

Strong dissipativity of generalized time-fractional derivatives and quasi-linear (stochastic) partial differential equations

Wei Liu

Michael Röckner

José Luís da Silva

Abstract

In this paper strong dissipativity of generalized time-fractional derivatives on Gelfand triples of properly in time weighted L^2 -path spaces is proved. In particular, as special cases the classical Caputo derivative and other fractional derivatives appearing in applications are included. As a consequence one obtains the existence and uniqueness of solutions to evolution equations on Gelfand triples with generalized time-fractional derivatives. These equations are of type

$$\frac{d}{dt}(k * u)(t) + A(t, u(t)) = f(t), \quad 0 < t < T,$$

with (in general nonlinear) operators $A(t, \cdot)$ satisfying general weak monotonicity conditions. Here k is a non-increasing locally Lebesgue-integrable nonnegative function on $[0, \infty)$ with $\lim_{s \rightarrow \infty} k(s) = 0$. Analogous results for the case, where f is replaced by a time-fractional additive noise, are obtained as well. Applications include generalized time-fractional quasi-linear (stochastic) partial differential equations. In particular, time-fractional (stochastic) porous medium and fast diffusion equations with ordinary or fractional Laplace operators and the time-fractional (stochastic) p -Laplace equation are covered.

Keywords. generalized time-fractional derivative; strong dissipativity; weak monotonicity; generalized porous medium equation; p -Laplace equation

1 Introduction

In this paper (see Theorem 2.2 below) we prove existence and uniqueness of solutions to non-local in time evolution equations of type

$$\partial_t^{*k}(u - u_0) + A(t, u(t)) = f(t), \quad 0 < t < T, \quad (1.1)$$

W. Liu: School of Mathematics and Statistics, Jiangsu Normal University, 221116 Xuzhou, China; e-mail: weiliu@jsnu.edu.cn

M. Röckner: Faculty of Mathematics, Bielefeld University, 33615 Bielefeld, Germany / Academy of Mathematics and Systems Science, CAS, 100190 Beijing, China; e-mail: roeckner@math.uni-bielefeld.de (*Corresponding author*)

J.L. da Silva: CIMA, University of Madeira, 9020-105 Funchal, Portugal; e-mail: joses@staff.uma.pt

Mathematics Subject Classification (2010): Primary 35R11, 45D05, 35K59, 26A33; Secondary 60H15, 76S05, 35K92

on a separable real Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, which is the pivôt space of a Gelfand triple

$$V \subseteq H (\cong H^*) \subseteq V^*,$$

where V is a reflexive Banach space with dual V^* . Here $T \in (0, \infty)$, u_0 is the initial condition and

$$A(t, \cdot): V \longrightarrow V^*, \quad t \in [0, \infty),$$

are (in general nonlinear) weakly-monotone operators satisfying (H1)–(H4) in Section 2 below. Furthermore, $f(t) \in V^*$, $t \in [0, \infty)$, and

$$\partial_t^{*k} u := \partial_t(k * u) := \frac{d}{dt} \int_0^t k(t-s)u(s) ds, \quad t \in [0, \infty), \quad (1.2)$$

for $k \in L^1_{\text{loc}}([0, \infty))$, $k \geq 0$, non-increasing and without loss of generality right-continuous. Here we also refer to (2.2) below, which is the integral form of (1.1) and follows from (1.1) under an additional assumption on k (see condition (\tilde{k}) in Section 2 below).

In [39] the special case

$$k(t) := g_{1-\beta}(t) := \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t \in [0, \infty), \beta \in (0, 1), \quad (1.3)$$

i.e. where $\partial_t^{*k}(u - u_0)$ is the Caputo time-fractional derivative of u , has been treated. For more examples of functions k , also called *kernels* in the literature, we refer to Section 6.

Although fractional calculus has a long history and its origins can be traced back to the end of seventeenth century (cf. [57]), this subject became very active only over the last few decades. One of the main reasons is that scientists and engineers have established a vast amount of new models (e.g. to describe anomalous diffusions) that naturally involve time-fractional differential equations, which have been applied successfully, e.g. in mechanics (cf. [43]), bio-chemistry (cf. [24, 25]), electrical engineering (cf. [23]), medical science (cf. [21]). For more applications and references we refer to [6, 22, 30, 50, 51, 52].

In [39], however, the stronger hypothesis that $A(t, \cdot): V \longrightarrow V^*$, $t \in [0, \infty)$, is monotone (that is, $C_1 = 0$ in (H2), see Section 2), was assumed, which excludes a number of important applications. Apart from this and the more general non-local time derivatives ∂_t^{*k} , which for distinction we call *generalized* time-fractional derivatives, in this paper we give a new and easy proof of uniqueness of solutions to (1.1). The proofs both for generalizing to weakly-monotone $A(t, \cdot)$, $t \in [0, \infty)$, and for uniqueness turn out to be consequences of a new result on (generalized) time-fractional derivatives, namely that ∂_t^{*k} considered as an operator (more precisely, $-\partial_t^{*k}$ is identified as a generator of a C_0 -operator semigroup) on a properly in time weighted L^2 -space is strongly dissipative (see Theorem 2.1 and Section 3 below), which together with its applications to uniquely solving (1.1) (see Theorem 2.2 and Section 4 below) can be considered as the main contribution of this work. In particular, our results are applicable to the time-fractional generalized porous medium and fast diffusion equations with ordinary or fractional Laplace operators

$$\partial_t^{*k}(u(t) - u_0) + (-\Delta)^\alpha(|u(t)|^{r-1}u(t)) = f(t, u(t))$$

and the time-fractional p -Laplace equation

$$\partial_t^{*k}(u(t) - u_0) - \operatorname{div} (|\nabla u(t)|^{p-2} \nabla u(t)) = f(t, u(t)).$$

We refer to Section 7 for details and more general types of these equations, which our results apply to.

Analogously to [39] as a consequence by a simple shift argument we obtain the unique solvability of the stochastically perturbed variant of (1.1), namely

$$\partial_t^{*k_1}(X(t) - X_0) + A(t, X(t)) = \partial_t^{*k_2} \int_0^t B(s) dW(s), \quad 0 < t < T, \quad (1.4)$$

where $W(t)$, $t \geq 0$, is a cylindrical Brownian motion in some other separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ and $B(s): U \rightarrow H$ is a Hilbert–Schmidt operator for every $s \in [0, \infty)$ (see Theorem 2.3 below).

At this point we would like to stress that, since the operator A is allowed to be nonlinear as e.g. a quasi-linear partial or pseudo differential operator (see Section 7 below for examples), the classical probabilistic “inverse subordination method” (see [4, 5, 47, 54] and also [19] as well as the references therein) to solve equation (1.1) does not work.

Let us now comment on the motivation of studying equations of type (1.1) with generalized time-fractional derivatives. There is substantial motivation from Physics and Mathematics as regards the use of generalized time-fractional derivatives (see e.g. [1, 22, 32, 50, 51, 52]). Here we mention a few examples and refer to Section 6 for the details. Starting from the seminal paper [12] the Caputo fractional derivative was introduced to properly handle initial value problems, namely to model waves in viscoelastic media. Later on it was generalized to the so called distributed order derivative (also called variable order derivative in [40]), see [13] and Example 6.3 below for details. Other successful applications of the distributed order derivative includes the kinetic theory (cf. [15, 16, 34, 35]) to describe ultra-slow diffusion or the theory of elasticity (see [40]) for the description of rheological properties of composite materials. Inverse stable subordinators arise (cf. [46, 48]) as scaling limits of continuous time random walks. In [49] it was shown that under certain technical conditions the probability density of the hitting time process $E(t)$ (that is the inverse of a certain subordinator) solve a distributed order time-fractional evolution equation. For more applications of the distributed order derivative we refer the reader to [3, 14, 28, 33, 44, 45].

When dealing with a particular anomalous diffusion process, it is often difficult to choose which model of the time-fractional diffusion equations is suitable for its mathematical description. Thus a general framework of time-fractional derivatives is needed. In [36], the author introduced a general fractional calculus for integral operators of convolution type with an arbitrary nonnegative locally integrable kernel k . He considered the initial value problem for both relaxation and diffusion equations with these general time-fractional derivatives. Since then many authors applied the generalized time-fractional derivative to solve in general linear fractional equations and nonlinear differential equations, see e.g. [41, 62, 39] and references therein. We want to remark that huge amount of existing literatures in this subject are mostly for the case of linear and semilinear type equations, however, to the best of our knowledge, there are only very few results that is applicable to the quasilinear case.

We should mention that time-fractional linear evolution equations in the Gelfand triple setting have first been investigated in [63]. Later on the author also proved the global solvability of a nondegenerate parabolic equation with time-fractional derivative in [64] (cf. [1] for more general cases). However, these results cannot be applied to quasilinear type equations like the porous medium or the p -Laplace equation. In [31] the authors investigate elliptic-parabolic integro-differential equations with L^1 -data. Their framework includes the time-fractional p -Laplace equation. However, the authors in [31] only obtain generalized solutions (*i.e.* entropy solutions). Therefore, the results of the current paper generalize or complement the corresponding results in [31, 39, 62, 63, 64] within the setting of time-fractional quasilinear PDE with weakly monotone coefficients. In particular, the authors in [62] derive very interesting decay estimates for the solutions of time-fractional porous medium and p -Laplace equations (by assuming the existence of solutions), and the decay behaviour is notably different from the case with usual time derivative. In [39], we give a positive answer to the question on the existence and uniqueness of solutions to the time-fractional porous medium equations and p -Laplace equations, which are left open in [62]. The current work further extend the results in [39] to both generalized fractional derivative and the weakly monotone case.

Recently, there has been also growing interest in time-fractional stochastic partial differential equations. For instance, the authors in [20] investigate the L^2 -theory for a class of semilinear SPDEs with time-fractional derivatives, which can be used to describe random effects on transport of particles in media with thermal memory, or particles subject to sticking and trapping. In [26, 53], the authors consider a space-time fractional stochastic heat type equation to model phenomena with random effects with thermal memory, and they prove the existence and uniqueness of mild solutions as well as some intermittency property. For a linear stochastic partial differential equation of fractional order both in the time and space variables with a different type of noise term, we refer to [18] (see also [2]). In [17] the authors investigate linear stochastic time-fractional partial differential equations for the type of heat equation and wave equation.

The list of references quoted above is far from being complete, but show the enormous interest in the subject. But to the best of our knowledge the results of this paper are new. In particular, the result on the strong dissipativity of generalized time-fractional derivatives is novel and we hope that it will be useful for the field.

The rest of the paper is organized as follows. In Section 2 we present the main results (Theorems 2.1, 2.2 and 2.3) on the existence and uniqueness of solutions to deterministic and stochastic nonlinear evolution equations with generalized time-fractional derivatives. Theorem 2.1 will be proved in Section 3. The proof of Theorem 2.2 is given in Section 4. It relies on Theorem 2.1 and an abstract perturbation result for (pseudo) monotone operators already used in [39]. For the convenience of the reader we include its proof in the Appendix of this paper. Because of its importance we give a more detailed proof than in [39], where this proof was presented in a very concise way. The proof of Theorem 2.3 will be given in Section 5. In Section 6 we give many examples of kernels k which appeared in literatures. In Section 7 we apply the main results to some concrete quasi-linear deterministic and stochastic PDE.

