SINGULAR PERTURBATIONS OF ORNSTEIN-UHLENBECK PROCESSES: INTEGRAL ESTIMATES AND GIRSANOV DENSITIES

MARIA GORDINA*, MICHAEL RÖCKNER‡, AND ALEXANDER TEPLYAEV†

Abstract. We consider a perturbation of an infinite-dimensional Ornstein–Uhlenbeck process by a class of singular nonlinear non-autonomous maximal monotone time-dependent drifts F_0. The only further assumption on F_0 is that it is bounded by a radially symmetric non-negative function of arbitrary growth. First we introduce a new notion of generalized solutions for such equations which we call pseudo-weak solutions and prove that they always exist and obtain some pathwise estimates in terms of the data of the equation. Then we prove that their laws are absolutely continuous with respect to the law of the original Ornstein–Uhlenbeck process. In particular, we show that pseudo-weak solutions always have continuous sample paths. In addition, we obtain higher integrability estimates of the associated Girsanov densities. Some of our results concern non-random equations as well, while probabilistic results are new even in finite-dimensional autonomous situations.

Contents

1991 *Mathematics Subject Classification.* Primary 60H10; Secondary 35R15, 60H15, 47D07, 47N30.

Key words and phrases. Ornstein-Uhlenbeck process, singular perturbation, nonlinear infinite-dimensional stochastic differential equations, non-Lipschitz monotone coefficients, Girsanov theorem.

∗Research was supported in part by NSF Grants DMS-1405169, DMS-1712427, and a Simons Fellowship.

‡ Research was supported in part by the German Science Foundation (DFG) through CRC 1283.

† Research was supported in part by NSF Grant DMS-1613025.

1. Introduction

The aim of this paper is to study solutions to the following stochastic differential equation on a real separable Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $| \cdot |_H$

$$dX_t = (AX_t + F_0(t, X_t))\, dt + \sigma dW_t, \ \ X_0 = x \in H. \tag{1.1}$$

Here W_t is a cylindrical Wiener process in H on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness for all $t \geqslant 0$. For the precise setting we refer to Section 2.1.

We consider equation (1.1) without the standard assumption on $F_0(t, \cdot)$ of being Lipschitz continuous. The motivation for our study includes a better understanding of equations such as (1.1) with time-dependent drifts of not necessarily polynomial growth.

Equation (1.1) can be viewed as a nonlinear non-autonomous perturbation of the stochastic differential equation corresponding to an Ornstein-Uhlenbeck process. In fact, it is a long standing open problem to find optimal or nearly optimal conditions on F_0 such that (1.1) has a solution under the usual assumption that A generates a C_0-semigroup on H (see e.g. [6, 9] and the references therein). A natural condition is to assume that $F_0(t, \cdot)$ is the minimal section of a maximal monotone multivalued map on some domain $D_F \subset H$ similarly to [6]. If F_0 is maximal monotone with $D_F = H$, then rewriting (1.1) as the random equation

$$dZ_t = \big(AZ_t + F_0(t, Z_t + W_{0,A,\sigma}(t))\big)dt, \quad Z_0 = x, \tag{1.2}$$

where $Z_t = X_t - W_{0,A,\sigma}(t)$ and the Ornstein-Uhlenbeck process $W_{0,A,\sigma}$ solves (1.1) for $F_0 \equiv 0$, $x = 0$. Moreover, in this case one can easily obtain a unique solution by classical results due to F. Browder, Kato, Komura and Rockafellar in [5, 14, 15, 25]. However, this case excludes many interesting examples, and therefore we include the case $D_F \subsetneq H$ following [6, 9]. The first main result of this paper is that under natural assumptions on F_0, including an arbitrary radial growth estimate, (1.1) always has a solution in a generalized sense. We introduce such generalized solutions and call them **pseudo-weak solutions** in Definition 2.5 and further in Section 3.1.

The main parts of the paper include a proof of existence of pseudo-weak solutions, pathwise *a priori* estimates of these solutions in Section 4, a proof of absolute continuity of the law of these solutions with respect to the law of the Ornstein-Uhlenbeck process, and finally integral estimates of the corresponding Radon-Nikodym density in Section 5. Our approach can be interpreted as an extension of the classical use of Girsanov transformation to find a solution for a stochastic differential equation with a nonzero (but at most linearly growing) drift. The main idea behind results such as Theorem 2.8 is that we can find a suitable finite generalized φ-type estimate of the solutions X_t by looking at the behavior of the nonlinearity F_0 at infinity.

We would like to comment on some of the previous results both in terms of the assumptions we make and the techniques we use. We describe the setting in Section 2.1 in detail, including the assumptions on the coefficients of the non-autonomous equation (1.1). The approach we use does not rely on an invariant measure (which is not available for non-autonomous equations), and therefore we do not use typical assumptions such as finite moments of the invariant measure and on integrability properties of the nonlinear drift with respect to this measure. The paper consists of three major parts which are intertwined: we introduce a notion of pseudo-weak solutions to (1.1) in Section 2.2, and prove their existence in Section 5.1. We use monotonicity of the coefficients of the equation to prove *a priori* pathwise bounds in Theorem 2.8.

In general one expects that Assumptions 2.1 and 2.3 imply uniqueness, by appealing to Gronwall's lemma, but this seems out of reach for now in a general setting such as ours. We refer to [3, 6] for a discussion of when and how martingale solutions to (1.1) can be constructed, and for more details on such solutions.

Some of our results on the Girsanov transform are closely related to the infinite-dimensional estimates by D. Gatarek and B. Gołdys in [12, 13]. They considered equations in Banach spaces, while we restrict our consideration to Hilbert space though for non-autonomous perturbations. In the future work we plan to extend our techniques to the reaction-diffusion equation in a Banach space. Our estimates of solutions and Girsanov densities are new even in finite dimensions such as the ones due to N. V. Krylov in [17, 19], [18, Chapter IV, §3].

2. Setting and main results

2.1. **Setting and assumptions.** Let H be a real separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|\cdot|_H$. We denote the space of bounded linear operators equipped with the operator norm $\|\cdot\|$ by $B(H)$. The Hilbert-Schmidt norm is denoted by $\|\cdot\|_{HS}$. We suppose that the coefficients A, F and B in Equation (1.1) satisfy the following assumptions.

Assumption 2.1. *The operator (A, D_A) generates a C_0-semigroup on H denoted by e^{tA}, $t \geqslant 0$. We assume that there is $\beta > 0$ such that for all $x \in D_A$*

$$\langle Ax, x \rangle \leqslant -\beta |x|_H^2.$$

Note that Assumption 2.1 implies that A is m-dissipative on H. Also note that some of our results hold true under weaker assumption $\beta \geqslant 0$, but this is not essential for our paper.

Assumption 2.2. *Both σ and σ^{-1} are in $B(H)$ with σ being self-adjoint and positive such that*

$$\int_0^T \|e^{tA}\sigma\|_{HS}^2 \, dt < \infty, \ \text{for all } T > 0.$$

Recall that under Assumption 2.2 the Ornstein-Uhlenbeck process

$$W_{x,A,\sigma}(t) := e^{tA}x + \int_0^t e^{(t-s)A} dW_s, \ t \geqslant 0, \tag{2.1}$$

is well-defined by [9, Section 5.1.2], and it is pathwise continuous by Assumption 2.1 and [16, Theorem 2].

Assumption 2.3. *Denote by 2^H the power set of the Hilbert space H. Let $F(t, \cdot) : [0, \infty) \times D_F \to 2^H$ be a family of maps such that D_F is a non-empty Borel set in H, and $dt \otimes \mathbb{P}$-almost surely the Ornstein-Uhlenbeck process $W_{x,A,\sigma} \in D_F$ for all $x \in D_F$. Furthermore, $F(t, \cdot)$ is an m-dissipative map, that is, for any $x_1, x_2 \in D_F$*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leqslant 0, \ \text{for any } y_1 \in F(t, x_1), y_2 \in F(t, x_2), t \in [0, \infty)$$

and for any $\alpha > 0$ and $t \in [0, \infty)$

$$\text{Range}\,(\alpha I - F(t, \cdot)) = H.$$

We refer to [1, Section II.3] and [2, Chapter 3] for basic facts about dissipative maps, as well as to the exposition in [29]. In particular, it is known that in a Hilbert space a map is m-dissipative if and only if it is maximal dissipative, that is, it has no proper dissipative extensions. By [1, Proposition 3.5(iv), Chapter II] for any $(t,x) \in [0,\infty) \times D_F$, the set $F(t,x)$ is non-empty, closed and convex, and so we can consider a well-defined map

$$F_0(t,x) := \{y \in F(t,x) : |y| = \inf\{|z|_H, z \in F(t,x)\}\}, \text{ for any } x \in D_F.$$

Using the Yosida approximation to F described in Section 3 we see that the function $F_0(t,x)$ is Borel-measurable.

The next assumption is similar to the ones introduced in [8, 13].

Assumption 2.4. *We assume that there is an increasing function $a : [0,\infty) \to [0,\infty)$ such that*

$$|F_0(t,x)|_H \leqslant a(|x|_H), (t,x) \in [0,\infty) \times D_F. \tag{2.2}$$

We are mostly interested in the case when $\lim_{u\to\infty} a(u) = \infty$.

