

SCHAUDER THEOREMS FOR A CLASS OF (PSEUDO-)DIFFERENTIAL OPERATORS ON FINITE AND INFINITE DIMENSIONAL STATE SPACES

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ABSTRACT. We prove maximal regularity results in Hölder and Zygmund spaces for linear stationary and evolution equations driven by a large class of differential and pseudo-differential operators L , both in finite and in infinite dimension. The assumptions are given in terms of the semigroup generated by L . We cover the cases of fractional Laplacians and Ornstein-Uhlenbeck operators with fractional diffusion in finite dimension, and several types of local and nonlocal Ornstein-Uhlenbeck operators, as well as the Gross Laplacian and its negative powers, in infinite dimension.

1. INTRODUCTION

This paper is devoted to maximal regularity results in Hölder and Zygmund spaces for linear stationary and evolution equations driven by a large class of differential and pseudo-differential operators L , both in finite and in infinite dimension. The underlying space X is any separable real Banach space, that may be either \mathbb{R}^N or infinite dimensional.

The operators L under consideration are the generators of the so called *generalized Mehler semigroups*, namely semigroups of operators in the space $C_b(X)$ of the continuous and bounded functions from X to \mathbb{R} that may be represented as

$$P_t f(x) = \int_X f(T_t x + y) \mu_t(dy), \quad t \geq 0, f \in C_b(X). \quad (1.1)$$

Here T_t is a strongly continuous semigroup of bounded operators on X , and $\{\mu_t : t \geq 0\}$ is a family of Borel probability measures in X such that $\mu_0 = \delta_0$ (the Dirac measure at $0 \in X$), $t \mapsto \mu_t$ is weakly continuous in $[0, +\infty)$ and

$$\mu_{t+s} = (\mu_t \circ T_s^{-1}) * \mu_s, \quad t, s > 0. \quad (1.2)$$

Such a condition is necessary and sufficient for P_t be a semigroup (namely, $P_{t+s} = P_t \circ P_s$ for $t, s \geq 0$), even in the space $\mathcal{B}_b(X)$ of the bounded, Borel measurable functions $f : X \rightarrow \mathbb{R}$.

Then for every $f \in C_b(X)$ the function $(t, x) \mapsto P_t f(x)$ is continuous in $[0, +\infty) \times X \mapsto \mathbb{R}$, and this allows to define a closed operator L in $C_b(X)$ through its resolvent,

$$R(\lambda, L)f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad \lambda > 0, f \in C_b(X), x \in X. \quad (1.3)$$

L is called the *generator* of P_t , although it is not the infinitesimal generator in the standard sense since P_t is not strongly continuous in $C_b(X)$, in general.

Though this paper, its results and techniques of proof are purely analytic, let us briefly recall the probabilistic framework in which generalized Mehler semigroups occur. In fact, they are the transition semigroups of solution processes to the following type of stochastic differential equations (meant in the weak or mild sense) on X :

$$dX(t) = AX(t)dt + dY(t), \quad t > 0; X(0) = x, \quad (1.4)$$

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where $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of T_t , and $Y(t)$, $t \geq 0$, is a Levy process in X , i.e. a stochastic process in X with cadlag paths starting at 0, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and having stationary and independent increments. It is characterized by a negative definite function $\lambda : X^* \rightarrow \mathbb{C}$ (where X^* is the dual space of X), satisfying

$$\int_{\Omega} e^{i\xi(Y(t)(\omega))} \mathbb{P}(d\omega) = \exp(-t\lambda(\xi)), \quad \xi \in X^*, t > 0. \quad (1.5)$$

Then the transition semigroup for the solution $X(t, x)$ of (1.5) (called ‘‘Ornstein-Uhlenbeck process on X ’’ in the case that $Y(t)$ is a Wiener process, and ‘‘Ornstein-Uhlenbeck process on X with jumps’’ if $Y(t)$ is a more general Levy process) is given by P_t as in (1.1), i.e., for $f \in \mathcal{B}_b(X)$, $x \in X$, $t > 0$,

$$\int_{\Omega} f(X(t, x)(\omega)) \mathbb{P}(d\omega) = P_t f(x), \quad (1.6)$$

where $X(t, x)$, $t \geq 0$, denotes the (weak or mild) solution of (1.5) with $X(0, x) = x$ \mathbb{P} -a.s. We then have an explicit formula for the Fourier transforms of μ_t , $t > 0$, in terms of λ and T_t , namely

$$\hat{\mu}_t(\xi) := \int_X \exp(i\xi(z)) \mu_t(dz) = \exp\left(-\int_0^t \lambda(T_s^* \xi) ds\right), \quad \xi \in X^*, t > 0, \quad (1.7)$$

where T_t^* denotes the dual semigroup of T_t .

There have been a number of papers on generalized Mehler semigroups and their related Ornstein-Uhlenbeck processes with jumps. We refer e.g. to [7, 18, 27, 28, 30, 37, 38, 39, 45, 46, 52] and the references therein.

Now let us come back to the main results of this paper, which are purely analytic. What we prove are maximal H\"older and Zygmund regularity results both for the stationary equation

$$\lambda u(x) - Lu(x) = f(x), \quad x \in X, \quad (1.8)$$

namely for the function $u = R(\lambda, L)f$ defined in (1.3), and for the mild solutions of evolution problems, given by

$$v(t, x) = P_t f(x) + \int_0^t P_{t-s} g(s, \cdot)(x) ds, \quad 0 \leq t \leq T, x \in X, \quad (1.9)$$

with continuous and bounded f, g .

Of course, we need some ‘‘regularity’’ hypothesis on the measures μ_t in connection with the semigroup T_t . Specifically, we assume that there exists a Banach space $H \subset X$ such that $T(t)(H) \subset H$, and such that each μ_t is Fomin differentiable along $T_t(H)$, namely for every $h \in H$, $t > 0$ there exists $\beta_{t,h} \in L^1(X, \mu_t)$ such that

$$\int_X \frac{\partial f}{\partial(T_t h)}(x) \mu_t(dx) = - \int_X \beta_{t,h}(x) f(x) \mu_t(dx), \quad f \in C_b^1(X). \quad (1.10)$$

Moreover we assume that there exist $C > 0$, $\omega \in \mathbb{R}$, $\theta > 0$ such that

$$\|T_t h\|_H \leq C e^{\omega t} \|h\|_H, \quad \|\beta_{t,h}\|_{L^1(X, \mu_t)} \leq \frac{C e^{\omega t}}{t^\theta} \|h\|_H, \quad t > 0, h \in H. \quad (1.11)$$

These assumptions are satisfied in several remarkable examples. We consider the following ones.

(a) In finite dimension, with $X = H = \mathbb{R}^N$, they are satisfied by the heat semigroup with $\theta = 1/2$, by the semigroups generated by the powers $-(-\Delta)^s$ for $s \in (0, 1)$, and more generally by Ornstein-Uhlenbeck semigroups with fractional diffusion,

$$\mathcal{L}u(x) = \frac{1}{2}(\text{Tr}^s(QD^2u))(x) - \langle Bx, \nabla u(x) \rangle, \quad x \in \mathbb{R}^N,$$

where Q is any symmetric positive definite matrix, B is any matrix, and $\text{Tr}^s(QD^2)$ is the pseudo-differential operator with symbol $\langle Q\xi, \xi \rangle^s$, $s \in (0, 1)$. The semigroup T_t is now e^{-tB} , and the measures μ_t are given by $\mu_t(dx) = g_t(x)dx$, with $g_t \in W^{1,1}(\mathbb{R}^N)$, so that μ_t is Fomin differentiable along all directions, and (1.11) holds with $H = \mathbb{R}^N$ and $\theta = 1/(2s)$. See Sections 4.1, 4.2.

(b) In infinite dimension they are satisfied by a class of smoothing (strong Feller) Ornstein-Uhlenbeck semigroups, still with $H = X$, that includes the ones considered in [26], and by a class of not strong Feller Ornstein-Uhlenbeck semigroups, that includes the classical Ornstein-Uhlenbeck semigroup used in the Malliavin Calculus, and other non symmetric Ornstein-Uhlenbeck semigroups such as in [57, 58]; here H is the Cameron-Martin space of a reference Gaussian measure μ . In all these cases the measures μ_t are Gaussian, and we have $\theta = 1/2$, see Section 5.1. In Section 5.2 we consider nonlocal perturbations of the generator of a specific strong Feller Ornstein-Uhlenbeck semigroup and show that (1.10) and (1.11) also hold in such a case, still with $H = X$ and $\theta = 1/2$. Moreover, when X is a Hilbert space endowed with a centered Gaussian measure μ and H is the Cameron-Martin space of μ , (1.10) and (1.11) are satisfied by the semigroup generated by the Gross Laplacian G , again with $\theta = 1/2$, and by the semigroups generated by $-(-G)^s$ with $s \in (0, 1)$ and $\theta = 1/(2s)$, in which case the measures μ_t are mixtures of measures. See Section 5.3. In Section 5.4 we show that some nonlocal versions of the classical Ornstein-Uhlenbeck semigroup from Malliavin calculus still satisfy our assumptions.

Our techniques are independent of the dimension of the state space X , and the most important and newest part of the paper is in the infinite dimensional case. Indeed, several familiar tools in finite dimension, such as Calderon-Zygmund theory, Fourier transform, and the uncountable consequences of local compactness, are not available in infinite dimension, as well as any translation invariant reference measure such as the Lebesgue measure.

Needless to say, maximal regularity results are very rare in infinite dimension. A few L^p maximal regularity results, with $p \in (1, +\infty)$, have been proved for certain Ornstein-Uhlenbeck stationary equations; in these cases the solution to (1.8) belongs to a suitable $W^{2,p}$ space with respect to an invariant Gaussian measure μ whenever $f \in L^p(X, \mu)$. After the pioneering Meyer inequalities for the classical Ornstein-Uhlenbeck operator ([44], see also [5, Sect. 5.6]), maximal L^p regularity for a more general class of Ornstein-Uhlenbeck equations was proved in [19, 20, 43]. Concerning non Gaussian measures, the only available results are for $p = 2$, about (nontrivial) perturbations of certain Ornstein-Uhlenbeck equations ([23, 12]); here μ is an invariant Gibbs (= weighted Gaussian) measure. For $p = 2$ some of the above results have been extended to the case where the whole X is replaced by a good domain $\mathcal{O} \subset X$, with generalized Dirichlet or Neumann boundary conditions ([24, 25, 11, 13]).

Also the literature about maximal Hölder regularity in infinite dimension is very scarce, dealing mainly with Ornstein-Uhlenbeck equations or with equations driven by the Gross Laplacian, see e.g. [26, 14, 16] and the references therein. More details are in Sections 5.1, 5.3. Moreover, Schauder estimates for some nontrivial perturbations of a specific Ornstein-Uhlenbeck operator in the space $X = C([0, 1])$ were proved in [15].

In our general setting, P_t is smoothing along H : for every $f \in C_b(X)$ and $t > 0$, $P_t f$ has continuous Gateaux derivatives of any order along H , and for every $(h_1, \dots, h_n) \in H^n$ we have

$$\left| \frac{\partial^n P_t f}{\partial h_1 \dots \partial h_n}(x) \right| \leq C_n \left(1 + \frac{1}{t^{n\theta}} \right) \prod_{j=1}^n \|h_j\|_H \|f\|_\infty, \quad t > 0, x \in X. \quad (1.12)$$

On the other hand, in general $P_t f$ is not Gateaux differentiable along other subspaces than H . Therefore, any regularity result is expressed in terms of regularity along H . The Hölder spaces

that we use are in fact defined by

$$C_H^\alpha(X) = \left\{ f \in C_b(X) : [f]_{C_H^\alpha(X)} := \sup_{x \in X, h \in H \setminus \{0\}} \frac{|f(x-h) - f(x)|}{\|h\|_H^\alpha} < +\infty \right\},$$

$$\|f\|_{C_H^\alpha(X)} = \|f\|_\infty + [f]_{C_H^\alpha(X)},$$

for $\alpha \in (0, 1)$. In the case $H = X$ this is the usual space of bounded and α -Hölder continuous functions from X to \mathbb{R} .

The Schauder type regularity results for (1.8) are the following,

(i) If $1/\theta \notin \mathbb{N}$, for every $\lambda > 0$ and $f \in C_b(X)$ the solution u to (1.8) belongs to $C_H^{1/\theta}(X)$, and there is $C(\lambda)$ independent of f such that $\|u\|_{C_H^{1/\theta}(X)} \leq C(\lambda)\|f\|_\infty$.

(ii) If $\alpha \in (0, 1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, for every $\lambda > 0$ and $f \in C_H^\alpha(X)$ the solution u to (1.8) belongs to $C_H^{\alpha+1/\theta}(X)$ and there is $C(\lambda, \alpha)$ independent of f such that $\|u\|_{C_H^{\alpha+1/\theta}(X)} \leq C(\lambda, \alpha)\|f\|_{C_H^\alpha(X)}$.

Here, for $\sigma \in (0, 1)$ and $k \in \mathbb{N}$, $C_H^{\sigma+k}(X)$ denotes the space of all continuous and bounded functions from X to \mathbb{R} that possess continuous and bounded Gateaux derivatives of any order $\leq k$ along H , and such that all the k -th order derivatives belong to $C_H^\sigma(X)$, endowed with its natural norm. If $H = X$ this is the space of the k times Gateaux differentiable functions with continuous and bounded Gateaux derivatives of any order $\leq k$, and such that all the k -th order derivatives are α -Hölder continuous.

The exponents $1/\theta$ in (i), and $\alpha + 1/\theta$ in (ii) are optimal, in the sense that they cannot be replaced by $1/\theta + \varepsilon$, $\alpha + 1/\theta + \varepsilon$ respectively, for any $\varepsilon > 0$.

In the critical cases $\alpha + 1/\theta = k \in \mathbb{N}$ (with $\alpha = 0$ in statement (i), $\alpha \in (0, 1)$ in statement (ii)) we do not expect that the solution to (1.8) has bounded partial derivatives of order k , in general. The simplest counterexample is given by the Laplacian in finite dimension. The heat semigroup in \mathbb{R}^N has the representation (1.1) with $T_t = I$, $\mu_t(dx) = (4\pi t)^{-N/2} \exp(-|x|^2/4t)dx$, so that it satisfies (1.10) and (1.11) with $X = H = \mathbb{R}^N$ and $\theta = 1/2$; however it is well known that for every $\lambda > 0$ the solution to $\lambda u - \Delta u = f$ with $f \in C_b(\mathbb{R}^N)$ is not twice continuously differentiable and its first order derivatives are not Lipschitz continuous in general, if $N > 1$. They belong to the Zygmund space $Z^1(\mathbb{R}^N)$, namely they satisfy $|D_k u(x+2h) - 2D_k u(x+h) + D_k u(x)| \leq C|h|$ for every $k = 1, \dots, N$, $x, h \in \mathbb{R}^N$, with C independent of x and h . We extend this result to our general setting, introducing the Zygmund spaces $Z_H^n(X)$ for $n \in \mathbb{N}$ and showing that in the above critical cases the solution to (1.8) belongs to $Z_H^k(X)$.

Similar results are proved for the mild solutions to Cauchy problems with continuous and bounded data,

$$\begin{cases} v_t(t, x) = Lv(t, \cdot)(x) + g(t, x), & t \in [0, T], x \in X, \\ v(0, \cdot) = f, \end{cases} \quad (1.13)$$

namely for the functions v defined by

$$v(t, x) = P_t f(x) + \int_0^t P_{t-s} g(s, \cdot)(x) ds, \quad t \in [0, T], x \in X,$$

with $f \in C_b(X)$, $g \in C_b([0, T] \times X)$. Our assumptions are not strong enough to guarantee that v is differentiable with respect to t , so it is just a mild solution and not a classical one. Moreover, time-space Schauder estimates such as the standard ones for the heat equation are not available in general. For instance, they are not available when L is the classical one-dimensional Ornstein-Uhlenbeck operator $Lu(x) = u''(x) - xu'(x)$, as a consequence of [22]. So, our Schauder and Zygmund regularity results concern only space regularity. More precisely, we introduce the space $C_H^{0, \alpha+k}([0, T] \times X)$ for

$\alpha \in (0, 1)$, $k \in \mathbb{N} \cup \{0\}$, consisting of the bounded continuous functions $g : [0, T] \times X \mapsto \mathbb{R}$ such that $g(t, \cdot) \in C_H^{\alpha+k}(X)$ for every $t \in [0, T]$, $\|g\|_{C_H^{\alpha+k}([0, T] \times X)} := \sup_{t \in [0, T]} \|g(t, \cdot)\|_{C_H^{\alpha+k}(X)} < +\infty$, and if $k \neq 0$, all the Gateaux derivatives $\partial^j g / \partial h_1 \dots \partial h_j$ with $j \leq k$ and $h_1, \dots, h_j \in H$ are continuous in $[0, T] \times X$. We prove that

(i) If $1/\theta \notin \mathbb{N}$, for every $f \in C_H^{1/\theta}(X)$ and $g \in C_b([0, T] \times X)$, the function v belongs to $C_H^{0, 1/\theta}([0, T] \times X)$, and there is $C(T)$ independent of f and g such that $\|v\|_{C_H^{0, 1/\theta}([0, T] \times X)} \leq C(T) (\|f\|_{C_H^{1/\theta}(X)} + \|g\|_\infty)$.

(ii) If $\alpha \in (0, 1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, for every $f \in C_H^{\alpha+1/\theta}(X)$ and $g \in C_H^{0, \alpha}([0, T] \times X)$ the function v belongs to $C_H^{0, \alpha+1/\theta}([0, T] \times X)$ and there is $C(T, \alpha)$ independent of f and g such that $\|v\|_{C_H^{0, \alpha+1/\theta}([0, T] \times X)} \leq C(T, \alpha) (\|f\|_{C_H^{\alpha+1/\theta}(X)} + \|g\|_{C_H^{0, \alpha}([0, T] \times X)})$.

In the critical cases $\alpha + 1/\theta \in \mathbb{N}$ we obtain Zygmund space regularity results, as in the stationary case.

The proofs rely on estimates (1.12) and on the better estimates for $\alpha \in (0, 1)$,

$$\left| \frac{\partial^n P_t f}{\partial h_1 \dots \partial h_n}(x) \right| \leq C_{n, \alpha} \left(1 + \frac{1}{t^{(n-\alpha)\theta}} \right) \prod_{j=1}^n \|h_j\|_H \|f\|_{C_H^\alpha(X)}, \quad t > 0, x \in X, f \in C_H^\alpha(X), \quad (1.14)$$

through a procedure that employs interpolation techniques such as in the recent paper [16] where the classical Ornstein-Uhlenbeck operator in infinite dimension is considered.

Both in the stationary and in the evolution case the general results are applied to the above mentioned examples, and yield old and new maximal regularity results. Comparisons with the literature dealing with Hölder and Zygmund maximal regularity are given in Sections 4.1, 4.2, 5.1, 5.3.

The structure of this paper is as follows. In Section 2 we mainly fix notations. In Section 3 we introduce our hypotheses, state and prove our main results described above. In particular, we prove an explicit formula for the n -th Gâteaux derivative of $P_t g$ for $g \in C_b(X)$ in Proposition 3.3. Sections 4 and 5 are devoted to examples in finite and infinite dimensions respectively.

2. NOTATION AND PRELIMINARIES

Below, X, Y are Banach spaces. If we write $X \subset Y$ this means that X is contained in Y with continuous embedding. By $\mathcal{L}(X)$, $\mathcal{L}(X, Y)$ we denote the spaces of the linear bounded operators from X to X , from X to Y , respectively.

Let $\mathcal{B}_b(X; Y)$ and $C_b(X; Y)$ denote the space of all bounded Borel measurable (resp. bounded continuous) functions $F : X \mapsto Y$, endowed with the sup norm $\|F\|_\infty := \sup_{x \in X} \|F(x)\|_Y$. If $X = \mathbb{R}$ we set $\mathcal{B}_b(X; \mathbb{R}) = \mathcal{B}_b(X)$ and $C_b(X; \mathbb{R}) := C_b(X)$.

We use the standard notation for partial derivatives along elements of X : for any fixed v , $x \in X$ and $F : X \mapsto Y$, we say that F is differentiable along v at x if there exists the limit $\lim_{t \rightarrow 0} (F(x + hv) - F(x))/t$. In this case the limit is denoted by $\partial f(x)/\partial v$.

In this paper we shall consider spaces of functions that enjoy regularity properties only along certain directions. They are defined as follows.

Let $H \subset X$ be a Banach space. If $F : X \mapsto Y$ is differentiable along every $h \in H$, and the mapping $h \mapsto \partial F / \partial h(x)$ belongs to $\mathcal{L}(H, Y)$, F is called H -Gateaux differentiable at x . Such a mapping is called H -Gateaux derivative of f at x , and denoted by $D_H F(x)$. If in addition

$$\lim_{\|h\|_H \rightarrow 0} \frac{\|F(x+h) - F(x) - D_H F(x)(h)\|_Y}{\|h\|_H} = 0,$$

F is called H -Fréchet differentiable at x .

Note that in the case $H = X$, these are the usual notions of Gateaux and Fréchet differentiable functions at x .

We shall consider also higher order derivatives. We identify $\mathcal{L}(H, \mathcal{L}(H, \mathbb{R}))$ with the space of the bilinear continuous functions from H^2 to \mathbb{R} ; more generally, denoting by $\mathcal{L}^n(H)$ the space of all n -linear continuous mappings from H^n to \mathbb{R} , we identify $\mathcal{L}(H, \mathcal{L}^{n-1}(H))$ with $\mathcal{L}^n(H)$.

Let $f : X \mapsto \mathbb{R}$ be H -Gateaux (resp. H -Fréchet) differentiable at every $x \in X$. If the mapping $D_H f : X \mapsto \mathcal{L}(H, \mathbb{R})$ is in turn H -Gateaux (resp. H -Fréchet) differentiable at x_0 , its H -Gateaux (resp. H -Fréchet) derivative belongs to $\mathcal{L}(H, \mathcal{L}(H, \mathbb{R})) = \mathcal{L}^2(H)$ and is denoted by $D_H^2 f(x_0)$. For $n \in \mathbb{N}$, $n \geq 2$, n times H -Gateaux (resp. H -Fréchet) differentiable functions are defined by recurrence. If $f : X \mapsto \mathbb{R}$ is $n-1$ times H -Gateaux (resp. H -Fréchet) differentiable at every $x \in X$, and the mapping $D_H^{n-1} f : X \mapsto \mathcal{L}^{n-1}(H)$ is H -Gateaux (resp. H -Fréchet) differentiable at x_0 , we say that f is n times H -Gateaux (resp. H -Fréchet) differentiable at x_0 , and the derivative of $D_H^{n-1} f$ at x_0 is denoted by $D_H^n f(x_0)$.

The space $C_H^n(X)$ consists of all continuous and bounded functions $f : X \mapsto \mathbb{R}$ having H -Gateaux derivatives up to the order n , such that for every $k = 1, \dots, n$ and for every $(h_1, \dots, h_k) \in H^k$, the mapping $X \mapsto \mathbb{R}$, $x \mapsto D_H^k f(x)(h_1, \dots, h_k)$ is continuous and bounded. It is endowed with the norm

$$\|f\|_{C_H^n(X)} = \|f\|_\infty + \sum_{k=1}^n \sup_{x \in X} \|D_H^k f(x)\|_{\mathcal{L}^k(H)},$$

where, for every n -linear continuous function $T : H^k \mapsto \mathbb{R}$,

$$\|T\|_{\mathcal{L}^k(H)} := \sup \left\{ \frac{|T(h_1, \dots, h_k)|}{\|h_1\|_H \cdots \|h_k\|_H} : h_i \in H \setminus \{0\} \right\}.$$

We note that by the multilinear version of the uniform bounded principle (see [50], [4]) we have that if $f \in C_H^n(X)$, then indeed $\|f\|_{C_H^n(X)} < \infty$. Furthermore, we remark that if $f \in C_H^n(X)$, for every $x \in X$ and $h \in H$ the function $t \mapsto f(x + th)$ is in $C^n(\mathbb{R})$, and we have

$$(i) \quad f(x+h) - f(x) = \int_0^1 D_H f(x + \sigma h)(h) d\sigma, \text{ if } n = 1,$$

$$(ii) \quad (D_h^{n-1} f(x+h) - D_H^{n-1} f(x))(h_1, \dots, h_{n-1}) = \int_0^1 D_H^n f(x + \sigma h)(h_1, \dots, h_{n-1}, h) d\sigma, \text{ if } n > 1. \quad (2.1)$$

For $\alpha \in (0, 1)$ we set

$$C_H^\alpha(X; Y) := \left\{ F \in C_b(X; Y) : [F]_{C_H^\alpha(X; Y)} := \sup_{x \in X, h \in H \setminus \{0\}} \frac{\|F(x+h) - F(x)\|_Y}{\|h\|_H^\alpha} < +\infty \right\}$$

and we endow $C_H^\alpha(X; Y)$ with the norm

$$\|F\|_{C_H^\alpha(X; Y)} := \|F\|_\infty + [F]_{C_H^\alpha(X; Y)}$$

For $\alpha = 1$, instead of Lipschitz continuity we shall consider a weaker condition, called Zygmund continuity. We set

$$Z_H^1(X, Y) :=$$

$$\left\{ F \in C_b(X; Y) : [F]_{Z_H^1(X; Y)} := \sup_{x \in X, h \in H \setminus \{0\}} \frac{\|F(x+2h) - 2F(x+h) + F(x)\|_Y}{\|h\|_H} < +\infty \right\}$$

and we endow $Z_H^1(X; Y)$ with the norm

$$\|F\|_{Z_H^1(X; Y)} := \sup_{x \in X} \|F(x)\|_Y + [F]_{Z_H^1(X; Y)}.$$

If $H = X$ we drop the subindex H and we use the more standard notations $C_b^n(X, Y)$ for $n \in \mathbb{N}$, $C_b^\alpha(X, Y)$ for $\alpha \in (0, 1)$, $Z_b^1(X, Y)$.