2 Framework and main results

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a real separable Hilbert space identified with its dual space H^* by the Riesz isomorphism. Let V be a real reflexive Banach space, continuously and densely embedded into H . Then we have the following Gelfand triple

$$V \subseteq H \cong H^* \subseteq V^*.$$

Let ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denote the dualization between V and its dual space V^* and let $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_{V^*}$ denote the respective norms. Then it is easy to show that

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \quad u \in H, v \in V.$$

Now, for $T \in [0, \infty)$ fixed, we consider the following general nonlinear evolution equation with generalized time-fractional derivative

$$\partial_t^{*k}(u - u_0) + A(t, u(t)) = f(t), \quad \text{for dt-a.e. } t \in [0, T], \quad (2.1)$$

where $k \in L^1_{\text{loc}}([0, \infty); \mathbb{R}, ds) =: L^1_{\text{loc}}([0, \infty))$ (with $ds = \text{Lebesgue measure}$) satisfies condition (k) below, $f \in L^1([0, \infty); V^*)$, ∂_t^{*k} is as in (1.2), $u_0 \in V$ is the initial condition, and we are seeking for solutions $u \in L^1([0, \infty); V)$. Therefore, the derivative $\frac{d}{dt}$ in the definition (1.2) of ∂_t^{*k} is understood in the weak sense. Consider the following conditions on k :

(k) $k \in L^1_{\text{loc}}([0, \infty))$, k is nonnegative, non-increasing and (hence without loss of generality) right continuous such that $\lim_{s \rightarrow \infty} k(s) = 0$.

(\tilde{k}) There exists $\tilde{k} \in L^1_{\text{loc}}([0, \infty))$, nonnegative, such that

$$(\tilde{k} * k)(t) = \int_0^t \tilde{k}(t-s)k(s) ds = 1 \quad \text{for dt-a.e. } t \in [0, \infty).$$

Here and below we consider k and \tilde{k} as functions on \mathbb{R} defining them to be zero on $(-\infty, 0)$. Obviously (k) and (\tilde{k}) hold for k as in (1.3).

If (k) and (\tilde{k}) hold, then (2.1) can be rewritten as

$$u(t) = u_0 - \int_0^t \tilde{k}(t-s)A(s, u(s)) ds + \int_0^t \tilde{k}(t-s)f(s) ds \quad \text{for dt-a.e. } t \in [0, \infty). \quad (2.2)$$

This can be easily seen by first integrating (2.1) with respect to dt and using the fact that the convolution with $k * \tilde{k} = \tilde{k} * k$ is just integration with respect to dt . Defining $\tilde{u}(t)$ to be equal to the right hand side of (2.2) for every $t \in [0, \infty)$, we have that \tilde{u} is a dt -version of u , hence still satisfies (2.2) with $\tilde{u}(0) = u_0$. In this sense u has u_0 as its initial condition. Now let us specify the conditions on the map

$$A: [0, \infty) \times V \longrightarrow V^*$$

which is first of all assumed to be $\mathcal{B}([0, \infty) \times V)/\mathcal{B}(V^*)$ measurable (where $\mathcal{B}(\cdot)$ means Borel σ -algebra of \cdot) and assumed to satisfy the following: There exist $\alpha \in (1, \infty)$, $\delta \in (0, \infty)$, $C_1, C_2 \in [0, \infty)$ and $g \in L^1([0, \infty); \mathbb{R})$ such that for all $t \in [0, \infty)$, $v, v_1, v_2 \in V$

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on \mathbb{R} .

(H2) (Weak Monotonicity)

$${}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \geq -C_1 \|v_1 - v_2\|_H^2.$$

(H3) (Coercivity)

$${}_{V^*}\langle A(t, v), v \rangle_V \geq \delta \|v\|_V^\alpha - C_2 \|v\|_H^2 - g(t).$$

(H4) (Growth)

$$\|A(t, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq g(t) + C_2 (\|v\|_V^\alpha + \|v\|_H^2).$$

We define the following spaces,

$$\begin{aligned} \mathcal{V} &= L^\alpha([0, \infty); V) \cap L^2([0, \infty); H), \\ \mathcal{H} &= L^2([0, \infty); H), \\ \mathcal{V}^* &= L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*) + L^2([0, \infty); H), \end{aligned}$$

where $\|\cdot\|_{\mathcal{V}} := \max(\|\cdot\|_{L^\alpha([0, \infty); V)}, \|\cdot\|_{\mathcal{H}})$ and for $u \in \mathcal{V}^*$

$$\|u\|_{\mathcal{V}^*} := \inf \left\{ \|u_1\|_{L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*)} + \|u_2\|_{\mathcal{H}} : u_1 \in L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*), u_2 \in \mathcal{H} \text{ s.t. } u = u_1 + u_2 \right\}.$$

Then for $u_0 = 0$ (the case for general initial conditions $u_0 \in V$ will then follow easily as we shall see below) the original equation (2.1) can be rewritten in the following form

$$\partial_t^{*k} u + \mathcal{A}u = f, \tag{2.3}$$

where

$$\mathcal{A}: \mathcal{V} \longrightarrow \mathcal{V}^*; (\mathcal{A}u)(t) = A(t, u(t)), \quad t \in [0, \infty).$$

It is easy to see that $\mathcal{A}: \mathcal{V} \longrightarrow \mathcal{V}^*$ is weakly monotone, coercive and bounded on bounded sets. Below we fix k and \mathcal{A} as above.

To formulate our main results we furthermore need to define the following ‘‘shift to the right’’ semigroup $U_t, t > 0$, on \mathcal{H} . Below we extend every $f \in \mathcal{H}$ by $f := 0$ on $(-\infty, 0)$ to a function $f: \mathbb{R} \longrightarrow H$. For $f \in \mathcal{H}, t \geq 0$, define

$$U_t f(r) := \mathbb{1}_{[0, \infty)}(r-t) f(r-t), \quad r \in [0, \infty). \tag{2.4}$$

Then it is trivial to check that $(U_t)_{t>0}$ is a strongly continuous (shortly: C_0 -)contraction semigroup on \mathcal{H} and it obviously can be restricted to a C_0 -semigroup on \mathcal{V} (even in this case consisting also of contractions on \mathcal{V}). Now for k as above and $\mu_t^k, t \geq 0$, as defined in (3.5) below, we define for $f \in \mathcal{H}$

$$U_t^k f := \int_0^\infty U_s f \mu_t^k(ds) = f * \mu_t^k, \quad t \geq 0, \tag{2.5}$$

It is a well-known fact (see e.g. [42, Chap. II, Sect. 4b]), that $(U_t^k)_{t>0}$ is also a C_0 -semigroup of contractions on \mathcal{H} . Let Λ^k with domain $D(\Lambda^k, \mathcal{H})$ be its infinitesimal generator on \mathcal{H} .

Obviously, $(U_t^k)_{t>0}$ can be restricted to a C_0 -semigroup on \mathcal{V} (again consisting of contractions). The generator of the latter is again Λ^k , but with domain

$$D(\Lambda^k, \mathcal{V}) := \{u \in D(\Lambda^k, \mathcal{H}) \cap \mathcal{V} \mid \Lambda^k u \in \mathcal{V}\}.$$

Then $D(\Lambda^k, \mathcal{V})$ is dense in \mathcal{V} , hence so is $D(\Lambda^k, \mathcal{H}) \cap \mathcal{V}$.

By [61, Lemma 2.3], $\Lambda^k: D(\Lambda^k, \mathcal{H}) \cap \mathcal{V} \rightarrow \mathcal{V}^*$ is closable as an operator from \mathcal{V} to \mathcal{V}^* . Denoting its closure by $(\Lambda^k, \mathcal{F}^k)$, \mathcal{F}^k is a Banach space with norm $\|u\|_{\mathcal{F}^k} := (\|u\|_{\mathcal{V}}^2 + \|\Lambda^k u\|_{\mathcal{V}^*}^2)^{\frac{1}{2}}$, $u \in \mathcal{F}^k$.

Finally, we define a convenient domain of ∂_t^{*k} , namely:

$$D(\partial_t^{*k}) := \{u \in \mathcal{V}^* \mid k * u \in W^{1,1}((0, T); V^*), \forall T \in (0, \infty)\}, \quad (2.6)$$

where $W^{1,1}((0, T); V^*)$ denotes the standard Sobolev space of order 1 in $L^1([0, T]; V^*)$.

We recall that for $u \in L^1([0, \infty); V^*)$ we set $u \equiv 0$ on $(\infty, 0)$, hence

$$(k * u)(t) = \int_0^t k(t-s)u(s) ds. \quad (2.7)$$

Then, obviously, for all $T \in (0, \infty)$

$$k * u = (\mathbb{1}_{[0, T]} k) * (\mathbb{1}_{[0, \infty)} u) \quad \text{on } [0, T], \quad (2.8)$$

where for $p \in [1, \infty)$ the function on the right hand side belongs to $L^p(\mathbb{R}; V^*)$ if so does $\mathbb{1}_{[0, \infty)} u$, and is in $C(\mathbb{R}; V^*)$, if in addition $k \in L_{\text{loc}}^{p'}([0, \infty))$, where $p' := \frac{p}{p-1}$.

For $\gamma \in (0, \infty)$, $p \in [1, \infty)$, and $E = \mathbb{R}, V, V^*$ or H we set

$$L_\gamma^p([0, \infty); E) := L^p([0, \infty); E, e^{-\gamma t} dt). \quad (2.9)$$

Theorem 2.1. *Suppose that k satisfies (k). Then:*

(i) $\mathcal{F}^k \subset D(\partial_t^{*k})$ and

$$\Lambda^k u = -\partial_t^{*k} u.$$

*In particular, $k * u \in C([0, \infty), H)$.*

(ii) (“strong dissipativity on $L_\gamma^2([0, \infty); H)$ ”). For every $\gamma \in (0, \infty)$ and all $u \in \mathcal{F}^k$

$$\int_0^\infty {}_{V^*} \langle \partial_t^{*k} u(s), u(s) \rangle_V e^{-\gamma s} ds \geq \frac{1}{2} \psi^k(\gamma) \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds, \quad (2.10)$$

where

$$\psi^k(\gamma) := \int_{(0, \infty)} (1 - e^{-\tau \gamma}) M^k(d\tau) (> 0!) \quad (2.11)$$

and M^k is the unique measure on $((0, \infty), \mathcal{B}(0, \infty))$ such that $k(s) = M^k((s, \infty))$, $s \in (0, \infty)$ (see the beginning of Section 3 for more details, in particular (3.3)).

Theorem 2.2. *Suppose that $T \in [0, \infty)$, k satisfies (k) and $A: [0, \infty) \times V \rightarrow V^*$ satisfies (H1)–(H4). Furthermore, assume that for C_1 from (H2) there exists $\gamma \in (0, \infty)$ such that $\psi^k(\gamma) > 2C_1$, which is always the case if $\lim_{s \rightarrow 0} k(s) = \infty$. Then:*

- (i) *For every $u_0 \in V$ and $f \in \mathcal{V}^*$, (2.1) has a unique solution u such that $u - u_0\varphi \in \mathcal{F}^k$ for every $\varphi \in L^\alpha([0, \infty); \mathbb{R})$ with $\varphi \equiv 1$ on $[0, T + 1)$. In particular,*

$$u - u_0\varphi \in L^\alpha([0, \infty); V); \partial_t^{*k}(u - u_0\varphi) \in L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*) \quad (2.12)$$

and $t \mapsto \int_0^t k(t-s)(u(s) - u_0\varphi(s)) ds$ has a continuous H -valued dt -version.

- (ii) *If, in addition, (\tilde{k}) holds, then for dt -a.e. $t \in [0, T]$,*

$$u(t) = u_0 - \int_0^t \tilde{k}(t-s)A(s, u(s)) ds + \int_0^t \tilde{k}(t-s)f(s) ds. \quad (2.13)$$

Furthermore, if $\tilde{k} \in L_{\text{loc}}^\alpha([0, \infty))$, $t \mapsto u(t)$ has a continuous V^ -valued dt -version.*

Now we turn to our last main result, namely the stochastic version of (2.1) and (2.2).

