

COLLECTIVE STOCHASTIC DYNAMICS OF THE CUCKER-SMALE ENSEMBLE UNDER UNCERTAIN COMMUNICATIONS

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ABSTRACT. We present collective dynamics of the Cucker-Smale (C-S) ensemble under random communications. As an effective modeling of the C-S ensemble, we introduce a stochastic kinetic C-S equation with a multiplicative white noise. For the proposed stochastic kinetic model with a multiplicative noise, we present a global well-posedness of strong solutions and their asymptotic flocking dynamics, when initial datum is sufficiently regular, and random communication weight function has a positive lower bound.

1. INTRODUCTION

Collective movements of self-propelled particles are ubiquitous in many biological systems in our nature, to name a few, flocking of birds, herding of sheep and swarming of fish, etc. Throughout the paper, we will use a terminology “*flocking*” to denote aforementioned coherent collective motions. More precisely, *flocking* phenomenon denotes a situation in which self-propelled particles adjust their motions into a self-organized ordered motion using only the environmental information based on simple rules [12, 13, 27, 28, 35, 37]. After Reynolds and Vicsek et al’s pioneering works in [32, 38], several mechanical models were introduced in literature [5, 7, 9, 10, 25, 27, 29, 30, 34, 37] to model such coherent collective motions. Among others, our main interest lies on the mean-field kinetic model, namely “kinetic C-S model [20, 22, 23]”. Let $f := f(t, x, v)$ be the one-particle distribution function for C-S ensemble at position x with microscopic velocity v at time t . Then, the dynamics of f is governed by the kinetic C-S equation:

$$(1.1) \quad \begin{aligned} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (\tilde{F}_a[f]f) &= 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}, \\ \tilde{F}_a[f](t, x, v) &:= \int_{\mathbb{R}^{2d}} \phi(x_* - x)(v_* - v)f(t, x_*, v_*) dv_* dx_*. \end{aligned}$$

Here $\tilde{F}_a[f]$ is a non-local operator measuring the attractive interactions between particles, and ϕ is a communication weight function which is nonnegative and radially symmetric:

$$\phi(x) = \bar{\phi}(|x|) \geq 0, \quad \forall x \in \mathbb{R}^d,$$

where $\bar{\phi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is Lipschitz continuous, bounded and monotonically decreasing.

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In the last decade, the mean-field equation (1.1) and its variants have been extensively studied from the various perspectives, e.g., well-posedness and emergent dynamics [4, 23], Fokker-Planck perturbation [19], local sensitivity analysis [18], etc. For more detailed discussion, we refer to a recent survey article [7]. In this paper, we are interested in the quantitative effects on the flocking dynamics of (1.1) due to the uncertain communication weights. Recently, the local sensitivity analysis for (1.1) has been discussed in an abstract and general framework in [18] and some quantitative pathwise estimates for the variations in f and its derivatives in random space were studied. However, authors in [18] could not provide interesting probabilistic estimates in relation with the emergent dynamics (see [2, 6, 11, 17, 26] for a related local sensitivity analysis in uncertainty quantification (UQ)). Thus, our goal of this paper is to address some probability estimate for (1.1) with uncertain communications. To fix the idea, we employ the following ansatz for $\tilde{F}_a[f]$ which is responsible for the flocking mechanism due to mutual communications:

$$\tilde{F}_a[f](t, x, v) := \int_{\mathbb{R}^{2d}} \left[\phi(x_* - x)(v_* - v) + \sigma(v_* - v) \circ \dot{W}_t \right] f(t, x_*, v_*) dv_* dx_*,$$

where \dot{W}_t is a one-dimensional white noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, σ denotes the strength of noise and the stochastic integration is taken in Stratonovich sense. Formally, under the unit mass assumption $\int_{\mathbb{R}^{2d}} f(t, x, v) dx dv = 1$, the non-local operator $\tilde{F}_a[f]$ can split into the deterministic part $F_a[f]$ and stochastic part involving with \dot{W}_t :

$$(1.2) \quad \tilde{F}_a[f] = \underbrace{\int_{\mathbb{R}^{2d}} \phi(x_* - x)(v_* - v) f(t, x_*, v_*) dv_* dx_*}_{=: F_a[f]} + \sigma(v_c - v) \circ \dot{W}_t.$$

Now, we combine (1.1) and (1.2) to derive the stochastic kinetic C-S equation:

$$(1.3) \quad \partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (F_a[f_t] f_t) = \sigma \nabla_v \cdot ((v - v_c) f_t) \circ \dot{W}_t.$$

Here we use the standard notation for random probability density function $f_t(x, v) := f(t, x, v)$.

At the particle level, the effects of white noise perturbations were discussed in [1, 14, 21]. However, as far as the authors know, the kinetic C-S equation (1.2) perturbed by a multiplicative white noise has not been addressed in literature yet. For other types of stochastic kinetic equations, we refer to [16, 31]. In this paper, we address the following two questions:

- (Well-posedness): Is the stochastic kinetic C-S equation (1.3) well-posed in a suitable function space?
- (Emergence of flocking): If so, does the solution to (1.3) exhibit asymptotic flocking dynamics?

Our main results in this paper are affirmative answers for the above two questions. First, we introduce a concept of a strong solution to (1.3) and then provide a global well-posedness for strong solutions by employing a suitable regularization method and stopping time argument. Second, we provide a stochastic flocking estimate by showing that the expectation of the second velocity moment decays to zero exponentially fast, when the communication weight function ϕ has a positive infimum $\phi_m := \inf_{x \in \mathbb{R}^d} \phi(x)$ and noise strength σ is sufficiently small compared to ϕ_m . The main difficulty in our analysis arises, when we prove

the existence of a solution to the regularized equation. Here, we obtain $W^{m,\infty}$ -estimates for the sequence of functions that approximates the regularized equation. Our $W^{m,\infty}$ -estimates contain terms with infinite expectation. Hence, even though we can find a limit function of the sequence from the pathwise estimates, it is not certain that the limit function becomes a solution to the regularized equation (see Remark 3.2 for detailed discussion). To cope with this problem, we used stopping time argument to get a solution to the regularized equation.

The rest of this paper is organized as follows. In Section 2, we briefly study several a priori estimates for (1.3) and then briefly discuss main results on the global well-posedness of strong solutions and asymptotic flocking estimate of classical solutions. In Section 3, we provide a global well-posedness of strong solutions to (1.3) with regularized initial datum. In Section 4, we derive a global well-posedness and emergent dynamics of strong solutions to (1.3). Finally, Section 5 is devoted to a brief summary of main results and discussion on future works.

2. PRESENTATION OF MAIN RESULTS

In this section, we provide our main results on the global well-posedness of (1.3) and emergent flocking dynamics.

First, we consider the Cauchy problem for (1.3):

$$(2.1) \quad \begin{aligned} \partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (F_a[f_t]f_t) &= \sigma \nabla_v \cdot ((v - v_c)f_t) \circ \dot{W}_t, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}, \\ f_0(x, v) &= f^{in}(x, v), \end{aligned}$$

where we assume that initial datum f^{in} is deterministic. Next, we provide a definition for a strong solution to the Cauchy problem (2.1) as follows.

Definition 2.1. *For a given $T \in (0, \infty]$, $f_t = f_t(x, v)$ is a strong solution to (2.1) on $[0, T]$ if it satisfies the following relations:*

- (1) (Regularity): For $k \geq 1$, $f_t \in \mathcal{C}([0, T]; W^{k,p}(\mathbb{R}^{2d}))$ for any $p \in [1, \infty)$.
- (2) (Integral relation): f_t satisfies the equation (1.3) in distribution sense: for $\psi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R}^{2d})$,

$$(2.2) \quad \begin{aligned} \int_{\mathbb{R}^{2d}} f_t \psi \, dv dx &= \int_{\mathbb{R}^{2d}} f^{in} \psi \, dv dx + \int_0^t \int_{\mathbb{R}^{2d}} f_s (v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi) \, dv dx ds \\ &\quad - \sigma \int_0^t \left(\int_{\mathbb{R}^{2d}} [(v - v_c)f_s] \cdot \nabla_v \psi \, dv dx \right) \circ dW_s, \quad a.s. \, \omega \in \Omega. \end{aligned}$$

Note that relation (2.2) can be reformulated in the Itô form under suitable conditions.

Lemma 2.1. *Suppose that for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ and a random process $f_t \in L^\infty(\Omega \times [0, T] \times \mathbb{R}^{2d})$, $\int_{\mathbb{R}^{2d}} f_t \psi \, dv dx$ has a continuous \mathcal{F}_t -adapted modification, where $\{\mathcal{F}_t\}$ is a family of σ -field generated by the Wiener process. Then, f_t is a \mathcal{F}_t -semimartingale satisfying relation*

(2.2) if and only if for every $\psi \in C_c^\infty(\mathbb{R}^{2d})$,

$$(2.3) \quad \begin{aligned} \int_{\mathbb{R}^{2d}} f_t \psi \, dv dx &= \int_{\mathbb{R}^{2d}} f^{in} \psi \, dv dx + \int_0^t \int_{\mathbb{R}^{2d}} f_s (v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi) \, dv dx ds \\ &- \sigma \int_0^t \left(\int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \, dv dx \right) dW_s \\ &+ \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} (v - v_c) f_s \cdot \left[\nabla_v ((v - v_c) \cdot \nabla_v \psi) \right] \, dv dx ds \quad a.s. \, \omega \in \Omega. \end{aligned}$$

Proof. The proof is almost the same as in Lemma 13 from [16], but we provide a proof for readers' convenience. Note that the following relation between Itô and Stratonovich integrals holds:

$$\int_0^t h_s \circ dW_s = \int_0^t h_s dW_s + \frac{1}{2} \langle h, W \rangle_t,$$

where $\langle \cdot, \cdot \rangle$ denotes the joint quadratic variation (see [24]). In our case, h_s corresponds to $\int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \, dv dx$. Then, to deal with $\langle h, W \rangle_t$, one needs to specify the stochastic part of h_s . Here, if we replace ψ in (3.3) by $(v - v_c) \cdot \nabla_v \psi$, we can find out that the stochastic part of h_s becomes $-\sigma \int_0^t \left[\int_{\mathbb{R}^{2d}} ((v - v_c) f_s) \cdot \nabla_v (v \cdot \nabla_v \psi) \, dv dx \right] dW_s$. This means

$$\left\langle \int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \, dv dx, W \right\rangle_t = -\sigma \int_0^t \int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v [(v - v_c) \cdot \nabla_v \psi] \, dv dx ds,$$

and we may conclude the proof here. \square

Once we reformulate relation (2.2) to Itô form, we can show that the process f_t satisfies the following pointwise relation if f is sufficiently smooth.

Lemma 2.2. *Suppose that $f_t \in L^\infty(\Omega; \mathcal{C}([0, T]; \mathcal{C}^2(\mathbb{R}^{2d})))$ has a continuous \mathcal{F}_t -adapted modification and has a compact support in x and v . Then, f_t satisfies relation (2.3) if and only if f_t satisfies the following relation:*

$$(2.4) \quad \begin{aligned} &f_t(x, v) \\ &= f^{in}(x, v) - \int_0^t \left(v \cdot \nabla_x f_s + \nabla_v \cdot (F_a[f_s] f_s) \right) ds + \sigma \int_0^t \left[\nabla_v \cdot ((v - v_c) f_s) \right] dW_s \\ &+ \frac{\sigma^2}{2} \int_0^t \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] ds, \quad \mathbb{P} \otimes dx \otimes dv\text{-a.s.} \end{aligned}$$

Proof. First, we assume that f satisfies (2.3). Since f_t is smooth and compactly supported, we use Fubini's theorem to show that (2.3) is equivalent to

$$(2.5) \quad \begin{aligned} \int_{\mathbb{R}^{2d}} f_t \psi \, dv dx &= \int_{\mathbb{R}^{2d}} f^{in} \psi \, dv dx - \int_0^t \int_{\mathbb{R}^{2d}} \left[v \cdot \nabla_x f_s + \nabla_v \cdot (F_a[f_s] f_s) \right] \psi \, dv dx ds \\ &+ \sigma \int_0^t \left(\int_{\mathbb{R}^{2d}} \nabla_v \cdot [(v - v_c) f_s] \psi \, dv dx \right) dW_s \\ &+ \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] \psi \, dv dx ds \quad a.s. \, \omega \in \Omega. \end{aligned}$$

Note that for each $\psi \in \mathcal{D}(\mathbb{R}^{2d})$, it satisfies the relation (2.5) outside \mathbb{P} -zero set that depends on the choice of ψ . We recall from standard functional analysis that $\mathcal{D}(\mathbb{R}^{2d})$ is separable,

i.e. there exists $\{\psi_i\}_{i=1}^\infty \subseteq \mathcal{D}(\mathbb{R}^{2d})$ which is dense in $\mathcal{D}(\mathbb{R}^{2d})$. Here, we choose $\Omega_i \subset \Omega$ such that $\mathbb{P}(\Omega_i) = 1$ and (2.2) holds for f_t and ψ_i over Ω_i . Let $\tilde{\Omega} := \cap_{i=1}^\infty \Omega_i$. Then $\mathbb{P}(\tilde{\Omega}) = 1$ and (2.2) holds for any ψ_i and f_t over $\tilde{\Omega}$.

