REPRESENTATIONS OF SOLUTIONS TO FOKKER–PLANCK–KOLMOGOROV EQUATIONS WITH COEFFICIENTS OF LOW REGULARITY

Vladimir I. Bogachev[1], Stanislav V. Shaposhnikov[2]

Abstract We prove a formula representing a solution to a parabolic Fokker–Planck–Kolmogorov with coefficients of low regularity.

Keywords Fokker–Planck–Kolmogorov equation, double divergence form equation, representation of solutions,

Mathematics Subject Classification Primary 35J15; Secondary 35B65

In the classical theory of second order elliptic and parabolic equations a number of integral convolution-type representations of solutions are known that involve under the integral sign suitable fundamental solutions and the represented solution itself. Such self-representations are useful in many respects, for example, in deriving various estimates. In case of smooth coefficients such representations are typically obtained by multiplying the regarded equation with variable coefficients by the fundamental solution for an equation with constant coefficients (that admits an explicit expression) and integrating by parts (below we consider an example of this type). However, in case of coefficients of low regularity, when also solutions need not be differentiable, such an integration by parts can be illegal, although it might lead to a meaningful resulting expression. Therefore, some other means of justification are needed. Here we derive a self-representation of this sort for solutions to parabolic Fokker–Planck–Kolmogorov equations with non-differentiable coefficients, when solutions need not be Sobolev differentiable. Our result extends some previously known representations obtained in the elliptic case in [11], [12], and [5].

Given a mapping $A = (a^{ij})_{i,j}$ on $\mathbb{R}^d \times (0, +\infty)$ with values in the space of positive definite symmetric $d \times d$-matrices with Borel measurable entries a^{ij}, a Borel measurable function c on $\mathbb{R}^d \times (0, +\infty)$ and a mapping

$$b = (b^i)_{i\le d} \colon \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}^d$$

with Borel components b^i, we consider the associated Fokker–Planck–Kolmogorov equation

$$\partial_t \mu = \partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu) + c\mu \qquad (1)$$

This research was supported by the RFBR grants 14-01-00237, 15-31-20082 and the SFB 701 at Bielefeld University.

[1]Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia, the National Research University Higher School of Economics, Moscow, and St. Tikhon's Orthodox Humanitarian University, Moscow

[2]Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia, the National Research University Higher School of Economics, Moscow, and St. Tikhon's Orthodox Humanitarian University, Moscow

with respect to locally bounded Borel measures μ on $\mathbb{R}^d \times (0,+\infty)$ with the usual convention about summation over repeated indices. This equation is understood in the sense of the identity

$$\int \partial_t\varphi\, d\mu = \int [a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi]\, d\mu \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d \times (0,+\infty))$$

under the assumption that a^{ij}, b^i, c are locally locally integrable with respect to $|\mu|$ (the total variation of μ), which is automatically the case if these functions are locally bounded. It is convenient to use the shortened expression

$$\partial_t\mu - L^*_{A,bc,}\mu = 0,$$

where

$$L_{A,b,c}\varphi := a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi.$$

It is known (see [3], [4]) that under our assumptions (the coefficients are locally $|\mu|$-integrable and $\det A > 0$) every solution μ possesses a density ϱ with respect to Lebesgue measure on $\mathbb{R}^d \times (0,+\infty)$. This density satisfies the equation

$$\partial_t\varrho = \partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) + c\varrho \tag{2}$$

interpreted in the same way. One can also consider the Cauchy problem for this equation complemented with initial data interpreted in the sense of distributions, but this is not used in this paper.

It follows from (2) that, for every continuous function φ on $\mathbb{R}^d \times [0,+\infty)$ with continuous derivatives $\partial_t\varphi$, $\partial_{x_i}\varphi$, $\partial_{x_i}\partial_{x_j}\varphi$ such that $\varphi(x,t) = 0$ for all x outside a ball Ω and for all $t \le t_0$ for some $t_0 > 0$, one has (see [4, Proposition 6.1.2])

$$\int_\Omega \varphi(x,s)\varrho(x,s)\, dx = \int_0^s \int_\Omega [\partial_t\varphi(x,t) + L_{A,b,c}\varphi(x,t)]\varrho(x,t)\, dx\, dt \tag{3}$$

for almost all $s > 0$. We can take a version of ϱ such that this is true for all $s > 0$.