Suppose that U is a Hilbert space and $W(t)$ is a U -valued cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with normal filtration \mathcal{F}_t , $t \geq 0$. Now we consider stochastic nonlinear evolution equations with generalized time-fractional derivative of type

$$\partial_t^{*k_1}(X(t) - x_0) + A(t, X(t)) = \partial_t^{*k_2} \int_0^t B(s) dW(s), \quad 0 < t < T, \quad (2.14)$$

where $x_0 \in V$ and $B: [0, T] \rightarrow L_{HS}(U; H)$ is measurable, here $(L_{HS}(U; H), \|\cdot\|_{HS})$ denotes the space of all Hilbert–Schmidt operators from U to H . Note that, if k_1 satisfies (\tilde{k}) , the integral form of (2.14) is as follows

$$X(t) = x_0 - \int_0^t \tilde{k}_1(t-s)A(s, X(s)) ds + \int_0^t (\tilde{k}_1 * k_2)(t-s)B(s) dW(s). \quad (2.15)$$

For this we need to assume more about k_1 and k_2 from above, namely that they satisfy

- (ks) (k) holds for k_1 and k_2 , and k_1 satisfies (\tilde{k}) such that $\tilde{k}_1 * k_2 \in L_{\text{loc}}^2([0, \infty))$.

Note that the stochastic integral term in (2.14)

$$F(t) := \int_0^t (\tilde{k}_1 * k_2)(t-s)B(s) dW(s)$$

is well-defined if e.g. $\|B\|_{HS} \in L_{\text{loc}}^\infty([0, \infty))$, because then

$$\int_0^t (\tilde{k}_1 * k_2)^2(t-s)\|B(s)\|_{HS}^2 ds < \infty.$$

If $k_1 = k_2$, then the stochastic integral term is even well-defined if merely $\|B\|_{HS} \in L_{\text{loc}}^2([0, \infty))$.

Theorem 2.3. *Suppose that (ks) holds, A satisfies (H1)–(H4) and $B \in L^\infty([0, T], L_{HS}(U; H))$. Assume also that $F \in V, dt \otimes \mathbb{P}$ -a.e. (which is e.g. the case if $B(t)$ is a Radonifying map from U to V). Then:*

(i) *For every $x_0 \in V$ the “shifted equation”*

$$\partial_t^{*k_1}(X(t) - F(t) - x_0\varphi) + A(t, X(t)) = 0, \quad dt\text{-a.e. } t \in [0, T], \quad (2.16)$$

has a unique (\mathcal{F}_t) -adapted solution X such that $X - F - x_0\varphi \in \mathcal{F}^{k_1}, \mathbb{P}$ -a.s. for every $\varphi \in L^\alpha([0, \infty); \mathbb{R})$ with $\varphi \equiv 1$ on $[0, T + 1)$. In particular,

$$X - F - x_0\varphi \in L^\alpha([0, \infty); V); \quad \partial_t^{*k_1}(X - F - x_0\varphi) \in L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*), \quad \mathbb{P}\text{-a.s.}$$

and $t \mapsto \int_0^t k_1(t-s)(X(s) - x_0\varphi(s)) ds$ \mathbb{P} -a.s. has a continuous H -valued dt -version.

(ii) *For dt -a.e. $t \in [0, T]$,*

$$X(t) = x_0 - \int_0^t \tilde{k}_1(t-s)A(s, X(s)) ds + \int_0^t (\tilde{k}_1 * k_2)(t-s)B(s) dW(s), \quad \mathbb{P}\text{-a.s.} \quad (2.17)$$

Furthermore, if $\tilde{k}_1 \in L_{loc}^\alpha([0, \infty))$, $t \mapsto X(t)$ \mathbb{P} -a.s. has a continuous V^ -valued dt -version.*

Remark 2.4. *In [39], we have investigated the case that $\partial_t^{*k_1} = \partial_t^\beta, \partial_t^{*k_2} = \partial_t^\gamma$ and A is monotone. Then it is easy to see that the assumption (ks) is equivalent to $\gamma < \beta + \frac{1}{2}$. We want to remark that the special case $\gamma = \beta$ or $\gamma = 1$ has been intensively investigated for some semilinear SPDE models (such as the stochastic heat equation or the stochastic wave equation), see e.g. [2, 17, 18, 26, 53] and more references therein.*

3 Generalized time-fractional derivatives as generators of C_0 -semigroups and their strong dissipativity

In this section we prove Theorem 2.1, so assume that k satisfies (k). By Caratheodory’s theorem there exists a σ -finite (nonnegative) measure M^k on $((0, \infty), \mathcal{B}((0, \infty)))$ such that

$$M^k((s, \infty)) = k(s), \quad s \in (0, \infty). \quad (3.1)$$

By Fubini’s theorem it is easy to show that

$$\int_{(0, \infty)} \tau \wedge 1 M^k(d\tau) < \infty. \quad (3.2)$$

Define

$$\mathbb{C}_{\geq 0} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$$

and

$$\mathbb{C}_{>0} := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

We define the following function $\psi^k : \mathbb{C}_{\geq 0} \rightarrow \mathbb{C}$ by

$$\psi^k(\lambda) := \int_{(0,\infty)} (1 - e^{-\lambda\tau}) M^k(d\tau), \quad \lambda \in \mathbb{C}_{\geq 0}, \quad (3.3)$$

which by (3.2) is well-defined and holomorphic on $\mathbb{C}_{>0}$, as well as continuous on $\mathbb{C}_{\geq 0}$ (see [59, p.25] for details). Hence the same is true for the function

$$\lambda \mapsto e^{-t\psi^k(\lambda)}, \quad \lambda \in \mathbb{C}_{\geq 0}, \quad (3.4)$$

for every $t \in [0, \infty)$. Furthermore for every $t \in [0, \infty)$, since $t\psi^k$ restricted to $(0, \infty)$ is a non-negative Bernstein function (see [59, Theorem 3.2]), there exists a unique probability measure μ_t^k on $([0, \infty), \mathcal{B}([0, \infty)))$ such that

$$\int_{[0,\infty)} e^{-\lambda s} \mu_t^k(ds) = e^{-t\psi^k(\lambda)}, \quad \lambda \in (0, \infty), \quad (3.5)$$

(see [59, Theorems 3.7 and 1.4]). Furthermore, since the Laplace transform

$$\mathcal{L}\mu_t^k(\lambda) := \int_{[0,\infty)} e^{-\lambda s} \mu_t^k(ds) \quad (3.6)$$

is defined for all $\lambda \in \mathbb{C}_{\geq 0}$ and is obviously holomorphic on $\mathbb{C}_{>0}$, as well as continuous on $\mathbb{C}_{\geq 0}$, (3.5) implies that

$$\mathcal{L}\mu_t^k(\lambda) = e^{-t\psi^k(\lambda)} \quad \text{for all } \lambda \in \mathbb{C}_{\geq 0}. \quad (3.7)$$

In particular, we have for every $t \in [0, \infty)$ for the Fourier transform $\hat{\mu}_t^k$ of μ_t^k

$$\hat{\mu}_t^k(\lambda) := \int_{(0,\infty)} e^{i\lambda s} \mu_t^k(ds) = e^{-t\psi^k(-i\lambda)} = e^{-t \int_{(0,\infty)} (1 - e^{i\lambda\tau}) M^k(d\tau)}, \quad \lambda \in \mathbb{R}. \quad (3.8)$$

By (3.2) the function $\mathbb{R} \ni \lambda \mapsto |\psi^k(-i\lambda)|$ is of at most linear growth.

Now let us consider the C_0 -semigroup $(U_t^k)_{t \geq 0}$ on \mathcal{H} with infinitesimal generator $(\Lambda^k, D(\Lambda^k, \mathcal{H}))$ introduced in Section 2. First we characterize this generator through its Fourier transform and as a corollary we prove that it coincides with ∂_t^{*k} on an operator core.

Proposition 3.1. *The generator $(\Lambda^k, D(\Lambda^k, \mathcal{H}))$ of $(U_t^k)_{t > 0}$ (on \mathcal{H}), defined in Section 2, is given as follows*

$$\begin{aligned} D(\Lambda^k, \mathcal{H}) &= \{u \in \mathcal{H} \mid r \mapsto |\psi^k(-ir)|\hat{u}(r) \in L^2(\mathbb{R}; H_{\mathbb{C}})\}, \\ (\Lambda^k u)^\wedge(r) &= -\psi^k(-ir)\hat{u}(r), \quad r \in \mathbb{R}, \end{aligned}$$

where $H_{\mathbb{C}}$ denotes the complexification of H and \hat{u} denotes the Fourier transform of u considered as a function from \mathbb{R} to H , i.e. $u := 0$ on $(-\infty, 0)$ and

$$\hat{u}(r) := \int_{\mathbb{R}} e^{irs} u(s) ds, \quad r \in \mathbb{R}.$$

Proof. Below we consider each $\mu_t^k(ds)$ as a measure on all of \mathbb{R} , by defining

$$\mu_t^k(A) := \mu_t^k(A \cap [0, \infty)), \quad A \in \mathcal{B}(\mathbb{R}).$$

Let $D := \{u \in \mathcal{H} \mid r \mapsto |\psi^k(-ir)|\hat{u}(r) \in L^2(\mathbb{R}; H_{\mathbb{C}})\}$. Then for $u \in D$, because $u * \mu_t^k \in L^2(\mathbb{R}; H)$ and $\hat{\mu}_t^k$ is bounded, we have

$$\begin{aligned} \frac{1}{t}(U_t^k u - u)^\wedge(r) &= \frac{1}{t} \left(\int_0^\infty U_s u \mu_t^k(ds) - u \right)^\wedge(r) \\ &= \frac{1}{t} \left(\int_{\mathbb{R}} u(\cdot - s) \mu_t^k(ds) - u \right)^\wedge(r) \\ &= \frac{1}{t} (u * \mu_t^k - u)^\wedge(r) \\ &= \frac{1}{t} \hat{u}(r) \left(e^{-t\psi^k(-ir)} - 1 \right) \xrightarrow[t \rightarrow 0]{} -\psi^k(-ir)\hat{u}(r) \end{aligned}$$

for dr -a.e. $r \in \mathbb{R}$. But since for all $r \in \mathbb{R}$ and $t > 0$,

$$\frac{1}{t} \left| e^{-t\psi^k(-ir)} - 1 \right| \leq 2|\psi^k(-ir)|,$$

the last convergence also holds in $L^2(\mathbb{R}; H_{\mathbb{C}})$. Hence $D \subset D(\Lambda^k, \mathcal{H})$ and

$$(\Lambda^k u)^\wedge(r) = -\psi^k(-ir)\hat{u}(r), \quad r \in \mathbb{R}. \quad (3.9)$$

Because $\hat{\mu}_t^k$ is bounded, one similarly checks that

$$U_t^k D \subseteq D \quad \forall t > 0, \quad (3.10)$$

and that (Λ^k, D) is closed as an operator from \mathcal{H} to \mathcal{H} . Since the function $\mathbb{R} \ni r \mapsto |\psi^k(-ir)|$ is at most of linear growth, D is dense in \mathcal{H} . Hence (3.9) implies (see [55, Theorem X.49]) that D is an operator core of $(\Lambda^k, D(\Lambda^k, \mathcal{H}))$, i.e. D is dense in $D(\Lambda^k, \mathcal{H})$ with respect to the graph norm given by Λ^k . Consequently, $D = D(\Lambda^k, \mathcal{H})$ and Λ^k is given by (3.9). \square

Proposition 3.2. $D(\Lambda^k, \mathcal{H}) \subset \mathcal{F}^k$ and for all $u \in D(\Lambda^k, \mathcal{H})$

$$\Lambda^k u = -\frac{d}{dt}(k * u). \quad (3.11)$$

Proof. First let $u \in D_0 := D(\Lambda^k, \mathcal{H}) \cap \mathcal{V} \cap D(\partial_t^{*k}) \cap L^\infty([0, \infty); H)$. Then for $T \in (0, \infty)$

$$\|(k * u)(t)\|_{V^*} \leq \text{ess sup}_{s \in [0, T]} \|u(s)\|_{V^*} \int_0^t k(s) ds \quad \text{for } dt\text{-a.e. } t \in [0, T], \quad (3.12)$$

and the same inequality holds with $\|\cdot\|_H$ replacing $\|\cdot\|_{V^*}$.

