UPPER ENVELOPES OF FAMILIES OF FELLER SEMIGROUPS AND VISCOSITY SOLUTIONS TO A CLASS OF NONLINEAR CAUCHY PROBLEMS

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ABSTRACT. In this paper we construct the smallest semigroup $\mathscr{S}$ that dominates a given family of linear Feller semigroups. The semigroup $\mathscr{S}$ will be referred to as the semigroup envelope or Nisio semigroup. In a second step we investigate strong continuity properties of the semigroup envelope and show that it is a viscosity solution to a nonlinear abstract Cauchy problem. We derive a condition for the existence of a Markov process under a nonlinear expectation for the case where the state space of the Feller processes is locally compact. The procedure is then applied to numerous examples, in particular nonlinear PDEs that arise from control problems for infinite dimensional Ornstein-Uhlenbeck processes and infinite dimensional Lévy processes.

Key words: Nisio semigroup, fully nonlinear PDE, viscosity solution, Feller process, nonlinear expectation

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1. Introduction and main results

For two (possibly nonlinear) semigroups $S = \big(S(t)\big)_{t\geq 0}$ and $T = \big(T(t)\big)_{t\geq 0}$ on a Banach lattice X we write $S \leq T$ if $S(t)x \leq T(t)x$ for all $t \geq 0$ and $x \in X$. For a nonempty index set Λ and a family $(S_\lambda)_{\lambda\in\Lambda}$ of semigroups on X we call a semigroup T an upper bound of S if $T \geq S_\lambda$ for all $\lambda \in \Lambda$. We call $\mathscr{S}$ a least upper bound of $(S_\lambda)_{\lambda\in\Lambda}$ if $\mathscr{S}$ is an upper bound of $(S_\lambda)_{\lambda\in\Lambda}$ and $\mathscr{S} \leq T$ for any other upper bound T of $(S_\lambda)_{\lambda\in\Lambda}$. Then, the very interesting question arises under which conditions the family $(S_\lambda)_{\lambda\in\Lambda}$ has a least upper bound. To the best of our knowledge this question has first been adressed by Nisio [11] for strongly continuous semigroups on the space of all bounded measurable functions, which is why we call the least upper bound $\mathscr{S}$ of $(S_\lambda)_{\lambda\in\Lambda}$ the *Nisio semigroup* or the *semigroup envelope* of $(S_\lambda)_{\lambda\in\Lambda}$. Due to a Theorem of Lotz [8] it is known that strongly continuous semigroups on the space of all bounded measurable functions always have a bounded generator, which is why the result of Nisio is not applicable for most semigroups related to partial differential equations. However, using a similar approach to the one by Nisio on the space of bounded and uniformly continuous functions, Denk et al. [3] proved the existence of a least upper bound for semigroups related to Lévy processes. In [9] the approach by Nisio has been used to characterize the generators of Markov chains with finite state space under

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nonlinear expectations. In the present paper, we use the idea of Nisio in a more general framework than Denk et al. in order to go beyond Lévy processes. Main examples will be Ornstein-Uhlenbeck processes on real separable Hilbert spaces, Geometric Brownian Motions, Lévy processes on real separable Hilbert spaces and Koopman semigroups with semiflows in real separable Banach spaces.

A fundamental result from semigroup theory is the fact that for a strongly continuous semigroup $S = \big(S(t)\big)_{t\geq 0}$ of linear operators with generator A the function $u(t) := S(t)x$ is a solution to the abstract Cauchy problem

$$u'(t) = Au(t) \quad \text{for } t \geq 0, \quad u(0) = u_0.$$

Similar as in the work by Denk et al. [3] we show that the Nisio semigroup yields a viscosity solution to the nonlinear Cauchy problem

$$u'(t) = \mathcal{A}u(t) \quad \text{for } t \geq 0, \quad u(0) = u_0,$$

where $\mathcal{A}u := \sup_{\lambda\in\Lambda} A_\lambda u$ is the pointwise supremum of the generators of the family of semigroups $(S_\lambda)_{\lambda\in\Lambda}$. On one hand, this is interesting from a structural point of view since it shows that there is still a relation between the least upper bound of a family of semigroups and the least upper bound of their generators. On the other hand, this shows that Nisio semigroups are closely related to solutions to optimal control problems in Mathematical Finance. Examples include solutions to BSDEs (see e.g. Coquet et al. [1]), 2BSDEs or the G-heat equation (cf. Peng [12],[13], Soner et al. [14],[15]), semigroups related to the geometric G-Brownian Motion (cf. Epstein and Ji [5], Vorbrink [16]) and G-Lévy processes (cf. Hu and Peng [6], Neufeld and Nutz [10], Denk et al. [3]). Against this background, under certain conditions, we derive a stochasic representation of Nisio semigroup via stochastic processes under a nonlinear expectation, which gives a link to robust finance and the pricing of contingent claims under ambiguity.

Throughout, we consider a nonempty index set Λ, a fixed separable metric space (M, d) and a fixed weight function $\kappa\colon M \to (0,\infty)$, which is assumed to be continuous and bounded. Let $\mathrm{C} = \mathrm{C}(M)$ be the space of all continuous functions $M \to \mathbb{R}$. We denote the space of all $u \in \mathrm{C}$ with norm

$$\|u\|_\infty := \sup_{x\in M} |u(x)| < \infty$$

by $\mathrm{C_b}$ and the space of all $u \in \mathrm{C}$ with seminorm

$$\|u\|_{\mathrm{Lip}} := \inf\big\{L \geq 0 \,\big|\, \forall x, y \in M : |u(x) - u(y)| \leq Ld(x, y)\big\} < \infty$$

by Lip. Finally, we denote the space of all $u \in C$ with norm

$$\|u\|_\kappa := \|\kappa u\|_\infty < \infty$$

by C_κ and the closure of $\mathrm{Lip_b} := \mathrm{Lip} \cap \mathrm{C_b}$ in the space C_κ by UC_κ. If κ is bounded below by some positive constant, then $C_\kappa = C_b$ and $\|\cdot\|_\kappa$ is equivalent to $\|\cdot\|_\infty$. In this case, UC_κ is the closure of Lip w.r.t. $\|\cdot\|_\infty$, which in most examples will be the space $\mathrm{UC_b}$ of all bounded and uniformly continuous functions $M \to \mathbb{R}$. For a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{UC}_\kappa$ and $u \in \mathrm{UC}_\kappa$ we write $u_n \nearrow u$ as $n \to \infty$ if $u_n \leq u_{n+1}$

for all $n \in \mathbb{N}$ and $u_n(x) \to u(x)$ as $n \to \infty$ for all $x \in M$. Analogously, we write $u_n \searrow u$ as $n \to \infty$ if $u_n \geq u_{n+1}$ for all $n \in \mathbb{N}$ and $u_n(x) \to u(x)$ as $n \to \infty$ for all $x \in M$.

Definition 1.1.

a) We call a family $\mathscr{S} = (\mathscr{S}(t))_{t\geq 0}$ of possibly nonlinear operators a *Feller semigroup* if the following conditions are satisfied:
   (i) $\mathscr{S}(t)\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is continuous for all $t \geq 0$,
   (ii) $\mathscr{S}(0)u = u$ and $\mathscr{S}(s+t)u = \mathscr{S}(s)\mathscr{S}(t)u$ for all $s, t \geq 0$ and $u \in \mathrm{UC}_\kappa$,
   (iii) $\mathscr{S}(t)$ is *monotone* and *continuous from below* for all $t \geq 0$, i.e. for any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{UC}_\kappa$ and $u \in \mathrm{UC}_\kappa$ with $u_n \nearrow u$ as $n \to \infty$ it holds $\mathscr{S}(t)u_n \nearrow \mathscr{S}(t)u$ as $n \to \infty$.

b) Let $D \subset \mathrm{UC}_\kappa$. We then say that a Feller semigroup $\mathscr{S}$ is *strongly continuous* on D if the map
$$[0,\infty) \to \mathrm{UC}_\kappa, \quad t \mapsto \mathscr{S}(t)u$$
is continuous for all $u \in D$. If $D = \mathrm{UC}_\kappa$, we say that $\mathscr{S}$ is *strongly continuous*.

Remark 1.2. Let $\mathscr{S}$ be a Feller semigroup and $D \subset \mathrm{UC}_\kappa$ the set of all $u \in \mathrm{UC}_\kappa$ such that
$$[0,\infty) \to \mathrm{UC}_\kappa, \quad t \mapsto \mathscr{S}(t)u$$
is continuous. Then, by the semigroup property (ii), the set D is invariant under $\mathscr{S}$, i.e. $\mathscr{S}(t)u \in D$ for all $u \in D$ and all $t \geq 0$.

Throughout this work, we assume the following setup:

(A1) For all $\lambda \in \Lambda$ let S_λ be a Feller semigroup of linear operators.

(A2) There exist constants $\alpha, \beta \in \mathbb{R}$ such that
$$\|S_\lambda(t)u\|_\kappa \leq e^{\alpha t}\|u\|_\kappa \quad \text{and} \quad \|S_\lambda(t)u\|_{\mathrm{Lip}} \leq e^{\beta t}\|u\|_{\mathrm{Lip}}$$
for all $u \in \mathrm{Lip}_{\mathrm{b}}$, $\lambda \in \Lambda$ and $t \geq 0$.

The paper is structured as follows. In Section 2 we show the existence of the semigroup envelope $\mathscr{S}$ of the family $S = (S_\lambda)_{\lambda\in\Lambda}$ under the assumptions (A1) and (A2). The main result of this section is Theorem 2.6. In Section 3 we first derive three conditions that guarantee the strong continuity of the semigroup envelope, which in turn yields viscosity solutions to a nonlinear abstract Cauchy problem. The main result of this section is Theorem 3.12. In Section 4 we derive some approximation results for the Nisio semigroup. In Section 5, we give a stochastic representation of the semigroup envelope via a stochastic process under a sublinear expectation. In Section 6 we apply the results from the sections 2, 3 and 5 to several examples.

2. Construction of the semigroup envelope

Throughout, we assume that the conditions (A1) and (A2) are satisfied. Let $u \in \mathrm{UC}_\kappa$, $\lambda \in \Lambda$ and $t \geq 0$. Then, since $S_\lambda(t)\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is continuous, we have that
$$\|S_\lambda(t)u\|_\kappa \leq e^{\alpha t}\|u\|_\kappa.$$

Hence,

$$\big(\mathcal{E}_h u\big)(x) := \sup_{\lambda\in\Lambda} \big(S_\lambda(h)u\big)(x)$$

is well-defined for all $x \in M$ and $h \geq 0$. Moreover, we have the following:

Lemma 2.1. *Let $h \geq 0$.*

a) *For all $u, v \in \mathrm{UC}_\kappa$,*

$$\|\mathcal{E}_h u - \mathcal{E}_h v\|_\kappa \leq e^{\alpha h}\|u - v\|_\kappa.$$

In particular, $\|\mathcal{E}_h u\|_\kappa \leq e^{\alpha h}\|u\|_\kappa$ for all $u \in \mathrm{UC}_\kappa$.

b) *$\|\mathcal{E}_h u\|_{\mathrm{Lip}} \leq e^{\beta h}\|u\|_{\mathrm{Lip}}$ for all $u \in \mathrm{Lip}_\mathrm{b}$.*

c) *The map $\mathcal{E}_h \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined and Lipschitz continuous with Lipschitz constant $e^{\alpha h}$.*

d) *$\mathcal{E}_h$ is sublinear, monotone and continuous from below for all $h \geq 0$.*

Proof.

a) Let $u, v \in \mathrm{UC}_\kappa$ and $h \geq 0$.

$$\begin{aligned}\kappa\big(S_\lambda(h)u - \mathcal{E}_h v\big) &\leq \kappa\big(S_\lambda(h)u - S_\lambda(h)v\big) = \kappa S_\lambda(h)(u-v)\\ &\leq \|S_\lambda(h)(u-v)\|_\kappa \leq e^{\alpha h}\|u-v\|_\kappa\end{aligned}$$

for all $\lambda \in \Lambda$. Taking the supremum over all $\lambda \in \Lambda$, we obtain that

$$\kappa\big(\mathcal{E}_h u - \mathcal{E}_h v\big) \leq e^{\alpha h}\|u - v\|_\kappa$$

and therefore, by a symmetry argument,

$$\|\mathcal{E}_h u - \mathcal{E}_h v\|_\kappa \leq e^{\alpha h}\|u - v\|_\kappa.$$

b) Let $u \in \mathrm{Lip}_\mathrm{b}$ and $x, y \in M$. Then,

$$(S_\lambda(h)u\big)(x) - \big(\mathcal{E}_h u\big)(y) \leq (S_\lambda(h)u\big)(x) - (S_\lambda(h)u\big)(y) \leq e^{\beta h}\|u\|_{\mathrm{Lip}} d(x,y).$$

Taking the supremum over all $\lambda \in \Lambda$, we obtain that

$$\big(\mathcal{E}_h u\big)(x) - \big(\mathcal{E}_h u\big)(y) \leq e^{\beta h}\|u\|_{\mathrm{Lip}} d(x,y)$$

and therefore, by a symmetry argument,

$$\big|\big(\mathcal{E}_h u\big)(x) - \big(\mathcal{E}_h u\big)(y)\big| \leq e^{\beta h}\|u\|_{\mathrm{Lip}} d(x,y).$$

This shows that $\|\mathcal{E}_h u\|_{\mathrm{Lip}} \leq e^{\beta h}\|u\|_{\mathrm{Lip}}$.

c) By part b), we have that $\mathcal{E}_h u \in \mathrm{UC}_\kappa$ for all $u \in \mathrm{Lip}_\mathrm{b}$. Since Lip_b is dense in UC_κ, part a) implies that $\mathcal{E}_h \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined and Lipschitz continuous with Lipschitz constant $e^{\alpha h}$.

d) All these properties diretly carry over to the supremum.