Similarly one defines stationary (elliptic) equations. For example, any stationary solution admits a density ϱ satisfying the so-called double divergence form equation

$$\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) + c\varrho = 0. \tag{4}$$

For such equations there is the following self-representation derived in [11], [12].

Let ω_d be the area of the unit sphere in $\mathbb{R}^d$. Given $d > 2$, set

$$H(x,y) = \frac{1}{(d-2)\omega_d}(\det A(y))^{-1/2}\langle A(y)^{-1}(x-y), x-y\rangle^{(2-d)/2}.$$

The function H, for each fixed y, serves as a fundamental solution for the equation with the constant diffusion matrix $A(y)$ and zero b and c, i.e., $L_{A(y),0,0}H(\cdot,y) = \delta_y$, or in the integral form

$$\int_{\mathbb{R}^d} a^{ij}(y)\partial_{x_i}\partial_{x_j}\varphi(x)\, H(x,y)\, dx = \varphi(y), \quad \varphi \in C_0^\infty(\mathbb{R}^d). \tag{5}$$

Let A have a modulus of continuity ω satisfying the Dini condition:

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty. \tag{6}$$

This condition holds if $\omega(t) \le ct^\delta$ with $\delta > 0$ or even $\omega(t) \le c|\ln t|^{-1-\delta}$ for $t < 1$. Let us observe that once $\omega(t)$ satisfies (6), then $\omega(\sqrt{t})$ does also, which is seen from the change of variable $s = t^2$.

It is known that in this case any solution to the stationary equation has a continuous density (see [11], [12], [5]). The proof of the following result from [11], [12] can be also found in [5].

Proposition 1. *Suppose that on every ball A has a modulus of continuity satisfying the Dini condition. Let* $\det A > 0$ *and* $|b|, c \in L^q_{loc}(\mathbb{R}^d)$*, where* $q > d$*. Let* $\eta \in C_0^\infty(\mathbb{R}^d)$ *be a fixed function equal to* 1 *in a neighborhood* U *of a point* $z \in \mathbb{R}^d$*. Then the continuous version of* ϱ *satisfying* (4) *admits the following representation*:

$$- \varrho(z) = \int_U \varrho(x) a^{ij}(z) H(x,z) \partial_{x_i}\partial_{x_j}\eta(x)\, dx + 2\int_U \varrho(x) a^{ij}(z)\partial_{x_i} H(x,z)\eta_j(x)\, dx$$
$$+ \int_U \varrho(x)(a^{ij}(x) - a^{ij}(z))\partial_{x_i}\partial_{x_j}(\eta(x)H(x,z))\, dx$$
$$+ \int_U \varrho(x) b^i(x)\partial_{x_i}(\eta(x)H(x,z))\, dx + \int_U \varrho(x)c(x)\eta(x)H(x,z)\, dx. \quad (7)$$

With the aid of this representation the Hölder continuity of densities of solutions to stationary equations with Hölder continuous A was proved in [11], [12].

Our goal is to obtain a parabolic analog of this representation. First we prove a result of independent interest: the solution density ϱ is locally integrable to any power provided that A belongs to the class VMO and is locally uniformly bounded along with A^{-1} and b and c are locally integrable to some power larger than $d+2$.

Let us recall that a function f on $\mathbb{R}^n$ belongs to the class VMO if there is a modulus of continuity ω_0 such that

$$\sup_a |U(x_0,r)|^{-1} \left| \int_{U(x_0,r)} f(x)\, dx - |U(x_0,r)|^{-1}\int_{U(x_0,r)} f(y)\, dy \right| \le \omega_0(r),$$

where $U(x_0,r)$ is the open ball of radius r centered at x_0. A function on a domain is said to belong to VMO if it has an extension of class VMO on the whole space.