Again we consider all appearing functions, originally only defined on $[0, \infty)$, as functions on all of \mathbb{R} by defining them to be equal to zero on $\mathbb{R} \setminus [0, \infty)$. As in the proof of Proposition 3.1, one can check

that D_0 is dense in \mathcal{H} and also that $U_t^k(D_0) \subset D_0$. Concerning the latter we note that all spaces in the intersection defining D_0 are obviously invariant under U_t^k except for $D(\partial_t^{*k})$. To see that this is also true for the latter, let $u \in D(\partial_t^{*k})$. Then $U_t^k u = u * \mu_t^k \in L^1([0, \infty); V^*)$ and for $T \in (0, \infty)$ there exist $h \in L^1([0, T]; V^*)$ and $v \in V^*$ such that for $r \in [0, T]$

$$(k * u)(r) = v + \int_0^r h(\tau) d\tau.$$

But again by setting $h \equiv 0$ on $(-\infty, 0)$ and using Fubini's theorem

$$\begin{aligned} (k * U_t^k u)(r) &= (k * u * \mu_t^k)(r) \\ &= v + \int_0^\infty \int_0^{r-s} h(\tau) d\tau \mu_t^k(ds) \\ &= v + \int_0^\infty \int_{-s}^{r-s} h(\tau) d\tau \mu_t^k(ds) \\ &= v + \int_0^\infty \int_0^r h(\tau - s) d\tau \mu_t^k(ds) \\ &= v + \int_0^r (h * \mu_t^k)(\tau) d\tau, \quad r \in [0, T]. \end{aligned}$$

Since $h * \mu_t^k \in L^1([0, T]; V^*)$, this implies that $U_t^k u \in D(\partial_t^{*k})$. Again applying Theorem X.49 from [55] we obtain that D_0 is an operator core of $(\Lambda^k, D(\Lambda^k, \mathcal{H}))$. Hence it remains to prove (3.11).

Let us start with calculating the Laplace transform \mathcal{L} of the right hand side of (3.11) for any $u \in D(\partial_t^{*k}) \cap L^\infty([0, \infty); V^*)$. So let $\lambda \in (0, \infty)$. Then integrating by parts, using (3.12) and Fubini's Theorem we obtain

$$\begin{aligned} \mathcal{L} \left(\frac{d}{dt} (k * u) \right) (\lambda) &= \int_0^\infty e^{-\lambda t} \frac{d}{dt} (k * u)(t) dt \\ &= \lim_{T \rightarrow \infty} \left(e^{-\lambda T} (k * u)(T) + \lambda \int_0^T e^{-\lambda t} (k * u)(t) dt \right) \\ &= \lambda \int_0^\infty e^{-\lambda t} \int_0^t k(t-s) u(s) ds dt \\ &= \lambda \int_0^\infty \int_s^\infty k(t-s) e^{-\lambda t} dt u(s) ds \\ &= \int_0^\infty e^{-\lambda s} u(s) ds \lambda \int_0^\infty M^k((t, \infty)) e^{-\lambda t} dt \\ &= \lambda \int_0^\infty \int_{(t, \infty)} M^k(ds) e^{-\lambda t} dt \mathcal{L}u(\lambda) \\ &= \int_{(0, \infty)} \int_0^s \lambda e^{-\lambda t} dt M^k(ds) \mathcal{L}u(\lambda) \\ &= \psi^k(\lambda) \mathcal{L}u(\lambda), \end{aligned}$$

where we used (3.1) in the fifth inequality and (3.3) in the last inequality.

For the left-hand side of (3.11) and $u \in D_0$ we find for all $h \in H$, $\lambda \in (0, \infty)$, because of (3.7)

$$\begin{aligned} \left\langle \int_0^\infty \Lambda^k u(r) e^{-\lambda r} dr, h \right\rangle_H &= \lim_{t \rightarrow 0} \frac{1}{t} \int_0^\infty \langle U_t^k u(r) - u(r), h \rangle_H e^{-\lambda r} dr \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\mathcal{L}(\langle u, h \rangle_H * \mu_t^k) - \mathcal{L}(\langle u, h \rangle_H) \right) (\lambda) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (e^{-t\psi^k(\lambda)} - 1) \mathcal{L}(\langle u, h \rangle_H) (\lambda) \\ &= -\psi^k(\lambda) \langle \mathcal{L}u(\lambda), h \rangle_H. \end{aligned}$$

Hence, $\mathcal{L}(\Lambda^k(u))(\lambda) = -\psi^k(\lambda) \mathcal{L}u(\lambda)$ and (3.11) follows for $u \in D_0$.

Now let $u \in D(\Lambda^k, \mathcal{H})$. Then, since D_0 is an operator core for $(\Lambda^k, D(\Lambda^k, \mathcal{H}))$, there exist $u_n \in D_0$, $n \in \mathbb{N}$, such that as $n \rightarrow \infty$

$$u_n \longrightarrow u \quad \text{and} \quad \Lambda^k u_n \longrightarrow \Lambda^k u \quad \text{in } \mathcal{H}.$$

Let $T \in (0, \infty)$. Then as $n \rightarrow \infty$ by (2.8) and, since (3.11) holds for u_n ,

$$k * u_n \longrightarrow k * u$$

and

$$-\frac{\partial}{\partial t}(k * u_n) \longrightarrow \Lambda^k u$$

in $L^1([0, T]; V^*)$. Hence, the last assertion follows by the completeness of $W^{1,1}([0, T]; V^*)$. \square

After these preparations we can prove the first part of Theorem 2.1.

Proof of Theorem 2.1(i). Let $u \in \mathcal{F}^k$. Then there exist $u_n \in \mathcal{V} \cap D(\Lambda^k, \mathcal{H})$, $n \in \mathbb{N}$, such that as $n \rightarrow \infty$

$$u_n \longrightarrow u \quad \text{in } \mathcal{V} \quad \text{and} \quad -\partial_t^{*k} u_n = \Lambda^k u_n \longrightarrow \Lambda^k u \quad \text{in } \mathcal{V}^*, \quad (3.13)$$

where we used Proposition 3.2. Let $T \in (0, \infty)$. By (2.8), $k * u_n \longrightarrow k * u$ in $L^\alpha([0, T]; V)$, hence in $L^1([0, T]; V^*)$, as $n \rightarrow \infty$ and for $p := \min\{2, \frac{\alpha}{\alpha-1}\}$ the latter part of (3.13) implies that $\partial_t^{*k} u_n$, $n \in \mathbb{N}$, are bounded in $L^p([0, T]; V^*)$. Hence the Cesaro mean of a subsequence of $(\partial_t^{*k} u_n)_{n \in \mathbb{N}}$ converges strongly in $L^p([0, T]; V^*)$, hence in $L^1([0, T]; V^*)$. Therefore, by completeness $k * u \in W^{1,1}([0, T]; V^*)$ and

$$\Lambda^k u = -\frac{d}{dt}(k * u) \quad \text{on } [0, T] \quad \text{dt-a.e.}$$

The last part of the assertion then follows by [7, Theorem 1.19, pp.25]. \square

To prove Theorem 2.1(ii) we need some preparations.

Lemma 3.3. *Let $\gamma \in (0, \infty)$. Then for all $u \in \mathcal{H}$, $t \geq 0$,*

$$\int_0^\infty \|U_t^k u(s)\|_H^2 e^{-\gamma s} ds \leq e^{-\psi^k(\gamma)t} \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds.$$

Proof. Let $u \in \mathcal{H}$, $t \geq 0$. Then

$$\int_0^\infty \|U_t^k u(s)\|_H^2 e^{-\gamma s} ds = \int_0^\infty \|(u * \mu_t^k)(s)\|_H^2 e^{-\gamma s} ds \leq \mathcal{L}(\|u\|_H^2)(\gamma) e^{-t\psi^k(\gamma)},$$

where we used Jensen's inequality and (3.7) in the last step. \square

Lemma 3.4. *Let $\gamma \in (0, \infty)$ and $u \in D(\Lambda^k, \mathcal{H})$. Then*

$$\int_0^\infty \langle \Lambda^k u(s), u(s) \rangle_H e^{-\gamma s} ds \leq -\frac{1}{2} \psi^k(\gamma) \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds.$$

Proof. Since

$$\Lambda^k u = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (U_\varepsilon^k u - u) \quad \text{in } \mathcal{H}, \text{ hence in } L_\gamma^2([0, \infty); H),$$

we have by Lemma 3.3 and the Cauchy–Schwarz inequality

$$\begin{aligned} & \int_0^\infty \langle \Lambda^k u(s), u(s) \rangle_H e^{-\gamma s} ds \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_0^\infty \langle U_\varepsilon^k u(s), u(s) \rangle_H e^{-\gamma s} ds - \int_0^\infty \langle u(s), u(s) \rangle_H e^{-\gamma s} ds \right] \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[e^{-\frac{\varepsilon}{2} \psi^k(\gamma)} - 1 \right] \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds \\ &= -\frac{1}{2} \psi^k(\gamma) \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds. \end{aligned}$$

\square

Now we can prove the second part of Theorem 2.1.

Proof of Theorem 2.1(ii). Let $u \in \mathcal{F}^k$. By definition of $(\Lambda^k, \mathcal{F}^k)$ there exist $u_n \in D(\Lambda^k, \mathcal{H}) \cap \mathcal{V}$ such that as $n \rightarrow \infty$

$$u_n \rightarrow u \quad \text{in } \mathcal{V} \quad \text{and} \quad \Lambda^k u_n \rightarrow \Lambda^k u \quad \text{in } \mathcal{V}^*.$$

Hence by Lemma 3.4

$$\begin{aligned} \int_0^\infty {}_{V^*} \langle \Lambda^k u(s), u(s) \rangle_V e^{-\gamma s} ds &= \lim_{n \rightarrow \infty} \int_0^\infty {}_{V^*} \langle \Lambda^k u_n(s), u_n(s) \rangle_V e^{-\gamma s} ds \\ &\leq \lim_{n \rightarrow \infty} -\frac{1}{2} \psi^k(\gamma) \int_0^\infty \|u_n(s)\|_H^2 e^{-\gamma s} ds \\ &= -\frac{1}{2} \psi^k(\gamma) \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds, \end{aligned}$$

since $u_n \rightarrow u$ in \mathcal{V} as $n \rightarrow \infty$, implies that $u_n \rightarrow u$ in \mathcal{H} , hence in $L_\gamma^2([0, \infty); H)$ as $n \rightarrow \infty$. Hence the assertion follows by Theorem 2.1(i). \square

4 Proof of main existence and uniqueness result: the deterministic case

In this section we proof Theorem 2.2, so assume that (k) and (H1)–(H4) hold.

As in [39] the proof heavily relies on a general perturbation result of operators \mathcal{A} of the type as in Theorem 2.2, which we briefly recall now.