□

In the sequel, we consider finite partitions $P := \{\pi \subset [0,\infty) \colon 0 \in \pi,\ |\pi| < \infty\}$ of the positive half line. The set of partitions with end-point t will be denoted by P_t, i.e. $P_t := \{\pi \in P : \max \pi = t\}$. Note that

$$P = \bigcup_{t\geq 0} P_t.$$

Let $u \in \mathrm{UC}_\kappa$ and $\pi \in P \setminus \big\{\{0\}\big\}$. Then, there exist $0 = t_0 < t_1 < \ldots < t_m$ such that $\pi = \{t_0, t_1, \ldots, t_m\}$ and we set

$$\mathcal{E}_\pi u := \mathcal{E}_{t_1 - t_0} \ldots \mathcal{E}_{t_m - t_{m-1}} u.$$

Moreover, we set $\mathcal{E}_{\{0\}} u := u$. Note that, by definition, $\mathcal{E}_h = \mathcal{E}_{\{0,h\}}$ for $h > 0$. Since $\mathcal{E}_h \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined, the map $\mathcal{E}_\pi \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined, too.

Lemma 2.2. *Let $\pi \in P$.*

a) *$\mathcal{E}_\pi$ is a sublinear, monotone and continuous from below.*
b) *For all $u, v \in \mathrm{UC}_\kappa$,*

$$\|\mathcal{E}_\pi u - \mathcal{E}_\pi v\|_\kappa \le e^{\alpha \max \pi} \|u - v\|_\kappa.$$

In particular, $\mathcal{E}_\pi \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined and Lipschitz continuous with Lipschitz constant $e^{\alpha \max \pi}$. Moreover,

$$\|\mathcal{E}_\pi u\|_{\mathrm{Lip}} \le e^{\beta \max \pi} \|u\|_{\mathrm{Lip}}$$

for all $u \in \mathrm{Lip}_{\mathrm{b}}$.

Proof.

a) Since $\mathcal{E}_h$ is a sublinear, monotone and continuous from below for all $h \ge 0$, the same holds for $\mathcal{E}_\pi$ as these properties are preserved under compositions.
b) This follows from Lemma 2.1 and the behaviour of Lipschitz constants under composition.

□

Let $u \in \mathrm{UC}_\kappa$. In the following, we consider the limit of $\mathcal{E}_\pi u$ when the mesh size of the partition $\pi \in P$ tends to zero. For this, first note that for $h_1, h_2 \ge 0$ and $x \in M$ we have that

$$\begin{aligned}\big(\mathcal{E}_{h_1+h_2} u\big)(x) &= \sup_{\lambda \in \Lambda} \big(S_\lambda(h_1 + h_2) u\big)(x) = \sup_{\lambda \in \Lambda} \big(S_\lambda(h_1) S_\lambda(h_2) u\big)(x) \\ &\le \sup_{\lambda \in \Lambda} \big(S_\lambda(h_1) \mathcal{E}_{h_2} u\big)(x) = \big(\mathcal{E}_{h_1} \mathcal{E}_{h_2} u\big)(x),\end{aligned}$$

which implies the pointwise inequality

$$\mathcal{E}_{\pi_1} u \le \mathcal{E}_{\pi_2} u \tag{2.1}$$

for $\pi_1, \pi_2 \in P$ with $\pi_1 \subset \pi_2$. In particular, for $\pi_1, \pi_2 \in P$ and $\pi := \pi_1 \cup \pi_2$ we have that $\pi \in P$ with

$$\big(\mathcal{E}_{\pi_1} u\big) \vee \big(\mathcal{E}_{\pi_2} u\big) \le \mathcal{E}_\pi u. \tag{2.2}$$

Recall that we denote by P_t the set of all finite partitions with end point $t \ge 0$. For $t \ge 0$, $x \in M$ and $u \in \mathrm{UC}_\kappa$ we define

$$\big(\mathscr{S}(t) u\big)(x) := \sup_{\pi \in P_t} \big(\mathcal{E}_\pi u\big)(x).$$

Note that $\mathscr{S}(0) u = u$ for all $u \in \mathrm{UC}_\kappa$. The family $\mathscr{S} = \big(\mathscr{S}(t)\big)_{t \ge 0}$ is called the *semigroup envelope* or *Nisio semigroup* of the family $(S_\lambda)_{\lambda \in \Lambda}$. Let $t \ge 0$. Then,

$$\|\mathscr{S}(t) u - \mathscr{S}(t) v\|_\kappa \le e^{\alpha t} \|u - v\|_\kappa$$

for all $u, v \in \mathrm{UC}_\kappa$ and

$$\|\mathscr{S}(t)\|_{\mathrm{Lip}} \leq e^{\beta t}\|u\|_{\mathrm{Lip}}$$

due to Lemma 2.2. Therefore, the map $\mathscr{S}(t)\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined and Lipschitz continuous with Lipschitz constant $e^{\alpha t}$. Moreover, $\mathscr{S}(t)$ is sublinear, monotone and continuous from below. In the following, we show that the Nisio semigroup $\mathscr{S}$ is in fact a semigroup. We start with the following lemma.

Lemma 2.3. *Let $u \in \mathrm{UC}_\kappa$ and $t > 0$. Then, there exists a sequence $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with*

$$\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u \quad \text{as } n \to \infty.$$

Proof. Let $(x_k)_{k\in\mathbb{N}} \subset M$ such that the set $\{x_k \,|\, k \in \mathbb{N}\}$ is dense in M. Then, for every $k \in \mathbb{N}$, there exists a sequence $(\pi_n^k)_{n\in\mathbb{N}} \subset P_t$ with $\pi_n^k \subset \pi_{n+1}^k$ for all $n \in \mathbb{N}$ and

$$\big(\mathcal{E}_{\pi_n^k} u\big)(x_k) \nearrow \big(\mathscr{S}(t)u\big)(x_k) \quad \text{as } n \to \infty.$$

Now, let

$$\pi_n := \bigcup_{k=1}^n \pi_n^k$$

for all $n \in \mathbb{N}$. Then, $\pi_n^k \subset \pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. Hence,

$$\mathcal{E}_{\pi_n^k} \leq \mathcal{E}_{\pi_n} u \leq \mathcal{E}_{\pi_{n+1}} \tag{2.3}$$

for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. Let

$$\big(\mathcal{E}_\infty v\big)(x) := \sup_{n\in\mathbb{N}} \big(\mathcal{E}_{\pi_n} v\big)(x)$$

for all $v \in \mathrm{UC}_\kappa$ and $x \in M$. Then, by Lemma 2.2, the map $\mathcal{E}_\infty\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined. In particular, $\mathcal{E}_\infty u\colon M \to \mathbb{R}$ is continuous and, by (2.3),

$$\mathcal{E}_{\pi_n} u \nearrow \mathcal{E}_\infty u \quad \text{as } n \to \infty.$$

Again, by (2.3),

$$\big(\mathscr{S}(t)u\big)(x_k) = \lim_{n\to\infty} \big(\mathcal{E}_{\pi_n^k} u\big)(x_k) \leq \lim_{n\to\infty} \big(\mathcal{E}_{\pi_n} u\big)(x_k) = \big(\mathcal{E}_\infty u\big)(x_k) \leq \big(\mathscr{S}(t)u\big)(x_k)$$

for all $k \in \mathbb{N}$. Since, $\mathscr{S}(t)u$ and $\mathcal{E}_\infty u$ are both continuous and the set $\{x_k \,|\, k \in \mathbb{N}\}$ is dense in M, it follows that $\mathscr{S}(t)u = \mathcal{E}_\infty u$, which shows that

$$\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u \quad \text{as } n \to \infty.$$

□

Proposition 2.4 (Dynamic programming principle)**.** *For all $s, t \geq 0$ we have that*

$$\mathscr{S}(s+t) = \mathscr{S}(s)\mathscr{S}(t). \tag{2.4}$$

Proof. Let $u \in \mathrm{UC}_\kappa$. If $s = 0$ or $t = 0$ the statement is trivial. Therefore, let $s, t > 0$, $\pi_0 \in P_{s+t}$ and $\pi := \pi_0 \cup \{s\}$. Then, we have that $\pi \in P_{s+t}$ with $\pi_0 \subset \pi$. Hence, by (2.1), we get that

$$\mathcal{E}_{\pi_0} u \leq \mathcal{E}_\pi u.$$

Let $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots t_m = s + t$ with $\pi = \{t_0, \dots, t_m\}$ and $i \in \{1, \dots, m\}$ with $t_i = s$. Then, we have that $\pi_1 := \{t_0, \dots, t_i\} \in P_s$ and $\pi_2 := \{t_i - s, \dots, t_n - s\} \in P_t$ with

$$\mathcal{E}_{\pi_1} = \mathcal{E}_{t_1 - t_0} \cdots \mathcal{E}_{t_i - t_{i-1}}$$

and

$$\mathcal{E}_{\pi_2} = \mathcal{E}_{t_{i+1} - t_i} \cdots \mathcal{E}_{t_m - t_{m-1}}.$$

We thus get that

$$\begin{aligned} \mathcal{E}_{\pi_0} u \le \mathcal{E}_\pi u = \mathcal{E}_{t_1 - t_0} \cdots \mathcal{E}_{t_m - t_{m-1}} u &= \big(\mathcal{E}_{t_1 - t_0} \cdots \mathcal{E}_{t_i - t_{i-1}}\big)\big(\mathcal{E}_{t_{i+1} - t_i} \cdots \mathcal{E}_{t_m - t_{m-1}} u\big) \\ = \mathcal{E}_{\pi_1} \mathcal{E}_{\pi_2} u \le \mathcal{E}_{\pi_1}\big(\mathscr{S}(t) u\big) \le \mathscr{S}(s)\mathscr{S}(t) u. \end{aligned}$$

Taking the supremum over all $\pi_0 \in P_{s+t}$, we get that $\mathscr{S}(s+t)u \le \mathscr{S}(s)\mathscr{S}(t)u$.

