gamma)}) - (\partial_j(a_{ij}\partial_i\phi))^{(\gamma)}|) \to 0$ pointwise. Hence if we denote

$$H^\gamma_{Q^n}(T,(2-r\eta)\nu,r) := \int_{Q^n} (h^\gamma)^r(t,x)I_{(0,T)}(t)e^{-(2-r\eta)\nu|x|^2}dtdx,$$

we have

$$\lim_{\gamma\to 0} H^\gamma_{Q^n}(T,(2-r\eta)\nu,r) \le H_{Q^n}(T,(2-r\eta)\nu,r).$$

Furthermore, the conditions $2D_t\phi^{(\gamma)} \le K_1\phi^{(\gamma)}$ also hold in a neighborhood of Q^n for sufficiently small γ.

We now apply Theorem 4.7 for $Q^n \cap Q_T$ in place of Q to conclude that

$$\begin{aligned}
E \sup_{t<\xi^n\wedge T} \exp(\mu\phi(t,X_t) + \mu\nu|X_t|^2) &= \lim_{\gamma\downarrow 0} E \sup_{t<\xi^n(X^\gamma_\cdot)\wedge T} \exp(\mu\phi^{(\gamma)}(t,X^\gamma_t) + \mu\nu|X^\gamma_t|^2) \\
&\le \lim_{\gamma\downarrow 0}(N + NH^\gamma_{Q^n}(T,(2-r\eta)\nu,r)) \\
&\le N + NH_{Q^n}(T,(2-r\eta)\nu,r) \\
&\le N + NH_Q(T,(2-r\eta)\nu,r),
\end{aligned}$$

where the values of all the parameters are specified in 4.7 and the constants N depend only on r, d, $p(m+1)$, $q(m+1)$, ϵ, T, K, K_1, $\|I_{Q^{m+1}}\nabla\phi\|_{L^{q(m+1)}_{p(m+1)}}$, and $\sup_{Q^{m+1}}\{\phi + h\}$.

We finally use condition (H) from Assumption 1 . Fix any $r_0 \in (1, 2/(2\delta+\epsilon))$, set $a = (2-r_0\eta)\nu$ (>0) and take $r = r(T,a)$ from condition (H). Hölder's inequality shows that if condition (H) is satisfied with $r = r'$ where $r' > 1$, then it is also satisfied with any $r \in (1, r']$. Hence without losing generality we may assume that $r = r(T,a) \in (1, r_0]$. Then $(2-r\eta)\nu \ge a$ and $H_Q(T,(2-r\eta)\nu,r) \le H_Q(T,a,r(T,a)) < \infty$. Thus, Theorem 4.7 yields (5.3). The theorem is proved.

□

Remark 5.1. We can add another drift term to (2.5), it does not have to be the gradient of a function. Under Assumption 1 take a Borel measurable locally bounded $\mathbb{R}^d$-valued function $b(t,x)$ defined on $\mathbb{R}^{d+1}$ satisfying the condition $|b(t,x)| \le c(1+|x|)$, where c is a finite positive constant, then it turns out that the first assertion of Theorem 2.4 still holds with the equation

$$\begin{aligned}
X_t = x + \int_0^t \sigma(s+r,X_r)dW_r + \int_0^t (-\sigma\sigma^*\nabla\phi)(s+r,X_r)dr + \int_0^t b(s+r,X_r)dr \\
+ (\int_0^t \frac{1}{2}\sum_{j=1}^d \partial_j a_{ij}(s+r,X_r)dr)_{1\le i\le d}, \quad t \ge 0 \qquad (5.6)
\end{aligned}$$

in place of (2.5). To prove this we follow the proof in [7] Remark 8.2. The only needed materials are the estimate (2.6) and the Markov property of solution to the equation (2.5), which we already get from the proof of Theorem 2.1. By applying Girsanov theorem we get the non-explosion result for the equation (5.6) from Theorem 2.4.