As in [61] we consider a generator Λ , with domain $D(\Lambda, \mathcal{H})$, of a C_0 -contraction semigroup of linear operators on \mathcal{H} whose restrictions to \mathcal{V} form a C_0 -semigroup of linear operators on \mathcal{V} . The generator of the latter is again Λ , but with domain $D(\Lambda, \mathcal{V}) := \{u \in \mathcal{V} \cap D(\Lambda, \mathcal{H}) \mid \Lambda u \in \mathcal{V}\}$. Then $D(\Lambda, \mathcal{V})$ is dense in \mathcal{V} , hence so is $D(\Lambda, \mathcal{H}) \cap \mathcal{V}$. By [61, Lemma 2.3], $\Lambda : \mathcal{D}(\Lambda, \mathcal{H}) \cap \mathcal{V} \rightarrow \mathcal{V}^*$ is closable as an operator from \mathcal{V} to \mathcal{V}^* . Denoting its closure by (Λ, \mathcal{F}) we obtain that \mathcal{F} is a Banach space with norm

$$\|u\|_{\mathcal{F}} := (\|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}^*}^2)^{\frac{1}{2}}, u \in \mathcal{F}.$$

Now we can formulate the following perturbation result.

Theorem 4.1. *Let conditions (H1)–(H4) hold. Assume that in (H2) we have $C_1 = 0$. Then for every $f \in \mathcal{V}^*$ there exists $u \in \mathcal{F}$ such that $\mathcal{A}u - \Lambda u = f$.*

This result is a generalization of [61, Proposition 3.2]. We replace the strong monotonicity assumption in [61, Proposition 3.2] by the classical monotonicity, i.e. (H2) with $C_1 = 0$, and consider a reflexive Banach space \mathcal{V} , while this space was assumed to be a Hilbert space in [61]. A rather concise proof in this more general case was given in [39]. Since this result is crucial for Theorem 2.2 and for the convenience of the reader we include a more detailed proof in the Appendix of this paper. Now we are prepared to prove the second main result of this paper.

Proof of Theorem 2.2(i). Existence:

Case 1: $u_0 = 0$.

Consider the operator

$$\tilde{A} := A + C_1 I,$$

where $I: V \rightarrow V^*$, $I(u) := u, u \in V$, and let $\tilde{\mathcal{A}}$ be defined as \mathcal{A} was for A . Then we can apply Theorem 4.1 with \mathcal{A} replaced by $\tilde{\mathcal{A}}$ and Λ replaced by $\Lambda^k = -\partial_t^{*k}$ with domain \mathcal{F}^k (see Theorem 2.1(i)). Hence for every $g \in \mathcal{H}(\subset \mathcal{V}^*)$ there exists $u_g \in \mathcal{F}^k$ such that

$$\partial_t^{*k} u_g + \mathcal{A}u_g + C_1 u_g = g + f \quad \text{in } \mathcal{V}^*. \quad (4.1)$$

Define: $\mathcal{H}_T := L^2([0, T]; H)$. Then $\mathcal{H}_T \hookrightarrow \mathcal{H}$ by the map $i(g) = \begin{cases} g \text{ on } [0, T] \\ 0 \text{ on } (T, \infty) \end{cases}$.

Consider the map $\mathcal{H} \supset \mathcal{H}_T \ni g \mapsto C_1 u_{g|_{[0, T]}} \in \mathcal{H}_T$, where for a function $h: [0, \infty) \rightarrow H$ we denote its restriction to $[0, T]$ by $h|_{[0, T]}$.

By (H2) and Theorem 2.1(ii) we have for all $\gamma \in (0, \infty)$; $g_1, g_2 \in \mathcal{H}_T$

$$\begin{aligned} & \frac{1}{2} \psi^k(\gamma) \int_0^T \|C_1 u_{g_1}(s) - C_1 u_{g_2}(s)\|_H^2 e^{-\gamma s} ds \\ & \leq C_1^2 \int_0^\infty \langle u_{g_1}(s) - u_{g_2}(s), \partial_t^{*k}(u_{g_1} - u_{g_2}) + A(s, u_{g_1}(s)) - A(s, u_{g_2}(s)) \\ & \quad + C_1(u_{g_1}(s) - u_{g_2}(s)) \rangle_{V^*} e^{-\gamma s} ds \\ & = C_1 \int_0^T \langle C_1 u_{g_1}(s) - C_1 u_{g_2}(s), g_1(s) - g_2(s) \rangle_H e^{-\gamma s} ds. \end{aligned}$$

Hence by the Cauchy–Schwarz inequality

$$\left(\int_0^T \|u_{g_1}(s) - u_{g_2}(s)\|_H^2 e^{-\gamma s} ds \right)^{\frac{1}{2}} \leq \frac{2C_1}{\psi^k(\gamma)} \left(\int_0^T \|g_1(s) - g_2(s)\|_H^2 e^{-\gamma s} ds \right)^{\frac{1}{2}}.$$

We recall that by assumption

$$\frac{2C_1}{\psi^k(\gamma)} < 1,$$

which can always be achieved for large enough γ by (2.11), if $M^k((0, \infty)) = \infty$, i.e. if $\lim_{s \rightarrow 0} k(s) = \infty$. Hence by Banach's fixed point theorem there exists $g \in \mathcal{H}_T$

$$C_1 u_g = g \quad dt\text{-a.e. on } [0, T].$$

But then by (4.1)

$$\partial_t^{*k} u_g(t) + A(t, u_g(t)) = f(t) \quad \text{for } dt\text{-a.e. } t \in [0, T],$$

so (2.1) holds for $u_0 = 0$. Furthermore by construction $u_g \in \mathcal{F}^k$. In particular, (2.12) holds for $u_0 = 0$.

Case 2: $u_0 \in V$.

Let φ be as in the assertion of the Theorem. Set $x := u_0$ and define \mathcal{A}_x as \mathcal{A} , but with

$$A_x(t, v) := A(t, v + x\varphi(t)), \quad t > 0, \quad v \in V,$$

replacing A . Then by Case 1 there exist $u_x \in \mathcal{F}^k$ such that

$$\frac{d}{dt}(k * u_x(t)) + A_x(t, u_x(t)) = f(t) \quad \text{for } dt\text{-a.e. } t \in [0, T].$$

Define $u := u_x + x\varphi$. Then $u - x\varphi (= u_x)$ satisfies (2.12) and

$$\partial_t^{*k}(u(t) - x) + A(t, u(t)) = f(t) \quad dt\text{-a.e. on } (0, T)$$

and (2.1) is solved.

The last part of assertion (i) of Theorem 2.2 follows by the last part of Theorem 2.1(i).

Uniqueness: Let u_1, u_2 be two solutions of (2.1) on $[0, T]$ such that $u_1 - \varphi u_0, u_2 - \varphi u_0 \in \mathcal{F}^k$ with φ as in the assertion. Then $u_1 - u_2 \in \mathcal{F}^k$ and by Theorem 2.1(ii)

$$\begin{aligned} 0 &= \int_0^\infty V \langle u_1(s) - u_2(s), \partial_t^{*k}(u_1(s) - u_2(s)) + A(s, u_1(s)) - A(s, u_2(s)) \rangle_{V^*} e^{-\gamma s} ds \\ &\geq \left(\frac{1}{2} \psi^k(\gamma) - C_1 \right) \int_0^\infty \|u_1(s) - u_2(s)\|_H^2 e^{-\gamma s} ds \\ &\geq 0, \end{aligned}$$

since by assumption $\psi^k(\gamma) > 2C_1$. Hence $u_1 = u_2$.

Proof of Theorem 2.2(ii). That under assumption (\tilde{k}) equation (2.1) can be rewritten as (2.13) was already explained in Section 2 of this paper. The last part of the assertion is an elementary fact about convolutions in Lebesgue L^p -spaces. \square

5 Proof of the stochastic case

The proof of Theorem 2.3 follows from Theorem 2.2 by a simple shift argument (cf. [39]).

Proof of Theorem 2.3. Let $u(t) = X(t) - F(t)$, then $u(t)$ satisfies the following equation

$$\partial_t^{*k_1}(u(t) - x) + A(t, u(t) + F(t)) = 0, \quad 0 < t < T. \quad (5.1)$$

Define

$$\tilde{A}(t, u) = A(t, u + F(t)), \quad u \in V.$$

Since $F \in V dt \otimes \mathbb{P}$ -a.e., it is easy to see that \tilde{A} still satisfies (H1)–(H4). Hence assertion (i) follows by Theorem 2.2(i). Assertion (ii) is then proved analogously to Theorem 2.2(ii).

The (\mathcal{F}_t) -adaptedness of the solution follows by the proofs of Theorem 4.1 and Lemma .6. The last two assertions are obvious. \square

6 Examples of Kernels

In the introduction we have mentioned the motivation of using different types of time-fractional derivatives in applications, and many related literatures have also been stated. In this section we give some examples of kernels k which satisfy both condition (k) and (\tilde{k}) needed to apply Theorems 2.1, 2.2 and 2.3 in Section 2.

Example 6.1 (Fractional Caputo derivative). *Let $0 < \beta < 1$ be given and define the function k on $[0, \infty)$ by*

$$k(t) := g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t \in [0, \infty).$$

Then k is nonnegative, nonincreasing function on $[0, \infty)$ and we have $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. It is well known that $\partial_t^{*k}(f - f(0))$ corresponds to the Caputo derivative of f and the problem stated in (1.1) has been treated in [39] in the case of monotone operator A . The associated Lévy measure is absolutely continuous with respect to the Lebesgue measure and is given by

$$M_\beta^k(dt) = \frac{\beta}{\Gamma(1-\beta)} t^{-(1+\beta)} dt. \quad (6.1)$$

It is simple to verify that k satisfies (k) and the corresponding $\tilde{k} \in L_{\text{loc}}^1([0, \infty))$ is given by

$$\tilde{k}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t \in [0, \infty).$$

Hence condition (\tilde{k}) is satisfied. The pair (k, \tilde{k}) is called Sonine kernels and $(\tilde{k} * k)(t) = 1$, $t \in [0, \infty)$ is known as Sonine condition, see [60] and [58] for a survey.

Example 6.2 (Truncated β -stable subordinator, cf. Example 2.1-(ii) in [19]). A process $S(t)$, $t \geq 0$ is called truncated β -stable subordinator if it is driftless and its Lévy measure is

$$M_\delta^k(dx) := \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} \mathbb{1}_{(0,\delta]}(x) dx, \quad \delta > 0.$$

The kernel k defined by

$$k(t) := M_\delta^k((t, \infty)) = \frac{\beta}{\Gamma(1-\beta)} \mathbb{1}_{(0,\delta]}(t) \int_t^\delta x^{-(1+\beta)} dx = \frac{\mathbb{1}_{(0,\delta]}(t)}{\Gamma(1-\beta)} (t^{-\beta} - \delta^{-\beta})$$

induces the following generalized time-fractional derivative

$$\partial_t^{*k}(f - f(0))(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t ((t-s)^{-\beta} - \delta^{-\beta})(f(s) - f(0)) ds.$$

Here for $a \in \mathbb{R}$, $a^+ := \max\{a, 0\}$. This is the generalized time-fractional derivative whose value at time t depends only on the δ -range of the past of f in contrast to the usual case which depends on the history of f on $(0, t)$. Notice that $\lim_{\delta \rightarrow 0} k(t) = \frac{1}{\Gamma(1-\beta)} t^{-\beta}$. We have $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. Hence, k satisfies condition (k), but also (\tilde{k}) , because M_δ^k is absolutely continuous with respect to the Lebesgue measure. Hence the existence of the kernel \tilde{k} follows from the theory of complete Bernstein functions, see Theorem 6.2 in [59].