Now, let $(\pi_n)_{n \in \mathbb{N}} \subset P_t$ with $\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t) u$ as $n \to \infty$ (see Lemma 2.3) and fix $\pi_0 \in P_s$. Then, for all $n \in \mathbb{N}$ we have that

$$\pi_n' := \pi_0 \cup \{s + \tau \colon \tau \in \pi_n\} \in P_{s+t}$$

with $\mathcal{E}_{\pi_n'} = \mathcal{E}_{\pi_0} \mathcal{E}_{\pi_n}$. As $\mathcal{E}_{\pi_0}$ is continuous from below, we get that

$$\mathcal{E}_{\pi_0}\big(\mathscr{S}(t)u\big) = \lim_{n \to \infty} \mathcal{E}_{\pi_0} \mathcal{E}_{\pi_n} u = \lim_{n \to \infty} \mathcal{E}_{\pi_n'} u \le \mathscr{S}(s+t) u.$$

Taking the supremum over all $\pi_0 \in P_s$, we get that $\mathscr{S}(s)\mathscr{S}(t) f \le \mathscr{S}(s+t) f$. $\square$

Remark 2.5. The semigroup $\mathscr{S}$ is the least upper bound of the family $S = (S_\lambda)_{\lambda \in \Lambda}$. In fact, let T be an upper bound of the family S, i.e.

$$\big(S_\lambda(t) u\big)(x) \le \big(T(t) u\big)(x)$$

for all $\lambda \in \Lambda$, $u \in \mathrm{UC}_\kappa$, $t \ge 0$ and $x \in M$. Then, we have that

$$\big(S_\lambda(h) u\big)(x) \le \big(\mathcal{E}_h u\big)(x) \le \big(T(h) u\big)(x)$$

for all $\lambda \in \Lambda$, $u \in \mathrm{UC}_\kappa$, $h \ge 0$ and $x \in M$. Consequently

$$\big(S_\lambda(t) u\big)(x) \le \big(\mathcal{E}_\pi u\big)(x) \le \big(T(t) u\big)(x)$$

for all $\lambda \in \Lambda$, $u \in \mathrm{UC}_\kappa$, $t \ge 0$, $\pi \in P_t$ and $x \in M$. Taking the supremum over all $\pi \in P_t$, we obtain that

$$\big(S_\lambda(t) u\big)(x) \le \big(\mathscr{S}(t) u\big)(x) \le \big(T(t) u\big)(x)$$

for all $\lambda \in \Lambda$, $u \in \mathrm{UC}_\kappa$, $t \ge 0$ and $x \in M$.

We conclude this section with the following main theorem.

Theorem 2.6. *The family $\mathscr{S}$ is a Feller semigroup of sublinear operators and the least upper bound of the family $S = (S_\lambda)_{\lambda \in \Lambda}$.*

Proof. All properties except for the semigroup property follow directly from the definition of $\mathscr{S}$ before Lemma 2.3. The semigroup property is exactly (2.4). The fact that $\mathscr{S}$ is the least upper bound of the family S has been shown in the previous remark. $\square$

3. Strong continuity and viscosity solutions

Let $\mathscr{S}$ be the Feller semigroup from the previous section, i.e. the semigroup envelope of the family $(S_\lambda)_{\lambda\in\Lambda}$.

Lemma 3.1. *Let $u \in \mathrm{UC}_\kappa$. Then, the following statements are equivalent:*

(i) $\lim_{h\to 0} \|\mathscr{S}(h)u - u\|_\kappa = 0$.

(ii) *The map*

$$[0,\infty) \to \mathrm{UC}_\kappa, \quad t \mapsto \mathscr{S}(t)u$$

is continuous.

Proof. Clearly, (ii) implies (i). Therefore, assume that $\lim_{h\to 0} \|\mathscr{S}(h)u-u\|_\kappa = 0$. Let $t \geq 0$ and $\varepsilon > 0$. W.l.o.g. we may assume that in (A2) we have $\alpha \geq 0$. By assumption, there exists some $\delta > 0$ such that

$$\|\mathscr{S}(h)u - u\|_\kappa < e^{-\alpha(t+1)}\varepsilon$$

for all $h \in [0,\delta)$. Now, let $s \in [0, t+1]$ with $|t-s| < \delta$. Then,

$$\begin{aligned}\|\mathscr{S}(t)u - \mathscr{S}(s)u\|_\kappa &= \big\|\mathscr{S}(t\wedge s)\mathscr{S}\big(|t-s|\big)u - \mathscr{S}(t\wedge s)u\big\|_\kappa \\ &\leq e^{\alpha(t+1)}\big\|\mathscr{S}\big(|t-s|\big)u - u\big\|_\kappa < \varepsilon.\end{aligned}$$

□

Remark 3.2. Let $D \subset \mathrm{UC}_\kappa$ and assume that $\mathscr{S}$ is stongly continuous on D. Then, $\mathscr{S}$ is also strongly continuous on the closure $\overline{D}$ of D. In order to see this, let $(u_n)_{n\in\mathbb{N}} \subset D$ and $u \in \mathrm{UC}_\kappa$ with $\|u_n - u\|_\kappa \to 0$ as $n \to \infty$. W.l.o.g. we may assume that $\alpha \geq 0$. Let $\varepsilon > 0$. Then, there exists some $n_0 \in \mathbb{N}$ such that $\|u_{n_0} - u\|_\kappa \leq \frac{\varepsilon}{3}e^{-\alpha}$. Since $u_{n_0} \in D$, there exists some $\delta \in (0,1]$ such that $\|\mathscr{S}(h)u_{n_0} - u_{n_0}\|_\kappa < \frac{\varepsilon}{3}$ for all $h \in [0,\delta)$. Hence, for $h \in [0,\delta)$, it follows that

$$\|\mathscr{S}(h)u - u\|_\kappa \leq \frac{2\varepsilon}{3} + \|\mathscr{S}(h)u_{n_0} - u_{n_0}\|_\kappa < \varepsilon.$$

By the previous lemma, it follows that the map $[0,\infty) \to \mathrm{UC}_\kappa$, $t \mapsto \mathscr{S}(t)u$ is continuous.

In the sequel, we will give three conditions that imply the strong continuity of the semigroup $\mathscr{S}$.

Proposition 3.3. *Assume that for every $\delta > 0$ there exists a family of functions $(\varphi_x)_{x\in M} \subset \mathrm{UC}_\kappa$ satisfying the following:*

(i) $0 \leq \varphi_x(y) \leq 1$ *for all* $y \in M$, $\varphi_x(x) = 1$, *and* $\varphi_x(y) = 0$ *for all* $y \in M$ *with* $d(x,y) \geq \delta$,

(ii) *There exists some $h_0 > 0$ and a continuous function $f\colon [0,h_0) \to [0,\infty)$ with $f(0) = 0$ such that, for all $h \in [0,h_0)$ and $x \in M$,*

$$\big(\mathscr{S}(h)(1-\varphi_x)\big)(x) \leq \frac{f(h)}{\kappa(x)}.$$

If, additionally, $S_\lambda(t)1 = 1$ for all $\lambda \in \Lambda$ and $t \geq 0$, where 1 denotes the constant 1-function, then $\mathscr{S}$ is strongly continuous.

Proof. Let $u \in \mathrm{Lip_b} \setminus \{0\}$ and $\varepsilon > 0$. Then, there exists some $\delta' > 0$ such that

$$|u(x) - u(y)| \leq \tfrac{\varepsilon}{2\|\kappa\|_\infty} \quad \text{for all } x, y \in M \text{ with } d(x,y) < \delta'.$$

By assumption, there exists a family $(\varphi_x)_{x\in M} \subset \mathrm{UC}_\kappa$ satisfying (i) and (ii) for $\delta = \delta'$. Since $f\colon [0, h_0) \to [0,\infty)$ is continuous, there exists some $h_1 \in (0, h_0)$ such that

$$f(h) < \frac{\varepsilon}{4\|u\|_\infty}$$

for all $h \in [0, h_1)$. Hence, for all $x \in M$ and all $h \in [0, h_1)$,

$$\begin{aligned}
\big|\big(\mathscr{S}(h)u\big)(x) - u(x)\big| &= \big|\big(\mathscr{S}(h)(u - u(x))\big)(x)\big| \leq \big(\mathscr{S}(h)|u - u(x)|\big)(x) \\
&\leq \big(\mathscr{S}(h)(\varphi_x|u - u(x)|)\big)(x) + \big(\mathscr{S}(h)(1-\varphi_x)|u - u(x)|\big)(x) \\
&\leq \frac{\varepsilon}{2\|\kappa\|_\infty} + 2\|u\|_\infty\big(\mathscr{S}(h)(1-\varphi_x)\big)(x) \\
&< \frac{\varepsilon}{\kappa(x)}.
\end{aligned}$$

This shows that

$$\big\|\mathscr{S}(h)u - u\big\|_\kappa < \varepsilon$$

for all $h \in [0, h_1)$ and therefore, $\mathscr{S}$ is strongly continuous on $\mathrm{Lip_b}$. Since $\mathrm{Lip_b}$ is dense in UC_κ, Remark 3.2 imples that $\mathscr{S}$ is strongly continuous. $\square$

Throughout the rest of this section, we denote by $D_\Lambda \subset \mathrm{UC}_\kappa$ the subspace of all $u \in \mathrm{UC}_\kappa$ for which there exist $L_u \geq 0$ and $h_u > 0$ such that

$$\sup_{\lambda\in\Lambda} \|S_\lambda(h)u - u\|_\kappa \leq L_u h$$

for all $h \in [0, h_u)$.

Proposition 3.4. *The semigroup $\mathscr{S}$ is strongly continuous on $\overline{D_\Lambda}$. If D_Λ is dense in UC_κ, then $\mathscr{S}$ is strongly continuous.*

Proof. Let $u \in D_\Lambda$ and $0 \leq h_1 < h_2$ with $h_2 - h_1 < h_u$. Then, for all $x \in M$ and $\lambda_0 \in \Lambda$, we have that

$$\big(S_{\lambda_0}(h_1)u\big)(x) - \big(\mathscr{E}_{h_2}u\big)(x) \leq \big(S_{\lambda_0}(h_1)u\big)(x) - \big(S_{\lambda_0}(h_2)u\big)(x).$$

Taking the supremum over $\lambda_0 \in \Lambda$, we get that

$$\big(\mathscr{E}_{h_1}u\big)(x) - \big(\mathscr{E}_{h_2}u\big)(x) \leq \sup_{\lambda\in\Lambda}\big|\big(S_\lambda(h_1)u\big)(x) - \big(S_\lambda(h_2)u\big)(x)\big|$$

for all $x \in M$. By a symmetry argument, multiplying by $\kappa(x)$ and taking the supremum over all $x \in M$, we thus get that

$$\|\mathscr{E}_{h_1}u - \mathscr{E}_{h_2}u\|_\kappa \leq \sup_{\lambda\in\Lambda}\|S_\lambda(h_1)u - S_\lambda(h_2)u\|_\kappa.$$

Further, by assumption (A3),

$$\|S_\lambda(h_1)u - S_\lambda(h_2)u\|_\kappa \leq e^{\alpha h_1}\|S_\lambda(h_2 - h_1)u - u\|_\kappa \leq L_u e^{\alpha h_1}(h_2 - h_1).$$

Taking the supremum over all $\lambda \in \Lambda$, we obtain that

$$\|\mathscr{E}_{h_1}u - \mathscr{E}_{h_2}u\|_\kappa \leq L_u e^{\alpha h_1}(h_2 - h_1). \tag{3.1}$$

Next, we show that

$$\|\mathcal{E}_\pi u - u\|_\kappa \leq L_u e^{\alpha \max \pi} \max \pi \tag{3.2}$$

for all $\pi \in P$ with $\max \pi \in [0, h_u)$ by an induction on $\#\pi \in \mathbb{N}$. First, let $\pi \in P$ with $\#\pi = 1$, i.e. $\pi = \{0\}$. Then, we have that

$$\|\mathcal{E}_\pi u - u\|_\kappa = \|\mathcal{E}_{\{0\}} u - u\|_\kappa = 0 = L_u e^{\alpha \max \pi} \max \pi.$$

Now, let $m \in \mathbb{N}$ and assume that (3.2) holds for all $\pi \in P$ with $\max \pi \in [0, h_u)$ and $\#\pi = m$. Let $\pi \in P$ with $\#\pi = m + 1$ and $t_m := \max \pi \in [0, h_u)$. Then $\pi' := \pi \setminus \{t_m\} \in P$ with $\#\pi' = m$ and $t_{m-1} := \max \pi' \in [0, t_m)$. We thus have that

$$\mathcal{E}_\pi u = \mathcal{E}_{\pi'} \mathcal{E}_{t_m - t_{m-1}} u \tag{3.3}$$

and therefore, by induction hypothesis, (3.3) and (3.1), we get that

$$\begin{aligned}
\|\mathcal{E}_\pi u - u\|_\kappa &\leq \|\mathcal{E}_\pi u - \mathcal{E}_{\pi'} u\|_\kappa + \|\mathcal{E}_{\pi'} u - u\|_\kappa \\
&= \|\mathcal{E}_{\pi'} \mathcal{E}_{t_m - t_{m-1}} u - \mathcal{E}_{\pi'} u\|_\kappa + \|\mathcal{E}_{\pi'} u - u\|_\kappa \\
&\leq e^{\alpha t_{m-1}} \|\mathcal{E}_{t_m - t_{m-1}} u - u\|_\kappa + \|\mathcal{E}_{\pi'} u - u\|_\kappa \\
&\leq L_u e^{\alpha t_{m-1}} (t_m - t_{m-1}) + L_u e^{\alpha t_{m-1}} t_{m-1} \\
&= L_u e^{\alpha t_{m-1}} t_m \leq L_u e^{\alpha \max \pi} \max \pi.
\end{aligned}$$

By definition of the semigroup $\mathscr{S}$ we thus obtain that

$$\|\mathscr{S}(h) u - u\|_\kappa \leq L_u e^{\alpha h} h \to 0 \quad \text{as } h \to 0.$$

□

Proposition 3.5. *Assume that for every $\delta > 0$ there exists a family of functions $(\varphi_x)_{x \in M} \subset \mathrm{UC}_\kappa$ satisfying the following:*

(i) *$0 \leq \varphi_x(y) \leq 1$ for all $y \in M$, $\varphi_x(x) = 1$, and $\varphi_x(y) = 0$ for all $y \in M$ with $d(x, y) \geq \delta$,*

(ii') *There exist $L \geq 0$ and $h_0 > 0$ such that*

$$\sup_{\lambda \in \Lambda} \|S_\lambda(h) \varphi_x - \varphi_x\|_\kappa \leq Lh$$

for all $h \in [0, h_0)$ and $x \in M$.