Further we can carry our results in Theorem 2.4 to the cases in which ϕ is not necessarily nonnegative but $\phi \geq -C(1+|x|^2)$, $C>0$. Since the equation (2.5) is equivalent to the following

$$X_t = x + \int_0^t \sigma(s+r, X_r)dW_r + (\frac{1}{2}\int_0^t \sum_{j=1}^d \partial_j a_{ij}(s+r, X_r)dr)_{1\leq i\leq d}$$
$$+ \int_0^t 2C\sigma\sigma^*(s+r, X_r)X_r dr - \int_0^t \sigma\sigma^*\nabla[C(1+|x|^2)+\phi](s+r, X_r)dr, \quad t\geq 0,$$

obviously $|\sigma\sigma^*(t,x)x| \leq K(1+|x|)$. We conclude that the SDE (2.5) has a unique solution defined for all times if $(s,x)\in Q$ provided that $\phi + C(1+|x|^2)$ rather than ϕ satisfies Assumption 1.

6 Examples and applications

In this section, we give several examples to show the local well-posedness and non-explosion of solution to the SDE that our results are applied.

6.1 Examples-Maximal local well-posedness

Example 6.1. *Consider the equation* (1.1) *when* $d=1$, $Q=\mathbb{R}_+\times(0,\infty)$, $Q^n=(0,n)\times\{x: 1/n<x<n\}$ *for* $n\in\mathbb{N}$, $b(t,x)=-x^{-1}$, $\sigma(t,x)=(1+x^2)^{-1}$.

For any $(s,x)\in Q$, *for any* $n\in\mathbb{N}$, *if we take* $q(n)=\infty$ *and* $p(n)\in(2,\infty)$, *then* $1/p(n)+2/q(n)<1$. *We can also easily check that* $\|bI_{Q^n}\|_{\mathbb{L}^\infty_{p(n)}}<\infty$, *and* $\|I_{Q^n}\nabla\sigma\|_{\mathbb{L}^\infty_{p(n)}}<\infty$. *Furthermore,* $\sigma(t,x)$ *is uniformly continuous in* x *uniformly with respect to* t *for* $(t,x)\in Q^n$, *and there exist positive constants* $\delta_n(=(1+n^2)^{-2})$ *such that for all* $(t,x)\in Q^n$,

$$|\sigma^*(t,x)\lambda|^2 \geq \delta_n|\lambda|^2, \quad \forall\lambda\in\mathbb{R}^d.$$

Hence by Theorem 2.1 there exists an $(\mathcal{F}_t)$*-stopping time* ξ *and a unique* $(\mathcal{F}_t)$*-adapted solution to the following equation*

$$X_t = x - \int_0^t \frac{1}{X_r}dr + \int_0^t (1+X_r^2)^{-1}dW_r, \quad t\in[0,\xi).$$

Example 6.2. *If* $d=2$ *with* $b(t,x)=x\ln|x^{(1)}|=(x^{(1)}\ln|x^{(1)}|, x^{(2)}\ln|x^{(1)}|)$, $\sigma(t,x)=I_2\cdot\ln(2+|x|^2)$ *on* $Q=\mathbb{R}_+\times\mathbb{R}^2\setminus\{x^{(1)}=0\}$ *and* $Q_n=(0,n)\times\{x\in\mathbb{R}^2: 1/n<|x^{(1)}|<n, |x^{(2)}|<n\}$, *where* $x^{(i)}$ *denotes the* $i-th$ *exponent of the vector* $x\in\mathbb{R}^d$ *and* I_2 *is the identity matrix in* $\mathbb{R}^2$. *Then by Theorem 2.1 for any* $(s,x)\in Q$, *there exist an* $(\mathcal{F}_t)$*-stopping time* ξ *and a unique* $(\mathcal{F}_t)$*-adapted solution to the following SDE*

$$\begin{cases} X_t^{(1)} = x^{(1)} + \int_0^t X_r^{(1)}\ln|X_r^{(1)}|dr + \int_0^t \ln(2+|X_r|^2)dW_r^{(1)}, \\ X_t^{(2)} = x^{(2)} + \int_0^t X_r^{(2)}\ln|X_r^{(1)}|dr + \int_0^t \ln(2+|X_r|^2)dW_r^{(2)}, \end{cases}$$

which can be rewritten as

$$X_t = x + \int_0^t X_r\ln|X_r^{(1)}|dr + \int_0^t I_2\ln(2+X_r^2)dW_r, \quad t\in[0,\xi).$$

More precisely, for $n \in \mathbb{N}$, we can take $p(n) \in (2, \infty)$ and $q(n) = \infty$, then $\|bI_{Q^n}\|_{\mathbb{L}^\infty_{p(n)}} < \infty$, and $\|\partial\sigma I_{Q^n}\|_{\mathbb{L}^\infty_{p(n)}} < \infty$. Put $0 < \delta_n < \ln^2 2$, then condition (ii) in Theorem 2.1 also is fulfilled.