Example 6.3 (Distributed order derivative). Let g_β as in (1.3) and define the kernel k by

$$k(t) := \int_0^1 g_\beta(t) d\beta, \quad t \geq 0.$$

The corresponding generalized time-fractional derivative is called distributed order derivative and it may be written as

$$\partial_t^{*k}(f - f(0))(t) = \int_0^1 \partial_t(g_\beta * (f - f(0)))(t) d\beta.$$

The kernel k is a nonincreasing, nonnegative function on $[0, \infty)$ which belongs to $L^1_{\text{loc}}([0, \infty))$. Moreover, $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. The associated nonnegative kernel \tilde{k} such that $\tilde{k} * k = 1$ has the form

$$\tilde{k}(t) = \int_0^\infty \frac{e^{-st}}{1+s} ds$$

and we have $\tilde{k} \in L^1_{\text{loc}}([0, \infty))$, so condition (\tilde{k}) is satisfied.

Example 6.4 (Exponential weight). For any $\gamma > 0$ and $0 < \beta < 1$ define the kernel k by

$$k(t) := g_{1-\beta}(t)e^{-\gamma t} = \frac{t^{\beta-1}}{\Gamma(\beta)}e^{-\gamma t}.$$

The kernel k is nonnegative, nonincreasing and $k \in L^1_{\text{loc}}([0, \infty))$, hence k satisfies condition (k) . We have $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. The associated nonnegative \tilde{k} such that $\tilde{k} * k = 1$ is given by

$$\tilde{k}(t) = \gamma^\beta + \frac{\beta}{\Gamma(1-\beta)} \int_t^\infty \frac{e^{-\gamma s}}{s^{1+\beta}} ds, \quad t \in [0, \infty).$$

The fact that $\tilde{k} * k = 1$ may be checked by applying the Laplace transform to both sides of the equation. Moreover, a simple integration shows that $\tilde{k} \in L^1_{\text{loc}}([0, \infty))$, hence condition (\tilde{k}) is satisfied.

Example 6.5 (Gamma subordinator). Let $a, b > 0$ be given and k the kernel defined by

$$k(t) := a\Gamma(0, bt), \quad t \in [0, \infty),$$

where $\Gamma(\nu, x) := \int_x^\infty t^{\nu-1}e^{-t} dt$ is the upper incomplete gamma function. It follows from the properties of $\Gamma(\nu, x)$ that k is a locally integrable, nonnegative, nonincreasing function on $[0, \infty)$ and we have $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. Hence, k satisfies condition (k) . The kernel k is related to the gamma subordinator (see for example [9, Ch. III]) through its Laplace transform, namely the process with Laplace exponent equal to

$$\lambda \int_0^\infty e^{-\lambda t} k(t) dt = a \log \left(1 + \frac{\lambda}{b} \right) = a \int_0^\infty (1 - e^{-\lambda t}) t^{-1} e^{-bt} dt, \quad \lambda > 0,$$

where the second equality stems from the Frullani integral. Hence, the Lévy measure is $M_{a,b}^k(dt) = at^{-1}e^{-bt} dt$. The existence of a positive $\tilde{k} \in L^1_{\text{loc}}([0, \infty))$ such that $\tilde{k} * k = 1$ is a consequence of the fact that $M_{a,b}^k$ is absolutely continuous with respect to the Lebesgue measure and the theory of complete Bernstein functions, see Theorem 6.2 in [59]. Hence condition (\tilde{k}) is satisfied.

Example 6.6 (Multi-term derivative). Let $0 < \beta < 1$ and $0 < \alpha < 1$ be given. Define the kernel k by

$$k(t) := g_{1-\beta}(t) + g_{1-\alpha}(t), \quad t > 0.$$

The kernel k is completely monotone, that is $k \in C^\infty((0, \infty))$ and $(-1)^n k^{(n)}(t) \geq 0$ for all $t > 0$ and $n \in \mathbb{N} \cup \{0\}$. The corresponding generalized time-fractional derivative ∂_t^{*k} is called multi-term

fractional derivative. We have $\lim_{t \rightarrow 0} k(t) = \infty$ and $\lim_{t \rightarrow \infty} k(t) = 0$. It follows from Example 6.1 that the Lévy measure $M_{\beta, \alpha}^k$ defining k is the sum of two Lévy measures of the type (6.1). It follows from Theorem 5.5 and Corollary 5.6 of [29] that there exists a nonnegative kernel $\tilde{k} \in L_{\text{loc}}^1([0, \infty))$ such that $\tilde{k} * k = 1$ and its Laplace transform is

$$\mathcal{L}\tilde{k}(\lambda) = \frac{1}{\lambda^\alpha + \lambda^\beta}.$$

Hence, the kernel k satisfies both conditions (k) and (\tilde{k}). This example may be generalized to kernels $k(t) := \sum_{j=1}^n a_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$ with $a_j > 0$ and $0 < \beta_1 < \dots < \beta_n < 1$.

7 Applications to quasi-linear (S)PDE

In this section we apply Theorems 2.2 and 2.3 to (stochastic) generalized porous medium equations, (stochastic) generalized p -Laplace equations, and (stochastic) generalized fast-diffusion equations (cf. [8, 38]) with time-fractional derivative. Here for simplicity we mainly concentrate on the deterministic case, the extension to the stochastic case is straightforward.

7.1 Generalized porous medium equations

We introduce the model as in [56]. Let $(E, \mathcal{B}, \mathbf{m})$ be a separable σ -finite measure space and $(L, \mathbb{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ having discrete spectrum. Let

$$(0 <) \lambda_1 \leq \lambda_2 \leq \dots$$

be all eigenvalues of $-L$ including multiplicities with unit eigenfunctions $\{e_i\}_{i \geq 1}$. Let H be the dual space of the $\mathbb{D}((-L)^{\frac{1}{2}})$ with respect to $L^2(\mathbf{m})$; i.e. H is the completion of $L^2(\mathbf{m})$ under the inner product

$$\langle x, y \rangle := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \mathbf{m}(xe_i) \mathbf{m}(ye_i),$$

where $\mathbf{m}(x) := \int_E x d\mathbf{m}$ for $x \in L^1(\mathbf{m})$. Let

$$\Psi, \Phi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$$

be measurable, and be continuous in the second variable. We consider the following generalized porous medium equation with generalized time-fractional derivative

$$\partial_t^{*k}(X_t - x_0) = L\Psi(t, X_t) + \Phi(t, X_t). \quad (7.1)$$

To verify conditions (H1), (H2), (H3) and (H4) for $A(t, v) := L\Psi(t, v) + \Phi(t, v)$, we assume that for a fixed constant $r \geq 1$,

$$\begin{aligned} |\Psi(t, s)| + |\Phi(t, s)| &\leq c(1 + |s|^r), \quad s \in \mathbb{R}, t \geq 0, \\ -\mathbf{m}((\Psi(t, x) - \Psi(t, y))(x - y)) + \mathbf{m}((\Phi(t, x) - \Phi(t, y))(-L)^{-1}(x - y)) \\ &\leq K\|x - y\|_H^2 - \delta\|x - y\|_{r+1}^{r+1}, \quad t \geq 0, \end{aligned} \quad (7.2)$$

hold for some constants $c, K, \delta > 0$ and all $x, y \in L^{r+1}(\mathbf{m})$, where $\|\cdot\|_{r+1}$ is the norm in $L^{r+1}(\mathbf{m})$. Obviously, the assumptions above are satisfied provided $\Psi(t, s) = h(t)|s|^{r-1}s$ and $\Phi(t, s) = g(t)s$, $t \in [0, T]$, $s \in \mathbb{R}$, with $0 < \inf h \leq \sup h < \infty$ and $\|g\|_\infty < \infty$.

Example 7.1 Let $V = L^{1+r}(\mathbf{m})$ and V^* be the dual space of V with respect to H . Then it is easy to see that (7.2) implies that (H1), (H2), (H3) and (H4) hold for (see [56, page 137])

$$A(t, v) := L\Psi(t, v) + \Phi(t, v).$$

Therefore, Theorem 2.2 is applicable to the time-fractional generalized porous medium equation (7.1) if k satisfies (k), (\tilde{k}) respectively.

Remark 7.1. (i) Let $r > 1$ and Δ be the Dirichlet Laplacian on an open domain $D \subset \mathbb{R}^d$. Let $L = \Delta$ if D is bounded and, in addition, $r \leq \frac{2d}{d+2}$, or $L = -(-\Delta)^\alpha$ for some constant $\alpha \in (0, \frac{d}{2}) \cap (0, 1]$ if $D = \mathbb{R}^d$ (the definition of V and H should be revised in the latter case, see [56]). Let

$$\Phi(t, s) = cs, \quad \Psi(t, s) = s|s|^{r-1},$$

for some constant $c \in \mathbb{R}$ (see [56, Example 3.4] for possible more general cases). Then the assertions in Theorem 2.2 hold.

(ii) Similarly, we could apply Theorem 2.3 to investigate the time-fractional stochastic generalized porous medium equation

$$\partial_t^{*k_1}(X_t - x_0) = \{L\Psi(t, X_t) + \Phi(t, X_t)\}dt + \partial_t^{*k_2} \int_0^t B(s)dW(s), \quad (7.3)$$

where $W(s)$, $s \geq 0$, is cylindrical Brownian motion on H and $B : [0, \infty) \rightarrow L_{HS}(H)$ is measurable and locally bounded.

7.2 Stochastic generalized p -Laplace equations

Let $D \subset \mathbb{R}^d$ be an open bounded domain, \mathbf{m} be the normalized volume measure on D , and $p \in [2, \infty)$. Let $H_0^{1,p}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm

$$\|f\|_{1,p} := \|f\|_p + \|\nabla f\|_p,$$

where $\|\cdot\|_p$ is the norm in $L^p(\mathbf{m})$. Let $H = L^2(\mathbf{m})$ and $V = H_0^{1,p}(D)$. By the Poincaré inequality, there exists a constant $C > 0$ such that $\|f\|_{1,p} \leq C\|\nabla f\|_p$. Now we consider the following time-fractional generalized p -Laplace equations

$$\partial_t^{*k}(X_t - x_0) = \operatorname{div}(\Phi(t, \nabla X_t)) + f(t, X_t), \quad (7.4)$$

where

$$\Phi : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d; \quad f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$$

are measurable, and continuous in the second variable.

To verify conditions (H1), (H2), (H3) and (H4) for $A(t, v) := \operatorname{div}(\Phi(t, \nabla v)) + f(t, v)$, we assume that for a fixed $p \in [2, \infty)$,

$$\begin{aligned} |\Phi(t, s)| &\leq K(1 + |s|^{p-1}), \quad s \in \mathbb{R}^d, t \geq 0, \\ \mathbf{m}((\Phi(t, x) - \Phi(t, y))(x - y)) &\geq \delta \|x - y\|_p^p, \quad t \geq 0, \\ \mathbf{m}((f(t, x) - f(t, y))(x - y)) &\leq K \|x - y\|_2^2, \quad t \geq 0, \\ |f(t, x)| &\leq K(1 + |x|^{p-1}), \quad t \geq 0, \end{aligned} \tag{7.5}$$

hold for some constants $K, \delta > 0$ and all $x, y \in L^p(\mathbf{m})$.

Example 7.2 Suppose that (7.5) holds, then (H1), (H2), (H3) and (H4) hold for (see e.g. [27, Example 4.1])

$$A(t, v) := \operatorname{div}(\Phi(t, \nabla v)) + f(t, v).$$

Therefore, Theorem 2.2 is applicable to the time-fractional generalized p -Laplace equations (7.4), if k satisfies (k), (\tilde{k}) respectively.

Remark 7.2. (i) Obviously, the assumptions above are satisfied provided $\Phi(t, s) = h(t)|s|^{p-2}s$ and $f(t, s) = f_1(t)s - f_2(t)|s|^{p-2}s$, $t \in [0, T]$, $s \in \mathbb{R}$, with $0 < \inf h \leq \sup h < \infty$ and $\|f_i\|_\infty < \infty$, which is the classical p -Laplace equation with polynomial type perturbation.