If $Y = \mathbb{R}$, we set $C_H^n(X; \mathbb{R}) =: C_H^n(X)$ for $n \in \mathbb{N}$, $C_H^\alpha(X; \mathbb{R}) =: C_H^\alpha(X)$ for $\alpha \in (0, 1)$, $Z_H^1(X; \mathbb{R}) =: Z_H^1(X)$. Higher order Hölder and Zygmund spaces of real valued functions will also be used; they are defined in a natural way, as follows.

For $\alpha \in (0, 1)$ and $n \in \mathbb{N}$ we set

$$\begin{aligned} C_H^{\alpha+n}(X) &:= \{f \in C_H^n(X) : D_H^n f \in C_H^\alpha(X, \mathcal{L}^n(H))\}, \\ \|f\|_{C_H^{\alpha+n}(X)} &:= \|f\|_{C_H^n(X)} + [D_H^n f]_{C_H^\alpha(X, \mathcal{L}^n(H))} \end{aligned} \quad (2.2)$$

and for $n \in \mathbb{N}$, $n \geq 2$,

$$\begin{aligned} Z_H^n(X) &:= \{f \in C_H^{n-1}(X) : D_H^{n-1} f \in Z_H^1(X, \mathcal{L}^{n-1}(H))\}, \\ \|f\|_{Z_H^n(X)} &:= \|f\|_{C_H^{n-1}(X)} + [D_H^{n-1} f]_{Z_H^1(X, \mathcal{L}^{n-1}(H))} \end{aligned} \quad (2.3)$$

In the next lemma we collect some properties of the above defined spaces, that are easy extensions of known properties in the case $H = X$, and that will be used later.

Lemma 2.1. *Let X, Y be Banach spaces.*

(i) *For every $\alpha \in (0, 1)$ and $F \in C_H^1(X; Y)$ we have*

$$\|F\|_{C_H^\alpha(X; Y)} \leq 2^{1-\alpha} \|D_H F\|_{C_b(X; \mathcal{L}(H; Y))}^\alpha \|F\|_{C_b(X; Y)}^{1-\alpha}. \quad (2.4)$$

(ii) *If $F : X \mapsto Y$ is H -Gateaux differentiable and $D_H F$ is continuous at $x \in X$, then F is H -Fréchet differentiable at x .*

(iii) *If $f \in C_H^2(X)$ we have*

$$|f(x+2h) - 2f(x+h) + f(x)| \leq \sup_{y \in X} \|D_H^2 f(y)\|_{\mathcal{L}^2(H)} \|h\|_H^2, \quad x \in X, h \in H. \quad (2.5)$$

Proof. Let $F \in C_H^1(X; Y)$. For every $x \in X$, $h \in H$, the function $\psi : \mathbb{R} \mapsto Y$, $\psi(t) := F(x+th)$ is continuously differentiable, and $\psi'(t) = D_H F(x+th)(h)$. Therefore we have

$$F(x+h) - F(x) = \int_0^1 D_H F(x+\sigma h)(h) d\sigma \quad (2.6)$$

so that

$$\|F(x+h) - F(x)\|_Y \leq \sup_{z \in X} \|D_H F(z)\|_{\mathcal{L}(H; Y)} \|h\|_H. \quad (2.7)$$

Of course, we also have

$$\|F(x+h) - F(x)\|_Y \leq 2 \sup_{z \in X} \|F(z)\|_Y.$$

Consequently,

$$\|F(x+h) - F(x)\|_Y \leq \left(\sup_{z \in X} \|D_H F(z)\|_{\mathcal{L}(H; Y)} \|h\|_H \right)^\alpha \left(2 \sup_{z \in X} \|F(z)\|_Y \right)^{1-\alpha}$$

and statement (i) follows.

Let us prove (ii). Using again (2.6) we get, for every $h \in H$,

$$\begin{aligned} \|F(x+h) - F(x) - D_H F(x)(h)\|_Y &= \left\| \int_0^1 (D_H F(x + \sigma h) - D_H F(x))(h) d\sigma \right\|_Y \\ &\leq \sup_{v \in H, \|v\|_H \leq \|h\|_H} \|D_H F(x+v) - D_H F(x)\|_{\mathcal{L}(X,Y)} \|h\|_H \end{aligned}$$

so that, recalling that $H \subset X$, $\|F(x+h) - F(x) - D_H F(x)(h)\|_Y = o(\|h\|_H)$ as $\|h\|_H \rightarrow 0$, and F is H -Fréchet differentiable at x .

Let now $f \in C_H^2(X)$. Applying thrice (2.6), for every $x \in X$ and $h \in H$ we get

$$\begin{aligned} f(x+2h) - 2f(x+h) + f(x) &= \int_0^1 (D_H f(x + (1+\sigma)h) - D_H f(x + \sigma h))(h) d\sigma \\ &= \int_0^1 \int_0^1 D_H^2 f(x + (\tau + \sigma)h)(h, h) d\tau d\sigma \end{aligned}$$

and estimating in an obvious way statement (iii) follows. \square

A Borel probability measure μ in X is called *Fomin differentiable* along $v \in X$ if for every Borel set A the incremental ratio $(\mu(A+tv) - \mu(A))/t$ has finite limit as $t \rightarrow 0$. Such a limit is called $d_v \mu(A)$; $d_v \mu$ is a signed measure and denoting the translated measure $\mu_v(A) := \mu(A+v)$ by μ_v we have

$$\lim_{s \rightarrow 0} \left\| \frac{\mu_{sv} - \mu}{s} - d_v \mu \right\| = 0, \quad (2.8)$$

where $\|\cdot\|$ denotes the total variation norm.

Moreover, $d_v \mu$ is absolutely continuous with respect to μ . The density $\beta_v^\mu \in L^1(X, \mu)$ is called *Fomin derivative* or *logarithmic derivative* of μ along v , and it satisfies

$$\int_X \frac{\partial f}{\partial v} \mu(dx) = - \int_X \beta_v^\mu f \mu(dx), \quad f \in C_b^1(X). \quad (2.9)$$

By [6, Thm. 3.6.8], this equality characterizes Fomin differentiability, in the sense that if (2.9) holds for some $\beta_v^\mu \in L^1(X, \mu)$ and for every $f \in C_b^1(X)$, then μ is Fomin differentiable along v .

If μ is Fomin differentiable along two vectors v, w , then it is Fomin differentiable along any linear combination of v and w , and we have $d_{\lambda_1 v + \lambda_2 w} \mu = \lambda_1 d_v \mu + \lambda_2 d_w \mu$; therefore

$$\beta_{\lambda_1 v + \lambda_2 w}^\mu = \lambda_1 \beta_v^\mu + \lambda_2 \beta_w^\mu, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \quad (2.10)$$

The proofs of these statements may be found in [6, Chapter 3]. We refer to [6] for the general theory of differentiable measures.

3. SCHAUDER AND ZYGMUND REGULARITY

Under the only assumptions that $T_t \in \mathcal{L}(X)$ and μ_t is a Borel probability measure for every t , the operators P_t defined in (1.1) map $C_b(X)$ into itself and we have

$$\|P_t f\|_\infty \leq \|f\|_\infty, \quad t > 0, f \in C_b(X). \quad (3.1)$$

The weak continuity of $t \mapsto \mu_t$ yields that for every $f \in C_b(X)$ the function $[0, +\infty) \times X, (t, x) \mapsto P_t f(x)$ is continuous, by [7, Lemma 2.1]. Consequently, the operators F_λ in right-hand side of (1.3) are one to one, and since P_t is a semigroup they satisfy the resolvent identity $F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu)$. By the general spectral theory, there exists a unique closed operator $L : D(L) \subset C_b(X) \mapsto C_b(X)$ such that $F(\lambda) = R(\lambda, L)$. The domain $D(L)$ is just the range of $F(\lambda)$, for every $\lambda > 0$.

The leading assumptions of the paper are the following.

Hypothesis 3.1. For every $t > 0$ there exists a subspace $\{0\} \neq H_t \subset X$ such that μ_t is Fomin differentiable along every $h \in H_t$.

According to the notation of Section 2, for every $v \in H_t$ we denote by $\beta_v^{\mu_t}$ the Fomin derivative of μ_t along v .

Hypothesis 3.2. There exists a Banach space $H \subset X$, and constants $M, C, \theta > 0, \omega \in \mathbb{R}$ such that

$$\begin{cases} (i) & T_t(H) \subset H, \quad \|T_t h\|_H \leq M e^{\omega t} \|h\|_H, \quad t > 0, h \in H, \\ (ii) & T_t(H) \subset H_t, \quad \|\beta_{T_t h}^{\mu_t}\|_{L^1(X, \mu_t)} \leq \frac{C e^{\omega t}}{t^\theta} \|h\|_H, \quad t > 0, h \in H. \end{cases} \quad (3.2)$$

3.1. Properties of P_t and estimates. The starting point of our analysis is the next proposition, which shows that each P_t is smoothing along suitable directions.

Proposition 3.3. (i) Let $g \in \mathcal{B}_b(X)$ and $t > 0$. Then $P_t g$ is H -Gateaux differentiable with bounded H -Gateaux derivative, and

$$D_H P_t g(x)(h) = - \int_X g(T_t x + y) \beta_{T_t h}^{\mu_t}(y) \mu_t(dy), \quad t > 0, h \in H. \quad (3.3)$$

(ii) Let $g \in C_b(X)$ and $t > 0, n \in \mathbb{N}$. Then $P_t g \in C_H^n(X)$; for all $h_1, \dots, h_n \in H$ we have

$$\begin{aligned} & D_H^n P_t g(x)(h_1, \dots, h_n) \\ &= (-1)^n \int_X \cdots \int_X g(T_t x + T_{\frac{n-1}{n}t} y_1 + \cdots + T_{\frac{1}{n}t} y_{n-1} + y_n) \\ & \quad \beta_{T_{\frac{1}{n}t} h_n}^{\mu_t/n}(y_n) \cdots \beta_{T_{\frac{1}{n}t} h_1}^{\mu_t/n}(y_1) \mu_{t/n}(dy_n) \cdots \mu_{t/n}(dy_1), \end{aligned} \quad (3.4)$$

and

$$\|D_H^n P_t g(x)\|_{\mathcal{L}^n(H)} \leq K_n \frac{e^{\omega t}}{t^{n\theta}} \|g\|_\infty, \quad (3.5)$$

with $K_n := C^n n^\theta$. If $\omega > 0$ a better estimate than (3.5) holds for large t , namely there exists $K'_n > 0$ such that

$$\|D_H^n P_t g(x)\|_{\mathcal{L}^n(H)} \leq K'_n \max\{1, t^{-n\theta}\} \|g\|_\infty, \quad t > 0, x \in X, g \in C_b(X). \quad (3.6)$$

(iii) Let $g \in C_H^1(X)$ and $t > 0$. Then

$$D_H P_t g(x)(h) = \int_X \frac{\partial g}{\partial(T_t h)}(T_t x + y) \mu_t(dy) = P_t \left(\frac{\partial g}{\partial(T_t h)} \right)(x), \quad t > 0, x \in X, h \in H. \quad (3.7)$$

If even $g \in C_H^n(X)$ for some $n \in \mathbb{N}$, then for all $t > 0, x \in X$ and $h_j \in H, j = 1, \dots, n$, we have

$$D_H^n P_t g(x)(h_1, \dots, h_n) = \int_X D_H^n g(T_t x + y)(T_t h_1, \dots, T_t h_n) \mu_t(dy) = P_t \left(\frac{\partial^n g}{\partial(T_t h_n) \cdots \partial(T_t h_1)} \right)(x), \quad (3.8)$$

and the function $(t, x) \mapsto D_H^n P_t g(x)(h_1, \dots, h_n)$ is continuous in $[0, +\infty) \times X$. Moreover,

$$\|P_t\|_{\mathcal{L}(C_H^n(X))} \leq \max\{1, M^n e^{n\omega t}\}, \quad t > 0, n \in \mathbb{N}. \quad (3.9)$$

Proof. (i) For every $x \in X, h \in H$ and $s \neq 0$ we have

$$\begin{aligned} \frac{P_t g(x + sh) - P_t g(x)}{s} &= \frac{1}{s} \int_X (g(T_t x + s T_t h + y) - g(T_t x + y)) \mu_t(dy) \\ &= \frac{1}{s} \left(\int_X g(T_t x + z) (\mu_t)_{s T_t h}(dz) - \int_X g(T_t x + y) \mu_t(dy) \right) \end{aligned}$$

so that

$$\left| \frac{P_t g(x + sh) - P_t g(x)}{s} + \int_X g(T_t x + y) \beta_{T_t h}^{\mu_t}(y) \mu_t(dy) \right| \leq \|g\|_\infty \left\| \frac{(\mu_t)_{sT_t h} - \mu_t}{s} - d_{T_t h} \mu_t \right\|$$

that vanishes as $s \rightarrow 0$ by (2.8). Therefore, $P_t g$ is differentiable along h at x , with derivative $\partial P_t g(x)/\partial h$ given by the right-hand side of (3.3). Such a derivative is linear in h by (2.10) and by the linearity of T_t , and by Hypothesis 3.2(ii) its modulus is bounded by $\|g\|_\infty C e^{\omega t} t^{-\theta} \|h\|_H$. Therefore, $P_t g$ is H -Gateaux differentiable at x and (3.3) holds. If $g \in C_b(X)$, then for every $x, x_0 \in X$, and $h \in H$,

$$|(D_H P_t g(x) - D_H P_t g(x_0))(h)| \leq \int_X |g(T_t x + y) - g(T_t x_0 + y)| |\beta_{T_t h}^{\mu_t}(y)| \mu_t(dy) \quad (3.10)$$

where the right-hand side vanishes as $x \rightarrow x_0$ by the Dominated Convergence Theorem. So, $D_H P_t g(\cdot)h$ is continuous on X , hence $P_t g \in C_H^1(X)$.

(ii) Now let us prove (3.4) for $g \in C_b(X)$, $t > 0$, by induction over $n \in \mathbb{N}$. We have just proved (3.4) for $n = 1$ above. Suppose that (3.4) holds for $n \in \mathbb{N}$. By the induction hypothesis applied to the n -step equipartition $0 < \frac{1}{n+1} < \dots < \frac{n}{n+1}$ of $[0, \frac{n}{n+1}]$ for $h_1, \dots, h_{n+1} \in H$ we have

$$\begin{aligned} D_H^n P_t g(x)(h_1, \dots, h_n) &= D_H^n P_{\frac{n}{n+1}t} (P_{\frac{t}{n+1}} g)(x)(h_1, \dots, h_n) \\ &= (-1)^n \int_X \dots \int_X P_{\frac{t}{n+1}} g(T_{\frac{n}{n+1}t} x + T_{\frac{n-1}{n+1}t} y_1 + \dots + T_{\frac{t}{n+1}} y_{n-1} + y_n) \\ &\quad \beta_{T_{t/(n+1)} h_n}^{\mu_{t/(n+1)}}(y_n) \dots \beta_{T_{t/(n+1)} h_1}^{\mu_{t/(n+1)}}(y_1) \mu_{t/(n+1)}(dy_n) \dots \mu_{t/(n+1)}(dy_1). \end{aligned}$$

Since we already know that $P_{\frac{t}{n+1}} g \in C_H^1(X)$, by Hypothesis 3.2(ii), (2.7) and the Dominated Convergence Theorem we can differentiate the right-hand side along h_{n+1} interchanging the partial derivative with the multiple integrals, and using (3.3) we obtain

$$\begin{aligned} \frac{\partial}{\partial h_{n+1}} D_H^n P_t g(x)(h_1, \dots, h_n) &= (-1)^{(n+1)} \int_X \dots \int_X g(T_t x + T_{\frac{nt}{n+1}} y_1 + \dots + T_{\frac{t}{n+1}} y_n + z) \\ &\quad \beta_{T_{t/(n+1)} h_{n+1}}^{\mu_{t/(n+1)}}(z) \beta_{T_{t/(n+1)} h_n}^{\mu_{t/(n+1)}}(y_n) \dots \beta_{T_{t/(n+1)} h_1}^{\mu_{t/(n+1)}}(y_1) \mu_{t/(n+1)}(dz) \mu_{t/(n+1)}(dy_n) \dots \mu_{t/(n+1)}(dy_1). \end{aligned}$$

The right-hand side is just $D_H^{n-1} P_t g(x)(h_1, \dots, h_{n+1})$, so that (3.4) holds for $n + 1$.

The continuity and boundedness on X of the map $x \mapsto D_H^n P_t g(x)(h_1, \dots, h_n)$ is obvious by (3.4), Hypothesis 3.2(ii) and the Dominated Convergence Theorem. Then also (3.5) follows immediately by Hypothesis 3.2(ii).

Assume now that $\omega > 0$. Using (3.5) we get for $0 < t \leq 2$

$$\|D_H^n P_t g(x)\|_{\mathcal{L}^n(H)} \leq K_n \frac{e^{2\omega}}{t^{n\theta}} \|g\|_\infty,$$

while for $t \geq 2$, writing $D_H^n P_t g = D_H^n P_1 (P_{t-1} g)$ and using (3.1) and (3.5) with $t = 1$, we get

$$\|D_H^n P_t g(x)\|_{\mathcal{L}^n(H)} \leq K_n e^\omega \|P_{t-1} g\|_\infty \leq K_n e^\omega \|g\|_\infty.$$

Putting together such estimates, we get (3.6).

(iii) Now we prove (3.7). If $g \in C_H^1(X)$, for every $s \neq 0$ and $x \in X$, as before,

$$\frac{P_t g(x + sh) - P_t g(x)}{s} = \int_X \frac{g(T_t x + sT_t h + y) - g(T_t x + y)}{s} \mu_t(dy)$$

and the right-hand side converges to $\int_X \frac{\partial g}{\partial T_t h}(T_t x + y) \mu_t(dy) = \int_X D_H g(T_t x + y)(h) \mu_t(dy)$ as $s \rightarrow 0$, by (2.7) and the Dominated Convergence Theorem. By the definition of P_t , such limit coincides with $P_t(\partial g / \partial(T_t h))(x)$.

If $g \in C_H^n(X)$, formula (3.8) follows applying several times (3.7). The proof of the continuity of $(t, x) \mapsto D_H^n P_t g(x)(h_1, \dots, h_n)$ is similar to the proof of the continuity of $(t, x) \mapsto P_t f(x)$ of [7, Lemma 2.1]. Here is the argument:

Let $t_k \rightarrow t \in [0, +\infty)$, $x_k \rightarrow x \in X$. Then,

$$\begin{aligned} & D_H^n P_{t_k} g(x_k)(h_1, \dots, h_n) - D_H^n P_t g(x)(h_1, \dots, h_n) \\ &= \int_X (D_H^n g(T_{t_k} x_k + y)(T_{t_k} h_1, \dots, T_{t_k} h_n) - D_H^n g(T_t x + y)(T_t h_1, \dots, T_t h_n)) \mu_{t_k}(dy) \\ &+ \int_X D_H^n g(T_t x + y)(T_t h_1, \dots, T_t h_n) (\mu_{t_k}(dy) - \mu_t(dy)) =: I_{1,k} + I_{2,k} \end{aligned}$$

Since μ_{t_k} weakly converges to μ_t as $k \rightarrow \infty$ and $D_H^n g(T_t x + \cdot)(T_t h_1, \dots, T_t h_n)$ is continuous and bounded, $I_{2,k} \rightarrow 0$ as $k \rightarrow \infty$. Still by the weak convergence, the measures μ_{t_k} are uniformly tight, namely for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ such that $\mu_{t_k}(X \setminus K_\varepsilon) \leq \varepsilon$ for every $k \in \mathbb{N}$. Splitting $I_{1,k}$ into the sum of the integral over K and the integral over $X \setminus K_\varepsilon$, and using the uniform continuity of $(t, z) \mapsto D_H^n g(z)(T_t h_1, \dots, T_t h_n)$ on compact sets, one gets $\lim_{k \rightarrow \infty} I_{1,k} = 0$, too.

By (3.8) and Hypothesis 3.2(i) we have, for every natural number $j \leq n$ and $x \in X$, $h_1, \dots, h_j \in H$,

$$|(D_H^j P_t g(x))(h_1, \dots, h_j)| \leq \sup_{y \in X} \|D_H^j f(y)\|_{\mathcal{L}^j(H)} \prod_{l=1}^j \|T_t h_l\|_H \leq M^j e^{j\omega t} \prod_{l=1}^j \|h_l\|_H, \quad g \in C_H^m(X),$$

which yields (3.9). □

Remark 3.4. Under our general assumptions we cannot prove that $D_H P_t g$ is continuous with values in H^* (and therefore that $P_t g$ is Fréchet differentiable, by Lemma 2.1(ii)) for every $t > 0$ and $g \in C_b(X)$. (3.10) implies immediately that $D_H P_t g$ is continuous for every uniformly continuous and bounded g , but we prefer to deal with merely continuous rather than uniformly continuous functions.

If in addition the functions $\beta_{T_t h}^{\mu_t}$ belong to $L^p(X, \mu_t)$ for some $p > 1$, and for every $t > 0$ there exists $C_t > 0$ such that $\|\beta_{T_t h}^{\mu_t}\|_{L^p(X, \mu_t)} \leq C_t \|h\|_H$ for every $h \in H$, using the Hölder inequality in the right-hand side of (3.10) and then the Dominated Convergence Theorem yields that $D_H P_t g$ is continuous with values in H^* . In this case, throughout the paper we could use stronger higher order Hölder and Zygmund spaces, obtained replacing the condition of H -Gateaux differentiability by H -Fréchet differentiability in the definition of the C_H^n spaces.

The behavior of P_t in the Hölder spaces $C_H^\alpha(X)$ and in the Zygmund spaces $Z_H^k(X)$ is coherent with its behavior in $C_b(X)$, as the next lemma shows.

Lemma 3.5. *For every $t > 0$ and $\alpha \in (0, +\infty)$, $k \in \mathbb{N} \cup \{0\}$, $P_t \in \mathcal{L}(C_H^{k+\alpha}(X))$ and there exists $c = c(k + \alpha) > 0$ such that*

$$\|P_t f\|_{C_H^{k+\alpha}(X)} \leq c \|f\|_{C_H^{k+\alpha}(X)}, \quad t > 0, f \in C_H^{k+\alpha}(X). \quad (3.11)$$

Moreover, for every $t > 0$ and $k \in \mathbb{N}$, $P_t \in \mathcal{L}(Z_H^k(X))$ and there exists $c = c(k) > 0$ such that

$$\|P_t f\|_{Z_H^k(X)} \leq c \|f\|_{Z_H^k(X)}, \quad t > 0, f \in Z_H^k(X). \quad (3.12)$$

Proof. Let $\alpha \in (0, 1)$ and $f \in C_H^\alpha(X)$, $t > 0$. From the representation formula (1.1) and Hypothesis 3.2(i) we get, for every $x \in X$ and $h \in H$,

$$\begin{aligned} |P_t f(x+h) - P_t f(x)| &= \left| \int_X (f(T_t x + T_t h + y) - f(T_t x + y)) \mu_t(dy) \right| \\ &\leq [f]_{C_H^\alpha(X)} \|T_t h\|_H^\alpha \leq M^\alpha e^{\alpha \omega t} [f]_{C_H^\alpha(X)} \|h\|_H^\alpha, \end{aligned} \quad (3.13)$$

so that

$$\|P_t f\|_{C_H^\alpha(X)} \leq \|f\|_\infty + M^\alpha e^{\alpha \omega t} [f]_{C_H^\alpha(X)}.$$

This proves (3.11) for $k = 0$, in the case $\omega \leq 0$.

If $f \in C_H^{k+\alpha}(X)$ for some $k \in \mathbb{N}$, we use (3.8) and again Hypothesis 3.2(i), that give, for every $x \in X$ and $h, h_1, \dots, h_k \in H$,

$$\begin{aligned} |(D_H^k P_t f(x+h) - D_H^k P_t f(x))(h_1, \dots, h_k)| &= \\ &= \left| \int_X (D_H^k f(T_t x + T_t h + y) - D_H^k f(T_t x + y))(T_t h_1, \dots, T_t h_k) \mu_t(dy) \right| \\ &\leq [D_H^k f]_{C_H^\alpha(X; \mathcal{L}^k(H))} \|T_t h\|_H^\alpha \prod_{j=1}^k \|T_t h_j\|_H \leq M^{k+\alpha} e^{(k+\alpha)\omega t} [D_H^k f]_{C_H^\alpha(X; \mathcal{L}^k(H))} \|h\|_H^\alpha \prod_{j=1}^k \|h_j\|_H. \end{aligned}$$

This estimate and (3.9) yield (3.11) for $k \in \mathbb{N}$, in the case $\omega \leq 0$.