Now, we show f_t satisfies the relation (2.4). Define functionals $\mathcal{L}_t[f]$ and $\mathcal{M}[f_t]$ as follows:

$$\begin{aligned} \mathcal{L}[f_t](x, v) &:= f_t - f^{in} + \int_0^t \left(v \cdot \nabla_x f_s + \nabla_v \cdot (F_a[f_s] f_s) \right) ds \\ &\quad - \frac{\sigma^2}{2} \int_0^t \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] ds, \\ \mathcal{M}[f_t] &:= \nabla_v \cdot [(v - v_c) f_t]. \end{aligned}$$

For given $(x^*, v^*) \in \mathbb{R}^{2d}$, we can choose a sequence $\{\rho_i\} \subset \mathcal{D}(\mathbb{R}^{2d})$, using standard mollifier technique or other tools, such that for any $i \in \mathbb{N}$,

$$\left| (\rho_i * \mathcal{L}[f_t])(x^*, v^*) - \mathcal{L}[f_t](x^*, v^*) \right| + \int_0^t \left| (\rho_i * \mathcal{M}[f_s])(x^*, v^*) - \mathcal{M}[f_s](x^*, v^*) \right|^2 ds \leq \frac{1}{2^{i+1}},$$

where the regularity and compact support of f can be used to guarantee the above inequality. We also use the denseness of $\{\psi_i\}$ to obtain $\{\tilde{\psi}_i\} \subseteq \{\psi_i\}$ which satisfies, for any $i \in \mathbb{N}$,

$$\left| (\rho_i - \tilde{\psi}_i) * \mathcal{L}_t[f](x^*, v^*) \right| + \int_0^t \left| (\rho_i - \tilde{\psi}_i) * \mathcal{M}[f_s](x^*, v^*) \right|^2 ds \leq \frac{1}{2^{i+1}},$$

Thus we have

$$(2.6) \quad \left(\tilde{\psi}_i * \mathcal{L}_t[f] \right)(x^*, v^*) \longrightarrow \mathcal{L}_t[f](x^*, v^*).$$

Moreover, we use Itô isometry to get

$$(2.7) \quad \mathbb{E} \left[\left(\int_0^t (\tilde{\psi}_i * \mathcal{M}[f_s] - \mathcal{M}[f_s]) dW_s \right)^2 \right] = \mathbb{E} \left[\int_0^t (\tilde{\psi}_i * \mathcal{M}[f_s] - \mathcal{M}[f_s])^2 ds \right] \longrightarrow 0.$$

Hence, we can obtain the convergence of (2.5) with $\psi = \tilde{\psi}_i(x^* - x, v^* - v)$ towards (2.4) at (x^*, v^*) as $i \rightarrow \infty$, by combining (2.6) and (2.7). We perform this procedure to obtain that for every $(x^*, v^*) \in \mathbb{R}^{2d}$, f satisfies relation (2.4) \mathbb{P} -a.s. and this gives

$$\mathbb{E} \left[\left| \mathcal{L}[f_t] - \int_0^t \mathcal{M}[f_s] dW_s \right| (x, v) \right] = 0,$$

for every $(x, v) \in \mathbb{R}^{2d}$. Thus, we use Fubini theorem to get

$$\mathbb{E} \left[\int_{\mathbb{R}^{2d}} \left| \mathcal{L}[f_t] - \int_0^t \mathcal{M}[f_s] dW_s \right| dv dx \right] = 0.$$

This implies our first assertion.

Next, we assume that f satisfies (2.4) $\mathbb{P} \otimes dx \otimes dv$ -a.s. Then by (deterministic) Fubini's theorem, the following relation is easily obtained: for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,

$$\int_{\mathbb{R}^{2d}} f_t \psi \, dv dx = \int_{\mathbb{R}^{2d}} f^{in} \psi \, dv dx + \int_0^t \int_{\mathbb{R}^{2d}} f_s (v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi) \, dv dx ds$$

$$\begin{aligned}
& + \sigma \int_{\mathbb{R}^{2d}} \left(\int_0^t \nabla_v \cdot [(v - v_c) f_s] \psi \, dW_s \right) dv dx \\
& + \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} (v - v_c) f_s \cdot \left[\nabla_v \left((v - v_c) \cdot \nabla_v \psi \right) \right] dv dx ds \quad \text{a.s. } \omega \in \Omega.
\end{aligned}$$

Since f_t is in $L^\infty(\Omega; \mathcal{C}([0, T]; \mathcal{C}^2(\mathbb{R}^{2d})))$ and compactly supported, we have

$$\int_{\mathbb{R}^{2d}} \left(\int_0^t \left| \nabla_v \cdot [(v - v_c) f_s] \psi \right|^2 ds \right)^{1/2} dv dx < \infty, \quad \text{a.s. } \omega \in \Omega.$$

Then, we can use the stochastic Fubini theorem (see [36] and references therein) and deterministic Fubini's theorem to get

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \left(\int_0^t \nabla_v \cdot [(v - v_c) f_s] \psi \, dW_s \right) dv dx & = \int_0^t \left(\int_{\mathbb{R}^{2d}} \nabla_v \cdot [(v - v_c) f_s] \psi \, dv dx \right) dW_s \\
& = - \int_0^t \left(\int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \, dv dx \right) dW_s,
\end{aligned}$$

which implies our desired result. \square

Remark 2.1. 1. If a strong solution f_t to (1.3) satisfies conditions in Lemma 2.2, then f_t satisfies the relation (2.4).

2. If f_t is a \mathcal{F}_t -semimartingale with the regularity $f \in L^\infty(\Omega; \mathcal{C}([0, T]; \mathcal{C}^{3,\delta}(\mathbb{R}^{2d})))$ for some $\delta \in (0, 1)$, we may use Lemma 2.2 in [8] to obtain that relation (2.4) is equivalent to (2.1).

3. We say f_t is a classical solution to (2.1) if it is a \mathcal{F}_t -semimartingale satisfying relation (2.1) pointwisely and the regularity condition $f_t \in L^\infty(\Omega; \mathcal{C}([0, T]; \mathcal{C}^{3,\delta}(\mathbb{R}^{2d})))$ for some $\delta \in (0, 1)$.

Next, we study the propagation of velocity moments along the stochastic flow (2.1)₁. For a random density function f_t , we set

$$(2.8) \quad M_0(t) := \int_{\mathbb{R}^{2d}} f_t dv dx, \quad M_1(t) := \int_{\mathbb{R}^{2d}} v f_t dv dx, \quad M_2(t) := \int_{\mathbb{R}^{2d}} |v|^2 f_t dv dx, \quad t \geq 0.$$

Lemma 2.3. Let f_t be a classical solution to (2.1) which is compactly supported in x and v and satisfies

$$M_0(0) = 1, \quad M_1(0) = 0.$$

Then, we have

$$M_0(t) = 1, \quad M_1(t) = 0, \quad M_2(t) \leq M_2(0) \exp(-2\phi_m t - 2\sigma W_t), \quad t \geq 0.$$

Proof. • (Conservation of mass): It follows from Remark 2.1 that

$$\begin{aligned}
(2.9) \quad f_t(x, v) & = f^{in}(x, v) - \int_0^t \left(v \cdot \nabla_x f_s + \nabla_v \cdot (F_a[f_s] f_s) \right) ds + \sigma \int_0^t \left(\nabla_v \cdot ((v - v_c) f_s) \right) dW_s \\
& \quad + \frac{\sigma^2}{2} \int_0^t \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] ds.
\end{aligned}$$

We integrate (2.9) over $(x, v) \in \mathbb{R}^{2d}$ to get

$$\begin{aligned}
 & \int_{\mathbb{R}^{2d}} f_t(x, v) dv dx \\
 &= \int_{\mathbb{R}^{2d}} f^{in}(x, v) dv dx - \int_{\mathbb{R}^{2d}} \int_0^t \left(v \cdot \nabla_x f_s + \nabla_v \cdot (F_a[f_s] f_s) \right) ds dv dx \\
 &+ \sigma \int_{\mathbb{R}^{2d}} \left[\int_0^t \left(\nabla_v \cdot ((v - v_c) f_t) \right) dW_s \right] dv dx \\
 &+ \frac{\sigma^2}{2} \int_{\mathbb{R}^{2d}} \int_0^t \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] ds dv dx \\
 &=: \int_{\mathbb{R}^{2d}} f^{in}(x, v) dv dx + \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
 \end{aligned}$$

Next, we show that the terms \mathcal{I}_{1i} are zero using deterministic and stochastic Fubini's theorems.

◇ (Estimate of \mathcal{I}_{11} and \mathcal{I}_{13}): Since f_t has a compact support in (x, v) , we can use deterministic Fubini's theorem to see

$$\begin{aligned}
 \mathcal{I}_{11} + \mathcal{I}_{13} &= - \int_0^t \int_{\mathbb{R}^{2d}} \left(\nabla_x \cdot (v f_s) + \nabla_v \cdot (F_a[f_s] f_s) \right) dv dx ds \\
 &+ \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} \nabla_v \cdot \left[(v - v_c) \nabla_v \cdot ((v - v_c) f_s) \right] dv dx ds \\
 &= 0.
 \end{aligned}$$

◇ (Estimate of \mathcal{I}_{12}): As in the proof of Lemma 2.2, we can use the stochastic Fubini theorem to get

$$\mathcal{I}_{12} = \int_0^t \left(\int_{\mathbb{R}^{2d}} \nabla_v \cdot ((v - v_c) f_t) dv dx \right) dW_s = 0.$$

• (Conservation of momentum): In this case, we multiply v to (2.9) and use the same argument for conservation of mass to derive

$$M_1(t) = M_1(0) = 0, \quad t \geq 0.$$

• (Dissipation estimate): We multiply (2.9) by $|v|^2$ and use stochastic Fubini's theorem to have

$$(2.10) \quad dM_2(t) = \left(2\sigma^2 M_2(t) + \int_{\mathbb{R}^{2d}} 2v \cdot F_a[f_s] f_s dv dx \right) dt - 2\sigma M_2(t) dW_t,$$

where we used the relation $M_1(t) = 0$.

We use (2.10) to get

$$\begin{aligned}
M_2(t) &= M_2(0) + \int_0^t \left[\left(\int_{\mathbb{R}^{2d}} 2v \cdot F_a[f_s] f_s dv dx \right) + 2\sigma^2 M_2(s) \right] ds - 2\sigma \int_0^t M_2(s) dW_s \\
&= M_2(0) + 2 \int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x) (v_* - v) \cdot v f_s(x_*, v_*) f_s(x, v) dv_* dx_* dv dx ds \\
&\quad + 2\sigma^2 \int_0^t M_2(s) ds - 2\sigma \int_0^t M_2(s) dW_s \\
&= M_2(0) - \int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x) |v - v_*|^2 f_s(x_*, v_*) f_s(x, v) dv_* dx_* dv dx ds \\
&\quad + 2\sigma^2 \int_0^t M_2(s) ds - 2\sigma \int_0^t M_2(s) dW_s \\
&\leq M_2(0) - \phi_m \int_0^t \int_{\mathbb{R}^{4d}} |v - v_*|^2 f_s(x_*, v_*) f_s(x, v) dv_* dx_* dv dx ds \\
&\quad + 2\sigma^2 \int_0^t M_2(s) ds - 2\sigma \int_0^t M_2(s) dW_s \\
&\leq M_2(0) - 2(\phi_m - \sigma^2) \int_0^t M_2(s) ds - 2\sigma \int_0^t M_2(s) dW_s.
\end{aligned}$$

Then we use Lemma A.1 and Lemma A.2 to get

$$M_2(t) \leq M_2(0) \exp(-2\phi_m t - 2\sigma W_t).$$

□

Remark 2.2. *It is well-known that*

$$\limsup_{t \rightarrow \infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1, \quad \text{for a.s. } \omega \in \Omega.$$

Thus, if $\phi_m > 0$, there exists $t^* = t^*(\omega)$ such that for a.s. ω ,

$$-2\phi_m t - 2\sigma W_t \leq -\phi_m t, \quad t \geq t^*(\omega).$$

Hence, we have an exponential decay of $M_2(t)$ for each sample path:

$$M_2(t) \leq M_2(0) \exp(-\phi_m t), \quad \text{for } t \geq t^*(\omega).$$

Moreover, if $\phi_m > \sigma^2$, we also obtain the emergence of asymptotic flocking discussed in [18]:

$$\mathbb{E}[M_2(t)] \leq M_2(0) \exp(-2(\phi_m - \sigma^2)t), \quad t \geq 0.$$

Finally, we are ready to provide a framework (\mathcal{F}) and main results below:

- ($\mathcal{F}1$): The initial datum f^{in} is nonnegative, compactly supported in x and v and independent of ω .
- ($\mathcal{F}2$): For $k \geq 1$, f^{in} and ϕ are assumed to be in $\mathcal{C}^k(\mathbb{R}^{2d})$ and $\mathcal{C}^\infty(\mathbb{R}^{2d})$, respectively.
- ($\mathcal{F}3$): The zeroth and first moment of f^{in} are normalized:

$$(2.11) \quad \int_{\mathbb{R}^{2d}} f^{in} dv dx = 1, \quad \int_{\mathbb{R}^{2d}} v f^{in} dv dx = 0.$$

Note that due to Lemma 2.1, the conditions (2.11) would imply

$$\int_{\mathbb{R}^{2d}} f_t dv dx = 1, \quad \int_{\mathbb{R}^{2d}} v f_t dv dx = 0, \quad t \geq 0.$$

Under the framework (\mathcal{F}) , our main results are summarized as follows.

Theorem 2.1. *Let $T \in (0, \infty)$ and assume that f^{in} and ϕ satisfies the framework (\mathcal{F}) . Then, there exists a strong solution f_t to (2.1) on $[0, T]$ such that*

$$\mathbb{E} \|f_t\|_{L^\infty} \leq \|f^{in}\|_{L^\infty} \exp \left\{ \left(d\phi_M + \frac{(\sigma d)^2}{2} \right) t \right\}, \quad \mathbb{E} M_2(t) \leq M_2(0) \exp(2\sigma^2 t), \quad t \in [0, T].$$

Moreover, if a strong solution f_t exists on $(0, \infty)$ and $\phi_m := \inf_{x \in \mathbb{R}^N} \phi(x) > \sigma^2$, then one obtains an asymptotic flocking estimate:

$$\mathbb{E} M_2(t) \leq M_2(0) \exp(-2(\phi_m - \sigma^2)t), \quad t > 0.$$

Proof. For the proof, we will first regularize the initial datum using standard mollification and then solve the linearized system for (2.1) to get a sequence of approximate solutions. Then, we use the stopping time argument to get a strong solution for (2.1) with the given initial datum. The detailed proof will be presented in Section 4. \square

Remark 2.3. *Note that for $k > 3$, a strong solution f_t to (2.1) can be shown to satisfy the equation (2.1) pointwisely under our framework. Moreover, we can use the stability estimates for the classical solutions to get the uniqueness of solutions.*

3. CONSTRUCTION OF APPROXIMATE SOLUTIONS

In this section, we provide a global well-posedness of (2.1) with regularized one for the given initial datum satisfying (\mathcal{F}) . In particular, the first velocity moment is assumed to be zero, i.e., $v_c(t) = 0$.