(ii) Similarly, we could apply Theorem 2.3 to the following time-fractional stochastic generalized p -Laplace equations

$$\partial_t^{*k_1}(X_t - x_0) = (\operatorname{div}(\Phi(t, \nabla X_t)) + f(t, X_t)) dt + \partial_t^{*k_2} \int_0^t B(s) dW(s), \tag{7.6}$$

where $W(s)$, $s \geq 0$, is cylindrical Brownian motion on H , $B : [0, \infty) \rightarrow L_{HS}(H)$ is measurable and locally bounded.

7.3 Stochastic generalized fast-diffusion equations

Let $(E, \mathcal{B}, \mathbf{m})$, $(L, \mathbb{D}(L))$, H (B and $W(s)$, $s \geq 0$) be as in Example 7.1. Suppose that $r \in (0, 1)$ and $\Psi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, continuous in the second variable and such that for some constant $\delta > 0$,

$$(\Psi(t, s_1) - \Psi(t, s_2))(s_1 - s_2) \geq \frac{\delta |s_1 - s_2|^2}{(|s_1| \vee |s_2|)^{1-r}}, \quad s_1, s_2 \in \mathbb{R}, t \geq 0, \tag{7.7}$$

$$s\Psi(t, s) \geq \delta |s|^{r+1}, \quad \sup_{t \in [0, T], s \geq 0} \frac{|\Psi(t, s)|}{1 + |s|^r} < \infty, \quad s \in \mathbb{R}, t \geq 0, \tag{7.8}$$

where $\frac{|s_1 - s_2|^2}{(|s_1| \vee |s_2|)^{1-r}} := 0$ for $s_1 = s_2 = 0$.

We consider the following time-fractional generalized fast-diffusion equations

$$\partial_t^{*k}(X(t) - x_0) = L\Psi(t, X(t)) + h(t)X(t), \quad (7.9)$$

where $h \in C([0, \infty))$.

Let $V = L^{r+1}(\mathbf{m}) \cap H$ with $\|v\|_V := \|v\|_{1+r} + \|v\|_H$. Then it is easy to show that (H1)-(H4) hold for (see [56, Theorem 3.9] for a more general result)

$$A(t, v) := L\Psi(t, v) + h(t)v, \quad v \in V.$$

Example 7.3 Suppose that (7.7) and (7.8) hold, then the assertions in Theorem 2.2 hold for (7.9), if k satisfies (k), (\tilde{k}) respectively.

Remark 7.3. (i) By the mean-valued theorem, one has for $r \in (0, 1)$

$$(s_1 - s_2)(s_1|s_1|^{r-1} - s_2|s_2|^{r-1}) \geq r|s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}, \quad s_1, s_2 \in \mathbb{R}.$$

So, a simple example of Ψ , so that (7.7) and (7.8) hold, is $\Psi(t, s) = cs|s|^{r-1}$ for some constant $c > 0$. This corresponds to the classical fast-diffusion equation.

(ii) Similar results also hold for the corresponding time-fractional stochastic equations

$$\partial_t^{*k_1}(X(t) - x_0) = \left\{ L\Psi(t, X(t)) + h(t)X(t) \right\} dt + \partial_t^{*k_2} \int_0^t B(s) dW(s). \quad (7.10)$$

Appendix A. Proof of Theorem 4.1

For the proof of Theorem 4.1 we need some preparations. We recall the definition of a pseudo-monotone operator, which is a very useful generalization of monotone operator and was first introduced by Brézis in [10]. We use the notation “ \rightharpoonup ” for weak convergence in Banach spaces.

Definition .4. An operator $M : \mathcal{V} \longrightarrow \mathcal{V}^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in \mathcal{V} as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \mathcal{V}^* \langle M(v_n), v_n - v \rangle_{\mathcal{V}} \leq 0$$

implies for all $u \in \mathcal{V}$

$$\mathcal{V}^* \langle M(v), v - u \rangle_{\mathcal{V}} \leq \liminf_{n \rightarrow \infty} \mathcal{V}^* \langle M(v_n), v_n - u \rangle_{\mathcal{V}}.$$

Remark .5. (i) Browder introduced a slightly different definition of a pseudo-monotone operator in [11]: An operator $M : \mathcal{V} \longrightarrow \mathcal{V}^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in \mathcal{V} as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \mathcal{V}^* \langle M(v_n), v_n - v \rangle_{\mathcal{V}} \leq 0$$

implies

$$M(v_n) \rightharpoonup M(v) \text{ and } \lim_{n \rightarrow \infty} \mathcal{V}^* \langle M(v_n), v_n \rangle_{\mathcal{V}} = \mathcal{V}^* \langle M(v), v \rangle_{\mathcal{V}}.$$

In particular, if M is bounded on bounded sets, then these two definitions are equivalent, we refer to [37, 38].

(ii) We recall that as mentioned before our operator $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$ in (2.3) is coercive and bounded as well as monotone if $C_1 = 0$ in (H2), hence in particular pseudo-monotone. If we add a continuous monotone linear operator $\tilde{\Lambda}: \mathcal{V} \rightarrow \mathcal{H}$ to it, it is easy to see that also $\mathcal{A} + \tilde{\Lambda}$ is pseudo-monotone.

Lemma .6. *If $M: \mathcal{V} \rightarrow \mathcal{V}^*$ is pseudo-monotone, bounded on bounded sets and coercive, then M is surjective, i.e. for any $f \in \mathcal{V}^*$, the equation $Mu = f$ has a solution.*

Proof. This is a classical result due to Brézis. For the proof we refer to [10] or [65, Theorem 27.A]. \square

Proof of Theorem 4.1. Step 1: Let $\alpha > 0$ and consider the Yosida approximation $\Lambda_\alpha: \mathcal{V} \rightarrow \mathcal{V}^*$ defined by

$$\mathcal{V}^* \langle \Lambda_\alpha u, \cdot \rangle_{\mathcal{V}} := \alpha \mathcal{V}^* \langle \alpha V_\alpha u - u, \cdot \rangle_{\mathcal{V}},$$

where $V_\alpha = (\alpha - \Lambda)^{-1}$, $\alpha > 0$, is the resolvent of $(\Lambda, D(\Lambda, \mathcal{H}))$ (on \mathcal{H}).

We note that since αV_α is a contraction on \mathcal{H} , we have

$$\mathcal{V}^* \langle \Lambda_\alpha u, u \rangle_{\mathcal{V}} = \langle \Lambda_\alpha u, u \rangle_{\mathcal{H}} \geq 0 \quad \text{for all } u \in V,$$

hence by Remark .5(ii) it follows that $\mathcal{A} - \Lambda_\alpha$ is pseudo-monotone, coercive and bounded on bounded sets. Therefore, by Lemma .6 there exists $u_\alpha \in \mathcal{V}$ such that $\mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha = f$.

Step 2: Note that

$$\mathcal{V}^* \langle \mathcal{A}u_\alpha, u_\alpha \rangle_{\mathcal{V}} \leq \mathcal{V}^* \langle \mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha, u_\alpha \rangle_{\mathcal{V}} = \mathcal{V}^* \langle f, u_\alpha \rangle_{\mathcal{V}} \leq \|f\|_{\mathcal{V}^*} \|u_\alpha\|_{\mathcal{V}}.$$

Hence, by the coercivity assumption (H3) we obtain that $\sup_{\alpha > 0} \|u_\alpha\|_{\mathcal{V}} < \infty$, and hence

$$\sup_{\alpha > 0} \|\mathcal{A}u_\alpha\|_{\mathcal{V}^*} < \infty$$

by (H4).

Since for any $v \in \mathcal{V}$

$$\begin{aligned} \mathcal{V}^* \langle \Lambda_\alpha u_\alpha, v \rangle_{\mathcal{V}} &= -\mathcal{V}^* \langle \mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha, v \rangle_{\mathcal{V}} + \mathcal{V}^* \langle \mathcal{A}u_\alpha, v \rangle_{\mathcal{V}} \\ &= -\mathcal{V}^* \langle f, v \rangle_{\mathcal{V}} + \mathcal{V}^* \langle \mathcal{A}u_\alpha, v \rangle_{\mathcal{V}} \\ &\leq (\|f\|_{\mathcal{V}^*} + \|\mathcal{A}u_\alpha\|_{\mathcal{V}^*}) \|v\|_{\mathcal{V}}, \end{aligned}$$

we have $\sup_{\alpha > 0} \|\Lambda_\alpha u_\alpha\|_{\mathcal{V}^*} < \infty$.

By the apriori estimates above we know there exists a subsequence $\alpha_n \rightarrow \infty$ such that

$$\begin{aligned} u_{\alpha_n} &\rightharpoonup u \quad \text{in } \mathcal{V}; \\ \mathcal{A}u_{\alpha_n} &\rightharpoonup h \quad \text{in } \mathcal{V}^*; \\ \Lambda_{\alpha_n} u_{\alpha_n} &\rightharpoonup g \quad \text{in } \mathcal{V}^*. \end{aligned}$$

So, it is easy to see that $h - g = f$.

By the strong continuity of the dual resolvent $(\hat{V}_\alpha)_{\alpha>\omega}$ in \mathcal{V}^* , we have for all $v \in \mathcal{V}^*$

$$\lim_{n \rightarrow \infty} \mathcal{V}^* \langle v, \alpha_n V_{\alpha_n} u_{\alpha_n} \rangle_{\mathcal{V}} = \lim_{n \rightarrow \infty} \mathcal{V}^* \langle \alpha_n \hat{V}_{\alpha_n} v, u_{\alpha_n} \rangle_{\mathcal{V}} = \mathcal{V}^* \langle v, u \rangle_{\mathcal{V}},$$

and, therefore,

$$\alpha_n V_{\alpha_n} u_{\alpha_n} \rightharpoonup u \text{ in } \mathcal{V}.$$

Since $\Lambda \alpha_n V_{\alpha_n} u_{\alpha_n} = \Lambda \alpha_n u_{\alpha_n}$, we also have

$$\Lambda \alpha_n V_{\alpha_n} u_{\alpha_n} \rightharpoonup g \text{ in } \mathcal{V}^*.$$

Since Λ is linear and (Λ, \mathcal{F}) is closed as an operator from \mathcal{V} to \mathcal{V}^* , this implies that $u \in \mathcal{F}$ and $\Lambda u = g$.