If, additionally, $S_\lambda(t)1 = 1$ for all $\lambda \in \Lambda$ and $t \geq 0$, where 1 denotes the constant 1-function, then $\mathscr{S}$ is strongly continuous.

Proof. By assumption the family $(\varphi_x)_{x \in M}$ satisfies condition (i) from Proposition 3.3. We now verify that (ii') implies condition (ii) from Proposition 3.3. Observe that

$$\big(\mathcal{E}_h(1 - \varphi_x)\big)(y) \leq 1 - \varphi_x(y) + \big|\big(\mathcal{E}_h \varphi_x\big)(y) - \varphi_x(y)\big|$$

for all $h \in [0, h_0)$ and $x, y \in M$. W.l.o.g. we assume that $\alpha \geq 0$ in (A2). Then, by (3.1), we thus obtain that

$$\begin{aligned}\big(\mathcal{E}_\pi \mathcal{E}_h(1-\varphi_x)\big)(x) &\leq \big(\mathcal{E}_\pi(1-\varphi_x)\big(x) + \big(\mathcal{E}_\pi\big|\mathcal{E}_h\varphi_x - \varphi_x\big|\big)(x) \\ &\leq \big(\mathcal{E}_\pi(1-\varphi_x)\big)(x) + \frac{e^{\alpha\delta}}{\kappa(x)}\|\mathcal{E}_h\varphi_x - \varphi_x\|_\kappa \\ &\leq \big(\mathcal{E}_\pi(1-\varphi_x)\big)(x) + \frac{Le^{\alpha\delta}h}{\kappa(x)}\end{aligned}$$

for all $\pi \in P$ with $\max \pi \in [0, h_0)$ and $h \in [0, h_0)$. Inductively, it follows that

$$\big(\mathcal{E}_\pi(1-\varphi_x)\big(x) \leq 1 - \varphi_x(x) + \frac{Le^{\alpha\delta}\max\pi}{\kappa(x)} = \frac{Le^{\alpha\delta}\max\pi}{\kappa(x)}$$

for all $\pi \in P$ with $\max \pi \in [0, h_0)$. Taking the supremum over all $\pi \in P_h$ for $h \in [0, h_0)$ yields that

$$\big(\mathscr{S}(h)(1-\varphi_x)\big)(x) \leq \frac{Le^{\alpha\delta}h}{\kappa(x)}.$$

Therefore, setting $f(h) := Le^{\alpha\delta}h$ for $h \in [0, h_0)$, condition (ii) from Proposition 3.3 is satisfied and the strong continuity of $\mathscr{S}$ follows. $\square$

Let $\lambda \in \Lambda$. Then, we denote by $D_\lambda \subset \mathrm{UC}_\kappa$ the space of all $u \in \mathrm{UC}_\kappa$ such that

$$[0, \infty) \to \mathrm{UC}_\kappa, \quad t \mapsto S_\lambda(t)u$$

is continuous. We further denote by $D(A_\lambda)$ the space of all $u \in \mathrm{UC}_\kappa$ for which

$$A_\lambda u := \lim_{h \searrow 0} \frac{S_\lambda(h)u - u}{h} \in \mathrm{UC}_\kappa$$

exists. Notice that, by definition, $D(A_\lambda) \subset D_\lambda$.

Remark 3.6. Let $u \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ with

$$C_u := \sup_{\lambda\in\Lambda} \|A_\lambda u\|_\kappa < \infty.$$

Then, it follows that (see e.g. [4, Lemma II.1.3])

$$\|S_\lambda(h)u - u\|_\kappa \leq \int_0^h \|S_\lambda(s)A_\lambda u\|_\kappa \,\mathrm{d}s \leq C_u e^{\alpha h} h$$

for all $\lambda \in \Lambda$. This shows that $u \in D_\Lambda$. Moreover, since $\sup_{\lambda\in\Lambda}\|A_\lambda u\|_\kappa < \infty$, it follows that

$$\big(\mathcal{A}u\big)(x) := \sup_{\lambda\in\Lambda} \big(A_\lambda u\big)(x)$$

is well-defined for all $x \in M$.

Lemma 3.7. *Let $u \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ with*

$$\sup_{\lambda\in\Lambda}\|A_\lambda u\|_\kappa < \infty \quad \text{and} \quad \sup_{\lambda\in\Lambda}\|S_\lambda(h)A_\lambda u - A_\lambda u\|_\kappa \to 0 \quad \text{as } h \to 0.$$

Then,

$$\lim_{h\searrow 0}\left\|\frac{\mathcal{E}_h u - u}{h} - \mathcal{A}u\right\|_\kappa = 0.$$

In particular, $\mathcal{A}u \in \mathrm{UC}_\kappa$*.*

Proof. Let $\varepsilon > 0$. Then, by assumption, there exists some $h_0 > 0$ such that

$$\sup_{\lambda\in\Lambda} \|S_\lambda(s)A_\lambda u - A_\lambda u\|_\kappa \leq \varepsilon$$

for all $s \in [0, h_0]$. Hence, for all $h \in (0, h_0]$ it follows that

$$\begin{aligned}
\left\| \frac{\mathcal{E}_h u - u}{h} - \mathcal{A}u \right\|_\kappa &\leq \sup_{\lambda\in\Lambda} \left\| \frac{S_\lambda(h)u - u}{h} - A_\lambda u \right\|_\kappa \\
&= \sup_{\lambda\in\Lambda} \frac{1}{h} \left\| \int_0^h S_\lambda(s)A_\lambda u - A_\lambda u \,\mathrm{d}s \right\|_\kappa \\
&\leq \sup_{\lambda\in\Lambda} \frac{1}{h} \int_0^h \|S_\lambda(s)A_\lambda u - A_\lambda u\|_\kappa \,\mathrm{d}s \\
&\leq \varepsilon.
\end{aligned}$$

□

Lemma 3.8. *Let* $T\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ *be sublinear and Lipschitz continuous. Further, let* $v\colon [0,\infty) \to \mathrm{UC}_\kappa$ *be continuous. Then,* $Tv\colon [0,\infty) \to \mathrm{UC}_\kappa,\ t \mapsto T\big(v(t)\big)$ *is again continuous and*

$$T\left(\int_0^t v(s)\,\mathrm{d}s\right) \leq \int_0^t Tv(s)\,\mathrm{d}s \quad \text{for all } t \geq 0.$$

Proof. It is clear that $Tv\colon [0,\infty) \to \mathrm{UC}_\kappa$ is continuous. Let $t \geq 0$ and $u\colon [0,t] \to \mathrm{UC}_\kappa$ be a step function, then $Tu\colon [0,t] \to \mathrm{UC}_\kappa,\ s \mapsto T\big(u(s)\big)$ is a step function and it follows that

$$T\left(\int_0^t u(s)\,\mathrm{d}s\right) \leq \int_0^t Tu(s)\,\mathrm{d}s,$$

where we used the fact that T is sublinear. Now, let $(v_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\lim_{n\to\infty} \sup_{s\in[0,t]} \|v_n(s) - v(s)\|_\kappa = 0$. The continuity of T implies that $\lim_{n\to\infty} \sup_{s\in[0,t]} \|Tv_n(s) - Tv(s)\|_\kappa = 0$ and therefore,

$$T\left(\int_0^t v(s)\,\mathrm{d}s\right) = \lim_{n\to\infty} T\left(\int_0^t v_n(s)\,\mathrm{d}s\right) \leq \lim_{n\to\infty} \int_0^t Tv_n(s)\,\mathrm{d}s = \int_0^t Tv(s)\,\mathrm{d}s.$$

□

Proposition 3.9. *Let* $u \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ *with*

$$\sup_{\lambda\in\Lambda} \|A_\lambda u\|_\kappa < \infty \quad \text{and} \quad \sup_{\lambda\in\Lambda} \|S_\lambda(h)A_\lambda u - A_\lambda u\|_\kappa \to 0 \quad \text{as } h \to 0.$$

Then, $\mathcal{A}u \in \mathrm{UC}_\kappa$ *and the following statements are equivalent:*

(i) *The map* $[0,\infty) \to \mathrm{UC}_\kappa,\ t \mapsto \mathscr{S}(t)\mathcal{A}u$ *is continuous,*
(ii) $\lim_{h\searrow 0} \left\| \frac{\mathscr{S}(h)u - u}{h} - \mathcal{A}u \right\|_\kappa = 0$.

Proof. By Lemma 3.7, we already know that $\mathcal{A}u \in \mathrm{UC}_\kappa$. Let D denote the set of all $v \in \mathrm{UC}_\kappa$ such that the map $[0,\infty) \to \mathrm{UC}_\kappa,\ t \mapsto \mathscr{S}(t)v$ is continuous. By Remark 3.6, $u \in D_\Lambda$. Therefore, by Proposition 3.4, $u \in D$ and, by Remark

1.2, $\mathscr{S}(t)u \in D$ for all $t \geq 0$. Therefore, by Remark 3.2, statement (ii) implies statement (i). Therefore, assume that (i) is satisfied. By Lemma 3.7,

$$\mathcal{A}u - \frac{\mathscr{S}(h)u - u}{h} \leq \mathcal{A}u - \frac{\mathcal{E}_h u - u}{h} \to 0, \quad \text{as } h \searrow 0$$

Since the map $[0,\infty) \to \mathrm{UC}_\kappa,\ t \mapsto \mathscr{S}(t)\mathcal{A}u$ is continuous, it follows that

$$\left\| \frac{1}{h}\int_0^h \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s - \mathcal{A}u \right\|_\kappa \to 0, \quad \text{as } h \searrow 0.$$

Hence, it is sufficient to show that

$$\mathscr{S}(t)u - u \leq \int_0^t \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s \tag{3.4}$$

for all $t \geq 0$. Let $t \geq 0$ and $h > 0$. Then,

$$\begin{aligned} \mathcal{E}_h u - u &= \sup_{\lambda\in\Lambda} S_\lambda(h)u - u = \sup_{\lambda\in\Lambda} \int_0^h S_\lambda(s)A_\lambda u \,\mathrm{d}s \\ &\leq \int_0^h \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s = \int_t^{t+h} \mathscr{S}(s-t)\mathcal{A}u \,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Next, we prove that

$$\mathcal{E}_\pi u - u \leq \int_0^{\max\pi} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s$$

for all $\pi \in P$ by an induction on $m = \#\pi$. If $m = 1$, i.e. if $\pi = \{0\}$, the statement is trivial. Hence, assume that

$$\mathcal{E}_{\pi'} u - u \leq \int_0^{\max\pi'} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s$$

for all $\pi' \in P$ with $\#\pi' = m$ for some $m \in \mathbb{N}$. Let $\pi = \{t_0, t_1, \ldots, t_m\}$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\pi' := \pi \setminus \{t_m\}$. Then, it follows from (3.5) that

$$\begin{aligned} \mathcal{E}_\pi u - \mathcal{E}_{\pi'} u &\leq \mathscr{S}(t_{m-1})\big(\mathcal{E}_{t_m - t_{m-1}} u - u\big) \\ &\leq \mathscr{S}(t_{m-1})\left(\int_{t_{m-1}}^{t_m} \mathscr{S}(s - t_{m-1})\mathcal{A}u \,\mathrm{d}s\right) \\ &\leq \int_{t_{m-1}}^{t_m} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s, \end{aligned}$$

where the last inequality follows from Lemma 3.8. By the induction hypothesis, we thus get that

$$\begin{aligned} \mathcal{E}_\pi u - u = \big(\mathcal{E}_\pi u - \mathcal{E}_{\pi'} u\big) + \big(\mathcal{E}_{\pi'} u - u\big) &\leq \int_{t_{m-1}}^{t_m} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s + \int_0^{t_{m-1}} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s \\ &= \int_0^{\max\pi} \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s. \end{aligned}$$

In particular, for every $\pi \in P_t$,

$$\mathcal{E}_\pi u - u \leq \int_0^t \mathscr{S}(s)\mathcal{A}u \,\mathrm{d}s.$$

Taking the supremum over all $\pi \in P_t$ yields the assertion. □

Let $\mathcal{D}$ denote the set of all $u \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ with

$$\sup_{\lambda\in\Lambda} \|A_\lambda u\|_\kappa < \infty \quad \text{and} \quad \left\| \frac{\mathscr{S}(h)u - u}{h} - \mathcal{A}u \right\|_\kappa \to 0 \quad \text{as } h \to 0.$$

In the sequel, we are interested in viscosity solutions to the abstract differential equation

$$u'(t) = \mathcal{A}u(t), \quad \text{for } t > 0,$$

where we use the following notion of a viscosity solution.

