

Conservative stochastic 2-dimensional Cahn-Hilliard equation ^{*}

Michael Röckner^{c,d}, Huanyu Yang^{a,c,d}, Rongchan Zhu^{b,c}, [†]

^aSchool of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China,

^bDepartment of Mathematics, Beijing Institute of Technology, Beijing 100081, China,

^c Department of Mathematics, University of Bielefeld, D-33615 Bielefeld, Germany,

^d Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China,

Abstract

We consider the stochastic 2-dimensional Cahn-Hilliard equation which is driven by the derivative in space of a space-time white noise. We use two different approaches to study this equation. First we prove that there exists a unique solution Y to the shifted equation (see (1.4) below). Then $X := Y + Z$ is the unique solution to the stochastic Cahn-Hilliard equation, where Z is to the corresponding O-U process. Moreover, we use the Dirichlet form approach in [AR91] to construct the probabilistically weak solution the original equation (1.1) below. By clarifying the precise relation between the two solutions, we also get the restricted Markov uniqueness of the generator and the uniqueness of the martingale solutions to the equation (1.1). Furthermore, we also obtain exponential ergodicity of the solutions.

Keywords: stochastic quantization problem, Dirichlet forms, space-time white noise, Wick power, non-linear stochastic PDE

1 Introduction

In this paper we show the well-posedness for the conservative stochastic Cahn-Hilliard equation

$$\begin{cases} dX_t = -\frac{1}{2}A (AX - :X^3:) dt + BdW_t, \\ X(0) = z \in V_0^{-1}, \end{cases} \quad (1.1)$$

on \mathbb{T}^2 in the probabilistically strong sense where $A = \Delta$, $B = \text{div}$. W_t is an $L_0^2(\mathbb{T}^2, \mathbb{R}^2)$ -cylindrical Wiener process, which is defined in Section 3. $:X^3:$ denotes the Wick power, which

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[†]E-mail address: roeckner@math.uni-bielefeld.de(M. Röckner), hyang@math.uni-bielefeld.de(H. Y. Yang), zhurongchan@126.com(R. C. Zhu),

is introduced in Section 3 and the space V_0^{-1} is similar to the Sobolev space of order -1 , which is introduced in Section 2.

The Cahn-Hilliard equation is given by

$$\partial_t u = -\Delta^2 u - \Delta f(u),$$

which was introduced by Cahn and Hilliard [CH58] to study the phase separation of binary alloys. Here f is the derivative of a free energy and generally f is chosen as $f(u) = u^3 - u$. The stochastic Cahn-Hilliard equation was first studied in [PM83], where Petschek and Metiu performed some numerical experiments for the stochastic Cahn-Hilliard equation driven by space-time white noise. In [EM91], Elezovic and Mikelic proved the existence and uniqueness of a strong solution to the stochastic Cahn-Hilliard equation driven by trace-class noise. Then Da Prato and Debussche [DPD96] proved existence and uniqueness of solutions for space-time white noise and obtained the existence and uniqueness of an invariant measure for trace-class noise. Later there are many papers in which the authors study the properties of the solutions to the stochastic Cahn-Hilliard equations driven by trace-class noise (e.g. [DG11, Sca17]).

For the conservative-type equation (1.1), the Gibbs measure ν is formally given by the following Φ_2^4 field:

$$\nu(d\phi) = c \exp\left(-\int_{\mathbb{T}^2} \frac{1}{4} : \phi^4 : dx\right) \mu(d\phi),$$

where μ is the Gaussian free field, c is a normalization constant, and $: \phi^4 :$ is the fourth order Wick power of ϕ . Equation (1.1) can be interpreted as the natural "Kawasaki" dynamics (see [GLP99]) associated to the Euclidean Φ_2^4 -quantum field. In [PW81] Parisi and Wu proposed a program for Euclidean quantum field theory based on getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study properties of the Gibbs states. This procedure is called stochastic field quantization (see [JLM85]). The equation (1.1) can be also viewed as a stochastic quantization equation for the Φ_2^4 -field.

Over the years, there is a lot of literature (see [JLM85, AR91, DPD03, MW15, RZZ17a, RZZ17b]) on the stochastic quantization of the Φ_2^4 -field. The authors in these papers considered the following non-conservative stochastic quantization equation:

$$dX_t = (AX - : X^3 :) dt + dW_t. \tag{1.2}$$

First results are due to Jona-Lasinio and Mitter [JLM85]. Using the Girsanov theorem, they constructed solutions to a modified equation on \mathbb{T}^2 :

$$dX_t = (-\Delta + 1)^{-\varepsilon} (\Delta X - : X^3 : + aX) + (-\Delta + 1)^{-\frac{\varepsilon}{2}} dW_t \tag{1.3}$$

for $\frac{9}{10} < \varepsilon < 1$. They also proved the ergodicity for (1.3). In [AR91] Albeverio and Röckner studied (1.2) using Dirichlet forms and constructed probabilistically weak solutions to (1.2). In [MR99], Mikulevicius and Rozovskii constructed martingale solutions to (1.2) but the uniqueness remained open. In [DPD03] Da Prato and Debussche considered the associated shifted equation to (1.2) on \mathbb{T}^2 and proved the local existence and uniqueness of solutions in the probabilistically strong sense via a fixed point argument and then showed the non-explosion

for almost every initial point by using the invariant measure. Recently Mourrat and Weber [MW15] showed the global existence and uniqueness for the shifted equation both on \mathbb{T}^2 and \mathbb{R}^2 for every initial point. Combining the results from the weak approach and strong approach, Röckner, Zhu and Zhu [RZZ17b] proved the restricted Markov uniqueness for the generator of (1.2) and the uniqueness of the martingale problem to (1.2) arised in [MR99] on \mathbb{T}^2 and \mathbb{R}^2 . Furthermore, the ergodicity of (1.2) on \mathbb{T}^2 has been obtained in [HM16, RZZ17a, TW16].

For the conservative case, Funaki [Fun89] proved the existence and uniqueness of equation (1.1) on \mathbb{R} and in [DZ07] Debussche and Zambotti studied equation (1.1) on $[0, 1]$ with reflection. But for the higher dimensional case, even though the linear operator Δ^2 gives much more regularity, the noise and hence the solutions are still so singular that the non-linear terms in (1.1) are not well-defined in the classical sense. This difficulty is similar as in equation (1.2).

To overcome this difficulty, we use two approaches to study (1.1). First we follow the idea in [DPD03], [MW15] and [RZZ17b] to split the solution to $X = Y + Z$, where $Z(t) = \int_0^t e^{-\frac{(t-s)}{2}A^2} B dW_s$. Similarly as in the Φ_2^4 case, Y has better regularity than the solution to (1.1) and satisfies the following *shifted equation*:

$$\begin{cases} \frac{dY}{dt} = -\frac{1}{2}A^2Y + \frac{1}{2}A \sum_{k=0}^3 C_3^k Y^{3-k} : Z^k : \\ Y(0) = z \end{cases} \quad (1.4)$$

where $Z(t) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s$. In this paper we obtain the existence and uniqueness of the solution to (1.4). The fixed point arguments for local well-posedness in [DPD03] and [MW15] only hold for initial values in $\mathcal{C}^{-\frac{4}{3}+}$. Due to the singularity of the noise and the lack of a maximum principle and a uniform L^p -estimate, we only have a uniform H^{-1} -estimate (see Theorem 4.1), which is not strong enough to combine it with local well-posedness (see Remark 4.5). Instead, our argument is based on a classical compactness argument. We obtain the existence of global solutions starting from the uniform H^{-1} -estimate directly. Moreover we consider the solutions in H^{-1} and use the L^4 -integrability to obtain uniqueness for (1.4).

In addition, we use the method in [AR91] to construct the Dirichlet form for (1.1) (see Theorem 5.4), which is given by

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle_{V_0^{-1}} d\nu, \varphi, \psi \in \mathcal{FC}_b^\infty,$$

where \mathcal{FC}_b^∞ is defined in Section 5. We note that the tangent space is chosen as V_0^{-1} and the gradient operator ∇ is also defined in H^{-1} . This is different from the Dirichlet form for (1.2), where the tangent space is chosen as L^2 and the gradient is the L^2 -derivative. By the integration by parts formula for ν we also obtain the closability for the bilinear form $(\mathcal{E}, \mathcal{FC}_b^\infty)$. The closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form, which enables us to construct a probabilistically weak solution to (1.1). Then by clarifying the relation between this solution and the solution to (1.4), we prove that $X - Z$, where X is the solution obtained by the Dirichlet form approach, also satisfies the shifted equation (1.4). It follows that Φ_2^4 field is an invariant measure for X . Then we obtain the Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to \mathcal{FC}_b^∞ and the uniqueness of probabilistically weak solutions to (1.1) having ν as an invariant measure.

We prove exponential ergodicity by two approaches. One simple and short way by the Dirichlet form approach is presented in Remark 6.9. Using a uniform estimate, an invariant measure can also be constructed by the Krylov-Bogoliubov method. We follow an idea from [TW16] to prove the strong Feller property of the semigroup of the solution to the equation (1.1). Then we obtain exponential convergence to the unique invariant measure of the semigroup for every starting point.

This paper is organized as follows: In Section 2 we collect some results related to Besov spaces. In Section 3 we study the solution to the linear equation and define the Wick power. In Section 4 we obtain the global existence and uniqueness of solutions to the shifted equation (1.4). In Section 5 we obtain existence of probabilistically weak solutions via the Dirichlet form approach. By clarifying the relation between the two solutions we obtain Φ_2^4 field ν is an invariant measure of X Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to $\mathcal{F}C_b^\infty$ and uniqueness of the probabilistically weak solutions to (1.1). Moreover, using the Yamada-Watanabe Theorem in [Kur07] we obtain a proababilistically strong solution to (1.1) in the stationary case. Finally we prove the strong Feller property and exponential ergodicity of the Markov semigroup associated to the solution to (1.1) in Section 6.

2 Preliminaries

In the following we recall the definition of Besov spaces. For a general introduction to the theory of Besov spaces we refer to [BCD11, Tri78, Tri06]. First we introduce the following notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or \mathcal{D} . The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform are denoted by \mathcal{F} and \mathcal{F}^{-1} , respectively.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on \mathbb{R}^d , such that

- i. the support of χ is contained in a ball and the support of θ is contained in an annulus;
- ii. $\chi(z) + \sum_{j \geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.
- iii. $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\text{supp}(\theta(2^{-i}\cdot)) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $|i - j| > 1$.

We call such a pair (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot) \mathcal{F}u).$$

Besov spaces

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{D}$ we define

$$\|u\|_{B_{p,q}^\alpha} := \left(\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p})^q \right)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. The Besov space $B_{p,q}^\alpha$ consists of the completion of \mathcal{D} with respect to this norm and the Hölder-Besov space \mathcal{C}^α is given by

$\mathcal{C}^\alpha(\mathbb{R}^d) = B_{\infty,\infty}^\alpha(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B_{p,q}^\alpha(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B_{p,q}^\alpha} < \infty\}.$$

$$\mathcal{C}^\alpha(\mathbb{R}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} < \infty\}.$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [Sic85], [SW72]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on \mathbb{T}^d . Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{B_{p,q}^\alpha(\mathbb{T}^d)} := \left(\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q \right)^{1/q},$$

and the Hölder-Besov space \mathcal{C}^α is given by $\mathcal{C}^\alpha = B_{\infty,\infty}^\alpha(\mathbb{T}^d)$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B_{\infty,\infty}^\alpha(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$

$$B_{p,q}^\alpha(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B_{p,q}^\alpha(\mathbb{T}^d)} < \infty\}.$$

$$\mathcal{C}^\alpha \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_\alpha < \infty\}. \quad (2.1)$$

Here we choose Besov spaces as completions of smooth functions, which ensures that the Besov spaces are separable which has a lot of advantages for our analysis below.

Wavelet analysis

We will also use wavelet analysis to determine the regularity of a distribution in a Besov space. In the following we briefly summarize wavelet analysis below and we refer to work of Meyer [Mey95], Daubechies [Dau92] and [Tri06] for more details on wavelet analysis. For every $r > 0$, there exists a compactly supported function $\varphi \in C^r(\mathbb{R})$ such that:

1. We have $\langle \varphi(\cdot), \varphi(\cdot - k) \rangle = \delta_{k,0}$ for every $k \in \mathbb{Z}$;
2. There exist $\tilde{a}_k, k \in \mathbb{Z}$ with only finitely many non-zero values, and such that $\varphi(x) = \sum_{k \in \mathbb{Z}} \tilde{a}_k \varphi(2x - k)$ for every $x \in \mathbb{R}$;
3. For every polynomial P of degree at most r and for every $x \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} \int P(y) \varphi(y - k) dy \varphi(x - k) = P(x)$.

Given such a function φ , we define for every $x \in \mathbb{R}^d$ the recentered and rescaled function φ_x^n as follows

$$\varphi_x^n(y) := \prod_{i=1}^d 2^{\frac{n}{2}} \varphi(2^n(y_i - x_i)).$$

Observe that this rescaling preserves the L^2 -norm. We let V_n be the subspace of $L^2(\mathbb{R}^d)$ generated by $\{\varphi_x^n : x \in \Lambda_n\}$, where

$$\Lambda_n := \{(2^{-n}k_1, \dots, 2^{-n}k_d) : k_i \in \mathbb{Z}\}.$$

An important property of wavelets is the existence of a finite set Ψ of compactly supported functions in C^r such that, for every $n \geq 0$, the orthogonal complement of V_n inside V_{n+1} is given by the linear span of all the $\psi_x^n, x \in \Lambda_n, \psi \in \Psi$. For every $n \geq 0$

$$\{\varphi_x^n, x \in \Lambda_n\} \cup \{\psi_x^m : m \geq n, \psi \in \Psi, x \in \Lambda_m\},$$

forms an orthonormal basis of $L^2(\mathbb{R}^d)$. This wavelet analysis allows one to identify a countable collection of conditions that determine the regularity of a distribution.

Setting $\Psi_\star = \Psi \cup \{\varphi\}$, by some methods in weighted Besov space (see [RZZ17b, (2.2), (2.3), (2.4)] and its reference for details), we know that for $p \in (1, \infty)$, $\alpha \in \mathbb{R}$, $f \in \mathcal{C}^\alpha$

$$\|f\|_\alpha^p \lesssim \sum_{n=0}^{\infty} 2^{n(\alpha+1)p} \sum_{\psi \in \Psi_\star} \sum_{x \in \Lambda_n} |\langle f, \psi_x^n \rangle|^p w(x)^p. \quad (2.2)$$

where $w(x) = (1 + |x|^2)^{-\frac{\sigma}{2}}$, $\sigma > 0$.

Estimates on the torus

In this part we give estimates on the torus for later use. Set $\Lambda = (I - \Delta)^{\frac{1}{2}}$. For $s \geq 0$, $p \in [1, +\infty]$ we use H_p^s to denote the subspace of $L^p(\mathbb{T}^d)$, consisting of all f which can be written in the form $f = \Lambda^{-s}g$, $g \in L^p(\mathbb{T}^d)$ and the H_p^s norm of f is defined to be the L^p norm of g , i.e. $\|f\|_{H_p^s} := \|\Lambda^s f\|_{L^p(\mathbb{T}^d)}$.

To study (1.1) in the finite volume case, we will need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus first (c.f. [Tri78, Theorem 4.6.1], [GIP15, Lemma A.2], [Tri92, Remark 3, Section 2.3.2]):

Lemma 2.1 (i) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B_{p_1, q_1}^\alpha(\mathbb{T}^d)$ is continuously embedded in $B_{p_2, q_2}^{\alpha-d(1/p_1-1/p_2)}(\mathbb{T}^d)$.*

(ii) *Let $s \geq 0$, $1 < p < \infty$, $\varepsilon > 0$. Then $H_p^{s+\varepsilon} \subset B_{p,1}^s(\mathbb{T}^d) \subset B_{1,1}^s(\mathbb{T}^d)$.*

(iii) *Let $1 \leq p_1 \leq p_2 < \infty$ and let $\alpha \in \mathbb{R}$. Then $H_{p_1}^\alpha$ is continuously embedded in $H_{p_2}^{\alpha-d(1/p_1-1/p_2)}$.*

(iv) *Let $0 < q \leq \infty$, $1 \leq p \leq \infty$ and $s > 0$. Then $B_{p,q}^s \subset L^p$.*

Here \subset means that the embedding is continuous and dense.

We recall the following Schauder estimates, i.e. the smoothing effect of the heat flow, for later use.