For $\omega > 0$ we argue as follows. For every $f \in C^{k+\alpha}(X)$ with $\alpha \in (0, 1)$ and $k \in \mathbb{N} \cup \{0\}$ the above estimates yield

$$\|P_t f\|_{C^{k+\alpha}(X)} \leq M^{k+\alpha} e^{(k+\alpha)\omega t} \|f\|_{C^{k+\alpha}(X)}, \quad t \leq 1,$$

while for $t > 1$ we write $P_t f = P_1 P_{t-1} f$, so that $\|P_t f\|_{C_H^{k+\alpha}(X)} \leq \|P_1\|_{\mathcal{L}(C_b(X), C_H^{k+\alpha}(X))} \|f\|_\infty$ by (3.1) (P_1 belongs to $\mathcal{L}(C_b(X), C_H^{k+\alpha}(X))$) because it belongs to $\mathcal{L}(C_b(X), C_H^{k+1}(X))$ by Proposition 3.3, and $C_H^{k+1}(X) \subset C_H^{k+\alpha}(X)$.

The proof of estimates (3.12) is similar, and it is left to the reader. \square

If f is H -Hölder continuous estimates (3.5) may be improved near $t = 0$. Such improvements are crucial in the proof of our Schauder theorems.

Proposition 3.6. *For every $\alpha \in (0, 1)$ and $n \in \mathbb{N}$ there are constants $K_{n,\alpha} > 0$ such that*

$$\|D_H^n P_t f(x)\|_{\mathcal{L}^n(H)} \leq K_{n,\alpha} \frac{e^{\omega t}}{t^{(n-\alpha)\theta}} [f]_{C_H^\alpha(X)}, \quad t > 0, x \in X, f \in C_H^\alpha(X). \quad (3.14)$$

Proof. The key step is to prove that (3.14) holds for $n = 1$. We use the same argument of [16]. Let $t > 0$, $f \in C_H^\alpha(X)$, $h \in H \setminus \{0\}$. For every $s > 0$ we have

$$\begin{aligned} D_H P_t f(x)(h) &= \left(D_H P_t f(x)(h) - \frac{P_t f(x+sh) - P_t f(x)}{s} \right) + \frac{P_t f(x+sh) - P_t f(x)}{s} \\ &= \left(\frac{1}{s} \int_0^s (D_H P_t f(x)(h) - D_H P_t f(x+\sigma h)(h)) d\sigma \right) + \frac{P_t f(x+sh) - P_t f(x)}{s} \\ &=: I_1(s) + I_2(s). \end{aligned}$$

To estimate $I_1(s)$ we remark that for every $k \in H$, by (3.3) we have

$$\begin{aligned} |(D_H P_t f(x+k) - D_H P_t f(x))(h)| &= \left| \int_X (f(T_t x + T_t k + y) - f(T_t x + y)) \beta_{T_t h}^{\mu_t}(y) \mu_t(dy) \right| \\ &\leq [f]_{C_H^\alpha(X)} \|T_t k\|_H^\alpha \|\beta_{T_t h}^{\mu_t}\|_{L^1(X, \mu_t)} \leq [f]_{C_H^\alpha(X)} M^\alpha e^{\alpha \omega t} \|k\|_H^\alpha \frac{C e^{\omega t}}{t^\theta} \|h\|_H. \end{aligned}$$

Using this estimate with $k = \sigma h$ we get

$$|I_1(s)| \leq \frac{1}{s} \int_0^s |D_H P_t f(x + \sigma h)(h) - D_H P_t f(x)(h)| d\sigma \leq \frac{1}{s} \frac{C M^\alpha}{t^\theta} e^{(\alpha+1)\omega t} \int_0^s \sigma^\alpha d\sigma \|h\|_H^{\alpha+1} [f]_{C_H^\alpha(X)}.$$

On the other hand, by (3.13) we get

$$|I_2(s)| \leq M^\alpha e^{\alpha \omega t} s^{\alpha-1} \|h\|_H^\alpha [f]_{C_H^\alpha(X)}.$$

Summing up,

$$|D_H P_t f(x)(h)| \leq \left(\frac{C M^\alpha}{\alpha+1} e^{(\alpha+1)\omega t} \|h\|_H^{\alpha+1} \frac{s^\alpha}{t^\theta} + M^\alpha e^{\alpha \omega t} s^{\alpha-1} \|h\|_H^\alpha \right) [f]_{C_H^\alpha(X)}, \quad s > 0.$$

Choosing now $s = t^\theta e^{-\omega t} / \|h\|_H$ we get

$$|D_H P_t f(x)(h)| \leq \left(\frac{C M^\alpha}{\alpha+1} + M^\alpha \right) \frac{1}{t^{(1-\alpha)\theta}} e^{\omega t} \|h\|_H [f]_{C_H^\alpha(X)},$$

which yields (3.14) for $n = 1$.

For $n > 1$ we have $D_H^n P_t f = D_H^n P_{t/2} g$, with $g = P_{t/2} f$. By (3.7),

$$\frac{\partial^n P_t f}{\partial h_n \cdots \partial h_1} = \frac{\partial^{n-1}}{\partial h_{n-1} \cdots \partial h_1} P_{t/2} \left(\frac{\partial P_{t/2} f}{\partial (T_{t/2} h_n)} \right)$$

so that (3.14) follows from (3.5) and (3.14) with $n = 1$. \square

3.2. Schauder and Zygmund estimates: stationary equations. In this section we use the smoothing properties of P_t to deduce regularity results for the elements of $D(L)$, namely for the functions u given by (1.3) for some $\lambda > 0$ and $f \in C_b(X)$. Estimate (3.1) yields immediately

$$\|u\|_\infty \leq \frac{1}{\lambda} \|f\|_\infty. \quad (3.15)$$

The first (not optimal) regularity result is a standard consequence of Propositions 3.3 and 3.6.

Proposition 3.7. *Given $\lambda > 0$ and $f \in C_b(X)$, let $u = R(\lambda, L)f$.*

- (i) *Let $\theta < 1$. For every $n \in \mathbb{N}$ such that $n < 1/\theta$, $u \in C_H^n(X)$. There exists $C = C(\lambda) > 0$, independent of f , such that*

$$\|u\|_{C_H^n(X)} \leq C \|f\|_\infty \quad (3.16)$$

- (ii) *Let $\alpha \in (0, 1)$ be such that $\alpha + 1/\theta > 1$. For every $f \in C_H^\alpha(X)$ and for every $n \in \mathbb{N}$ such that $n < \alpha + 1/\theta$, $u \in C_H^n(X)$. There exists $C = C(\lambda, \alpha) > 0$, independent of f , such that*

$$\|u\|_{C_H^n(X)} \leq C \|f\|_{C_H^\alpha(X)} \quad (3.17)$$

Proof. The proof is in two steps. First we consider the case $\lambda > \omega$, and then, if $\omega > 0$, the case $\lambda \in (0, \omega]$.

First step: $\lambda > \omega$. Estimate (3.5) yields, for every $k \in \{1, \dots, n\}$,

$$e^{-\lambda t} \|D_H^k P_t f(x)\|_{\mathcal{L}^k(H)} \leq e^{-\lambda t} K_k \frac{e^{\omega t}}{t^{k\theta}} \|f\|_\infty, \quad t > 0, x \in X, f \in C_b(X), \quad (3.18)$$

and if $\alpha \in (0, 1)$, (3.14) yields, for every $k \in \{1, \dots, n\}$,

$$e^{-\lambda t} \|D_H^k P_t f(x)\|_{\mathcal{L}^k(H)} \leq e^{-\lambda t} K_{k,\alpha} \frac{e^{\omega t}}{t^{(k-\alpha)\theta}} [f]_{C_H^\alpha(X)}, \quad t > 0, x \in X, f \in C_H^\alpha(X). \quad (3.19)$$

The right-hand sides of (3.18) and (3.19) belong to $L^1(0, +\infty)$ because $\lambda > \omega$, and $k\theta \in (0, 1)$ in (3.18), $(k-\alpha)\theta \in (0, 1)$ in (3.19). Therefore u is n times H -Gateaux differentiable at every $x \in X$, and for every $h_1, \dots, h_k \in H$ with $k \in \{1, \dots, n\}$ we have

$$D_H^k u(x)(h_1, \dots, h_k) = \int_0^\infty e^{-\lambda t} D_H^k P_t f(x)(h_1, \dots, h_k) dt.$$

(3.18) and (3.19) imply respectively, for every $x \in X$ and $k \in \{1, \dots, n\}$,

$$\|D_H^k u(x)\|_{\mathcal{L}^k(H)} \leq K_k \int_0^\infty \frac{e^{\omega t - \lambda t}}{t^{k\theta}} dt \|f\|_\infty \leq \frac{K_k \Gamma(1 - k\theta)}{(\lambda - \omega)^{1 - k\theta}} \|f\|_\infty,$$

and

$$\|D_H^k u(x)\|_{\mathcal{L}^k(H)} \leq K_{k,\alpha} \int_0^\infty \frac{e^{\omega t - \lambda t}}{t^{(k-\alpha)\theta}} dt [f]_{C_H^\alpha(X)} \leq \frac{K_{k,\alpha} \Gamma(1 - (k-\alpha)\theta)}{(\lambda - \omega)^{1 - (k-\alpha)\theta}} [f]_{C_H^\alpha(X)},$$

for $\alpha > 0$, $f \in C_H^\alpha(X)$ (here, Γ is the Euler function). In both cases, since for every $t > 0$ the function $x \mapsto D_H^k P_t f(x)(h_1, \dots, h_k)$ is continuous by Proposition 3.3, estimates (3.18), (3.19) and the Dominated Convergence Theorem imply that $x \mapsto D_H^k u(x)(h_1, \dots, h_k)$ is continuous for $k = 1, \dots, n$. Therefore, $u \in C_H^n(X)$ and

$$\|u\|_{C_H^n(X)} \leq \|f\|_\infty + \left(\sum_{k=1}^n K_k \frac{\Gamma(1 - k\theta)}{(\lambda - \omega)^{1 - k\theta}} \right) \|f\|_\infty, \quad (3.20)$$

so that (3.16) holds with $C = 1 + \sum_{k=1}^n K_k \Gamma(1 - k\theta) / (\lambda - \omega)^{1 - k\theta}$. In the case that $\alpha \in (0, 1)$ and $f \in C_H^\alpha(X)$, we get

$$\|u\|_{C_H^n(X)} \leq \|f\|_\infty + \left(\sum_{k=1}^n K_{k,\alpha} \frac{\Gamma(1 - (k-\alpha)\theta)}{(\lambda - \omega)^{1 - (k-\alpha)\theta}} \right) [f]_{C_H^\alpha(X)}, \quad (3.21)$$

so that (3.17) holds with $C = 1 + \sum_{k=1}^n K_{k,\alpha} \Gamma(1 - (k-\alpha)\theta) / (\lambda - \omega)^{1 - (k-\alpha)\theta}$.

Second step: $\omega > 0$, $\lambda \in (0, \omega]$.

In this case the statement follows from Step 1 by a perturbation argument. Indeed, since $\lambda u - Lu = f$, we have $(\omega + 1)u - Lu = (\omega + 1 - \lambda)u + f$. The right-hand side belongs to $C_b(X)$, and its sup norm is bounded by $((\omega + 1 - \lambda)/\lambda + 1)\|f\|_\infty$, by (3.15). So, statement (i) follows from Step 1.

Concerning statement (ii), it is sufficient to prove that $u \in C_H^\alpha(X)$, with $\|u\|_{C_H^\alpha(X)} \leq C\|f\|_{C_H^\alpha(X)}$ for some $C > 0$, and to use Step 1 as above. This is a simple consequence of Lemma 3.5. Indeed, using (3.11) with $k = 0$ we get

$$|u(x+h) - u(x)| \leq \int_0^\infty e^{-\lambda t} |P_t f(x+h) - P_t f(x)| dt \leq \int_0^\infty e^{-\lambda t} c \|h\|_H^\alpha [f]_{C_H^\alpha(X)} dt = \frac{c}{\lambda} \|h\|_H^\alpha [f]_{C_H^\alpha(X)}. \quad \square$$

Notice that for $n \geq 1/\theta$ in case (i) and for $n \geq \alpha + 1/\theta$ in case (ii), the arguments used above do not work, since the functions $t \mapsto t^{-n\theta}$, $t \mapsto t^{-(n-\alpha)\theta}$, respectively, are not integrable near 0, and (3.18), (3.19) are not helpful to conclude that $D_H^n u(x)$ exists.

Optimal regularity results are provided by the next theorems. The first one deals with Hölder regularity, and the second one with Zygmund regularity.

Theorem 3.8. *Let $\lambda > 0$, $f \in C_b(X)$ and let $u = R(\lambda, L)f$. The following statements hold.*

(i) If $1/\theta \notin \mathbb{N}$ then $u \in C_H^{1/\theta}(X)$. There exists $C = C(\lambda) > 0$, independent of f , such that

$$\|u\|_{C_H^{1/\theta}(X)} \leq C \|f\|_\infty. \quad (3.22)$$

(ii) If $\alpha \in (0, 1)$ with $\alpha + 1/\theta \notin \mathbb{N}$ and $f \in C_H^\alpha(X)$ then $u \in C_H^{\alpha+1/\theta}(X)$ and there exists $C = C(\lambda, \alpha) > 0$, independent of f , such that

$$\|u\|_{C_H^{\alpha+1/\theta}(X)} \leq C \|f\|_{C_H^\alpha(X)}. \quad (3.23)$$

Proof. Let $n \in \mathbb{N} \cup \{0\}$ be the integral part of $\alpha + 1/\theta$, with $\alpha = 0$ in the case of statement (i) and $\alpha \in (0, 1)$ in the case of statement (ii). If $n = 0$, $u \in C_b(X)$ and (3.15) holds. If $n > 0$, we already know, by Proposition 3.7, that $u \in C_H^n(X)$, and that estimate (3.20) (resp. estimate (3.21)) holds.

We have to prove that $D_H^n u$ belongs to $C_H^{\alpha+1/\theta-n}(X, \mathcal{L}^n(H))$. As in Proposition 3.7, it is sufficient to consider the case $\lambda > \omega$. If $\omega > 0$, the case $\lambda \in (0, \omega]$ is recovered by the same argument used in Step 2 of Proposition 3.7.

We treat separately the cases $n > 0$ and $n = 0$.

Let $n = 0$. This implies that $\theta > 1$ in statement (i), and $(1 - \alpha)\theta > 1$ in statement (ii). For every fixed h , we split $u = a_h + b_h$, where

$$a_h(y) = \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} P_t f(y) dt, \quad b_h(y) = \int_{\|h\|_H^{1/\theta}}^\infty e^{-\lambda t} P_t f(y) dt, \quad y \in X. \quad (3.24)$$

So, for every $x \in X$ we have

$$|a_h(x+h) - a_h(x)| \leq \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} |P_t f(x+h) - P_t f(x)| dt \leq \int_0^{\|h\|_H^{1/\theta}} 2\|f\|_\infty dt = 2\|h\|_H^{1/\theta} \|f\|_\infty.$$

To estimate $|b_h(x+h) - b_h(x)|$ we remark that by (2.1)(i) and (3.5) with $n = 1$ for every $t > 0$ we have

$$\|P_t f(x+h) - P_t f(x)\| \leq \sup_{y \in X} \|D_H P_t f(y)\|_{H^*} \|h\|_H \leq K_1 \frac{e^{\omega t}}{t^\theta} \|h\|_H \|f\|_\infty, \quad (3.25)$$

which yields

$$\begin{aligned} |b_h(x+h) - b_h(x)| &\leq \int_{\|h\|_H^{1/\theta}}^\infty e^{-\lambda t} |P_t f(x+h) - P_t f(x)| dt \leq \int_{\|h\|_H^{1/\theta}}^\infty \frac{K_1}{t^\theta} dt \|h\|_H \|f\|_\infty \\ &\leq \frac{K_1}{\theta - 1} \|h\|_H^{1/\theta} \|f\|_\infty. \end{aligned}$$

Summing up, $u \in C_H^{1/\theta}(X)$, and

$$[u]_{C_H^{1/\theta}(X)} \leq \left(2 + \frac{K_1}{\theta - 1}\right) \|f\|_\infty.$$

This estimate and (3.15) give (3.22) with $C(\lambda) = 2 + K_1/(\theta - 1) + 1/\lambda$, in the case that $\theta > 1$.

If $\alpha \in (0, 1)$ and $f \in C_H^\alpha(X)$ we use (3.11)(i) and we get

$$\begin{aligned} |a_h(x+h) - a_h(x)| &\leq \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} |P_t f(x+h) - P_t f(x)| dt \\ &\leq \int_0^{\|h\|_H^{1/\theta}} e^{-(\lambda - \alpha\omega)t} M^\alpha \|h\|_H^\alpha [f]_{C_H^\alpha(X)} dt \leq M^\alpha \|h\|_H^{\alpha+1/\theta} [f]_{C_H^\alpha(X)}. \end{aligned}$$

To estimate $|b_h(x+h) - b_h(x)|$ we use (3.14) with $n = 1$, that gives

$$\|P_t f(x+h) - P_t f(x)\| \leq \sup_{y \in X} \|D_H P_t f(y)\|_{H^*} \|h\|_H \leq K_{1,\alpha} \frac{e^{\omega t}}{t^{(1-\alpha)\theta}} \|h\|_H [f]_{C_H^\alpha(X)}, \quad (3.26)$$

which yields

$$|b_h(x+h) - b_h(x)| \leq \int_{\|h\|_H^{1/\theta}}^{\infty} \frac{K_{1,\alpha}}{t^{(1-\alpha)\theta}} dt \|h\|_H [f]_{C_H^\alpha(X)} \leq \frac{K_{1,\alpha}}{(1-\alpha)\theta - 1} \|h\|_H^{1/\theta + \alpha} [f]_{C_H^\alpha(X)}.$$

Summing up, we obtain $u \in C_H^{\alpha+1/\theta}(X)$, and

$$[u]_{C_H^{\alpha+1/\theta}(X)} \leq \left(M^\alpha + \frac{K_{1,\alpha}}{(1-\alpha)\theta - 1} \right) [f]_{C_H^\alpha(X)}.$$

This estimate, together with (3.15), yield (3.23), with $C(\lambda) = M^\alpha + K_{1,\alpha}/((1-\alpha)\theta - 1) + 1/\lambda$, in the case that $\alpha + 1/\theta < 1$.

For $n = [\alpha + 1/\theta] \geq 1$ the procedure is similar, just with different notations and constants. We already know from Proposition 3.7 that $u \in C_H^n(X)$, and we have to show that $D_H^n u \in C_H^{\alpha+1/\theta-n}(X, \mathcal{L}^n(H))$, with $\alpha = 0$ as far as statement (i) is concerned, and $\alpha \in (0, 1)$ as far as statement (ii) is concerned.

For every fixed $h, h_1, \dots, h_n \in H$ we split $D_H^n u(y)(h_1, \dots, h_n) = a_h(y) + b_h(y)$, where now

$$\begin{cases} a_h(y) & := \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} D_H^n P_t f(y)(h_1, \dots, h_n) dt, & y \in X, \\ b_h(y) & := \int_{\|h\|_H^{1/\theta}}^{\infty} e^{-\lambda t} D_H^n P_t f(y)(h_1, \dots, h_n) dt, & y \in X. \end{cases} \quad (3.27)$$

Let us prove that statement (i) holds. In this case we have $f \in C_b(X)$, $n\theta \in (0, 1)$, $(n+1)\theta > 1$. Recalling that $\omega - \lambda < 0$, estimate (3.5) yields

$$\begin{aligned} |a_h(x+h) - a_h(x)| &\leq |a_h(x+h)| + |a_h(x)| \leq 2K_n \int_0^{\|h\|_H^{1/\theta}} \frac{e^{(\omega-\lambda)t}}{t^{n\theta}} dt \prod_{j=1}^n \|h_j\|_H \|f\|_\infty \\ &\leq \frac{2K_n}{1-n\theta} \|h\|_H^{(1-n\theta)/\theta} \prod_{j=1}^n \|h_j\|_H \|f\|_\infty. \end{aligned}$$

To estimate $|b_h(x+h) - b_h(x)|$ we apply (2.1)(ii) to the function $P_t f$, and using (3.5) we get

$$\|D_H^n P_t f(x+h) - D_H^n P_t f(x)\|_{\mathcal{L}^n(H)} \leq \sup_{y \in X} \|D_H^{n+1} P_t f(y)\|_{\mathcal{L}^n(H)} \|h\|_H \leq K_{n+1} \frac{e^{\omega t}}{t^{(n+1)\theta}} \|f\|_\infty \|h\|_H, \quad (3.28)$$

which yields (since $\omega - \lambda < 0$)

$$|b_h(x+h) - b_h(x)| \leq \int_{\|h\|_H^{1/\theta}}^{\infty} \frac{K_{n+1} \|h\|_H}{t^{(n+1)\theta}} dt \prod_{j=1}^n \|h_j\|_H \|f\|_\infty \leq \frac{K_{n+1} \|h\|_H^{1/\theta-n}}{(n+1)\theta - 1} \prod_{j=1}^n \|h_j\|_H \|f\|_\infty.$$

Summing up we get

$$|(D_H^n u(x+h) - D_H^n u(x))(h_1, \dots, h_n)| \leq C_1 \|h\|_H^{1/\theta-n} \prod_{j=1}^n \|h_j\|_H \|f\|_\infty \quad (3.29)$$

with

$$C_1 = \frac{2K_n}{1 - n\theta} + \frac{K_{n+1}}{(n+1)\theta - 1}.$$

Therefore, $D_H^n u \in C_H^{1/\theta-n}(X; \mathcal{L}^n(H))$ and $[D_H^n u]_{C_H^{1/\theta-n}(X; \mathcal{L}^n(H))} \leq C_1 \|f\|_\infty$. This estimate and (3.20) give (3.22) for $n \geq 1$.

Let us prove that statement (ii) holds. Now we have $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$, $(n - \alpha)\theta \in (0, 1)$, $(n + 1 - \alpha)\theta > 1$. Estimate (3.14) yields

$$\begin{aligned} |a_h(x+h) - a_h(x)| &\leq |a_h(x+h)| + |a_h(x)| \leq 2K_{n,\alpha} \int_0^{\|h\|_H^{1/\theta}} \frac{e^{(\omega-\lambda)t}}{t^{(n-\alpha)\theta}} dt \prod_{j=1}^n \|h_j\|_H [f]_{C_H^\alpha(X)} \\ &= \frac{2K_{n,\alpha}}{1 - (n - \alpha)\theta} \|h\|_H^{(1-(n-\alpha)\theta)/\theta} \prod_{j=1}^n \|h_j\|_H [f]_{C_H^\alpha(X)}. \end{aligned}$$

To estimate $|b_h(x+h) - b_h(x)|$ we use again (2.1)(ii) and by (3.5) we get

$$\|D_H^n P_t f(x+h) - D_H^n P_t f(x)\|_{\mathcal{L}^n(H)} \leq \sup_{y \in X} \|D_H^{n+1} P_t f(y)\|_{\mathcal{L}^n(H)} \|h\|_H \leq \frac{K_{n+1,\alpha} e^{\omega t}}{t^{(n+1-\alpha)\theta}} [f]_{C_H^\alpha(X)} \|h\|_H, \quad (3.30)$$

which yields

$$\begin{aligned} |b_h(x+h) - b_h(x)| &\leq \int_{\|h\|_H^{1/\theta}}^\infty \frac{K_{n+1,\alpha}}{t^{(n+1-\alpha)\theta}} dt \|h\|_H \prod_{j=1}^n \|h_j\|_H [f]_{C_H^\alpha(X)} \\ &\leq \frac{K_{n+1}}{(n+1-\alpha)\theta - 1} \|h\|_H^{1/\theta - n + \alpha} \prod_{j=1}^n \|h_j\|_H [f]_{C_H^\alpha(X)}. \end{aligned}$$

Summing up we get

$$|(D_H^n u(x+h) - D_H^n u(x))(h_1, \dots, h_n)| \leq C_2 \|h\|_H^{1/\theta - n + \alpha} \prod_{j=1}^n \|h_j\|_H [f]_{C_H^\alpha(X)} \quad (3.31)$$

with

$$C_2 = \frac{2K_{n,\alpha}}{1 - (n - \alpha)\theta} + \frac{K_{n+1,\alpha}}{(n + 1 - \alpha)\theta - 1}.$$

Therefore, $D_H^n u \in C_H^{1/\theta-n+\alpha}(X; \mathcal{L}^n(H))$ and $[D_H^n u]_{C_H^{1/\theta-n+\alpha}(X; \mathcal{L}^n(H))} \leq C_2 [f]_{C_H^\alpha(X)}$. This estimate and (3.21) give (3.23) in the case $n \geq 1$. \square

If $1/\theta = k \in \mathbb{N}$ we do not expect that $u \in C_H^{1/\theta}(X)$ whenever $f \in C_b(X)$. The simplest counterexample is $X = H = \mathbb{R}^N$ with $N > 1$, $L = \Delta$. In this case (3.2) is satisfied with $\theta = 1/2$ (see Sect. 4.1) and it is well known that the equation $\lambda u - \Delta u = f$ has not solutions in $C_b^2(\mathbb{R}^N)$ (and even not in $C_b^1(\mathbb{R}^N)$ with Lipschitz gradient) for every $f \in C_b(\mathbb{R}^N)$. The best regularity result in this scale of spaces is in Zygmund spaces.