Definition 3.10. We say that u is a *viscosity subsolution* of the abstract differential equation

$$u'(t) = \mathcal{A}u(t), \quad \text{for } t > 0, \tag{3.6}$$

if $u\colon [0,\infty) \to \mathrm{UC}_\kappa$ is continuous and for every $t > 0$ and $x \in M$ we have

$$\big(\psi'(t)\big)(x) \le \big(\mathcal{A}\psi(t)\big)(x)$$

for every differentiable $\psi\colon (0,\infty) \to \mathrm{UC}_\kappa$ which satisfies $\psi(t) \in \mathcal{D}$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \le u(s)$ for all $s > 0$.

Analogously, u is called a *viscosity supersolution* of (3.6) if $u\colon [0,\infty) \to \mathrm{UC}_\kappa$ is continuous and for every $t > 0$ and $x \in M$ we have

$$\big(\psi'(t)\big)(x) \ge \big(\mathcal{A}\psi(t)\big)(x)$$

for every differentiable $\psi\colon (0,\infty) \to \mathrm{UC}_\kappa$ which satisfies $\psi(t) \in \mathcal{D}$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \le u(s)$ for all $s > 0$.

We say that u is a *viscosity solution* of (3.6) if u is a viscosity subsolution and a viscosity supersolution.

Remark 3.11. In general it is not clear how rich the class of test functions for a viscosity solution from the previous definition is. However, in the examples in Section 6, we will see that often $\mathrm{Lip}_b^k \subset \mathcal{D}$ with $k \in \{0,1,2\}$. For $\psi\colon (0,\infty)\times M \to \mathbb{R}$ differentiable w.r.t. t and $\partial_t\psi\colon (0,\infty) \times M \to \mathbb{R}$ uniformly w.r.t. x Lipschitz continuous in t with Lipschitz constant $L \ge 0$, we have

$$\sup_{x\in M} \left| \frac{\psi(t+h,x) - \psi(t,x)}{h} - \partial_t\psi(t,x) \right| \le Lh \to 0 \quad \text{as } h \searrow 0$$

for all $t > 0$. Hence, if $\mathrm{Lip}_b^k \subset \mathcal{D}$ for some $k \in \mathbb{N}_0$, then every function $\psi \in \mathrm{Lip}_b^{1,k}\big((0,\infty)\times M\big)$ is differentiable as a map $(0,\infty) \to \mathrm{UC}_\kappa$ and satisfies $\psi(t) \in \mathcal{D}$ for all $t > 0$. In most applications the class $\mathrm{Lip}_b^{1,k}\big((0,\infty) \times M\big)$ of test functions is sufficiently large in order to obtain uniqueness of a viscosity solution.

We conclude this section with the following main theorem.

Theorem 3.12. *Assume that $\mathscr{S}$ is strongly continuous. Then, for every $u_0 \in \mathrm{UC}_\kappa$, the function*

$$u(t) := \mathscr{S}(t)u_0, \quad \textit{for } t \ge 0,$$

is a viscosity solution to the abstract initial value problem

$$\begin{aligned} u'(t) &= \mathcal{A}u(t), \quad \text{for } t > 0, \\ u(0) &= u_0. \end{aligned}$$

Proof. Fix $t > 0$ and $x \in M$. We first show that u is a viscosity subsolution. Let $\psi\colon (0,\infty) \to \mathrm{UC}_\kappa$ differentiable with $\psi(t) \in \mathcal{D}$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \le u(s)$ for all $s > 0$. Then, for every $h \in (0,t)$, it follows from Theorem 2.4 that

$$\begin{aligned} 0 &= \frac{\mathscr{S}(h)\mathscr{S}(t-h)u_0 - \mathscr{S}(t)u_0}{h} = \frac{\mathscr{S}(h)u(t-h) - u(t)}{h} \\ &\le \frac{\mathscr{S}(h)\psi(t-h) - u(t)}{h} \le \frac{\mathscr{S}(h)\big(\psi(t-h) - \psi(t)\big) + \mathscr{S}(h)\psi(t) - u(t)}{h} \\ &= \mathscr{S}(h)\left(\frac{\psi(t-h) - \psi(t)}{h}\right) + \frac{\mathscr{S}(h)\psi(t) - \psi(t)}{h} + \frac{\psi(t) - u(t)}{h}. \end{aligned}$$

Let $\varepsilon > 0$. Then,

$$\lim_{h \searrow 0} \left\| \mathscr{S}(h)\left(\frac{\psi(t-h) - \psi(t)}{h}\right) + \psi'(t) \right\|_\kappa$$

and

$$\lim_{h \searrow 0} \left\| \frac{\mathscr{S}(h)\psi(t)\psi(t)}{h} - \mathcal{A}\psi(t) \right\|_\kappa.$$

Since $\big(u(t)\big)(x) = \big(\psi(t)\big)(x)$, it follows that

$$0 \le -\big(\psi'(t)\big)(x) + \big(\mathcal{A}\psi(t)\big)(x).$$

To show that u is a viscosity supersolution, let $\psi\colon (0,\infty) \to \mathrm{UC}_\kappa$ differentiable with $\psi(t) \in \mathcal{D}$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \le u(s)$ for all $s > 0$. By Theorem 2.4, for all $h > 0$ with $0 < h < t$ we get

$$\begin{aligned} 0 &= \frac{\mathscr{S}(t)u_0 - \mathscr{S}(h)\mathscr{S}(t-h)u_0}{h} = \frac{u(t) - \mathscr{S}(h)u(t-h)}{h} \le \frac{u(t) - \mathscr{S}(h)\psi(t-h)}{h} \\ &= \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} + \frac{\mathscr{S}(h)\psi(t) - \mathscr{S}(h)\psi(t-h)}{h} \\ &\le \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} + \mathscr{S}(h)\left(\frac{\psi(t) - \psi(t-h)}{h}\right). \end{aligned}$$

Let $\varepsilon > 0$. Then,

$$\lim_{h \searrow 0} \left\| \mathscr{S}(h)\left(\frac{\psi(t) - \psi(t-h)}{h}\right) - \psi'(t) \right\|_\kappa$$

and

$$\lim_{h \searrow 0} \left\| \frac{\psi(t) - \mathscr{S}(h)\psi(t)}{h} + \mathcal{A}\psi(t) \right\|_\kappa.$$

Since $\big(u(t)\big)(x) = \big(\psi(t)\big)(x)$, we obtain that

$$0 \le -\big(\mathcal{A}\psi(t)\big)(x) + \big(\psi'(t)\big)(x).$$

□

4. Approximation of the semigroup envelope

Throughout this section, we assume that the map

$$[0,\infty) \to \mathrm{UC}_\kappa, \quad h \mapsto \mathcal{E}_h u$$

is continuous for all $u \in \mathrm{UC}_\kappa$. Note that this assumption is, for example, implied by the condition that

$$\sup_{\lambda\in\Lambda} \|S_\lambda(h)u - u\|_\kappa \to 0 \quad \text{as } h \to 0,$$

for all $u \in \mathrm{Lip_b}$, which, in most applications, is satisfied. The following lemma shows that $\mathcal{E}_\pi$ depends continuously on the partition $\pi \in P$.

Lemma 4.1. *Let $m \in \mathbb{N}$ and $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < \ldots < t_m$. For each $n \in \mathbb{N}$ let $\pi_n = \{t_0^n, t_1^n, \ldots, t_m^n\} \in P$ with $0 = t_0^n < t_1^n < \ldots < t_m^n$ and $t_i^n \to t_i$ as $n \to \infty$ for all $i \in \{1, \ldots, m\}$. Then, for all $u \in \mathrm{UC}_\kappa$ we have that*

$$\|\mathcal{E}_\pi u - \mathcal{E}_{\pi_n} u\|_\infty \to 0, \quad n \to \infty.$$

Proof. First note that the set of all partitions with cardinality $m+1$ can be identified with the set

$$S^m := \big\{(s_1, \ldots, s_m) \in \mathbb{R}^m \,\big|\, 0 < s_1 < \ldots < s_m\big\} \subset \mathbb{R}^m.$$

Therefore, the assertion is equivalent to the continuity of the map

$$S^m \to \mathrm{UC}_\kappa, \quad (s_1, \ldots, s_m) \to \mathcal{E}_{\{0,s_1,\ldots,s_m\}} u. \tag{4.1}$$

Since the mapping

$$[0,\infty) \to \mathrm{UC}_\kappa, \quad h \mapsto \mathcal{E}_h u$$

is continuous for all $u \in \mathrm{UC}_\kappa$ and $\|\mathcal{E}_h u - \mathcal{E}_h v\|_\kappa \le e^{\alpha h}\|u - v\|_\kappa$ for all $h \ge 0$ and $u, v \in \mathrm{UC}_\kappa$, it follows that (4.1) is continuous. □

Let $u \in \mathrm{UC}_\kappa$. In the following, we now consider the limit of $\mathcal{E}_\pi u$ when the mesh size

$$|\pi|_\infty := \max_{j=1,\ldots,m} (t_j - t_{j-1})$$

of the partition $\pi = \{t_0, t_1, \ldots, t_m\} \in P$ with $0 = t_0 < t_1 < \ldots < t_m$ tends to zero. For the sake of completeness, we define $|\{0\}|_\infty := 0$. The following lemma shows that $\mathscr{S}(t)u$ can be obtained by a pointwise monotone approximation with finite partitions letting the mesh size tend to zero.

Lemma 4.2. *Let $t \ge 0$ and $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with $\pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$ and $|\pi_n|_\infty \searrow 0$ as $n \to \infty$. Then, for all $u \in \mathrm{UC}_\kappa$,*

$$\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u \quad \text{as } n \to \infty.$$

Proof. For $t = 0$ the statement is trivial. Therefore, assume that $t > 0$ and let

$$\big(\mathcal{E}_\infty u\big)(x) := \sup_{n\in\mathbb{N}} \big(\mathcal{E}_{\pi_n} u\big)(x)$$

for $u \in \mathrm{UC}_\kappa$ and $x \in M$. As in the proof of Lemma 2.3, the map $\mathcal{E}_\infty \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ is well-defined. Let $u \in \mathrm{UC}_\kappa$. Since $\pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$, it follows

$$\mathcal{E}_{\pi_n} u \nearrow \mathcal{E}_\infty u \quad \text{as } n \to \infty.$$

Since $(\pi_n)_{n\in\mathbb{N}} \subset P_t$, we obtain that

$$\mathcal{E}_\infty u \leq \mathscr{S}(t)u.$$

Let $\pi = \{t_0, t_1, \ldots, t_m\} \in P_t$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_m = t$. Since $|\pi_n|_\infty \searrow 0$ as $n \to \infty$, w.l.o.g. we may assume that $\#\pi_n \geq m+1$ for all $n \in \mathbb{N}$. Moreover, let $0 = t_0^n < t_1^n < \ldots < t_m^n = t$ for all $n \in \mathbb{N}$ with $\pi_n' := \{t_0^n, t_1^n, \ldots, t_m^n\} \subset \pi_n$ and $t_i^n \to t_i$ as $n \to \infty$ for all $i \in \{1, \ldots, m\}$. Then, by Lemma 4.1 b), we have that

$$\|\mathcal{E}_\pi u - \mathcal{E}_{\pi_n'} u\|_\kappa \to 0 \quad \text{as } n \to \infty.$$

and therefore,

$$\mathcal{E}_\infty u - \mathcal{E}_\pi u \geq \mathcal{E}_{\pi_n} u - \mathcal{E}_\pi u \geq \mathcal{E}_{\pi_n'} u - \mathcal{E}_\pi u \to 0 \quad \text{as } n \to \infty.$$

This shows that $\mathcal{E}_\infty u \geq \mathcal{E}_\pi u$. Taking the supremum over all $\pi \in P_t$ we thus get that $\mathcal{E}_\infty u = \mathscr{S}(t)u$. □

Corollary 4.3. *For all $t > 0$ there exists a sequence $(\pi_n)_{n\in\mathbb{N}} \subset P_t$ with*

$$\mathcal{E}_{\pi_n} u \nearrow \mathscr{S}(t)u$$

as $n \to \infty$ for all $u \in \mathrm{UC}_\kappa$.