Lemma 2.2 ([GIP15, Lemma A.7]) *Let $u \in B_{p,q}^\alpha(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. Then for every $\delta \geq 0$*

$$\|e^{-tA^2} u\|_{B_{p,q}^{\alpha+\delta}(\mathbb{T}^d)} \lesssim t^{-\delta/4} \|u\|_{B_{p,q}^\alpha(\mathbb{T}^d)}.$$

One can extend the multiplication on suitable Besov spaces and also have the duality properties of Besov spaces from [Tri78, Chapter 4]:

Lemma 2.3 (i) *The bilinear map $(u; v) \mapsto uv$ extends to a continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$ if and only if $\alpha + \beta > 0$.*

(ii) *Let $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u; v) \mapsto \int uv dx$ extends to a continuous bilinear form on $B_{p,q}^\alpha(\mathbb{T}^d) \times B_{p',q'}^{-\alpha}(\mathbb{T}^d)$.*

We recall the following interpolation inequality and multiplicative inequality for the elements in H_p^s , which is required for the a-priori estimate in section 4 (cf. [Tri78, Theorem 4.3.1], [RZZ15, Lemma 2.1], [BCD11, Theorem 2.80]):

Lemma 2.4 (i) Suppose that $s \in (0, 1)$ and $p \in (1, \infty)$. Then for $u \in H_p^1$

$$\|u\|_{H_p^s} \lesssim \|u\|_{L^p(\mathbb{T}^d)}^{1-s} \|u\|_{H_p^1}^s.$$

(ii) Suppose that $s > 0$ and $p \in (1, \infty)$. If $u, v \in C^\infty(\mathbb{T}^2)$ then

$$\|\Lambda^s(uv)\|_{L^p(\mathbb{T}^d)} \lesssim \|u\|_{L^{p_1}(\mathbb{T}^d)} \|\Lambda^s v\|_{L^{p_2}(\mathbb{T}^d)} + \|v\|_{L^{p_3}(\mathbb{T}^d)} \|\Lambda^s u\|_{L^{p_4}(\mathbb{T}^d)},$$

with $p_i \in (1, \infty], i = 1, \dots, 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

(iii) Suppose that $s_1 < s_2$ and $1 \leq p, q \leq \infty$. Then for $u \in B_{p,q}^{s_2}$ and $\forall \theta \in (0, 1)$

$$\|u\|_{B_{p,q}^{\theta s_1 + (1-\theta)s_2}} \leq \|u\|_{B_{p,q}^{\theta}} \|u\|_{B_{p,q}^{s_2}}^{1-\theta}.$$

We also collect some important properties for the multiplicative structure of Besov spaces from [MW15] and [Tri06].

Lemma 2.5 ([MW15, Corollary 3.19, Corollary 3.21]) (1) For $\alpha > 0, p_1, p_2, p, q \in [1, \infty], \frac{1}{p} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a continuous bilinear map from $B_{p_1,q}^\alpha \times B_{p_2,q}^\alpha$ to $B_{p,q}^\alpha$.

(2) For $\alpha < 0, \alpha + \beta > 0, p_1, p_2, p, q \in [1, \infty], \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a continuous bilinear map from $B_{p_1,q}^\alpha \times B_{p_2,q}^\beta$ to $B_{p,q}^\alpha$.

Notations

Let L denote the space $L^2(\mathbb{T}^2) = [0, 1]^2$, where \mathbb{T}^2 is the 2 dimensional torus and we use $\langle \cdot, \cdot \rangle$ to denote the inner product in L . A is the Laplacian operator on L , that is,

$$D(A) = H_2^2(\mathbb{T}^2), A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (2.3)$$

A is a self-adjoint operator in L , with complete orthonormal system $(e_n)_n$ of eigenvectors in L , given by

$$\begin{aligned} e_0(x) &:= 1, e_{(k_1,0)}(x) = \sqrt{2}e^{i\pi k_1 x_1}, e_{(0,k_2)}(x) = \sqrt{2}e^{i\pi k_2 x_2}, \\ e_k(x) &:= 2e^{i\pi(k_1 x_1 + k_2 x_2)}, k_1 k_2 \neq 0. \end{aligned}$$

Then we have $Ae_k = -\lambda_k e_k$, where $\lambda_k = |k|^2 \pi^2, k = (k_1, k_2) \in \mathbb{Z}^2, |k|^2 = k_1^2 + k_2^2$. We also introduce a notation for the average of $h \in \mathcal{S}'(\mathbb{T}^2)$:

$$m(h) := \mathcal{S}' \langle h, e_0 \rangle_{\mathcal{S}}.$$

For any $\alpha \in \mathbb{R}$, we define

$$V^\alpha := \{u \in \mathcal{S}' : \sum_k \lambda_k^\alpha |\mathcal{S}' \langle u, e_k \rangle_{\mathcal{S}}|^2 < \infty\}.$$

For any $u, v \in V^\alpha$, define

$$\langle u, v \rangle_{V^\alpha} := m(u)m(v) + \sum_k \lambda_k^\alpha \langle u, e_k \rangle_{\mathcal{S}} \langle v, e_k \rangle_{\mathcal{S}}.$$

It's easy to see that $(V^\alpha, \langle \cdot, \cdot \rangle_{V^\alpha})$ is a Hilbert space and $V^\alpha \simeq H_2^\alpha$. Then for any $s, \alpha \in \mathbb{R}$, we can define a bounded operator $(-A)^s : V^\alpha \rightarrow V^{\alpha-2s}$ by:

$$(-A)^s u = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \lambda_k^s u_k e_k,$$

where $u = \sum_k u_k e_k \in V^\alpha$. In particular, we set $Q := (-A)^{-1}$ and extend it to a one-to-one bounded operator \bar{Q} by

$$\bar{Q}h := Qh + m(h)e_0. \quad (2.4)$$

Note that

$$Qe_k = \begin{cases} \frac{1}{\lambda_k} e_k & k \neq (0, 0), \\ 0 & k = (0, 0), \end{cases} \quad (2.5)$$

and

$$\bar{Q}e_k = \begin{cases} \frac{1}{\lambda_k} e_k & k \neq (0, 0), \\ e_0 & k = (0, 0). \end{cases} \quad (2.6)$$

Then we have

$$\langle u, v \rangle_{V^\alpha} := \langle \bar{Q}^{-\alpha/2} u, \bar{Q}^{-\alpha/2} v \rangle,$$

and $\bar{Q}^s : V^\alpha \rightarrow V^{\alpha+2s}$ is an isomorphism for any $\alpha, s \in \mathbb{R}$, since

$$\langle \bar{Q}^s u, \bar{Q}^s v \rangle_{V^{\alpha+2s}} = \langle u, v \rangle_{V^\alpha}.$$

We also set

$$V_0^\alpha := \{h \in V^\alpha : \langle h, e_0 \rangle_{V^\alpha} = 0\},$$

and denote $L_0^2 := V_0^0$. Let Π denote the symmetric projector of V^α on V_0^α , that is,

$$\Pi : V^\alpha \rightarrow V_0^\alpha, \Pi h := h - m(h). \quad (2.7)$$

Moreover, we define

$$V^\alpha(\mathbb{T}^2, \mathbb{R}^2) := \{f = (f_1, f_2) : f_i \in V^\alpha, i = 1, 2\},$$

and similarly

$$V_0^\alpha(\mathbb{T}^2, \mathbb{R}^2) := \{f = (f_1, f_2) : f_i \in V_0^\alpha, i = 1, 2\}.$$

In this paper, we consider the initial value and the reference measure on V_0^α for simplicity. For general case, we refer to [DZ07].

3 The Linear Equation and Wick Powers

We consider the O-U process

$$\begin{cases} dZ_t = -\frac{1}{2}A^2 Z dt + B dW_t, \\ Z(0) = 0, \end{cases} \quad (3.1)$$

where W is a U -cylindrical Wiener process and $U := L_0^2(\mathbb{T}^2, \mathbb{R}^2)$. For $f \in L_0^2(\mathbb{T}^2, \mathbb{R}^2)$ we denote its component functions by $f_1, f_2 \in L_0^2(\mathbb{T}^2)$ i.e. $f(x) = (f_1(x), f_2(x)), \forall x \in \mathbb{T}^2$. There exist two independent $L^2(\mathbb{T}^2)$ -cylindrical Wiener processes W^1 and W^2 such that $W = (W^1, W^2)$. Set

$$D(B) = H^1(\mathbb{T}^2, \mathbb{R}^2), B = \text{div}, D(B^*) = H_2^1(\mathbb{T}^2), B^* = -\nabla. \quad (3.2)$$

We know that

$$Z_t(x) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s = \int_0^t \langle K(t-s, x - \cdot), dW_s \rangle_U,$$

where $K(t, x) = -\nabla_x M(t, x) = (K^1, K^2)$, and $M(t, x)$ is the kernel of $e^{-\frac{t}{2}A^2}$, that is, $M(t, x) = \sum_k e^{-\frac{t}{2}\lambda_k^2} e_k(x)$.

For any function f on \mathbb{T}^2 , we can view it as a periodic function on \mathbb{R}^2 by defining $\bar{f}(x) := f(x+m)$, when $x+m \in \mathbb{T}^2$, $x \in \mathbb{R}^2$, $m \in \mathbb{Z}^2$. Moreover, define

$$\bar{K}^j(t, x) := -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\pi \xi|^4})(x), j = 1, 2.$$

and $\bar{K} := (\bar{K}^1, \bar{K}^2)$. By the Poisson summation formula (see [SW72, Section VII.2]) we know that

$$K(t, x) = \sum_m \bar{K}(t, x+m), \forall t \quad (3.3)$$

and for any $f \in L^2(\mathbb{T}^2)$, $j = 1, 2$, $x \in \mathbb{T}^2$

$$\begin{aligned} \partial_j e^{-\frac{t}{2}A^2} f(x) &= \int_{\mathbb{T}^2} K^j(t, x-y) f(y) dy \\ &= \int_{\mathbb{R}^2} K^j(t, x-y) f(y) \mathbb{1}_{\mathbb{T}^2}(y) dy \\ &= \sum_m \int_{\mathbb{R}^2} \bar{K}^j(t, x-y+m) f(y) \mathbb{1}_{\mathbb{T}^2}(y) dy, \\ &= \int_{\mathbb{R}^2} \bar{K}^j(t, x-y) \sum_m \mathbb{1}_{\mathbb{T}^2}(y+m) f(y+m) dy \\ &= (\bar{K}^j(t, \cdot) * \bar{f})(x) \end{aligned} \quad (3.4)$$

where we used (3.3) in the third inequality and $\mathbb{1}_{\mathbb{T}^2}$ is the indicator function of \mathbb{T}^2 . Since

$$\bar{K}^j(t, x) = -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\pi \xi|^4})(x) = t^{-\frac{3}{4}} \bar{K}^j(1, t^{-\frac{1}{4}}x)$$

and

$$|\bar{K}^j(1, t^{-\frac{1}{4}}x)| \lesssim |\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{1}{2}|\pi \xi|^4})(t^{-\frac{1}{4}}x)| \lesssim |1 + t^{-\frac{1}{4}}x|^{-3},$$

we have the following estimate:

$$|\bar{K}(t, x)| \lesssim t^{-\frac{3}{4}} |x|^{-3+\varepsilon}, \forall \varepsilon \in [0, 3]. \quad (3.5)$$

Lemma 3.1 $Z \in C([0, T]; \mathcal{C}^{-\alpha})$ \mathbb{P} -almost-surely, for all $\alpha > 0$.

Proof By the factorization method in [DP04] we have that for $\kappa \in (0, 1)$

$$Z(t) = \frac{\sin(\pi\kappa)}{\pi} \int_0^t (t-s)^{\kappa-1} \langle M(t-s, x-\cdot), U(s) \rangle ds,$$

and

$$U(s, \cdot) = \int_0^s (s-r)^{-\kappa} e^{-\frac{s-r}{2}A^2} BdW_r.$$

A similar argument as in the proof of Lemma 2.7 in [DP04] implies that it suffices to prove that for $p > 1/(2\kappa)$,

$$\mathbf{E}\|U\|_{L^{2p}(0, T; \mathcal{C}^{-\alpha})} < \infty. \quad (3.6)$$

In fact, by (2.2) we have that

$$\begin{aligned} \mathbf{E}\|U(s)\|_{-\alpha}^{2p} &\lesssim \sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} \mathbf{E} 2^{-2\alpha pn + 2np} |\langle U(s), \psi_x^n \rangle|^{2p} w(x)^{2p} \\ &\lesssim \sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} 2^{-2\alpha pn + 2np} (\mathbf{E} |\langle U(s), \psi_x^n \rangle|^2)^p w(x)^{2p}. \end{aligned}$$

Here $\sigma > 0$ in $w(x)$ and we used that $\langle U(s), \psi_x^n \rangle$ belongs to the first order Wiener-chaos and Gaussian hypercontractivity (cf. [Nua13, Section 1.4.3] and [Nel73]) in the second inequality. Moreover, we obtain that

$$\begin{aligned} \mathbf{E} |\langle U(s), \psi_x^n \rangle|^2 &= \mathbf{E} |\langle U^1(s), \psi_x^n \rangle|^2 + \mathbf{E} |\langle U^2(s), \psi_x^n \rangle|^2 \\ &\leq \sum_{j=1}^2 \int \int |\psi_x^n(y) \psi_x^n(\bar{y})| \int_0^s (s-r)^{-2\kappa} \bar{K}^j * \bar{K}^j(s-r, y-\bar{y}) dr dy d\bar{y} \\ &\lesssim \int \int |\psi_x^n(y) \psi_x^n(\bar{y})| \int_0^s (s-r)^{1-\frac{\varepsilon}{2}-2\kappa} |y-\bar{y}|^{-4+2\varepsilon} dr dy d\bar{y} \\ &\lesssim 2^{2n-2\varepsilon n} s^{2-2\kappa-\frac{\varepsilon}{2}}, \end{aligned}$$

where

$$U^j(y) = \int_0^s (t-s)^{\kappa-1} \langle K^j(s-r, y-\cdot), dW_r^j \rangle, j = 1, 2$$

and we used (3.4) in the second inequality and we also used [Hai14, Lemma 10.17] and (3.5) to deduce that $|\bar{K}^j * \bar{K}^j(s-r, y-\bar{y})| \lesssim |s-r|^{1-\frac{\varepsilon}{2}} |y-\bar{y}|^{-4+2\varepsilon}$ in the second inequality.

In fact, we can decompose \bar{K} into $\bar{K} := \bar{K}_\delta + \bar{K}_\delta^c$, where \bar{K}_δ is a compactly supported function and satisfies (3.5), \bar{K}_δ^c is a Schwartz function. Then $\bar{K} * \bar{K} = \bar{K}_\delta * \bar{K}_\delta + H$, where H is a Schwartz function. By [Hai14, Lemma 10.17] we have $\bar{K}_\delta * \bar{K}_\delta(t, x) \lesssim t^{1-\frac{\varepsilon}{2}} |x|^{-4+2\varepsilon}$ and $\bar{K} * \bar{K}$ satisfies the same inequality.

Thus, we have

$$\mathbf{E}\|U(s)\|_{-\alpha}^{2p} \lesssim \sum_{n \geq 0} 2^{(4-2\varepsilon-2\alpha)pn} s^{(2-2\kappa-\frac{\varepsilon}{2})p}.$$

Let κ be so small that $2 - \alpha < \varepsilon < 4 - 2(2\kappa - \frac{1}{p})$, which implies that

$$4 - 2\varepsilon - 2\alpha < 0, (2 - 2\kappa - \frac{\varepsilon}{2})p > -1.$$

Then (3.6) follows. □

Note that $BB^* = -A$. Then by Fourier expansion it is easy to see that $Z_t \sim \mathcal{N}(0, Q_t)$, i.e. for any $h \in \mathcal{S}(\mathbb{T}^2)$

$$Ee^{is\langle h, Z_t \rangle_{S'}} = \exp(-\frac{1}{2}\langle Q_t h, h \rangle),$$

where $Q_t = (-A)^{-1}(I - e^{-\frac{t}{2}A^2})$.

According to the definition of V^α and Lemma 2.1, we have $\mathcal{C}^{-\alpha} \subset V^{-\alpha-\varepsilon}$ for any $\alpha, \varepsilon > 0$. Then by Lemma 3.1, μ_t is supported on $V_0^{-\alpha}$ for any $\alpha > 0$ and letting $t \rightarrow \infty$, by [Bog98, 3.8.13, Example], the law of Z_t converges to the Gaussian measure $\mu = \mathcal{N}(0, Q)$, which is also supported on $V_0^{-\alpha}$.

In the following we are going to define the Wick powers both in the state space and the path space.

Firstly, we define the Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$.

Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$

μ is of course also a measure supported on $\mathcal{S}'(\mathbb{T}^2)$. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}'(\mathbb{T}^2), \mu) = \bigoplus_{n \geq 0} \mathcal{H}_n,$$

where \mathcal{H}_n is the Wiener chaos of order n (cf. [Nua13, Section 1.1.1]). Now we define the Wick powers by using approximations: for $\phi \in \mathcal{S}'(\mathbb{T}^2)$ define

$$\phi_\varepsilon := \rho_\varepsilon * \phi,$$

with ρ_ε an approximate delta function on \mathbb{R}^2 given by

$$\rho_\varepsilon(x) = \varepsilon^{-2} \rho(\frac{x}{\varepsilon}) \in \mathcal{D}, \int \rho = 1.$$

Here the convolution means that we view ϕ as a periodic distribution in $\mathcal{S}'(\mathbb{R}^2)$ and convolve on \mathbb{R}^2 . For every $n \in \mathbb{N}$ we set

$$: \phi_\varepsilon^n := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2} \phi_\varepsilon),$$

where $P_n, n = 0, 1, \dots$, are the Hermite polynomials defined by the formula

$$P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{(n-2j)! j! 2^j} x^{n-2j},$$

and $c_\varepsilon = \int \phi_\varepsilon^2 \mu(d\phi) = \int \int G(z-y) \rho_\varepsilon(y) dy \rho_\varepsilon(z) dz$. Then

$$: \phi_\varepsilon^n := \in \mathcal{H}_n.$$

Here and in the following G is the Green function associated with $-A$ on \mathbb{T}^2 . In fact by [SW72, Section 6.1, Chapter VII],

$$G(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\lambda_k} e_k(x) \simeq -\log|x|, |x| \rightarrow 0,$$

and G is continuously differentiable except outside $\{0\}$.

For Hermite polynomial P_n we have that for $s, t \in \mathbb{R}$

$$P_n(s+t) = \sum_{m=0}^n C_n^m P_m(s) t^{n-m}, \quad (3.7)$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

A direct calculation yields the following:

Lemma 3.2 *Let $\alpha > 0$, $n \in \mathbb{N}$ and $p > 1$. $:\phi_\varepsilon^n$ converges to some element in $L^p(\mathcal{S}'(\mathbb{T}^2), \mu; \mathcal{C}^{-\alpha})$ as $\varepsilon \rightarrow 0$. This limit is called the n -th Wick power of ϕ with respect to the covariance Q and denoted by $:\phi^n$.*

Proof The proof is similar to that of [RZZ17b, Lemma 3.1] since the Green function G has the same regularity. Therefore we omit it here for simplicity. \square

Wick powers on a fixed probability space

Now we fix a probability space (Ω, \mathcal{F}, P) and consider a U -cylindrical Wiener process W . In the following we assume that \mathcal{F} is the σ -field generated by $\{\langle W_t, h \rangle, h \in U, t \in \mathbb{R}^+\}$. We also have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n \geq 0} \mathcal{H}'_n,$$

where \mathcal{H}'_n is the Wiener chaos of order n (cf. [Nua13, Section 1.1.1]). We can define Wick powers of $Z(t)$ with respect to different covariances by approximations: Let

$$\begin{aligned} Z_\varepsilon(t, x) &= \rho_\varepsilon * Z_t = \int_0^t \langle B^* e^{-\frac{t-s}{2}A^2} \rho_{\varepsilon, x}, dW_s \rangle_U \\ &= \int_0^t \langle K_\varepsilon(t-s, x - \cdot), dW_s \rangle_U, \end{aligned}$$

where $\rho_{\varepsilon, x} = \rho_\varepsilon(x - \cdot)$, $K_\varepsilon(t, x) = (\rho_\varepsilon * K_t^1, \rho_\varepsilon * K_t^2)$ and

$$K_t^j = - \sum_k (i\pi k_j) e^{-\frac{t}{2}\lambda_k^2} e_k, \quad j = 1, 2.$$

Lemma 3.3 ([RZZ17b, Lemma 3.4]) *For $\alpha > 0$, $p > 1$, $n \in \mathbb{N}$, $:\mathcal{Z}_\varepsilon^n$ converges in $L^p(\Omega, C((0, T]; \mathcal{C}^{-\alpha}))$. Here $C((0, T]; \mathcal{C}^{-\alpha})$ is equipped with the norm $\sup_{t \in [0, T]} t^\rho \|\cdot\|_{-\alpha}$ for $\rho > 0$. The limit is called Wick power of $Z(t)$ of order n with respect to the covariance Q and is denoted by $:\mathcal{Z}^n(t)$.*

Proof The kernel K is a little different from the kernel in [RZZ17b]. But (3.5) satisfies the condition in [RZZ17b, Lemma 3.2] and [ZZ15, Lemma 4.1] which yields a similar proof as [RZZ17b, Lemma 3.3], so we omit it here. \square

Remark 3.4 *Here we do not combine the initial value with the Wick powers as in [MW15, RZZ17b], since we can obtain existence of solutions to the shifted equation (4.1) for any initial value in V_0^{-1} (see Section 4).*

Relations between two different Wick powers

We introduce the following probability measure. Set $: q(\phi) := \frac{1}{4} : \phi^4 :$, $: p(\phi) := : \phi^3 :$. Let

$$\nu = c \exp(-N)\mu,$$

where c is a normalization constant and $N = \mathcal{S}' \langle q : e_0 \rangle_{\mathcal{S}}$. Then according to [Sim74, Lemma V.5 and Theorem V.7] we have for every $p \in [1, \infty)$, $\varphi(\phi) := e^{-N} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$.

The following result is about the relation between the two different Wick powers.

Lemma 3.5 *Let ϕ be a measurable map from $(\Omega, \mathcal{F}, \mathbb{P})$ to $C([0, T], B_{2,2}^{-\gamma})$ with $\gamma > 2$, $\mathbb{P} \circ \phi(t)^{-1} = \nu$ for every $t \in [0, T]$ and let $Z(t)$ be defined as above. Assume in addition that $y = \phi - Z \in C((0, T]; \mathcal{C}^\beta)$ \mathbb{P} -a.s. for some $\beta > \alpha > 0$. Here $C((0, T]; \mathcal{C}^\beta)$ is equipped with the norm $\sup_{t \in [0, T]} t^{\frac{\beta+\alpha}{4}} \|\cdot\|_\beta$. Then for every $t > 0$, $n \in \mathbb{N}$*

$$: \phi^n(t) := \sum_{k=0}^n C_n^k y^{n-k}(t) : Z^k(t) : \quad \mathbb{P} - a.s..$$

Here the Wick power on the left hand side is the limit obtained and defined in Lemma 3.2.

Proof By Lemma 3.3 it follows that for every $k \in \mathbb{N}$, $p > 1$

$$: Z_\varepsilon^k : \rightarrow : Z^k : \quad \text{in } L^p(\Omega, C((0, T]; \mathcal{C}^{-\alpha})), \text{ as } \varepsilon \rightarrow 0.$$

Since $y_\varepsilon = \phi_\varepsilon - Z_\varepsilon = \rho_\varepsilon * y$ and $y \in C((0, T]; \mathcal{C}^\beta)$ \mathbb{P} -a.s., it is obvious that $y_\varepsilon \rightarrow y$ in $C((0, T]; \mathcal{C}^{\beta-\kappa})$ \mathbb{P} -a.s. for every $\kappa > 0$ with $\beta - \kappa - \alpha > 0$, which combined with Lemma 2.3 implies that for $k \in \mathbb{N}$, $k \leq n$,

$$y_\varepsilon^{n-k} : Z_\varepsilon^k : \rightarrow^{\mathbb{P}} y^{n-k} : Z^k : \quad \text{in } C((0, T]; \mathcal{C}^{-\alpha}), \text{ as } \varepsilon \rightarrow 0.$$

Here $\rightarrow^{\mathbb{P}}$ means convergence in probability. Since $e^{-N} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$ for every $p \geq 1$, by Hölder's inequality and Lemma 3.2 we get that for $t > 0$ and $p > 1$

$$: \phi_\varepsilon^n(t) : \rightarrow : \phi^n(t) : \quad \text{in } L^p(\Omega, \mathcal{C}^{-\alpha}), \text{ as } \varepsilon \rightarrow 0.$$

Moreover, by (3.7) we have

$$\begin{aligned} & : \phi_\varepsilon^n := (y_\varepsilon + Z_\varepsilon)^n := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}(y_\varepsilon + Z_\varepsilon)) \\ & = \sum_{k=0}^n C_n^k c_\varepsilon^{n/2} P_k(c_\varepsilon^{-1/2} Z_\varepsilon) (c_\varepsilon^{-1/2} y_\varepsilon)^{n-k} \\ & = \sum_{k=0}^n C_n^k : Z_\varepsilon^k : y_\varepsilon^{n-k}, \end{aligned}$$

which implies the result by letting $\varepsilon \rightarrow 0$. \square

4 The Solution to the Shifted Equation

Now we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, \mathbb{P})$ and on it a U -cylindrical Wiener process W . Define $Z(t) = \int_0^t e^{-(t-s)A^2/2} B dW(s)$ as in Section 3. Now we consider the following shifted equation:

$$\begin{cases} \frac{dY}{dt} = -\frac{1}{2}A^2Y + \frac{1}{2}A \sum_{k=0}^3 C_3^k Y^{3-k} : Z^k :, \\ Y(0) = x. \end{cases} \quad (4.1)$$

Generally we consider initial data x that are \mathcal{F}_0 measurable and belong to V_0^{-1} , *a.s.*. To prove the existence of the solution to equation (4.1), we use a smooth approximation on each path:

$$\begin{cases} \frac{dY_\varepsilon}{dt} = -\frac{1}{2}A^2Y_\varepsilon + \frac{1}{2}A \sum_{k=0}^3 C_3^k Y_\varepsilon^{3-k} : Z_\varepsilon^k :, \\ Y_\varepsilon(0) = x_\varepsilon, \end{cases} \quad (4.2)$$

where $Z_\varepsilon = Z * \rho_\varepsilon$, $x_\varepsilon = x * \rho_\varepsilon$, and ρ_ε is as introduced in Section 3. Note that the solution Y to equation (4.1) and the solution Y_ε to (4.2) also satisfy:

$$\frac{dm(Y(t))}{dt} = 0, m(Y(0)) = 0, \quad (4.3)$$

which means that $m(Y(t)) = m(Y_\varepsilon(t)) = 0$.

From Lemma 3.2 we know that there exists a $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, such that for any $\omega \in \Omega'$, $Z(\omega), : Z^n : (\omega) \in C((0, T]; \mathcal{C}^{-\alpha})$, $n = 2, 3$, $\forall \alpha > 0$. Since $Z_\varepsilon(\omega)$ is smooth, by monotone trick in [LR15, Theorem 5.2.2 and Theorem 5.2.4], there exists a unique solution $Y_\varepsilon(\omega)$ to equation (4.2) in $L^2(0, T; V_0^2) \cap C([0, T]; L_0^2)$ for each $\omega \in \Omega'$. We are going to find a convergent subsequence of $\{Y_\varepsilon(\omega)\}$, which converge to a solution to equation (4.1) and prove uniqueness of solutions to (4.1). Then we obtain a unique \mathcal{F}_t -adapted solution to equation (4.1).

In this section we never distinguish V^α , H_2^α and $B_{2,2}^\alpha$ since they have equivalent norms. For convenience we denote all of them as H^α .

Theorem 4.1 (a-priori estimate) *If Y is a solution to equation (4.1), then there exists a constant C_T which only depends on T and $Z(\omega)$, such that for $\forall t \in [0, T]$*

$$\|Y\|_{H^{-1}}^2 - \|x\|_{H^{-1}}^2 + \frac{1}{2} \int_0^t (\|Y(s)\|_{H^1}^2 + \|Y(s)\|_{L^4}^4) ds \leq C_T. \quad (4.4)$$

Moreover there exist constants $C > 0$, $\gamma_k > 0$, $k = 1, 2, 3$, for every $t \in (0, T]$

$$\|Y_t\|_{H^{-1}}^2 \leq C \left(t^{-1} \vee \left(\sum_{k=1}^3 t^{-\rho\gamma_k} \sup_{0 \leq r \leq t} (r^{\rho\gamma_k} \| : Z_r : \|_{-\alpha}^{\gamma_k}) \right)^{\frac{1}{2}} \right), \quad (4.5)$$

where $\rho > 0$ is a small enough constant introduced in Lemma 3.3.

Proof Since

$$\frac{dY}{dt} = -\frac{1}{2}A\left(AY - \sum_{k=0}^3 C_3^k Y^{3-k} : Z^k :\right),$$

and $m(Y) = 0$, taking scalar product with $(-A)^{-1}Y$ we obtain that

$$\frac{d}{dt}\|Y\|_{H^{-1}}^2 + \|Y\|_{H^1}^2 + \|Y\|_{L^4}^4 = -\left\langle \sum_{k=1}^3 C_3^k Y^{3-k} : Z^k :, Y \right\rangle,$$

that is

$$\frac{d}{dt}\|Y\|_{H^{-1}}^2 + \|Y\|_{H^1}^2 + \|Y\|_{L^4}^4 \lesssim |\langle Y, : Z^3 : \rangle| + |\langle Y^2, : Z^2 : \rangle| + |\langle Y^3, Z \rangle|. \quad (4.6)$$

So, we only need to estimate the right hand side of (4.6). We only consider $|\langle Y^3, Z \rangle|$. The other terms can be estimated similarly. Lemma 2.3 implies

$$|\langle Y^3, Z \rangle| \lesssim \|Z\|_{-\alpha} \|Y^3\|_{B_{1,1}^\alpha}, \quad \forall \alpha > 0.$$

Moreover, by Lemma 2.1 and Lemma 2.4. Then

$$\|Y^3\|_{B_{1,1}^\alpha} \lesssim \|\Lambda^{\beta_0} Y^3\|_{L^{p_0}} \lesssim \|\Lambda^{\beta_0} Y^{\frac{3}{2}}\|_{L^{p_1}} \|Y^{\frac{3}{2}}\|_{L^{q_1}},$$

where $\beta_0 > \alpha$, $p_0 > 1$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_1}$.

Choose $q_1 \leq \frac{8}{3}$ and $p_1 > \frac{8}{5}$, we have

$$\|Y^{\frac{3}{2}}\|_{L^{q_1}} = \|Y\|_{L^{\frac{2}{3}q_1}}^{\frac{3}{2}} \lesssim \|Y\|_{L^4}^{\frac{3}{2}}.$$

For $\|\Lambda^{\beta_0} Y^{\frac{3}{2}}\|_{L^{p_1}}$, we have

$$\|\Lambda^{\beta_0} Y^{\frac{3}{2}}\|_{L^{p_1}} \lesssim \|\Lambda^{\beta_1} Y^{\frac{3}{2}}\|_{L^{p_2}} \lesssim \|\Lambda Y^{\frac{3}{2}}\|_{L^{p_2}}^{\beta_1} \|Y^{\frac{3}{2}}\|_{L^{p_2}}^{1-\beta_1},$$

where $1 < p_2 < p_1 < 2$, $\beta_0 = \beta_1 - 2(\frac{1}{p_2} - \frac{1}{p_1})$, $\beta_1 < 1$ and we used Lemma 2.1 in the first inequality and Lemma 2.5 in the second inequality. For $\|\Lambda Y^{\frac{3}{2}}\|_{L^{p_2}}$, let $p_2 < \frac{8}{5}$, we have

$$\|\Lambda Y^{\frac{3}{2}}\|_{L^{p_2}} \lesssim \|Y^{\frac{1}{2}} \nabla Y\|_{L^{p_2}} \lesssim \|Y\|_{H^1} \|Y^{\frac{1}{2}}\|_{L^{\frac{2p_2}{2-p_2}}} \lesssim \|Y\|_{H^1} \|Y\|_{L^4}^{\frac{1}{2}} \lesssim \|Y\|_{H^1} \|Y\|_{L^4}^{\frac{1}{2}},$$

where we used Hölder's inequality in the second inequality. Furthermore

$$\|Y^{\frac{3}{2}}\|_{L^{p_2}} \lesssim \|Y\|_{L^{\frac{3p_2}{2}}}^{\frac{3}{2}} \lesssim \|Y\|_{L^4}^{\frac{3}{2}}.$$

Combining the above estimates we get

$$\|Y^3\|_{B_{1,1}^\alpha} \lesssim \|Y\|_{L^4}^{3-\beta_1} \|Y\|_{H^1}^{\beta_1}.$$

Combining this with Lemma 3.3, we have

$$|\langle Y^3, Z \rangle| \lesssim \|Y\|_{L^4}^{3-\beta_1} \|Y\|_{H^1}^{\beta_1} t^{-\frac{\ell}{4}} \lesssim t^{-\frac{\ell}{4}\gamma} + \kappa (\|Y\|_{L^4}^4 + \|Y\|_{H^1}^2),$$

where $\gamma = \frac{4}{1-\beta_1}$ and we used the Young's inequality. Choosing ρ to be so small that $\frac{\rho}{4}\gamma < 1$, we can conclude that there exists $\gamma_k > 0$, $k = 1, 2, 3$ such that $\frac{\gamma_k \rho}{4} < 1$

$$\frac{d}{dt} \|Y\|_{H^{-1}}^2 + \frac{1}{2} (\|Y\|_{H^1}^2 + \|Y\|_{L^4}^4) \lesssim \sum_{k=1}^3 \| : Z^k : \|_{-\alpha}^{\gamma_k} \lesssim \sum_{k=1}^3 t^{-\frac{\gamma_k \rho}{4}},$$

and (4.4) follows. Moreover, since $\|Y\|_{H^{-1}} \lesssim \|Y\|_{L^4}$ we have that

$$\frac{d}{dt} \|Y\|_{H^{-1}}^2 + \frac{1}{2} \|Y\|_{H^{-1}}^4 \lesssim \sum_{k=1}^3 \| : Z^k : \|_{-\alpha}^{\gamma_k}.$$

By [TW16, Lemma 3.8] we have

$$\|Y_t\|_{H^{-1}}^2 \lesssim t^{-1} \vee \left(\sum_{k=1}^3 t^{-\rho \gamma_k} \sup_{0 \leq r \leq t} (r^{\rho \gamma_k} \| : Z_r : \|_{-\alpha}^{\gamma_k}) \right)^{\frac{1}{2}}.$$

□

Since the approximation equation (4.2) have the same prior estimate as (4.1). By (4.4) we deduce that the sequence $\{Y_\varepsilon\}$ is bounded in $L^\infty(0, T; H^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; H^1)$. This implies that $\{AY_\varepsilon\}$ is bounded in $L^2(0, T; H^{-1})$ and $\{Y_\varepsilon^3\}$ is bounded in $L^{4/3}([0, T] \times \mathbb{T}^2)$. Moreover, Lemma 2.1 and Lemma 3.3 imply that $\{ : Z_\varepsilon^3 : \}$ is bounded in $L^p(0, T; H^{-\alpha})$ for any $\alpha > 0, \varepsilon > 0$ and $p > 1$. Then we can prove the following lemma:

Lemma 4.2 $\{ \frac{dY_\varepsilon}{dt} \}$ is bounded in $L^p(0, T; H^{-3})$, where $p \in (1, \frac{4}{3})$.