6.2 Example-Non-explosion

Example 6.3. *Consider $d = 1$, $Q = \mathbb{R}_+ \times (0, \infty)$, and $Q^n = (0, n) \times \{x : 1/n < x < n\}$, for $\delta > 0$, let $\phi(t, x) = |x|^{-\delta} + |x|$, $\sigma(t, x) = 2 + \sin x$.*

We can find that ϕ is a nonnegative continuous function on Q and blows up near the parabolic boundary of Q. For $n \in \mathbb{N}$, take $q(n) = \infty$, $p(n) \in (2, \infty)$, then $1/p(n) + 2/q(n) < 1$ and $\|(-\sigma^2\nabla\phi + \sigma\nabla\sigma)I_{Q^n}\|_{\mathbb{L}^\infty_{p(n)}} < \infty$. Besides,

$$\nabla(\sigma^2\nabla\phi)(t, x) \leq Ce^{3/2\phi(t,x)}$$

with constant $C \in (0, \infty)$. For σ, it can be easily checked that the conditions in Assumption 1 are satisfied. Then by Theorem 2.4 the following SDE has a unique $(\mathcal{F}_t)$-adapted solution on Q:

$$X_t = x + \int_0^t (2 + \sin X_s)dW_s + \int_0^t (\delta X_s|X_s|^{-\delta-2} - \frac{X_s}{|X_s|})(2 + \sin X_s)^2 ds + \int_0^t (2 + \sin X_s)\cos X_s ds, \quad t \geq 0.$$

6.3 Diffusions in random media

We apply our results to a particle which performs a random motion in $\mathbb{R}^d$, $d \geq 2$, interacting with impurities which are randomly distributed according to a Gibbs measure of Ruelle type. So, the impurities form a locally finite subset $\gamma = \{x_k | k \in \mathbb{N}\} \subset \mathbb{R}^d$. The interaction is given by a pair potential V and diffusion coefficient σ to be specified below defined on $\{x \in \mathbb{R}^d : |x| > \rho\}$, where $\rho \geq 0$ is a given constant. The stochastic dynamics of the particle is then determined by a stochastic equation type (2.5) as in Theorem 2.4 above with

$$Q := \mathbb{R}_+ \times (\mathbb{R}^d \backslash \gamma^\rho), \quad \phi(t, x) := \sum_{y\in\gamma} V(x - y), \quad (t, x) \in Q, \tag{6.1}$$

where γ^ρ is the closed $\rho-$neighborhood of the set γ, i.e., the random path $(X_t)_{t\geq 0}$ of the particle should be the strong solution of

$$X_t = x + \int_0^t \sigma(X_s)dW_s + (\frac{1}{2}\sum_{j=1}^d \int_0^t \partial_j a_{ij}(X_s)ds)_{1\leq i\leq d} - \sum_{w\in\gamma}\int_0^t (\sigma\sigma^*)(X_s)\nabla V(X_s - w)ds. \tag{6.2}$$

Below we shall give conditions on the pair potential V and diffusion coefficient σ which imply that this is indeed the case, i.e. that Theorem 2.4 above applies, for all γ outside a set of measure zero for the Gibbs measure. Here the original case is from [7] section 9.1, we generalize it to the multiplicative noise case. Similarly the set of admissible impurities γ we can treat is

$$\Gamma_{ad} := \left\{\gamma \subset \mathbb{R}^d | \forall r > 0 \exists c(\gamma, r) > 0 : |\gamma \cap B_r(x)| \leq c(\gamma, r)\log(1 + |x|), \forall x \in \mathbb{R}^d\right\}, \tag{6.3}$$

where $B_r(x)$ denotes the open ball with center x and radius r, $|A|$ denotes the cardinality of a set A. From [7] we know that for essentially all classes of Gibbs measure in equilibrium statistical mechanics of interacting infinite particle systems in $\mathbb{R}^d$ the set Γ_{ad} has measure one, this is also true for Ruelle measures.