Proof. Choose

$$\pi_n = \left\{ \frac{kt}{2^n} \,\middle|\, k \in \{0, \ldots, 2^n\} \right\} \quad \text{or} \quad \pi_n = \left\{ \frac{kt}{n!} \,\middle|\, k \in \{0, \ldots, n!\} \right\}$$

in Lemma 4.2. □

Corollary 4.4. *For all $t \geq 0$ and $u \in \mathrm{UC}_\kappa$ we have that*

$$\mathscr{S}(t)u = \sup_{n\in\mathbb{N}} \mathcal{E}_{\frac{t}{n}}^n u = \lim_{n\to\infty} \mathcal{E}_{2^{-n}t}^{2^n} u,$$

where the supremum and the limit are to be understood pointwise.

5. Stochastic representation

In this section, we derive a stochastic representation for the semigroup envelope $\mathscr{S}$ using sublinear expectations. We again assume that the conditions (A1) and (A2) are satisfied. We start with a short introdution into the theory of nonlinear expectations. For a measurable space $(\Omega, \mathcal{F})$, we denote the space of all bounded $\mathcal{F}$-measurable functions $\Omega \to \mathbb{R}$ by $\mathcal{L}^\infty(\Omega, \mathcal{F})$. For two bounded random variables $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we write $X \leq Y$ if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. For a constant $\alpha \in \mathbb{R}$, we do not distinguish between α and the constant function taking the value α. Throughout, we assume that $S_\lambda(t)1 = 1$ for all $t \geq 0$ and $\lambda \in \Lambda$.

Definition 5.1. Let $(\Omega, \mathcal{F})$ be a measurable space. A functional $\mathcal{E}\colon \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is called a *sublinear expectation* if for all $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $\lambda > 0$

(i) $\mathcal{E}(X) \leq \mathcal{E}(Y)$ if $X \leq Y$,
(ii) $\mathcal{E}(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$,
(iii) $\mathcal{E}(X+Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$ and $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$.

We say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *sublinear expectation space* if there exists a set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ such that

$$\mathcal{E}(X) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) \quad \text{for all } X \in \mathcal{L}^\infty(\Omega, \mathcal{F}),$$

where $\mathbb{E}_{\mathbb{P}}(\cdot)$ denotes the expectation w.r.t. to the probability measure $\mathbb{P}$.

Definition 5.2. We say that $\mathscr{S}$ is *continuous from above* on UC_b if $\mathscr{S}(t)u_n \searrow \mathscr{S}(t)u$ for all $t \geq 0$ and all $(u_n)_{n\in\mathbb{N}} \subset \mathrm{UC}_b$ with $u_n \searrow u \in \mathrm{UC}_b$ as $n \to \infty$.

Remark 5.3.

a) Assume that M is compact. Then, by Dini's lemma $\mathscr{S}$ is continuous from above.
b) Assume that $\mathscr{S}$ is continuous from above on UC_b. Then, by [2, Remark 5.4 c)], $\mathscr{S}(t)$ uniquely extends to an operator $\mathscr{S}(t)\colon \mathrm{C_b} \to \mathrm{C_b}$, which is again continuous from above. Moreover, for every $n \in \mathbb{N}$, $v \in \mathrm{C_b}(M^{n+1})$ the mapping

$$M^{n+1} \to \mathbb{R}, \quad (x_1, \ldots, x_n, x_{n+1}) \mapsto \big(\mathscr{S}(t)v(x_1, \ldots, x_n, \,\cdot\,)\big)(x_{n+1})$$

is bounded and continuous.

Continuity from above on UC_b will be crucial for the existence of a stochastc representation. The following proposition gives a sufficitont condition for the continuity from above on UC_b. Let UC_0 be the closure of the space Lip_c of all Lipschitz continuous functions with compact support w.r.t. the supremum norm $\|\cdot\|_\infty$.

Proposition 5.4. *Suppose that for every $\delta > 0$ there exists a family of functions $(\varphi_x)_{x\in M} \subset \mathrm{UC}_0$ satisfying the following for all $x \in M$:*

(i) $\varphi_x(x) = 1$ *and* $0 \leq \varphi_x \leq 1$,
(ii) $\varphi_x \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ *with* $\sup_{\lambda\in\Lambda} \|A_\lambda \varphi_x\|_\kappa \leq \delta$.

Then, $\mathscr{S}$ is continuous from above on UC_b.

Proof. Fix $t > 0$, $x \in M$ and $\delta > 0$. Since $\varphi_x(x) = 1$ and $1 - \varphi_x \in \bigcap_{\lambda\in\Lambda} D(A_\lambda)$ with $A_\lambda(1-\varphi) = -A_\lambda\varphi$, it follows that

$$\begin{aligned}\big(\mathscr{S}(t)(1-\varphi_x)\big)(x) &\leq \frac{1}{\kappa(x)}\|\mathscr{S}(t)(1-\varphi_x) - (1-\varphi_x)\|_\kappa \\ &\leq t \sup_{\lambda\in\Lambda} \|A_\lambda\varphi_x\|_\kappa \leq \frac{\delta t}{k(x)}.\end{aligned}$$

Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{UC}_b$ with $u_n \searrow 0$ as $n \to \infty$ and $\varepsilon > 0$. Then, there exists some $\varphi_x \in \mathrm{UC}_0$ satisfying (i) and (ii) with $\delta = \frac{\varepsilon\kappa(x)}{2tc}$, where $c := \max\big\{1, \|u_1\|_\infty\big\}$. Then,

$$\|u_n\|_\infty \big(\mathscr{S}(t)(1-\varphi_x)\big)(x) \leq \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there exists some $n \in \mathbb{N}$ such that $\|u_n\varphi_x\|_\kappa < \frac{\varepsilon}{2}$ since $\varphi_x \in \mathrm{UC}_0$. Hence,

$$\big(\mathscr{S}(t)u_n\big)(x) \leq \|u_n\|_\infty \big(\mathscr{S}(t)(1-\varphi_x)\big)(x) + \big(\mathscr{S}(t)(u_n\varphi_x)\big)(x) < \varepsilon.$$

This shows that $\mathscr{S}(t)u_n \searrow 0$ as $n \to \infty$. Now, let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{UC}_b$ and $u \in \mathrm{UC}_b$ with $u_n \searrow u$ as $n \to \infty$. Then,

$$|\mathscr{S}(t)u_n - \mathscr{S}(t)u| \le \mathscr{S}(t)(u_n - u) \searrow 0 \quad \text{as } n \to \infty.$$

□

Note that the existence of a function in $\varphi_x \in \mathrm{UC}_0$ with $\varphi_x(x) \neq 0$ for all $x \in M$ implies that M is locally compact. Thus, Proposition 5.4 is only applicable for locally compact M. The following theorem is a direct consequence of [2, Theorem 5.6].

Theorem 5.5. *Assume that M is a Polish space and that the semigroup $\mathscr{S}$ is continuous from above on UC_b. Then, there exists a quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x\in M}, (X_t)_{t\ge 0})$ such that*

(i) *$X_t\colon \Omega \to M$ is $\mathcal{F}$-$\mathcal{B}$-measurable for all $t \ge 0$,*
(ii) *$(\Omega, \mathcal{F}, \mathcal{E}^x)$ is a sublinear expectation space with $\mathcal{E}^x(u(X_0)) = u(x)$ for all $x \in M$ and $u \in \mathrm{C_b}$,*
(iii) *For all $0 \le s < t$, $n \in \mathbb{N}$, $0 \le t_1 < \ldots < t_n \le s$ and $v \in \mathrm{C_b}(M^{n+1})$,*
$$\mathcal{E}^x\big(v(X_{t_1}, \ldots, X_{t_n}, X_t)\big) = \mathcal{E}^x\left(\big(\mathscr{S}(t-s)v(X_{t_1}, \ldots, X_{t_n}, \cdot\,)\big)(X_s)\right).$$

In particular,
$$\big(\mathscr{S}(t)u\big)(x) = \mathcal{E}^x(u(X_t)).$$
for all $t \ge 0$, $x \in M$ and $u \in \mathrm{C_b}$.

The quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x\in M}, (X_t)_{t\ge 0})$ can be seen as a nonlinear Markov process, where (iii) is the nonlinear analogue of the Markov property.

6. Examples

For $k \in \mathbb{N}_0$, let $\mathrm{Lip}_\mathrm{b}^k$ denote the space of all k-times differentiable functions with bounded and Lipschitz continuous derivatives up to order k.

Example 6.1 (Koopman semigroups on real separable Banach spaces)**.** We consider the case, where the state space $M = X$ is a real separable Banach space. Let $F\colon X \to X$ be Lipschitz continuous with Lipschitz constant $L > 0$. Then, we denote by $\Phi_F\colon [0,\infty) \times X \to X$ the solution to the initial value problem

$$\partial_t \Phi_F(t,x) = F(\Phi_F(t,x)), \quad t \ge 0, \tag{6.1}$$
$$\Phi_F(0,x) = x \tag{6.2}$$

for all $x \in X$. Then, Φ_F defines a so-called *continuous semiflow*. Let $\beta \in (0,\infty)$ with $\beta \ge L$. Then,

$$\|\Phi_F(t,x) - \Phi_F(t,y)\| \le \|x-y\| + \beta \int_0^t \|\Phi_F(s,x) - \Phi_F(s,y)\|\,\mathrm{d}s$$

for all $t \ge 0$ and $x, y \in X$. Hence, by Gronwall's lemma,

$$\|\Phi_F(t,x) - \Phi_F(t,y)\| \le \|x-y\| e^{\beta t} \tag{6.3}$$

for all $t \geq 0$ and $x, y \in X$. Let $C \in [1, \infty)$ with $C \geq \|F(0)\|$ and $\alpha = \max\{1, \beta\}$. Then, for all $t \geq 0$ and $x \in X$,

$$C + \|x\| + \|\Phi_F(t,x) - x\| \leq C + \|x\| + \alpha \int_0^t C + \|x\| + \|\Phi_F(s,x) - x\| \, \mathrm{d}s.$$