Let $f^{in, \varepsilon}$ be a smooth mollification of the given initial datum f^{in} satisfying the framework (\mathcal{F}) . Then, with this regularized initial datum, consider the Cauchy problem (2.1) with regularized initial datum:

$$(3.1) \quad \begin{aligned} \partial_t f_t^\varepsilon + v \cdot \nabla_x f_t^\varepsilon + \nabla_v \cdot (F_a[f_t^\varepsilon] f_t^\varepsilon) &= \sigma \nabla_v \cdot (v f_t^\varepsilon) \circ \dot{W}_t, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}, \\ f_0^\varepsilon(x, v) &= f^{in, \varepsilon}(x, v). \end{aligned}$$

Note that due to the framework (\mathcal{F}) , the initial datum f^{in} and its partial derivatives up to order k are uniformly continuous on \mathbb{R}^{2d} and there exists a constant $R_0 > 0$, such that

$$\text{supp } f^{in} \subseteq B_{R_0}(0),$$

where $B_{R_0}(0)$ is a ball of radius R_0 centered at $0 \in \mathbb{R}^{2d}$. As mentioned above, we use a mollifier to obtain a family of regularized initial data $f^{in, \varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, $\varepsilon \in (0, 1)$, so that the regularized datum satisfies the following conditions:

- $(\mathcal{F}^\varepsilon 1)$: $\{f^{in, \varepsilon}\}$ are nonnegative, compactly supported, converge to f^{in} in $\mathcal{C}^k(\mathbb{R}^{2d})$ and

$$\|f^{in, \varepsilon}\|_{W^{k, \infty}} \leq \|f^{in}\|_{W^{k, \infty}}.$$

- ($\mathcal{F}^\varepsilon 2$): $\{M_2^\varepsilon\}(0)$ is uniformly bounded with respect to ε and converges to $M_2(0)$ as $\varepsilon \rightarrow 0$.

- ($\mathcal{F}^\varepsilon 3$): The zeroth and first moment of $f^{in,\varepsilon}$ are initially constrained:

$$\int_{\mathbb{R}^{2d}} f^{in,\varepsilon} dx dv = 1, \quad \int_{\mathbb{R}^{2d}} v f^{in,\varepsilon} dv dx = 0.$$

- ($\mathcal{F}^\varepsilon 4$): $f^{in,\varepsilon}$ has a compact support in x and v , and satisfy

$$\text{supp} f^{in,\varepsilon} \subseteq B_{R_0+1}(0).$$

In the following three subsections, we will provide a global well-posedness for system (3.1).

3.1. Construction of approximate solutions. In this subsection, we provide a sequence of approximate solutions to (3.1) using the successive approximations.

First, the zeroth iterate $f_t^{0,\varepsilon}$ is simply defined as the mollified initial datum:

$$f_t^{0,\varepsilon}(x, v) := f^{in,\varepsilon}(x, v), \quad (x, v) \in \mathbb{R}^{2d}.$$

For $n \geq 1$, suppose that the $(n-1)$ -th iterate $f_t^{n-1,\varepsilon}$ is given. Then, n -th iterate is defined as the solution to the linear equation with fixed initial datum:

$$(3.2) \quad \begin{cases} \partial_t f_t^{n,\varepsilon} + v \cdot \nabla_x f_t^{n,\varepsilon} + \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) = \sigma \nabla_v \cdot (v f_t^{n,\varepsilon}) \circ \dot{W}_t, & n \geq 1, \\ f_0^{n,\varepsilon}(x, v) = f^{in,\varepsilon}(x, v). \end{cases}$$

The linear system (3.2) can be solved by the method of stochastic characteristics. Let $\varphi_t^{n,\varepsilon}(x, v) := (X_t^{n,\varepsilon}(x, v), V_t^{n,\varepsilon}(x, v))$ be the forward stochastic characteristics, which is a solution to the following SDE:

$$(3.3) \quad \begin{cases} dX_t^{n,\varepsilon} = V_t^{n,\varepsilon} dt, \\ dV_t^{n,\varepsilon} = F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon}) dt - \sigma V_t^{n,\varepsilon} \circ dW_t, \\ (X_t^{n,\varepsilon}(0), V_t^{n,\varepsilon}(0)) = (x, v) \in \text{supp} f^{in,\varepsilon}. \end{cases}$$

Note that the SDE (3.3) is equivalent to the following Itô SDE [15] :

$$(3.4) \quad \begin{cases} dX_t^{n,\varepsilon} = V_t^{n,\varepsilon} dt, \\ dV_t^{n,\varepsilon} = \left(F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon}) + \frac{\sigma^2}{2} V_t^{n,\varepsilon} \right) dt - \sigma V_t^{n,\varepsilon} dW_t, \\ (X_t^{n,\varepsilon}(0), V_t^{n,\varepsilon}(0)) = (x, v) \in \text{supp} f^{in,\varepsilon}. \end{cases}$$

Here, we can deduce from our framework, Lemma 3.1 and Theorem 3.2 in [8] that for any $m \geq 3$, (3.3) has a unique solution $f_t^{n,\varepsilon}$ which is a \mathcal{C}^m -semimartingale for every $n \geq 0$ and the characteristics (3.3) becomes a \mathcal{C}^m -diffeomorphism. Then, $f_t^{n,\varepsilon}$ can also be represented by the following integral formula:

$$(3.5) \quad f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}(x, v)) = f^{in,\varepsilon}(x, v) \exp \left[- \int_0^t \nabla_v \cdot F_a[f^{n-1,\varepsilon}](s, \varphi_s^{n,\varepsilon}(x, v)) ds + d\sigma W_t \right].$$

Note that if $f^{in,\varepsilon}$ is nonnegative, then surely $f_t^{n,\varepsilon}$ is also nonnegative as well. Before we finish this subsection, we also remark that the linear, first-order Stratonovich equation (3.2) is equivalent to the following parabolic Itô equation (see Corollary 3.3. in [8]):

$$(3.6) \quad \begin{cases} \partial_t f_t^{n,\varepsilon} + v \cdot \nabla_x f_t^{n,\varepsilon} + \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) \\ \quad = \sigma \nabla_v \cdot (v f_t^{n,\varepsilon}) \dot{W}_t + \frac{\sigma^2}{2} \nabla_v \cdot [v \nabla_v \cdot (v f_t)], \quad n \geq 1, \\ f_0^{n,\varepsilon}(x, v) = f^{in,\varepsilon}(x, v). \end{cases}$$

3.2. Estimates on approximate solutions. In this subsection, we provide several estimates for the approximate solutions for (3.2). To be more precise, we would try to obtain n and ε -independent estimates for the later sections. Before we move on, we define p -th velocity moments $M_p^{n,\varepsilon}(t)$, $p = 0, 1, 2$:

$$\begin{aligned} M_0^{n,\varepsilon}(t) &:= \int_{\mathbb{R}^{2d}} f_t^{n,\varepsilon} dv dx, & M_1^{n,\varepsilon}(t) &:= \int_{\mathbb{R}^{2d}} v f_t^{n,\varepsilon} dv dx, \\ M_2^{n,\varepsilon}(t) &:= \int_{\mathbb{R}^{2d}} |v|^2 f_t^{n,\varepsilon} dv dx, & M_p^{n,\varepsilon}(0) &:= M_{p0}^\varepsilon. \end{aligned}$$

Before we provide the uniform estimates for the p -th ($p = 0, 1, 2$) moments, we set

$$(3.7) \quad M_{20}^\infty := \sup_{\varepsilon \in (0,1)} M_2^\varepsilon(0), \quad \phi_M := \sup_{x \in \mathbb{R}^N} \phi(x) < \infty, \quad \gamma := \max\{M_{20}^\infty, \phi_M\}.$$

We also present a technical lemma from [3] for later discussion.

Lemma 3.1. [3] *Let $T \in (0, \infty]$ and $(a_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative continuous functions on $[0, T]$ satisfying*

$$a_n(t) \leq A + B \int_0^t a_{n-1}(s) ds + C \int_0^t a_n(s) ds, \quad t \in [0, T], \quad n \geq 1,$$

where A, B and C are nonnegative constants.

(1) *If $A = 0$, there exists a constant $\Lambda \geq 0$ depending on B, C and $\sup_{t \in [0, T]} a_0(t)$ such that*

$$a_n(t) \leq \frac{(\Lambda t)^n}{n!}, \quad t \in [0, T], \quad n \in \mathbb{N}.$$

(2) *If $A > 0$ and $C = 0$, there exists a constant $\Lambda \geq 0$ depending on A, B and $\sup_{t \in [0, T]} a_0(t)$ such that*

$$a_n(t) \leq \Lambda \exp(\Lambda t), \quad t \in [0, T], \quad n \in \mathbb{N}.$$

Remark 3.1. 1. *In (2) of Lemma 3.1, Λ can be explicitly written as*

$$\Lambda := \max \left\{ A, B, \sup_{t \in [0, T]} a_0(t) \right\}.$$

2. *We can also use the similar argument to obtain the following estimate for (2):*

$$a_n(t) \leq (\Lambda + \kappa_t) \exp(\Lambda t), \quad t \in [0, T], \quad n \in \mathbb{N},$$

where $\Lambda := \max\{A, B\}$ and $\kappa_t := \sup_{0 \leq s \leq t} a_0(s)$.

Proposition 3.1. *For every $n \in \mathbb{N}$ and $T \in (0, \infty)$, let $f_t^{n,\varepsilon}$ be a solution to (3.2). Then, for any $t \in (0, T)$ we have*

$$M_0^{n,\varepsilon}(t) = 1, \quad M_1^{n,\varepsilon}(t) = 0, \quad M_2^{n,\varepsilon}(t) \leq (\gamma + K_t) \exp\{(\gamma + \phi_M)t - 2\sigma W_t\},$$

where γ is a constant in (3.7) and K_t is defined as

$$K_t := M_{20}^\infty \sup_{0 \leq s \leq t} \exp(-\phi_M s + 2\sigma W_s).$$

Proof. We will use the same arguments as in Lemma 2.3. First note that $f_t^{n,\varepsilon}$ satisfies relation (3.6) and $f_t^{n,\varepsilon}$ is compactly supported in x and v , since $f^{in,\varepsilon}$ is compactly supported in the phase space and $\varphi_t^{n,\varepsilon}$ is a C^m -diffeomorphism. Thus, we may follow the arguments in Lemma 2.3 to derive the conservation estimates.

For the dissipation estimate of $M_2^{n,\varepsilon}$, we use a similar argument to Lemma 2.3 to have

$$\begin{aligned} M_2^{n,\varepsilon}(t) &= M_2^\varepsilon(0) + 2 \int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x)(v_* - v) \cdot v f_s^{n-1,\varepsilon}(x_*, v_*) f_s^{n,\varepsilon}(x, v) dv_* dx_* dv dx ds \\ &\quad + 2\sigma^2 \int_0^t M_2^{n,\varepsilon}(s) ds - 2\sigma \int_0^t M_2^{n,\varepsilon}(s) dW_s \\ &\leq M_2^\varepsilon(0) + 2 \int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x) v_* \cdot v f_s^{n-1,\varepsilon}(x_*, v_*) f_s^{n,\varepsilon}(x, v) dv_* dx_* dv dx \\ &\quad + 2\sigma^2 \int_0^t M_2^{n,\varepsilon}(s) ds - 2\sigma \int_0^t M_2^{n,\varepsilon}(s) dW_s \\ &\leq M_2^\varepsilon(0) + \phi_M \int_0^t M_2^{n-1,\varepsilon}(s) ds + (\phi_M + 2\sigma^2) \int_0^t M_2^{n,\varepsilon}(s) ds - 2\sigma \int_0^t M_2^{n,\varepsilon}(s) dW_s, \end{aligned}$$

where we used Young's inequality on the second inequality. In differential form, we have

$$(3.8) \quad dM_2^{n,\varepsilon}(t) \leq \left\{ \phi_M M_2^{n-1,\varepsilon}(t) + (\phi_M + 2\sigma^2) M_2^{n,\varepsilon}(t) \right\} dt - 2\sigma M_2^{n,\varepsilon}(t) dW_t.$$

Then, it follows from (3.8) and comparison theorem (in Lemma A.2) that

$$M_2^{n,\varepsilon}(t) \leq X_t,$$

where the process X_t satisfies

$$\begin{cases} dX_t = \left\{ \phi_M M_2^{n-1,\varepsilon}(t) + (\phi_M + 2\sigma^2) X_t \right\} dt - 2\sigma X_t dW_t, & t > 0, \\ X_0 = M_2^\varepsilon(0). \end{cases}$$

It follows from Lemma A.1 that X_t can be represented as

$$X_t = X_0 \exp(\phi_M t - 2\sigma W_t) + \phi_M \int_0^t \exp\{\phi_M(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s) ds.$$

This implies

$$M_2^{n,\varepsilon}(t) \leq M_2^\varepsilon(0) \exp(\phi_M t - 2\sigma W_t) + \phi_M \int_0^t \exp\{\phi_M(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s) ds.$$

Now, we set

$$a_n(t) := M_2^{n,\varepsilon}(t) \exp\{-\phi_M t + 2\sigma W_t\}.$$

Then, it satisfies

$$a_{n+1}(t) \leq M_{20}^\varepsilon + \phi_M \int_0^t a_n(s) ds.$$

We use Lemma 3.1 in the way from Remark 3.1 to get

$$a_n(t) \leq (\gamma + K_t)e^{\gamma t}, \quad t \in (0, T).$$

This yields the desired result. \square

We also provide uniform estimates for the stochastic characteristic flows.

Proposition 3.2. *For each $n \in \mathbb{N}$ and $T \in (0, \infty)$, let $(X_t^{n,\varepsilon}, V_t^{n,\varepsilon})$ be the stochastic characteristic flow for (3.2) with this initial data:*

$$(X_0^{n,\varepsilon}, V_0^{n,\varepsilon}) = (x, v) \in \text{supp} f^{in,\varepsilon}.$$

Then for $t \in (0, T)$, we have

$$\begin{aligned} (i) \quad |V_t^{n,\varepsilon}|^2 &\leq \left\{ |v|^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds \right\} \exp(\phi_M t - 2\sigma W_t). \\ (ii) \quad |X_t^{n,\varepsilon}|^2 &\leq 2 \left(|x|^2 + t \int_0^t \left\{ |v|^2 + \phi_M \int_0^s (\gamma + K_\tau) \exp(\gamma \tau) d\tau \right\} \exp(\phi_M s - 2\sigma W_s) ds \right). \end{aligned}$$

Proof. (i) It follows from Itô's lemma and (3.4) that

$$\begin{aligned} d|V_t^{n,\varepsilon}|^2 &= 2V_t^{n,\varepsilon} \cdot dV_t^{n,\varepsilon} + dV_t^{n,\varepsilon} \cdot dV_t^{n,\varepsilon} \\ &= 2 \left(F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon}) \cdot V_t^{n,\varepsilon} + \sigma^2 |V_t^{n,\varepsilon}|^2 \right) dt - 2\sigma |V_t^{n,\varepsilon}|^2 dW_t \\ &\leq \left(2 \int_{\mathbb{R}^{2d}} \phi(x_* - X_t^{n,\varepsilon})(v_* \cdot V_t^{n,\varepsilon}) f_t^{n-1,\varepsilon}(x_*, v_*) dv_* dx_* + \sigma^2 |V_t^{n,\varepsilon}|^2 \right) dt - 2\sigma |V_t^{n,\varepsilon}|^2 dW_t \\ &\leq \left(\phi_M M_2^{n-1,\varepsilon}(t) + (\phi_M + 2\sigma^2) |V_t^{n,\varepsilon}|^2 \right) dt - 2\sigma |V_t^{n,\varepsilon}|^2 dW_t, \end{aligned}$$

where $dV_t^{n,\varepsilon} \cdot dV_t^{n,\varepsilon}$ denotes a handy notation for a quadratic variation of $V_t^{n,\varepsilon}$.