Theorem 2. *Suppose that the matrices A and A^{-1} are locally bounded, on every ball the functions a^{ij} belong to the class VMO and $b^i, c \in L^q_{loc}$ with some $q > d+2$. If ϱ satisfies* (2)*, then $\varrho \in L^r_{loc}$ for all $r \in [1, +\infty)$.*

Proof. It is proved in [4, Theorem 6.3.1] that $\varrho \in L^r_{loc}$ for all $r \in [1, (d+2)')$ provided that A is locally Hölder continuous in x uniformly in t. It is readily seen from the proof of the cited theorem that the same reasoning remains in force for A in the class VMO if we use the results of [8] or [9] to bound gradients of direct equations considered in the proof. As in the elliptic case in [5], we raise the local integrability of ϱ by iterations applying the parabolic embedding theorem.

The proof is based on the following known fact (see [8], [9]). Let us fix a ball Ω in $\mathbb{R}^d$, say, of radius 1, and consider our equation in the cylinder $U := \Omega \times (0,1)$. There are constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \mathrm{I} \le A(x,t) \le \lambda_2 \mathrm{I} \quad \forall\, (x,t) \in U.$$

Set

$$L_A := L_{A,0,0}.$$

According to [9, Theorem 22], for any function $f \in C_0^\infty(U)$, there is a function v with Sobolev derivatives $\partial_t v, \partial_{x_i} v, \partial_{x_i}\partial_{x_j} v \in L^p(U)$ for every $p \in [1, +\infty)$ such that it vanishes on the parabolic boundary of U (i.e., the union of $\partial\Omega\times[0,1)$ and $\Omega\times\{0\}$), satisfies the equation $\partial_t v - L_A v = f$ and the inequality

$$\|\partial_t v\|_{L^p(U)} + \|D_x^2 v\|_{L^p(U)} \le C(p,\omega_0,d,\lambda_1,\lambda_2)\|f\|_p,$$

where ω_0 is a fixed modulus of continuity determining membership of A in VMO. It follows also that (with another constant) we have

$$\|\partial_t v\|_{L^p(U)} + \|D_x v\|_{L^p(U)} + \|D_x^2 v\|_{L^p(U)} \le C\|f\|_p.$$

In case $p<d+2$, by the parabolic embedding theorem, see, e.g., [7, Corollary 7.6], [6, Theorem 7.2] (where the restriction $p>2$ was needed only in the case of stochastic Sobolev spaces), [1, Chapter III], and [10, Theorem 7.1], we have

$$\|D_x v\|_{L^s(U)} \le C\|f\|_{L^p(U)} \quad \forall\, s < \frac{p(d+2)}{d+2-p},$$

where the constant C depends also on p and s (in addition to A and the ball), but is independent of f.

We now describe our iterations. We can take for ϱ some initial order of local integrability $p_1<(d+2)'$ as close to $(d+2)'$ as we wish, in particular, we can start with $p_1<(d+2)'$ such that $p_1'<q$.

Let us take a function $\zeta\in C_0^\infty(U)$ equal to 1 on a closed ball in U such that $0\le\zeta\le 1$. For the function u considered above we have

$$L_{A,b,c}(\zeta u) = \zeta L_A u + u L_A\zeta + 2\langle AD_x\zeta, D_x u\rangle + \zeta\langle b, D_x u\rangle + u\langle b, D_x\zeta\rangle + c\zeta u,$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the inner product in $\mathbb{R}^d$. In addition,

$$\sup_U\Bigl[|L_A\zeta| + 2|AD_x\zeta| + |D_x\zeta| + |\partial_t\zeta|\Bigr] \le M$$

with some number M. Hence by the generalized Hölder inequality applied to the products $(|b|+M)\,|\varrho D_x u|$, $(|c|+M|b|)\,|u\varrho|$, $|\partial_t\zeta u\varrho|$ and the exponents q, p_1, and s with

$$\frac{1}{s} = 1 - \frac{1}{q} - \frac{1}{p_1}$$

(we recall that $p_1'<q$), we obtain

$$\int_0^s\int_\Omega [\partial_t u + L_A u]\,\zeta\varrho\,dx \le C\|D_x u\|_s + C\|u\|_s, \quad s = \frac{p_1' q}{q-p_1'}, \tag{8}$$

where C does not depend on u, but depends on ϱ.