Proof According to the argument before, we only need to show that $\{Y_\varepsilon^2 Z_\varepsilon\}$ and $\{Y_\varepsilon : Z_\varepsilon^2 : \}$ are bounded in $L^p(0, T; H^{-1})$ when $p \in (1, \frac{3}{4})$.

We omit ε if there is no confusion in this proof.

For $Y^2 Z$ we have

$$\|Y^2 Z\|_{B_{2,\infty}^{-\alpha}} \lesssim \|Y^2\|_{B_{2,\infty}^{\beta_0}} \|Z\|_{-\alpha} \lesssim \|Y^2\|_{B_{2,1}^{\beta_0}} \|Z\|_{-\alpha},$$

where $\beta_0 > \alpha > 0$, we used Lemma 2.5 in the first inequality and Lemma 2.1 in the second inequality. Furthermore,

$$\|Y^2\|_{B_{2,1}^{\beta_0}} \lesssim \|\Lambda^{\beta_1} Y^2\|_{L^2} \lesssim \|\Lambda^{\beta_1} Y\|_{L^{p_0}} \|Y\|_{L^{q_0}},$$

where $\beta_1 > \beta_0$, $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{2}$, we used Lemma 2.1 in the first inequality and Lemma 2.4 in the second inequality. By Lemma 2.1, $B_{q,2}^s \subset L^q$ for any $q \geq 1$ and $s > 0$. Since $H^\delta \simeq B_{2,2}^\delta \subset B_{q,2}^{\delta-1+\frac{2}{q}}$ for $q \geq 2$, the Besov interpolation in Lemma 2.4 implies that

$$\|Y\|_{L^{q_0}} \lesssim \|Y\|_{B_{q_0,2}^s} \lesssim \|Y\|_{B_{q_0,2}^{\frac{2}{q_0}}}^{1-\frac{1}{q_0}+\frac{s}{2}} \|Y\|_{B_{q_0,2}^{\frac{1}{q_0}-\frac{s}{2}}} \lesssim \|Y\|_{H^1}^{1-\frac{1}{q_0}+\frac{s}{2}} \|Y\|_{H^{-1}}^{\frac{1}{q_0}-\frac{s}{2}}. \quad (4.7)$$

For $\|\Lambda^{\beta_1} Y\|_{L^{p_0}}$, let $p_0 \geq 2$. Then we use Lemma 2.1 and Sobolev interpolation to get

$$\|\Lambda^{\beta_1} Y\|_{L^{p_0}} \lesssim \|Y\|_{H^{\beta_2}} \lesssim \|Y\|_{H^1}^{\frac{1+\beta_2}{2}} \|Y\|_{H^{-1}}^{\frac{1-\beta_2}{2}},$$

where $\beta_1 = \beta_2 + \frac{2}{p_0} - 1 = \beta_2 - \frac{2}{q_0}$. Thus we have

$$\|Y^2\|_{B_{2,1}^{\beta_0}} \lesssim \|Y\|_{H^1}^{\frac{3}{2} + \frac{\beta_1}{2} + \frac{s}{2}} \|Y\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_1}{2} - \frac{s}{2}}. \quad (4.8)$$

By Lemma 3.3 we deduce that

$$\|Y^2 Z\|_{B_{2,\infty}^{-\alpha}} \lesssim \|Y\|_{H^1}^{\frac{3}{2} + \frac{\beta_1}{2} + \frac{s}{2}} \|Y\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_1}{2} - \frac{s}{2}} t^{-\frac{\rho}{4}}.$$

For any $p \in (1, \frac{4}{3})$, let β_1 and s be small enough such that $(\beta_1 + s + 3)p < 4$. Then Young's inequality implies that there exists $\gamma > 0$ such that

$$\|Y^2 Z\|_{B_{2,\infty}^{-\alpha}}^p \lesssim \|Y\|_{H^1}^2 + \|Y\|_{H^{-1}}^{\frac{4}{3}\gamma(\frac{1}{2} - \frac{\beta_1}{2} - \frac{s}{2})} t^{-\frac{\rho}{3}\gamma}.$$

For ρ small enough, $\{Y_\varepsilon^2 Z_\varepsilon\}$ is bounded in $L^p(0, T; B_{2,\infty}^{-\alpha})$.

On the other hand,

$$\|Y : Z^2 : \|_{B_{2,\infty}^{-\alpha}} \lesssim \|Y\|_{B_{2,\infty}^1} \| : Z^2 : \|_{-\alpha} \lesssim \|Y\|_{H^1} t^{-\frac{\rho}{4}},$$

where we used Lemma 2.5 in the first inequality and Lemma 2.1, Lemma 3.3 in the second inequality. Then by Young's inequality

$$\|Y : Z^2 : \|_{B_{2,\infty}^{-\alpha}}^{\frac{4}{3}} \lesssim \|Y\|_{H^1}^2 + t^{-\rho}.$$

Choosing ρ small enough we deduce that $\{Y_\varepsilon : Z_\varepsilon^2 : \}$ is bounded in $L^{\frac{4}{3}}(0, T; B_{2,\infty}^{-\alpha})$. By Lemma 2.1 we have $B_{2,\infty}^{-\alpha} \subset H_2^{-\alpha-\delta}$ for any $\delta > 0$. Hence $\{Y_\varepsilon^2 Z_\varepsilon\}$ and $\{Y_\varepsilon : Z_\varepsilon^2 : \}$ are bounded in $L^p(0, T; H^{-1})$, $\forall p \in (1, \frac{4}{3})$, which implies the results. \square

Theorem 4.3 *For every $x \in V_0^{-1}$, there exists at least one solution to equation (4.1) in $C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1)$.*

Proof Since $H^1 \subset H^\delta$ compactly for any $\delta < 1$ (see [Tri06, Proposition 4.6]), a classical compactness argument (cf. [GRZ09, Lemma C.2] or [Tem01, Theorem 2.1, Chapter III]) implies that there exists a sequence $\{\varepsilon_k\}$ and $Y \in L^\infty(0, T, H^{-1}) \cap L^2(0, T; H^1) \cap L^4([0, T] \times \mathbb{T}^2)$, such that $Y_{\varepsilon_k} \rightarrow Y$ in $L^2(0, T; H^\delta) \cap C([0, T]; H^{-3})$, $\forall \delta < 1$.

It is sufficient to show that for a suitable $\delta \in (0, 1)$, the limit Y we obtained above is a solution in H^{-3} .

In fact, if Y is a solution in H^{-3} , i.e. for any $h \in H^3$

$${}_{H^{-3}}\langle Y_t - Y_0, h \rangle_{H^3} = -\frac{1}{2} \int_0^t {}_{H^{-1}}\langle A^2 h, Y_s \rangle_{H^1} ds + \frac{1}{2} \int_0^t {}_{H^{-1}}\langle \sum_{k=0}^3 C_3^k Y_s^{3-k} : Z_s^k : , Ah \rangle_{H^1} ds. \quad (4.9)$$

Y is in $L^\infty(0, T, H^{-1}) \cap L^2(0, T; H^1) \cap L^4([0, T] \times \mathbb{T}^2)$. Then we take the scalar product of $\frac{dY}{dt}$ and $(-A)^{-1}Y$, which is just the duality in H^{-3} and H^3 . Hence

$$\frac{d}{dt} \|Y\|_{H^{-1}}^2 + \|Y\|_{H^1}^2 + \|Y\|_{L^4}^4 = -\left\langle \sum_{k=1}^3 C_3^k Y^{3-k} : Z^k :, Y \right\rangle.$$

Thus $\|Y\|_{H^{-1}}$ is continuous w.r.t t . Moreover, [Tem01, Lemma 1.4, Chapter III] implies that Y is weakly continuous in H^{-1} , then $Y \in C([0, T]; H^{-1})$.

We still write ε instead of ε_k if there is no confusion. Since Y_ε is a solution to equation (4.2), letting $\varepsilon \rightarrow 0$, it's easy to see that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} {}_{H^{-3}}\langle Y_\varepsilon, h \rangle_{H^3} &= {}_{H^{-3}}\langle Y, h \rangle_{H^3}, & \lim_{\varepsilon \rightarrow 0} {}_{H^{-1}}\langle A^2 h, Y_\varepsilon \rangle_{H^1} &= {}_{H^{-1}}\langle A^2 h, Y \rangle_{H^1}, \\ \lim_{\varepsilon \rightarrow 0} {}_{H^{-1}}\langle \bar{Z}_\varepsilon^3 :, Ah \rangle_{H^1} &= {}_{H^{-1}}\langle \bar{Z}^3 :, Ah \rangle_{H^1}. \end{aligned}$$

It remains to show for any $h \in H^1$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \langle Y_\varepsilon^3(s) - Y^3(s), h \rangle ds \right| = 0, \quad (4.10)$$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \langle Y_\varepsilon^2(s) Z_\varepsilon(s) - Y^2(s) Z(s), h \rangle ds \right| = 0, \quad (4.11)$$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \langle Y_\varepsilon(s) : Z_\varepsilon^2 : (s) - Y(s) : Z^2 : (s), h \rangle ds \right| = 0. \quad (4.12)$$

Since $Y_\varepsilon \rightarrow Y$ in $L^4([0, T] \times \mathbb{T}^2)$, which is equivalent to $\|Y_\varepsilon\|_{L^4([0, T] \times \mathbb{T}^2)} \rightarrow \|Y\|_{L^4([0, T] \times \mathbb{T}^2)}$ and $Y_\varepsilon \Rightarrow^m Y$, where \Rightarrow^m means convergence in Lebesgue measure m on $[0, T] \times \mathbb{T}^2$, we have $\|Y_\varepsilon^3\|_{L^{\frac{4}{3}}([0, T] \times \mathbb{T}^2)} \rightarrow \|Y^3\|_{L^{\frac{4}{3}}([0, T] \times \mathbb{T}^2)}$ and $Y_\varepsilon^3 \Rightarrow^m Y^3$. Then (4.10) holds by uniform integrability.

For (4.11), let $R_\varepsilon = Y_\varepsilon - Y$. By the triangle inequality

$$|\langle Y_\varepsilon^2 Z_\varepsilon - Y^2 Z, h \rangle| \lesssim |\langle R_\varepsilon(Y + Y_\varepsilon)h, Z \rangle| + |\langle Z_\varepsilon - Z, Y^2 h \rangle|.$$

For the second term on the right hand side of the above inequality, we have

$$|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim \|Z_\varepsilon - Z\|_{-\alpha} \|Y^2 h\|_{B_{1,1}^\alpha} \lesssim \|Z_\varepsilon - Z\|_{-\alpha} \|Y^2\|_{B_{2,1}^\alpha} \|h\|_{B_{2,1}^\alpha},$$

where we used Lemma 2.3 in the first inequality and Lemma 2.5 in the second inequality. By [Tri92, Remark 2, Section 3.2, Chapter 2] we have $H^1 \subset B_{2,1}^\alpha$ for any $\alpha < 1$. Hence

$$|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim \|Z_\varepsilon - Z\|_{-\alpha} \|Y^2\|_{B_{2,1}^\alpha} \|h\|_{H^1}.$$

Combining with (4.8), we have

$$|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim \|Z_\varepsilon - Z\|_{-\alpha} \|h\|_{H^1} \|Y\|_{H^1}^{\frac{3}{2} + \frac{\beta_3}{2} + \frac{s}{2}} \|Y\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_3}{2} - \frac{s}{2}},$$

where $\beta_3 > \alpha > 0, s > 0$. Let $\frac{3}{2} + \frac{\beta_3}{2} + \frac{s}{2} < 2$. Then Lemma 3.3 and Hölder's inequality imply that

$$\left| \int_0^t \langle Z_\varepsilon - Z, Y^2 h \rangle ds \right| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Similarly

$$|\langle R_\varepsilon Y h, Z \rangle| \lesssim \|R_\varepsilon Y\|_{B_{2,1}^\alpha} \|h\|_{H^1} \|Z\|_{-\alpha}.$$

For $\|R_\varepsilon Y\|_{B_{2,1}^\alpha}$, we have

$$\|R_\varepsilon Y\|_{B_{2,1}^\alpha} \lesssim \|R_\varepsilon Y\|_{B_{2,2}^{\beta_0}} \lesssim \|\Lambda^{\beta_0} R_\varepsilon\|_{L^4} \|Y\|_{L^4} + \|\Lambda^{\beta_0} Y\|_{L^4} \|R_\varepsilon\|_{L^4},$$

where $\beta_0 > \alpha > 0$ and we used Lemma 2.1 in the first inequality and Lemma 2.4 in the second inequality. By Lemma 2.1 we have the Sobolev embedding $H_2^{\beta_0 + \frac{1}{2}} \subset H_4^\beta$. Hence

$$\|R_\varepsilon Y\|_{B_{2,1}^\alpha} \lesssim \|R_\varepsilon\|_{H^{\beta_0 + \frac{1}{2}}} \|Y\|_{L^4} + \|Y\|_{H^{\beta_0 + \frac{1}{2}}} \|R_\varepsilon\|_{H^{\frac{1}{2}}}.$$

By Sobolev interpolation, choosing $\delta > \frac{1}{2} + \beta_0$, we have

$$\|Y\|_{H^{\beta_0 + \frac{1}{2}}} \lesssim \|Y\|_{H^1}^{\frac{3}{4} + \frac{\beta_0}{2}} \|Y\|_{H^{-1}}^{\frac{1}{4} - \frac{\beta_0}{2}}.$$

Moreover, since $\delta > \frac{1}{2} + \beta_0$, we have $\|R_\varepsilon\|_{H^{\frac{1}{2}}} \lesssim \|R_\varepsilon\|_{H^\delta}$ and $\|Y\|_{H^{\frac{1}{2} + \beta_0}} \lesssim \|Y\|_{H^\delta}$. Then we deduce that

$$\|R_\varepsilon Y\|_{B_{2,1}^\alpha} \lesssim \|R_\varepsilon\|_{H^\delta} \|Y\|_{L^4} + \|Y\|_{H^1}^{\frac{3}{4} + \frac{\beta_0}{2}} \|R_\varepsilon\|_{H^\delta} \|Y\|_{H^{-1}}^{\frac{1}{4} - \frac{\beta_0}{2}}.$$

Let $\beta_0 < \frac{1}{2}$ such that

$$\frac{3}{4} + \frac{\beta_0}{2} + 1 < 2.$$

Then by Hölder inequality, we get

$$\int_0^t \|R_\varepsilon Y\|_{B_{2,1}^\alpha} \|h\|_{H^1} \|\bar{Z}\|_{-\alpha} ds \lesssim \left(\int_0^t \|R_\varepsilon\|_{H^\delta}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t (\|Y\|_{H^1}^2 F) ds \right)^{\frac{1}{2}} \left(\int_0^t \|Y\|_{L^4}^4 ds \right)^{\frac{1}{4}} \rightarrow 0,$$

where $F \in L^\infty(0, T)$.