Again, by Gronwall's lemma, it follows

$$C + \|\Phi_F(t,x)\| \leq C + \|x\| + \|\Phi_F(t,x) - x\| \leq \big(C + \|x\|\big)e^{\alpha t} \tag{6.4}$$

for all $t \geq 0$ and $x \in X$. Let $\kappa(x) := (C + \|x\|)^{-2}$ for all $x \in X$. For $u \in C_\kappa$, $t \geq 0$ and $x \in X$, we then define

$$\big(S_F(t)u\big)(x) := u(\Phi_F(t,x)).$$

Then, for $u \in \mathrm{UC}_\kappa$, $t \geq 0$ and $x \in X$,

$$\big|\big(S_F(t)u\big)(x)\big| \leq \|u\|_\kappa \big(C + \|\Phi_F(t,x)\|\big)^2 \leq \|u\|_\kappa \big(C + \|x\|\big)^2 e^{2\alpha t}$$

and therefore, $\|S_F(t)u\|_\kappa \leq e^{2\alpha t}\|u\|_\kappa$. On the other hand, by (6.4),

$$\big|\big(S_F(t)u\big)(x) - u(x)\big| \leq \|u\|_{\mathrm{Lip}} \|\Phi_F(t,x) - x\| \leq \|u\|_{\mathrm{Lip}} \big(C + \|x\|\big)\big(e^{\alpha t} - 1\big)$$

for all $u \in \mathrm{Lip}_\mathrm{b}$. Therefore, $\|S_F(t)u - u\|_\kappa \leq \|u\|_{\mathrm{Lip}}\big(e^{\alpha t} - 1\big)$ for $u \in \mathrm{Lip}_\mathrm{b}$. By (6.3),

$$\big|\big(S_F(t)u\big)(x) - \big(S_F(t)u\big)(y)\big| \leq e^{\beta t}\|u\|_{\mathrm{Lip}}\|x - y\|$$

for all $x, y \in X$ and $u \in \mathrm{Lip}_\mathrm{b}$. That is, $\|S_F(t)u\|_{\mathrm{Lip}} \leq e^{\beta t}\|u\|_{\mathrm{Lip}}$. For $u \in \mathrm{Lip}_\mathrm{b}^1$, let

$$C_u := \|u'\|_\infty + \|u'\|_{\mathrm{Lip}}$$

and $A_F u \in \mathrm{C}_\kappa$ be given by

$$\big(A_F u\big)(x) := u'(x)F(x) \quad \text{for } x \in X.$$

Let $u \in \mathrm{Lip}_\mathrm{b}^1$. Then, for all $t \geq 0$ and $x \in X$,

$$\big|\big(S_F(t)A_F u\big)(x) - \big(A_F u\big)(x)\big| \leq \tfrac{1}{\kappa(x)} C_u \alpha \big(e^{\alpha t} - 1\big)$$

and therefore,

$$\|S_F(t)A_F u - A_F u\|_\kappa \leq C_u \alpha \big(e^{\alpha t} - 1\big).$$

By the chain rule and the fundamental theorem of infinitesimal calculus, it follows that

$$\frac{\big(S_F(h)u\big)(x) - u(x)}{h} = \frac{1}{h}\int_0^h \big(S_F(s)A_F u\big)(x) \, \mathrm{d}s$$

for all $h > 0$, which implies that

$$\left\| \frac{S_F(h)u - u}{h} - A_F u \right\|_\kappa \leq C_u \alpha \big(e^{\alpha h} - 1\big) \to 0 \quad \text{as } h \searrow 0.$$

Hence, for any nonempty set Λ of Lipschitz continuous functions $F \colon X \to X$ with

$$\sup_{F \in \Lambda} \left(\|F(0)\| + \sup_{x,y \in M} \frac{\|F(x) - F(y)\|}{\|x - y\|} \right) < \infty$$

the assumptions (A1) and (A2) are satisfied, the semigroup envelope $\mathscr{S}$ is strongly continuous and $\mathrm{Lip}_\mathrm{b}^1 \subset \mathcal{D}$. By Theorem 3.12, we thus obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the fully nonlinear PDE

$$\begin{aligned} u_t(t,x) &= \sup_{F\in\Lambda} \nabla u(t,x)F(x), \quad (t,x) \in (0,\infty)\times X, \\ u(0,x) &= u_0(x), \quad x \in X. \end{aligned}$$

Moreover, if $X = \mathbb{R}^d$, the semigroup envelope $\mathscr{S}$ is continuous from above by Proposition 5.4. In this case, Theorem 5.5 implies the existence of a Markov process under a nonlinear expectation related to $\mathscr{S}$. This Markov process can be viewed as a nonlinear drift process.

Example 6.2 (Geometric Brownian Motion)**.** Let $M = \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$ and W a Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$X_t^x := \exp\left(t\big(\mu - \tfrac{\sigma^2}{2}\big) + \sigma W_t \right) x$$

for $t \geq 0$ and $x \in \mathbb{R}$. Then, for $x \in \mathbb{R}$, the stochastic processes $(X_t^x)_{t\geq 0}$ is a solution to the SDE

$$dX_t^x = \mu X_t \,\mathrm{d}t + \sigma X_t^x \mathrm{d}W_t, \quad X_0^x = x.$$

Let $\beta \geq 0$ with $\beta \geq \mu + \sigma^2$. Then,

$$\mathbb{E}(|X_t^x|^p)^{\frac{1}{p}} \leq |x| e^{\left(\mu + \frac{(p-1)\sigma^2}{2}\right)t} \leq |x| e^{\beta t}$$

for $p \in [2,3]$ and $x \in \mathbb{R}$. By Young's inequality and Ito's isometry,

$$1 + \mathbb{E}\big(|X_t^1 - 1|^2\big) \leq 1 + 4\beta \int_0^t 1 + \mathbb{E}\big(|X_s^1 - 1|^2\big)\,\mathrm{d}s.$$

By Gronwall's lemma, it follows that

$$\mathbb{E}\big(|X_t^1 - 1|^2\big) \leq e^{4\beta t} - 1.$$

Let $\kappa(x) := (1+|x|)^3$ for $x \in \mathbb{R}$ and $S = S_{\mu,\sigma}$ be given by

$$\big(S(t)u\big)(x) := \mathbb{E}\big(u(X_t^x)\big)$$

for $u \in \mathrm{UC}_\kappa$, $t \geq 0$ and $x \in \mathbb{R}$. Then, it follows that

$$\|S(t)u\|_\kappa \leq \|u\|_\kappa e^{3\beta t}$$

for $t \geq 0$ and $u \in \mathrm{UC}_\kappa$. Moreover, for $u \in \mathrm{Lip}_\mathrm{b}$,

$$\|u\|_{\mathrm{Lip}} \leq e^{\beta t}\|u\|_{\mathrm{Lip}}.$$

and

$$\|S(t)u - u\|_\kappa \leq \|u\|_{\mathrm{Lip}} \mathbb{E}\big(|X_t^1 - 1|\big) \leq \sqrt{e^{4\beta t} - 1} \to 0 \quad \text{as } t \to 0.$$

Therefore, S is a strongly continuous Feller semigroup. For $u \in \mathrm{Lip}_\mathrm{b}^2$, let

$$C_u := \max\{\|u'\|_\infty, \|u''\|_\infty, \|u''\|_{\mathrm{Lip}}\}$$

and $Au \in \mathrm{UC}_\kappa$ be given by

$$\big(Au\big)(x) := \mu x u'(x) + \frac{\sigma^2 x^2}{2} u''(x) \quad \text{for } x \in \mathbb{R}.$$

Let $u \in \mathrm{Lip}_{\mathrm{b}}^2$. Then, for $h > 0$,

$$\big|\big(S(h)Au\big)(x) - \big(Au\big)(x)\big| \leq \kappa(x) C_u \alpha \max\left\{\sqrt{e^{4\beta h} - 1}, e^{4\beta h} - 1\right\}$$

and therefore,

$$\|S(h)Au - Au\|_\kappa \leq C_u \alpha \max\left\{\sqrt{e^{4\beta h} - 1}, e^{4\beta h} - 1\right\}.$$

By Ito's formula, it follows that

$$\frac{\big(S(h)u\big)(x) - u(x)}{h} = \frac{1}{h}\int_0^h \big(S(s)Au\big)(x)\,\mathrm{d}s$$

for all $h > 0$ and $x \in \mathbb{R}$, which implies that

$$\left\|\frac{S(h)u - u}{h} - Au\right\|_\kappa \leq C_u \alpha \max\left\{\sqrt{e^{4\beta h} - 1}, e^{4\beta h} - 1\right\} \to 0 \quad \text{as } h \searrow 0.$$

Hence, for any nonempty set Λ of tuples (μ, σ) with

$$\sup_{(\mu,\sigma)\in\Lambda} \big(\mu + \sigma^2\big) < \infty$$

the assumptions (A1) and (A2) are satisfied, the semigroup envelope $\mathscr{S}$ is strongly continuous and $\mathrm{Lip}_{\mathrm{b}}^2 \subset \mathcal{D}$. By Theorem 3.12, we thus obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the fully nonlinear Cauchy problem

$$\begin{aligned} u_t(t,x) &= \sup_{(\mu,\sigma)\in\Lambda} \mu x \partial_x u(t,x) + \frac{\sigma^2 x^2}{2}\partial_{xx} u(t,x), \quad (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

Moreover, the semigroup $\mathscr{S}$ is continuous from above by Proposition 5.4. The nonlinear Markov process related to $\mathscr{S}$ can be seen as a geometric G-Brownian Motion (cf. Theorem 5.5).

Example 6.3 (Ornstein-Uhlenbeck processes on separable Hilbert spaces)**.** We consider the case where $M = H$ is a real separable Hilbert space and $\Gamma = \emptyset$. Let $m \in H$, $B \in L(H)$, $T(t) := e^{tB}$ for all $t \geq 0$, $C \in L(H)$ a trace class operator and W^C a Brownian Motion with covariance operator C on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define

$$X_t^x := T(t)x + \int_0^t T(t-s)m\,\mathrm{d}s + \int^t T(t-s)\,\mathrm{d}W_s^C$$

for $t \geq 0$ and $x \in H$. Then, for $x \in H$, the stochastic process $(X_t^x)_{t\geq 0}$ is a mild solution to the infinite-dimensional SDE

$$dX_t^x = \big(BX_t^x + m\big)\,\mathrm{d}t + \mathrm{d}W_t^C, \quad X_0^x = x.$$

Let $\alpha \geq 0$ with $\alpha \geq e^{2\|B\|}\big(\|B\|^2 + \|m\|^2 + \|C\|_{\mathrm{tr}}\big)$. Then, by Young's inequality,

$$\begin{aligned} 1 + \mathbb{E}(\|X_t^x\|^2) &\leq 1 + \|x\|^2 + 4e^{2\|B\|}\big(\|B\|^2\|x\|^2 t^2 + \|m\|t^2 + \|C\|_{\mathrm{tr}} t\big) \\ &\leq (1 + \|x\|^2)e^{4\alpha t} \end{aligned}$$

for all $x \in H$ and $0 \leq t \leq 1$. Moreover,

$$\mathbb{E}\big(\|X_t - x\|\big) \leq e^{\|B\|}\|B\|\|x\|t + e^{\|B\|}\|m\|t + e^{\|B\|}\sqrt{\|C\|_{\mathrm{tr}}t} \leq (1 + \|x\|)3\sqrt{\alpha t}$$

for all $x \in H$ and $0 \leq t \leq 1$. Let $\kappa := (1 + \|x\|^2)^{-1}$ and define $S = S_{B,m,C}$ by

$$\big(S(t)u\big)(x) := \mathbb{E}\big(u(X_t^x)\big)$$

for $x \in H$, $t \geq 0$ and $u \in \mathrm{C}_\kappa$. Then, $\|S(t)u\|_\kappa \leq e^{4\alpha t}\|u\|_\kappa$ for all $u \in C_\kappa$ and $t \geq 0$ and

$$\|S(t)u\|_{\mathrm{Lip}} \leq e^{\|B\|t}\|u\|_{\mathrm{Lip}}.$$

For $u \in \mathrm{Lip}_{\mathrm{b}}^2$ let

$$C_u := \max\{\|\nabla u\|, \|\nabla^2 u\|_\infty, \|\nabla^2 u\|_{\mathrm{Lip}}\}$$

and $Au \in C_\kappa$ be given by

$$\big(Au\big)(x) = \nabla u(x)(Bx + m) + \frac{1}{2}\operatorname{tr}\big(C\nabla^2 u(x)\big)$$

for $x \in H$. Then, for $h \in [0, 1]$,

$$\|S(h)Au - Au\|_\kappa \leq 3C_u\sqrt{\alpha^3 h}.$$

By Ito's formula it follows that

$$\frac{\big(S(h)u\big)(x) - u(x)}{h} = \frac{1}{h}\int_0^h \big(S(s)Au\big)(x)\,\mathrm{d}s$$

for all $h > 0$ and $x \in H$, which implies that

$$\left\|\frac{S(h)u - u}{h} - Au\right\|_\kappa \leq 3C_u\sqrt{\alpha^3 h} \to 0 \quad \text{as } h \searrow 0.$$