Theorem 3.9. *Let $\lambda > 0$, $f \in C_b(X)$ and let $u = R(\lambda, L)f$. Then*

(i) *If $1/\theta = k \in \mathbb{N}$, $u \in Z_H^k(X)$, and there exists $C = C(\lambda) > 0$, independent of f , such that*

$$\|u\|_{Z_H^k(X)} \leq C \|f\|_\infty. \quad (3.32)$$

(ii) If $\alpha \in (0, 1)$ and $\alpha + 1/\theta = k \in \mathbb{N}$, for every $f \in C_H^\alpha(X)$ the function u belongs to $Z_H^k(X)$, and there exists $C = C(\lambda, \alpha) > 0$, independent of f , such that

$$\|u\|_{Z_H^k(X)} \leq C \|f\|_{C_H^\alpha(X)}. \quad (3.33)$$

Proof. We proceed as in the proof of Theorem 3.8, with due modifications. So, it is enough to prove that the statement holds if $\lambda > \omega$. The case where $\omega > 0$ and $\lambda \in (0, \omega]$ will follow as in Step 2 of Proposition 3.7.

First we prove statements (i) and (ii) in the case $k = 1$.

We already know that $u \in C_b(X)$, with $\|u\|_\infty \leq \|f\|_\infty/\lambda$. To show that $u \in Z_H^1(X)$, for every fixed $h \in H$ we consider again the functions a_h and b_h defined in (3.24), such that $u = a_h + b_h$.

Let us prove statement (i), in the case $\theta = k = 1$. For every $x \in X$ we have

$$\begin{aligned} |a_h(x+2h) - 2a_h(x+h) + a_h(x)| &\leq \int_0^{\|h\|_H} e^{-\lambda t} |P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| dt \\ &\leq \int_0^{\|h\|_H} e^{-\lambda t} |P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| dt \leq 4 \int_0^{\|h\|_H} e^{-\lambda t} \|f\|_\infty dt = 4\|h\|_H \|f\|_\infty. \end{aligned}$$

To estimate $b_h(x+2h) - 2b_h(x+h) + b_h(x)$ we use (2.1) twice, that gives

$$\begin{aligned} P_t f(x+2h) - 2P_t f(x+h) + P_t f(x) &= \int_0^1 D_H P_t f(x + (1+\sigma)h)(h) d\sigma - \int_0^1 D_H P_t f(x + \sigma h)(h) d\sigma \\ &= \int_0^1 \int_0^1 D_H^2 P_t f(x + (\tau + \sigma)h)(h, h) d\tau d\sigma \end{aligned}$$

so that, by (3.5) with $n = 2$,

$$|P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| \leq \sup_{y \in X} \|D_H^2 P_t f(y)\|_{\mathcal{L}^2(H)} \|h\|_H^2 \leq K_2 \frac{e^{\omega t}}{t^2} \|f\|_\infty \|h\|_H^2. \quad (3.34)$$

Therefore,

$$\begin{aligned} |b_h(x+2h) - 2b_h(x+h) + b_h(x)| &\leq \int_{\|h\|_H}^\infty e^{-\lambda t} |P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| dt \\ &\leq \int_{\|h\|_H}^\infty e^{-\lambda t} K_2 \frac{e^{\omega t}}{t^2} \|f\|_\infty dt \leq K_2 \|h\|_H \|f\|_\infty. \end{aligned}$$

Summing up,

$$|u(x+2h) - 2u(x+h) + u(x)| \leq (4 + K_2) \|h\|_H \|f\|_\infty,$$

so that $u \in Z_H^1(X)$ and (3.32) holds with $C = 1/\lambda + 4 + K_2$. So, statement (i) is proved for $\theta = 1$. Concerning statement (ii), when $\alpha + 1/\theta = 1$ and $f \in C_H^\alpha(X)$ we have by (3.11)(i)

$$\begin{aligned} &|a_h(x+2h) - 2a_h(x+h) + a_h(x)| \\ &\leq \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} (|P_t f(x+2h) - P_t f(x+h)| + |P_t f(x+h) - P_t f(x)|) dt \\ &\leq 2 \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} M^\alpha e^{\alpha \omega t} [f]_{C_H^\alpha(X)} \|h\|_H^\alpha dt \leq 2M^\alpha \|h\|_H [f]_{C_H^\alpha(X)} \end{aligned}$$

while (3.34) has to be replaced (using (3.14) with $n = 2$) by

$$|P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| \leq \sup_{y \in X} \|D_H^2 P_t f(y)\|_{\mathcal{L}^2(H)} \|h\|_H^2 \leq \frac{K_{2,\alpha} e^{\omega t}}{t^{(2-\alpha)\theta}} [f]_{C_H^\alpha(X)} \|h\|_H^2, \quad (3.35)$$

and therefore, recalling that $(2 - \alpha)\theta = 1 + \theta$,

$$\begin{aligned} |b_h(x+2h) - 2b_h(x+h) + b_h(x)| &\leq \int_{\|h\|_H^{1/\theta}}^{\infty} e^{-\lambda t} |P_t f(x+2h) - 2P_t f(x+h) + P_t f(x)| dt \\ &\leq \int_{\|h\|_H^{1/\theta}}^{\infty} e^{-\lambda t} K_{2,\alpha} \frac{e^{\omega t}}{t^{(2-\alpha)\theta}} [f]_{C_H^\alpha(X)} \|h\|_H^2 dt \leq \frac{K_{2,\alpha}}{\theta} \|h\|_H [f]_{C_H^\alpha(X)}. \end{aligned}$$

Summing up,

$$|u(x+2h) - 2u(x+h) + u(x)| \leq \left(2M^\alpha + \frac{K_{2,\alpha}}{\theta} \right) \|h\|_H [f]_{C_H^\alpha(X)},$$

so that $u \in Z_H^1(X)$ and (3.33) holds with $C = 1/\lambda + 2M^\alpha + K_{2,\alpha}/\theta$. So, statement (ii) is proved for $\alpha + 1/\theta = 1$.

In the case that $k > 1$ (we recall that $k = 1/\theta$ in statement (i), $k = \alpha + 1/\theta$ in statement (ii)), we know from Proposition 3.7 that $u \in C_H^{k-1}(X)$ and that estimates (3.16), (3.17) hold with $n = k - 1$. What we have to prove is that $D_H^{k-1}u \in Z^1(X, \mathcal{L}^{k-1}(H))$, and to estimate its Z^1 norm in terms of f . To this aim, fixed any $h, h_1, \dots, h_{k-1} \in H$, for every $y \in X$ we split $D_H^{k-1}u(y)(h_1, \dots, h_{k-1})$ as $a_h(y) + b_h(y)$, where now

$$\begin{aligned} a_h(y) &:= \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} D_H^{k-1} P_t f(y)(h_1, \dots, h_k) dt, \quad y \in X, \\ b_h(y) &:= \int_{\|h\|_H^{1/\theta}}^{\infty} e^{-\lambda t} D_H^{k-1} P_t f(y)(h_1, \dots, h_k) dt, \quad y \in X. \end{aligned} \quad (3.36)$$

So, for every $x \in X$ we have

$$\begin{aligned} &|(D_H^{k-1}u(x+2h) - 2D_H^{k-1}u(x+h) + D_H^{k-1}u(x))(h_1, \dots, h_{k-1})| \leq \\ &= |a_h(x+2h) - 2a_h(x+h) + a_h(x)| + |b_h(x+2h) - 2b_h(x+h) + b_h(x)|. \end{aligned} \quad (3.37)$$

By the definition of a_h we get

$$\begin{aligned} &|a_h(x+2h) - 2a_h(x+h) + a_h(x)| \\ &\leq \int_0^{\|h\|_H^{1/\theta}} e^{-\lambda t} |(D_H^{k-1} P_t f(x+2h) - 2D_H^{k-1} P_t f(x+h) + D_H^{k-1} P_t f(x))(h_1, \dots, h_{k-1})| dt, \end{aligned} \quad (3.38)$$

To estimate the right-hand side we observe that

$$\begin{aligned} &|(D_H^{k-1} P_t f(x+2h) - 2D_H^{k-1} P_t f(x+h) + D_H^{k-1} P_t f(x))(h_1, \dots, h_{k-1})| \\ &\leq 4 \sup_{y \in X} \|D_H^{k-1} P_t f(y)\|_{\mathcal{L}^{k-1}(H)} \prod_{j=1}^{k-1} \|h_j\|_H, \end{aligned}$$

which is bounded by $4K_{k-1} e^{\omega t} t^{-(k-1)\theta} \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty$ thanks to (3.5), and by $4K_{k-1,\alpha} e^{\omega t} t^{-(k-1-\alpha)\theta} \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)}$ thanks to (3.14) if $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$. Therefore, the

right-hand side of (3.38) is bounded by

$$\int_0^{\|h\|_H^{1/\theta}} \frac{4K_{k-1}}{t^{(k-1)\theta}} dt \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty = 4kK_{k-1} \|h\| \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty$$

if $k = 1/\theta$, and by

$$\int_0^{\|h\|_H^{1/\theta}} \frac{4K_{k-1,\alpha}}{t^{(k-1-\alpha)\theta}} dt \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)} = 4(k-\alpha)K_{k-1,\alpha} \|h\|_H \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)},$$

if $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$ and $k = \alpha + 1/\theta$. Moreover, by the definition of b_h we get

$$\begin{aligned} & |b_h(x+2h) - 2b_h(x+h) + b_h(x)| \\ & \leq \int_{\|h\|_H^{1/\theta}}^\infty e^{-\lambda t} |(D_H^{k-1} P_t f(x+2h) - 2D_H^{k-1} P_t f(x+h) + D_H^{k-1} P_t f(x))(h_1, \dots, h_{k-1})| dt. \end{aligned} \quad (3.39)$$

To estimate the right-hand side we recall that for every $t > 0$, $x \in X$, $h \in H$, by (2.5) we have

$$\begin{aligned} & |(D_H^{k-1} P_t f(x+2h) - 2D_H^{k-1} P_t f(x+h) + D_H^{k-1} P_t f(x))(h_1, \dots, h_{k-1})| \\ & \leq \sup_{y \in X} \|D_H^{k+1} P_t f(y)\|_{\mathcal{L}^2(H)} \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H \end{aligned}$$

which is respectively bounded by $K_{k+1} e^{\omega t} t^{-(k+1)\theta} \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty$ due to (3.5), and by $K_{k+1,\alpha} e^{\omega t} t^{-(k+1-\alpha)\theta} \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)}$ if $f \in C_H^\alpha(X)$, due to (3.14). Therefore, the right-hand side of (3.39) is bounded by

$$\int_{\|h\|_H^{1/\theta}}^\infty \frac{K_{k+1}}{t^{(k+1)\theta}} dt \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty = kK_{k+1} \|h\| \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty,$$

if $k = 1/\theta$, and by

$$\int_{\|h\|_H^{1/\theta}}^\infty \frac{K_{k+1,\alpha}}{t^{(k+1-\alpha)\theta}} dt \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)} = (k-\alpha)K_{k+1,\alpha} \|h\|_H \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)},$$

if $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$ and $k = \alpha + 1/\theta$. Summing up, the left-hand side of (3.37) is bounded by

$$k(4K_{k-1} + K_{k+1}) \prod_{j=1}^{k-1} \|h_j\|_H \|f\|_\infty \|h\|,$$

if $1/\theta = k$, and by

$$(k-\alpha)(4K_{k-1,\alpha} + K_{k+1,\alpha}) \prod_{j=1}^{k-1} \|h_j\|_H [f]_{C_H^\alpha(X)} \|h\|,$$

if $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$ and $\alpha + 1/\theta = k$. In both cases, this implies that $D^{k-1}u \in Z_H^1(X, \mathcal{L}^{k-1}(H))$ (so that $u \in Z_H^k(X)$) with Zygmund seminorm bounded by $k(4K_{k-1} + K_{k+1})\|f\|_\infty$ in the first case, and by $(k-\alpha)(4K_{k-1,\alpha} + K_{k+1,\alpha})[f]_{C_H^\alpha(X)}\|h\|$, in the second case. Such estimates and (3.16), (3.17) with $n = k-1$ yield (3.32) and (3.33), respectively. \square

3.3. Schauder and Zygmund estimates: evolution equations. This section deals with mild solutions to Cauchy problems,

$$\begin{cases} v_t(t, x) = Lv(t, x) + g(t, x), & t \in [0, T], x \in X, \\ v(0, \cdot) = f, \end{cases} \quad (3.40)$$

where L is the operator defined in (1.3), and $f \mapsto \mathbb{R}$, $g : [0, T] \times X \mapsto \mathbb{R}$ are bounded continuous functions. Mild solutions are defined by

$$v(t, x) = P_t f(x) + \int_0^t P_{t-s} g(s, \cdot)(x) ds, \quad t \in [0, T], x \in X. \quad (3.41)$$

We already know that $(t, x) \mapsto P_t f(x)$ is continuous and bounded in $[0, +\infty) \times X$; if in addition $f \in C_H^n(X)$ for some $n \in \mathbb{N}$ all the derivatives $\partial^k / \partial h_1 \dots \partial h_k (P_t f)$ with $k \leq n$ enjoy the same property, by Proposition 3.3. We still have to study the function

$$v_0(t, x) := \int_0^t P_{t-s} g(s, \cdot)(x) ds = \int_0^t P_s g(t-s, \cdot)(x) ds, \quad t \in [0, T], x \in X, \quad (3.42)$$

with $g \in C_b([0, T] \times X)$. Our final aim are maximal regularity results in Hölder and Zygmund spaces with respect to the x variable, so we introduce the relevant functional spaces.

Definition 3.10. Let $T > 0$, $\alpha > 0$. We denote by $C_H^{0,\alpha}([0, T] \times X)$ the space of the bounded continuous functions $g : [0, T] \times X \mapsto \mathbb{R}$ such that $g(t, \cdot) \in C_H^\alpha(X)$ for every $t \in [0, T]$ and

$$\|g\|_{C_H^{0,\alpha}([0,T] \times X)} := \sup_{t \in [0, T]} \|g(t, \cdot)\|_{C_H^\alpha(X)} < +\infty,$$

and moreover, if $\alpha > 1$, for every $(h_1, \dots, h_k) \in H^k$, with $k \leq [\alpha]$, the functions $(t, x) \mapsto (\partial^k g / \partial h_1 \dots \partial h_k)(t, x)$ are continuous in $[0, +\infty) \times X$.

For $k \in \mathbb{N}$ we denote by $Z_H^{0,k}([0, T] \times X)$ the space of the bounded continuous functions $g : [0, T] \times X \mapsto \mathbb{R}$ such that $g(t, \cdot) \in Z_H^k(X)$ for every $t \in [0, T]$ and

$$\|g\|_{Z_H^{0,k}([0,T] \times X)} := \sup_{t \in [0, T]} \|g(t, \cdot)\|_{Z_H^k(X)} < +\infty,$$

and, if $k \geq 2$, $g \in C_H^{0,k-1}([0, T] \times X)$.

If $H = X$ we drop the subindex H , setting $C_b^{0,\alpha}([0, T] \times X) := C_X^{0,\alpha}([0, T] \times X)$, $Z_b^{0,k}([0, T] \times X) := Z_H^{0,k}([0, T] \times X)$.

The next proposition is the evolution counterpart of Proposition 3.7.

Proposition 3.11. For every $g \in C_b([0, T] \times X)$ the function v_0 defined in (3.42) is continuous, and we have

$$\|v_0\|_\infty \leq T \|g\|_\infty. \quad (3.43)$$

Moreover the following statements hold.

- (i) Let $\theta < 1$. For every $n \in \mathbb{N}$ such that $n < 1/\theta$, $v_0 \in C_H^{0,n}([0, T] \times X)$. There exists $C = C(T) > 0$, independent of g , such that

$$\|v_0\|_{C_H^{0,n}([0,T] \times X)} \leq C \|g\|_\infty. \quad (3.44)$$

- (ii) Let $\alpha \in (0, 1)$ be such that $\alpha + 1/\theta > 1$. For every $f \in C_H^\alpha(X)$ and for every $n \in \mathbb{N}$ such that $n < \alpha + 1/\theta$, $v_0 \in C_H^{0,n}([0, T] \times X)$. There exists $C = C(T, \alpha) > 0$, independent of g , such that

$$\|v_0\|_{C_H^{0,n}([0,T] \times X)} \leq C \|g\|_{C_H^{0,\alpha}([0,T] \times X)} \quad (3.45)$$

Proof. Fix $t, t_0 \in [0, T]$ and $x, x_0 \in X$. If $t > t_0$ we have

$$|v_0(t, x) - v_0(t_0, x_0)| \leq \int_0^{t_0} |P_{t-s}g(s, \cdot)(x) - P_{t_0-s}g(s, \cdot)(x_0)| ds + \int_{t_0}^t |P_{t-s}g(s, \cdot)(x)| ds.$$

Since for every $s \geq 0$ the function $(t, x) \mapsto P_{t-s}g(s, \cdot)(x)$ is continuous in $[s, +\infty) \times X$, and $|P_{t-s}g(s, \cdot)(x) - P_{t_0-s}g(s, \cdot)(x_0)| \leq 2\|g\|_\infty$, by the Dominated Convergence Theorem the first integral vanishes as $t \rightarrow t_0, x \rightarrow x_0$. The second integral is bounded by $(t - t_0)\|g\|_\infty$, so that it vanishes too as $t \rightarrow t_0, x \rightarrow x_0$. If $t < t_0$ we split $v_0(t, x) - v_0(t_0, x_0) = \int_0^t (P_{t-s}g(s, \cdot)(x) - P_{t_0-s}g(s, \cdot)(x_0)) ds + \int_t^{t_0} P_{t_0-s}g(s, \cdot)(x_0) ds$ and we argue in the same way. So, v_0 is continuous. Estimate (3.43) is immediate.

Concerning statements (i) and (ii), the proof of the fact that $v_0(t, \cdot) \in C_H^n(X)$ for every $t \in [0, T]$, and that

$$\frac{\partial^k v_0}{\partial h_1 \dots \partial h_k}(t, x) = \int_0^t D_H^k P_{t-s}g(s, \cdot)(x)(h_1, \dots, h_k) ds, \quad k \in \{1, \dots, n\}, t \in [0, T], x \in X,$$

is quite analogous to the corresponding proof of Proposition 3.7, and it is omitted. Estimates (3.44) and (3.45) follow as well as in the proof of Proposition 3.7.

It remains to prove that $(t, x) \mapsto D_H^k v_0(t, \cdot)(x)(h_1, \dots, h_k)$ is continuous in $[0, T] \times X$ for every $k \in \{1, \dots, n\}, h_1, \dots, h_k \in H$, and this is similar to the proof of the continuity of v_0 . For $t > t_0 \in [0, T]$ and $x, x_0 \in X$ we split $\frac{\partial^k v_0}{\partial h_1 \dots \partial h_k}(t, x) - \frac{\partial^k v_0}{\partial h_1 \dots \partial h_k}(t_0, x_0) = I_1 + I_2$, where

$$I_1 = \int_0^{t_0} (D_H^k P_{t-s}g(s, \cdot)(x) - D_H^k P_{t_0-s}g(s, \cdot)(x_0))(h_1, \dots, h_k) ds,$$

$$I_2 = \int_{t_0}^t D_H^k P_{t-s}g(s, \cdot)(x)(h_1, \dots, h_k) ds.$$

Concerning I_1 , by Proposition 3.3 for every $s \in [0, T]$ the function $(t, x) \mapsto D_H^k P_{t-s}g(s, \cdot)(x)(h_1, \dots, h_k)$ is continuous in $(s, +\infty) \times X$, moreover for $0 < s < t_0$ we have

$$|D_H^k P_{t-s}g(s, \cdot)(x) - D_H^k P_{t_0-s}g(s, \cdot)(x_0)|(h_1, \dots, h_k) \leq \varphi(s),$$

where $\varphi(s) = 2K_k \max\{e^{2\omega T}, 1\}(t_0 - s)^{k\theta}\|g\|_\infty \prod_{j=1}^k \|h_j\|_H$ if $g \in C_b([0, T] \times X)$ by (3.5), and $\varphi(s) = 2K_{k,\alpha} \max\{e^{2\omega T}, 1\}(t_0 - s)^{(k-\alpha)\theta}\|g\|_{C_H^{0,\alpha}([0, T] \times X)} \prod_{j=1}^k \|h_j\|_H$ if $g \in C_H^{0,\alpha}([0, T] \times X)$ by (3.14). Both in case of statement (i) and of statement (ii), $\varphi \in L^1(0, t_0)$ and the Dominated Convergence Theorem yields that I_1 vanishes as $t \rightarrow t_0, x \rightarrow x_0$.

Moreover we have $|I_2| \leq \int_{t_0}^t \psi(s) ds$, where $\psi(s) = K_k \max\{e^{2\omega T}, 1\}(t - s)^{k\theta} \|g\|_\infty \prod_{j=1}^k \|h_j\|_H$ if $g \in C_b([0, T] \times X)$ by (3.5), and $\psi(s) = K_{k,\alpha} \max\{e^{2\omega T}, 1\}(t - s)^{(k-\alpha)\theta} \|g\|_{C_H^{0,\alpha}([0, T] \times X)} \prod_{j=1}^k \|h_j\|_H$ if $g \in C_H^{0,\alpha}([0, T] \times X)$, by (3.14). So we get $|I_2| \leq C(t - t_0)^{1-k\theta}$ in the first case, $|I_2| \leq C(t - t_0)^{1-(k-\alpha)\theta}$ in the second case; in both cases I_2 vanishes as $t \rightarrow t_0, x \rightarrow x_0$.

If $t < t_0 \in [0, T]$ and $x, x_0 \in X$ we split $D_H^k v_0(t, \cdot)(x)(h_1, \dots, h_k) - D_H^k v_0(t_0, \cdot)(x_0)(h_1, \dots, h_k)$ as above, replacing $\int_0^t, \int_{t_0}^t$ by $\int_0^t, \int_t^{t_0}$, respectively, and arguing in the same way. This ends the proof. \square

Theorem 3.12. *Let $f \in C_b(X)$, $g \in C_b([0, T] \times X)$ and let v be defined by (3.41). The following statements hold.*

- (i) *If $1/\theta \notin \mathbb{N}$ and $f \in C_H^{1/\theta}(X)$, $g \in C_b([0, T] \times X)$, then $v \in C_H^{0,1/\theta}([0, T] \times X)$. There exists $C = C(T) > 0$, independent of f and g , such that*

$$\|v\|_{C_H^{0,1/\theta}([0, T] \times X)} \leq C(\|f\|_{C_H^{1/\theta}(X)} + \|g\|_\infty). \quad (3.46)$$

(ii) If $\alpha \in (0, 1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, $f \in C_H^{\alpha+1/\theta}(X)$ and $g \in C_H^{0,\alpha}([0, T] \times X)$, then $v \in C_H^{0,\alpha+1/\theta}([0, T] \times X)$. There exists $C = C(T, \alpha) > 0$, independent of f and g , such that

$$\|v\|_{C_H^{0,\alpha+1/\theta}([0, T] \times X)} \leq C(\|f\|_{C_H^{\alpha+1/\theta}(X)} + \|g\|_{C_H^{0,\alpha}([0, T] \times X)}). \quad (3.47)$$

Proof. Both for $\alpha = 0$ and for $\alpha \in (0, 1)$, the function $(t, x) \mapsto P_t f(x)$ belongs to $C_H^{0,\alpha+1/\theta}([0, T] \times X)$ provided $f \in C_H^{\alpha+1/\theta}(X)$, by Lemma 3.5. Therefore it is sufficient to prove that the statements hold in the case $f \equiv 0$, namely when $v = v_0$. Taking proposition 3.11 into account, it remains to be checked that for every $t \in [0, T]$, $v(t, \cdot) \in C_H^{1/\theta}(X)$ in case of statement (i), $v_0(t, \cdot) \in C_H^{\alpha+1/\theta}(X)$ in case of statement (ii), with Hölder norm bounded by a constant independent of t . The proof is quite similar to the proof of Theorem 3.8. Let $n \in \mathbb{N} \cup \{0\}$ be the integral part of $\alpha + 1/\theta$, with $\alpha = 0$ in the case of statement (i) and $\alpha \in (0, 1)$ in the case of statement (ii); we treat separately the cases $n > 0$ and $n = 0$.