2.2. **Pseudo-weak solutions and their properties.** Throughout this paper we assume that Assumptions 2.1, 2.2, 2.3 and 2.4 hold. The first step in defining pseudo-weak solutions to Equation (1.1) requires suitable approximations to F. We use the Yosida approximation F_α described in Section 3.2 below.

Let $Z^x_{\alpha,t}$ be the continuous H-valued processes that are the mild solutions to a family of regularized random ordinary differential equations

$$dZ_{\alpha,t} = (AZ_{\alpha,t} + F_\alpha(t, Z_{\alpha,t} + W_{0,A,\sigma}(t)))\,dt, \qquad Z_{\alpha,0} = x, \tag{2.3}$$

where $W_{0,A,\sigma}$ is the pathwise continuous Ornstein-Uhlenbeck process defined by (2.1) with $x = 0$. One can use [24, Chapter 6, Theorem 1.2, page 184] to justify the existence of solutions to (2.3) in the mild sense. We note that technically speaking [24] assumes that F_α is continuous in time, but it is clear that this assumption is not essential, and it is enough to assume joint measurability in time and space, and Lipschitz continuity in space, with the Lipschitz constant uniform in time, which holds for F_α as we describe in Section 3.2.

The stochastic differential equation

$$\begin{aligned} dX_t &= (AX_t + F_\alpha(t, X_t))\,dt + \sigma dW_t, \\ X_0 &= x \in H \end{aligned} \tag{2.4}$$

has a mild solution $X^x_{\alpha,t} = X_\alpha(t,x), t \geqslant 0$. Even though we have dependence on α in this equation, we prove bounds (2.5) below with the right-hand side not depending on α, so that we can take the supremum over all α.

$Z^x_{\alpha,t}$ is obviously a mild solution to the random ordinary differential equation (2.3) if and only if

$$X^x_{\alpha,t} := Z^x_{\alpha,t} + W_{0,A,\sigma}(t)$$

is a mild solution to (2.4).

In what follows, unless stated otherwise, a pseudo-weak limit means a ψ-pseudo-weak limit, in the sense of Definition 3.1 and Remark 3.2 below.

Definition 2.5. An adapted H-valued process X^x_t is a *pseudo-weak solution* to (1.1) if it is a pseudo-weak limit point of the approximating processes $X^x_{\alpha,t}$ defined by (2.4).

Remark 2.6. Obviously, such pseudo-weak limit points are automatically adapted. Surprisingly, by Theorem 2.10 below they are also automatically continuous $\mathbb{P}$-a.s. in H.

The main results of our paper are summarized in the following three theorems. We start with pathwise *a priori* estimates. For this purpose we introduce the space $\mathcal{M}$ as follows.

Definition 2.7. Let $\mathcal{M}$ denote the space of all continuous functions $\varphi : [0,\infty) \to (0,\infty)$ such that

(1) φ is a strictly increasing convex function which is C^2 on $(0,\infty)$;
(2) the limit $\frac{u\varphi'(u)}{\varphi(u)} \xrightarrow[u\to\infty]{} L_\varphi$ exists, and $L_\varphi \in [1,\infty]$.

For the properties of functions in the space $\mathcal{M}$ we refer to the statements and examples in Section 4.

Theorem 2.8 (Uniform pathwise *a priori* φ-estimates)**.** *Under Assumptions 2.1, 2.2, 2.3, 2.4, for every $\varphi \in \mathcal{M}$ we have the following estimates for any pseudo-weak solution X_t^x to Equation* (1.1)

$$\varphi\left(|X_t^x|_H^2\right) \leqslant \frac{e^{-\beta t}}{2}\varphi\left(4|x|_H^2\right) + \frac{1}{2}K_\varphi\left(t\right) + \frac{\beta t}{2}K_{\varphi,\beta,a}\left(t\right) < \infty \quad a.s. \tag{2.5}$$

Here $K_\varphi(t)$ and $K_{\varphi,\beta,a}(t)$ are random functions defined in Notation 4.5 below. These functions only depend on β, σ, A and a.

Theorem 2.9 (Pseudo-weak solutions)**.** *Under Assumptions 2.1, 2.2, 2.3, 2.4, there exists a pseudo-weak solution X_t^x, $t \geqslant 0$, to Equation* (1.1)*, i.e.*

$$X_t^x = Z_t^x + W_{0,A,\sigma}\left(t\right) \quad \mathbb{P}\text{-}a.s., \tag{2.6}$$

where the process Z_t^x is a pseudo-weak limit point of $Z_{\alpha,t}^x$, as $\alpha \to 0$, and $Z_{\alpha,t}^x$ are solutions to equations (2.3)*. Moreover, $\mathbb{P}$-a.s. we have the following estimate*

$$|X_t^x|_H \leqslant |x|_H e^{-\beta t} + \int_0^t e^{-\beta(t-s)}a\left(|W_{0,A,\sigma}\left(s\right)|_H\right)ds + |W_{0,A,\sigma}\left(t\right)|_H. \tag{2.7}$$

In the next theorem we prove a Girsanov-type result with respect to the law of the Ornstein–Uhlenbeck process $W_{x,A,\sigma}$ defined by (4.1) below.

Theorem 2.10. *Suppose Assumptions 2.1, 2.2, 2.3, 2.4 hold with $\lim_{u\to\infty} a(u) = \infty$. Let $x \in D_F$, $T > 0$. Then on any finite time interval $[0,T]$ the law of any pseudo-weak solution X_t^x to Equation* (1.1) *is absolutely continuous with respect to the law of $W_{x,A,\sigma}$ on $L^2([0,T];H)$. In particular, X^x has $\mathbb{P}$-a.s continuous sample paths in H. Moreover, there exists an increasing positive unbounded function $\Psi : [0,\infty) \to [0,\infty)$ such that the corresponding density ρ^x satisfies*

$$\mathbb{E}\rho^x \Psi\left(\rho^x\right) < \infty. \tag{2.8}$$

We prove Theorem 2.8 in Section 4 and Theorem 2.9 in Section 5, where we provide more detailed statements as well. These results are illustrated by Examples 4.2, 4.3, 4.4 and 5.8. Note that Theorem 2.10 addresses the absolute continuity of the laws which is a long-standing question that has been implicitly stated in a number of publications such as [27, 28].

3. Preliminaries: Pseudo-weak convergence and Yosida approximations

3.1. **Pseudo-weak convergence.** Let $(S, \mathcal{F}, \mu)$ be a σ-finite measure space and H a separable real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|\cdot|_H$. Let $L^2(S; H, \mu)$ denote the space of H-valued square-integrable functions on S. Below for $A \in \mathcal{F}$ we set

$$\mu_A := \mathbb{1}_A \mu.$$

Definition 3.1. Suppose $F, F_n \in L^0(S; H, \mu)$, i.e. $F, F_n : S \longrightarrow H$, $n \in \mathbb{N}$, are $\mathcal{F}$-measurable. We say that $\{F_n\}_{n=1}^{\infty}$ is *pseudo-weakly convergent* to F, denoted by

$$F_n \xrightharpoonup[n\to\infty]{\psi} F,$$

if there exists a strictly increasing continuous function $\psi_0 : \mathbb{R} \to \mathbb{R}$ such that $\psi_0(0) = 0$ and for $\psi : H \to H$ defined by

$$\psi(h) := \begin{cases} \frac{h}{|h|_H} \psi_0(|h|_H), & \text{if } h \neq 0, \\ 0, & \text{if } h = 0, \end{cases} \tag{3.1}$$

$$\psi(F_n) \xrightharpoonup[n\to\infty]{} \psi(F) \text{ in } L^2(S; H, \mu_A) \tag{3.2}$$

for any $A \in \mathcal{F}$ with $\mu(A) < \infty$, where "$\xrightharpoonup[n\to\infty]{}$" as usual denotes weak convergence in a Banach space. In this case we say that F is an *ψ-pseudo-weak limit* of the sequence $\{F_n\}_{n=1}^{\infty}$.

Remark 3.2. Typical examples for ψ_0 above are $\psi_0(r) = r$ or

$$\psi_0(r) = \frac{r}{1 + |r|}, \tag{3.3}$$

$r \in \mathbb{R}$. In particular weak convergence is the same as pseudo-weak convergence if $\psi_0(r) = r$. For us the most interesting case is when ψ is bounded, and most of the time we assume that (3.3) holds. When we say "pseudo-weak" we mean ψ-pseudo-weak for ψ_0 given by (3.3).

Proposition 3.3. *Suppose $F, F_n \in L^0(S; H, \mu)$, i.e. $F, F_n : S \longrightarrow H$, $n \in \mathbb{N}$, are $\mathcal{F}$-measurable. Then, for a bounded ψ we have that $F_n \xrightharpoonup[n\to\infty]{\psi} F$ if and only if*

$$\int_A \langle \psi(F_n) - \psi(F), h \rangle \, d\mu \xrightarrow[n\to\infty]{} 0 \tag{3.4}$$

for any $h \in H$ and any $A \in \mathcal{F}$ with $\mu(A) < \infty$.

Remark 3.4. Observe that the pseudo-weak limit is *unique*, that is, if

$$F_n \xrightharpoonup[n\to\infty]{\psi} F,$$
$$F_n \xrightharpoonup[n\to\infty]{\psi} G,$$

then $F = G$ μ-a.e.