We use Lemma A.1 to get

$$\begin{aligned} |V_t^{n,\varepsilon}|^2 &\leq |v|^2 \exp(\phi_M t - 2\sigma W_t) + \phi_M \int_0^t \exp\{\phi_M(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s) ds \\ &\leq \left\{ |v|^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds \right\} \exp(\phi_M t - 2\sigma W_t). \end{aligned}$$

(ii) For the estimate of spatial process, we use Cauchy-Schwarz inequality to get

$$\begin{aligned} |X_t^{n,\varepsilon}|^2 &\leq \left(|x|^2 + \int_0^t |V_s^{n,\varepsilon}|^2 ds \right)^2 \\ &\leq 2 \left(|x|^2 + t \int_0^t |V_s^{n,\varepsilon}|^2 ds \right) \\ &\leq 2 \left(|x|^2 + t \int_0^t \left\{ |v|^2 + \phi_M \int_0^s (\gamma + K_\tau) \exp(\gamma \tau) d\tau \right\} \exp(\phi_M s - 2\sigma W_s) ds \right). \end{aligned}$$

This yields the desired result.

□

As a corollary of Proposition 3.2, we have estimates for the sizes of velocity and spatial supports: We set

$$\begin{aligned}\mathcal{X}^{n,\varepsilon}(t) &:= \sup\{|x| : f_t^{n,\varepsilon}(x, v) \neq 0 \text{ for some } v \in \mathbb{R}^d\}, \\ \mathcal{V}^{n,\varepsilon}(t) &:= \sup\{|v| : f_t^{n,\varepsilon}(x, v) \neq 0 \text{ for some } x \in \mathbb{R}^d\}.\end{aligned}$$

Corollary 3.1. *For each $n \in \mathbb{N}$ and $T \in (0, \infty]$, let $(X_t^{n,\varepsilon}, V_t^{n,\varepsilon})$ be the stochastic characteristic flow for (3.2) with the initial data:*

$$(X_0^{n,\varepsilon}, V_0^{n,\varepsilon}) = (x, v) \in \text{supp}f^{in,\varepsilon}.$$

Then for $t \in (0, T)$, we have

$$|\mathcal{V}^{n,\varepsilon}(t)| \leq |\mathcal{V}^\infty(t)| \quad \text{and} \quad |\mathcal{X}^{n,\varepsilon}(t)| \leq |\mathcal{X}^\infty(t)|,$$

where $\mathcal{X}^\infty(t)$ and $\mathcal{V}^\infty(t)$ are given by the following relations:

$$\begin{aligned}|\mathcal{X}^\infty(t)|^2 &:= 2 \left((R_0 + 1)^2 + t \int_0^t \left\{ (R_0 + 1)^2 + \phi_M \int_0^s (\gamma + K_\tau) \exp(\gamma\tau) d\tau \right\} \exp(\phi_M s - 2\sigma W_s) ds \right), \\ |\mathcal{V}^\infty(t)|^2 &:= \left\{ (R_0 + 1)^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds \right\} \exp(\phi_M t - 2\sigma W_t).\end{aligned}$$

Proof. It follows from Proposition 3.2 that

$$\begin{aligned}|\mathcal{V}^{n,\varepsilon}(t)|^2 &\leq \left\{ |\mathcal{V}^{n,\varepsilon}(0)|^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds \right\} \exp(\phi_M t - 2\sigma W_t) \\ &\leq \left\{ (R_0 + 1)^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds \right\} \exp(\phi_M t - 2\sigma W_t) = |\mathcal{V}^\infty(t)|^2.\end{aligned}$$

This yields the first estimate for velocity support. On the other hand, we also use Proposition 3.2 to get

$$\begin{aligned}|\mathcal{X}^{n,\varepsilon}(t)|^2 &\leq 2 \left(|\mathcal{X}^{n,\varepsilon}(0)|^2 + t \int_0^t \left\{ |\mathcal{V}^{n,\varepsilon}(0)|^2 + \phi_M \int_0^s (\gamma + K_\tau) \exp(\gamma\tau) d\tau \right\} \exp(\phi_M s - 2\sigma W_s) ds \right) \\ &\leq 2 \left((R_0 + 1)^2 + t \int_0^t \left\{ (R_0 + 1)^2 + \phi_M \int_0^s (\gamma + K_\tau) \exp(\gamma\tau) d\tau \right\} \exp(\phi_M s - 2\sigma W_s) ds \right) \\ &=: |\mathcal{X}^\infty(t)|^2.\end{aligned}$$

□

Remark 3.2. *Note that $f_t^{n,\varepsilon}$ has compact supports in x and v for every sample path which are bounded uniformly in n and ε .*

Now, we are ready to state the results on the uniform bound for the sequence $\{f_t^{n,\varepsilon}\}$.

Proposition 3.3. *For every $n, m \in \mathbb{N}$ and $t \in (0, T)$, there exists a nonnegative process \mathcal{A}_t^m which has continuous sample paths and is independent of n and ε such that*

$$\|f_t^{n,\varepsilon}\|_{W^{m,\infty}} \leq \mathcal{A}_t^m \cdot \|f^{in,\varepsilon}\|_{W^{m,\infty}}.$$

Proof. Since the proof is rather lengthy, we postpone it to Appendix B. \square

Remark 3.3. *It is easy to see that for fixed t and ω , A_t^n is monotonically increasing with respect to m .*

Next, we prove that the sample paths of approximate solutions become a Cauchy sequence in a suitable functional space. For this, we set

$$\|\varphi_t^{n,\varepsilon}\|_{L^\infty} := \sup_{(x,v) \in \text{supp} f^{in,\varepsilon}} |\varphi_t^{n,\varepsilon}(x,v)|.$$

Proposition 3.4. *For every n and $t \in (0, T)$, there exists a nonnegative process \mathcal{B}_t which has continuous sample paths and is independent of n and ε such that*

$$(3.9) \quad \begin{aligned} & \|f_t^{n,\varepsilon} - f_t^{n-1}\|_{L^\infty}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2 \\ & \leq \mathcal{B}_t \left[\int_0^t \left(\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 \right) ds \right], \quad n \geq 2. \end{aligned}$$

Proof. For the estimate (3.9), we claim:

$$(3.10) \quad \begin{aligned} (i) & \|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{L^\infty}^2 \\ & \leq \mathcal{B}_t^1 \left[\|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2 + \int_0^t \mathcal{C}_s^1 \left(\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 \right) ds \right]. \\ (ii) & \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2 \\ & \leq \mathcal{B}_t^2 \left(\int_0^t \mathcal{C}_s^2 \left(\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 \right) ds \right), \end{aligned}$$

where \mathcal{B}_t^i and \mathcal{C}_t^i ($i = 1, 2$) are nonnegative processes which has continuous sample paths and is independent of n and ε , respectively.

• (Estimate for (i)): By direct estimates, we have

$$\begin{aligned} & f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) - f_t^{n-1,\varepsilon}(\varphi_t^{n,\varepsilon}) \\ & = \{f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) - f_t^{n-1,\varepsilon}(\varphi_t^{n-1,\varepsilon})\} - \{f_t^{n-1,\varepsilon}(\varphi_t^{n,\varepsilon}) - f_t^{n-1,\varepsilon}(\varphi_t^{n-1,\varepsilon})\} \\ & =: \mathcal{I}_{21} + \mathcal{I}_{22}. \end{aligned}$$

Below, we estimate the terms \mathcal{I}_{2i} separately.

◇ (Estimate on \mathcal{I}_{21}): We use mean-value theorem, unit mass, $(\mathcal{F}^\varepsilon 1)$ and Corollary 3.1 to obtain

$$\begin{aligned} |\mathcal{I}_{21}| & = |f^{in,\varepsilon} \exp(d\sigma W_t)| \\ & \times \left| \exp \left(- \int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}) ds \right) - \exp \left(- \int_0^t \nabla_v \cdot F_a[f_s^{n-2,\varepsilon}](\varphi_s^{n-1,\varepsilon}) ds \right) \right| \\ & \leq \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \\ & \times \left| \int_0^t \left(\nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}) - \nabla_v \cdot F_a[f_s^{n-2,\varepsilon}](\varphi_s^{n-1,\varepsilon}) \right) ds \right| \\ & \leq d \|f^{in}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \end{aligned}$$

$$\begin{aligned}
& \times \int_0^t \int_{\mathbb{R}^{2d}} |\phi(x_* - X_s^{n,\varepsilon}) f_s^{n-1,\varepsilon} - \phi(x_* - X_s^{n-1,\varepsilon}) f_s^{n-2,\varepsilon}| dv_* dx_* ds \\
& \leq d \|f^{in}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \\
& \quad \times \left[\int_0^t \int_{\mathbb{R}^{2d}} |\phi(x_* - X_s^{n,\varepsilon}) - \phi(x_* - X_s^{n-1,\varepsilon})| f_s^{n-1,\varepsilon} \right. \\
& \quad \quad \left. + \phi(x_* - X_s^{n-1,\varepsilon}) |f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}| dv_* dx_* ds \right] \\
& \leq d \|\phi\|_{C^1} \|f^{in}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \\
& \quad \times \left[\int_0^t |X_s^{n,\varepsilon} - X_s^{n-1,\varepsilon}| ds + \int_0^t (4\mathcal{X}^\infty(s)\mathcal{V}^\infty(s))^d \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty} ds \right].
\end{aligned}$$

◇ (Estimate on \mathcal{I}_{22}): By the mean-value theorem, we have

$$\begin{aligned}
|\mathcal{I}_{22}| & \leq \|f_t^{n-1,\varepsilon}\|_{W^{1,\infty}} \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty} \leq \|f^{in,\varepsilon}\|_{W^{1,\infty}} \mathcal{A}_t^1 \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty} \\
& \leq \|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1 \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}.
\end{aligned}$$

Now, we collect the estimates for \mathcal{I}_{21} and \mathcal{I}_{22} , use Young's inequality, Cauchy-Schwarz inequality and take a supremum over all characteristic flows to get

$$\begin{aligned}
& \|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{L^\infty}^2 \\
& \leq 2 \sup_{\varphi_t^{n,\varepsilon}} (|\mathcal{I}_{21}|^2 + |\mathcal{I}_{22}|^2) \\
& \leq 2 \left(d \|\phi\|_{C^1} \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \right)^2 \\
& \quad \times \left[\int_0^t |X_s^{n,\varepsilon} - X_s^{n-1,\varepsilon}| ds + \int_0^t (4\mathcal{X}^\infty(s)\mathcal{V}^\infty(s))^d \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty} ds \right]^2 \\
& \quad + 2 \left(\|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1 \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty} \right)^2 \\
& \leq 4 \left(d \|\phi\|_{C^1} \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \right)^2 \\
& \quad \times \left[\left(\int_0^t |X_s^{n,\varepsilon} - X_s^{n-1,\varepsilon}| ds \right)^2 + \left(\int_0^t (4\mathcal{X}^\infty(s)\mathcal{V}^\infty(s))^d \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty} ds \right)^2 \right] \\
& \quad + 2 \left(\|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1 \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty} \right)^2 \\
& \leq 4T \left(d \|\phi\|_{C^1} \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t) \right)^2 \\
& \quad \times \left[\int_0^t |X_s^{n,\varepsilon} - X_s^{n-1,\varepsilon}|^2 ds + \int_0^t (4\mathcal{X}^\infty(s)\mathcal{V}^\infty(s))^{2d} \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 ds \right] \\
& \quad + 2 \left(\|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1 \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty} \right)^2 \\
& \leq \mathcal{B}_t^1 \left[\|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2 + \int_0^t \mathcal{C}_s^1 \left(\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 \right) ds \right],
\end{aligned}$$

where \mathcal{B}_t^1 and \mathcal{C}_t^1 are given by

$$\begin{aligned}\mathcal{B}_t^1 &:= 4T \left\{ d\|\phi\|_1 \|f^{in}\|_{L^\infty} \exp(d\phi_M t + \sigma dW_t) \right\}^2 + 2 \left(\|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1 \right)^2, \\ \mathcal{C}_t^1 &:= 1 + (4\mathcal{X}^\infty(t)\mathcal{V}^\infty(t))^{2d}.\end{aligned}$$

- (Estimate for (ii)): We use (3.3) and Itô's lemma to see

$$\begin{aligned}d|V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 &= 2(V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon})d(V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) + d(V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) \cdot d(V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) \\ &= 2 \left\{ \underbrace{(F_a[f_t^{n-1,\varepsilon}](\varphi_t^{n,\varepsilon}) - F_a[f_t^{n-2,\varepsilon}](\varphi_t^{n-1,\varepsilon})) \cdot (V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon})}_{=: \mathcal{I}_{31}} + \sigma^2 |V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 \right\} dt \\ &\quad - 2\sigma |V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 dW_t.\end{aligned}$$

Next, we estimate the term \mathcal{I}_{31} as follows.

$$\begin{aligned}\mathcal{I}_{31} &= \left(F_a[f_t^{n-1,\varepsilon}](\varphi_t^{n,\varepsilon}) - F_a[f_t^{n-2,\varepsilon}](\varphi_t^{n-1,\varepsilon}) \right) \cdot (V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) \\ &\leq \int_{\mathbb{R}^{2d}} \left[\left(\phi(x_* - X_t^{n,\varepsilon}) - \phi(x_* - X_t^{n-1,\varepsilon}) \right) (v_* - V_t^{n,\varepsilon}) f_t^{n-1,\varepsilon} \right. \\ &\quad \left. - \phi(x_* - X_t^{n-1,\varepsilon}) (V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) f_t^{n-1,\varepsilon} \right. \\ &\quad \left. + \phi(x_* - X_t^{n-1,\varepsilon}) (v_* - V_t^{n-1,\varepsilon}) (f_t^{n-1,\varepsilon} - f_t^{n-2,\varepsilon}) \right] \cdot (V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}) dv_* dx_* \\ &\leq \left(\|\phi\|_{C^1} + (2\mathcal{V}^\infty(t))^{d+1} (2\mathcal{X}^\infty(t))^d + 2\mathcal{V}^\infty(t) \right) \\ &\quad \times \left(|V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 + |X_t^{n,\varepsilon} - X_t^{n-1,\varepsilon}|^2 + \|f_t^{n-1,\varepsilon} - f_t^{n-2,\varepsilon}\|_{L^\infty}^2 \right).\end{aligned}$$