Therefore, we arrive at the estimates

$$\int_U f\,\zeta\varrho\,dx \le C\|f\|_p \quad \forall\, p > \frac{s(d+2)}{s+d+2} \tag{9}$$

with constants depending on p. If we could take p equal to $s(d+2)/(s+d+2)$, then we would obtain $\zeta\varrho\in L^{p_2}(U)$ with

$$p_2 = p' = \frac{s(d+2)}{s(d+2)-s-d-2}.$$

Let us express p_2 in terms of p_1:

$$p_2 = p_1\frac{p_1'-1}{p_1'(1-(q-d-2)/(q(d+2)))-1} = p_1\Bigl(1+p_1'\frac{q-d-2}{q(d+2)}\Bigr) > p_1\Bigl(1+\frac{q-d-2}{q(d+2)}\Bigr).$$

Therefore, although we cannot take for p_2 exactly this number, starting with some p_1 sufficiently close to $(d+2)'$ and repeating this procedure, we increase the order of integrability p_n with factor separated from 1. This works unless $s(d+2) > s+d+2$, i.e., $s > (d+2)/(d+1) = (d+2)'$, or, in terms of p_1,

$$p_1 < \frac{q(d+2)}{q-d-2}.$$

If we can take s close to $(d+2)'$, then we can make p_2 as large as we wish in one step starting from a suitable value of p_1. The calculations above show that this is possible indeed. □

We now turn to our chief goal, the integral representation.

Suppose that A satisfies the condition

$$|A(x,t) - A(y,s)| \le \omega(|x-y|) + \omega(|t-s|), \tag{10}$$

where ω is a modulus of continuity satisfying the Dini condition (6).

It follows from our assumptions (including that that A nondegenerate) that A and A^{-1} are locally bounded.

Whenever $s > t$, $x, y \in \mathbb{R}^d$, we set

$$G(x,t,y,s) = (4\pi(s-t))^{-d/2}(\det A(y,s))^{-1/2} \exp\Bigl(-\frac{\langle A^{-1}(y,s)(x-y), x-y\rangle}{4(s-t)}\Bigr).$$

If $t < s$ we have

$$\partial_t G(x,t,y,s) + a^{ij}(y,s)\partial_{x_i}\partial_{x_j} G(x,t,y,s) = 0.$$

For each function $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,+\infty))$ we have the equality

$$\int_0^s \int_{\mathbb{R}^d} [\partial_t\varphi(x,t) - a^{ij}(y,s)\partial_{x_i}\partial_{x_j}\varphi(x,t)] G(x,t,y,s)\, dx\, dt = \varphi(y,s). \tag{11}$$

Let (z,s) be a fixed point.

Let us fix a function $\eta \in C_0^\infty(\mathbb{R}^d \times (0,+\infty))$ with support in an open ball U containing (z,s) (one can take the unit ball) that equals 1 on a ball containing (z,s) and satisfies the bounds $0 \le \eta \le 1$. To shorten long formulas below, we use occasionally the notation

$$\eta_i = \partial_{x_i}\eta,\ \ \eta_{ij} = \partial_{x_j}\partial_{x_i}\eta,\ \ \eta_t = \partial_t\eta.$$

Here is our parabolic representation.