Moreover, we have

$$|\langle Y_\varepsilon : Z_\varepsilon^2 : -Y : Z^2 :, h \rangle| \lesssim |\langle Y_\varepsilon(: Z_\varepsilon^2 : - : Z^2 :), h \rangle| + |\langle R_\varepsilon : Z^2 :, h \rangle|.$$

By essentially the same argument as above, (4.12) also follows.

Then we have got a solution Y in $C([0, T]; H^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; H^1)$. Combining this with (4.3), we have $Y \in C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1)$. \square

Now we have obtained the existence of solutions of equation (4.1). The following is the uniqueness result.

Theorem 4.4 *For every $x \in V_0^{-1}$, there exists a unique solution to equation (4.1) in $C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1)$.*

Proof Suppose u, v are two solutions of (4.1) with the same initial value. Let $r = u - v$, then r satisfies:

$$\begin{cases} \frac{dr}{dt} = -\frac{1}{2} A^2 r + \frac{1}{2} A \sum_{k=0}^3 C_3^k (u^{3-k} - v^{3-k}) : Z^k :, \\ r(0) = 0. \end{cases}$$

Similarly to (4.6) we have:

$$\frac{d}{dt} \|r\|_{H^{-1}}^2 + \|r\|_{H^1}^2 \lesssim |\langle r^2(u+v), Z \rangle| + |\langle r^2, : Z^2 : \rangle|. \quad (4.13)$$

By Lemma 2.3 and Lemma 3.3 we know

$$|\langle r^2, : Z^2 : \rangle| \lesssim \|r^2\|_{B_{1,1}^\alpha} t^{-\rho},$$

where $\beta > \alpha > 0$. Then Lemma 2.1 and Lemma 2.4 imply that

$$\|r^2\|_{B_{1,1}^\alpha} \lesssim \|\Lambda^{\beta_0} r^2\|_{L^{\frac{4}{3}}} \lesssim \|\Lambda^{\beta_0} r\|_{L^2} \|r\|_{L^4} \lesssim \|r\|_{H^1}^{\frac{\beta_0+3}{2}} \|r\|_{H^{-1}}^{\frac{1-\beta_0}{2}},$$

where $1 > \beta_0 > \alpha > 0$ and we used the Sobolev interpolation and Sobolev embedding theorem in the last inequality. Then by Young's inequality, there exists a $\gamma_1 > 0$ such that for any $\varepsilon > 0$

$$|\langle r^2, : Z^2 : \rangle| \lesssim \varepsilon \|r\|_{H^1}^2 + \|r\|_{H^{-1}}^2 t^{-\rho\gamma_1}. \quad (4.14)$$

Let ρ be small enough. Then $g := t^{-\rho\gamma_1} \in L^1(0, T)$.

For $|\langle r^2(u+v), Z \rangle|$, we similarly obtain that

$$|\langle r^2(u+v), Z \rangle| \lesssim \|r^2(u+v)\|_{B_{1,1}^\alpha} \|Z\|_{-\alpha} \lesssim \left(\|ur^2\|_{B_{1,1}^\alpha} + \|vr^2\|_{B_{1,1}^\alpha} \right) t^{-\rho}.$$

For $\|ur^2\|_{B_{1,1}^\alpha}$, we have

$$\|ur^2\|_{B_{1,1}^\alpha} \lesssim \|\Lambda^{\beta_0}(ur^2)\|_{L^{p_0}} \lesssim \|\Lambda^{\beta_0} u\|_{L^{p_1}} \|r^2\|_{L^{q_1}} + \|\Lambda^{\beta_0} r^2\|_{L^{p_2}} \|u\|_{L^{q_2}} := (I) + (II),$$

with $p_0 > 1$, $\beta_0 > \alpha > 0$, and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, $p_i, q_i > p_0, i = 1, 2$. Here we used Lemma 2.1 in the first inequality and Lemma 2.5 in the second inequality.

For (I), according to (4.7) we know that for any $s > 0$

$$\|r^2\|_{L^{q_1}} = \|r\|_{L^{2q_1}}^2 \lesssim \|r\|_{H^1}^{2-\frac{1}{q_1}+s} \|r\|_{H^{-1}}^{\frac{1}{q_1}-s}.$$

Moreover, let $p_1 \geq 4$. Then

$$\|\Lambda^{\beta_0} u\|_{L^{p_1}} \lesssim \|\Lambda^{\beta_1} u\|_{L^4} \lesssim \|u\|_{L^4}^{1-2\beta_1} \|u\|_{H^{\frac{1}{2}}}^{2\beta_1} \lesssim \|u\|_{L^4}^{1-2\beta_1} \|u\|_{H^1}^{2\beta_1},$$

where $\beta_1 = \beta_0 + \frac{1}{2} - \frac{2}{p_1}$ and we used Lemma 2.1 in the first inequality and Sobolev interpolation in the second inequality and Besov embedding Lemma 2.1 in the last inequality. Combining these estimates above we have

$$(I) \lesssim \|r\|_{H^1}^{2-\frac{1}{q_1}+s} \|r\|_{H^{-1}}^{\frac{1}{q_1}-s} \|u\|_{L^4}^{1-2\beta_1} \|u\|_{H^1}^{2\beta_1}.$$

Hence by Young's inequality

$$t^{-\rho}(I) \lesssim \varepsilon \|r\|_{H^1}^2 + \|r\|_{H^{-1}}^2 \|u\|_{H^1}^{\frac{4\beta_1}{1-s}} \|u\|_{L^4}^{\frac{2(1-2\beta_1)}{1-s}} t^{-\frac{2\rho}{1-s}}.$$

Let p_0 be close to 1 and β_0, s be so small enough such that $\frac{1}{p_1} > 1 - \frac{1}{p_0} + \beta_0 + s$, which is equivalent to $\frac{2\beta_1}{\frac{1}{q_1}-s} + \frac{(1-2\beta_1)\frac{1}{2}}{\frac{1}{q_1}-s} < 1$. Then the Hölder inequality yields for ρ small enough

$$\int_0^t \|u\|_{H^1}^{\frac{4\beta_1}{\frac{1}{q_1}-s}} \|u\|_{L^4}^{\frac{2(1-\beta_1)}{\frac{1}{q_1}-s}} \tau^{-\frac{2\rho}{\frac{1}{q_1}-s}} d\tau \lesssim \left(\int_0^t \|u\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u\|_{L^4}^4 d\tau \right)^{\frac{1}{4}}.$$

Then we get

$$f_1^u := \|u\|_{H^1}^{\frac{4\beta_1}{\frac{1}{q_1}-s}} \|u\|_{L^4}^{\frac{2(1-\beta_1)}{\frac{1}{q_1}-s}} t^{-\frac{2\rho}{\frac{1}{q_1}-s}} \in L^1(0, T),$$

and for any $\varepsilon > 0$,

$$t^{-\rho}(I) \lesssim \varepsilon \|r\|_{H^1}^2 + f_1^u \|r\|_{H^{-1}}^2. \quad (4.15)$$

For (II), let $q_2 = 4$. Then we have $\frac{1}{p_2} + \frac{1}{4} = \frac{1}{p_0} \in (\frac{3}{4}, 1)$, which implies that $p_2 \in (\frac{4}{3}, 2)$. Similarly by Lemma 2.5

$$\|\Lambda^{\beta_0} r^2\|_{L^{p_2}} \lesssim \|\Lambda^{\beta_0} r\|_{L^{p_3}} \|r\|_{L^{q_3}},$$

where $\frac{1}{p_3} + \frac{1}{q_3} = \frac{1}{p_2}$, $p_3, q_3 > p_2$. From (4.7) we know that for every $s > 0$

$$\|r\|_{L^{q_3}} \lesssim \|r\|_{H^1}^{1-\frac{1}{q_3}+\frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{q_3}-\frac{s}{2}}.$$

Let $p_3 \geq 2$. Then by Lemma 2.1 we have

$$\|\Lambda^{\beta_0} r\|_{L^{p_3}} \lesssim \|r\|_{H^{\beta_2}} \lesssim \|r\|_{H^1}^{\frac{1+\beta_2}{2}} \|r\|_{H^{-1}}^{\frac{1-\beta_2}{2}},$$

where we used Sobolev interpolation in the second inequality and that $\beta_0 = \beta_2 - 1 + \frac{2}{p_3}$. Hence

$$\|\Lambda^{\beta_0} r^2\|_{L^{p_2}} \lesssim \|r\|_{H^1}^{\frac{3}{2}+\frac{\beta_2}{2}-\frac{1}{q_3}+\frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{2}-\frac{\beta_2}{2}+\frac{1}{q_3}-\frac{s}{2}} = \|r\|_{H^1}^{2+\frac{\beta_0}{2}-\frac{1}{p_2}+\frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}.$$

Thus, we have

$$(II) \lesssim \|r\|_{H^1}^{2+\frac{\beta_0}{2}-\frac{1}{p_2}+\frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}} \|u\|_{L^4}.$$

Then by Young's inequality we have

$$t^{-\rho}(II) \lesssim \varepsilon \|r\|_{H^1}^2 + \|r\|_{H^{-1}}^2 \|u\|_{L^4}^{\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} t^{-\frac{2\rho}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}}.$$

It is easy to see that $p_2 < 2$ yields $\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}} \leq 4$ when s, β_0 are small enough. Then for small

enough ρ we have $f_2^u := \|u\|_{L^4}^{\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} t^{-\frac{2\rho}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} \in L^1(0, T)$.

Then we obtain that for any $\varepsilon > 0$

$$|\langle r^2 u, Z \rangle| \lesssim \varepsilon \|r\|_{H^1}^2 + f_2^u \|r\|_{H^{-1}}^2,$$

where $f^u := f_1^u + f_2^u \in L^1(0, T)$.

The same holds with u replaced by v . Let $f = f^u + f^v \in L^1(0, T)$. Then

$$|\langle r^2(u+v), \bar{Z} \rangle| \lesssim \varepsilon \|r\|_{H^1}^2 + f \|r\|_{H^{-1}}^2.$$

Hence we get

$$\frac{d}{dt} \|r\|_{H^{-1}}^2 + \|r\|_{H^1}^2 \lesssim \varepsilon \|r\|_{H^1}^2 + (f+g) \|r\|_{H^{-1}}^2.$$

Choose a suitable $\varepsilon > 0$ such that

$$\frac{d}{dt} \|r\|_{H^{-1}}^2 \lesssim (f+g) \|r\|_{H^{-1}}^2.$$

Then by Gronwall's inequality we have

$$\|r(t)\|_{H^{-1}}^2 \lesssim \|r(0)\|_{H^{-1}}^2 \exp\left(\int_0^t f(s) + g(s) ds\right) = 0.$$

Since V_0^{-1} is a subspace of H^{-1} , we obtain the uniqueness. \square

Remark 4.5 *We emphasize that we cannot obtain global well-posedness of equation (4.1) by combining (4.4) with fixed point argument in [DPD03] and [MW15] since we only have an H^{-1} -estimate. In fact, in order to use fixed point arguments to obtain local solutions, the initial value should be in $C^{-\frac{4}{3}+}$. An initial value in H^{-1} -norm is not enough to use mild formulation to obtain local solution.*

5 Relation to the solution given by Dirichlet forms

In this section, we are going to obtain a probabilistically weak solution of equation (1.1) via the Dirichlet form approach and compare this solution with the solution we obtain in Section 4.

According to the definition of V_0^α and [Hid80, Theorem 3.1], μ is supported on V_0^{-s} for any $s > 1$. So we fix a small enough $s_0 > 0$ and $V_0^{-1-s_0}$ as the state space and denote it by E for convenience. By indentifying V_0^1 and V_0^{-1} via the Riesz isomorphism we have the following Gelfand triple:

$$E^* \subset V_0^{-1} \subset E \quad (5.1)$$

where $E^* = V_0^{s_0-1}$ and the dualization between E^* and E is ${}_{E^*}\langle u, v \rangle_E := {}_{V_0^{1+s_0}}\langle Qu, v \rangle_{V_0^{-1-s_0}}$ for any $u \in E^*, v \in E$. Here ${}_{V^s}\langle \cdot, \cdot \rangle_{V^{-s}}$ is denoted by

$${}_{V_0^s}\langle u, v \rangle_{V_0^{-s}} := \sum_k {}_{S'}\langle u, e_k \rangle_{S'} {}_S\langle v, e_k \rangle_S, u \in V_0^s, v \in V_0^{-s}. \quad (5.2)$$

Then we have that

$${}_{E^*}\langle u, v \rangle_E = \langle u, v \rangle_{V_0^{-1}}, \forall u \in E^*, \forall v \in V_0^{-1}. \quad (5.3)$$

Moreover we define $\mathcal{FC}_b^\infty := \{f({}_{E^*}\langle l_1, \cdot \rangle_E, \dots, {}_{E^*}\langle l_m, \cdot \rangle_E) : m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \dots, l_m \in E^*\}$. For all $\varphi = f({}_{E^*}\langle l_1, \cdot \rangle_E, \dots, {}_{E^*}\langle l_m, \cdot \rangle_E) \in \mathcal{FC}_b^\infty$, we can define the directional derivative for $h \in V_0^{-1}$:

$$\partial_h \varphi(z) := \lim_{t \rightarrow 0} \frac{\varphi(z + th) - \varphi(z)}{t} = \sum_{i=1}^m \partial_i f({}_{E^*}\langle l_1, \cdot \rangle_E, \dots, {}_{E^*}\langle l_m, \cdot \rangle_E) \langle l_i, h \rangle_{V_0^{-1}}.$$

Then by the Riesz representation theorem, there exists a map $\nabla \varphi : E \rightarrow V_0^{-1}$ such that

$$\langle \nabla \varphi(z), h \rangle_{V_0^{-1}} = \partial_h \varphi(z), h \in V_0^{-1}.$$

5.1 Solution given by Dirichlet forms

Since $Q^{-1-s_0} : V_0^{1+s_0} \rightarrow V_0^{-1-s_0}$ is the Riesz isomorphism for $V_0^{1+s_0}$, i.e.

$$V_0^{1+s_0} \langle h, k \rangle_{V_0^{-1-s_0}} = \langle Q^{-1-s_0} h, k \rangle_{V_0^{-1-s_0}},$$

μ is in fact a Gaussian measure on Hilbert space $V_0^{-1-s_0}$, with covariance operator $C := Q^{2+s_0}$, that is

$$\int_{V_0^{-1-s_0}} e^{i \langle h, z \rangle_{V_0^{-1-s_0}}} \mu(dz) = \langle Ch, h \rangle_{V_0^{-1-s_0}}.$$

Then we have the following integration by parts formula for μ :

Proposition 5.1 *For all $F \in \mathcal{FC}_b^\infty$, $h \in V_0^{3+s_0}$, we have*

$$\int \partial_h F d\mu = \int_{E^*} \langle A^2 h, \phi \rangle_E F(\phi) \mu(d\phi). \quad (5.4)$$

Proof First, by [DPZ02, Section 1.2.4] we know the reproducing kernel of $(V_0^{-1-s_0}, \mu)$ is $V_\mu := C^{1/2} V_0^{-1-s_0} = V_0^1$. Then by [MR92, Theorem 3.1, Chapter II] we have

$$\begin{aligned} \int \partial_h F d\mu &= \int \langle C^{-1} h, \phi \rangle_{V_0^{-1-s_0}} F(\phi) \mu(d\phi) \\ &= \int \langle Q^{-2-s_0} h, \phi \rangle_{V_0^{-1-s_0}} F(\phi) \mu(d\phi) \\ &= - \int_{V_0^{1+s_0}} \langle Ah, \phi \rangle_{V_0^{-1-s_0}} F(\phi) \mu(d\phi) \\ &= - \int_{E^*} \langle Q^{-1} Ah, \phi \rangle_E F(\phi) \mu(dz) \\ &= \int_{E^*} \langle A^2 h, \phi \rangle_E F(\phi) \mu(dz). \end{aligned}$$

□

Remark 5.2 *In fact, by a similar argument in [GJ12, (9.1.32)], (5.4) still holds for $F \exp(-N)$, where $N = \mathcal{S}' \langle : q :, e_0 \rangle_{\mathcal{S}}$ i.e. for all $F \in \mathcal{FC}_b^\infty$, $h \in V_0^{3+s_0}$*

$$\int \partial_h (F \exp(-N)) d\mu = \int_{E^*} \langle A^2 h, \phi \rangle_E F(\phi) \exp(-N(\phi)) \mu(d\phi)$$

Then for the Gibbs measure ν defined in Section 3, we have the following integration by parts formula:

Proposition 5.3 *For all $F \in \mathcal{FC}_b^\infty$, $h \in V_0^{3+s_0}$, we have*

$$\int \partial_h F d\nu = \int ({}_{E^*} \langle A^2 h, \phi \rangle_E - {}_{E^*} \langle Ah, : \phi^3 : \rangle_E) F(\phi) \nu(d\phi). \quad (5.5)$$

Proof According to Proposition 5.1 and Remark 5.2

$$\begin{aligned}
\int \partial_h F d\nu &= c \int (\partial_h F) \exp(-N) d\mu \\
&= c \int [\partial_h(F \exp(-N)) + F \exp(-N) \partial_h N] d\mu \\
&= \int F(\phi) ({}_{E^*} \langle A^2 h, \phi \rangle_E - \partial_h N(\phi)) \nu(d\phi)
\end{aligned}$$

By [Oba94, Theorem 4.1.1],

$$\partial_h : \phi_\varepsilon^n(x) := n : \phi_\varepsilon^{n-1}(x) : (\rho_\varepsilon * h)(x).$$

Here $\partial_h : \phi_\varepsilon^n(x) :$ is defined as the directional derivative of the function $\phi \rightarrow : \phi_\varepsilon^n(x) :$. Then

$$\partial_h N_\varepsilon(\phi) = \langle : \phi_\varepsilon^3 :, h * \rho_\varepsilon \rangle,$$

where $N_\varepsilon(\phi) := \langle \frac{1}{4} : \phi_\varepsilon^4 :, e_0 \rangle$. Letting $\varepsilon \rightarrow 0$, due to the closability of $\partial_{\Pi h}$ in $L^2(E, \mu)$,

$$\partial_h N(\phi) = \langle : \phi^3 :, h \rangle = -{}_{E^*} \langle Ah, : \phi^3 : \rangle_E,$$

which implies

$$\int \partial_h F d\nu = \int ({}_{E^*} \langle A^2 h, \phi \rangle_E - {}_{E^*} \langle Ah, : \phi^3 : \rangle_E) F(\phi) \nu(d\phi).$$

□

Theorem 5.4 *The bilinear form*

$$\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle_{V_0^{-1}} d\nu, \forall \varphi, \psi \in \mathcal{FC}_b^\infty,$$

is closable in $L^2(E, \nu)$. Its closure is a symmetric quasi-regular Dirichlet form denoted by $(\mathcal{E}, D(\mathcal{E}))$.