Let $n = 0$. For every fixed h , we split $v = a_h + b_h$, where for every $t \in [0, T]$, $y \in X$

$$a_h(t, y) = \int_0^{\|h\|_H^{1/\theta} \wedge t} P_s g(t-s, y) ds, \quad b_h(t, y) = \int_{\|h\|_H^{1/\theta} \wedge t}^t P_s g(t-s, y) ds. \quad (3.48)$$

So, for every $x \in X$ and $t \in [0, T]$ we have

$$\begin{aligned} |a_h(t, x+h) - a_h(t, x)| &\leq \int_0^{\|h\|_H^{1/\theta}} |P_s g(t-s, x+h) - P_s g(t-s, x)| ds \\ &\leq \int_0^{\|h\|_H^{1/\theta}} 2\|g\|_\infty dt = 2\|h\|_H^{1/\theta} \|g\|_\infty. \end{aligned}$$

If $\|h\|_H^{1/\theta} \geq t$, we have $b_h(t, x+h) - b_h(t, x) = 0$. If $\|h\|_H^{1/\theta} < t$ to estimate $|b_h(t, x+h) - b_h(t, x)|$ we use (3.25), which yields

$$\begin{aligned} |b_h(t, x+h) - b_h(t, x)| &\leq \int_{\|h\|_H^{1/\theta}}^t |P_s g(t-s, x+h) - P_s g(t-s, x)| dt \\ &\leq \int_{\|h\|_H^{1/\theta}}^\infty \frac{K_1}{s^\theta} dt \|h\|_H \|g\|_\infty \leq \frac{K_1}{\theta-1} \|h\|_H^{1/\theta} \|g\|_\infty. \end{aligned}$$

Summing up, $v(t, \cdot) \in C_H^{1/\theta}(X)$, and

$$[v_0(t, \cdot)]_{C_H^{1/\theta}(X)} \leq \left(2 + \frac{K_1}{\theta-1}\right) \|g\|_\infty, \quad 0 \leq t \leq T.$$

This estimate and (3.43) give (3.46) with $C(T) = 2 + K_1/(\theta-1) + T$, in the case that $\theta > 1$.

If $\alpha \in (0, 1)$, $\alpha + 1/\theta < 1$, and $g \in C_H^{0,\alpha}([0, T] \times X)$, we use (3.11)(i) and we get

$$\begin{aligned} |a_h(t, x+h) - a_h(t, x)| &\leq \int_0^{\|h\|_H^{1/\theta} \wedge t} |P_s g(t-s, x+h) - P_s g(t-s, x)| ds \\ &\leq \int_0^{\|h\|_H^{1/\theta} \wedge t} e^{\alpha\omega s} M^\alpha \|h\|_H^\alpha [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \\ &\leq \max\{e^{\alpha\omega T}, 1\} M^\alpha \|h\|_H^{\alpha+1/\theta} \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}. \end{aligned}$$

As before, if $\|h\|_H^{1/\theta} \geq t$ we have $b_h(t, x+h) - b_h(t, x) = 0$. If $\|h\|_H^{1/\theta} < t$, to estimate $|b_h(t, x+h) - b_h(t, x)|$ we use (3.26), that yields

$$\begin{aligned} |b_h(t, x+h) - b_h(t, x)| &\leq \int_{\|h\|_H^{1/\theta}}^t \frac{K_{1,\alpha} e^{\omega s}}{s^{(1-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \|h\|_H \\ &\leq \max\{e^{\alpha\omega T}, 1\} \frac{K_{1,\alpha}}{(1-\alpha)\theta - 1} \|h\|_H^{1/\theta + \alpha} \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}. \end{aligned}$$

Summing up, we obtain $v(t, \cdot) \in C_H^{\alpha+1/\theta}(X)$, and

$$[v_0(t, \cdot)]_{C_H^{\alpha+1/\theta}(X)} \leq \max\{e^{\alpha\omega T}, 1\} \left(M^\alpha + \frac{K_{1,\alpha}}{(1-\alpha)\theta - 1} \right) \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}, \quad 0 \leq t \leq T.$$

This estimate, together with (3.43), yields (3.23), with $C(T) = T + \max\{e^{\alpha\omega T}, 1\} (M^\alpha + K_{1,\alpha}/((1-\alpha)\theta - 1))$, in the case that $\alpha + 1/\theta < 1$.

Let us consider now the case $n > 0$. By Proposition 3.11 we already know that $v_0 \in C_H^{0,n}([0, T] \times X)$. It remains to prove that $D_H^n v(t, \cdot)$ is H -Hölder continuous with values in $\mathcal{L}^n(H)$, with exponent $1/\theta - n$ as far as statement (i) is concerned, and with exponent $\alpha + 1/\theta - n$ as far as statement (ii) is concerned. Once again, this is done as in Theorem 3.8, splitting every partial derivative $D_H^n v(t, y)(h_1, \dots, h_n) = a_h(t, y) + b_h(t, y)$, where now for every we set

$$a_h(t, y) := \int_0^{\|h\|_H^{1/\theta} \wedge t} D_H^n P_s g(t-s, \cdot)(y)(h_1, \dots, h_n) ds, \quad t \in [0, T], y \in X, \quad (3.49)$$

$$b_h(t, y) = \int_{\|h\|_H^{1/\theta} \wedge t}^t D_H^n P_s g(t-s, \cdot)(y)(h_1, \dots, h_n) ds, \quad t \in [0, T], y \in X. \quad (3.50)$$

Let us consider statement (i). We recall that in this case we have $g \in C_b([0, T] \times X)$, $n\theta \in (0, 1)$, $(n+1)\theta > 1$. Estimate (3.5) yields

$$\begin{aligned} |a_h(t, x+h) - a_h(t, x)| &\leq |a_h(t, x+h)| + |a_h(t, x)| \leq 2K_n \int_0^{\|h\|_H^{1/\theta} \wedge t} \frac{e^{\omega s}}{s^{n\theta}} ds \prod_{j=1}^n \|h_j\|_H \|g\|_\infty \\ &\leq \max\{e^{2\omega T}, 1\} \frac{2K_n}{1-n\theta} \|h\|_H^{(1-n\theta)/\theta} \prod_{j=1}^n \|h_j\|_H \|g\|_\infty. \end{aligned}$$

To estimate $|b_h(t, x+h) - b_h(t, x)|$ when $\|h\|_H^{1/\theta} < t$ we use (3.28), which yields

$$\begin{aligned} |b_h(t, x+h) - b_h(t, x)| &\leq \int_{\|h\|_H^{1/\theta}}^t \frac{K_{n+1} e^{\omega s}}{s^{(n+1)\theta}} ds \|h\|_H \prod_{j=1}^n \|h_j\|_H \|g\|_\infty \\ &\leq \max\{e^{2\omega T}, 1\} \frac{K_{n+1}}{(n+1)\theta - 1} \|h\|_H^{1-\theta} \prod_{j=1}^n \|h_j\|_H \|g\|_\infty. \end{aligned}$$

Summing up we get

$$|(D_H^n v_0(t, x+h) - D_H^n v_0(t, x))(h_1, \dots, h_n)| \leq C_3 \|h\|_H^{1/\theta - n} \prod_{j=1}^n \|h_j\|_H \|f\|_\infty, \quad 0 \leq t \leq T,$$

with

$$C_3 = \max\{e^{2\omega T}, 1\} \frac{2K_n}{1-n\theta} + \frac{K_{n+1}}{(n+1)\theta - 1}.$$

Therefore, $[D_H^n v_0(t, \cdot)]_{C_H^{1/\theta-n}(X; \mathcal{L}^n(H))} \leq C_3 \|f\|_\infty$ for every $t \in [0, T]$. This estimate and (3.43) give (3.46) for $n \geq 1$.

Let us consider statement (ii). Now we have $g \in C_H^{0,\alpha}([0, T] \times X)$ with $\alpha \in (0, 1)$, $(n-\alpha)\theta \in (0, 1)$, $(n+1-\alpha)\theta > 1$. Estimate (3.14) yields

$$\begin{aligned} |a_h(t, x+h) - a_h(t, x)| &\leq 2K_{n,\alpha} \int_0^{\|h\|_H^{1/\theta} \wedge t} \frac{e^{\omega s}}{s^{(n-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \prod_{j=1}^n \|h_j\|_H \\ &= \max\{e^{2\omega T}, 1\} \frac{2K_{n,\alpha}}{1-(n-\alpha)\theta} \|h\|_H^{(1-(n-\alpha)\theta)/\theta} \prod_{j=1}^n \|h_j\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}. \end{aligned}$$

To estimate $|b_h(t, x+h) - b_h(t, x)|$ for $\|h\|_H^{1/\theta} < t$ we use (3.30), which yields

$$\begin{aligned} |b_h(t, x+h) - b_h(t, x)| &\leq \int_{\|h\|_H^{1/\theta}}^t \frac{K_{n+1,\alpha} e^{\omega s}}{s^{(n+1-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \|h\|_H \prod_{j=1}^n \|h_j\|_H \\ &\leq \max\{e^{2\omega T}, 1\} \frac{K_{n+1,\alpha}}{(n+1-\alpha)\theta - 1} \|h\|_H^{1/\theta - n + \alpha} \prod_{j=1}^n \|h_j\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}. \end{aligned}$$

Summing up we get

$$|(D_H^n v_0(t, x+h) - D_H^n v_0(t, x))(h_1, \dots, h_n)| \leq C_4 \|h\|_H^{1/\theta - n + \alpha} \prod_{j=1}^n \|h_j\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}$$

with

$$C_4 = \max\{e^{2\omega T}, 1\} \frac{2K_{n,\alpha}}{1-(n-\alpha)\theta} + \frac{K_{n+1,\alpha}}{(n+1-\alpha)\theta - 1}.$$

Therefore, $[D_H^n v_0(t, \cdot)]_{C_H^{1/\theta-n+\alpha}(X; \mathcal{L}^n(H))} \leq C_4 [f]_{C_H^\alpha(X)}$ for every $t \in [0, T]$. This estimate and (3.43) give (3.47) in the case $n \geq 1$. \square

Theorem 3.13. *Let $f \in C_b(X)$, $g \in C_b([0, T] \times X)$ and let v be defined by (3.41). The following statements hold.*

- (i) *If $1/\theta = k \in \mathbb{N}$ and $f \in Z_H^k(X)$, then $v \in Z_H^{0,k}([0, T] \times X)$ and there exists $C = C(T) > 0$, independent of f and g , such that*

$$\|v\|_{Z_H^{0,k}([0,T] \times X)} \leq C(\|f\|_{Z_H^k(X)} + \|g\|_\infty). \quad (3.51)$$

- (ii) *If $\alpha \in (0, 1)$, $\alpha + 1/\theta = k \in \mathbb{N}$, $f \in Z_H^k(X)$ and $g \in C_H^{0,\alpha}([0, T] \times X)$, then $v \in Z_H^{0,k}([0, T] \times X)$, and there exists $C = C(T, \alpha) > 0$, independent of f and g , such that*

$$\|v\|_{Z_H^{0,k}([0,T] \times X)} \leq C(\|f\|_{Z_H^k(X)} + \|g\|_{C_H^{0,\alpha}([0,T] \times X)}). \quad (3.52)$$

Proof. We know by Lemma 3.5 that for every $f \in Z_H^k(X)$ the function $(t, x) \mapsto P_t f(x)$ belongs to $Z_H^{0,k}([0, T] \times X)$, and estimate (3.12) holds. So it is enough to prove that the statements hold for $f \equiv 0$, in which case $v = v_0$ defined by (3.42).

First we prove statements (i) and (ii) in the case $k = 1$.

By Proposition 3.11 we already know that $v_0 \in C_b([0, T] \times X)$, with $\|v_0\|_\infty \leq T \|g\|_\infty$. To show that $v_0(t, \cdot) \in Z_H^1(X)$ for every $t \in [0, T]$, for every fixed $h \in H$ we consider again the functions a_h and b_h defined in (3.48), such that $v_0 = a_h + b_h$.

Let us prove statement (i), in the case $\theta = k = 1$. For every $x \in X$ we have

$$\begin{aligned} & |a_h(t, x + 2h) - 2a_h(t, x + h) + a_h(t, x)| \\ & \leq \int_0^{\|h\|_H \wedge t} |P_s g(t-s, \cdot)(x + 2h) - 2P_s g(t-s, \cdot)(x + h) + P_s g(t-s, \cdot)(x)| dt \\ & \leq 4 \int_0^{\|h\|_H} \|g\|_\infty dt = 4\|h\|_H \|g\|_\infty. \end{aligned}$$

We recall that $b_h(t, x + 2h) - 2b_h(t, x + h) + b_h(t, x) = 0$ if $\|h\|_H \geq t$. To estimate $b_h(t, x + 2h) - 2b_h(t, x + h) + b_h(t, x)$ if $\|h\|_H < t$ we use (3.34), that yields

$$\begin{aligned} & |b_h(t, x + 2h) - 2b_h(t, x + h) + b_h(t, x)| \\ & \leq \int_{\|h\|_H}^t |P_s g(t-s, \cdot)(x + 2h) - 2P_s g(t-s, \cdot)(x + h) + P_s g(t-s, \cdot)(x)| ds \\ & \leq \int_{\|h\|_H}^t K_2 \frac{e^{\omega s}}{s^2} \|g\|_\infty \leq \max\{e^{2\omega T}, 1\} K_2 \|h\|_H \|g\|_\infty. \end{aligned}$$

Summing up,

$$|v_0(t, x + 2h) - 2v_0(t, x + h) + v_0(t, x)| \leq (4 + \max\{e^{2\omega T}, 1\} K_2) \|h\|_H \|g\|_\infty,$$

so that $v_0 \in Z_H^1(X)$ and (3.51) holds with $C = T + 4 + \max\{e^{2\omega T}, 1\} K_2$. So, statement (i) is proved for $\theta = 1$. Concerning statement (ii), when $\alpha + 1/\theta = 1$ and $g \in C_H^{0,\alpha}([0, T] \times X)$ we have by (3.11)(i)

$$\begin{aligned} & |a_h(t, x + 2h) - 2a_h(t, x + h) + a_h(t, x)| \\ & \leq \int_0^{\|h\|_H^{1/\theta} \wedge t} |P_s g(t-s, \cdot)(x + 2h) - 2P_s g(t-s, \cdot)(x + h) + P_s g(t-s, \cdot)(x)| ds \\ & \leq 2 \int_0^{\|h\|_H^{1/\theta} \wedge t} M^\alpha e^{\alpha \omega s} [g(t-s, \cdot)]_{C_H^\alpha(X)} \|h\|_H^\alpha dt \leq 2 \max\{e^{\alpha \omega T}, 1\} M^\alpha \|h\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)} \end{aligned}$$

while to estimate $|b_h(t, x + 2h) - 2b_h(t, x + h) + b_h(t, x)|$ for $\|h\|_H^{1/\theta} < t$ we use (3.35), that gives (recalling that $(2 - \alpha)\theta = 1 + \theta$),

$$\begin{aligned} & |b_h(t, x + 2h) - 2b_h(t, x + h) + b_h(t, x)| \\ & \leq \int_{\|h\|_H^{1/\theta}}^t |P_s g(t-s, \cdot)(x + 2h) - 2P_s g(t-s, \cdot)(x + h) + P_s g(t-s, \cdot)(x)| ds \\ & \leq \int_{\|h\|_H^{1/\theta}}^t K_{2,\alpha} \frac{e^{\omega s}}{s^{(2-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} \|h\|_H^2 ds \leq \max\{e^{2\omega T}, 1\} \frac{K_{2,\alpha}}{\theta} \|h\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}. \end{aligned}$$

Summing up,

$$\begin{aligned} & |v_0(t, x + 2h) - 2v_0(t, x + h) + v_0(t, x)| \leq \\ & \leq \left(2 \max\{e^{\omega T}, 1\} M^\alpha + \max\{e^{2\omega T}, 1\} \frac{K_{2,\alpha}}{\theta} \right) \|h\|_H \sup_{0 \leq r \leq T} [g(r, \cdot)]_{C_H^\alpha(X)}, \end{aligned}$$

so that $u \in Z_H^1(X)$ and (3.33) follows. So, statement (ii) is proved for $\alpha + 1/\theta = 1$.

In the case that $k > 1$ (we recall that $k = 1/\theta$ in statement (i), $k = \alpha + 1/\theta$ in statement (ii)), Proposition 3.11 yields $v_0 \in C_H^{0,k-1}([0, T] \times X)$. We have to prove that $[D_H^{k-1}v_0(t, \cdot)]_{Z^1(X, \mathcal{L}^{k-1}(H))}$ is bounded by a constant independent of t . To this aim, fixed any $h, h_1, \dots, h_{k-1} \in H$, for every $t \in [0, T]$ and $y \in X$ we split $D_H^{k-1}v_0(t, y)(h_1, \dots, h_{k-1})$ as $a_h(t, y) + b_h(t, y)$, where now

$$a_h(t, y) = \int_0^{\|h\|_H^{1/\theta} \wedge t} D_H^{k-1}P_s g(t-s, \cdot)(y)(h_1, \dots, h_k) ds$$

$$b_h(t, y) = \int_{\|h\|_H^{1/\theta} \wedge t}^t D_H^{k-1}P_s g(t-s, \cdot)(y)(h_1, \dots, h_k) ds.$$

We have

$$\begin{aligned} & |a_h(t, x+2h) - 2a_h(t, x+h) + a_h(t, x)| \\ & \leq \int_0^{\|h\|_H^{1/\theta} \wedge t} |(D_H^{k-1}P_s g(t-s, \cdot)(x+2h) - 2D_H^{k-1}P_s g(t-s, \cdot)(x+h) \\ & \quad + D_H^{k-1}P_s g(t-s, \cdot)(x))(h_1, \dots, h_{k-1})| dt, \end{aligned}$$

and arguing as in the proof of Theorem 3.9, we see that the right-hand side is bounded by

$$\int_0^{\|h\|_H^{1/\theta}} e^{\omega s} \frac{4K_{k-1}}{s^{(k-1)\theta}} ds \prod_{j=1}^{k-1} \|h_j\|_H \|g\|_\infty \leq \max\{e^{2\omega T}, 1\} 4kK_{k-1} \|h\| \prod_{j=1}^{k-1} \|h_j\|_H \|g\|_\infty$$

if $k = 1/\theta$, and by

$$\begin{aligned} & \int_0^{\|h\|_H^{1/\theta}} e^{\omega s} \frac{4K_{k-1, \alpha}}{s^{(k-1-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \prod_{j=1}^{k-1} \|h_j\|_H \\ & \leq \max\{e^{2\omega T}, 1\} 4(k-\alpha)K_{k-1, \alpha} \|h\|_H \prod_{j=1}^{k-1} \|h_j\|_H \sup_{0 \leq r \leq 1} [g(r, \cdot)]_{C_H^\alpha(X)}, \end{aligned}$$

if $g \in C_H^{0, \alpha}([0, T] \times X)$ with $\alpha \in (0, 1)$ and $k = \alpha + 1/\theta$. If $\|h\|_H^{1/\theta} < t$, we estimate

$$\begin{aligned} & |b_h(t, x+2h) - 2b_h(t, x+h) + b_h(t, x)| \\ & \leq \int_{\|h\|_H^{1/\theta}}^t e^{-\lambda t} |(D_H^{k-1}P_t f(x+2h) - 2D_H^{k-1}P_t f(x+h) + D_H^{k-1}P_t f(x))(h_1, \dots, h_{k-1})| dt \end{aligned}$$

and arguing again as in the proof of Theorem 3.9 we see that the right-hand side is bounded by

$$\int_{\|h\|_H^{1/\theta}}^t e^{\omega s} \frac{K_{k+1}}{s^{(k+1)\theta}} ds \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H \|g\|_\infty \leq \max\{e^{2\omega T}, 1\} kK_{k+1} \|h\| \prod_{j=1}^{k-1} \|h_j\|_H \|g\|_\infty,$$

if $k = 1/\theta$, and by

$$\begin{aligned} & \int_{\|h\|_H^{1/\theta}}^t e^{\omega s} \frac{K_{k+1,\alpha}}{s^{(k+1-\alpha)\theta}} [g(t-s, \cdot)]_{C_H^\alpha(X)} ds \|h\|_H^2 \prod_{j=1}^{k-1} \|h_j\|_H \\ & \leq \max\{e^{2\omega T}, 1\} (k-\alpha) K_{k+1,\alpha} \|h\|_H \prod_{j=1}^{k-1} \|h_j\|_H \sup_{0 \leq r \leq 1} [g(r, \cdot)]_{C_H^\alpha(X)}, \end{aligned}$$

if $g \in C_H^{0,\alpha}([0, T] \times X)$ with $\alpha \in (0, 1)$ and $k = \alpha + 1/\theta$. Summing up, we estimate $[D_H^{k-1} v_0(t, \cdot)(x + 2h) - 2D_H^{k-1} v_0(t, \cdot)(x+h) + D_H^{k-1} v_0(t, \cdot)(x)](h_1, \dots, h_{k-1})$ by

$$\max\{e^{2\omega T}, 1\} k (4K_{k-1} + K_{k+1}) \prod_{j=1}^{k-1} \|h_j\|_H \|g\|_\infty \|h\|,$$

if $1/\theta = k$, and by

$$\max\{e^{2\omega T}, 1\} (k-\alpha) (4K_{k-1,\alpha} + K_{k+1,\alpha}) \prod_{j=1}^{k-1} \|h_j\|_H \sup_{0 \leq r \leq 1} [g(r, \cdot)]_{C_H^\alpha(X)} \|h\|,$$

if $g \in C_H^{0,\alpha}([0, T] \times X)$ with $\alpha \in (0, 1)$ and $\alpha + 1/\theta = k$. This implies that $v_0(t, \cdot) \in Z^k(X)$ with Zygmund seminorm bounded by $\max\{e^{2\omega T}, 1\} k (4K_{k-1} + K_{k+1}) \|f\|_\infty$ in the first case, and by $\max\{e^{2\omega T}, 1\} (k-\alpha) (4K_{k-1,\alpha} + K_{k+1,\alpha}) [f]_{C_H^\alpha(X)} \|h\|$, in the second case. Such estimates and (3.12) yield (3.51) and (3.52), respectively. \square

4. EXAMPLES IN FINITE DIMENSION

In this section $X = \mathbb{R}^N$ and $T_t = e^{tB}$ for every t , where B is any $N \times N$ matrix, so that

$$P_t f(x) = \int_{\mathbb{R}^N} f(e^{tB} x + y) \mu_t(dy), \quad t > 0, f \in C_b(\mathbb{R}^N), x \in \mathbb{R}^N. \quad (4.1)$$

The measures μ_t are given by

$$\mu_t(dy) = g_t(y) dy, \quad t > 0, \quad (4.2)$$

where the nonnegative functions $g_t \in L^1(\mathbb{R}^N)$ satisfy $g_{t+s}(x) = \int_{\mathbb{R}^N} g_s(x - e^{sB} y) g_t(y) dy$ for $t, s > 0$, a.e. $x \in \mathbb{R}^N$, and $\|g_t\|_{L^1(\mathbb{R}^N)} = 1$, for every $t > 0$. If $B = 0$ this condition is simply $g_{t+s} = g_t \star g_s$ for $s, t > 0$.

Hypotheses 3.1 and 3.2 are satisfied with $H = H_t = \mathbb{R}^N$ provided g_t is weakly differentiable in all directions and

$$\sup_{t>0} t^\theta \left\| \frac{\partial g_t}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)} < +\infty, \quad k = 1, \dots, n. \quad (4.3)$$

4.1. The Laplacian and the fractional Laplacian. Strictly speaking, the results of this section are contained in the ones of both sections 4.2 and 5.3, but we prefer to isolate them because checking our assumptions is particularly simple in this case and does not involve the technicalities needed in the more complicated situations of the next sections.