We fix a $\gamma \in \Gamma_{ad}$. The necessary conditions on the pair potential V and diffusion coefficient σ go as follows (the typical case when $\rho = 0$ is also included):
(V1) The function V is positive and once continuously differentiable in $\mathbb{R}^d \cap \{|x| > \rho\}$, $\lim_{|x|\downarrow\rho} V(x) = \infty$.
(V2) There exist finite constants $\alpha > d/2$, $C \geq 0$, $\epsilon \in [1, 2)$ such that with $U(x) =: C(1 + |x|^2)^{-\alpha}$ we have

$$|V(x)| + |\nabla V(x)| \leq U(x) \quad \text{for} \quad |x| > \rho, \tag{6.4}$$

and for any $|y| > \rho$

$$\sum_{i,j=1}^{d} (\partial_j a_{ij}(x)\partial_i V(y) + a_{ij}(x)\partial_i\partial_j V(y)) \leq C(e^{\epsilon(V+U)(y)} - 1) \tag{6.5}$$

in the sense of distributions on $\left\{x \in \mathbb{R}^d : |x| > \rho\right\}$ where $\sigma(x) = (\sigma_{ij}(x))_{1\leq i,j\leq d} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ satisfies the following conditions:
(σ1) There exists a positive constant K such that for all $x \in \mathbb{R}^d$

$$\frac{1}{K}|\lambda|^2 \leq \langle(\sigma\sigma^*)(x)\lambda, \lambda\rangle \leq K|\lambda|^2, \quad \forall\lambda \in \mathbb{R}^d. \tag{6.6}$$

(σ2) For $1 \leq i,\ j \leq d$, $\sigma_{ij} \in \mathcal{C}_b^2(\mathbb{R}^d)$.

We emphasize that above conditions are fulfilled for essentially all potentials of interests in statistical physics.

Introduce $\bar{V}(x) = V(x) + 2U(x)$, $|x| > \rho$, and for $(t, x) \in Q$ let

$$\bar{\phi}(t, x) := \sum_{y\in\gamma} \bar{V}(x - y), \quad (a_{ij})_{1\leq i,j\leq d} := \sigma\sigma^*,$$

$$b(t, x) := 2\sum_{w\in\gamma} (\sigma\sigma^*)(x)\nabla U(x - w).$$

Owing to (6.4), (6.6) and the fact that $\gamma \in \Gamma_{ad}$, the function ϕ is continuously differentiable in Q and $|b(t, x)| \leq NK \log(2 + |x|)$, where N is independent of (t, x) (See [7] Section 9.1). Meanwhile for appropriate constants N on Q we have for $|y| > \rho$

$$2\sum_{i,j=1}^{d} (\partial_j a_{ij}(x)\partial_i U(y) + a_{ij}(x)\partial_j\partial_i U(y)) \leq N(e^{\epsilon U(y)} - 1)$$

because of conditions (σ1) and (σ2). Combing this with the fact that $V + U$ is positive and $\sum(e^{a_k} - 1) \leq e^{\sum a_k} - 1$, $a_k \geq 0$, we find that there exists a constant $N' > 0$ independent of (t, x) such that

$$\sum_{i,j=1}^{d} \partial_j(a_{ij}\partial_i\bar{\phi})(x) = \sum_{i,j=1}^{d}\sum_{w\in\gamma} \partial_j(a_{ij}(x)\partial_i(V(x - w) + 2U(x - w)))$$

$$\leq N \sum_{w\in\gamma} \Big((e^{\epsilon(V(x-w)+2U(x-w))} - 1) + (e^{\epsilon U(x-w)} - 1) \Big)$$
$$\leq N'(e^{\epsilon\bar{\phi}(x)} - 1).$$

It shows that all conditions on $\bar{\phi}$ and σ in Theorem 2.4 are fulfilled and therefore by Remark 5.1 the equation

$$X_t = x + \int_0^t \sigma(X_s)dW_s - \int_0^t (\sigma\sigma^*\nabla\bar{\phi})(X_s)ds + (\frac{1}{2}\sum_{j=1}^d \int_0^t \partial_j a_{ij}(X_s)ds)_{1\leq i\leq d} + \int_0^t b(X_s)ds \tag{6.7}$$

has a unique strong solution defined for all times if $x \in \mathbb{R}^d \backslash \gamma^\rho$. Since equation (6.7) coincides with SDE (6.2), we get the desired conclusion.