This implies

$$\begin{aligned}d|V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 &\leq \tilde{\mathcal{B}}_t \left(|V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 + |X_t^{n,\varepsilon} - X_t^{n-1,\varepsilon}|^2 + \|f_t^{n-1,\varepsilon} - f_t^{n-2,\varepsilon}\|_{L^\infty}^2 \right) dt \\ &\quad - 2\sigma |V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 dW_t,\end{aligned}$$

where the process $\tilde{\mathcal{B}}_t$ is defined as follows.

$$\tilde{\mathcal{B}}_t := 2 \left(\|\phi\|_{C^1} + (2\mathcal{V}^\infty(t))^{d+1} (2\mathcal{X}^\infty(t))^d + \mathcal{V}^\infty(t) + \sigma^2 \right).$$

We apply Lemma A.1 and Lemma A.2 to get

$$(3.11) \quad \begin{aligned}|V_t^{n,\varepsilon} - V_t^{n-1,\varepsilon}|^2 &\leq \int_0^t \exp \left\{ \int_s^t \tilde{\mathcal{B}}_\tau d\tau - 2\sigma^2(t-s) - 2\sigma(W_t - W_s) \right\} \\ &\quad \times \tilde{\mathcal{B}}_s \left(|X_s^{n,\varepsilon} - X_s^{n-1,\varepsilon}|^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2 \right) ds.\end{aligned}$$

On the other hand, it is easy to see that

$$(3.12) \quad |X_t^{n,\varepsilon} - X_t^{n-1,\varepsilon}|^2 \leq \int_0^t |X_s^n - X_s^{n-1,\varepsilon}|^2 ds + \int_0^t |V_s^{n,\varepsilon} - V_s^{n-1,\varepsilon}|^2 ds.$$

We combine all estimates (3.11) and (3.12) to derive (ii), which is satisfied by the following processes:

$$\begin{aligned}\mathcal{B}_t^2 &:= 1 + \exp\left(\int_0^t \tilde{\mathcal{B}}_\tau d\tau - 2\sigma^2 t - 2\sigma W_t\right), \\ \mathcal{C}_t^2 &:= 1 + (\mathcal{B}_t^2 - 1)^{-1}.\end{aligned}$$

Finally, for (i) and (ii) in (3.10), the combination (i) + $(\mathcal{B}_t^1 + 1) \cdot$ (ii) gives

$$\begin{aligned}& \|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{L^\infty}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2 \\ & \leq \mathcal{B}_t^1 \left(\int_0^t \mathcal{C}_s^1 (\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2) ds \right) \\ & \quad + (\mathcal{B}_t^1 + 1) \mathcal{B}_t^2 \left(\int_0^t \mathcal{C}_s^2 (\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2) ds \right) \\ & \leq \mathcal{B}_t \left(\int_0^t (\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{L^\infty}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{L^\infty}^2) ds \right),\end{aligned}$$

where the process \mathcal{B}_t is defined as follows.

$$\mathcal{B}_t := (\mathcal{B}_t^1 + (\mathcal{B}_t^1 + 1)\mathcal{B}_t^2) \left(\sup_{0 \leq s \leq t} \mathcal{C}_s^1 + \sup_{0 \leq s \leq t} \mathcal{C}_s^2 \right).$$

This implies our desired result. \square

For each t and $\omega \in \Omega$, we define

$$\Delta_n^\varepsilon(t, \omega) := \|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{L^\infty}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{L^\infty}^2.$$

Corollary 3.2. *The functional $\Delta_n^\varepsilon(t)$ satisfies*

$$\Delta_n^\varepsilon(t, \omega) \leq \frac{(\mathcal{K}(\omega)t)^n}{n!}, \quad \text{for each } t \in [0, T] \quad \text{and a.s. } \omega \in \Omega,$$

where $\mathcal{K} = \mathcal{K}(\omega)$ is a nonnegative random variable.

Proof. It follows from Proposition 3.4 that

$$\Delta_{n+1}^\varepsilon(t) \leq \mathcal{B}_t \left(\int_0^t (\Delta_n^\varepsilon(s) + \Delta_{n+1}^\varepsilon(s)) ds \right).$$

Since \mathcal{B}_t is a nonnegative process with continuous sample paths, there exists a nonnegative random variable $B = B(\omega)$ such that

$$\sup_{0 \leq t \leq T} \mathcal{B}_t(\omega) \leq B(\omega) < \infty, \quad \text{for each } \omega \in \Omega.$$

Thus, we can use the Grönwall-type lemma in Lemma 3.1 to deduce

$$\Delta_n(t, \omega) \leq \frac{(\mathcal{K}(\omega)t)^n}{n!}, \quad \text{for each } t \in [0, T], \omega \in \Omega,$$

where $\mathcal{K} = \mathcal{K}(\omega)$ depends on $B(\omega)$. \square

Remark 3.4. *Corollary 3.2 implies that for every ω , $f_t^{n,\varepsilon}(\omega) \rightarrow f_t^\varepsilon(\omega)$ in $\mathcal{C}([0, T]; L^\infty(\mathbb{R}^{2d}))$. Since $f_t^{n,\varepsilon}$ is \mathcal{F}_t -adapted (where \mathcal{F}_t is a filtration generated by the Wiener process) and f_t^ε is a pointwise limit of $f_t^{n,\varepsilon}$ over Ω , we have f is \mathcal{F}_t -adapted. Moreover, we have a uniform boundedness of $f_t^{n,\varepsilon}$ in $L^\infty([0, T]; W^{m,p}(\mathbb{R}^{2d}))$ for any $p \in [1, \infty)$.*

By the property of reflexive Banach space, there exists a subsequence $\{f^{n_k, \varepsilon}(\omega)\} \subseteq \{f^{n, \varepsilon}(\omega)\}$ which is weakly convergent to $\tilde{f}_t(\omega)$ in $L^\infty([0, T]; W^{m, p}(\mathbb{R}^{2d}))$ for each $\omega \in \Omega$ and every $p \in [1, \infty)$. Since we have already a strong convergence in the lower order, we can conclude that $f_t^\varepsilon(\omega) = \tilde{f}_t(\omega)$.

However, we can not proceed further, since it is not clear whether f_t^ε satisfies the equation (1.3) at this moment. This is due to the noise term in the right-hand side of (3.2). It is not certain whether the Stratonovich integral of f_t^ε can be defined or not. In addition, even if the noise term can be well-defined, it is also not clear whether the Stratonovich integral of $f^{n, \varepsilon}$ converges to that of f_t^ε or not.

3.3. Convergence estimate. In this subsection, we provide a global existence of solution to system (3.2) by showing that the limit of the sequence $\{f_t^{n, \varepsilon}\}$ exists as $n \rightarrow \infty$ for each ε , and this limit is indeed a strong solution to (2.8) corresponding to the regularized initial datum $f^{in, \varepsilon}$.

Theorem 3.1. *Suppose that f^{in} and ϕ satisfy the framework (\mathcal{F}) in Section 2. Then, for given $T \in (0, \infty)$, there exists a strong solution f_t^ε to (3.2).*

Proof. In order to cope with problems discussed in Remark 3.4, we employ the stopping time argument. First, for each $m \geq \max\{4, k\}$, we define a sequence of stopping times $\{\tau_{M, m}\}_{M \in \mathbb{N}}$ as follows:

$$\begin{aligned}\tau_{M, m}^1(\omega) &:= \inf\{t \geq 0 \mid \mathcal{A}_t^m(\omega) > M\} \wedge T, \\ \tau_{M, m}^2(\omega) &:= \inf\{t \geq 0 \mid \mathcal{B}_t(\omega) > M\} \wedge T, \\ \tau_{M, m} &:= \tau_{M, m}^1 \wedge \tau_{M, m}^2.\end{aligned}$$

Next, we verify the existence of solutions step by step.

- (Step A: Extracting a limit function): We can find out that for each $n \in \mathbb{N}$,

$$\begin{aligned}(i) \quad & \|f_{t \wedge \tau_{M, m}}^{n, \varepsilon}\|_{W^{m, \infty}} \leq M \|f^{in, \varepsilon}\|_{W^{m, \infty}}. \\ (ii) \quad & \|f_{t \wedge \tau_{M, m}}^{n, \varepsilon} - f_{t \wedge \tau_{M, m}}^{n-1, \varepsilon}\|_{L^\infty}^2 + \|\varphi_{t \wedge \tau_{M, m}}^{n, \varepsilon} - \varphi_{t \wedge \tau_{M, m}}^{n-1, \varepsilon}\|_{L^\infty}^2 \\ & \leq M \left[\int_0^t \left(\|\varphi_{s \wedge \tau_{M, m}}^{n, \varepsilon} - \varphi_{s \wedge \tau_{M, m}}^{n-1, \varepsilon}\|_{L^\infty}^2 + \|f_{s \wedge \tau_{M, m}}^{n-1, \varepsilon} - f_{s \wedge \tau_{M, m}}^{n-2, \varepsilon}\|_{L^\infty}^2 \right) ds \right].\end{aligned}$$

Thus, we can use the same argument as in Corollary 3.2 to yield that as $n \rightarrow \infty$, there exists a limit function $f_{t \wedge \tau_{M, m}}^\varepsilon$ such that

$$\begin{aligned}f_{t \wedge \tau_{M, m}}^{n, \varepsilon} &\rightarrow f_{t \wedge \tau_{M, m}}^\varepsilon \quad \text{in } L^\infty(\Omega; \mathcal{C}([0, T]; L^\infty(\mathbb{R}^{2d}))), \\ f_{t \wedge \tau_{M, m}}^{n_k, \varepsilon} &\rightharpoonup f_{t \wedge \tau_{M, m}}^\varepsilon \quad \text{in } L^\infty(\Omega \times [0, T]; W^{m, p}(\mathbb{R}^{2d})), \quad \forall p \in [1, \infty).\end{aligned}$$

- (Step B: Verification of relation (2.2)): Now, we need to show that $f_{t \wedge \tau_{M, m}}^\varepsilon$ satisfies (3.2) in the sense of Definition 2.1. Since $f_{t \wedge \tau_{M, m}}^{n, \varepsilon}$ satisfies (3.6) and conditions of Lemma 2.2, it satisfies the following relation:

$$\begin{aligned}
& \int_{\Sigma} f_{t \wedge \tau_{M,m}}^{n,\varepsilon} \psi dz \\
(3.13) \quad &= \int_{\Sigma} f^{in,\varepsilon} \psi dz + \int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{n,\varepsilon} \left(v \cdot \nabla_x \psi + \left(F_a[f_{s \wedge \tau_{M,m}}^{n-1,\varepsilon}] + \frac{1}{2} \sigma^2 v \right) \cdot \nabla_v \psi \right) dz ds \\
& \quad + \frac{1}{2} \sigma^2 \int_0^t \int_{\Sigma} v f_{s \wedge \tau_{M,m}}^{n,\varepsilon} \cdot (D_v^2 \psi) v dz ds - \sigma \int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{n,\varepsilon} v \cdot \nabla_v \psi dz dW_s,
\end{aligned}$$

where $\Sigma := \mathbb{R}^{2d}$ and $dz = dv dx$. Next, our job is to pass $n \rightarrow \infty$ in the integral relation (3.13) to derive an integral relation (2.2) for $f_{t \wedge \tau_{M,m}}^{\varepsilon}$. For this, note that the x - and v -supports of $f_{t \wedge \tau_{M,m}}^{n,\varepsilon}$ and $f_{t \wedge \tau_{M,m}}^{\varepsilon}$ are uniformly bounded by $|\mathcal{X}_{t \wedge \tau_M}^{\infty}|$ and $|\mathcal{V}_{t \wedge \tau_M}^{\infty}|$ (see Corollary 3.1). Moreover, we can find out that $|\mathcal{X}_{t \wedge \tau_M}^{\infty}|$ and $|\mathcal{V}_{t \wedge \tau_M}^{\infty}|$ are bounded by $\mathcal{A}_{t \wedge \tau_M}^m$, and hence by M . We combine the strong convergence on the lower order with these facts to yield

$$\begin{aligned}
(i) \quad & \int_{\Sigma} (f_{t \wedge \tau_{M,m}}^{n,\varepsilon} - f_{t \wedge \tau_{M,m}}^{\varepsilon}) \psi dz \longrightarrow 0. \\
(ii) \quad & \int_0^t \int_{\Sigma} (f_{s \wedge \tau_{M,m}}^{n,\varepsilon} - f_{s \wedge \tau_{M,m}}^{\varepsilon}) \left(v \cdot \nabla_x \psi + \left(F_a[f_{s \wedge \tau_{M,m}}^{n-1,\varepsilon}] + \frac{1}{2} \sigma^2 v \right) \cdot \nabla_v \psi \right) dz ds \longrightarrow 0. \\
(iii) \quad & \int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{\varepsilon} \left(F_a[f_{s \wedge \tau_{M,m}}^{n-1,\varepsilon}] - F_a[f_{s \wedge \tau_{M,m}}^{\varepsilon}] \right) \nabla_v \psi dz ds \longrightarrow 0. \\
(iv) \quad & \frac{1}{2} \sigma^2 \int_0^t \int_{\Sigma} v (f_{s \wedge \tau_{M,m}}^{n,\varepsilon} - f_{s \wedge \tau_{M,m}}^{\varepsilon}) \cdot (D_v^2 \psi) v dz ds \longrightarrow 0,
\end{aligned}$$

uniformly in ω , as n goes to infinity.