Remark 3.5. In addition, $L^0(S; H, \mu)$-convergence, i.e. convergence in measure

$$\lim_{n\to\infty} \mu\left(\left\{|F_n - F|_H > \varepsilon\right\} \cap A\right) = 0 \quad \text{for all } \varepsilon > 0,\ A \in \mathcal{F},\ \mu(A) < \infty$$

implies pseudo-weak convergence, but these two types of convergence are not equivalent in general.

Proposition 3.6. *Suppose $F, F_n \in L^2(S; H, \mu)$, $n \in \mathbb{N}$, and*

$$F_n \xrightarrow[n\to\infty]{} F. \tag{3.5}$$

then

$$|F|_H \leqslant \limsup_{n\to\infty} |F_n|_H \quad \mu\text{-a.e.}$$

Proof. The assertion is an easy consequence of the Banach-Saks Theorem applied to the Hilbert space $L^2(S; H, \mu)$. Here, however, we include a more elementary proof based on Fatou's Lemma. Recall that H is assumed to be separable, therefore there exists a sequence $h_n \in H, n \in \mathbb{N}$ such that

$$\begin{aligned} |h_n|_H &= 1 \text{ for all } n \in \mathbb{N}, \\ |h|_H &= \sup_n \langle h_n, h\rangle \text{ for all } h \in H. \end{aligned}$$

Then for all non-negative $f \in L^\infty(S; \mathbb{R}, \mu)$ such that $\mu(\{f > 0\}) < \infty$, by Fatou's lemma and (3.5) for all $k \in \mathbb{N}$

$$\begin{aligned} \int_S \langle h_k, F(y)\rangle f(y)\, \mu(dy) &= \lim_{n\to\infty} \int_S \langle h_k, F_n(y)\rangle f(y)\, \mu(dy) \\ &\leqslant \int_S \limsup_{n\to\infty} |\langle h_k, F_n(y)\rangle| f(y)\, \mu(dy) \\ &\leqslant \int_S \limsup_{n\to\infty} |F_n(y)|_H f(y)\, \mu(dy). \end{aligned}$$

Then we have that for μ-a.e. $y \in S$ and all $k \in \mathbb{N}$

$$\langle h_k, F(y)\rangle \leqslant \limsup_{n\to\infty} |F_n(y)|_H$$

and the assertion follows. □

Corollary 3.7. *Let $F, F_n \in L^0(S; H, \mu)$, $n \in \mathbb{N}$, such that*

$$F_n \xrightarrow[n\to\infty]{\psi} F.$$

Then

$$|F|_H \leqslant \limsup_{n\to\infty} |F_n|_H \quad \mu\text{-a.e.}$$

Proof. Let $A \in \mathcal{F}$, $\mu(A) < \infty$ and ψ_0, ψ as in Definition 3.1. Then by Proposition 3.6 applied with μ_A replacing μ we have that on the set

$$\left\{F \neq 0, \limsup_{n\to\infty} |F_n|_H < \infty\right\}$$

we have μ_A-a.e.

$$0 < \psi_0(|F|_H) = |\psi(F)|_H \leqslant \limsup_{n\to\infty} |\psi(F_n)|_H = \limsup_{n\to\infty} \psi_0(|F_n|_H) \leqslant \psi_0\Big(\limsup_{n\to\infty} |F_n|_H\Big)$$

Applying the inverse of ψ_0 to both sides of this inequality and using that μ is σ-finite proves the desired result. □

Proposition 3.8. *If $F_n \in L^0(S; H, \mu)$, $n \in \mathbb{N}$, are such that*

$$\sup_{n \in \mathbb{N}} |F_n|_H < \infty \quad \mu - a.e.,$$

then there exists $F \in L^0(S; H, \mu)$ such that for some subsequence $\{n_k\}_{k \in \mathbb{N}}$

$$F_{n_k} \xrightarrow[k \to \infty]{\psi} F.$$

Proof. Let ψ_0 be any ψ_0 as in Definition 3.1 which is bounded and let $B_R(0)$ denote the open ball in H with center 0 and radius $R \in (0, \infty)$. Define $\psi^{-1} : B_{|\psi_0|_\infty} \longrightarrow H$, defined by

$$\psi^{-1}(h) := \begin{cases} \frac{h}{|h|_H} \psi_0^{-1}(|h|_H), & \text{if } h \neq 0, \\ 0, & \text{if } h = 0, \end{cases}$$

where ψ_0^{-1} is the inverse function of ψ_0. Then ψ^{-1} is easy to check to be the inverse map of ψ with ψ as in (3.1). Now let $A \in \mathcal{F}$, $\mu(A) < \infty$, and

$$V_n := \psi(F_n), \quad n \in \mathbb{N}.$$

Then $(V_n)_{n \in \mathbb{N}}$ is bounded in $L^2(S; H, \mu_A)$. Hence there exists $V \in L^2(S; H, \mu_A)$ such that for some subsequence $(n_k)_{k \in \mathbb{N}}$

$$V_{n_k} \xrightarrow[k \to \infty]{} V$$

in $L^2(S; H, \mu_A)$. Since a subsequence of the Cesaro mean of a subsequence of $\{V_{n_k}\}_{k \in \mathbb{N}}$ converges μ_A-a.e. to V and since μ is σ-finite, V does not depend on A. By Proposition 3.6 we have on $\{V \neq 0\}$

$$|V|_H \leqslant \limsup_{k \to \infty} |V_{n_k}|_H = \limsup_{k \to \infty} \psi_0(|F_{n_k}|_H) \leqslant \psi_0(\sup_{n \in \mathbb{N}} |F_n|_H), \ \mu_A\text{-a.e.},$$

hence μ-a.e. by the σ-finiteness of μ, and thus by assumption $V \in B_{|\psi_0|_\infty}(0)$ and

$$F := \psi^{-1}(V)$$

is well-defined. Obviously, by definition

$$F_{n_k} \xrightarrow[k \to \infty]{\psi} F.$$

□

3.2. **Yosida approximations to F and A.** Recall that to define pseudo-weak solutions in Definition 2.5, we used the Yosida approximation to F satisfying Assumption 2.3. While there are standard references for this approximation such as [1, 2, 4]), and in the setting similar to the one considered in this paper in [6, 7, 29], we include details for completeness: Fix $t \in [0, \infty)$ and set $F := F(t, \cdot)$. Then for any $\alpha > 0$ we define

$$F_\alpha := \frac{1}{\alpha}\left(J_\alpha(x) - x\right), x \in H, \tag{3.6}$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \ I(x) = x.$$

Then each F_α is single-valued, dissipative, Lipschitz continuous with Lipschitz constant less than $\frac{2}{\alpha}$ and satisfies

$$\lim_{\alpha\to 0} F_\alpha(x) = F_0(x), x \in D_F, \tag{3.7}$$

$$|F_\alpha(x)|_H \leqslant |F_0(x)|_H, x \in D_F. \tag{3.8}$$

It is clear from the last inequality that for each $x_0 \in D_F$

$$|F_\alpha(t,x)|_H \le |F_0(t,x_0)|_H + \frac{2}{\alpha}|x|_H \le a(|x_0|_H) + \frac{2}{\alpha}|x|_H, \quad x \in H. \tag{3.9}$$

In addition, we need the Yosida approximations A_λ to A for large enough λ, in particular, we will use the fact that such A_λ satisfy Assumption 2.1. Surprisingly, it is not easy to find a reference to this fact, so again we include it for completeness.

We start by recalling some standard facts about C_0-semigroups and their generators, most of this goes back to Hille and Yosida. We refer to [11, Chapter II] for most of the material below. Let $\rho(A)$ be the resolvent set, then the resolvent of A is defined as

$$\begin{aligned} &R_\lambda(A) := (\lambda I - A)^{-1}, \quad \lambda \in \rho(A) \in B(H), \\ &R_\lambda(A) : H \longrightarrow D_A. \end{aligned}$$

Recall that for $\lambda > 0$ we have $\|R_\lambda(A)\| \leqslant 1/\lambda$. In addition,

$$\lambda R_\lambda(A)\, x \xrightarrow[\lambda\to\infty]{} x, \quad x \in H. \tag{3.10}$$

Finally the Yosida approximations to A are defined by

$$A_\lambda x := \lambda A R_\lambda(A)\, x, x \in H. \tag{3.11}$$

Since (A, D_A) as a generator of a contractive C_0-semigroup is m-dissipative, A_λ is a special case of F_α in (3.6), more presicely $A_\lambda = F_{\frac{1}{\lambda}}$. The Yosida approximations to A satisfy the following properties.

$$\begin{aligned} &A_\lambda \in B(H), \\ &\lambda A R_\lambda(A)\, x = \lambda R_\lambda(A)\, A x, \quad x \in D_A, \\ &A_\lambda x = -\lambda (I - \lambda R_\lambda(A))\, x, \quad x \in D_A, \\ &A_\lambda x \xrightarrow[\lambda\to\infty]{} A x, \quad x \in D_A. \end{aligned} \tag{3.12}$$