6.4 M-particle systems with gradient dynamics

In this subsection we consider a model of M particles in $\mathbb{R}^d$ interacting via a pair potential V and diffusion coefficient σ satisfying the following conditions:
(V1) The function V is once continuously differentiable in $\mathbb{R}^d\backslash\{0\}$, $\lim_{|x|\to 0} V(x) = \infty$, and on $\mathbb{R}^d\backslash\{0\}$ we assume that $V \geq -U$, where $U(x) := C(1+|x|^2)$, C is a constant.
(V2) There exists a constant $\epsilon \in [1,2)$ such that for arbitrary $x, y \in \mathbb{R}^d\backslash\{0\}$,

$$\sum_{i,j=1}^d (\partial_j a_{i,j}(x)\partial_i V(y) + a_{i,j}(x)\partial_i\partial_j V(y)) \leq Ce^{\epsilon(V+U)(y)} \tag{6.8}$$

in the sense of distributions.
Here $(a_{i,j})_{1\leq i,j\leq d} := \sigma\sigma^*$ and $\sigma(x) = (\sigma_{i,j}(x))_{1\leq i,j\leq d} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the diffusion coefficient satisfying:
(σ1) There exists a positive constant K such that for all $x \in \mathbb{R}^d$

$$\frac{1}{K}|\lambda|^2 \leq \langle(\sigma\sigma^*)(x)\lambda, \lambda\rangle \leq K|\lambda|^2, \quad \forall\lambda \in \mathbb{R}^d,$$

(σ2) For $1 \leq i, j \leq d$, $\sigma_{i,j} \in \mathcal{C}_b^2(\mathbb{R}^d)$.

Introduce $\bar{V} := V + 2U$,

$$Q := \mathbb{R}_+ \times \left(\mathbb{R}^{Md}\backslash \cup_{1\leq k<j\leq M} \left\{x = (x^{(1)}, ..., x^{(M)}) \in \mathbb{R}^{Md} : x^{(k)} = x^{(j)}\right\}\right),$$

$$Q^n := (0,n) \times \left\{x = (x^{(1)}, ..., x^{(M)}) \in \mathbb{R}^{Md} : |x| < n, x^{(k)} \neq x^{(j)} \text{ for } 1 \leq k < j \leq M\right\},$$

and let the function ϕ, $\bar{\phi}$, $\bar{\sigma}$, $\bar{a}$ and b be defined on Q by

$$\phi(t,x) := \sum_{1\leq k<j\leq M} V(x^{(k)} - x^{(j)}), \quad \bar{\phi}(t,x) := \sum_{1\leq k<j\leq M} \bar{V}(x^{(k)} - x^{(j)}),$$

$$\bar{\sigma}(t,x) := \begin{bmatrix} \sigma(x^{(1)}) & 0 & 0 \\ 0 & \sigma(x^{(2)}) & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \sigma(x^{(M)}) \end{bmatrix}, \quad \bar{a}(t,x) := \begin{bmatrix} (\sigma\sigma^*)(x^{(1)}) & 0 & 0 \\ 0 & (\sigma\sigma^*)(x^{(2)}) & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & (\sigma\sigma^*)(x^{(M)}) \end{bmatrix},$$

$$b := (b^{(1)}, ..., b^{(M)}), \quad b^{(k)}(t,x) := 4C(\sigma\sigma^*)(x^{(k)}) \sum_{1\le j\ne k\le M} (x^{(k)} - x^{(j)}), \quad k = 1, \cdots, M.$$