Now it remains to check with the stochastic integral term in (3.13). For this term, one has

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t \int_{\Sigma} (f_{s \wedge \tau_{M,m}}^{n,\varepsilon} - f_{s \wedge \tau_{M,m}}^{\varepsilon}) v \cdot \nabla_v \psi dz dW_s \right)^2 \right] \\
&= \mathbb{E} \left[\int_0^t \left(\int_{\Sigma} (f_{s \wedge \tau_{M,m}}^n - f_{s \wedge \tau_{M,m}}^{\varepsilon}) v \cdot \nabla_v \psi dz \right)^2 ds \right] \\
&\leq \mathbb{E} \left[\int_0^t \|f_{s \wedge \tau_{M,m}}^{n,\varepsilon} - f_{s \wedge \tau_{M,m}}^{\varepsilon}\|_{L^\infty}^2 ds \right] \left(\int_{\Sigma} |v \cdot \nabla_v \psi| dz \right)^2 \longrightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

This L^2 -convergence over Ω implies that there exists a subsequence $\{f_{t \wedge \tau_{M,m}}^{n_l,\varepsilon}\}$ such that

$$\left(\int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{n_l,\varepsilon} v \cdot \nabla_v \psi dz dW_s \right) (\omega) \longrightarrow \left(\int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{\varepsilon} v \cdot \nabla_v \psi dz dW_s \right) (\omega),$$

for a.s. ω , as l goes to infinity. Thus, we can conclude that for a.s. $\omega \in \Omega$, $f_{s \wedge \tau_{M,m}}^{\varepsilon}$ satisfies

$$\begin{aligned}
\int_{\Sigma} f_{t \wedge \tau_{M,m}}^{\varepsilon} \psi dz &= \int_{\Sigma} f^{in,\varepsilon} \psi dz - \int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{\varepsilon} \left(v \cdot \nabla_x \psi + \left(F_a[f_{s \wedge \tau_{M,m}}^{\varepsilon}] + \frac{1}{2} \sigma^2 v \right) \cdot \nabla_v \psi \right) dz ds \\
& \quad - \frac{1}{2} \sigma^2 \int_0^t \int_{\Sigma} v f_{s \wedge \tau_{M,m}}^{\varepsilon} \cdot (D_v^2 \psi) v dz ds + \int_0^t \int_{\Sigma} f_{s \wedge \tau_{M,m}}^{\varepsilon} v \cdot \nabla_v \psi dz dW_s,
\end{aligned}$$

for every $\psi \in \mathcal{D}(\mathbb{R}^{2d})$. One also has $f_{t \wedge \tau_{M,m}}^\varepsilon$ is a \mathcal{F}_t -semimartingale. Here, we use Lemma 2.1 to obtain that $f_{t \wedge \tau_{M,m}}^\varepsilon$ satisfies (1.3) in the sense of distribution.

• (Step C): Due to the continuity in time of the involved processes \mathcal{A}_t^m and \mathcal{B}_t , it is obvious that

$$\tau_{M,m}(\omega) \rightarrow T \quad \text{as } M \rightarrow \infty \text{ for a.s. } \omega.$$

Thus, if we choose M sufficiently large for each $\omega \in \Omega$, $f_{t \wedge \tau_{M,m}}^\varepsilon$ satisfies relation (2.2) on $[0, T]$. Here, note that the strong convergence and (B.2) give

$$\|f_{t \wedge \tau_{M,m}}^\varepsilon\|_{L^\infty} \leq \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t \wedge \tau_{M,m} + d\sigma W_{t \wedge \tau_{M,m}}).$$

Since $m \geq 4$, one uses Sobolev embedding theorem to get $f_{t \wedge \tau_{M,m}}^\varepsilon \in L^\infty(\Omega; \mathcal{C}([0, T]; \mathcal{C}^{3,\delta}(\mathbb{R}^{2d})))$. Thus, it follows from Remark 2.1 that $f_{t \wedge \tau_{M,m}}^\varepsilon$ becomes a classical solution to (3.1) corresponding to the regularized initial datum $f^{in,\varepsilon}$. Hence, we can also obtain

$$M_2^\varepsilon(t \wedge \tau_{M,m}) \leq M_2(0) \exp(-2\phi_m t \wedge \tau_{M,m} - 2\sigma W_{t \wedge \tau_{M,m}}).$$

□

4. GLOBAL EXISTENCE OF STRONG SOLUTIONS

In this section, we study properties of classical solutions to (1.3) which could be obtained in Section 3. Moreover, as a corollary, we provide the uniqueness of regularized solutions and the existence of a solution to (1.3) corresponding to the original initial datum.

4.1. Quantitative estimates for classical solutions. We provide several properties of classical solutions f to (1.3). Here we recall that a stopped, regularized solution $f_{t \wedge \tau_{M,m}}^\varepsilon$ becomes a classical solution to (1.3). From now on, without further mentioning, we will denote f_t^ε by a classical solution to (1.3), corresponding to the initial datum $f^{in,\varepsilon}$.

First, we discuss the size of spatial and velocity supports of f_t^ε . We define

$$\begin{aligned} \mathcal{X}^\varepsilon(t) &:= \sup\{|x| : f_t^\varepsilon(x, v) \neq 0 \text{ for some } v \in \mathbb{R}^d\}, \\ \mathcal{V}^\varepsilon(t) &:= \sup\{|v| : f_t^\varepsilon(x, v) \neq 0 \text{ for some } x \in \mathbb{R}^d\}. \end{aligned}$$

Lemma 4.1. *The support functionals \mathcal{X}^ε and \mathcal{V}^ε satisfy the following estimates:*

(1) *If $\phi_m = 0$, then,*

$$\begin{aligned} \mathcal{X}^\varepsilon(t) &\leq \mathcal{X}_0^\varepsilon + (\mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)}) \int_0^t \exp\left(\frac{\phi_M}{2}s - \sigma W_s\right) ds, \quad t \geq 0, \\ \mathcal{V}^\varepsilon(t) &\leq \left(\mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)}\right) \exp\left(\frac{\phi_M}{2}t - \sigma W_t\right). \end{aligned}$$

(2) *If $\phi_m > 0$, then*

$$\begin{aligned} \mathcal{X}^\varepsilon(t) &\leq \mathcal{X}_0^\varepsilon + \left\{ \mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)} \left(\frac{\phi_M}{\phi_m}\right) \right\} \int_0^t \exp\left(-\frac{\phi_m}{2}s - \sigma W_s\right) ds, \\ \mathcal{V}^\varepsilon(t) &\leq \left\{ \mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)} \left(\frac{\phi_M}{\phi_m}\right) \right\} \exp\left(-\frac{\phi_m}{2}t - \sigma W_t\right). \end{aligned}$$

Proof. (i) Consider the case $\phi_m = 0$.

◇ (Estimate of \mathcal{V}^ε): Note that the stochastic characteristics $(X_t^\varepsilon, V_t^\varepsilon)$ starting from $(x, v) \in \text{supp} f^{in, \varepsilon}$ satisfies

$$(4.1) \quad \begin{cases} dX_t^\varepsilon = V_t^\varepsilon dt, \\ dV_t^\varepsilon = \left(F_a[f_t^\varepsilon](X_t^\varepsilon, V_t^\varepsilon) + \frac{1}{2}\sigma^2 V_t^\varepsilon \right) dt - \sigma V_t^\varepsilon dW_t. \end{cases}$$

Now, we use Itô's lemma and (4.1) to get

$$(4.2) \quad \begin{aligned} d|V_t^\varepsilon|^2 &= 2V_t^\varepsilon \cdot dV_t^\varepsilon + dV_t^\varepsilon \cdot dV_t^\varepsilon \\ &\leq 2 \left(\int_{\mathbb{R}^{2d}} \phi(x_* - X_t) (v_* - V_t^\varepsilon) \cdot V_t^\varepsilon f_t(x_*, v_*) dv_* dx_* + \sigma^2 |V_t^\varepsilon|^2 \right) dt \\ &\quad - 2\sigma |V_t^\varepsilon|^2 dW_t \\ &\leq 2 \left(\int_{\mathbb{R}^{2d}} \phi(x_* - X_t^\varepsilon) v_* \cdot V_t^\varepsilon f_t^\varepsilon(x_*, v_*) dv_* dx_* + \sigma^2 |V_t^\varepsilon|^2 \right) dt - 2\sigma |V_t^\varepsilon|^2 dW_t \\ &\leq \left(\phi_M M_2^\varepsilon(t) + (\phi_M + 2\sigma^2) |V_t^\varepsilon|^2 \right) dt - 2\sigma |V_t^\varepsilon|^2 dW_t. \end{aligned}$$

It follows from the same argument in Proposition 3.1 to get the desired estimate for V_t^ε :

$$(4.3) \quad \begin{aligned} |V_t^\varepsilon|^2 &\leq |v|^2 \exp(\phi_M t - 2\sigma W_t) + \phi_M \int_0^t \exp\{\phi_M(t-s) - 2\sigma(W_t - W_s)\} M_2^\varepsilon(s) ds \\ &\leq \left\{ |v|^2 + \phi_M M_2^\varepsilon(0) \int_0^t \exp(-\phi_M s) ds \right\} \exp(\phi_M t - 2\sigma W_t) \\ &= (|v|^2 + M_2^\varepsilon(0)(1 - \exp(-\phi_M t))) \exp(\phi_M t - 2\sigma W_t) \\ &\leq ((\mathcal{V}_0^\varepsilon)^2 + M_2^\varepsilon(0)) \exp(\phi_M t - 2\sigma W_t), \end{aligned}$$

i.e., we have

$$\mathcal{V}^\varepsilon(t) \leq \sqrt{((\mathcal{V}_0^\varepsilon)^2 + M_2^\varepsilon(0))} \exp\left(\frac{\phi_M t}{2} - \sigma W_t\right) \leq \left(\mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)}\right) \exp\left(\frac{\phi_M t}{2} - \sigma W_t\right).$$

This yields the desired estimate.

◇ (Estimate of \mathcal{X}^ε): We use Itô's formula and Cauchy-Schwarz inequality to get

$$d|X_t^\varepsilon|^2 = 2X_t^\varepsilon \cdot dX_t^\varepsilon + dX_t^\varepsilon \cdot dX_t^\varepsilon = 2X_t^\varepsilon \cdot V_t^\varepsilon dt \leq 2|X_t^\varepsilon| \cdot |V_t^\varepsilon| dt.$$

This and (4.3) yield

$$\frac{d|X_t^\varepsilon|}{dt} \leq |V_t^\varepsilon| \leq \left(\mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)}\right) \exp\left(\frac{\phi_M t}{2} - \sigma W_t\right).$$

We integrate the above differential inequality to get

$$|X_t^\varepsilon| \leq |X_0^\varepsilon| + \left(\mathcal{V}_0^\varepsilon + \sqrt{M_2^\varepsilon(0)}\right) \int_0^t \exp\left(\frac{\phi_M}{2}s - \sigma W_s\right) ds.$$

(ii) Now we consider the case $\phi_m > 0$.

In this case, we use the first line relation in (4.2) and Young's inequality to get

$$\begin{aligned} d|V_t^\varepsilon|^2 &\leq 2 \left(\int_{\mathbb{R}^{2d}} \phi(x_* - X_t)(v_* - V_t^\varepsilon) \cdot V_t^\varepsilon f_t(x_*, v_*) dv_* dx_* + \sigma^2 |V_t^\varepsilon|^2 \right) dt - 2\sigma |V_t^\varepsilon|^2 dW_t \\ &\leq 2 \left(\int_{\mathbb{R}^{2d}} \phi(x_* - X_t) v_* \cdot V_t^\varepsilon f_t(x_*, v_*) dv_* dx_* + (-\phi_m + \sigma^2) |V_t^\varepsilon|^2 \right) dt - 2\sigma |V_t^\varepsilon|^2 dW_t \\ &\leq \left[(-\phi_m + 2\sigma^2) |V_t^\varepsilon|^2 + \frac{\phi_M^2}{\phi_m} M_2^\varepsilon(t) \right] dt - 2\sigma |V_t^\varepsilon|^2 dW_t. \end{aligned}$$

Again, we use Lemma A.1 and Lemma A.2 to obtain

$$\begin{aligned} |V_t^\varepsilon|^2 &\leq |v|^2 \exp(-\phi_m t - 2\sigma W_t) + \frac{\phi_M^2}{\phi_m} \int_0^t \exp\{-\phi_m(t-s) - 2\sigma(W_t - W_s)\} M_2^\varepsilon(s) ds \\ &\leq \left\{ |v|^2 + M_2^\varepsilon(0) \frac{\phi_M^2}{\phi_m} \int_0^t \exp(-\phi_m s) ds \right\} \exp(-\phi_m t - 2\sigma W_t) \\ &\leq \left\{ |v|^2 + M_2^\varepsilon(0) \left(\frac{\phi_M}{\phi_m} \right)^2 \right\} \exp(-\phi_m t - 2\sigma W_t). \end{aligned}$$

This gives the desired estimate for \mathcal{V}^ε . For \mathcal{X}^ε , we use the same argument as above to conclude the proof. \square

Remark 4.1. *If we apply Proposition 3.2 to the regularized solution we obtained in Section 3, it is not difficult to find out that*

$$\mathcal{X}^\varepsilon(t) \lesssim \mathcal{X}^\infty(t), \quad \mathcal{V}^\varepsilon(t) \lesssim \mathcal{V}^\infty(t),$$

where \mathcal{X}^∞ and \mathcal{V}^∞ are defined in Corollary 3.1.

Lemma 4.2. *Let f_t be a classical solution to (1.3). Then, one has*

$$\|f_t\|_{W^{1,\infty}} \leq \|f^{in}\|_{W^{1,\infty}} \mathcal{A}_t^1,$$

where \mathcal{A}_t^1 is a nonnegative process defined in Theorem 3.1.

Proof. The proof is similar to that of Theorem 3.1. So we omit its details. \square

Now, we are ready to state the stability results for (1.3), where the pathwise uniqueness result can be obtained.