We include some of the proofs of these properties, as we use them to show stability of Assumption 2.1 under Yosida approximations. For example, to show (3.12) we can use that for any $x \in H$ we have

$$x = (\lambda I - A)\, R_\lambda(A)\, x, \tag{3.13}$$

therefore for all $x \in H$

$$\begin{aligned} &-\lambda (I - \lambda R_\lambda(A))\, x = -\lambda ((\lambda I - A)\, R_\lambda(A)\, x - \lambda R_\lambda(A)\, x) = \\ &-\lambda ((\lambda R_\lambda(A)\, x - A R_\lambda(A)\, x) - \lambda R_\lambda(A)\, x) = \lambda A R_\lambda(A)\, x = A_\lambda x. \end{aligned}$$

Thus for all $x \in H$ by (3.12)

$$
\begin{aligned}
\langle A_\lambda x, x\rangle &= \langle A_\lambda x, x-\lambda R_\lambda(A)x\rangle + \langle A_\lambda x, \lambda R_\lambda(A)x\rangle \\
&= -\frac{1}{\lambda}\langle A_\lambda x, A_\lambda x\rangle + \langle A_\lambda x, \lambda R_\lambda(A)x\rangle \\
&= -\frac{1}{\lambda}|A_\lambda x|_H^2 + \lambda^2\langle AR_\lambda(A)x, R_\lambda(A)x\rangle \\
&\leqslant \lambda^2\langle AR_\lambda(A)x, R_\lambda(A)x\rangle \leqslant -\lambda^2\beta|R_\lambda(A)x|_H^2.
\end{aligned}
\tag{3.14}
$$

Now we can use (3.10) to see that for $\lambda \geq 1$ the Yosida approximations A_λ satisfy Assumption 2.1.

4. Almost sure φ-type estimates of solutions X_t

First we recall that

$$
W_{x,A,\sigma}(t) = e^{tA}x + \int_0^t e^{(t-s)A}\sigma dW(s), t \geqslant 0. \tag{4.1}
$$

Moreover,

$$
W_{0,A,\sigma}(t) = W_{x,A,\sigma}(t) - e^{tA}x
$$

is a Gaussian random variable with values in H with mean 0 and its covariance operator Q_t given by

$$
Q_t x = \int_0^t e^{sA}\sigma^2 e^{sA^*}x ds.
$$

We will use the following notation for the maximum process

$$
W^*_{x,A,\sigma}(t) := \sup_{s\in[0,t]} |W_{x,A,\sigma}(s)|_H. \tag{4.2}
$$

Since for $T>0$ the law of $W_{0,A,\sigma}$ is a Gaussian (mean zero) measure on $C([0,T],H)$ (see e.g. [20, Proposition I.0.7]) by Fernique's Theorem there is a (small) $\gamma>0$ such that

$$
\mathbb{E}\left(e^{\gamma W^*_{0,A,\sigma}(T)}\right) < \infty. \tag{4.3}
$$

Next we establish some properties of the functions in $\mathcal{M}$ depending on the value of L_φ. We shall see that functions in $\mathcal{M}$ satisfy the standard condition in the de la Vallée-Poussin Theorem. We also find sharp constants that might be useful for finding φ-moments depending on the growth of F_0 (as measured by the function a).

Lemma 4.1. *Let $\varphi \in \mathcal{M}$. Then:*

(i) *For any $c>0$, $\beta>0$ and any $0<B<\beta L_\varphi$ there is a constant $C \geqslant 0$ such that*

$$
\varphi(u)\left[\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u}-\beta u\right)+B\right] \leqslant C, \text{ for all } u \in (0,\infty).
$$

The constant C can be chosen as follows.

$$
\begin{aligned}
C(c,\beta,B) := \max_{u\in[0,\infty)}\left(\varphi'(u)\left(c\sqrt{u}-\beta u\right)+B\varphi(u)\right) = \\
\max_{u\in[0,u_0]}\left(\varphi'(u)\left(c\sqrt{u}-\beta u\right)+B\varphi(u)\right),
\end{aligned}
\tag{4.4}
$$

where $u_0 := \max\left\{\frac{c^2}{\beta^2}, \frac{c^2}{4(\beta-B)^2}\right\}$. *In particular, for* $B = \frac{\beta}{2}$

$$C\left(c, \beta, \frac{\beta}{2}\right) = \frac{\beta}{2}\varphi\left(\frac{c^2}{\beta^2}\right).$$

(ii) *If* $L_\varphi > 1$, *then*

$$\frac{\varphi(u)}{u} \xrightarrow[u\to\infty]{} \infty.$$

Proof (ii). First, observe

$$\left(\frac{\varphi(u)}{u}\right)' = \left(\frac{u\varphi'(u)}{\varphi(u)} - 1\right) \cdot \frac{\varphi(u)}{u^2}. \tag{4.5}$$

Since $L_\varphi > 1$, we see that there exists $K > 0$ such that

$$\left(\frac{\varphi(u)}{u}\right)' > \frac{K}{u}\frac{\varphi(u)}{u} > 0$$

for all large enough u.

Define $H : (0, \infty) \longrightarrow (0, \infty)$ by $H(u) := \frac{\varphi(u)}{u}$. Then

$$\frac{H'(u)}{H(u)} > \frac{K}{u}$$

for all large enough u. Then for some $M > 0$

$$H(u) = \frac{\varphi(u)}{u} > Mu^K \text{ for all large enough } u,$$

which implies that $\frac{\varphi(u)}{u} \xrightarrow[u\to\infty]{} \infty$.

(i): It is enough to check that for $B \in (0, \beta L_\varphi)$

$$\varphi(u)\left[\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u} - \beta u\right) + B\right] \xrightarrow[u\to\infty]{} -\infty,$$

and so there is a $u_0 > 0$ such that

$$\varphi'(u)\left(c\sqrt{u} - \beta u\right) + B\varphi(u) < 0 \text{ for all } u > u_0.$$

Then we can choose

$$C := \max_{u\in[0,u_0]}\left(\varphi'(u)\left(c\sqrt{u} - \beta u\right) + B\varphi(u)\right) (> 0). \tag{4.6}$$

Observe that

$$\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u} - \beta u\right) = \frac{u\varphi'(u)}{\varphi(u)}\left(\frac{c}{\sqrt{u}} - \beta\right) \xrightarrow[u\to\infty]{} -\beta L_\varphi \quad \left(:= -\infty, \text{ if } L_\varphi = \infty\right),$$

and so

$$\varphi(u)\left[\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u} - \beta u\right) + B\right] \xrightarrow[u\to\infty]{} -\infty.$$

Recall that we can take C to be the maximum of the following function

$$f(u) := \varphi'(u)\left(c\sqrt{u} - \beta u\right) + B\varphi(u).$$

First we take the derivative of this function

$$f'(u) = \varphi''(u)\left(c\sqrt{u} - \beta u\right) + \varphi'(u)\left(\frac{c}{2\sqrt{u}} - \beta\right) + B\varphi'(u) =$$
$$\varphi''(u)\sqrt{u}\left(c - \beta\sqrt{u}\right) + \varphi'(u)\left(\frac{c}{2\sqrt{u}} - (\beta - B)\right).$$

By assumption φ is an increasing convex function, and therefore φ'' and φ' are non-negative, so, since $\beta - B > 0$, $f'(u) \leqslant 0$ for any $u \geqslant u_0 = \max\left\{\frac{c^2}{\beta^2}, \frac{c^2}{4(\beta-B)^2}\right\}$. Therefore we can choose

$$C(c,\beta,B) = \max_{u\in[0,\infty)} f(u) = \max_{u\in[0,u_0]} f(u).$$

Finally, if $B = \beta/2$, then $u_0 = \frac{c^2}{\beta^2}$, and $f'(u) \geq 0$ on $[0, u_0]$, so

$$C(c,\beta,\beta/2) = f(u_0) = \frac{\beta}{2}\varphi\left(\frac{c^2}{\beta^2}\right).$$

□

Example 4.2. *Suppose $\varphi(u) = u^p, p \geqslant 1$, then $\varphi \in \mathcal{M}$. In this case $L_\varphi = p$.*
To see how we can find C in (4.6), observe that for any $0 < B < p\beta$

$$f(u) := \varphi(u)\left[\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u} - \beta u\right) + B\right] =$$
$$cpu^{p-1/2} + (B - p\beta)\,u^p,$$

for which

$$f'(u) = cp\left(p - \frac{1}{2}\right)u^{p-3/2} + (B - p\beta)\,pu^{p-1} =$$
$$pu^{p-3/2}\left(c\left(p - \frac{1}{2}\right) - (p\beta - B)\sqrt{u}\right).$$

Then the maximum of f is attained at $u_0 = \left(\frac{c\left(p-\frac{1}{2}\right)}{p\beta-B}\right)^2$. Therefore

$$C_p(c,\beta,B) := \frac{c}{2}\left(\frac{c\left(p-\frac{1}{2}\right)}{p\beta - B}\right)^{2p-1} = \frac{c^{2p}}{2}\left(\frac{\left(p-\frac{1}{2}\right)}{p\beta - B}\right)^{2p-1}.$$