Observe that for arbitrary y, $x \in \mathbb{R}^d \setminus \{0\}$,

$$2\sum_{i,j=1}^{d} (\partial_j a_{i,j}(x)\partial_i U(y) + a_{i,j}(x)\partial_j\partial_i U(y)) \le N e^{\epsilon U(y)}$$

for an appropriate constant N which is independent of y, x. Besides ϕ and $\bar{\phi}$ are continuously differentiable on Q. If we use the notation

$$\partial_r^k f(x) := \partial_r^k f((x^{(1)}, \cdots, x^{(M)})) := \frac{\partial f((x^{(1)}, \cdots, x^{(M)}))}{\partial x_r^{(k)}}$$

for $k = 1, \cdots, M$ and $r = 1, \cdots, d$, then for $x \in \mathbb{R}^{Md}$,

$$\bar{a}_{i,j}(t,x) = \sum_{k=1}^{M} a_{i-(k-1)d,j-(k-1)d}(x^{(k)}) I_{(k-1)d<i,j\le kd}, \tag{6.9}$$

$$\partial_r^k \bar{a}_{i,j}(t,x) = \partial_r^k a_{i-(k-1)d,j-(k-1)d}(x^{(k)}) I_{(k-1)d<i,j\le kd} = \partial_r a_{i-(k-1)d,j-(k-1)d}(x^{(k)}) I_{(k-1)d<i,j\le kd}, \tag{6.10}$$

where $1 \le i,j \le Md$, and

$$\partial_r^k \bar{\phi}(t,x) = \sum_{1\le q\ne k\le M} \partial_r V((x^{(k)} - x^{(q)}) sign(q-k)) sign(q-k) + 4C \sum_{1\le q\ne k\le M} (x_r^{(k)} - x_r^{(q)}),$$

furthermore,

$$\partial_n^m \partial_r^k \bar{\phi}(t,x) = \sum_{1\le q\ne k\le M} \Big(I_{m=k} \partial_n \partial_r V((x^{(k)} - x^{(q)}) sign(q-k)) \\ - I_{m=q} \partial_n \partial_r V((x^{(k)} - x^{(q)}) sign(q-k)) \Big) + 4C(I_{m=k,n=r} - I_{m\ne k,n=r}).$$

Combining the above equalities with our assumptions of V and σ, by algebraic calculation we get that on Q there exists a large number $C_{M,d}$ depending on Md and a constant $C' \in (0,\infty)$ such that

$$2D_t\bar{\phi}(t,x) + \sum_{i,j=1}^{Md} \partial_j(\bar{a}_{i,j}\partial_i\bar{\phi})(t,x) \\ = \sum_{i,j=1}^{d}\sum_{k=1}^{M} \Big(\partial_j^k a_{i,j}(x^{(k)}) \partial_i^k \bar{\phi}(t,x) + a_{i,j}(x^{(k)}) \partial_j^k \partial_i^k \bar{\phi}(t,x) \Big) \\ = \sum_{i,j=1}^{d}\sum_{k=1}^{M}\sum_{1\le q\ne k\le M} \Big(\partial_j a_{i,j}(x^{(k)}) [\partial_i V((x^{(k)} - x^{(q)}) sign(q-k)) sign(q-k) + 4C(x_i^{(k)} - x_i^{(q)})] \\ + a_{i,j}(x^{(k)}) [\partial_j \partial_i V((x^{(k)} - x^{(q)}) sign(q-k))] \Big) + \sum_{i,j=1}^{d}\sum_{k=1}^{M} a_{i,j}(x^{(k)}) 4C I_{i=j}$$

$$\le C_{M,d} \sum_{1\le q<g\le M} (Ce^{\epsilon(V(x^{(q)}-x^{(g)})+U(x^{(q)}-x^{(g)}))} + Ne^{\epsilon(U(x^{(q)}-x^{(g)}))})$$
$$\le C' e^{\epsilon\bar{\phi}(t,x)}.$$