Theorem 4.1. (L^∞ -stability) *Let f_t and \tilde{f}_t be two classical solutions to (1.3) corresponding to regular initial data f^{in} and \tilde{f}^{in} , respectively, which are compactly supported in x and v . Then, we have*

$$\|f_t - \tilde{f}_t\|_{L^\infty} \leq \mathcal{D}_t \|f^{in} - \tilde{f}^{in}\|_{L^\infty},$$

where \mathcal{D}_t is a non-negative process with continuous sample paths:

$$\begin{aligned} \mathcal{D}_t &:= \exp \left[d\phi_M t + d\sigma W_t \right. \\ &\quad \left. + 2\phi_M \max\{\|f^{in}\|_{W^{1,\infty}}, \|\tilde{f}^{in}\|_{W^{1,\infty}}\} \int_0^t \left((2\mathcal{R}(s))^d (2\mathcal{P}(s))^{d+1} + d(4\mathcal{R}(s)\mathcal{P}(s))^d \right) ds \right], \end{aligned}$$

and $\mathcal{R}(t)$ and $\mathcal{P}(t)$ denotes, respectively,

$$\begin{aligned} \mathcal{R}(t) &:= \sup \left\{ |x| : f_t(x, v) \neq 0 \quad \text{or} \quad \tilde{f}_t(x, v) \neq 0 \quad \text{for some } v \in \mathbb{R}^d \right\}, \\ \mathcal{P}(t) &:= \sup \left\{ |v| : f_t(x, v) \neq 0 \quad \text{or} \quad \tilde{f}_t(x, v) \neq 0 \quad \text{for some } x \in \mathbb{R}^d \right\}. \end{aligned}$$

Proof. It follows from (1.3) that

$$\begin{aligned} & (f_t - \tilde{f}_t) - (f^{in} - \tilde{f}^{in}) \\ & + \int_0^t \left[v \cdot \nabla_x (f_s - \tilde{f}_s) + F_a[f_s] \cdot \nabla_v (f_s - \tilde{f}_s) \right] ds \\ & = \sigma \int_0^t \nabla_v \cdot (v(f_s - \tilde{f}_s)) \circ dW_s - \int_0^t \left[\nabla_v \cdot F_a[f_s]((f_s - \tilde{f}_s) + \nabla_v \cdot ((F_a[f_s] - F_a[\tilde{f}_s])\tilde{f}_s)) \right] ds. \end{aligned}$$

We integrate the above relation along the characteristic flow φ_t of f to yield

$$\begin{aligned} & (f_t - \tilde{f}_t)(\varphi_t) \\ & = (f^{in} - \tilde{f}^{in}) \exp \left\{ - \int_0^t \nabla_v \cdot F_a[f_s](\varphi_s) ds + d\sigma W_t \right\} \\ & \quad - \int_0^t \nabla_v \cdot ((F_a[f_s] - F_a[\tilde{f}_s])\tilde{f}_s) \exp \left\{ - \int_s^t \nabla_v \cdot F_a[f_\tau](\varphi_\tau) d\tau + \sigma d(W_t - W_s) \right\} ds. \end{aligned}$$

On the other hand, note that

$$\begin{aligned} |F_a[f_t] - F_a[\tilde{f}_t]| & \leq 2\phi_M \|f_t - \tilde{f}_t\|_{L^\infty} (2\mathcal{R}(t))^d (2\mathcal{P}(t))^{d+1}, \\ |\nabla_v \cdot (F_a[f_t] - F_a[\tilde{f}_t])| & \leq 2d\phi_M \|f_t - \tilde{f}_t\|_{L^\infty} (4\mathcal{R}(t)\mathcal{P}(t))^d. \end{aligned}$$

We combine the above estimates with Lemma 4.1 and take supremum over the characteristic flow as in Appendix B to get

$$\begin{aligned} \|f_t - \tilde{f}_t\|_{L^\infty} & \leq \|f^{in} - \tilde{f}^{in}\|_{L^\infty} \exp(d\phi_M t + \sigma dW_t) \\ & \quad + \int_0^t \tilde{\mathcal{D}}_s \|f_s - \tilde{f}_s\|_{L^\infty} \exp(d\phi_M(t-s) + \sigma d(W_t - W_s)) ds, \end{aligned}$$

where $\tilde{\mathcal{D}}_t$ is a nonnegative process with continuous sample paths:

$$\tilde{\mathcal{D}}_t := 2\phi_M \max\{\|f_0\|_{W^{1,\infty}}, \|\tilde{f}_0\|_{W^{1,\infty}}\} \left((2\mathcal{R}(t))^d (2\mathcal{P}(t))^{d+1} + d(4\mathcal{R}(t)\mathcal{P}(t))^d \right).$$

We set

$$a(t) := \exp(-d\phi_M t - \sigma dW_t) \|f_t - \tilde{f}_t\|_{L^\infty}.$$

Then, we have

$$a(t) \leq a_0 + \int_0^t \tilde{\mathcal{D}}_s a(s) ds.$$

Finally, we use Grönwall's lemma to complete the proof. \square

Remark 4.2. *We comment on several remarks.*

- (1) *Theorem 4.1 gives a uniqueness result for classical solutions to (1.3).*
- (2) *Using the relation*

$$\sup_{\varepsilon \in (0,1)} \|f^{in,\varepsilon}\|_{W^{1,\infty}} \leq \|f^{in}\|_{W^{1,\infty}},$$

the following modified process \mathcal{D}_t can be used in the statement of Theorem 4.1 when we estimate $\|f_{t \wedge \tau_{M,m}}^\varepsilon - \tilde{f}_{t \wedge \tau_{M,m}}^\varepsilon\|_{L^\infty}$:

$$\mathcal{D}_t := \exp \left[d\phi_M t + d\sigma W_t \right]$$

$$+ 2\phi_M \int_0^t \|f^{in}\|_{W^{1,\infty}} \left[(2\mathcal{X}^\infty(s))^d (2\mathcal{V}^\infty(s))^{d+1} + d(4\mathcal{X}^\infty(s)\mathcal{V}^\infty(s))^d \right] ds \Big].$$

4.2. Proof of Theorem 2.1. In this subsection, we proceed to the proof of the existence of a strong solution to (1.3) corresponding to the original initial data f^{in} and its emergent dynamics.

• (Existence part): For this, we define a sequence of stopping times $\{\tau_M\}_{M \in \mathbb{N}}$ as follows:

$$\begin{aligned} \tau_M^1(\omega) &:= \inf\{t \geq 0 \mid \mathcal{A}_t^{m_*}(\omega) > M\} \wedge T, & \tau_M^2(\omega) &:= \inf\{t \geq 0 \mid \mathcal{B}_t(\omega) > M\} \wedge T, \\ \tau_M^3(\omega) &:= \inf\{t \geq 0 \mid \mathcal{D}_t(\omega) > M\} \wedge T, & \tau_M &:= \tau_M^1 \wedge \tau_M^2 \wedge \tau_M^3, \end{aligned}$$

where $m_* := \max\{k, 4\}$. Then, we provide a global existence of a strong solution in three steps:

◊ (Step A: Passing $n \rightarrow \infty$). Let $f_{t \wedge \tau_M}^{n,\varepsilon}(\omega) := f^{n,\varepsilon}(t \wedge \tau_M, \omega)$. In this case, we proceed as Section 3.2 to derive a limit function:

$$\begin{aligned} f_{t \wedge \tau_M}^{n,\varepsilon} &\rightarrow f_{t \wedge \tau_M}^\varepsilon && \text{in } L^\infty(\Omega; \mathcal{C}([0, T] \times L^\infty(\mathbb{R}^{2d}))), \\ f_{t \wedge \tau_M}^{n_k,\varepsilon} &\rightharpoonup f_{t \wedge \tau_M}^\varepsilon && \text{in } L^\infty(\Omega \times [0, T]; W^{m_*,p}(\mathbb{R}^{2d})), \quad \forall p \in [1, \infty), \end{aligned}$$

and $f_{t \wedge \tau_M}^\varepsilon$ is a strong solution to (1.3) on the interval $[0, \tau_M]$. Moreover, it can be shown that $f_{t \wedge \tau_M}^\varepsilon$ is a classical solution to (1.3) on $[0, \tau_M]$ which is compactly supported in x and v .

◊ (Step B: Passing $\varepsilon \rightarrow 0$): We use the stability estimate in Theorem 4.1 to get

$$(4.4) \quad \|f_{t \wedge \tau_M}^\varepsilon - f_{t \wedge \tau_M}^{\varepsilon'}\|_{L^\infty} \leq \mathcal{D}_{t \wedge \tau_M} \|f^{in,\varepsilon} - f^{in,\varepsilon'}\|_{L^\infty} \leq M \|f^{in,\varepsilon} - f^{in,\varepsilon'}\|_{L^\infty}.$$

Since $f^{in,\varepsilon}$ converges uniformly to f^{in} , it follows from the stability estimate (4.4) that there exists $f_{t \wedge \tau_M}$ such that

$$f_{t \wedge \tau_M}^\varepsilon \rightarrow f_{t \wedge \tau_M} \quad \text{in } L^\infty(\Omega; \mathcal{C}([0, T]; L^\infty(\mathbb{R}^{2d}))).$$

Moreover, it follows from Theorem 4.1 and (A1) that

$$\|f_{t \wedge \tau_M}^\varepsilon\|_{W^{k,\infty}} \leq \mathcal{A}_t^k \|f^{in,\varepsilon}\|_{W^{k,\infty}} \leq M \|f^{in}\|_{W^{k,\infty}}.$$

This implies

$$f_{t \wedge \tau_M}^{\varepsilon_k} \rightharpoonup f_{t \wedge \tau_M}, \quad \text{in } L^\infty(\Omega \times [0, T]; W^{k,p}(\mathbb{R}^{2d})), \quad \forall p \in [1, \infty).$$

Hence, we can follow the arguments in Section 3 to yield that $f_{t \wedge \tau_M}$ satisfies (2.3) with the desired regularity for the strong solution, corresponding to initial data f^{in} , and hence (2.2). Moreover, $f_{t \wedge \tau_M}$ is compactly supported in x and v .

◊ (Step C): It is obvious that

$$\tau_M(\omega) \rightarrow T \quad \text{as } M \rightarrow \infty \text{ for a.s. } \omega.$$

Thus, choosing a sufficiently large M for each $\omega \in \Omega$ gives $f_{t \wedge \tau_M}(\omega)$ satisfies the relation (2.2) on $[0, T]$ and the strong convergence implies

$$\begin{aligned} \|f_{t \wedge \tau_M}\|_{L^\infty} &\leq \|f^{in}\|_{L^\infty} \exp(d\phi_M t \wedge \tau_M + d\sigma W_{t \wedge \tau_M}), \\ M_2(t \wedge \tau_M) &\leq M_2(0) \exp(-2\phi_m t \wedge \tau_M - 2\sigma W_{t \wedge \tau_M}). \end{aligned}$$

For the L^∞ -estimate of solution, we use Fatou's lemma to get, for any $p \in (1, \infty)$,

$$\begin{aligned}
& \mathbb{E} \|f_t\|_{L^\infty} \\
& \leq \liminf_{M \rightarrow \infty} \mathbb{E} \|f_{t \wedge \tau_M}\|_{L^\infty} \\
& \leq \liminf_{M \rightarrow \infty} \|f^{in}\|_{L^\infty} \mathbb{E} \left[\exp(d\phi_M t \wedge \tau_M + d\sigma W_{t \wedge \tau_M}) \right] \\
& = \liminf_{M \rightarrow \infty} \|f^{in}\|_{L^\infty} \mathbb{E} \left[\exp \left(\left(d\phi_M + \frac{(d\sigma)^2}{2} \right) t \wedge \tau_M \right) \right] \\
& = \liminf_{M \rightarrow \infty} \|f^{in}\|_{L^\infty} \mathbb{E} \left[\exp \left(d\sigma W_{t \wedge \tau_M} - \frac{p(d\sigma)^2}{2} t \wedge \tau_M \right) \exp \left(\left(d\phi_M + \frac{p(d\sigma)^2}{2} \right) t \wedge \tau_M \right) \right] \\
& \leq \liminf_{M \rightarrow \infty} \|f^{in}\|_{L^\infty} \mathbb{E} \left[\exp \left(\frac{p}{p-1} \left(d\phi_M + \frac{p(d\sigma)^2}{2} \right) t \wedge \tau_M \right) \right]^{(p-1)/p} \\
& = \|f^{in}\|_{L^\infty} \exp \left(\left(d\phi_M + \frac{p(d\sigma)^2}{2} \right) t \right),
\end{aligned}$$

where we used the fact $X_t = \exp(aW_t - a^2t/2)$ is a martingale and Hölder inequality. Then we take the limit $p \rightarrow 1$ on both sides to yield the desired result. For the dissipation of second velocity moment, we use a similar argument to get the desired estimate.

5. CONCLUSION

In this paper, we presented a global well-posedness of strong solutions and their asymptotic emergent dynamics for the stochastic kinetic Cucker-Smale equation perturbed by a multiplicative white noise. For a global well-posedness, we first derive a sequence of classical solutions to the stochastic kinetic C-S equation with regularized initial data. Then, using the properties of classical solutions, we obtained the existence of a strong solution corresponding to the original initial data and asymptotic emergent stochastic dynamics of strong solutions. Of course, there are lots of interesting issues to be addressed in a future work, e.g., a global existence of weak solutions, emergent dynamics under other types of random perturbations and zero noise limit, etc. These topics will be discussed in future works.

APPENDIX A. ELEMENTARY LEMMAS

In this appendix, we provide two useful lemmas used in previous sections. First, we begin with estimate on a variant of geometric brownian motion.

Lemma A.1. *Let $\{X_t\}_{t \geq 0}$ be a solution satisfying the following Cauchy problem:*

$$\begin{cases} dX_t = (a_t + b_t X_t)dt + cX_t dW_t, & t > 0, \\ X_0 = x \geq 0, \end{cases}$$

where $\{a_t\}_{t \geq 0}$ and $\{b_t\}_{t \geq 0}$ are stochastic processes with continuous sample paths, and c is a constant. Then one has

$$X_t = x \exp \left[\int_0^t \left(b_s - \frac{c^2}{2} \right) ds + cW_t \right] + \int_0^t a_s \exp \left[\int_s^t \left(b_\tau - \frac{c^2}{2} \right) d\tau + c(W_t - W_s) \right] ds.$$

Proof. The proof is exactly given in Example 19.7 from [33]. So, we refer to [33] for its proof. □

Lemma A.2. (Comparison principle) *Suppose that two stochastic processes $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ satisfy*

$$\begin{aligned} dX_t &\leq (a_t + bX_t)dt + cX_t dW_t, & X_0 &= x \geq 0, \\ dY_t &= (a_t + bY_t)dt + cY_t dW_t, & Y_0 &= x, \end{aligned}$$

where $\{a_t\}_{t \geq 0}$ is a stochastic process with continuous sample paths. Then, we have

$$X_t \leq Y_t, \quad \forall t \geq 0.$$

Proof. Let $\{Y_t^\delta\}_{t \geq 0}$, ($\delta > 0$) be a stochastic process satisfying

$$\begin{cases} dY_t^\delta = (a_t + bY_t^\delta)dt + cY_t^\delta dW_t, & t > 0, \\ Y_0^\delta = x + \delta, \end{cases}$$

and we set

$$Z_t^\delta := Y_t^\delta - X_t.$$

Then, we have

$$dZ_t^\delta \geq bZ_t^\delta dt + cZ_t^\delta dW_t, \quad t > 0 \quad \text{and} \quad Z_0 = \delta, \quad t = 0.$$

We use Itô's lemma to get

$$d(\ln Z_t^\delta) = \frac{dZ_t^\delta}{Z_t^\delta} - \frac{1}{2} \frac{1}{(Z_t^\delta)^2} (dZ_t^\delta) \cdot (dZ_t^\delta) \geq \left(b_t - \frac{c^2}{2} \right) dt + c dW_t.$$