In this example $L_\varphi = p$, and so by Lemma 4.1 for any $0 < B < \beta$ we can choose

$$C_1(c,\beta,B) := \frac{c^2}{4(\beta - B)}.$$

Example 4.3. *Suppose $\varphi(u) = e^u$, then $\varphi \in \mathcal{M}$. In this case $L_\varphi = \infty$, so we can take any positive constant B. For example, if $B = \beta/2$, then for*

$$f(u) := e^u\left[c\sqrt{u} - \beta u + B\right] = e^u\left[c\sqrt{u} - \beta u + \frac{\beta}{2}\right]$$

we have

$$f'(u) = e^{u}\left[c\sqrt{u} + \frac{c}{2\sqrt{u}} - \beta u - \frac{\beta}{2}\right] = e^{u}\left(\frac{c}{\sqrt{u}} - \beta\right)\left(u + \frac{1}{2}\right)$$

and we can take

$$C = f\left(\frac{c^2}{\beta^2}\right) = \frac{\beta}{2} e^{\frac{c^2}{\beta^2}}.$$

Example 4.4. *Suppose $\varphi(u) = u\ln(u+1)$, then $\varphi \in \mathcal{M}$. In this case $L_\varphi = 1$, so we can take any $0 < B < \beta$ and then C can be chosen by finding the maximum of the function*

$$\begin{aligned}&f(u) := \varphi(u)\left[\frac{\varphi'(u)}{\varphi(u)}\left(c\sqrt{u} - \beta u\right) + B\right] = \\ &\ln(u+1)\left(c\sqrt{u} - (\beta - B)u\right) + \frac{u}{u+1}\left(c\sqrt{u} - \beta u\right).\end{aligned}$$

Note that for $u > \left(\frac{c}{\beta - B}\right)^2$ the function $f(u)$ is negative. Therefore it is enough to find the maximum of f on $\left(0, \left(\frac{c}{\beta-B}\right)^2\right)$. We will use a rough estimate for $u \in \left(0, \left(\frac{c}{\beta-B}\right)^2\right)$

$$\begin{aligned}&\ln(u+1)\left(c\sqrt{u} - (\beta - B)u\right) + \frac{u}{u+1}\left(c\sqrt{u} - \beta u\right) \leqslant \\ &\frac{c^2}{4(\beta - B)}\ln(u+1) + \frac{c^2}{4\beta}\frac{u}{u+1} \leqslant \\ &\frac{c^2}{4(\beta - B)}u + \frac{c^2}{4\beta} \leqslant \frac{c^2}{4(\beta - B)}\left(\frac{c}{\beta - B}\right)^2 + \frac{c^2}{4\beta}.\end{aligned}$$

Thus we can take

$$C(c, \beta, B) := \frac{c^2}{4}\left(\frac{c^2}{(\beta - B)^3} + \frac{1}{\beta}\right).$$

Notation 4.5. *For any $\varphi \in \mathcal{M}$ and for all $t > 0$ we denote the following random functions by*

$$\begin{aligned}&K_{\varphi,\beta,a}(t) := \varphi\left(\frac{2\left[a\left(W^*_{0,A,\sigma}(t)\right)\right]^2}{\beta^2}\right), \\ &K_\varphi(t) := \varphi\left(2\left|W_{0,A,\sigma}(t)\right|_H^2\right)\end{aligned}$$

Note that these functions are finite a.s.

Remark 4.6. We will make use of the following elementary inequalities: for any $a, b \geqslant 0$, and $p \geqslant 1$

$$(a+b)^p \leqslant 2^{p-1}\left(a^p + b^p\right), \tag{4.7}$$

$$e^{(a+b)^2} \leqslant \frac{e^{4a^2}}{2} + \frac{e^{4b^2}}{2}.$$

We are now in position to prove pathwise estimates in Theorem 2.8.

Proof of Theorem 2.8. Suppose $\varphi \in \mathcal{M}$ and let for $\lambda \geq 1$, $\alpha > 0$, $Z^x_{\lambda,\alpha,t}$ be the (continuous in t) solution to

$$\begin{aligned} dZ^x_{\lambda,\alpha,t} &= (A_\lambda Z^x_{\lambda,\alpha,t} + F_\alpha(t, Z^x_{\lambda,\alpha,t} + W_{0,A,\sigma}(t))dt, \\ Z^x_{\lambda,\alpha,0} &= x, \end{aligned} \tag{4.8}$$

(which exists and is unique since the coefficients on the right hand side of (4.8) are Lipschitz). Then for Lebesgue-a.e. $t > 0$

$$\begin{aligned} &\frac{d}{dt}\varphi\left(|Z^x_{\lambda,\alpha,t}|^2_H\right) \\ &\quad = 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle \left(Z^x_{\lambda,\alpha,t}\right)', Z^x_{\lambda,\alpha,t}\rangle \\ &\quad = 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle A_\lambda Z^x_{\lambda,\alpha,t} + F_\alpha\left(t, Z^x_{\lambda,\alpha,t} + W_{0,A,\sigma}(t)\right), Z^x_{\lambda,\alpha,t}\rangle \\ &\quad = 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle A_\lambda Z^x_{\lambda,\alpha,t} + \left(F_\alpha\left(t, Z^x_{\lambda,\alpha,t} + W_{0,A_\alpha,\sigma}(t)\right) - F_\alpha\left(t, W_{0,A,\sigma}(t)\right)\right), Z^x_{\lambda,\alpha,t}\rangle \\ &\qquad + 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle F_\alpha\left(t, W_{0,A,\sigma}(t)\right), Z^x_{\lambda,\alpha,t}\rangle \\ &\quad \leqslant 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle A_\lambda Z^x_{\lambda,\alpha,t}, Z^x_{\lambda,\alpha,t}\rangle + 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\langle F_\alpha\left(t, W_{0,A,\sigma}(t)\right), Z^x_{\lambda,\alpha,t}\rangle \\ &\quad \leqslant 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\left(-\beta|Z^x_{\lambda,\alpha,t}|^2_H + |Z^x_{\lambda,\alpha,t}|_H\,|F_\alpha\left(t, W_{0,A,\sigma}(t)\right)|_H\right) \\ &\quad \leqslant 2\varphi'\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\left(a\left(|W_{0,A,\sigma}(t)|_H\right)|Z^x_{\lambda,\alpha,t}|_H - \beta|Z^x_{\lambda,\alpha,t}|^2_H\right), \end{aligned}$$

where we used (3.14), (3.8), Assumption 2.3 and (2.2). By Lemma 4.1 taking $B = \beta/2$ and

$$C := C\left(a\left(|W_{0,A,\sigma}(t)|_H\right), \beta, \frac{\beta}{2}\right) = \frac{\beta}{2}\varphi\left(\frac{\left[a\left(|W_{0,A,\sigma}(t)|_H\right)\right]^2}{\beta^2}\right),$$

we obtain for all $u \in (0, \infty)$

$$\varphi'(u)\left(a\left(|W_{0,A,\sigma}(t)|_H\right)\sqrt{u} - \beta u\right) \leqslant C - \frac{\beta}{2}\varphi(u),$$

therefore

$$\frac{d}{dt}\varphi\left(|Z^x_{\lambda,\alpha,t}|^2_H\right) \leqslant \beta\left(\varphi\left(\frac{\left[a\left(|W_{0,A,\sigma}(t)|_H\right)\right]^2}{\beta^2}\right) - \varphi\left(|Z^x_{\lambda,\alpha,t}|^2_H\right)\right).$$

Now by Gronwall's inequality we see that for all $t \geqslant 0$

$$\varphi\left(|Z^x_{\lambda,\alpha,t}|^2_H\right) \leqslant \varphi\left(|x|^2_H\right)e^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)}\varphi\left(\frac{\left[a\left(|W_{0,A,\sigma}(s)|_H\right)\right]^2}{\beta^2}\right)ds. \tag{4.9}$$

It is well-known and easy to show that $Z^x_{\lambda,\alpha,t} \longrightarrow Z^x_{\alpha,t}$ locally uniformly in $t \in [0, \infty)$ (for $\mathbb{P}$-a.e. $\omega \in \Omega$). So, since φ is continuous (4.9) holds for $Z^x_{\alpha,t}$ replacing $Z^x_{\lambda,\alpha,t}$.

Now we can use (4.7) and the fact that φ is convex to see that for the solution $X^x_{\alpha,t}$ to (2.4) we have (replacing $\varphi(\cdot)$ by $\varphi(2\,\cdot)$ which is again in the space $\mathcal{M}$)

$$\begin{aligned}
&\varphi\left(|X^x_{\alpha,t}|^2_H\right) \\
&\leqslant \frac{1}{2}\varphi\left(2|Z^x_{\alpha,t}|^2_H\right) + \frac{1}{2}\varphi\left(2|W_{0,A,\sigma}(t)|^2_H\right) \\
&\leqslant \frac{1}{2}\varphi\left(2|x|^2_H\right)e^{-\beta t} + \frac{\beta}{2}\int_0^t e^{-\beta(t-s)}\varphi\left(\frac{2\left[a\left(|W_{0,A,\sigma}(s)|_H\right)\right]^2}{\beta^2}\right)ds + \frac{1}{2}\varphi\left(2|W_{0,A,\sigma}(t)|^2_H\right) \\
&\leqslant \frac{1}{2}\varphi\left(2|x|^2_H\right)e^{-\beta t} + \frac{1}{2}\varphi\left(2|W_{0,A,\sigma}(t)|^2_H\right) + \varphi\left(\frac{2\left[a\left(W^*_{0,A,\sigma}(t)\right)\right]^2}{\beta^2}\right)\frac{\beta}{2}\int_0^t e^{-\beta(t-s)}ds \\
&\leqslant \frac{e^{-\beta t}}{2}\varphi\left(2|x|^2_H\right) + \frac{1}{2}\varphi\left(2|W_{0,A,\sigma}(t)|^2_H\right) + \frac{\beta t}{2}\varphi\left(\frac{2\left[a\left(W^*_{0,A,\sigma}(t)\right)\right]^2}{\beta^2}\right).
\end{aligned}$$

Now we apply φ^{-1} to the above inequality and use Corollary 3.7 to pass to the limit $\alpha \to 0$ along a subsequence. Subsequently, we apply φ to the resulting inequality to obtain (2.5). □

5. Uniform integrability of Girsanov densities

5.1. **Further *a priori* pathwise estimates of X_t.** Below we prove more bounds on X_t which in particular imply Theorem 2.9. Thus we work in the setting of Theorem 2.9, and in particular we assume that Assumptions 2.1–2.4 hold.