The continuity of $\bar{a}_{i,j}(t,x)$ on Q and $\partial_j^k \bar{a}_{i,j}(t,x)$ on Q^n can be easily checked from (6.9) and (6.10) and the conditions about σ. In order to reduce the lengthy algebraic computation, we only show the part for $\bar{a}_{i,j}(t,x)$, similarly we can get the desired continuity for $\partial_j^k \bar{a}_{i,j}(t,x)$ on Q^n. For any (t,x) and $(s,y) \in Q$, by (6.9) we have for $1 \le i,j \le Md$,

$$\begin{aligned}
|\bar{a}_{i,j}(t,x) - \bar{a}_{i,j}(s,y)| &\le C_{Md} \sum_{k=1}^{M} |a_{i-(k-1)d,j-(k-1)d}(x^{(k)}) - a_{i-(k-1)d,j-(k-1)d}(y^{(k)})| I_{(k-1)d<i,j\le kd} \\
&\le C_{Md} \sum_{k=1}^{M} |x^{(k)} - y^{(k)}| \\
&\le C''|x-y|.
\end{aligned}$$

We can adjust constants C'' and K such that there is still a positive constant such condition $(\sigma 1)$ satisfied.

It follows that all conditions on $\bar{\phi}$ and $\bar{\sigma}$ in Theorem 2.4 are fulfilled and therefore by Remark 5.1 the corresponding stochastic equation for a process $(X_t)_{t\ge 0} = (X_t^{(1)}, ..., X_t^{(M)})_{t\ge 0}$ has a unique strong solution defined for all times whenever for the initial condition x we have $(0,x) \in Q$. The corresponding equation is the following system

$$\begin{aligned}
X_t^{(k)} = x^{(k)} + \int_0^t \sigma(X_s^{(k)}) dW_s^{(k)} - &\int_0^t (\sigma\sigma^*)(X_s^{(k)}) \partial_k \bar{\phi}(s, X_s) ds \\
&+ (\frac{1}{2} \sum_{j=1}^{d} \int_0^t \partial_j a_{i,j}(X_s^{(k)}) ds)_{1\le i\le d} + \int_0^t b^{(k)}(s, X_s) ds.
\end{aligned}$$

We rewrite it as following with $k = 1, ..., M$

$$\begin{aligned}
X_t^{(k)} = x^{(k)} + &\int_0^t \sigma(X_s^{(k)}) dW_s^{(k)} \\
&- \int_0^t (\sigma\sigma^*)(X_s^{(k)}) \sum_{j=1, j\ne k}^{M} \nabla V((X_s^{(k)} - X_s^{(j)}) sign(j-k)) sign(j-k) ds \\
&+ (\frac{1}{2} \sum_{j=1}^{d} \int_0^t \partial_j a_{i,j}(X_s^{(k)}) ds)_{1\le j\le d},
\end{aligned}$$

which has a unique strong solution defined for all times whenever $(0, (x^{(1)}, ..., x^{(M)})) \in Q$.

A Appendix

Lemma A.1. *([13, P. 1 Lemma 1.1.]) Let $\{\beta(t)\}_{t\in[0,T]}$ be a nonnegative measurable $(\mathcal{F}_t)_{t\ge 0}-$adapted process. Assume that for all $0 \le s \le t \le T$,*

$$E\left(\int_s^t \beta(r) dr \middle| \mathcal{F}_s \right) \le \Gamma(s,t),$$

where $\Gamma(s,t)$ *is a nonrandom interval function satisfying the following conditions:*
(i) $\Gamma(t_1,t_2)\le\Gamma(t_3,t_4)$ *if* $(t_1,t_2)\subset(t_3,t_4)$*;*
(ii) $\lim_{h\downarrow 0}\sup_{0\le s<t\le T,|t-s|\le h}\Gamma(s,t)=\lambda,\ \lambda\ge 0$. *Then for any real* $\kappa<\lambda^{-1}$ *(if* $\lambda=0$*, then* $\lambda^{-1}=\infty$*),*

$$Eexp\left\{\kappa\int_0^T\beta(r)dr\right\}\le C=C(\kappa,\Gamma,T)<\infty.$$

For the convenience of the reader, we include the $\mathcal{C}^\infty$-Urysohn Lemma here.

Lemma A.2. *([3, 8.18]) If* $K\subset\mathbb{R}^n$ *is compact and* U *is an open set containing* K*, there exists a smooth function* f *such that* $0\le f\le 1$*,* $f=1$ *on* K*, and* $supp(f)\subset U$*.*

The following lemma is based on a consequence of 7.6.4 in [11]. We use this result a couple of times and hence for the sake of completeness we state it here precisely.