We recall that the heat semigroup is given by (1.1), with $T_t = I$ for every t (namely, $B = 0$) and $\mu_t(dx) = g_t(x) dx$, where g_t is the Gaussian kernel

$$g_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^N, t > 0,$$

that satisfies (4.3) with $\theta = 1/2$. The operator L is the realization of the Laplacian in $C_b(\mathbb{R}^N)$, whose domain is $\{f \in C_b(\mathbb{R}^N) \cap_{p>1} W_{loc}^{2,p}(\mathbb{R}^N) : \Delta f \in C_b(\mathbb{R}^N)\}$. Schauder and Zygmund regularity results

have several independent proofs by now, the present approach was outlined in [42]. Concerning the fractional Laplacian $-(-\Delta)^s$, $s \in (0, 1)$, Schauder and Zygmund regularity results for stationary equations are already available. The first proof of the Schauder estimates seems to be in [53, Cor. 2.9]; for more general classes of pseudodifferential operators including the fractional Laplacian see [29, 36] and the references therein. However, a proof through our approach is very simple. Indeed, the associated semigroup is given by the classical subordination formula,

$$e^{-t(-\Delta)^s} f(x) = \int_0^\infty T_\sigma f(x) \eta_t^{(s)}(\sigma) d\sigma, \quad t > 0, x \in \mathbb{R}^N,$$

where $\eta_t^{(s)}$ is the inverse Laplace transform of $\lambda \mapsto e^{-t\lambda^s}$. Setting $\eta^{(s)} := \eta_1^{(s)}$, we get

$$\eta_t^{(s)}(\sigma) = t^{-1/s} \eta^{(s)}(t^{-1/s} \sigma), \quad t, \sigma > 0.$$

Moreover, $\eta^{(s)}$ is smooth in $(0, +\infty)$, it has positive values and it belongs to $L^\infty(0, +\infty) \cap W^{1,1}(0, +\infty)$. This is easily seen modifying the integral that defines $\eta^{(s)}$, to get (see e.g. [60])

$$\eta^{(s)}(\sigma) = \frac{1}{\pi} \int_0^\infty e^{-\sigma r - r^{-s} \cos(s\pi)} \sin(r^s \sin(s\pi)) dr, \quad \sigma > 0. \quad (4.4)$$

Therefore, $e^{-t(-\Delta)^s}$ takes the form (4.1), with $B = 0$ and

$$\mu_t(dy) = p_{s,t}(y) dy,$$

where

$$p_{s,t}(y) = \frac{1}{t^{1/s}} \int_0^\infty g_\xi(y) \eta^{(s)}(t^{-1/s} \xi) d\xi, \quad y \in \mathbb{R}^N, t > 0, \quad (4.5)$$

By homogeneity, we get

$$p_{s,t}(y) = t^{-N/2s} p_{s,1}(t^{-1/(2s)} y), \quad t > 0, y \in \mathbb{R}^N,$$

and such equality easily yields that $t \mapsto \mu_t$ is weakly continuous in $[0, +\infty)$. Moreover,

$$\frac{\partial}{\partial y_k} p_{s,t}(y) = t^{-2(N+1)/s} \frac{\partial}{\partial y_k} p_{s,1}(t^{-2/s} y), \quad t > 0, y \in \mathbb{R}^N,$$

which implies

$$\int_{\mathbb{R}^N} \left| \frac{\partial}{\partial y_k} p_{s,t}(y) \right| dy = t^{-(N+1)/(2s)} \int_{\mathbb{R}^N} \left| \frac{\partial}{\partial y_k} p_{s,1}(t^{-1/2s} y) \right| dy = t^{-1/2s} \int_{\mathbb{R}^N} \left| \frac{\partial}{\partial z_k} p_{s,1}(z) \right| dz.$$

From the representation formula (4.5) we get

$$\begin{aligned} \int_{\mathbb{R}^N} \left| \frac{\partial}{\partial z_k} p_{s,1}(z) \right| dz &= \int_{\mathbb{R}^N} \left| \int_0^\infty \frac{z_k e^{-|z|^2/4\xi}}{2\xi(4\pi\xi)^{N/2}} \eta^{(s)}(\xi) d\xi \right| dz \\ &= \int_0^\infty \frac{\eta^{(s)}(\xi)}{2\xi(4\pi\xi)^{N/2}} \int_{\mathbb{R}^N} |z_k| e^{-|z|^2/4\xi} dz d\xi = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta^{(s)}(\xi)}{\sqrt{\xi}} d\xi. \end{aligned}$$

The last integral is finite, since $\eta^{(s)}$ is bounded and it belongs to $L^1(0, +\infty)$. Therefore, there is $C > 0$ such that

$$\left\| \frac{\partial}{\partial y_k} p_{s,t} \right\|_{L^1(\mathbb{R}^N)} \leq \frac{C}{t^{1/(2s)}}, \quad t > 0, k = 1, \dots, N,$$

so that Hypotheses 3.1 and 3.2 are satisfied with $X = H = \mathbb{R}^N$ and $\omega = 0$, $\theta = 1/(2s)$. Theorems 3.8 and 3.9 yield

Theorem 4.1. *Let $f \in C_b(\mathbb{R}^N)$ and $\lambda > 0$, $s \in (0, 1) \setminus \{1/2\}$. Then the equation*

$$\lambda u + (-\Delta)^s u = f \quad (4.6)$$

has a unique solution $u \in C_b^{2s}(\mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{2s}(\mathbb{R}^N)} \leq C \|f\|_\infty.$$

If $s = 1/2$, equation (4.6) has a unique solution in $Z^1(\mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{Z^1(\mathbb{R}^N)} \leq C \|f\|_\infty.$$

If in addition $f \in C_b^\alpha(\mathbb{R}^N)$ with $\alpha \in (0, 1)$ and $\alpha + 2s \notin \{1, 2\}$, then $u \in C_b^{\alpha+2s}(\mathbb{R}^N)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{\alpha+2s}(\mathbb{R}^N)} \leq C \|f\|_{C_b^\alpha(\mathbb{R}^N)}.$$

If $\alpha + 2s = k \in \{1, 2\}$, then $u \in Z_b^k(\mathbb{R}^N)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_b^k(\mathbb{R}^N)} \leq C \|f\|_{C_b^\alpha(\mathbb{R}^N)}.$$

Theorems 3.12 and 3.13 yield

Theorem 4.2. *Let $s \in (0, 1)$, $\alpha \in [0, 1)$ be such that $\alpha + 2s \notin \{1, 2\}$, and let $f \in C_b^{\alpha+2s}(\mathbb{R}^N)$, $g \in C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)$ ⁽¹⁾. The mild solution to*

$$\begin{cases} v_t(t, x) + (-\Delta)^s v(t, \cdot)(x) = g(t, x), & 0 \leq t \leq T, x \in \mathbb{R}^N, \\ v(0, x) = f(x), & x \in \mathbb{R}^N, \end{cases} \quad (4.7)$$

belongs to $C_b^{0,\alpha+2s}([0, T] \times \mathbb{R}^N)$, and there is $C > 0$, independent of f and g , such that

$$\|v\|_{C_b^{0,\alpha+2s}([0, T] \times \mathbb{R}^N)} \leq C (\|f\|_{C_b^{\alpha+2s}(\mathbb{R}^N)} + \|g\|_{C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)}).$$

Let $s \in (0, 1)$, $\alpha \in [0, 1)$ be such that $\alpha + 2s := k \in \{1, 2\}$. Then for every $f \in Z_b^k(\mathbb{R}^N)$, $g \in C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)$ the mild solution to (4.7) belongs to $Z_b^{0,k}([0, T] \times \mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|v\|_{Z_b^{0,k}(\mathbb{R}^N)} \leq C (\|f\|_{Z_b^k(\mathbb{R}^N)} + \|g\|_{C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)}).$$

In the non-fractional case $s = 1$ the first part of the theorem is known since many years ([35]). For $s \in (0, 1)$ it seems to be new.

4.2. Ornstein-Uhlenbeck operators with fractional diffusion. Ornstein-Uhlenbeck operators are expressed by

$$(Lu)(x) = \frac{1}{2}(\text{Tr}(QD^2u))(x) - \langle Bx, \nabla u(x) \rangle, \quad x \in \mathbb{R}^N,$$

where Q is a symmetric nonnegative definite matrix and B is any matrix. Under ellipticity or hypoellipticity conditions (respectively, $\det Q > 0$ or $\det \int_0^t e^{sB} Q e^{sB^*} ds > 0$ for every $t > 0$) we already have maximal Hölder and Zygmund regularity results, first proved in [22] in the elliptic case and then in [41] in the hypoelliptic case.

Here we consider modified Ornstein-Uhlenbeck operators which are the object of very recent studies (e.g., [33, 3, 17]), heuristically given by

$$(\mathcal{L}u)(x) = \frac{1}{2}(\text{Tr}^s(QD^2u))(x) - \langle Bx, \nabla u(x) \rangle, \quad x \in \mathbb{R}^N.$$

with $s \in (0, 1)$ and $Q > 0$. $\text{Tr}^s(QD^2)$ is the pseudo-differential operator with symbol $\langle Q\xi, \xi \rangle^s$.

⁽¹⁾For $\alpha = 0$ we mean $C_b^{0,0}([0, T] \times \mathbb{R}^N) = C_b([0, T] \times \mathbb{R}^N)$.

The realization of \mathcal{L} in $L^2(\mathbb{R}^N)$ has been studied in [3] even in the hypoelliptic case, using smoothing properties of the relevant semigroup, expressed through Fourier and inverse Fourier transform as

$$\widehat{P_t f} = e^{t\text{Tr}B} \exp\left(-\frac{1}{2} \int_0^t |Q^{1/2} e^{\tau B^*} \cdot|^{2s} d\tau\right) \widehat{f}(e^{tB^*} \cdot), \quad t > 0,$$

where $\widehat{\cdot}$ denotes the Fourier transform \mathcal{F} ,

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} f(x) dx.$$

Now we rewrite P_t in the form (4.1). Applying the inverse Fourier transform we get, for every $f \in L^2(\mathbb{R}^N)$,

$$P_t f = e^{t\text{Tr}B} \mathcal{F}^{-1} \left(\exp\left(-\frac{1}{2} \int_0^t |Q^{1/2} e^{\tau B^*} \cdot|^{2s} d\tau\right) \right) * \mathcal{F}^{-1}(\mathcal{F}f(e^{tB^*} \cdot))$$

where

$$\mathcal{F}^{-1}(g((e^{tB^*} \cdot))) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} g(e^{tB^*} \xi) e^{i\langle \xi, y \rangle} dy = e^{-t\text{Tr}B} (\mathcal{F}^{-1}g)(e^{-tB} y),$$

so that

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}^N} f(e^{-tB} x - e^{-tB} y) \mathcal{F}^{-1} \left(e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{\tau B^*} \cdot|^{2s} d\tau} \right) (y) dy \\ &= e^{t\text{Tr}B} \int_{\mathbb{R}^N} f(e^{-tB} x - z) \mathcal{F}^{-1} \left(e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{\tau B^*} \cdot|^{2s} d\tau} \right) (e^{tB} z) dz \\ &= \int_{\mathbb{R}^N} f(e^{-tB} x + z) g_t(z) dz, \end{aligned}$$

with

$$\begin{aligned} g_t(z) &= e^{t\text{Tr}B} \mathcal{F}^{-1} \left(e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{\tau B^*} \cdot|^{2s} d\tau} \right) (e^{tB} z) = \mathcal{F}^{-1} \left(e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{(\tau-t)B^*} \cdot|^{2s} d\tau} \right) (z) \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{-\sigma B^*} \xi|^{2s} d\sigma} e^{i\langle \xi, z \rangle} d\xi. \end{aligned}$$

Set

$$\varphi_t(\xi) = e^{-\frac{1}{2} \int_0^t |Q^{1/2} e^{-\sigma B^*} \xi|^{2s} d\sigma}, \quad \xi \in \mathbb{R}^N,$$

so that $g_t = \mathcal{F}^{-1}(\varphi_t)$.

Since $\hat{\mu}_t = \varphi_t$ and the function $(t, \xi) \mapsto \varphi_t(\xi)$ is continuous in $[0, +\infty) \times \mathbb{R}^N$, with $\varphi_0(\xi) = 1$ for every ξ , by the Lévy Theorem $t \mapsto \mu_t$ is weakly continuous, and it weakly converges to δ_0 as $t \rightarrow 0$. Therefore, P_t is well defined in $C_b(\mathbb{R}^N)$ and satisfies our assumptions with $\theta = 1/(2s)$ provided

- (i) for every $t > 0$, $g_t(x) \geq 0$, for a.e. $x \in \mathbb{R}^N$,
- (ii) for every $t > 0$, $g_t \in L^1(\mathbb{R}^N)$, $\|g_t\|_{L^1} = 1$,
- (iii) there exist $\partial g_t / \partial x_k \in L^1(\mathbb{R}^N)$, for each $k = 1, \dots, N$, and there are $C > 0$, $\omega \in \mathbb{R}$ such that $\|\partial g_t / \partial x_k\|_{L^1} \leq C t^{-1/(2s)} e^{\omega t}$ for every $t > 0$,

Properties (i) and (ii) are consequences of general results of Fourier analysis. First of all, since $\varphi_t(\xi) = \varphi_t(-\xi)$ for every ξ , $g_t(x) \in \mathbb{R}$ for every $x \in \mathbb{R}^N$. Moreover, since φ_t is continuous and $\varphi_t(0) = 1$, (i) implies (ii) (e.g., [54, Cor. 1.2.6]).

To prove that (i) holds it is enough to show that $g_t(x) \neq 0$ for every x . Indeed, since $\varphi_t \in L^1(\mathbb{R}^N)$, g_t is continuous and $g_t(0) > 0$. That g_t does not vanish at any z is a consequence of (a part of) the Wiener's Tauberian Theorem such as in [49, Thm. 9.5]: given $\varphi \in L^1(\mathbb{R}^N)$, its Fourier transform does not vanish at any point if and only if the subspace $V \subset L^1(\mathbb{R}^N)$ generated by the translated

functions $x \mapsto \varphi(x+a)$, $a \in \mathbb{R}^N$, is dense in $L^1(\mathbb{R}^N)$. In our case, the closure of V in $L^1(\mathbb{R}^N)$ contains all the convolutions $\varphi_t^\varepsilon \star f$, with $f \in L^1(\mathbb{R}^N)$ and $\varphi_t^\varepsilon(x) = \varepsilon^{-N} \varphi_t(x/\varepsilon)$, and it is not difficult to check that for every $f \in L^1(\mathbb{R}^N)$ we have $\lim_{\varepsilon \rightarrow 0} \varphi_t^\varepsilon \star f = f/A$, with $A = \int_{\mathbb{R}^N} \varphi(x) dx \neq 0$. Therefore, V is dense in $L^1(\mathbb{R}^N)$ and by [49, Thm. 9.5], $\mathcal{F}(\varphi)(x) \neq 0$ for every $x \in \mathbb{R}^N$. Since $\mathcal{F}^{-1}\varphi(x) = (2\pi)^{-N} \mathcal{F}(\varphi)(-x)$, $g_t(x) \neq 0$ for every $x \in \mathbb{R}^N$.

The proof of (iii) is in the next lemma.

Lemma 4.3. $g_t \in W^{1,1}(\mathbb{R}^N)$ for every $t > 0$, and we have

$$\sup_{0 < t \leq 1} t^{1/(2s)} \left\| \frac{\partial g_t}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)} < +\infty, \quad k = 1, \dots, N, \quad (4.8)$$

$$\sup_{t > 1} \left\| \frac{\partial g_t}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)} < +\infty, \quad k = 1, \dots, N. \quad (4.9)$$

Proof. The main step is to prove that $g_t \in W^{1,1}(\mathbb{R}^N)$ for every $t \in (0, 1]$ and that (4.8) holds. The remaining part of the statement will be a consequence, thanks to the algebraic relations among the functions g_t .

It is convenient to rewrite g_t as

$$g_t(x) = \frac{1}{t^{N/(2s)}} \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-\frac{1}{2t} \int_0^t |Q^{1/2} e^{-\sigma B^*} \eta|^{2s} d\sigma} e^{i(t^{-1/(2s)} \eta, x)} d\eta = \frac{1}{t^{N/2s}} \tilde{g}_t \left(\frac{x}{t^{1/2s}} \right), \quad (4.10)$$

where

$$\tilde{g}_t := \mathcal{F}^{-1} \left(e^{-\frac{1}{2t} \int_0^t |Q^{1/2} e^{-\sigma B^*} \cdot|^{2s} d\sigma} \right) = \mathcal{F}^{-1}(\tilde{\varphi}_t), \quad t > 0,$$

with $\tilde{\varphi}_t = (\varphi_t)^{1/t}$. Our aim is to show that \tilde{g}_t is C^1 , and that $\sup_{t \in (0, 1]} \|\partial \tilde{g}_t / \partial x_k\|_{L^1(\mathbb{R}^N)} < +\infty$. In this case, by (4.11) g_t is C^1 too, and $\partial g_t / \partial x_k(x) = t^{-(N+1)/(2s)} \partial \tilde{g}_t / \partial x_k(t^{-1/(2s)} x)$, which yields (4.8).

To prove that \tilde{g}_t is continuously differentiable and it has L^1 derivatives it is enough to show that $\xi \mapsto \xi_k \tilde{\varphi}_t(\xi)$ belongs to $L^1(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$ for every $k = 1, \dots, N$, with $m > N/2$. In this case $\partial \tilde{g}_t / \partial x_k = i \mathcal{F}^{-1} \psi_t$ with $\psi_t(\xi) := \xi_k \tilde{\varphi}_t(\xi)$ and

$$\begin{aligned} \int_{\mathbb{R}^N} \left| \frac{\partial \tilde{g}_t}{\partial x_k} \right| dx &= \int_{\mathbb{R}^N} |\mathcal{F}^{-1} \psi_t(x)| dx = \frac{1}{(2\pi)^N} |\mathcal{F} \psi_t(-x)| dx \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\mathcal{F} \psi_t(x)| (1 + |x|^2)^{m/2} \frac{1}{(1 + |x|^2)^{m/2}} dx \\ &\leq \frac{1}{(2\pi)^N} \|\mathcal{F} \psi_t(1 + |\cdot|^2)^{m/2}\|_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^m} dx \right)^{1/2} \\ &\leq C_N \|\psi_t\|_{H^m(\mathbb{R}^N)}. \end{aligned}$$

So, the rest of the proof of the differentiability of \tilde{g}_t for $t \in (0, 1]$ and of (4.8) is devoted to show that $\psi_t \in L^1(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$ with $m > N/2$, and with H^m norm bounded by a constant independent of t . As a first step, we observe that there exists $c > 0$ such that

$$\tilde{\varphi}_t(\xi) \leq e^{-c|\xi|^{2s}}, \quad 0 < t \leq 1, \quad \xi \in \mathbb{R}^N. \quad (4.11)$$

Indeed, let $M_B > 0$, $\omega_B \in \mathbb{R}$ be such that $\|e^{tB^*}\| \leq M_B e^{\omega_B t}$ for every $t > 0$. For every $\xi \in \mathbb{R}^N$ and $\sigma \in [0, 1]$ we have $\xi = e^{\sigma B^*} Q^{-1/2} Q^{1/2} e^{-\sigma B^*} \xi$, so that $\|\xi\| \leq M_B e^{\omega_B \sigma} \|Q^{-1/2}\| \|Q^{1/2} e^{-\sigma B^*} \xi\|$, and therefore $\|Q^{1/2} e^{-\sigma B^*} \xi\| \geq \|\xi\| / \kappa$, with $\kappa = \min_{\tau \in [0, 1]} M_B e^{\omega_B \tau} \|Q^{-1/2}\|$. Estimate (4.12) holds with

$c = 1/(2\kappa^{2s})$; it implies that $\psi_t \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, with L^1 and L^2 norms bounded by constants independent of t .

To estimate the derivatives of ψ_t we write it as $\psi_t(\xi) = \xi_k e^{f_t(\xi)}$, where

$$f_t(\xi) := -\frac{1}{2t} \int_0^t |Q^{1/2} e^{-\sigma B^*} \xi|^{2s} d\tau, \quad \xi \in \mathbb{R}^N.$$

The function $\theta(\sigma, \xi) := |Q^{1/2} e^{-\sigma B^*} \xi|^{2s}$ belongs to $C^\infty(\mathbb{R} \times (\mathbb{R}^N \setminus \{0\}))$, and therefore $f_t \in C^\infty(\mathbb{R}^N \setminus \{0\})$ for every $t > 0$, and for every multi-index α we have

$$D^\alpha f_t = -\frac{1}{2t} \int_0^t D^\alpha (|Q^{1/2} e^{-\sigma B^*} \cdot|^{2s}) d\tau.$$

Since for every $\sigma \in \mathbb{R}$ the function $\theta(\sigma, \cdot)$ is homogeneous with degree $2s$, its j -th order derivatives are homogeneous with degree $2s - j$; therefore for every multi-index α and $\xi \neq 0$ we have

$$|D_\xi^\alpha \theta(\sigma, \xi)| = \left| D_\xi^\alpha \theta \left(\sigma, \frac{\xi}{|\xi|} \right) \right| |\xi|^{2s-|\alpha|} \leq \max\{|D_\xi^\alpha \theta(\sigma, y)| : |y| = 1\} |\xi|^{2s-|\alpha|},$$

and consequently, for every $t \in (0, 1]$,

$$|D^\alpha f_t(\xi)| \leq \sup_{0 \leq \sigma \leq 1} |D_\xi^\alpha \theta(\sigma, \xi)| \leq |\xi|^{2s-|\alpha|} \max\{|D_\xi^\alpha \theta(\sigma, y)| : \sigma \in [0, 1], |y| = 1\} =: K_\alpha |\xi|^{2s-|\alpha|}.$$

For every multi-index α , $D^\alpha \tilde{\varphi}_t = D^\alpha e^{f_t}$ is a linear combination of functions such as $e^{f_t} D^{\alpha_1} f_t \dots D^{\alpha_j} f_t$, where $j \in \{1, \dots, |\alpha|\}$, $\alpha_1, \dots, \alpha_j \in \mathbb{N}$ and $\sum \alpha_j = |\alpha|$. By the above estimates,

$$e^{f_t(\xi)} D^{\alpha_1} f_t(\xi) \dots D^{\alpha_j} f_t(\xi) \leq K_{\alpha_1} |\xi|^{2s-\alpha_1} \dots K_{\alpha_j} |\xi|^{2s-\alpha_j} \tilde{\varphi}_t(\xi)$$

and therefore

$$|D^\alpha \tilde{\varphi}_t(\xi)| \leq \sum_{j=1}^{|\alpha|} c_j |\xi|^{2sj-|\alpha|} e^{-c|\xi|^{2s}}, \quad \xi \neq 0,$$

with suitable coefficients c_j . It follows that for every $\varepsilon \in (0, c)$ there exists $c_{\varepsilon, |\alpha|} > 0$ such that

$$|D^\alpha \tilde{\varphi}_t(\xi)| \leq c_{\varepsilon, |\alpha|} e^{-\varepsilon|\xi|^{2s}} |\xi|^{2s-|\alpha|}, \quad \xi \neq 0. \quad (4.12)$$

Now, $D^\alpha \psi_t(\xi)$ is equal to $\xi_k D^\alpha \tilde{\varphi}_t(\xi)$ plus a linear combination of derivatives of $\tilde{\varphi}_t$ of order $|\alpha| - 1$. Therefore,

$$|D^\alpha \psi_t(\xi)| \leq c_{\varepsilon, |\alpha|} e^{-\varepsilon|\xi|^{2s}} |\xi|^{2s-|\alpha|+1} + C e^{-\varepsilon|\xi|^{2s}} |\xi|^{2s-|\alpha|} |\xi| = \tilde{C}_{\varepsilon, |\alpha|} e^{-\varepsilon|\xi|^{2s}} |\xi|^{2s-|\alpha|+1}.$$

Consequently, $D^\alpha \psi_t \in L^2(\mathbb{R}^N)$ provided $\xi \mapsto e^{-\varepsilon|\xi|^{2s}} |\xi|^{2s-|\alpha|+1} \in L^2(\mathbb{R}^N)$, which is satisfied if $2(2s-|\alpha|+1) > -N$. It follows that $\psi_t \in H^m(\mathbb{R}^n)$ if $2(2s-m+1) > -N$, namely $m < 2s+1+N/2$. We recall that we need $m > N/2$. Since $2s+1 > 1$, the interval $(N/2, 2s+1+N/2)$ contains at least one integer m . For such m , $\psi_t \in H^m(\mathbb{R}^n)$ and $\|\psi_t\|_{H^m(\mathbb{R}^n)}$ is bounded by a constant independent of t . Recalling that also $\|\psi_t\|_{L^1(\mathbb{R}^N)}$ is bounded by a constant independent of t , (4.8) follows.

To prove (4.9) we argue as in the proof of estimates (3.6), (3.11) for large t . We use the semigroup property $P_t \circ P_s = P_{t+s}$ for $t, s > 0$, which may be rewritten as

$$g_{t+s}(x) = \int_{\mathbb{R}^N} g_s(x - e^{-sB} y) g_t(y) dy, \quad t, s > 0, x \in \mathbb{R}^N.$$

In particular, for $t > 1$ we get

$$g_t(x) = \int_{\mathbb{R}^N} g_1(x - e^{-B} y) g_{t-1}(y) dy, \quad x \in \mathbb{R}^N,$$

so that g_t is continuously differentiable and for every $k = 1, \dots, N$ we have

$$\frac{\partial g_t}{\partial x_k}(x) = \int_{\mathbb{R}^N} \frac{\partial g_1}{\partial x_k}(x - e^{-B}y)g_{t-1}(y)dy, \quad x \in \mathbb{R}^N,$$

which implies (recalling that $\|g_{t-1}\|_{L^1(\mathbb{R}^N)} = 1$)

$$\left\| \frac{\partial g_t}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)} \leq \left\| \frac{\partial g_1}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)}, \quad t > 1,$$

and (4.9) follows. □

Applying Theorems 3.8 and 3.9 we extend the results of Theorem 4.1.