Proposition 5.1. *Let $Z^x_{\alpha,t}$ be solutions to the regularized equations* (2.3)*. Suppose Z^x_t is a pseudo weak limit point of $Z^x_{\alpha,t}, \alpha \to 0$. Then almost surely for all $\alpha > 0$*

$$|Z^x_{\alpha,t}|_H \leqslant |x|_H e^{-\beta t} + \frac{1}{2}\int_0^t e^{-\beta(t-s)}a\left(|W_{0,A,\sigma}(s)|_H\right)ds =: Z^{*,x}_t \tag{5.1}$$

for all $t \geqslant 0$ and thus

$$|Z^x_t|_H \leqslant Z^{*,x}_t. \tag{5.2}$$

Proof. One of the observations in the proof of Theorem 2.8 was that (4.9) holds for $Z^x_{\lambda,\alpha,t}$ instead of $Z^x_{\alpha,t}$. Now we can take φ to be the identity map and apply [10, Theorem 5] to obtain (5.1). Equation (5.2) then follows by Corollary 3.7. □

Proof of Theorem 2.9. The assertion follows from (5.1) and Proposition 3.8. □

5.2. **Stopping times and Girsanov transforms.** Recall that by Definition 2.5 of pseudo-weak solutions we have

$$\begin{aligned}
X^x_t &= Z^x_t + W_{0,A,\sigma}(t) \\
&= \left(Z^x_t - e^{At}x\right) + \left(W_{0,A,\sigma}(t) + e^{At}x\right) =: Z^{0,x}_t + W_{x,A,\sigma}(t).
\end{aligned} \tag{5.3}$$

From this and (5.1) we have

$$|Z^{0,x}_t|_H \leqslant Z^{*,x}_t + |e^{At}x|_H. \tag{5.4}$$

Fix $T > 0$ and define stopping times by

$$\tau^x_n := \inf\{t \in [0,T] : Z^{*,x}_t + |e^{At}x|_H + |W_{x,A,\sigma}(t)|_H \geqslant n\} \wedge T, \tag{5.5}$$

where $W_{x,A,\sigma}$ is defined by (4.1). Note that the definition of τ_n^x does not depend on α and that $\mathbb{P}$-almost surely

$$\lim_{n\to\infty} \tau_n^x = T. \tag{5.6}$$

The estimates given above imply the following lemma, which is used in the proof of Theorem 2.10 in Section 5.3. By (5.3) and (5.1) we have

Lemma 5.2. *$\mathbb{P}$-almost surely, if $t \in (0, \tau_n^x]$ then $|X_t^x|_H \leqslant n$ and $|X_{t,\alpha}^x|_H \leqslant n$ for all $\alpha > 0$.*

Now we consider Girsanov transforms for the Yosida regularized equations as follows. For $x \in H$ let

$$\rho_\alpha(x,t) := \exp(\zeta_\alpha(x,t)), \tag{5.7}$$

where

$$\begin{aligned}\zeta_\alpha(x,t) = &\int_0^t \langle \sigma^{-1} F_\alpha(s, W_{x,A,\sigma}(s)), dW(s)\rangle \\ &- \frac{1}{2}\int_0^t |\sigma^{-1} F_\alpha(s, W_{x,A,\sigma}(s))|_H^2 ds.\end{aligned} \tag{5.8}$$

We define the measure $\mathbb{P}_\alpha^x$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ by

$$\frac{d\mathbb{P}_\alpha^x}{d\mathbb{P}} = \rho_\alpha(x,T) =: \rho_\alpha^x, \tag{5.9}$$

and we denote by $\mathbb{E}_\alpha^x$ the expectation with respect to the probability measure $\mathbb{P}_\alpha^x$ given by (5.9). Note that this gives a probabilistically weak mild solution to (2.4) according to [20, Appendix I]. More precisely, we define

$$\widetilde{X}_x(t) := W_{x,A,\sigma}(t). \tag{5.10}$$

Note that this process does not depend on α although the measure $\mathbb{P}_\alpha^x$ does depend on α which is important in (5.12) below, and

$$\widetilde{W}_{x,\alpha}(t) = W_t - \int_0^t \sigma^{-1} F_\alpha(s, W_{x,A,\sigma}(s))\, ds. \tag{5.11}$$

Then $\widetilde{W}_{x,\alpha}$ is a cylindrical Wiener process under $\mathbb{P}_\alpha^x$ and

$$\begin{aligned} d\widetilde{X}_x(t) &= dW_{x,A,\sigma}(t) = A\widetilde{X}_x dt + \sigma dW_t \\ &= A\widetilde{X}_x dt + F_\alpha(t, \widetilde{X}_x(t))dt + \sigma d\widetilde{W}_{x,\alpha}(t), \end{aligned} \tag{5.12}$$

in the mild sense.

Remark 5.3 (On localization)**.** As a side remark we would like to mention that in infinite dimensions the processes in (5.12) are not semimartingales in general (unlike in [22]), one might want to use localization to introduce

$$\widetilde{W}_{x,\alpha}^n(t) = W_t - \int_0^{t\wedge\tau_n} \sigma^{-1} F_\alpha(s, W_{x,A,\sigma}(s))ds, \tag{5.13}$$

$$\rho_\alpha^n(x,t) = \exp(\zeta_\alpha(x, t\wedge\tau_n^x)). \tag{5.14}$$

Then we can define $\rho_\alpha(x,t)$ as a limit as $n \to \infty$, if the limit exists. However, the localization can not be used easily for the equations with non-smooth coefficients because interchanging the limits as $n \to \infty$ and $\alpha \to 0$ may be problematic. We use stopping times in a different way in (5.21).

5.3. **Estimates of the Girsanov densities.**

Proof of Theorem 2.10. In this proof we assume that $r, T > 0$ are fixed, and $|x| < r$, $t \in [0, T]$. Therefore we abuse notation, and drop dependence on r, T although our estimates do depend on r, T.

By (5.9) we have for all Borel-measurable $\Psi : [0, \infty) \to [0, \infty)$

$$\mathbb{E}\rho_\alpha^x \Psi\left(\rho_\alpha^x\right) = \mathbb{E}_\alpha^x \Psi\left(\rho_\alpha^x\right), \tag{5.15}$$

where ρ_α^x is the density defined by (5.9). Note that by [23] the distribution of $(\widetilde{W}_{x,\alpha}, \widetilde{X}_x)$ under the measure $\mathbb{P}_\alpha^x$ is the same as the distribution of (W, X_α^x) under the measure $\mathbb{P}$, see Subsection 5.2. In particular, by Assumption 2.3

$$X_\alpha^x \in D_F \quad dt \times \mathbb{P} - \text{a.s.} \tag{5.16}$$

Recall that $(\widetilde{W}_{x,\alpha}, \widetilde{X}_x) = (\widetilde{W}_{x,\alpha}, W_{x,A,\sigma})$. Then

$$\mathbb{E}_\alpha^x \Psi\left(\rho_\alpha^x\right) = \mathbb{E}\Psi\left(\widetilde{\rho_\alpha^x}\right), \tag{5.17}$$

where

$$\widetilde{\rho_\alpha^x} := \exp\left(\int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha^x(s)), dW(s)\rangle + \frac{1}{2}\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2\, ds\right). \tag{5.18}$$

We can estimate $\mathbb{E}\mathbb{1}_A|\widetilde{\rho_\alpha^x}|^p$ for $A \in \mathcal{F}$ as follows.