Lemma A.3. *Let* σ *and* $b^{(i)}$*,* $i=1,2$ *satisfy the conditions in Lemma 3.1. Let* $(X_t^{(i)},W_t^{(i)})_{t\ge 0}$ *satisfy:*

$$X_t^{(i)}=x+\int_0^t b^{(i)}(s,X_s^{(i)})ds+\int_0^t\sigma(s,X_s^{(i)})dW_s^{(i)}.$$

Then for any bounded Borel function f *given on* $\mathcal{C}=:\mathcal{C}([0,\infty),\mathbb{R}^d)$ *we have*

$$Ef(X_\cdot^{(2)})=Ef(X_\cdot^{(1)})\overline{\rho}_\infty$$

if

$$E\exp\Big(\frac{1}{2}\int_0^\infty(\Delta b^*(s,X_s^{(1)})(\sigma\sigma^*)^{-1}(s,X_s^{(1)})\Delta b(s,Xs^{(1)}))ds\Big)<\infty,\tag{7.1}$$

where $\Delta b(t,X_t^{(1)}):=b^{(2)}(t,X_t^{(1)})-b^{(1)}(t,X_t^{(1)})$ *and*

$$\begin{aligned}\overline{\rho}_t:=\exp(&\int_0^t\Delta b^*(s,X_s^{(1)})(\sigma^*)^{-1}(s,X_s^{(1)})dW_s^{(1)}\\&-\frac{1}{2}\int_0^t(\Delta b^*(s,X_s^{(1)})(\sigma\sigma^*)^{-1}(s,X_s^{(1)})\Delta b)(s,X_s^{(1)})ds),\quad t\ge 0.\end{aligned}$$

Proof. Theorem 6.1 in [11] says if (7.1) (Novikov condition) holds, then $(\overline{\rho}_t)_{t\ge0}$ is an $(\mathcal{F}_t)-$ martingale. Let $\hat{P}=\overline{\rho}_\infty P$, then $\hat{P}$ is also a probability on $(\Omega,\mathcal{F})$. By Theorem 4.1 in [6],

$$\hat{W}_t=W_t^{(1)}-\int_0^t\sigma^{-1}(s,X_s^{(1)})\Delta b(s,X_s^{(1)})ds,\quad t\ge 0$$

is a $(\mathcal{F}_t)$-Brownian motion on the probability space $(\Omega,\mathcal{F},\hat{P})$. So we can wirte

$$\begin{aligned}X_t^{(1)}&=x+\int_0^t b^{(1)}(s,X_s^{(1)})ds+\int_0^t\sigma(s,X_s^{(1)})d\hat{W}_s+\int_0^t\sigma(s,X_s^{(1)})\sigma^{-1}(s,X_s^{(1)})\Delta b(s,X_s^{(1)})ds\\&=x+\int_0^t b^{(1)}(s,X_s^{(1)})ds+\int_0^t\sigma(s,X_s^{(1)})d\hat{W}_s+\int_0^t\Delta b(s,X_s^{(1)})ds\end{aligned}$$

$$= x + \int_0^t b^{(2)}(s, X_s^{(1)})ds + \int_0^t \sigma(s, X_s^{(1)})d\hat{W}_s, \quad t \geq 0.$$

This implies that $(X_t^{(1)}, \hat{W}_t)_{t\geq 0}$ is a solution to the SDE

$$X_t^{(2)} = x + \int_0^t b^{(2)}(s, X_s^{(2)})ds + \int_0^t \sigma(s, X_s^{(2)})dW_s^{(2)}, \quad t \geq 0, \tag{7.2}$$

on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \hat{P})$. From Lemma 3.1 we know that the solution to SDE (7.2) is unique, hence for any bounded Borel function $f(x)$, given on $\mathcal{C} =: \mathcal{C}([0,\infty), \mathbb{R}^d)$ we have

$$Ef(X_\cdot^{(2)}) = \hat{E}f(X_\cdot^{(1)}) = E\overline{\rho}_\infty f(X_\cdot^{(1)}).$$

$\square$

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