Theorem 4.4. *Let $f \in C_b(\mathbb{R}^N)$ and $\lambda > 0$, $s \in (0, 1) \setminus \{1/2\}$. Then the equation*

$$\lambda u - Lu = f \tag{4.13}$$

has a unique solution $u \in C_b^{2s}(\mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{2s}(\mathbb{R}^N)} \leq C\|f\|_\infty.$$

If $s = 1/2$, equation (4.14) has a unique solution in $Z^1(\mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{Z^1(\mathbb{R}^N)} \leq C\|f\|_\infty.$$

If in addition $f \in C_b^\alpha(\mathbb{R}^N)$ with $\alpha \in (0, 1)$ and $\alpha + 2s \notin \{1, 2\}$, then $u \in C_b^{\alpha+2s}(\mathbb{R}^N)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{\alpha+2s}(\mathbb{R}^N)} \leq C\|f\|_{C_b^\alpha(\mathbb{R}^N)}.$$

If $\alpha + 2s = k \in \{1, 2\}$, then $u \in Z_b^k(\mathbb{R}^N)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_b^k(\mathbb{R}^N)} \leq C\|f\|_{C_b^\alpha(\mathbb{R}^N)}.$$

Applying Theorems 3.12 and 3.13 we extend the results of Theorem 4.2.

Theorem 4.5. *Let $s \in (0, 1)$, $\alpha \in [0, 1)$ be such that $\alpha + 2s \notin \{1, 2\}$, and let $f \in C_b^{\alpha+2s}(\mathbb{R}^N)$, $g \in C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)$. The mild solution to*

$$\begin{cases} v_t(t, x) = Lv(t, \cdot)(x) + g(t, x), & 0 \leq t \leq T, x \in \mathbb{R}^N, \\ v(0, x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.14}$$

belongs to $C_b^{0,\alpha+2s}([0, T] \times \mathbb{R}^N)$, and there is $C > 0$, independent of f and g , such that

$$\|v\|_{C_b^{0,\alpha+2s}([0, T] \times \mathbb{R}^N)} \leq C(\|f\|_{C_b^{\alpha+2s}(\mathbb{R}^N)} + \|g\|_{C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)}).$$

Let $s \in (0, 1)$, $\alpha \in [0, 1)$ be such that $\alpha + 2s := k \in \{1, 2\}$. Then for every $f \in Z_b^k(\mathbb{R}^N)$, $g \in C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)$ the mild solution to (4.15) belongs to $Z_b^{0,k}([0, T] \times \mathbb{R}^N)$, and there is $C > 0$, independent of f , such that

$$\|v\|_{Z_b^{0,k}(\mathbb{R}^N)} \leq C(\|f\|_{Z_b^k(\mathbb{R}^N)} + \|g\|_{C_b^{0,\alpha}([0, T] \times \mathbb{R}^N)}).$$

The results of Theorem 4.4 seem to be new. A part of them, in the case $\alpha \in (0, 1)$, $s \geq 1/2$, $1 < \alpha + 2s < 2$, was proved in [47] for a similar operator \mathcal{L} , with Bx replaced by $b(x)$ in the drift, $b \in C_b^\alpha(\mathbb{R}^N; \mathbb{R}^N)$. Concerning Theorem 4.5, in the case that $\alpha \in (0, 1)$, $s < 1/2$, $\alpha + 2s \in (1, 2)$, a similar result has been recently obtained in [17] for a more general class of operators with suitable nonlinear and time dependent drift coefficients.

5. EXAMPLES IN INFINITE DIMENSION

5.1. Ornstein-Uhlenbeck operators. In this section we deal with the case that X is an infinite dimensional separable Banach space and the measures μ_t are Gaussian and centered (i.e. with zero mean).

For the general theory of Gaussian measures in Banach spaces we refer to [5]. In particular, we recall that every centered Gaussian measure γ is Fomin differentiable along every h in the Cameron-Martin space H_γ , and the Fomin derivative β_h^γ belongs to $L^p(X, \gamma)$ for every $p \in [1, +\infty)$ and satisfies

$$\|\beta_h^\gamma\|_{L^p(X, \gamma)} = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^p e^{-\xi^2 \|h\|_{H_\gamma}^2 / 2} d\xi \right)^{1/p} =: c_p \|h\|_{H_\gamma}, \quad (5.1)$$

with $c_1 = \sqrt{2/\pi}$.

The first Schauder type theorems in the literature are in [9], [26, Ch. 6], concerning smoothing Ornstein-Uhlenbeck operators in a Hilbert setting. We recall that if X is a Hilbert space, for every centered Gaussian measure γ with covariance Q , the relevant Cameron-Martin space H_γ is the range of $Q^{1/2}$, with norm $\|h\|_{H_\gamma} = \|Q^{-1/2}h\|$ where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$.

The assumptions to obtain (in all directions) smoothing Ornstein-Uhlenbeck semigroups are the following.

Hypothesis 5.1. *X is a separable Hilbert space, $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} , and $Q \in \mathcal{L}(X)$ is a self-adjoint nonnegative operator, such that the operators defined by*

$$Q_t := \int_0^t e^{sA} Q e^{sA^*} ds, \quad t > 0 \quad (5.2)$$

have finite trace for every $t > 0$. Moreover, e^{tA} maps X into $Q_t^{1/2}(X)$ for every $t > 0$.

The relevant Ornstein-Uhlenbeck semigroup is given by

$$P_t f(x) = \int_X f(e^{tA}x + y) \mu_t(dy), \quad f \in \mathcal{B}_b(X), x \in X, \quad (5.3)$$

where

$$\mu_t = \mathcal{N}(0, Q_t), \quad t > 0,$$

is the Gaussian measure in X with mean 0 and covariance Q_t . In this case P_t is strong Feller, namely it maps $\mathcal{B}_b(X)$ (the space of the Borel bounded functions from X to \mathbb{R}) into $C_b(X)$. In fact, it maps $\mathcal{B}_b(X)$ into $C_b^k(X)$ for every $k \in \mathbb{N}$ ([26, Thm. 6.2.2]). Our L is a realization of the operator \mathcal{L} defined by

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(Q D^2 u(x)) + \langle x, A^* \nabla u(x) \rangle, \quad (5.4)$$

see [26, Sect. 6.1].

Under Hypothesis 5.1, Hypothesis 3.1 is satisfied with $H = X$, $H_t = Q_t^{1/2}(X)$, and Hypothesis 3.2(i) holds, since e^{tA} is a strongly continuous semigroup on X . But also Hypothesis 3.2(ii) is satisfied provided there exist $\omega \in \mathbb{R}$, $C, M, \theta > 0$ such that

$$\|Q_t^{-1/2} e^{tA}\|_{\mathcal{L}(X)} \leq \frac{C e^{\omega t}}{t^\theta}, \quad t > 0. \quad (5.5)$$

Indeed, in this case for every $h \in X$ and $t > 0$, $p \geq 1$ we have $\|\beta_{e^{tA}h}^{\mu_t}\|_{L^p(X, \mu_t)} \leq c_p \|e^{tA}h\|_{Q_t^{1/2}(X)} \leq c_p C e^{\omega t} t^{-\theta} \|h\|$, thanks to (5.1) and (5.5). Taking $p = 1$ yields that Hypothesis 3.2(ii) is satisfied; taking $p > 1$ by Remark 3.4 the space derivatives in the statements of next Theorems 5.2 and 5.3 are Fréchet derivatives instead of mere Gateaux derivatives.

Examples where (5.5) is satisfied are in [26, Appendix B]; one of them is considered in the next subsection.

The corresponding Schauder and Zygmund regularity results in the stationary case are the following.

Theorem 5.2. *Let Hypotheses 5.1 and (5.5) hold, and assume that $1/\theta \notin \mathbb{N}$. For every $f \in C_b(X)$ and $\lambda > 0$, the equation*

$$\lambda u - Lu = f \quad (5.6)$$

has a unique solution $u \in C_b^{1/\theta}(X)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{1/\theta}(X)} \leq C\|f\|_\infty.$$

If Hypotheses 5.1 and (5.5) hold and $1/\theta \in \mathbb{N}$, equation (5.6) has a unique solution in $Z^{1/\theta}(X)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{Z^{1/\theta}(X)} \leq C\|f\|_\infty.$$

If in addition $f \in C_b^\alpha(X)$ with $\alpha \in (0, 1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, then $u \in C_b^{\alpha+1/\theta}(X)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{\alpha+1/\theta}(X)} \leq C\|f\|_{C_b^\alpha(X)}.$$

If $\alpha + 1/\theta = k \in \mathbb{N}$, then $u \in Z_b^k(X)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_b^k(X)} \leq C\|f\|_{C_b^\alpha(X)}.$$

The Schauder part of this result was stated in [9], [26, Sect. 6.4.1] in the case $Q = I$, A of negative type, and $\theta = 1/2$; see also [21] for further estimates in such a case. It was extended in [14] to Ornstein-Uhlenbeck semigroups arising as transition semigroups of some stochastic PDEs, with $X = L^2(\Omega)$, Ω being a bounded open set in \mathbb{R}^N . In this case, A is the realization of a second order elliptic differential operator in X and $\theta = 1/2$.

In the evolution case Theorems 3.12 and 3.13 yield

Theorem 5.3. *Let Hypotheses 5.1 and (5.5) hold, and let $T > 0$. For every $f \in C_b(X)$, $g \in C_b([0, T] \times X)$ let v be the mild solution to*

$$\begin{cases} v_t(t, x) = Lv(t, x) + g(t, x), & t \in [0, T], x \in X, \\ v(0, \cdot) = f. \end{cases}$$

(i) *If $1/\theta \notin \mathbb{N}$ and $f \in C_b^{1/\theta}(X)$ then $v \in C_b^{0,1/\theta}([0, T] \times X)$. There exists $C = C(T) > 0$, independent of f and g , such that*

$$\|v\|_{C_b^{0,1/\theta}([0, T] \times X)} \leq C(\|f\|_{C_b^{1/\theta}(X)} + \|g\|_\infty).$$

(ii) *If $\alpha \in (0, 1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, $f \in C_b^{\alpha+1/\theta}(X)$ and $g \in C^{0,\alpha}([0, T] \times X)$ then $v \in C_b^{0,\alpha+1/\theta}([0, T] \times X)$. There exists $C = C(T, \alpha) > 0$, independent of f and g , such that*

$$\|v\|_{C_b^{0,\alpha+1/\theta}([0, T] \times X)} \leq C(\|f\|_{C_b^{\alpha+1/\theta}(X)} + \|g\|_{C_b^{0,\alpha}([0, T] \times X)}).$$

Let us go back to the case where X is a Banach space. The classical Ornstein-Uhlenbeck semigroup,

$$P_t f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\mu(dy), \quad t > 0, f \in C_b(X), x \in X, \quad (5.7)$$

where μ is any centered Gaussian measure in X , is not strong Feller. It is smoothing only along the directions of the Cameron-Martin space H_μ . However, by the changement of variables $z = \sqrt{1 - e^{-2t}}y$ in the integral it may be rewritten in the form (1.1), with $T_t = e^{-t}I$ and $\mu_t = \mu \circ (\sqrt{1 - e^{-2t}}I)^{-1}$, which is the centered Gaussian measure in X with covariance $Q_t = (1 - e^{-2t})Q$, if $Q : X^* \mapsto X$ is the covariance of μ . For the case where μ is non-Gaussian see Subsection 5.4 below.

The generator L of P_t is a realization of $\text{div}_\mu \nabla_{H_\mu}$, where div_μ is the Gaussian divergence and ∇_{H_μ} is the gradient along H_μ , see [5, Sect. 5.8].

As we mentioned at the beginning of the section, μ_t is Fomin differentiable along every $h \in H_{\mu_t}$, and Hypothesis 3.1 is satisfied with $H_t = H_{\mu_t}$. Since Q_t is a multiple of Q , the elements of H_{μ_t} coincide with those of H_μ , but the norms of these spaces are different, and precisely we have

$$\|h\|_{H_{\mu_t}} = \frac{1}{\sqrt{1 - e^{-2t}}} \|h\|_{H_\mu}, \quad h \in H_\mu, \quad t > 0.$$

The semigroup $T_t = e^{-t}I$ maps obviously H_μ into itself and into H_{μ_t} for every $t > 0$; moreover by (5.1) we have

$$\|\beta_{T_t}^\gamma h\|_{L^1(X, \mu_t)} = \sqrt{\frac{2}{\pi}} \|e^{-t}h\|_{H_{\mu_t}} = \sqrt{\frac{2}{\pi}} \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \|h\|_{H_\mu} \leq \frac{c e^{-t}}{t^{1/2}} \|h\|_{H_\mu}, \quad t > 0, \quad h \in H_\mu,$$

with $c = (2/\pi)^{1/2} \sup_{t>0} t^{1/2}/\sqrt{1 - e^{-2t}}$. Therefore, Hypothesis 3.2 is satisfied with $H = H_\mu$, $\omega = -1$, $\theta = 1/2$. Applying Theorems 3.8 and 3.9 gives the same results of [16], namely

Theorem 5.4. *Let $\lambda > 0$, $f \in C_b(X)$, and set $H = H_\mu$. Then the unique solution to*

$$\lambda u - Lu = f$$

belongs to $Z_H^2(X)$, and there is $C > 0$ such that

$$\|u\|_{Z_H^2(X)} \leq C \|f\|_\infty.$$

If in addition $f \in C_H^\alpha(X)$ with $0 < \alpha < 1$, u belongs to $C_H^{2+\alpha}(X)$, and there is $C > 0$ such that

$$\|u\|_{C_H^{2+\alpha}(X)} \leq C \|f\|_{C_H^\alpha(X)}.$$

Let now $T > 0$, $f \in Z_H^2(X)$, $g \in C_b([0, T] \times X)$. Then the mild solution to

$$\begin{cases} v_t(t, x) = Lv(t, x) + g(t, x), & t \in [0, T], \quad x \in X, \\ v(0, \cdot) = f, \end{cases}$$

belongs to $Z_H^{0,2}([0, T] \times X)$, and there exists $C = C(T) > 0$ such that $\|v\|_{Z_H^{0,2}([0, T] \times X)} \leq C(\|f\|_{Z_H^2(X)} + \|g\|_\infty)$.

If in addition $f \in C_H^{2+\alpha}(X)$, $g \in C_H^{0,\alpha}([0, T] \times X)$ with $\alpha \in (0, 1)$, then $v \in C_H^{0,2+\alpha}([0, T] \times X)$ and there exists $C = C(T, \alpha) > 0$ such that $\|v\|_{C_H^{0,2+\alpha}([0, T] \times X)} \leq C(\|f\|_{C_H^{2+\alpha}(X)} + \|g\|_{C_H^{0,\alpha}([0, T] \times X)})$.

Theorem 5.4 can be extended to the wider class of Ornstein-Uhlenbeck operators considered in [32]. There, $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} , and μ is a fixed centered Gaussian measure in X with covariance $Q \in \mathcal{L}(X^*, X)$. Moreover, the operators Q_t defined through a Pettis integral,

$$Q_t x^* := \int_0^t e^{sA} Q e^{sA^*} x^* ds, \quad x^* \in X^*, \quad t > 0,$$

are assumed to be the covariances of centered Gaussian measures μ_t in X . We recall that if X is a Hilbert space, Q_t is the covariance operator of a Gaussian measure if and only if its trace is finite.

If X is just a Banach space, and $Q = BB^*$ with $B \in \mathcal{L}(H, X)$, (necessary and) sufficient conditions for Q_t be the covariance of a Gaussian measure are in [58, Thm. 7.1]. References for sufficient conditions are also in [59, Remark 2].

In this case, P_t defined by (1.1) with $T_t = e^{tA}$ is the transition semigroup of a stochastic evolution equation,

$$\begin{cases} dX(t) = AX(t)dt + BdW_H(t), & t > 0, \\ X(0) = x \end{cases}$$

where $W_H(t)$ is a cylindrical white noise with values in the Hilbert space $H := H_\mu$, see [8, 58] for precise definitions and more details. Moreover, it was proved in [32, Thm. 6.2] that for every $f \in C_b(X)$ and $x \in X$ the function $t \mapsto P_t f(x)$ is continuous in $[0, +\infty)$. Even more, the semigroup P_t is strongly continuous in the mixed topology on $C_b(X)$, which is the finest locally convex topology on $C_b(X)$ which agrees on every bounded set with the topology of uniform convergence on compact sets.

Hypothesis 3.1 is satisfied with $T_t = e^{tA}$, $H = H_\mu$ ([32, Thm. 3.4]) if and only if there exists $\omega \in \mathbb{R}$ such that for every $x^* \in D(A^*)$ we have $(A^* - \omega I)x^*(Qx^*) \leq 0$, or equivalently if for every $x^* \in X^*$ the function $t \mapsto \|i^* e^{-\omega t} (e^{tA})^* x^*\|_H$ is nonincreasing in $[0, +\infty)$ (here i is the embedding $i : H \mapsto X^*$). In this case, by [32, Thm. 3.5], all the Cameron-Martin spaces H_{μ_t} coincide and have equivalent norms for every $t > 0$, and e^{tA} maps H_μ into H_{μ_t} , with

$$\|e^{tA}h\|_{H_{\mu_t}}^2 \leq \frac{1}{t^2} \int_0^t \|e^{sA}h\|_{H_\mu}^2 ds, \quad t > 0.$$

Therefore, if M, ω are such that $\|e^{tA}\|_{\mathcal{L}(H_\mu)} \leq Me^{\omega t}$ for $t > 0$, we get

$$\|e^{tA}h\|_{H_{\mu_t}} \leq \frac{M}{t^{1/2}} e^{\max\{\omega, 0\}t} \|h\|_{H_\mu}, \quad t > 0.$$

Recalling (5.1), we obtain that Hypothesis 3.2 is satisfied with $\theta = 1/2$ and ω replaced by $\max\{\omega, 0\}$. The statements of Theorem 5.4 hold in this case too. Notice that, still by (5.1) and Remark 3.4, the space derivatives in the statements are Fréchet derivatives.

5.2. Nonlocal Ornstein-Uhlenbeck operators. As mentioned in the introduction, semigroups of type (1.1) arise as transition semigroups of Ornstein-Uhlenbeck processes with Levy noise (see [30], [37], [38]) in finite or infinite dimensional state spaces, i.e., a stochastic process $X(t)$, $t \geq 0$, solving a stochastic differential equation on X of type

$$dX(t) = AX(t)dt + dY(t),$$

where $Y(t)$, $t \geq 0$, is a Levy process. We have seen examples of this type to which our results apply in finite dimensions in Subsection 4.2. In this subsection we shall discuss such a “nonlocal” example in infinite dimensions. More precisely, in the situation of the previous subsection we take $X = L^2(0, 1) := L^2((0, 1), dt)$, where dt denotes Lebesgue measure on $(0, 1)$. Let A be the Laplace operator Δ on $L^2(0, 1)$ with Dirichlet boundary conditions. Since we do not want to use too much theory of Levy processes (see [1, 40, 51]), we just mention here that such a process is determined by a negative definite function $\lambda : L^2(0, 1) \rightarrow \mathbb{C}$, which in our case we take concretely to be

$$\lambda(x) := \|x\|_{L^2(0,1)}^2 + c\|x\|_{L^2(0,1)}^{2s}, \quad x \in L^2(0, 1), \quad (5.8)$$

where $c \geq 0$ and $s \in (0, 1)$. The first summand corresponds to the Wiener process part and the second to the pure jump part of $Y(t)$, $t \geq 0$, in its Levy-Itô-decomposition (see [1, 40, 51]). The corresponding transition semigroup of $X(t)$, $t \geq 0$, is then given by

$$P_t f(x) = \int f(e^{t\Delta}x + y) \mu_t(dy), \quad t > 0, f \in \mathcal{B}_b(L^2(0, 1)), x \in L^2(0, 1),$$

where μ_t , $t \geq 0$, are probability measures with $\mu_0 = \delta_0$ and with Fourier transforms given by

$$\begin{aligned} \hat{\mu}_t(x) &:= \int_{L^2(0,1)} e^{i\langle x,y \rangle_{L^2(0,1)}} \mu_t(dy) \\ &= \exp \left\{ - \int_0^t \|e^{r\Delta} x\|_{L^2(0,1)}^2 + c \|e^{r\Delta} x\|_{L^2(0,1)}^{2s} dr \right\}, \end{aligned} \quad (5.9)$$

for $t > 0$ and $x \in L^2(0, 1)$; see Section 8 in [38].

In fact, it follows from the proof of Proposition 8.1 in [38] that there exists probability measures μ_t^c on $L^2(0, 1)$ such that

$$\hat{\mu}_t^c(x) = \exp \left\{ - \int_0^t c \|e^{r\Delta} x\|_{L^2(0,1)}^{2s} dr \right\}, \quad t > 0,$$

while $\exp\{-\int_0^t \|e^{r\Delta} x\|_{L^2(0,1)}^2 dr\}$ is the Fourier transform of the Gaussian measure \mathcal{N}_{0, Q_t} , where

$$Q_t = 2 \int_0^t e^{2r\Delta} dr = (-\Delta)^{-1} (I - e^{2t\Delta}), \quad t > 0, \quad (5.10)$$

has finite trace, because the eigenvalues λ_k , $k \in \mathbb{N}$, of Δ are proportional to $-k^2$. Therefore,

$$\mu_t = \mathcal{N}(0, Q_t) * \mu_t^c(x), \quad t > 0. \quad (5.11)$$

Furthermore, it follows immediately from the proof of Proposition 8.1 in [38] that the functions in (5.9) are equicontinuous in 0 with respect to the Sazonov topology on $L^2(0, 1)$ (namely, the topology generated by the seminorms $x \mapsto Tx$, where T is any Hilbert-Schmidt operator in $L^2(0, 1)$). This implies that $t \mapsto \mu_t$ is weakly continuous (see e.g. Proposition 1.1 in [56, Chap. IV.1.2]).

The generator L of P_t is a realization in $C_b(L^2(0, 1))$ of the operator \mathcal{L} that reads as

$$\mathcal{L}u(x) = \int_{L^2(0,1)} \left(i\langle x, \Delta y \rangle_{L^2(0,1)} - \lambda(y) \right) e^{i\langle x,y \rangle_{L^2(0,1)}} \nu(dy), \quad (5.12)$$

for all smooth cylindrical functions u such that $u = \hat{\nu}$ for some probability measure ν on $L^2(0, 1)$. We refer to [37] for details and a rigorous analysis.

Clearly, if $c = 0$, μ_t is the Gaussian measure $\mathcal{N}(0, Q_t)$ above, which is given by (5.2) with $Q = 2I$ and $A = \Delta$. In this case, $e^{t\Delta}$ maps $L^2(0, 1)$ into $Q_t^{1/2}(L^2(0, 1))$, and by elementary spectral theory we get

$$\|Q_t^{-1/2} e^{t\Delta}\|_{\mathcal{L}(L^2(0,1))} \leq \frac{c}{t^{1/2}}, \quad t > 0,$$

so (5.5) holds. So, P_t is just the semigroup (5.3) with $Q = 2I$ and $A = \Delta$, and \mathcal{L} has the representation (5.4).

For $c > 0$ we can apply our approach to our realization L of the operator \mathcal{L} in (5.12). So, let us check our Hypotheses 3.1 and 3.2 with $H = X = L^2(0, 1)$, $H_t = Q_t^{1/2}(L^2(0, 1))$, and $\theta = 1/2$. Obviously the only thing to check is Hypothesis 3.2(ii).

Let us start with proving the Fomin differentiability of μ_t along $e^{t\Delta}h$, for every $h \in L^2(0, 1)$ and $t > 0$. By the previous subsection we know that $\mathcal{N}(0, Q_t)$ is Fomin differentiable along $e^{t\Delta}h$ for every $t > 0$ and $h \in L^2(0, 1)$, with

$$\|\beta_{e^{t\Delta}h}^{\mathcal{N}(0, Q_t)}\|_{L^1(L^2(0,1), \mathcal{N}(0, Q_t))} \leq \frac{c}{t^{1/2}} \|h\|_{L^2(0,1)}, \quad t > 0, h \in L^2(0, 1).$$

Now (5.11) and the following lemma ensure that Hypothesis 3.2(ii) also holds for the measures μ_t , still with $\theta = 1/2$.