Proof Let $h_k = \sqrt{\lambda_k} e_k$, $\{h_k\}_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}}$ is an orthonormal basis of V_0^{-1} . Then

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \int \frac{\partial \varphi}{\partial h_k} \frac{\partial \psi}{\partial h_k} d\nu, \forall \varphi, \psi \in \mathcal{FC}_b^\infty,$$

By Proposition 5.3 we have $\int \frac{\partial \varphi}{\partial h_k} d\nu = - \int \varphi \beta_{h_k} d\nu$, where $\beta_{h_k} \in L^2(E, \nu)$ and

$$\beta_{h_k}(\phi) = -{}_{E^*} \langle A^2 h_k, \phi \rangle_E + {}_{E^*} \langle Ah_k, : \phi^3 : \rangle_E, \forall k \neq (0, 0).$$

According to [MR92, Proposition 3.5, Chapter II], $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable on $L^2(E, \nu)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form. Moreover by [MR92, Proposition 4.2, Chapter IV], the capacity of $(\mathcal{E}, D(\mathcal{E}))$ is tight, and according to the fact that \mathcal{FC}_b^∞ separates the points in $L^2(E, \nu)$, we obtain that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form. □

Since $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E, \nu)$, it is well-known that there is a conservative Markov diffusion processes

$$M = (\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t \geq 0}, (\mathbb{P}^z)_{z \in E}),$$

which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. for $u \in L^2(E, \nu) \cap \mathcal{B}_b(E)$, the transition semigroup $P_t u(z) := E^z[u(X(t))]$ is \mathcal{E} -quasi-continuous for all $t > 0$ and is a ν -version of $T_t u$ where T_t is the semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$. For the notion of \mathcal{E} -quasi-continuity we refer to [MR92, Chapter III, Definition 3.2]. Then we have the following Fukushima decomposition for $X(t)$ under \mathbb{P}^z :

Theorem 5.5 *There exists a map $W : \Omega \rightarrow C([0, \infty); C([0, \infty); V_0^{-1-s_0}(\mathbb{T}^2, \mathbb{R}^2)))$, and a properly \mathcal{E} -exceptional set $S \subset E$, i.e. $\nu(S) = 0$ and $\mathbb{P}^z(X(t) \in E \setminus S, \forall t \geq 0) = 1$ for $z \in E \setminus S$, such that $\forall z \in E \setminus S$, W is a U -cylindrical Wiener process on $(\Omega, \mathcal{M}_t, \mathbb{P}^z)$ and the sample paths of the associated process $M = (\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t \geq 0}, (\mathbb{P}^z)_{z \in E})$ on E satisfy the following: for $h \in V^{3+s_0}$,*

$$\begin{aligned} E^* \langle h, X(t) - X(0) \rangle_E &= -\frac{1}{2} \int_0^t E^* \langle A^2 h, X(s) \rangle_E ds \\ &\quad + \frac{1}{2} \int_0^t E^* \langle Ah, : X(s)^3 : \rangle_E ds \\ &\quad + \int_0^t \langle B^* h, dW_s \rangle_{V_0^{-1}(\mathbb{T}^2, \mathbb{R}^2)}, \forall t \geq 0, \mathbb{P}^z - a.s., \end{aligned} \tag{5.6}$$

where B, B^* are defined as in (3.2). Moreover, ν is an invariant measure for M in the sense that $\int P_t u d\nu = \int u d\nu$ for $u \in L^2(E, \nu) \cap \mathcal{B}_b(E)$.

Proof Let $u_h(\phi) = E^* \langle h, \phi \rangle_E, h \in V_0^{3+s_0}$, and let \mathcal{L} be the generator of $(\mathcal{E}, D(\mathcal{E}))$. For any $v \in D(\mathcal{E})$

$$\begin{aligned} -\int \mathcal{L} u_h v d\nu &= \frac{1}{2} \int \langle \nabla u_h, \nabla v \rangle_{V_0^{-1}} d\nu \\ &= -\frac{1}{2} \int \partial_h v(\phi) \nu(d\phi) \\ &= \frac{1}{2} \int (E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, : \phi^3 : \rangle_E) v(\phi) \nu(d\phi). \end{aligned}$$

Hence $u_h \in D(\mathcal{L})$ and $\mathcal{L} u_h(\phi) = -\frac{1}{2} (E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, : \phi^3 : \rangle_E)$.

By Fukushima's decomposition, we have for q.e. $z \in E$,

$$u_h(X_t) - u_h(X_0) = M_t^h + \int_0^t \mathcal{L} u_h(X_s) ds = M_t^h - \frac{1}{2} \int_0^t (E^* \langle A^2 h, X_s \rangle_E - E^* \langle Ah, : X_s^3 : \rangle_E) ds,$$

where M^h is a martingale additive functional with $\langle M^h \rangle_t = t \|h\|_{V_0^{-1}}^2$.

In fact, by [AR91, Proposition 4.5],

$$\langle M^h \rangle_t = \int_0^t \langle \nabla u_h(X_s), \nabla u_h(X_s) \rangle_{V_0^{-1}} ds = t \|h\|_{V_0^{-1}}^2.$$

For $f = B^* \bar{Q} h \in U$, with $h \in V_0^{-1}$, define $W_t^f := M_t^h$ and let $D := \text{span}\{B^* Q e_k : k \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$. Since $\|B^* Q h\|_U^2 = \|h\|_{V_0^{-1}}^2$, it is easy to check that $\langle W^f, W^g \rangle_t = t \langle f, g \rangle_U$ for $f, g \in D$, where $\langle W^f, W^g \rangle_t$ is the bracket process of W^f and W^g . Moreover D is dense in U and W_t is \mathbb{Q} -linear on D , since the embedding $U \rightarrow V_0^{-1-s}(\mathbb{T}^2, \mathbb{R}^2)$ is Hilbert-Schmidt for any $s > 0$. By [AR91, Theorem 6.2], there exist a map $W : \Omega \rightarrow C([0, \infty); V_0^{-1-s}(\mathbb{T}^2, \mathbb{R}^2))$, and a properly \mathcal{E} -exceptional set $S \subset E$, i.e. $\nu(S) = 0$ and $\mathbb{P}^z(X(t) \in E \setminus S, \forall t \geq 0) = 1$ for $z \in E \setminus S$, such that $\forall z \in E \setminus S$, W is a U -cylindrical Wiener process on $(\Omega, \mathcal{M}_t, \mathbb{P}^z)$ such that for any $f \in D$

$${}_{V_0^{-1-s}} \langle W, f \rangle_{V_0^{1+s}} = W^f, \mathbb{P}^z - a.s.,$$

where ${}_{V_0^{-1-s}} \langle \cdot, \cdot \rangle_{V_0^{1+s}}$ is defined by (5.2). In particular,

$$\langle B^* h, W_t \rangle_{V_0^{-1}(\mathbb{T}^2, \mathbb{R}^2)} = \langle W_t, B^* Q h \rangle_U = M_t^h,$$

and $W = (W^1, W^2)$, where $W^i : \Omega \rightarrow C([0, \infty); E)$, $i = 1, 2$ are two independent L_0^2 -cylindrical Wiener processes under \mathbb{P}^z for any $z \in E \setminus S$. \square

5.2 Relation between the two solutions

In the following we discuss the relation between M constructed above and the shifted equation (1.4). In fact, by Lemma 2.1 we have that $\mathcal{C}^{-\alpha} \subset V^{-1-s_0}$ for $\alpha \in (0, 1)$, $\mathcal{C}^{-\alpha} \in \mathcal{B}(V^{-1-s_0})$ and $\nu(\mathcal{C}^{-\alpha} \cap E) = 1$. For W constructed in Theorem 5.5 define $Z(t) := \int_0^t e^{-(t-s)A^2/2} B dW_s$.

Theorem 5.6 *Let $\alpha \in (0, \frac{1}{3})$, $\alpha < \beta < 2 - \alpha$. There exists a properly \mathcal{E} -exceptional set $S_2 \subset E$ in the sense of Theorem 5.5 such that for every $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$ under \mathbb{P}^z , $Y := X - Z \in C((0, T]; \mathcal{C}^\beta) \cap C([0, T]; \mathcal{C}^{-\alpha})$ is a solution to the following equation:*

$$Y(t) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^3 C_3^l Y(s)^l : Z(s)^{3-l} : ds + e^{-\frac{t}{2}A^2} X(0). \quad (5.7)$$

Here $C((0, T]; \mathcal{C}^\beta)$ is equipped with the norm $\sup_{t \in [0, T]} t^{\frac{\beta+\alpha}{4}} \|\cdot\|_\beta$. Moreover,

$$\mathbb{P}^z[X(t) \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2.$$

Proof For $z \in E \setminus S$ under \mathbb{P}^z we have that

$$X(t) = \frac{1}{2} \int_0^t e^{-(t-\tau)A^2/2} A : X(\tau)^3 : d\tau + Z(t) + e^{-\frac{t}{2}A^2} X(0).$$

Since ν is an invariant measure for X , by Lemma 2.1 and Lemma 3.2 we conclude that for every $T \geq 0$, $p > 1$, $\delta > 0$, with $2\delta - \alpha < 0$, and $p_0 > 1$ large enough

$$\begin{aligned} & \int E^z \int_0^T \|\cdot\| : X(\tau)^3 : \|_{-\alpha}^p d\tau \nu(dz) \lesssim \int E^z \int_0^T \|\cdot\| : X(\tau)^3 : \|_{B_{p_0, p_0}^{\delta-\alpha}}^p d\tau \nu(dz) \\ & = T \int \|\cdot\| : \phi^3 : \|_{B_{p_0, p_0}^{\delta-\alpha}}^p \nu(d\phi) \lesssim T \int \|\cdot\| : \phi^3 : \|_{2\delta-\alpha}^p \nu(d\phi) < \infty, \end{aligned}$$

which implies that there exists a properly \mathcal{E} -exceptional set $S_1 \supset S$ such that for $z \in E \setminus S_1$ \mathbb{P}^z -a.s.

$$: X(\cdot)^3 : \in L^p(0, T; \mathcal{C}^{-\alpha}), \quad E^z \int_0^T \| : X(\tau)^3 : \|_{-\alpha}^p d\tau < \infty, \quad \forall p > 1.$$

Here we used Lemma 2.1 to deduce the first result. The second, however, does not imply the first directly because of (2.1). Lemma 2.2 implies that for $\alpha < \beta < 2 - \alpha$

$$\int_0^t e^{-(t-\tau)A^2/2} A : X(\tau)^3 : d\tau \in C([0, \infty); \mathcal{C}^\beta) \quad \mathbb{P}^z - a.s..$$

Now by Lemma 2.2 we conclude that for $z \in \mathcal{C}^{-\alpha} \setminus S_1$, $e^{-\frac{t}{2}A^2} X(0) \in C([0, T], \mathcal{C}^{-\alpha}) \cap C((0, T], \mathcal{C}^\beta)$. Thus,

$$X - Z \in C([0, T], \mathcal{C}^{-\alpha}) \cap C((0, T], \mathcal{C}^\beta) \quad \mathbb{P}^z - a.s..$$

Since $\mathbb{P}^\nu \circ X(t)^{-1} = \nu$, by Lemma 3.5 we conclude that under \mathbb{P}^ν , by Fubini's theorem $Y := X - Z$ satisfies (5.7) and for ν -a.e. $z \in E$ under \mathbb{P}^z , $Y := X - Z$ satisfies (5.7). In the following we prove that these results hold under \mathbb{P}^z for z outside a properly \mathcal{E} -exceptional set. First we have $Z \in C([0, \infty); \mathcal{C}^{-\alpha})$ \mathbb{P}^ν -a.s., which combined with $X - Z \in C([0, T], \mathcal{C}^{-\alpha})$ implies

$$\mathbb{P}^\nu[X \in C([0, \infty), \mathcal{C}^{-\alpha})] = 1.$$

We also have

$$\begin{aligned} \bar{Y}(s, t_0) := X(s + t_0) - Z(s + t_0) &= \frac{1}{2} \int_{t_0}^{t_0+s} e^{-(t_0+s-\tau)A^2/2} A : X(\tau)^3 : d\tau \\ &+ e^{-sA^2/2} (X(t_0) - Z(t_0)) \in C((0, \infty)^2; \mathcal{C}^\beta) \quad \mathbb{P}^\nu - a.s.. \end{aligned}$$

Similar arguments as in the proof of Lemma 3.5 imply that $\forall s > 0, t_0 \geq 0$

$$\begin{aligned} \mathbb{P}^\nu(: X(s + t_0)^3 : &:= \sum_{l=0}^3 C_3^l \bar{Y}(s, t_0)^l : Z(s + t_0)^{3-l} :), \\ X \in C([0, \infty), \mathcal{C}^{-\alpha}), \bar{Y} \in C((0, \infty)^2; \mathcal{C}^\beta)) &= 1, \end{aligned}$$

In the following we use I_{t, t_0} to denote the equality

$$\begin{aligned} &\int_0^t e^{-(t-s)A^2/2} A : X(s + t_0)^3 : ds \\ &= \sum_{l=0}^3 \int_0^t e^{-(t-s)A^2/2} A C_3^l \bar{Y}(s, t_0)^l : Z(s + t_0)^{3-l} : ds. \end{aligned}$$

Then using Fubini's theorem we know that

$$\mathbb{P}^\nu(I_{t, t_0} \text{ holds } \forall t \geq 0, a.e. t_0 \geq 0, X \in C([0, \infty); \mathcal{C}^{-\alpha}), \bar{Y} \in C((0, \infty)^2; \mathcal{C}^\beta)) = 1.$$

Here we used $X \in C([0, \infty); \mathcal{C}^{-\alpha})$ for $\alpha < \frac{1}{3}$ to make the right hand side of I_{t, t_0} meaningful. It is obvious that the right hand side of the first equality is continuous with respect to t_0 .