Step 3: Now we only need to show $\mathcal{A}u = h$. Since $u_{\alpha_n} \rightharpoonup u$ in \mathcal{V} and for all $v \in D(\Lambda, \mathcal{V})$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} &= \limsup_{n \rightarrow \infty} (\mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} - v \rangle_{\mathcal{V}} + \mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n}, v \rangle_{\mathcal{V}}) \\ &\leq \limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \Lambda_{\alpha_n} v, u_{\alpha_n} - v \rangle_{\mathcal{V}} + \mathcal{V}^* \langle \Lambda u, v \rangle_{\mathcal{V}} \\ &= \mathcal{V}^* \langle \Lambda v, u - v \rangle_{\mathcal{V}} + \mathcal{V}^* \langle \Lambda u, v \rangle_{\mathcal{V}}, \end{aligned}$$

where the inequality follows from $\mathcal{V}^* \langle \Lambda_{\alpha_n} (u_{\alpha_n} - u), u_{\alpha_n} - u \rangle_{\mathcal{V}} \leq 0$, since each Λ_{α_n} is a contraction on \mathcal{H} . Since $D(\Lambda, \mathcal{V})$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$, the above inequality extends to all $v \in \mathcal{F}$. In particular, we may take $v = u$, to obtain that

$$\limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} \leq \mathcal{V}^* \langle \Lambda u, u \rangle_{\mathcal{V}}.$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} - u \rangle_{\mathcal{V}} &= \limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n} + f, u_{\alpha_n} - u \rangle_{\mathcal{V}} \\ &= \limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} - u \rangle_{\mathcal{V}} \\ &\leq \mathcal{V}^* \langle \Lambda u, u - u \rangle_{\mathcal{V}} = 0. \end{aligned}$$

So, we have

$$\limsup_{n \rightarrow \infty} \mathcal{V}^* \langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} \leq \mathcal{V}^* \langle h, u \rangle_{\mathcal{V}}.$$

Hence, by the pseudo-monotonicity, we have for any $w \in \mathcal{V}$

$$\begin{aligned} \mathcal{V}^* \langle \mathcal{A}u, u - w \rangle_{\mathcal{V}} &\leq \liminf_{n \rightarrow \infty} \mathcal{V}^* \langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} - w \rangle_{\mathcal{V}} \\ &\leq \liminf_{n \rightarrow \infty} \mathcal{V}^* \langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} - \mathcal{V}^* \langle h, w \rangle_{\mathcal{V}} \\ &\leq \mathcal{V}^* \langle h, u - w \rangle_{\mathcal{V}}, \end{aligned}$$

which implies $\mathcal{A}u = h$ since $w \in \mathcal{V}$ was arbitrary. □

Acknowledgements

Financial support by the DFG through the CRC 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications” is acknowledged. W.L. is supported by NSFC (No. 11571147, 11822103, 11831014), NSF of Jiangsu Province (No. BK20160004) and PAPD of Jiangsu Higher Education Institutions, J.L.S. is supported by project I&D: UID/MAT/04674/2019.

The second named author would like to thank his hosts at Madeira University for a very pleasant stay in May 2018 and Summer 2019, where a part of this work was done. He would also like to thank the Isaac Newton Institute for a very stimulating stay in November 2018, where substantial progress was made on this paper.

References

- [1] M. Allen, L. Caffarelli, and A. Vasseur. A parabolic problem with a fractional time derivative. *Arch. Ration. Mech. Anal.*, 221(2):603–630, 2016.
- [2] H. Allouba and Y. Xiao. L-Kuramoto-Sivashinsky SPDEs vs. time-fractional SPIDEs: exact continuity and gradient moduli, $1/2$ -derivative criticality, and laws. *J. Differential Equations*, 263(2):1552–1610, 2017.
- [3] T. M. Atanackovic, S. Pilipovic, and D. Zorica. Time distributed-order diffusion-wave equation. I., II. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 465, pages 1869–1891, 1893–1917. The Royal Society, 2009.
- [4] B. Baeumer and M. M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.*, 4(4):481–500, 2001.
- [5] B. Baeumer, M. M. Meerschaert, and E. Nane. Brownian subordinators and fractional Cauchy problems. *Trans. Amer. Math. Soc.*, 361(7):3915–3930, 2009.
- [6] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo. *Fractional calculus*, volume 3 of *Series on Complexity, Nonlinearity and Chaos*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. Models and numerical methods.
- [7] V. Barbu. *Nonlinear differential equations of monotone types in Banach spaces*. Springer Monographs in Mathematics. Springer, New York, 2010.
- [8] V. Barbu and M. Röckner. An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise. *J. Eur. Math. Soc. (JEMS)*, 17(7):1789–1815, 2015.
- [9] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

- [10] H. Brezis. Équations et inéquations non linéaires dans les espaces vectoriels en dualité. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 1):115–175, 1968.
- [11] F. E. Browder. Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains. *Proc. Nat. Acad. Sci. U.S.A.*, 74(7):2659–2661, 1977.
- [12] M. Caputo. Linear Models of Dissipation whose Q is almost Frequency Independent-II. *Geophys. J. R. astr. Soc.*, 13:529–539, Jan. 1967.
- [13] M. Caputo. Mean fractional-order-derivatives differential equations and filters. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 41(1):73–84, 1995.
- [14] M. Caputo. Distributed order differential equations modelling dielectric induction and diffusion. *Fract. Calc. Appl. Anal.*, 4(4):421–442, 2001.
- [15] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, and V. Y. Gonchar. Distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 6(3):259–280, 2003.
- [16] A. V. Chechkin, J. Klafter, and I. M. Sokolov. Fractional Fokker-Planck equation for ultraslow kinetics. *EPL (Europhysics Letters)*, 63(3):326, 2003.
- [17] L. Chen. Nonlinear stochastic time-fractional diffusion equations on \mathbb{R} : moments, Hölder regularity and intermittency. *Trans. Amer. Math. Soc.*, 369(12):8497–8535, 2017.
- [18] L. Chen, G. Hu, Y. Hu, and J. Huang. Space-time fractional diffusions in Gaussian noisy environment. *Stochastics*, 89(1):171–206, 2017.
- [19] Z.-Q. Chen. Time fractional equations and probabilistic representation. *Chaos Solitons Fractals*, 102:168–174, 2017.
- [20] Z.-Q. Chen, K.-H. Kim, and P. Kim. Fractional time stochastic partial differential equations. *Stochastic Process. Appl.*, 125(4):1470–1499, 2015.
- [21] D. Craiem, F. J. R. Rojo, J. M. Atienza, R. L. Armentano, and G. V. Guinea. Fractional-order viscoelasticity applied to describe uniaxial stress relaxation of human arteries. *Physics in medicine and biology*, 53 17:4543–54, 2008.
- [22] D. del Castillo-Negrete, B. A. Carreras, V. E.: Lynch. Nondiffusive transport in plasma turbulence: a fractional diffusion approach. *Phys. Rev. Lett.* 94: 065003, 2005.
- [23] K. Diethelm. *The analysis of fractional differential equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.

- [24] M. Doi and S. Edwards. *The Theory of Polymer Dynamics*. Comparative Pathobiology - Studies in the Postmodern Theory of Education. Clarendon Press, 1988.
- [25] J. Ferry. *Viscoelastic Properties of Polymers*. Wiley, 1980.
- [26] M. Foondun and E. Nane. Asymptotic properties of some space-time fractional stochastic equations. *Math. Z.*, 287(1-2):493–519, 2017.
- [27] B. Gess. Random attractors for degenerate stochastic partial differential equations. *J. Dynam. Differential Equations*, 25(1):121–157, 2013.
- [28] R. Gorenflo and F. Mainardi. Fractional relaxation of distributed order. In *Complexus Mundi: Emergent Patterns in Nature*, pages 33–42. World Scientific, 2006.
- [29] G. Gripenberg, S.-O. Londen, and O. Steffans. *Volterra Integral and Functional Equations*, volume 34. Cambridge University Press, Cambridge, Aug. 1990.
- [30] R. Herrmann. *Fractional Calculus – An Introduction for Physicists (3rd revised and extended Edition)*, World Scientific Publishing, Singapore, September 2018, 636 pp, ISBN: 978-981-3274-57-0. 09 2018.
- [31] V. G. Jakubowski and P. Wittbold. On a nonlinear elliptic-parabolic integro-differential equation with L^1 -data. *J. Differential Equations*, 197(2):427–445, 2004.
- [32] I. Kim, K.-H. Kim, and S. Lim. An $L_q(L_p)$ -theory for the time fractional evolution equations with variable coefficients. *Adv. Math.*, 306:123–176, 2017.
- [33] A. N. Kochubei. Distributed-order calculus: An operator-theoretic interpretation. *Ukrainian Math. J.*, 60(4):551, 2008.
- [34] A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340(1):252–281, 2008.
- [35] A. N. Kochubei. Distributed order derivatives and relaxation patterns. *J. Phys. A*, 42(3):5203, Aug. 2009.
- [36] A. N. Kochubei. General fractional calculus, evolution equations, and renewal processes. *Integral Equations Operator Theory*, 71(4):583–600, Oct. 2011.
- [37] W. Liu and M. Röckner. Local and global well-posedness of SPDE with generalized coercivity conditions. *J. Differential Equations*, 254(2):725–755, 2013.
- [38] W. Liu and M. Röckner. *Stochastic partial differential equations: an introduction*. Universitext. Springer, Cham, 2015.

- [39] W. Liu, M. Röckner, and J. L. da Silva. Quasi-linear (stochastic) partial differential equations with time-fractional derivatives. *SIAM J. Math. Anal.*, 50(3):2588–2607, 2018.
- [40] C. F. Lorenzo and T. T. Hartley. Variable order and distributed order fractional operators. *Nonlinear Dynam.*, 29(1-4):57–98, 2002.
- [41] Y. Luchko and M. Yamamoto. General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.*, 19(3):676–695, 2016.
- [42] Z. M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.
- [43] F. Mainardi. *Fractional calculus and waves in linear viscoelasticity*. Imperial College Press, London, 2010. An introduction to mathematical models.
- [44] F. Mainardi, A. Mura, R. Gorenflo, and M. Stojanovic. The two forms of fractional relaxation of distributed order. *J. Vib. Control*, 13(9-10):1249–1268, Sep-Oct 2007.
- [45] F. Mainardi, A. Mura, G. Pagnini, and R. Gorenflo. Time-fractional diffusion of distributed order. *J. Vib. Control*, 14(9-10):1267–1290, Sept. 2008.
- [46] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler, and B. Baeumer. Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E*, 65(4):041103, 2002.
- [47] M. M. Meerschaert, E. Nane, and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *Ann. Probab.*, 37(3):979–1007, 2009.
- [48] M. M. Meerschaert and H.-P. Scheffler. Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.*, 41(3):623–638, Sept. 2004.
- [49] M. M. Meerschaert and H.-P. Scheffler. Stochastic model for ultraslow diffusion. *Stochastic Process. Appl.*, 116(9):1215–1235, 2006.
- [50] M. M. Meerschaert and A. Sikorskii. *Stochastic models for fractional calculus*, volume 43 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2012.
- [51] R. Metzler, E. Barkai, and J. Klafter. Anomalous diffusion and relaxation close to thermal equilibrium: A fractional Fokker-Planck equation approach. *Phys. Rev. Lett.*, 82:3563–3567, 1999.
- [52] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000.
- [53] J. B. Mijena and E. Nane. Space-time fractional stochastic partial differential equations. *Stochastic Process. Appl.*, 125(9):3301–3326, 2015.

- [54] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.*, 37(1):206–249, 2009.
- [55] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.
- [56] J. Ren, M. Röckner, and F.-Y. Wang. Stochastic generalized porous media and fast diffusion equations. *J. Differential Equations*, 238(1):118–152, 2007.
- [57] B. Ross. The development of fractional calculus 1695–1900. *Historia Math.*, 4:75–89, 1977.
- [58] S. G. Samko and R. P. Cardoso. Integral equations of the first kind of Sonine type. *Int. J. Math. Math. Sci.*, 57:3609–3632, 2003.
- [59] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions: Theory and applications*, volume 37 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2012.
- [60] N. Sonine. Sur la généralisation d’une formule d’Abel. *Acta Math.*, 4:171–176, 1884.
- [61] W. Stannat. The theory of generalized Dirichlet forms and its applications in analysis and stochastics. *Mem. Amer. Math. Soc.*, 142(678):viii+101, 1999.
- [62] V. Vergara and R. Zacher. Optimal Decay Estimates for Time-Fractional and Other NonLocal Subdiffusion Equations via Energy Methods. *SIAM J. Math. Anal.*, 47(1):210–239, Jan. 2015.
- [63] R. Zacher. Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. *Funkcial. Ekvac.*, 52(1):1–18, 2009.
- [64] R. Zacher. Global strong solvability of a quasilinear subdiffusion problem. *J. Evol. Equ.*, 12(4):813–831, 2012.
- [65] E. Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.