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_A\left|\widetilde{\rho_\alpha}\right|^p = \\
&\mathbb{E}\mathbb{1}_A \exp\left(p\int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha^x(s)), dW_s\rangle - p^2\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2 ds\right) \\
&\times \exp\left(\left(p^2 + \frac{p}{2}\right)\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2 ds\right) \\
&\leqslant \left(\mathbb{E}\exp\left(2p\int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha^x(s)), dW_s\rangle - 2p^2\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2 ds\right)\right)^{1/2} \\
&\times \left(\mathbb{E}\mathbb{1}_A \exp\left(\left(2p^2 + p\right)\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2 ds\right)\right)^{1/2}.
\end{aligned}$$

Note that the first term in the last formula is equal to the expectation of the stochastic exponential for the martingale $2p\int_0^t \langle \sigma^{-1} F_\alpha(s, X_\alpha^x(s)), dW_s\rangle$, and so its expectation is 1. Therefore,

$$\mathbb{E}\mathbb{1}_A\left|\widetilde{\rho_\alpha^x}\right|^p \leqslant \left(\mathbb{E}\mathbb{1}_A \exp\left(\left(2p^2 + p\right)\int_0^t |\sigma^{-1} F_\alpha(s, X_\alpha^x(s))|_H^2 ds\right)\right)^{1/2}. \tag{5.19}$$

Now we take $p = 2$ and use Equation (5.16), Proposition 5.1 and Lemma 5.2, to deduce that

$$\mathbb{E}\left|\widetilde{\rho_\alpha^x}\right|^2 \mathbb{1}_{(\tau_n^x \geqslant t)} \leqslant \exp\left(5\left(a(n)\|\sigma^{-1}\|\right)^2 T\right). \tag{5.20}$$

Then by (5.19) and Chebyshev's inequality we have

$$\mathbb{P}\left(\widetilde{\rho_\alpha^x} > y\right) \leqslant \mathbb{P}\left(\widetilde{\rho_\alpha^x} > y, \tau_n^x \geqslant t\right) + \mathbb{P}\left(\tau_n^x < t\right)$$

$$\leqslant \exp\left(5\left(a(n)\|\sigma^{-1}\|\right)^2 T\right)/y^2 + \mathbb{P}\left(\tau_n^x < T\right). \tag{5.21}$$

For $y \in \left(\exp(5(a(0)\|\sigma^{-1}\|)^2 T), \infty\right)$ we define $n(y)$ as the maximal natural number such that

$$\exp\left(5\left(a(n(y))\|\sigma^{-1}\|\right)^2 T\right) < y$$

or

$$a(n(y)) < \frac{1}{\|\sigma^{-1}\|}\left(\frac{\log(y)}{5T}\right)^{1/2}. \tag{5.22}$$

Since we assume that $a(n)$ is increasing and unbounded, we have that $n(y)$ is increasing in y and

$$\lim_{y\to\infty} n(y) = \infty$$

Then define the decreasing function

$$p(y) := p(y, \sigma, A, a, x, T) := \min\left\{1,\ 1/y + \mathbb{P}\left(\tau_{n(y)}^x < T\right)\right\},\ y \in (0,\infty). \tag{5.23}$$

Observe that by (5.6)

$$\lim_{y\to\infty} p(y, \sigma, A, a, x, T) = 0.$$

Let $\Psi : [0,\infty) \to [0,\infty)$ be increasing and C^1 on $(0,\infty)$ with $\Psi(0) = 0$. Then obviously

$$\mathbb{E}\Psi(Y) = \mathbb{E}\int_0^\infty \Psi'(y)\mathbb{1}_{\{0<y<Y\}}dy = \int_0^\infty \Psi'(y)\mathbb{P}\{Y > y\}dy \tag{5.24}$$

for any non-negative random variable Y. We use this formula for $Y = \widetilde{\rho}_\alpha^x$ to obtain by (5.21), (5.23) that for all $a > 0$

$$\mathbb{E}\Psi(\widetilde{\rho}_\alpha^x) \leqslant \int_0^\infty \Psi'(y)p(y)dy. \tag{5.25}$$

So, by (5.15) and (5.17) this implies uniform $L^1(\Omega, \mathbb{P})$-integrability of ρ_α^x, $\alpha > 0$, if we can find Ψ as above with the following two properties

$$\int_0^\infty \Psi'(y)p(y)dy < \infty \tag{5.26}$$

and

$$\lim_{y\to\infty} \Psi(y) = \infty. \tag{5.27}$$

The existence of such a Ψ can be seen as follows: since $y \mapsto p(y)$ decreases to zero as $y \to \infty$, we can find a sequence $(y_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ such that

$$y_k + 3 < y_{k+1},\ k \in \mathbb{N},$$

and $p(y) \leqslant \frac{1}{k^2}$ for $y \geqslant y_k$. Now define $g : [0, \infty) \to [0, \infty)$ by

$$g = \sum_{k=1}^{\infty} \Psi_k,$$

where $\Psi_k \in C^\infty((0, \infty))$ such that

$$\mathbb{1}_{[y_k+1, y_k+2]} \leqslant \Psi_k \leqslant \mathbb{1}_{[y_k, y_k+3]}.$$

Define

$$\Psi(y) = \int_0^y g(s)ds, \; y \geqslant 0.$$

Then $\Psi : [0, \infty) \to [0, \infty)$ is continuous, increasing and C^∞ on $(0, \infty)$. Furthermore, obviously (5.26) and (5.27) hold. Another construction of the function Ψ is given in Proposition 5.4.

Now let $X^x \in L^2(\Omega; L^2([0, T]; H))$ $(= L^2([0, T] \times \Omega; H))$ be the pseudo-weak limit of $X^x_{\alpha_n}$ as $\alpha_n \xrightarrow[n\to\infty]{} 0$, with corresponding function $\psi : H \to H$ as in (3.1). By the above and the Dunford–Pettis theorem, selecting another subsequence if necessary we have

$$\rho^x_{\alpha_n} \xrightarrow[n\to\infty]{} \rho^x \text{ in } L^1(\Omega; \mathbb{P})$$

and $\mathbb{P}$-a.s. for some $\rho^x \in L^1(\Omega, \mathbb{P})$.

Let $\mathcal{X} := L^2([0, T]; H)$ and let $G : \mathcal{X} \to \mathbb{R}$ be bounded and sequentially weakly continuous. Then for $\mathbb{Q}_x := \mathbb{P} \circ W^{-1}_{x,A,\sigma}$

$$\begin{aligned}
\int_{\mathcal{X}} G \circ \psi \; d\big(\mathbb{P} \circ (X^x)^{-1}\big) &= \int_\Omega G(\psi(X^x)) \; d\mathbb{P} = \lim_{n\to\infty} \int_\Omega G(\psi(X^x_{\alpha_n})) \; d\mathbb{P} \\
&= \lim_{n\to\infty} \int_\Omega (G \circ \psi)(W_{x,A,\sigma}) \rho^x_{\alpha_n} d\mathbb{P} = \int_\Omega (G \circ \psi)(W_{x,A,\sigma}) \rho^x \; d\mathbb{P} \qquad (5.28) \\
&= \int_\Omega (G \circ \psi)(W_{x,A,\sigma}) \, \mathbb{E}_\mathbb{P}[\rho^x \,|\, W_{x,A,\sigma}] \; d\mathbb{P} = \int_{\mathcal{X}} G \circ \psi \; \overline{\rho}^x \; d\mathbb{Q}_x,
\end{aligned}$$

where

$$\overline{\rho}^x := \mathbb{E}_\mathbb{P}[\rho^x \,|\, W_{x,A,\sigma} = \cdot].$$

Let $\mathcal{M}$ denote the set of all such functions $G \circ \psi : H \longrightarrow \mathbb{R}$ from above. Since ψ in (3.1) is one-to-one, we can find a countable set $\mathcal{M}_0 \subset \mathcal{M}$, which separates the points in $\mathcal{X}$. Indeed, let $\{e_i, i \in \mathbb{N}\}$ and $\{g_i, i \in \mathbb{N}\}$ be orthonormal bases of H and $L^2([0, T]; \mathbb{R}, dt)$ respectively. Define maps $G_{ij} : \mathcal{X} \to \mathbb{R}$, $i, j \in \mathbb{N}$,

$$G_{ij}(w) := \int_0^T g_i(t) \langle e_j, w(t) \rangle_H dt, \; w \in \mathcal{X}.$$

Then obviously $\{G_{ij}, i, j \in \mathbb{N}\}$ separates the points of $\mathcal{X}$ and hence so does $\mathcal{M}_0 := \{(N \wedge G_{ij} \vee (-N)) \circ \psi, i, j, N \in \mathbb{N}\}$. Clearly, each $N \wedge G_{ij} \vee (-N)$ is weakly continuous, so $\mathcal{M}_0 \subset \mathcal{M}$. Furthermore, obviously $\mathcal{M}$ is closed under multiplication and consists of bounded Borel measurable functions on $\mathcal{X}$. Therefore, (5.28) implies that

$$\mathbb{P} \circ (X^x)^{-1} = \overline{\rho}^x \mathbb{Q}_x.$$

Note that by Kuratowski's Theorem (e.g. [**?**, Section I.3]) $C([0,T];H)$ is a Borel subset of $\mathcal{X}$ such that $\mathbb{Q}_x(C([0,T];H)) = 1$, therefore

$$(\mathbb{P}\circ (X^x)^{-1})(C([0,T];H)) = 1,$$

so X^x has continuous sample paths $\mathbb{P}$-a.s. $\square$

5.4. **Quantitative estimates and examples.** Although the proof of (2.8) in Theorem 2.10 allows to obtain a formula for function $\Psi(\cdot)$, it seems not possible to find an optimal form of the function $\Psi(\cdot)$ under conditions of Theorem 2.10. Below we present some explicit results which may not me optimal, but are good enough in most applications.

We begin with the following elementary proposition which, to the best of our knowledge, is not available in the existing mathematical literature. In this proposition we consider a standard mollifier on $\mathbb{R}$, which is a smooth non-negative function $m(\cdot) : \mathbb{R} \to [0,\infty)$ with support in $[-1,1]$ and its integral equal to 1.