Lemma 5.5. *Let μ, ν be probability measures on a separable Banach space X , such that μ is Fomin differentiable along $v \in X$. Then $\mu * \nu$ is Fomin differentiable along v and*

$$\|\beta_v^{\mu * \nu}\|_{L^1(X, \mu * \nu)} \leq \|\beta_v^\mu\|_{L^1(X, \mu)}$$

Proof. Let $f \in C_b^1(X)$. Then, defining $\text{Ad}: X \times X \rightarrow X$ by $\text{Ad}(x, y) := x + y$ and $\pi_1: X \times X \rightarrow X$ by $\pi_1(x, y) = x$, we have

$$\begin{aligned} \int_X \frac{\partial f}{\partial v} d(\mu * \nu) &= \int_X \int_X \frac{\partial f}{\partial v}(x + y) \mu(dx) \nu(dy) \\ &= \int_X \int_X f(x + y) \beta_v^\mu(x) \mu(dx) \nu(dy) \\ &= \int_X \int_X f(\text{Ad}(x, y)) \mathbb{E}_{\mu \otimes \nu} [\beta_v^\mu \circ \pi_1 \mid \sigma(\text{Ad})] \mu(dx) \nu(dy) \\ &= \int_X f(z) \mathbb{E}_{\mu \otimes \nu} [\beta_v^\mu \circ \pi_1 \mid \text{Ad} = z] (\mu * \nu)(dz). \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_X \left| \mathbb{E}_{\mu \otimes \nu} [\beta_v^\mu \circ \pi_1 \mid \text{Ad} = z] \right| (\mu * \nu)(dz) &\leq \int_X \mathbb{E}_{\mu \otimes \nu} [|\beta_v^\mu \circ \pi_1| \mid \text{Ad} = z] (\mu * \nu)(dz) \\ &= \int_X \int_X \mathbb{E}_{\mu \otimes \nu} [|\beta_v^\mu \circ \pi_1| \mid \sigma(\text{Ad})] d\mu \otimes \nu \\ &= \int_X |\beta_v^\mu| d\mu. \end{aligned}$$

The statement follows, with $\beta_v^{\mu * \nu}(z) = \mathbb{E}_{\mu \otimes \nu} [\beta_v^\mu \circ \pi_1 \mid \text{Ad} = z]$. \square

Applying Theorems 3.8 and 3.9 yields

Theorem 5.6. *For every $f \in C_b(L^2(0, 1))$ and $\lambda > 0$, the equation*

$$\lambda u - Lu = f \tag{5.13}$$

has a unique solution $u \in Z_b^2(L^2(0, 1))$, and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_b^2(L^2(0, 1))} \leq C \|f\|_\infty.$$

If in addition $f \in C_b^\alpha(L^2(0, 1))$ with $\alpha \in (0, 1)$, then $u \in C_b^{2+\alpha}(X)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{C_b^{2+\alpha}(L^2(0, 1))} \leq C \|f\|_{C_b^\alpha(L^2(0, 1))}.$$

Applying Theorems 3.12 and 3.13 yields

Theorem 5.7. *Let $T > 0$. For every $f \in C_b(L^2(0, 1))$, $g \in C_b([0, T] \times L^2(0, 1))$, let v be the mild solution to*

$$\begin{cases} v_t(t, x) = Lv(t, x) + g(t, x), & t \in [0, T], x \in L^2(0, 1), \\ v(0, \cdot) = f. \end{cases}$$

(i) *If $f \in Z_b^2(L^2(0, 1))$ and $g \in C_b([0, T] \times L^2(0, 1))$, then $v \in Z_b^{0,2}([0, T] \times L^2(0, 1))$, and there exists $C = C(T) > 0$, independent of f and g , such that*

$$\|v\|_{Z_b^{0,2}([0, T] \times L^2(0, 1))} \leq C (\|f\|_{Z_b^2(L^2(0, 1))} + \|g\|_\infty).$$

(ii) If $\alpha \in (0, 1)$ and $f \in C_b^{2+\alpha}(L^2(0, 1))$, $g \in C_b^{0,\alpha}([0, T] \times L^2(0, 1))$, then $v \in C_b^{0,2+\alpha}([0, T] \times L^2(0, 1))$. There exists $C = C(T, \alpha) > 0$, independent of f and g , such that

$$\|v\|_{C_b^{0,2+\alpha}([0,T] \times L^2(0,1))} \leq C(\|f\|_{C_b^{2+2\alpha}(L^2(0,1))} + \|g\|_{C_b^{0,\alpha}([0,T] \times L^2(0,1))}).$$

5.3. The Gross Laplacian and its powers. Here X is a separable Hilbert space and $Q \in \mathcal{L}(X)$ is a self-adjoint positive operator with finite trace. The semigroup P_t is defined by (1.1) with $T_t = I$ for every $t > 0$, and $\mu_t = \mathcal{N}(0, tQ)$ is the centered Gaussian measure in X with covariance tQ . Therefore we have

$$P_t f(x) = \int_X f(x+y) \mu_t(dy) = \int_X f(x + \sqrt{t}z) \mu(dz), \quad f \in C_b(X), \quad t > 0. \quad (5.14)$$

with $\mu = \mu_1 = \mathcal{N}(0, Q)$. That P_t is a semigroup (namely, $\mu_t \star \mu_s = \mu_{t+s}$ for every $s, t > 0$) is a consequence of standard properties of Gaussian measures, e.g. [5, Prop. 2.2.10]. The operator L defined in (1.3) is a realization of the differential operator

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(QD^2u(x)).$$

See [31], [26, Ch. 3] and the references therein. We choose as H_t the Cameron-Martin space of μ_t . So, Hypothesis 3.1 is satisfied. Moreover we take $H = H_1$ = the Cameron-Martin space of μ . We have $H_t = Q^{1/2}(X) = H$ for every $t > 0$, with norm depending on t ,

$$\|h\|_{H_t} = \frac{1}{t^{1/2}} \|h\|_H, \quad h \in H, \quad t > 0.$$

Consequently, by (5.1),

$$\|\beta_{T_t h}^{\mu_t}\|_{L^p(X, \mu_t)} \leq \frac{C_p}{t^{1/2}} \|h\|_H, \quad h \in H, \quad t > 0, \quad (5.15)$$

and taking $p = 1$, Hypothesis 3.2 is satisfied with $\theta = 1/2$, $\omega = 0$. Therefore Theorems 3.8 and 3.9 yield that the statement of Theorem 5.4 holds in this case too, and in this case too the space derivatives in the statement are Fréchet derivatives, by (5.15) and Remark 3.4.

The Schauder part of Theorem 5.4 in the stationary case was already stated in [10, 26]; see also [2] for a related result.

Now let us consider the powers $(-L)^s$ with $s \in (0, 1)$. As in the finite dimensional case (see (4.5)) we define it as minus the generator of the subordinated semigroup S_t of P_t on $C_b(X)$ with subordinator $\{\eta_t^{(s)}(r)dr, t > 0\}$, where as in Subsection 4.1, $\eta_t^{(s)}(r)$, $r > 0$, is given as the inverse Laplace transform of $[0, \infty) \ni \lambda \mapsto e^{-t\lambda^s}$, i.e.

$$S_t f(x) = \int_0^\infty (P_\sigma f)(x) \eta_t^{(s)}(\sigma) d\sigma = \int_0^\infty \int_X f(x+y) \mathcal{N}(0, \sigma Q)(dy) \eta_t^{(s)}(\sigma) d\sigma = \int_X f(x+y) \nu_t(dy), \quad t > 0,$$

where P_t is the semigroup in (5.14), and the measures ν_t are defined by

$$\nu_0(B) = \delta_0(B); \quad \nu_t(B) = \int_0^\infty \mathcal{N}(0, \sigma Q)(B) \eta_t^{(s)}(\sigma) d\sigma = \int_0^\infty \mathcal{N}(0, Q)(B/\sigma^{1/2}) \eta_t^{(s)}(\sigma) d\sigma, \quad t > 0, \quad B \in \mathcal{B}(X), \quad (5.16)$$

where $\mathcal{B}(X)$ denotes the Borel σ -algebra of X . According to the terminology of [6, Ch. 4], ν_t is called ‘‘mixture of measures’’.

Lemma 5.8. $t \mapsto \nu_t$ is weakly continuous in $[0, +\infty)$. The generator of S_t is the operator whose resolvent is given by

$$\frac{\sin(s\pi)}{\pi} \int_0^\infty R(\xi, L) \frac{\xi^s}{\lambda^2 - 2\xi^s \cos(s\pi) + \xi^{2s}} d\xi, \quad \lambda > 0. \quad (5.17)$$

Proof. Let us check that $t \mapsto \nu_t$ is weakly continuous. For every $f \in C_b(X)$ and $t > 0$ we have

$$\int_X f(x) \nu_t(dx) = \int_X \int_0^{+\infty} f(x) \mathcal{N}(0, \sigma Q)(dx) \eta_t^{(s)}(\sigma) d\sigma = \int_X \int_0^{+\infty} f(t^{1/2s} \tau^{1/2} z) N_{0,Q}(dz) \eta^{(s)}(\tau) d\tau$$

For $t_0 > 0$ the right-hand side goes to $\int_X f(x) \nu_{t_0}(dx)$ as $t \rightarrow t_0$, by the Dominated Convergence Theorem. The same holds for $t_0 = 0$, recalling that $\int_0^{+\infty} \eta^{(s)}(\tau) d\tau = 1$.

Concerning the second assertion, using (4.4) for every $\lambda > 0$ and $f \in C_b(X)$ we get

$$\begin{aligned} \int_0^\infty e^{-\lambda t} S_t f(x) dt &= \int_0^\infty e^{-\lambda t} \int_0^\infty P_\sigma f(x) \frac{t^{-1/s}}{\pi} \int_0^\infty e^{-\sigma t^{-1/s} r - r^s \cos(s\pi)} \sin(r^s \sin(s\pi)) dr d\sigma dt \\ &= \frac{1}{\pi} \int_0^\infty d\xi \left(\int_0^\infty e^{-\lambda t - t\xi^s \cos(s\pi)} \sin(t\xi^s \sin(s\pi)) dt \int_0^\infty P_\sigma f(x) e^{-\sigma\xi} d\sigma \right) \\ &= \frac{\sin(s\pi)}{\pi} \int_0^\infty R(\xi, L) f(x) \frac{\xi^s}{\lambda^2 - 2\xi^s \cos(s\pi) + \xi^{2s}} d\xi \end{aligned}$$

(the last equality follows from $\int_0^\infty e^{-at} \sin(bt) dt = b/(b^2 + a^2)$). \square

We recall that if L is the infinitesimal generator of a bounded strongly continuous semigroup in a Banach space, formula (5.17) coincides with the Kato representation formula for the resolvent of $-(-L)^s$ for $s \in (0, 1)$, which may be taken as a definition of $-(-L)^s$ ([34]). In our case P_t is a contraction semigroup in $C_b(X)$ but it is not strongly continuous, whereas it is strongly continuous in $BUC(X)$. Therefore, the operator whose resolvent is given by (5.17) is an extension to $C_b(X)$ of $-(-L_0)^s$, where L_0 is the part of L in $BUC(X)$, and it may be called $-(-L)^s$, although our case is not covered by the standard theory of powers of (noninvertible) operators.

The following easy lemma will be used here and in the following.

Lemma 5.9. *Let ν be a probability measure in a Banach space X that is Fomin differentiable along some h , and let $c > 0$. Then the measure $\nu_c := \nu \circ (cI)^{-1}$ (namely, $\nu_c(A) = \nu(A/c)$) is Fomin differentiable along h , and*

$$\begin{cases} (i) & \beta_h^{\nu_c}(y) = \frac{1}{c} \beta_h^\nu\left(\frac{y}{c}\right), \quad \nu_c - a.e. y \in X; \\ (ii) & \|\beta_h^{\nu_c}\|_{L^1(X, \nu_c)} = \frac{1}{c} \|\beta_h^\nu\|_{L^1(X, \nu)}. \end{cases} \quad (5.18)$$

Proof. For every $f \in C_b^1(X)$ and $t > 0$ we have

$$\begin{aligned} \int_X \frac{\partial f}{\partial h}(y) \nu_c(dy) &= \int_X \frac{\partial f}{\partial h}(cz) \nu(dz) = \int_X \frac{1}{c} \frac{\partial}{\partial h} f(c \cdot)(z) \nu(dz) \\ &= \frac{1}{c} \int_X f(cz) \beta_h^\nu(z) \nu(dz) = \frac{1}{c} \int_X f(y) \beta_h^\nu\left(\frac{y}{c}\right) \nu_c(dy), \end{aligned}$$

and (5.18)(i) follows. Moreover,

$$\|\beta_h^{\nu_c}\|_{L^1(X, \nu_c)} = \frac{1}{c} \int_X \left| \beta_h^\nu\left(\frac{y}{c}\right) \right| \nu_c(dy) = \frac{1}{c} \int_X |\beta_h^\nu(y)| \nu(dy)$$

which is (5.18)(ii). \square

Proposition 5.10. *Let $H = Q^{1/2}(X)$, $T_t = I$ for every $t > 0$. The measures ν_t defined in (5.16) satisfy Hypothesis 3.2, with $\omega = 0$ and $\theta = 1/(2s)$.*

Proof. We have to check that ν_t is Fomin differentiable along every $h \in H$, and that there exists $C > 0$ such that

$$\|\beta_h^{\nu_t}\|_{L^1(X, \nu_t)} \leq \frac{C}{t^{1/(2s)}}, \quad t > 0, h \in H. \quad (5.19)$$

Setting as before $\mu_t := \mathcal{N}(0, tQ) = \mu \circ (t^{1/2}I)^{-1}$, $\mu := \mathcal{N}(0, Q)$ we get from Lemma 5.9

$$\beta_h^{\mu_t}(y) = \frac{1}{t^{1/2}} \beta_h^\mu\left(\frac{y}{t^{1/2}}\right), \quad t > 0, h \in H, y \in X.$$

Consequently, we get

$$\begin{aligned} \int_X \frac{\partial f}{\partial h}(y) \nu_t(dy) &= \int_X \frac{\partial f}{\partial h}(y) \int_0^\infty \mathcal{N}(0, \sigma Q)(dy) \eta_t^{(s)}(\sigma) d\sigma = \int_0^\infty \int_X \frac{\partial f}{\partial h}(y) \mathcal{N}(0, \sigma Q)(dy) \eta_t^{(s)}(\sigma) d\sigma \\ &= \int_0^\infty \int_X f(y) \frac{1}{\sigma^{1/2}} \beta_h^\mu\left(\frac{y}{\sigma^{1/2}}\right) \mu_\sigma(dy) \eta_t^{(s)}(\sigma) d\sigma = \int_X f(y) \gamma_t(dy) \end{aligned}$$

where the measures γ_t are defined by

$$\begin{aligned} \gamma_t(A) &:= \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_A \beta_h^\mu\left(\frac{y}{\sigma^{1/2}}\right) \mu_\sigma(dy) \right) \eta_t^{(s)}(\sigma) d\sigma, \\ &= \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_{A/\sigma^{1/2}} \beta_h^\mu(z) \mu(dz) \right) \eta_t^{(s)}(\sigma) d\sigma, \quad A \in \mathcal{B}(X). \end{aligned}$$

Now we prove that each γ_t is absolutely continuous with respect to ν_t . This will be done showing that the positive and negative parts of γ_t are respectively given by

$$\gamma_t^+(A) = \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_{A/\sigma^{1/2}} (\beta_h^\mu)^+(z) \mu(dz) \right) \eta_t^{(s)}(\sigma) d\sigma, \quad A \in \mathcal{B}(X), \quad (5.20)$$

$$\gamma_t^-(A) = \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_{A/\sigma^{1/2}} (\beta_h^\mu)^-(z) \mu(dz) \right) \eta_t^{(s)}(\sigma) d\sigma, \quad A \in \mathcal{B}(X). \quad (5.21)$$

Such representations yield that both γ_t^+ and γ_t^- are absolutely continuous with respect to ν_t , because for every ν_t -negligible A we have by definition $\int_0^\infty \mu(A/\sigma^{1/2}) \eta_t^{(s)}(\sigma) d\sigma = 0$, and since $\eta_t^{(s)}(\sigma) > 0$ for every $\sigma > 0$ we get $\mu(A/\sigma^{1/2}) = 0$ for a.e. $\sigma > 0$ and therefore $\gamma_t^+(A) = \gamma_t^-(A) = 0$.

By [5, Sect. 2.10] there exists a μ -version f_0 of β_h^μ which is linear on a full measure subspace of X . We set

$$X^+ := \{x \in X : f_0(x) \geq 0\}, \quad X^- := \{x \in X : f_0(x) < 0\},$$

and we check that $X = X^+ \cup X^-$ is a Hahn decomposition of (X, γ_t) , namely $X^+ \cap X^- = \emptyset$ (which is obvious) and

$$\gamma_t(A \cap X^+) \geq 0, \quad \gamma_t(A \cap X^-) \leq 0, \quad A \in \mathcal{B}(X).$$

Indeed, for every $A \in \mathcal{B}(X)$ we have

$$\begin{aligned} \gamma_t(A \cap X^+) &= \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_{(A \cap X^+)/\sigma^{1/2}} (\beta_h^\mu)^+(z) \mu(dz) \right) \eta_t^{(s)}(\sigma) d\sigma \\ &= \int_0^\infty \frac{1}{\sigma^{1/2}} \left(\int_{(A \cap X^+)/\sigma^{1/2}} f_0(z) \mu(dz) \right) \eta_t^{(s)}(\sigma) d\sigma. \end{aligned}$$

Since f_0 is linear on a μ -full measure subspace, then for every $\sigma > 0$ the sets $X^+/\sigma^{1/2}$ and X^+ may differ only by a μ -negligible set. Therefore, $f_0(z) \geq 0$ for μ -a.e. $z \in X^+/\sigma^{1/2}$, so that

$\int_{(A \cap X^+)/\sigma^{1/2}} f_0(z) \mu(dz) \geq 0$ and therefore $\gamma_t(A \cap X^+) \geq 0$. The same argument yields $\gamma_t(A \cap X^-) \leq 0$, and (5.20), (5.21) follow.

Therefore, γ_t is absolutely continuous with respect to ν_t and its density is the Fomin derivative $\beta_h^{\nu_t}$ of ν_t along h . Let us estimate its $L^1(X, \nu_t)$ norm. We have

$$\|\beta_h^{\nu_t}\|_{L^1(X, \nu_t)} = \sup \left\{ \frac{1}{\|f\|_\infty} \int_X f(y) \beta_h^{\nu_t}(y) \nu_t(dy), f \in L^\infty(X, \nu_t) \setminus \{0\} \right\},$$

and for every $f \in L^\infty(X, \nu_t)$ we have

$$\begin{aligned} \int_X f(y) \beta_h^{\nu_t}(y) \nu_t(dy) &= \int_X f(y) \gamma_t(dy) = \int_X f(y) \int_0^\infty \frac{1}{\sigma^{1/2}} \beta_h^\mu \left(\frac{y}{\sigma^{1/2}} \right) \mu_\sigma(dy) \eta_t^{(s)}(\sigma) d\sigma \\ &= \int_0^\infty \left(\frac{1}{\sigma^{1/2}} \int_X f(\sigma^{1/2}x) \beta_h^\mu(x) \mu(dx) \right) \eta_t^{(s)}(\sigma) d\sigma \\ &\leq \int_0^\infty \frac{1}{\sigma^{1/2}} \|f\|_\infty \|\beta_h^\mu\|_{L^1(X, \mu)} \eta_t^{(s)}(\sigma) d\sigma \leq \|f\|_\infty \|h\|_H \int_0^\infty \frac{1}{\sigma^{1/2}} \eta_t^{(s)}(\sigma) d\sigma, \end{aligned}$$

where

$$\int_0^\infty \frac{1}{\sigma^{1/2}} \eta_t^{(s)}(\sigma) d\sigma = \int_0^\infty \frac{1}{\sigma^{1/2}} \eta^{(s)} \left(\frac{\sigma}{t^{1/s}} \right) d\sigma = \frac{1}{t^{1/2s}} \int_0^\infty \frac{\eta^{(s)}(\tau)}{\tau^{1/2}} d\tau.$$

Therefore, (5.19) follows with $C = \int_0^\infty \eta(\tau) \tau^{-1/2} d\tau$. \square

Thanks to Lemma 5.8 and Proposition 5.10 we can apply Theorems 3.8 and 3.9, that give

Theorem 5.11. *Let $f \in C_b(X)$ and $\lambda > 0$, $s \in (0, 1) \setminus \{1/2\}$. Then the equation*

$$\lambda u + (-L)^s u = f \tag{5.22}$$

has a unique solution $u \in C_H^{2s}(X)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{C_H^{2s}(X)} \leq C \|f\|_\infty.$$

If $s = 1/2$, equation (5.22) has a unique solution in $Z_H^1(X)$, and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_H^1(X)} \leq C \|f\|_\infty.$$

If in addition $f \in C_H^\alpha(X)$ with $\alpha \in (0, 1)$ and $\alpha + 2s \notin \{1, 2\}$, then $u \in C_H^{\alpha+2s}(X)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{C_H^{\alpha+2s}(X)} \leq C \|f\|_{C_H^\alpha(X)}.$$

If $\alpha + 2s = k \in \{1, 2\}$, then $u \in Z_H^k(X)$ and there is $C > 0$, independent of f , such that

$$\|u\|_{Z_H^k(X)} \leq C \|f\|_{C_H^\alpha(X)}.$$

Applying Theorems 3.12 and 3.13 we obtain

Theorem 5.12. *Let $s \in (0, 1)$, $\alpha \in [0, 1)$ be such that $\alpha + 2s \notin \{1, 2\}$, and let $f \in C_H^{\alpha+2s}(X)$, $g \in C_H^{0, \alpha}([0, T] \times X)$ ⁽²⁾. The mild solution to*

$$\begin{cases} v_t(t, x) + (-L)^s v(t, \cdot)(x) + g(t, x), & 0 \leq t \leq T, x \in X, \\ v(0, x) = f(x), & x \in X, \end{cases} \tag{5.23}$$

⁽²⁾For $\alpha = 0$ we mean $C_H^{0,0}([0, T] \times X) = C_b([0, T] \times X)$.

belongs to $C_H^{0,\alpha+2s}([0,T] \times X)$, and there is $C > 0$, independent of f and g , such that

$$\|v\|_{C_H^{0,\alpha+2s}([0,T] \times X)} \leq C(\|f\|_{C_H^{\alpha+2s}(X)} + \|g\|_{C_H^{0,\alpha}([0,T] \times X)}).$$

Let $s \in (0,1)$, $\alpha \in [0,1)$ be such that $\alpha + 2s =: k \in \{1,2\}$. Then for every $f \in Z_H^k(X)$, $g \in C_H^{0,\alpha}([0,T] \times X)$ the mild solution to (5.23) belongs to $Z_H^{0,k}([0,T] \times X)$, and there is $C > 0$, independent of f , such that

$$\|v\|_{Z_H^{0,k}(X)} \leq C(\|f\|_{Z_H^k(X)} + \|g\|_{C_H^{0,\alpha}([0,T] \times X)}).$$

5.4. Non-Gaussian classical Ornstein-Uhlenbeck semigroups. In this section, as announced earlier, we come back to (5.7), more precisely to its non-Gaussian analogue considered in [30, Sect. 7], for which the semigroup P_t is given by

$$P_t f(x) = \int_X f(e^{-t}x + (1 - e^{-pt})^{1/p}y) \mu(dy), \quad t > 0, f \in C_b(X), x \in X, \quad (5.24)$$

where μ is a suitable Borel probability measure in a Hilbert space X . P_t may be written in the form (1.1), with $T_t = e^{-t}I$ and

$$\mu_t = \mu \circ [(1 - e^{-pt})^{1/p}I]^{-1}, \quad t > 0. \quad (5.25)$$

If μ is a centered Gaussian measure and $p = 2$, P_t is the classical Ornstein-Uhlenbeck semigroup considered before in (5.7). For P_t to be a semigroup, μ cannot be any Borel measure: indeed, we need that condition (1.2) is satisfied. It is satisfied provided

$$\hat{\mu}(a) = e^{-\lambda(a)/p}, \quad a \in X^*,$$

and $\lambda : X^* \mapsto \mathbb{C}$ is a negative definite function, which is Sazonov continuous, and such that

$$\lambda(ta) = t^p \lambda(a), \quad a \in X^*, t > 0.$$

The weak continuity of $t \mapsto \mu_t$ follows immediately from the equality $\int_X f(y) \mu_t(dy) = \int_X f((1 - e^{-pt})^{1/p}y) \mu(dy)$, for every $f \in C_b(X)$ and $t \geq 0$.

We fix now a Banach space $H \subset X$ such that μ is Fomin differentiable along every $h \in H$. (H may be the whole space $D(\mu)$ of all $h \in X$ such that μ is Fomin differentiable along h , or a smaller space continuously embedded in $D(\mu)$). In the case where X is e.g. a separable real Hilbert space, an easy example for such a probability measure μ with $D(\mu) \supset Q^{\frac{1}{2}}(X)$ is the measure ν_t defined in (5.16) with $t = \frac{1}{p}$ and $s = \frac{p}{2}$ (recall that ν_t in (5.16) also depends on s); in this case it is easy to check that $\lambda(a) = \langle Qa, a \rangle^{p/2}/2$.

Then, $T_t = e^{-t}I$ maps obviously H into itself. Moreover, by Lemma 5.9 μ_t is Fomin differentiable along every $h \in H$ and we have $\|\beta_h^{\mu_t}\|_{L^1(X,\mu_t)} = (1 - e^{-pt})^{-1/p} \|\beta_h^\mu\|_{L^1(X,\mu)}$. Therefore, for every $t > 0$ and $h \in D(\mu)$ we have

$$\|\beta_{T_t h}^\mu\|_{L^1(X,\mu_t)} = \frac{e^{-t}}{(1 - e^{-pt})^{1/p}} \|\beta_h^\mu\|_{L^1(X,\mu)} = \frac{e^{-t}}{(1 - e^{-pt})^{1/p}} \|h\|_{D(\mu)} \leq C \frac{e^{-t}}{t^{1/p}} \|h\|_{D(\mu)},$$

with $C = \sup_{t>0} t^{1/p} (1 - e^{-pt})^{-1/p}$. Since $H \subset D(\mu)$, Hypothesis 3.2 is satisfied with $\omega = -1$, $\theta = 1/p$ and our approach applies. Hence Theorems 3.8 and 3.9 hold for the generator L of the semigroup in (5.24), with $\theta = 1/p$, as well as Theorems 3.12 and 3.13.

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