Since $\int_0^t e^{-(t-s)A^2/2} A : X(s+t_0)^3 : ds = \int_{t_0}^{t+t_0} e^{-(t-s+t_0)A^2/2} A : X(s)^3 : ds$ we know that $\int_0^t e^{-(t-s)A^2/2} A : X(s+t_0)^3 : ds$ is also continuous with respect to t_0 and we obtain that

$$\mathbb{P}^\nu(I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0, X \in C([0, \infty); \mathcal{C}^{-\alpha}), \bar{Y} \in C((0, \infty)^2; \mathcal{C}^\beta)) = 1.$$

This implies that there exists a properly \mathcal{E} -exceptional set $S_2 \supset S_1$ such that for $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$ under \mathbb{P}^z

$$\mathbb{P}^z(X \in C([0, \infty); \mathcal{C}^{-\alpha}), I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0) = 1.$$

Indeed, define

$$\Omega_0 := \{\omega : X \in C([0, \infty); \mathcal{C}^{-\alpha}), : Z^k : \in C((0, \infty); \mathcal{C}^{-\alpha}), k = 1, 2, 3, I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0\},$$

and let $\Theta_t : \Omega \rightarrow \Omega, t > 0$, be the canonical shift, i.e. $\Theta_t(\omega) = \omega(\cdot + t), \omega \in \Omega$. Then it is easy to check that

$$\Theta_t^{-1}\Omega_0 \supset \Omega_0, \quad t \in \mathbb{R}^+,$$

and

$$\Omega_0 = \bigcap_{t>0, t \in \mathbb{Q}} \Theta_t^{-1}\Omega_0.$$

On the other hand, by the Markov property we know that

$$\mathbb{P}^z(\Theta_t^{-1}\Omega_0) = P_t(1_{\Omega_0})(z), \forall z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$$

which by [MR92, Chapter IV Theorem 3.5] is \mathcal{E} -quasi-continuous in the sense of [MR92, Chapter III Definition 3.2] on E . It follows that for every $t > 0$

$$\mathbb{P}^z(\Theta_t^{-1}\Omega_0) = 1 \quad q.e.z \in E,$$

which yields that

$$\mathbb{P}^z(\Omega_0) = 1 \quad q.e.z \in E.$$

Here q.e. means that there exists a properly \mathcal{E} -exceptional set such that outside this exceptional set the result holds. Now Y satisfies (5.7) \mathbb{P}^z -a.s. for $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$. Moreover, for $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$ $Y \in C([0, \infty); \mathcal{C}^{-\alpha}) \cap C([0, T], \mathcal{C}^\beta), Z \in C([0, \infty); \mathcal{C}^{-\alpha})$ \mathbb{P}^z -a.s., which implies that

$$\mathbb{P}^z[X(t) \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2.$$

□

Corollary 5.7 *Let $\bar{X} = \bar{Y} + Z$ where \bar{Y} is the unique solution to (4.1). ν is an invariant measure of \bar{X} .*

Proof By Theorem 5.6 and the uniqueness of the solution to (4.1) we know that $X \stackrel{d}{=} \bar{X}$, \mathbb{P}^z -a.s. $\forall z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$, which combined with $\nu(\mathcal{C}^{-\alpha} \cap E) = 1$ implies that ν is an invariant measure of \bar{X} . □

5.3 Markov uniqueness in the restricted sense

In this subsection we prove Markov uniqueness in the restricted sense and the uniqueness of the martingale (probabilistically weak) solutions to (1.1) if the solution has ν as an invariant measure.

By [MR92, Chapter 4, Section 4b] it follows that there is a point separating countable \mathbb{Q} -vector space $D \subset \mathcal{FC}_b^\infty$ such that $D \subset D(L(\mathcal{E}))$. Let $\mathcal{E}^{q.r.}$ be the set of all quasi-regular Dirichlet forms $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ (cf. [MR92]) on $L^2(E; \nu)$ such that $D \subset D(L(\tilde{\mathcal{E}}))$ and $\tilde{\mathcal{E}} = \mathcal{E}$ on $D \times D$. Here for a Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ we denote its generator by $(L(\tilde{\mathcal{E}}), D(L(\tilde{\mathcal{E}})))$.

In the following we consider the martingale problem in the sense of [AR94] and probabilistically weak solutions to (1.1):

Definition 5.8 (i) A ν -special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (\mathbb{P}^z))$ in the sense of [MR92, Chapter IV] with state space E is said to solve the martingale problem for $(L(\mathcal{E}), D)$ if for all $u \in D$, $u(X(t)) - u(X(0)) - \int_0^t L(\mathcal{E})u(X(s))ds$, $t \geq 0$, is an (\mathcal{M}_t) -martingale under \mathbb{P}^ν .

(ii) A ν -special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (P^z))$ with state space E is called a probabilistically weak solution to (1.1) if there exists two map $W^i : \Omega \rightarrow C([0, \infty); E)$ $i = 1, 2$ such that for ν -a.e. z under \mathbb{P}^z , $W := (W^1, W^2)$ is an $L^2(\mathbb{T}^2, \mathbb{R}^2)$ -cylindrical Wiener process with respect to (\mathcal{M}_t) and the sample paths of the associated process satisfy (5.6) for all $h \in V^{3+s_0}$.

Remark 5.9 If M is a probabilistically weak solution to (1.1), we can easily check that it also solves the martingale problem. Conversely, if M solves the martingale problem, then with the same argument in Theorem 5.5, there exists an $L^2_0(\mathbb{T}^2, \mathbb{R}^2)$ -cylindrical Wiener process W such that (X, W) satisfies (5.6) for $h \in V^{3+s_0}$. That is to say, these two definitions are equivalent.

To explain the uniqueness result below we also introduce the following concept:

Two strong Markov processes M and M' with state space E and transition semigroups $(p_t)_{t>0}$ and $(p'_t)_{t>0}$ are called ν -equivalent if there exists $S \in \mathcal{B}(E)$ such that (i) $\nu(E \setminus S) = 0$, (ii) $\mathbb{P}^z[X(t) \in S, \forall t \geq 0] = \mathbb{P}^{z'}[X'(t) \in S, \forall t \geq 0] = 1, z \in S$, (iii) $p_t f(z) = p'_t f(z)$ for all $f \in \mathcal{B}_b(E), t > 0$ and $z \in S$.

Combining Theorem 3.9 and Theorem 3.10, we obtain Markov uniqueness in the restricted sense for $(L(\mathcal{E}), D)$ (see part (iii)) and the uniqueness of martingale (probabilistically weak) solutions to (1.1) if the solution has ν as an invariant measure (see part (i), (ii)):

Theorem 5.10 (i) There exists (up to ν -equivalence) exactly one probabilistically weak solution M to (1.1) satisfying $\mathbb{P}^z(X \in C([0, \infty); E)) = 1$ for ν -a.e. and having ν as an invariant measure, i.e. for the transition semigroup $(p_t)_{t \geq 0}$, $\int p_t f d\nu = \int f d\nu$ for $f \in L^2(E; \nu)$.

(ii) There exists (up to ν -equivalence) exactly one ν -special standard process M with state space E solving the martingale problem for $(L(\mathcal{E}), D)$ and satisfying $\mathbb{P}^z(X \in C([0, \infty); E)) = 1$ for ν -a.e. and having ν as an invariant measure.

(iii) $\#\mathcal{E}^{q.r.} = 1$. Moreover, there exists (up to ν -equivalence) exactly one ν -special standard process M with state space E associated with a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ solving the martingale problem for $(L(\mathcal{E}), D)$.

Proof The proof is the same as [RZZ17b, Theorem 3.12] □

5.4 Stationary solution

Now we consider the stationary case. In this case, we can obtain a probabilistically strong solution to 1.1. Take two different stationary solutions X_1, X_2 to 1.1 with the same initial condition $\eta \in \mathcal{C}^{-\alpha} \cap E$, $\alpha > 0$, α small enough, having the distribution ν . We have

$$X_i(t) = e^{-\frac{t}{2}A^2} \eta + \frac{1}{2} \int_0^t e^{-\frac{t-\tau}{2}A^2} A : X_i(\tau)^3 : d\tau + Z(t),$$

where Z is the stochastic convolution

$$Z(t) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s.$$

By a similar argument as in the proof of Theorem 5.6 and using Lemma 3.2 we have that for every $p > 1$

$$E \int_0^T \| : X_i(\tau)^3 : \|_{-\alpha}^p d\tau = T \int \| : \phi^3 : \|_{-\alpha}^p \nu(d\phi) < \infty.$$

Then Lemma 2.2 implies that for $\alpha > 0$, $\alpha < \beta < 2 - \alpha$

$$\int_0^t e^{-\frac{t-\tau}{2}A^2} A : X_i(\tau)^3 : d\tau \in C([0, T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s..$$

Thus by Lemma 2.2 we conclude that

$$X_i - Z \in C((0, T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s.,$$

where $C((0, T]; \mathcal{C}^\beta)$ is equipped with the norm $\sup_{t \in [0, T]} t^{\frac{\beta+\alpha}{4}} \| \cdot \|_\beta$. Moreover, similar arguments as in the proof of Theorem 3.5 yield that if $\alpha > 0$ with α small enough, $X_i - Z$ is a solution to the following equation

$$Y(t) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^3 C_3^l Y(s)^l : Z(s)^{3-l} : ds + e^{-\frac{t}{2}A^2} \eta. \quad (5.8)$$

Here the Wick powers of Z are defined as in Lemma 3.3.

Now by [LR15, Proposition G.0.5] we know the solutions to equation (5.8) are also the solutions to (4.1) and by uniqueness of the solutions to (4.1) in Theorem 4.4, this implies that

$$X_1 - Z = X_2 - Z \text{ on } [0, T] \quad \mathbb{P} - a.s..$$

Then the pathwise uniqueness holds for the stationary solutions to (1.1). Now by the existence of the stationary martingale solution (cf. [MR99]) and the Yamada-Watanabe Theorem in [Kur07] we obtain:

Theorem 5.11 *For any initial condition $X(0) \in \mathcal{C}^{-\alpha} \cap E$ with distribution ν and $\alpha > 0$, α small enough, there exists a unique probabilistically strong solution X to (1.1) such that X is a stationary process, i.e. for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ with a U -Wiener process W , there exists an \mathcal{F}_t -adapted stationary process $X : [0, T] \times \Omega \rightarrow E$ such that for $\mathbb{P} - a.s.$ $\omega \in \Omega$, X satisfies (1.1). Moreover, for $0 < \beta < 2 - \alpha$*

$$X - Z \in C((0, T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s..$$

6 Ergodicity

Let $X = Y + Z$ where Y is the solution to equation (4.1). By the uniqueness of the solution Y we have that X is a Markov process. Let P_t be the semigroup of X , i.e

$$P_t\Phi(x) = E\Phi(X(t, x)), \quad \forall \Phi \in C_b(V_0^{-1}).$$

We recall that the U -cylindrical Wiener process W take values in $C([0, T], V_0^{-1-s_0}(\mathbb{T}^2, \mathbb{R}^2))$, \mathbb{P} -a.s., for any $s_0 > 0$. Let \mathcal{D} denote the Fréchet derivative of functions on $C([0, T], V_0^{-1-s_0}(\mathbb{T}^2, \mathbb{R}^2))$ (i.e. with respect to the noise). We also denote the Cameron-Martin space by $\mathcal{CM} := \{\omega : \partial_t\omega \in L^2([0, T], L_0^2(\mathbb{T}^2; \mathbb{R}^2)), \omega(0) = (0, 0)\}$. Here we view $\partial_t\omega$ as a function on $[0, T] \times \mathbb{T}^2$ rather than lying in the tangent space of \mathbb{T}^2 .

Proposition 6.1 *For a fixed $x \in V_0^{-1}$, let $\mathfrak{X}_t^x := X(t, x) = Z_t + Y(t, x)$ be a map from $C([0, T], V_0^{-1-s_0})$ to V_0^{-1} . For any $\omega \in \mathcal{CM}$ its directional derivative $\mathcal{D}\mathfrak{X}_t^x(\omega)$ is given in mild form as*

$$\mathcal{D}\mathfrak{X}_t^x(\omega) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^2 3C_2^l Y^{2-l}(s) : Z_s^l : \mathcal{D}\mathfrak{X}_s^x(\omega) ds + \int_0^t e^{-(t-s)A^2/2} B d\omega_s. \quad (6.1)$$

The proof of Proposition 6.1 can be obtained by using approximation or the implicit function theorem (see [Dri03, Theorem 19.28], [HM16], [TW16])

Let D denote the Fréchet derivative of functions on V_0^{-1} . We also consider the following equation:

$$\begin{cases} \partial_t J_{s,t} h = -\frac{1}{2} A^2 J_{s,t} h + \frac{1}{2} A \left(\sum_{l=0}^2 3C_2^l Y^{2-l}(t) : Z_t^l : J_{s,t} h \right) \\ J_{s,s} h = h \in V_0^{-1} \end{cases}. \quad (6.2)$$

Then $J_{0,t} h = DX(t, x)(h)$, i.e. it is the derivative of $X(t, \cdot)$ in the direction h . For $\omega \in \mathcal{CM}$, by Duhamel's principle

$$\mathcal{D}\mathfrak{X}_t^x(\omega) = \int_0^t J_{s,t} B \partial_s \omega(s) ds. \quad (6.3)$$

We define the stopping time

$$\tau^r := \inf\{t \in (0, T) : t^\rho \| : Z_t^k : \|_{-\alpha} > r, k = 1, 2, 3\}, \quad (6.4)$$

where $\rho > 0$ is a small enough constant introduced in Lemma 3.3.

Proposition 6.2 *For any $x \in V_0^{-1}$, there exists constants C_1, C_2 which only depend on $r, \|x\|_{H^{-1}}$, such that for all $t \leq \tau^r$*

$$\sup_{s \leq t} \|Y_s\|_{H^{-1}} \vee \int_0^t \|Y_s\|_{L^4}^4 ds \vee \int_0^t \|Y_s\|_{H^1}^2 ds \leq C_1 \quad \text{and} \quad \sup_{s \leq t} \|J_{0,s} h\|_{H^{-1}} \leq C_2 \|h\|_{H^{-1}}$$

Proof The first bound with constant C_1 follows from the proof of Theorem 4.1.

For the second bound, we note that $J_{0,t}h$ satisfies the following equation:

$$\begin{cases} \frac{du}{dt} = -\frac{1}{2}A^2u + \frac{1}{2}A \left(\sum_{l=0}^2 3C_2^l Y^{2-l}(t) : Z_t^l : u \right) \\ u(0) = h \end{cases}.$$

Taking scalar product with $(-A)^{-1}u$, we obtain that

$$\frac{d}{dt} \|u\|_{H^{-1}}^2 + \|u\|_{H^1}^2 = -3 \langle Y^2 + 2YZ_+ : Z^2 : , u^2 \rangle,$$

that is

$$\frac{d}{dt} \|u\|_{H^{-1}}^2 + \|u\|_{H^1}^2 \leq 6 |\langle YZ, u^2 \rangle| + 3 |\langle : Z^2 : , u^2 \rangle|.$$

Following the same argument that we used to estimate (4.13) and using the first bound, we use Grönwall's inequality to obtain the second bound. \square

Let $\chi_r \in C^\infty(\mathbb{R})$ such that $\chi_r(\zeta) \in [0, 1]$ for all $\zeta \in \mathbb{R}$, and

$$\chi_r(\zeta) = \begin{cases} 1, & |\zeta| \leq \frac{r}{2} \\ 0, & |\zeta| \geq r \end{cases}.$$

Following the notation in [TW16], we set

$$C^{3,-\alpha}(0, T) := C([0, T]; \mathcal{C}^{-\alpha}) \times C((0, T]; \mathcal{C}^{-\alpha})^2, \quad (6.5)$$

and $\underline{Z} := (Z, : Z^2 : , : Z^3 :) \in C^{3,-\alpha}(0, T)$. We also define

$$\|\underline{Z}\|_t := \max_{k=1,2,3} \left\{ \sup_{0 \leq s \leq T} s^k \| : Z_s^k : \|_{-\alpha} \right\}.$$

Theorem 6.3 (*Bismut-Elworthy-Li Formula*) *Let $x \in V_0^{-1}$, $\Phi \in C_b^1(V_0^{-1})$ and ω be a process taking values in the Cameron-Martin space \mathcal{CM} with $\partial_s \omega$ adapted. Assume that there exists a deterministic constant $C \equiv C(t)$ such that $\|\partial_s \omega\|_{L^2(0,t;U)} \leq C$ \mathbb{P} -a.s.. Then we have*

$$\begin{aligned} E[D\Phi(\mathfrak{x}_t^x)(\mathcal{D}\mathfrak{x}_t^x(\omega))\chi_r(\|\underline{Z}\|_t)] &= E \left(\Phi(\mathfrak{x}_t^x)\chi_r(\|\underline{Z}\|_t) \int_0^t \partial_s \omega(s) \cdot dW_s \right), \\ &\quad - E(\Phi(\mathfrak{x}_t^x)\partial_+\chi_r(\|\underline{Z}\|_t)(\omega)) \end{aligned} \quad (6.6)$$

where

$$\begin{aligned} \partial_+\chi_r(\|\underline{Z}\|_t)(\omega) &= \partial_\zeta \chi_r(\|\underline{Z}\|_t)\partial_+\|\underline{Z}\|_t(\underline{Y}) \\ \partial_+\|\underline{Z}\|_t(\underline{Y}) &= \lim_{\delta \rightarrow 0^+} \frac{\|\underline{Z} + \delta \underline{Y}\|_t - \|\underline{Z}\|_t}{\delta}, \end{aligned} \quad (6.7)$$

$\underline{Y} = (Q_\omega(\cdot), 2ZQ_\omega(\cdot), 3 : Z^2 : Q_\omega(\cdot)) \in C^{3,-\alpha}(0, t)$ and

$$Q_\omega(\cdot) := \int_0^\cdot e^{-(\cdot-s)A^2/2} B \partial_s \omega(s) ds.$$

Proof This is proved by the same calculation as that in the proof of [TW16, Theorem 5.4]. \square

We use (6.6) to prove the following proposition.