Proposition 5.4.

(1) *Assume that*

$$p(\cdot) : [0,\infty) \to (0,\infty)$$

is a non-increasing function continuous at zero and such that

$$\lim_{y\to\infty} p(y) = 0,$$

and that

$$\Psi(\cdot) : [0,\infty) \to [0,\infty)$$

is a non-decreasing absolutely continuous function. If

$$\Psi(y) \leqslant \frac{1}{\sqrt{p(y)}} \tag{5.29}$$

for all $y \in [0,\infty)$, then

$$\int_0^\infty \Psi'(y)p(y)dy \leqslant \sqrt{p(0)}. \tag{5.30}$$

(2) *Moreover, for a given $p(\cdot)$ and any $\epsilon > 0$ there exists $\delta > 0$ such that the function $\Psi_\delta(\cdot)$ defined by*

$$\Psi_\delta(y) := \int_{\mathbb{R}} \frac{1}{\delta\sqrt{p(s-\delta)}}\, m\left(\frac{s-y}{\delta}\right) ds \tag{5.31}$$

is in $C^\infty[0,\infty)$ and satisfies

$$\lim_{y\to\infty} \Psi_\delta(y) = \infty \tag{5.32}$$

and

$$\int_0^\infty \Psi_\delta'(y)p(y)dy \geqslant \sqrt{p(0)} - \varepsilon. \tag{5.33}$$

Proof. To prove (5.30), note that

$$\int_0^\infty \Psi'(y)p(y)dy \leqslant \int_0^\infty \frac{\Psi'(y)}{(\Psi(y))^2}dy = \frac{1}{\Psi(0)} - \lim_{y\to\infty}\frac{1}{\Psi(y)}. \tag{5.34}$$

It is straightforward to verify that assumptions of Proposition 5.4 and (5.32) are satisfied for any $\delta > 0$, and (5.33) holds if δ is sufficiently small. □

Remark 5.5. For fixed σ, A, a, x, T Proposition 5.4 allows to estimate $\Psi(y)$ in Theorem 2.10. However in the most general form this computation is cumbersome and is not presented in our paper. Instead we give illustrative examples below under natural extra assumptions.

Corollary 5.6. *Assume that σ, A, x, T are fixed and denote*

$$p_0(s) = \sup_{t\in[0,T]} \mathbb{P}\left(|W_{0,A,\sigma}(t)| > s\right) \tag{5.35}$$

for $s \in [0,\infty)$. Then there are constants $C_i \in (0,\infty)$ such that for any increasing function $a(\cdot) : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{x\to\infty}\frac{a(x)}{x} = \infty \tag{5.36}$$

(2.8) *in Theorem 2.10 holds if*

$$\Psi(y) < \frac{1}{\sqrt{p_0\left(C_1\,a^{-1}\left(C_2\,a^{-1}\left(\sqrt{\log(C_3+y)}\right)\right)\right)}} \tag{5.37}$$

where $a^{-1}(\cdot)$ is the inverse function of the function $a(\cdot)$.

Proof. The proof is a simple combination of (5.1), (5.5), (5.22) and Proposition 5.4, where we assume that $p(y)$ is given by (5.23). □

Remark 5.7. With more tedious computation estimate (5.37) can be improved to estimates of the type

$$\Psi(y) < \left(p_0\left(C_1\,a^{-1}\left(C_2\,a^{-1}\left(\sqrt{\log(C_3+y)}\right)\right)\right)\right)^{\epsilon-1} \tag{5.38}$$

with constants depending on $\epsilon > 0$.

Example 5.8. *In interesting examples the function $p_0(s)$ in* (5.35) *decays either exponentially or with a Gaussian type tail estimate and $a(y)$ is a polynomial of an arbitrary degree. In this case*

$$\Psi(y) = \exp\left(\left(\log(1+y)\right)^\delta\right) \tag{5.39}$$

for some $\delta > 0$ satisfies (2.8). *If $a(y)$ increases exponentially, then we need to replace* $\log(1+y)$ *by a triple-*log *function.*

Proof. If $p_0(s)$ in (5.35) decays either exponentially or with a Gaussian estimate, and $a(y)$ is a positive polynomial, then for small enough $\delta_i > 0$ we have $\delta_1\sqrt{p_0(y)} < \exp(\delta_2 y)$ and $\delta_3 a^{-1}(y) < (1+y)^{\delta_4}$ for all $y > 0$. This shows that if $\delta > 0$ is small enough, then (5.39) implies (5.37) up to a constant which is not essential. □

Acknowledgement. The authors thank G. Da Prato, M. Hairer, M. Hinz and N. Krylov for helpful discussions and suggestions.

References

1. Viorel Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Editura Academiei Republicii Socialiste România, Bucharest, 1976, Translated from the Romanian. MR 0390843 (52 #11666)
2. ———, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics, Springer, New York, 2010. MR 2582280 (2011d:34001)
3. Vladimir I. Bogachev, Giuseppe Da Prato, and Michael Röckner, *Regularity of invariant measures for a class of perturbed Ornstein-Uhlenbeck operators*, NoDEA Nonlinear Differential Equations Appl. **3** (1996), no. 2, 261–268. MR 1385887 (97c:60189)
4. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). MR 0348562
5. Felix E. Browder, *Nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 470–476. MR 0212626
6. Giuseppe Da Prato and Michael Röckner, *Singular dissipative stochastic equations in Hilbert spaces*, Probab. Theory Related Fields **124** (2002), no. 2, 261–303. MR 1936019 (2003k:60151)
7. Giuseppe Da Prato, Michael Röckner, and Feng-Yu Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. **257** (2009), no. 4, 992–1017. MR 2535460 (2010k:60211)
8. Giuseppe Da Prato and Jerzy Zabczyk, *Nonexplosion, boundedness, and ergodicity for stochastic semilinear equations*, J. Differential Equations **98** (1992), no. 1, 181–195. MR 1168978
9. ———, *Stochastic equations in infinite dimensions*, second ed., Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014. MR 3236753
10. Sever Silvestru Dragomir, *Some Gronwall type inequalities and applications*, Nova Science Publishers, Inc., Hauppauge, NY, 2003. MR 2016992
11. Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR 1721989 (2000i:47075)
12. Dariusz Gątarek and Beniamin Gołdys, *On solving stochastic evolution equations by the change of drift with application to optimal control*, Stochastic partial differential equations and applications (Trento, 1990), Pitman Res. Notes Math. Ser., vol. 268, Longman Sci. Tech., Harlow, 1992, pp. 180–190. MR 1222696 (94d:60096)
13. ———, *On invariant measures for diffusions on Banach spaces*, Potential Anal. **7** (1997), no. 2, 539–553. MR 1467205 (98k:60102)
14. Tosio Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520. MR 0226230
15. Yukio Kōmura, *Nonlinear semi-groups in Hilbert space*, J. Math. Soc. Japan **19** (1967), 493–507. MR 0216342
16. Peter Kotelenez, *A submartingale type inequality with applications to stochastic evolution equations*, Stochastics **8** (1982/83), no. 2, 139–151. MR 686575
17. N. V. Krylov, *A simple proof of the existence of a solution to the Itô equation with monotone coefficients*, Teor. Veroyatnost. i Primenen. **35** (1990), no. 3, 576–580. MR 1091217
18. ———, *Introduction to the theory of diffusion processes*, Translations of Mathematical Monographs, vol. 142, American Mathematical Society, Providence, RI, 1995, Translated from the Russian manuscript by Valim Khidekel and Gennady Pasechnik. MR 1311478
19. ———, *A simple proof of a result of A. Novikov*, an arxiv preprint, 2002.
20. Wei Liu and Michael Röckner, *Stochastic partial differential equations: an introduction*, Universitext, Springer, Cham, 2015. MR 3410409
21. Carlo Marinelli and Michael Röckner, *On uniqueness of mild solutions for dissipative stochastic evolution equations*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), no. 3, 363–376. MR 2729590
22. Michel Métivier, *Semimartingales*, de Gruyter Studies in Mathematics, vol. 2, Walter de Gruyter & Co., Berlin, 1982, A course on stochastic processes. MR MR688144 (84i:60002)

23. Martin Ondreját, *Uniqueness for stochastic evolution equations in Banach spaces*, Dissertationes Math. (Rozprawy Mat.) **426** (2004), 63. MR 2067962 (2005e:60133)
24. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR MR710486 (85g:47061)
25. R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88. MR 0282272
26. Michael Röckner, Rongchan Zhu, and Xiangchan Zhu, *A note on stochastic semilinear equations and their associated Fokker-Planck equations*, J. Math. Anal. Appl. **415** (2014), no. 1, 83–109. MR 3173156
27. Wilhelm Stannat, *(Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 99–140. MR 1679079
28. ———, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics*, Mem. Amer. Math. Soc. **142** (1999), no. 678, viii+101. MR 1632609
29. Sven Wiesinger, *Uniqueness of solutions to Fokker-Planck equations related to singular SPDE driven by Lévy and cylindrical Wiener noise*, Ph.D. thesis, University of Bielefeld, 2011.

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.
Email address: maria.gordina@uconn.edu

‡ Department of Mathematics, Bielefeld University, D-33501 Bielefeld, Germany
Email address: roeckner@math.uni-bielefeld.de

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.
Email address: teplyaev@uconn.edu