Hence, for any nonempty set Λ of triplets (B, m, C) with

$$e^{2\|B\|}\big(\|B\|^2 + \|m\|^2 + \|C\|_{\mathrm{tr}}\big) \leq \alpha$$

the assumptions (A1) and (A2) are satisfied, the semigroup envelope $\mathscr{S}$ is strongly continuous on $\mathrm{Lip}_{\mathrm{b}}^2$. In order to show that $\mathrm{Lip}_{\mathrm{b}}^2 \subset \mathcal{D}$, by the previous computations, it suffices to show that $\mathscr{S}$ is stongly continuous. For this we invoke Proposition 3.5. Notice that $\mathrm{Lip}_{\mathrm{b}}^2$ is not dense in $\mathrm{Lip}_{\mathrm{b}}$ if H is infinite dimensional. Let $\delta > 0$ and $\varphi\colon [0, \infty) \to [0, 1]$ infinitely smooth with $\varphi(s) = 1$ for $x \in \big[0, \frac{\delta}{2}\big]$ and $\varphi(s) = 0$ for $s \in [\delta, \infty)$. For $x, y \in H$, let $\varphi_x(y) := \varphi(\|y - x\|)$. Then, $\varphi_x \in \mathrm{Lip}_{\mathrm{b}}^2$ with

$$\|\nabla\varphi_x\|_\infty \leq \|\varphi'\|_\infty \quad \text{and} \quad \|\nabla^2\varphi_x\|_\infty \leq \frac{3}{\delta}\|\varphi'\|_\infty + \|\varphi''\|_\infty \quad \text{for all } x \in M.$$

Hence,

$$\|A\varphi_x\|_\kappa \leq \frac{3\sqrt{\alpha}}{2\delta}\max\big\{\|\varphi'\|_\infty, \|\varphi''\|_\infty\big\} =: L$$

for all $x \in M$. Therefore, by Remark 3.6 and Proposition 3.5, the semigroup $\mathscr{S}$ is strongly continuous. Altogether, we have shown that, for any nonempty set Λ of triplets (B, m, C) with

$$\sup_{(B,m,C)\in\Lambda}\big(\|B\|^2 + \|m\|^2 + \|C\|_{\mathrm{tr}}\big) < \infty,$$

the assumptions (A1) and (A2) are satisfied, the semigroup envelope $\mathscr{S}$ is strongly continuous and $\mathrm{Lip}_\mathrm{b}^2 \subset \mathcal{D}$. By Theorem 3.12, we thus obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the fully nonlinear PDE

$$\begin{aligned} u_t(t,x) &= \sup_{(B,m,C)\in\Lambda} \nabla u(t,x)(Bx+m) + \frac{1}{2}\operatorname{tr}\big(C\nabla^2 u(t,x)\big), \ (t,x) \in (0,\infty)\times H, \\ u(0,x) &= u_0(x), \quad x \in H. \end{aligned}$$

If $H = \mathbb{R}^d$, the semigroup $\mathscr{S}$ is continuous from above by Proposition 5.4, which implies the existence of an O-U-process under a nonlinear expectation which represents $\mathscr{S}$ (cf. Theorem 5.5).

Example 6.4 (Lévy Processes on abelian groups)**.** Let $M = G$ be an abelian group with a translation invariant metric d and $\kappa(x) := 1$ for all $x \in M$. Let $(S(t))_{t\geq 0}$ be a Markovian convolution semigroup, i.e. a semigroup arising from a Lévy process. Then, $(S(t))_{t\geq 0}$ is a strongly continuous Feller semigroup of linear contractions (cf. [3]). Moreoever, due to the translation invariance, $\|S(t)u\|_{\mathrm{Lip}} \leq \|u\|_{\mathrm{Lip}}$ for all $t \geq 0$ and $u \in \mathrm{Lip}_\mathrm{b}$. Now, let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of Markovian convolution semigroups with generators $(A_\lambda)_{\lambda\in\Lambda}$. Then, the assumptions (A1) - (A2) are satisfied. We refer to [3] for examples, where the semigroup envelope is strongly continuous. In particular, all examples from [3] fall into our theory. In the case, where $G = H$ is a real separable Hilbert space, we can improve the result obtained in [3, Example 3.3]. In this case, by the Lévy-Khintchine formula (see e.g. [7, Theorem 5.7.3]), every generator A of a Markovian convolution semigroup is characterized by a Lévy triplet (b, Σ, μ), where $b \in H$, $\Sigma \in L(H)$ is a self-adjoint positive semidefinite trace-class operator and μ is a Lévy measure on H. For $u \in \mathrm{Lip}_b^2(H)$ and a Lévy triplet (b, Σ, μ), the generator $A_{b,\Sigma,\mu}$ is given by

$$\begin{aligned} \big(A_{b,\Sigma,\mu}u\big)(x) = \langle b, \nabla u(x)\rangle + \frac{1}{2}\operatorname{tr}\big(\Sigma\nabla^2 u(x)\big) \\ + \int_H u(x+y) - u(x) - \langle \nabla u(x), h(y)\rangle \,\mathrm{d}\mu(y) \end{aligned}$$

for $x \in H$. Here, the function $h\colon H \to H$ is defined by $h(y) = y$ for $\|y\| \leq 1$, and $h(y) = 0$ whenever $\|y\| > 1$. Let Λ be a nonempty set of Lévy triplets. We assume that

$$C := \sup_{(b,\Sigma,\mu)\in\Lambda} \left(\|b\| + \|\Sigma\|_{\mathrm{tr}} + \int_H 1 \wedge \|y\|^2 \,\mathrm{d}\mu(y) \right) < \infty. \tag{6.5}$$

Notice that (6.5) does not exclude any Lévy triplet a priori. Under (6.5), the semigroup envelope $\mathscr{S}$ is strongly continuous on $\mathrm{Lip}_\mathrm{b}^2$. In order to show that $\mathrm{Lip}_\mathrm{b}^2 \subset \mathcal{D}$, by the computations in [3], it suffices to show that $\mathscr{S}$ is strongly continuous. For this we invoke Proposition 3.3. For $\delta > 0$, we choose the family $(\varphi_x)_{x\in H}$ as in the previous example. Since $\big(\mathscr{S}(t)v\big)(x) = \big(\mathscr{S}(t)v(x+\cdot)\big)(0)$ for all $v \in \mathrm{UC}_\kappa$, $x \in H$ and $t \geq 0$, it follows that

$$\big(\mathscr{S}(t)(1-\varphi_x)\big)(x) = \big(\mathscr{S}(t)(1-\varphi_0)\big)(0)$$

for all $x \in H$ and $t \geq 0$. Defining $f(t) := \big(\mathscr{S}(t)(1-\varphi_0)\big)(0)$ for $t \geq 0$, it follows that f is continuous with $f(0) = 0$. Therefore, by Proposition 3.3, the semigroup $\mathscr{S}$ is strongly continuous. Altogether, we have shown that under the condition (6.5), the assumptions (A1) and (A2) are satisfied, the semigroup envelope $\mathscr{S}$ is strongly continuous and $\mathrm{Lip}_{\mathrm{b}}^2 \subset \mathcal{D}$. By Theorem 3.12, we thus obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the fully nonlinear Cauchy problem

$$\begin{aligned} u_t(t,x) &= \sup_{(b,\Sigma,\mu)\in\Lambda} \big(A_{b,\Sigma,\mu}u(t)\big)(x), \quad (t,x) \in (0,\infty)\times H,\\ u(0,x) &= u_0(x), \quad x \in H. \end{aligned}$$

If $H = \mathbb{R}^d$ and the set of Lévy measures within the set of Lévy triplets Λ is tight, the semigroup envelope $\mathscr{S}$ is continuous from above, which implies the existence of a nonlinear Lévy process related to $\mathscr{S}$. However, due to the translation invariance of the semigroups, the continuity from above is not necessary in order to obtain the existence of a Lévy process under a nonlinear expectation. The nonlinear Lévy process can be explicitly constructed via stochastic integrals w.r.t. Lévy processes with Lévy triplet contained in Λ. We refer to [3, Proposition 5.12] for the details of the construction.

Example 6.5 (α-stable Lévy processes)**.** Consider the setup of the previous example, with $G = \mathbb{R}^d$ for some $d \in \mathbb{N}$ and let $A_\alpha := -(-\Delta)^\alpha$ be fractional Laplacian for $0 < \alpha < 1$. Then, for any compact subset $\Lambda \subset (0,1)$, condition (6.5) is satisfied. Hence, the assumptions (A1) and (A2) are satisfied and the semigroup envelope $\mathscr{S}$ is strongly continuous with $\mathrm{Lip}_{\mathrm{b}}^2 \subset \mathcal{D}$. By Theorem 3.12, we thus obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the nonlinear Cauchy problem

$$\begin{aligned} u_t(t,x) &= \sup_{\alpha\in\Lambda} -(-\Delta)^\alpha u(t,x), \quad (t,x) \in (0,\infty)\times \mathbb{R}^d,\\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

The related nonlinear Lévy process can be interpreted as a Λ-stable Lévy process.

Example 6.6 (Mehler semigroups)**.** Consider the case, where the state space $M = H$ is a real separable Hilbert space and $\kappa = 1$. Let (T,μ) be a tuple consisting of a C_0-semigroup $T = (T(t))_{t\geq 0}$ of linear operators on H with $\|T(t)\| \leq e^{\alpha t}$ for all $t \geq 0$ and some $\alpha \in \mathbb{R}$ and a family $\mu = (\mu_t)_{t\geq 0}$ of probability measures on H such that

$$\mu_0 = \delta_0 \quad \text{and} \quad \mu_{t+s} = \mu_s * \mu_t \circ T(s)^{-1} \quad \text{for all } s,t \geq 0.$$

We then define the semigroup $S = S_{(T,\mu)}$ by

$$\big(S(t)u\big)(x) := \int_H u(T(t)x + y)\,\mathrm{d}\mu_t(y)$$

for $u \in \mathrm{UC_b}$, $t \geq 0$ and $x \in H$. Then, $\|S(t)u\|_\infty \leq \|u\|_\infty$ for all $u \in \mathrm{C_b}$ and $\|S(t)u\|_{\mathrm{Lip}} \leq e^{\alpha t}\|u\|_{\mathrm{Lip}}$ for $u \in \mathrm{Lip_b}$. Hence, for any nonempty family Λ of tuples (T,μ) with $\|T(t)\| \leq e^{\alpha t}$ for all $t \geq 0$ the assumptions (A1) and (A2) are satisfied.

Example 6.7 (Bounded generators on ℓ^∞)**.** Let $M = \mathbb{N}$ and $\kappa(i) = 1$ for all $i \in \mathbb{N}$. Let $(A_\lambda)_{\lambda\in\Lambda} \subset L(\ell^\infty)$ be a family of operators satisfying the positive maximum principle and

$$\sup_{\lambda\in\Lambda} \|A_\lambda\|_{L(\ell^\infty)} < \infty.$$

Here, we say that an operator $A \in L(\ell^\infty)$ satisfies the positive maximum principle if $A_{ii} < 0$ for all $i \in \mathbb{N}$ and $A_{ij} \geq 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Then, the family $(A_\lambda)_{\lambda\in\Lambda}$ satisfies the assumptions (A1) and (A2) with $\mathcal{D} = \ell^\infty$. In particular, the semigroup envelope is strongly continuous. If $A_\lambda 1 = 0$ for all $\lambda \in \Lambda$, then the semigroup envelope admits a stochasic representation. This representation can be seen as a nonlinear Markov chain with state space $\mathbb{N}$.

Example 6.8 (Multiples of generators of Feller semigroups)**.** Let A be the generator of a strongly continuous Feller semigroup $(S(t))_{t\geq 0}$ of linear operators. Assume that there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$\|S(t)u\|_\kappa \leq e^{\alpha t}\|u\|_\kappa \quad \text{and} \quad \|S(t)u\|_{\mathrm{Lip}} \leq e^{\beta t}\|u\|_{\mathrm{Lip}}$$

for all $u \in \mathrm{Lip_b}$ and $t \geq 0$. For $\lambda \geq 0$ let $A_\lambda := \lambda A$ for all λ. Then, A_λ generates the semigroup S_λ given by $S_\lambda(t) := S(\lambda t)$ for all $t \geq 0$ and $\lambda \geq 0$. Then, for any compact set $\Lambda \subset [0, \infty)$ the family $(S_\lambda)_{\lambda\in\Lambda}$ satisfies the assumptions (A1) and (A2) with $D(A) \subset \mathcal{D}$ and the semigroup envelope is strongly continuous. Hence, by Theorem 3.12, we obtain that $u(t) := \mathscr{S}(t)u_0$, for $t \geq 0$, defines a viscosity solution to the abstract Cauchy problem

$$\begin{aligned} u'(t) &= \sup_{\lambda\in\Lambda} \lambda A u(t), \quad \text{for } t > 0, \\ u(0) &= u_0. \end{aligned}$$

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