Proposition 6.4 *There exists a universal constant $\theta_1 > 0$ such that for every $x \in V_0^{-1}$, there exists a constant $C(x)$, such that*

$$|P_t\Phi(x) - P_t\Phi(y)| \leq C(x) \frac{1}{t^{\theta_1}} \|\Phi\|_\infty \|x - y\|_{H^{-1}} + 2\|\Phi\|_\infty \mathbb{P}(t \geq \tau^{\frac{r}{2}}) \quad (6.8)$$

for every $y \in V_0^{-1}$, $\|x - y\|_{H^{-1}} \leq 1$, $t \leq T$ and $\Phi \in C_b^1(V_0^{-1})$

Proof Let $\Phi \in C_b^1(V_0^{-1})$. Then

$$|P_t\Phi(x) - P_t\Phi(y)| = |E[\Phi(X(t, x)) - \Phi(X(t, y))]| \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= |E[\Phi(X(t, x)) - \Phi(X(t, y)) \chi_r(\|\underline{Z}\|_t)]| \\ I_2 &:= |E[\Phi(X(t, x)) - \Phi(X(t, y)) (1 - \chi_r(\|\underline{Z}\|_t))]|. \end{aligned}$$

For the second term we have that $I_2 \leq 2\|\Phi\|_\infty \mathbb{P}(t \geq \tau^{\frac{r}{2}})$. By the mean value theorem we get that

$$\begin{aligned} I_1 &= \left| E \left(\int_0^1 D\Phi(\mathfrak{X}_t^{z_\lambda}(y-x)) d\lambda \cdot \chi_r(\|\underline{Z}\|_t) \right) \right| \\ &= \left| \int_0^1 E(D\Phi(\mathfrak{X}_t^{z_\lambda})(y-x) \chi_r(\|\underline{Z}\|_t)) d\lambda \right|, \end{aligned}$$

where $z_\lambda := x + \lambda(y-x)$. For any $h \in V_0^{-1}$, let ω be such that $B\partial_s\omega(s) = J_{0,s}h$ for $s \leq \tau^r$ and 0 otherwise. Then $\partial_s\omega(s)$ satisfies the condition in Theorem 6.3. Furthermore, by (6.3) and $J_{0,s}J_{s,t} = J_{0,t}$ we have $\mathcal{D}\mathfrak{X}_t^{z_\lambda}(\omega) = tD\mathfrak{X}_t^{z_\lambda}(h)$. Then we can use (6.6) to obtain that

$$\begin{aligned} E(D[\Phi(\mathfrak{X}_t^{z_\lambda})](h) \chi_r(\|\underline{Z}\|_t)) &= \frac{1}{t} E \left(\Phi(\mathfrak{X}_t^{z_\lambda}) \int_0^t \partial_s\omega(s) \cdot dW_s \chi_r(\|\underline{Z}\|_t) \right) \\ &\quad - \frac{1}{t} E(\Phi(\mathfrak{X}_t^{z_\lambda}) \partial_+ \chi_r(\|\underline{Z}\|_t)(\omega)). \end{aligned}$$

Then we have

$$I_1 \leq \frac{1}{t} \|\Phi\|_\infty \int_0^1 E \left| \int_0^t \partial_s\omega(s) dW_s \chi_r(\|\underline{Z}\|_t) \right| d\lambda + \frac{1}{t} \|\Phi\|_\infty \int_0^1 E \left| \partial_+ \chi_r(\|\underline{Z}\|_t)(\omega) \right| d\lambda.$$

For the first term we have

$$\begin{aligned} E \left| \int_0^t \partial_s\omega(s) dW_s \chi_r(\|\underline{Z}\|_t) \right| &\leq E \left| \int_0^{t \wedge \tau^r} \partial_s\omega(s) \cdot dW_s \right| \\ &\leq \left(\int_0^{t \wedge \tau^r} \|\partial_s\omega(s)\|_{\mathcal{U}}^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^{t \wedge \tau^r} \|J_{0,s}h\|_{H^{-1}}^2 ds \right)^{\frac{1}{2}} \\ &\leq C_2 t \|h\|_{H^{-1}}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality and Itô's isometry in the second step and Proposition 6.2 in the last step.

By the definition of $\partial_+ \chi_r(\|\underline{Z}\|_t)(\omega)$, we have

$$\left| \partial_+ \chi_r(\|\underline{Z}\|_t)(\omega) \right| \leq \partial_+ \|\underline{Z}\|_t(\underline{Y}) \leq \|\underline{Y}\|_t \lesssim \|\underline{Z}\|_t \|Q_\omega(t)\|_\beta,$$

where \underline{Y} is as introduced in Theorem 6.3 and we used Lemma 2.3 in the last inequality. Moreover, we use Lemma 2.2 and Lemma 2.5 to obtain

$$\|Q_\omega(t)\|_\beta \lesssim \int_0^t (t-s)^{-\frac{\beta+2}{4}} \|J_{0,s}h\|_{-2} ds \lesssim \int_0^t (t-s)^{-\frac{\beta+2}{4}} \|J_{0,s}h\|_{H^{-1}} ds \lesssim C_2 t^{\frac{2-\beta}{4}} \|h\|_{H^{-1}}.$$

Choosing β small enough, we deduce that there exists a constant $\theta_1 \in (0, \frac{1}{2})$, such that

$$I_1 \lesssim C_2 \frac{1}{t^{\theta_1}} \|\Phi\|_\infty \|h\|_{H^{-1}}.$$

Letting $h = y - x$ we finish the proof. \square

We denote by $\|\mu_1 - \mu_2\|_{TV}$ the total variation distance of two probability measures μ_1, μ_2 on V_0^{-1} given by

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{\|\Phi\|_\infty \leq 1} \left| \int \Phi d\mu_1 - \int \Phi d\mu_2 \right|.$$

Theorem 6.5 *There exists $\theta \in (0, 1)$ and $\sigma > 0$ such that for any $x, y \in V_0^{-1}$ and $\|x - y\|_{H^{-1}} \leq 1$ there exists a constant $C(x) > 0$ which only depends on x , and*

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{TV} \leq C(x)(1 + \|x\|_{H^{-1}})^\sigma \|x - y\|_{H^{-1}}^\theta,$$

for every $t \geq 1$.

Proof The proof is the same as that of [TW16, Theorem 5.8]. We omit it for simplicity. \square

In order to use Krylov-Bogoliubov method to prove the existence of an invariant measure, the H^{-1} uniform estimate is not enough. We need to find a space compactly embedded in H^{-1} where the solution is bounded in probability. We make use of the integrability on a smaller space, which is compactly embedded in H^{-1} . Thus we have

Theorem 6.6 *For every $x \in V_0^{-1}$, there exists a probability Borel measure ν_x on V_0^{-1} such that ν_x is an invariant measure for the semigroup $\{P_t, t \geq 0\}$ on V_0^{-1} .*

Proof By (4.5) and a similar argument as in the proof of [TW16, Corollary 3.10] we have that

$$\sup_{x \in V_0^{-1}} \sup_{t > 0} (t \wedge 1) E \|X(t, x)\|_{H^{-1}}^2 < \infty. \quad (6.9)$$

By the uniqueness of the solution, we know $X(t, x) = Z_{t-1,t} + Y_{t-1,t}$, where $Z_{s,t} := \int_s^t e^{-(t-r)A^2/2} B dW_r$ and $Y_{s,r}$, $r \geq t-1$ solves the equation

$$\begin{cases} \frac{dY_{s,r}}{dr} = -\frac{1}{2} A^2 Y_{s,r} + \frac{1}{2} A \sum_{k=0}^3 C_3^k Y_{s,r}^{3-k} : Z_{s,r}^k :, \\ Y_{s,s} = X(s, x). \end{cases} \quad (6.10)$$

Applying Theorem 4.1 with $Y_{t,r}$ replacing Y_r we have

$$E \int_t^{t+1} \|Y_{t,r}\|_{H^1}^2 dr \lesssim 1 + E\|Y_{t,t}\|_{H^{-1}}^2 = 1 + E\|X(t,x)\|_{H^{-1}}^2.$$

Combining this with (6.9) we deduce that for $\alpha \in (0, 1)$,

$$E \int_t^{t+1} \|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds \leq E \int_t^{t+1} \|Y_{t,s}\|_{H^1}^2 ds + E \int_t^{t+1} \|Z_{t,s}\|_{\mathcal{C}^{-\alpha}}^2 ds \lesssim 1 + \frac{1}{1 \wedge t},$$

where we used a similar argument as in the proof of [TW16, Theorem 2.1] in the last inequality. Then we obtain that for $t \geq 1$

$$E \int_1^t \|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds \lesssim t.$$

Moreover, by (4.4) we have

$$E \int_0^1 \|Y_s\|_{H^1}^2 ds \lesssim 1 + \|x\|_{H^{-1}}^2.$$

Thus for $t \geq 1$

$$\int_0^t E\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds \leq \int_0^1 E\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds + \int_1^t E\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds \lesssim 1 + \|x\|_{H^{-1}}^2 + t.$$

By Chebeshev's inequality, for any $K > 0$

$$\mathbb{P}(\|X(t,x)\|_{\mathcal{C}^{-\alpha}} > K) \leq \frac{1}{K^2} E\|X(t,x)\|_{\mathcal{C}^{-\alpha}}^2.$$

Thus there exists a constant $C > 0$, such that

$$\begin{aligned} \int_0^t \mathbb{P}(\|X(s,x)\|_{\mathcal{C}^{-\alpha}} > K) &\leq \frac{C}{K^2} \int_0^t E\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2 ds \\ &\leq \frac{C}{K^2} (1 + \|x\|_{H^{-1}}^2 + t). \end{aligned}$$

Letting $R_t(x, \cdot) = \frac{1}{t} \int_0^t P_s(x, \cdot) ds$, for $K_\varepsilon^2 = \frac{C}{\varepsilon}$ we get

$$R_t(f \in \mathcal{C}^{-\alpha} \cap V_0^1 : \|f\|_{\mathcal{C}^{-\alpha}} > K_\varepsilon) \leq R_t(f \in V_0^1 : \|f\|_{\mathcal{C}^{-\alpha}} > K_\varepsilon) \leq (1 + \frac{1 + \|x\|_{H^{-1}}}{t})\varepsilon.$$

By [Tri06, Proposition 4.6] we know that $\{f \in \mathcal{C}^{-\alpha} \cap V_0^1 : \|f\|_{\mathcal{C}^{-\alpha}} > K_\varepsilon\}$ is a compact subset of V_0^{-1} since the embedding $\mathcal{C}^{-\alpha} \subset V^{-1}$ is compact. This implies the tightness of $\{R_t\}_{t \geq 0}$ in V_0^{-1} . By the Krylov-Bogoliubov existence theorem (see [DPZ96, Corollary 3.1.2]), there exists a sequence $t_k \nearrow \infty$ and a measure ν_x such that $R_{t_k} \rightarrow \nu_x$ weakly in V_0^{-1} and ν_x is an invariant measure for the semigroup $\{P_t\}_{t \geq 0}$. □

To prove the exponential mixing property, we make use of the irreducibility of \underline{Z} and a uniform estimate, which is slightly different from that in the proof of [TW16, Theorem 6.3].

Theorem 6.7 *There exists a constant $\lambda \in (0, 1)$ and $T_0 > 0$ such that*

$$\|P_t(x) - P_t(y)\|_{TV} \leq 1 - \lambda,$$

for every $x, y \in V_0^{-1}$, $t \geq T_0 + 1$.

Proof From (4.5) we know that for any fixed $r > 0$, there exist $T_0, M > 0$ which are independent of ω, x , such that for any initial value $x \in V_0^{-1}$, we have that $\{\omega : \|\underline{Z}\|_{T_0} \leq M\} \subset \{\|Y(T_0)\|_{V_0^{-1}} < \frac{r}{2}\} \cap \{\|Z(T_0)\|_{V_0^{-1}} < \frac{r}{2}\}$.

By Theorem 6.5 for every $a \in (0, 1)$ there exists $r \equiv r(a) > 0$ such that for every $x, y \in \bar{B}_r(0)$ and $t \geq 1$

$$\|P_t(x) - P_t(y)\|_{TV} \leq 1 - a, \quad (6.11)$$

where $B_r(u) := \{x \in V_0^{-1} : \|x - u\|_{V_0^{-1}} < r\}$. Then by (4.5) for any initial value $x \in V_0^{-1}$, there exists $b \equiv b(r) \in (0, 1)$ such that

$$\begin{aligned} \mathbb{P}(\|X(T_0)\|_{V_0^{-1}} \leq r) &\geq \mathbb{P}\left(\{\|Y(T_0)\|_{V_0^{-1}} \leq \frac{r}{2}\} \cap \{\|Z(T_0)\|_{V_0^{-1}} \leq \frac{r}{2}\}\right) \\ &\geq \mathbb{P}(\|\underline{Z}\|_{T_0} \leq M) \\ &\geq b, \end{aligned} \quad (6.12)$$

where in the last step we used the irreducibility of the law of \underline{Z} . Here we omit the proof of the irreducibility of \underline{Z} , since it is the same as that of [TW16, Theorem 6.3]. Moreover, by (6.12) for any $R > 0$

$$\inf_{x \in V_0^{-1}} P_{T_0}(x, \bar{B}_R(0)) \geq b. \quad (6.13)$$

Then combining (6.11)-(6.13) and Markov property by the same argument as in the proof of [TW16, Theorem 6.5], we can complete the proof. \square

The following corollary gives the exponential convergence to a unique invariant measure.

Corollary 6.8 *There exists a unique invariant measure $\bar{\nu}$ for the semigroup $\{P_t\}_{t \geq 0}$ such that*

$$\|P_t - \bar{\nu}\|_{TV} \leq (1 - \lambda)^{\lfloor \frac{t}{T} \rfloor} \|\delta_x - \bar{\nu}\|_{TV},$$

for every $x \in V_0^{-1}$, $t \geq T_0 + 1$. Moreover, $\bar{\nu} = \nu$.

Proof For the first result, see the proof of [TW16, Corollary 6.6]. By Corollary 5.7, ν is an invariant measure of X . Hence $\bar{\nu} = \nu$. \square

Remark 6.9 *In the following we give a simple and short proof for exponential convergence by the theory of Dirichlet forms.*

Similarly to [DPDT04], by comparing the two Dirichlet forms for Cahn-Hilliard equation and the dynamical Φ_2^4 model, we can obtain the spectral gap of equation (1.1). Indeed, by the same arguments in [RZZ17b] and [TW16] we know that ν is also the invariant measure for the solution to the dynamical Φ_2^4 model. We denote the Dirichlet form associated with the dynamical Φ_2^4 model by $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$, i.e.

$$\bar{\mathcal{E}}(f, g) = \frac{1}{2} \int_E \langle Df, Dg \rangle_{L^2} d\nu, f, g \in D(\bar{\mathcal{E}}),$$

where D denotes the gradient operator in $L^2(\mathbb{T}^2)$ (see [RZZ17b]). In [TW16] the exponential convergence for the dynamical Φ_2^4 model in total variation is proved. This implies the exponential convergence in $L^2(E, \nu)$ -norm. By [Wan06, Theory 1.1, Example 1.1.2] this is equivalent to the Poincaré inequality

$$\int f^2 d\nu - \left(\int f d\nu\right)^2 \leq C\bar{\mathcal{E}}(f, f), f \in D(\bar{\mathcal{E}}).$$

From the proof of Theorem 5.4 we know that

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_k \int \left| \frac{\partial f}{\partial h_k} \right|^2 d\nu = \frac{1}{2} \sum_k \lambda_k \int \left| \frac{\partial f}{\partial e_k} \right|^2 d\nu \geq \frac{1}{2} \sum_k \int \left| \frac{\partial f}{\partial e_k} \right|^2 d\nu = \bar{\mathcal{E}}(f, f),$$

where $h_k = \sqrt{\lambda_k} e_k$, $\{h_k\}_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}}$ is an orthonormal basis of V_0^{-1} . Then by [Wan06, Theory 1.1, Example 1.1.2] we have

$$\|P_t f - \int f d\nu\|_{L^2(E, \nu)} \leq e^{-\frac{t}{C}} \|f - \int f d\nu\|_{L^2(E, \nu)}.$$

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