Again, we integrate the above relation to get

$$Z_t^\delta \geq \delta \exp \left\{ \int_0^t \left(b_s - \frac{c^2}{2} \right) ds + cW_t \right\} \geq 0.$$

This yields

$$X_t \leq Y_t^\delta \quad \text{for all } t \geq 0.$$

It follows from the representation formula in Lemma A.1 that

$$Y_t^\delta = (x + \delta) \exp \left\{ \int_0^t \left(b_s - \frac{c^2}{2} \right) ds + cW_t \right\} + \int_0^t a_s \exp \left[\int_s^t \left(b_\tau - \frac{c^2}{2} \right) d\tau + c(W_t - W_s) \right] ds,$$

$$Y_t = x \exp \left\{ \int_0^t \left(b_s - \frac{c^2}{2} \right) ds + cW_t \right\} + \int_0^t a_s \exp \left[\int_s^t \left(b_\tau - \frac{c^2}{2} \right) d\tau + c(W_t - W_s) \right] ds.$$

This yields the desired result:

$$Y_t = \liminf_{\delta \rightarrow 0} Y_t^\delta \geq X_t.$$

□

APPENDIX B. A PROOF OF PROPOSITION 3.3

Recall that $f_t^{n,\varepsilon}$ satisfies a differential form:

$$\partial_t f_t^{n,\varepsilon} = -v \cdot \nabla_x f_t^{n,\varepsilon} - \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) + \sigma \nabla_v \cdot (v f_t^{n,\varepsilon}) \circ \dot{W}_t,$$

i.e., it satisfies

$$(B.1) \quad f_t^{n,\varepsilon} = f^{in,\varepsilon} - \int_0^t v \cdot \nabla_x f_s^{n,\varepsilon} ds - \int_0^t \nabla_v \cdot (F_a[f_s^{n-1,\varepsilon}] f_s^{n,\varepsilon}) ds$$

$$+ \sigma \int_0^t \nabla_v \cdot (v f_s^{n,\varepsilon}) \circ dW_s.$$

Next, we claim: there exists a nonnegative process \mathcal{A}_t^m with continuous sample paths and independent of n and ε such that

$$\|f_t^{n,\varepsilon}\|_{W^{m,\infty}} \leq \|f^{in,\varepsilon}\|_{W^{m,\infty}} \mathcal{A}_t^m.$$

In the sequel, we provide L^∞ -estimate of f_t and its derivatives to provide a proof of Proposition 3.3.

- (Zeroth-order estimate): It follows the formula (3.5) that

$$f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}(x, v)) = f^{in,\varepsilon}(x, v) \exp \left\{ - \int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}(x, v)) ds + d\sigma W_t \right\}$$

$$\leq \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t).$$

This implies zeroth-order estimate:

$$(B.2) \quad \|f_t^{n,\varepsilon}\|_{L^\infty} \leq \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\phi_M t + d\sigma W_t).$$

- (Higher-order estimates): Let α and β be multi-indices satisfying

$$1 \leq |\alpha| + |\beta| \leq m.$$

Then, we apply $\partial_x^\alpha \partial_v^\beta$ to the relation (B.1) using Theorem 3.1.2 in [24]:

$$\begin{aligned}
 \partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon} &= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon} - \sum_{|\mu_1| \leq 1} \binom{\beta}{\mu_1} \int_0^t \partial_v^{\mu_1}(v) \cdot \nabla_x (\partial_x^\alpha \partial_v^{\beta-\mu_1} f_s^{n,\varepsilon}) ds \\
 &\quad - \sum_{\substack{\mu_2 \leq \alpha \\ |\mu_3| \leq 1}} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \int_0^t \nabla_v \cdot (\partial_x^{\mu_2} \partial_v^{\mu_3} F_a[f_s^{n-1,\varepsilon}]) \partial_x^{\alpha-\mu_2} \partial_v^{\beta-\mu_3} f_s^{n,\varepsilon} ds \\
 &\quad + \sigma \sum_{|\mu_4| \leq 1} \binom{\beta}{\mu_4} \int_0^t \nabla_v \cdot (\partial_v^{\mu_4}(v) \partial_x^\alpha \partial_v^{\beta-\mu_4} f_s^{n,\varepsilon}) \circ dW_s,
 \end{aligned}
 \tag{B.3}$$

where we used the relation:

$$\partial_v^{\mu_3} F_a[f_t^{n-1,\varepsilon}] = 0, \quad \text{for } |\mu_3| \geq 2.$$

Note that the differentiation equality (B.3) is only true outside a \mathbb{P} -zero set in Ω which depends on (x, v) , according to Theorem 3.1.2 in [24]. However, we can use the argument in Lemma 2.2 to obtain that the equality also holds $\mathbb{P} \otimes dx \otimes dv$ -a.s. Now, we rearrange the previous relation to obtain

$$\begin{aligned}
 &\partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon} \\
 &= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon} - \int_0^t \left[v \cdot \nabla_x (\partial_x^\alpha \partial_v^\beta f_s^{n,\varepsilon}) + F_a[f_s^{n-1,\varepsilon}] \cdot \nabla_v (\partial_x^\alpha \partial_v^\beta f_s^{n,\varepsilon}) \right] ds \\
 &\quad + \sigma \int_0^t v \cdot \nabla_v (\partial_x^\alpha \partial_v^\beta f_s^{n,\varepsilon}) \circ dW_s - \frac{d+|\beta|}{d} \int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}] \partial_x^\alpha \partial_v^\beta f_s^{n,\varepsilon} ds \\
 &\quad + \sigma(d+|\beta|) \int_0^t \partial_x^\alpha \partial_v^\beta f_s^{n,\varepsilon} \circ dW_s - \int_0^t \mathcal{L}_{\alpha,\beta}(s) ds, \quad \mathbb{P} \otimes dx \otimes dv\text{-a.s.},
 \end{aligned}
 \tag{B.4}$$

where the process $\mathcal{L}_{\alpha,\beta}$ is given by the following relation:

$$\begin{aligned}
 \mathcal{L}_{\alpha,\beta} &:= \sum_{|\mu_1|=1} \binom{\beta}{\mu_1} \partial_v^{\mu_1}(v) \cdot \nabla_x (\partial_x^\alpha \partial_v^{\beta-\mu_1} f_s^{n,\varepsilon}) \\
 &\quad + \sum_{0 \neq \mu_2 \leq \alpha} \binom{\alpha}{\mu_2} \nabla_v \cdot (\partial_x^{\mu_2} F_a[f_s^{n-1,\varepsilon}]) \partial_x^{\alpha-\mu_2} \partial_v^\beta f_s^{n,\varepsilon} \\
 &\quad + \sum_{\substack{0 \neq \mu_2 \leq \alpha \\ |\mu_3|=1}} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \partial_x^{\mu_2} \partial_v^{\mu_3} F_a[f_s^{n-1,\varepsilon}] \cdot \nabla_v (\partial_x^{\alpha-\mu_2} \partial_v^{\beta-\mu_3} f_s^{n,\varepsilon}) \\
 &\quad + \sum_{0 \neq \mu_2 \leq \alpha} \binom{\alpha}{\mu_2} \partial_x^{\mu_2} F_a[f_s^{n-1,\varepsilon}] \cdot \nabla_v (\partial_x^{\alpha-\mu_2} \partial_v^\beta f_s^{n,\varepsilon}).
 \end{aligned}$$

Next, we define λ and $\tilde{\lambda}$ as follows:

$$\begin{aligned}
 \lambda_t(x, v) &:= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon}(x, v) - \frac{d+|\beta|}{d} \int_0^t \lambda_s(x, v) (\nabla_v \cdot F_a[f_s^{n-1,\varepsilon}]) (\varphi_s^{n,\varepsilon}) ds \\
 &\quad + \sigma(d+|\beta|) \int_0^t \lambda_s(x, v) \circ dW_s - \int_0^t \mathcal{L}_{\alpha,\beta}(\varphi_s^{n,\varepsilon}) ds, \\
 \tilde{\lambda}_t(x, v) &:= \lambda_t((\varphi_t^{n,\varepsilon})^{-1}).
 \end{aligned}$$

By using generalized Itô's formula from Theorem 3.3.2 in [24], $\tilde{\lambda}_t$ satisfies the relation (B.4). Thus, by the uniqueness,

$$\tilde{\lambda}_t = \partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon},$$

and we use Itô's formula on λ_t to get

$$\begin{aligned} & \partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) \\ &= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon}(x, v) \exp \left[-\frac{d+|\beta|}{d} \int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}) ds + \sigma(d+|\beta|)W_t \right] \\ & \quad - \int_0^t \exp \left[-\frac{d+|\beta|}{d} \int_s^t \nabla_v \cdot F_a[f_\tau^{n-1,\varepsilon}](\varphi_\tau^{n,\varepsilon}) d\tau + \sigma(d+|\beta|)(W_t - W_s) \right] \\ & \quad \times \mathcal{L}_{\alpha,\beta}(s, \varphi_s^{n,\varepsilon}) ds. \end{aligned}$$

For detailed explanation for the above relation, we refer to the proof of Theorem 3.2 in [8].

Note that the following estimates hold:

- If $|\beta| = 1$, one has

$$|\partial_x^\alpha \partial_v^\beta F_a[f_t^{n-1,\varepsilon}]| \leq \|\phi\|_{\mathcal{C}^m}.$$

- If $|\alpha| \geq 1$, one gets

$$\begin{aligned} & |\partial_x^\alpha F_a[f_t^{n-1,\varepsilon}](\varphi_t^{n,\varepsilon})| \\ & \leq \|\phi\|_{\mathcal{C}^m} \int_{\mathbb{R}^{2d}} |v_* \cdot V_t^{n,\varepsilon}| f_t^{n-1,\varepsilon}(x_*, v_*) dv_* dx_* \leq \|\phi\|_{\mathcal{C}^m} \left(|V_t^{n,\varepsilon}|^2 + M_2^{n-1,\varepsilon}(t) \right) \\ & \leq \|\phi\|_{\mathcal{C}^m} \left((R_0 + 1)^2 + \phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds + (\gamma + K_s) \exp(\gamma t) \right) \exp(\phi_M t - 2\sigma W_t). \end{aligned}$$

We set $C_{\alpha,\beta}(t)$ to be

$$\begin{aligned} C_{\alpha,\beta}(t) &:= \left(\sum_{|\mu_1|=1} \binom{\beta}{\mu_1} + \sum_{0 \neq \mu_2 \leq \alpha} \binom{\alpha}{\mu_2} + \sum_{\substack{0 \leq \mu_2 \leq \alpha \\ |\mu_3|=1}} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \right) \\ & \quad \times \left[1 + 2\|\phi\|_{\mathcal{C}^m} + \|\phi\|_{\mathcal{C}^m} (R_0 + 1)^2 \exp(\phi_M t - 2\sigma W_t) \right. \\ & \quad \left. + \left(\phi_M \int_0^t (\gamma + K_s) \exp(\gamma s) ds + (\gamma + K_t) \exp(\gamma t) \right) \exp(\phi_M t - 2\sigma W_t) \right]. \end{aligned}$$

This yields

$$|\mathcal{L}_{\alpha,\beta}(t, \varphi_t^{n,\varepsilon})| \leq C_{\alpha,\beta}(t) \|f_t^{n,\varepsilon}\|_{W^{m,\infty}}.$$

Thus, we have

$$\begin{aligned} & \partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) \\ (B.5) \quad & \leq \|\partial_x^\alpha \partial_v^\beta f^{in,\varepsilon}\|_{L^\infty} \exp((d+|\beta|)(\phi_M t + \sigma W_t)) \\ & \quad + \int_0^t \exp((d+|\beta|)\{\phi_M(t-s) + \sigma(W_t - W_s)\}) C_{\alpha,\beta}(s) \|f_s^{n,\varepsilon}\|_{W^{m,\infty}} ds. \end{aligned}$$

Now, we take the supremum over all characteristic flow, sum (B.5) over all $1 \leq |\alpha| + |\beta| \leq m$ and combine this with (B.2) to obtain

$$\begin{aligned} \|f_t^{n,\varepsilon}\|_{W^{m,\infty}} &\leq \|f^{in,\varepsilon}\|_{W^{m,\infty}} \mathcal{M}_t^m + \mathcal{M}_t^m \int_0^t \left[\exp(-(d+m)\phi_M s) \right. \\ &\quad \left. \times \sum_{|\alpha|+|\beta|\leq m} \exp(-\sigma(d+|\beta|)W_s) C_{\alpha,\beta}(s) \|f_s^{n,\varepsilon}\|_{W^{m,\infty}} \right] ds, \end{aligned}$$

where the process \mathcal{M}_t^m is given by the following relation:

$$\mathcal{M}_t^m := \exp((d+m)\phi_M t) \sum_{|\beta|\leq m} \exp(\sigma(d+|\beta|)W_t).$$

Note that \mathcal{M}_t^m is independent of n and ε . We set

$$b_n(t) := \|f_t^{n,\varepsilon}\|_{W^{m,\infty}} (\mathcal{M}_t^m)^{-1}.$$

Then, one gets

$$b_{n+1}(t) \leq b_0 + \int_0^t \tilde{\mathcal{N}}_s^m b_{n+1}(s) ds,$$

where the process $\tilde{\mathcal{N}}_s^m$ is

$$\tilde{\mathcal{N}}_s^m := \left\{ \sum_{|\beta|\leq m} \exp(\sigma(N+|\beta|)W_s) \right\} \left\{ \sum_{|\beta|\leq m} \exp(-\sigma(N+|\beta|)W_s) \right\} \left(\sum_{|\alpha|+|\beta|\leq m} C_{\alpha,\beta}(s) \right).$$

Thus, we can use Grönwall's lemma to obtain

$$\|f_t^{n,\varepsilon}\|_{W^{m,\infty}} \leq \|f^{in,\varepsilon}\|_{W^{m,\infty}} \mathcal{A}_t^m,$$

where the process \mathcal{A}_t^m is given by the following relation:

$$\begin{aligned} \mathcal{A}_t^m &:= \exp((d+m)\phi_M t) \sum_{|\beta|\leq m} \exp(\sigma(d+|\beta|)W_t) \\ &\quad \times \exp \left[\left\{ \sum_{|\beta|\leq m} \exp(\sigma(d+|\beta|)W_s) \right\} \left\{ \sum_{|\beta|\leq m} \exp(-\sigma(d+|\beta|)W_s) \right\} \left(\sum_{|\alpha|+|\beta|\leq m} C_{\alpha,\beta}(s) \right) \right]. \end{aligned}$$

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