Theorem 3. *Let ϱ satisfy (2), where A is nondegenerate and satisfies (10) with ω satisfying Dini's condition, b^i and c belong to $L^q_{loc}(\mathbb{R}\times(0,+\infty))$ with some $q > d+2$. Then we have almost everywhere*

$$\begin{aligned}\eta(z,s)\varrho(z,s) = \int_0^s \int_{\mathbb{R}^d} &[G(x,t,z,s)\partial_t\eta(x,t) + G(x,t,z,s)a^{ij}(z,s)\eta_{x_ix_j}(x,t)\\ &+ 2a^{ij}(z,s)\eta_i(x,t)\partial_{x_i}G(x,t,z,s) + (a^{ij}(x,t) - a^{ij}(z,s))\partial_{x_i}\partial_{x_j}(\eta(x,t)G(x,t,z,s))\\ &+ b^i(x,t)\partial_{x_i}(\eta(x,t)G(x,t,z,s)) + c(x,t)\eta(x,t)G(x,t,z,s)]\varrho(x,t)\, dx\, dt. \end{aligned}\tag{12}$$

In particular,

$$\varrho(z,s) = \int_0^s \int_{\mathbb{R}^d} [G(x,t,z,s)\partial_t\eta(x,t) + G(x,t,z,s)a^{ij}(z,s)\eta_{x_ix_j}(x,t) + 2a^{ij}(z,s)\eta_i(x,t)\partial_{x_i}G(x,t,z,s) + (a^{ij}(x,t) - a^{ij}(z,s))\partial_{x_i}\partial_{x_j}(\eta(x,t)G(x,t,z,s)) + b^i(x,t)\partial_{x_i}(\eta(x,t)G(x,t,z,s)) + c(x,t)\eta(x,t)G(x,t,z,s)]\varrho(x,t)\,dx\,dt. \quad (13)$$

Proof. To derive the desired formula we multiply (2) by ηG, integrate over the strip $\mathbb{R}^d \times (0,s)$ and then formally integrate by parts with respect to x. However, under our assumptions, the solution ϱ need not be Sobolev differentiable, hence a different justification is needed.

We first observe that the right-hand side is meaningful. Indeed, we know that ϱ is locally integrable to any power, the derivatives of G locally admit the bounds

$$|\partial_{x_i}G(x,t,z,s)| \le C(s-t)^{-d/2-1}|x-z|\exp(-M|x-z|^2/(s-t)),$$

$$|\partial_{x_i}\partial_{x_j}G(x,t,z,s)| \le C[(s-t)^{-d/2-1} + (s-t)^{-d/2-2}|x-z|^2]\exp(-M|x-z|^2/(s-t)).$$

Hence the first derivative is locally integrable to any power less than $1+2/d$ and by Hölder's inequality its product with b^i is locally integrable to some power larger than 1. Similarly, the terms with G are integrable. Let us consider the term with the second derivative of G. We have

$$|a^{ij}(x,t) - a^{ij}(z,s)\partial_{x_i}\partial_{x_j}G(x,t,z,s)| \le C(\omega(|x-z|)+\omega(s-t))[(s-t)^{-d/2-1}+(s-t)^{-d/2-2}|x-z|^2]\exp(-M|x-z|^2/(s-t)).$$

The function on the right is integrable. Indeed, changing variables

$$u = (x-z)(s-t)^{-1/2},$$

we arrive at the integral of

$$C(\omega((s-t)^{1/2}|u|) + \omega(s-t))(s-t)^{-1}[1+|u|^2]\exp(-M|u|^2).$$

The integral of $\omega(s-t)(s-t)^{-1}[1+|u|^2]\exp(-M|u|^2)$ is finite by Dini's condition. The integral of $\omega((s-t)^{1/2}|u|)(s-t)^{-1}[1+|u|^2]\exp(-M|u|^2)$ is finite as well, which is seen from convergence of the integral of $\omega(\sqrt{t})/t$ at the origin (noted above). Therefore, the desired integrability follows by the integrability of the function $f_1(z-x)f_2(x)$ for almost every fixed z in case of integrable f_1 and f_2.

We show that equality (12) holds at every point at which the integral over the unit ball of the function

$$(x,t) \mapsto |\varrho(x,t)|(\omega(|x-z|) + \omega(s-t)) \times [(s-t)^{-d/2-1} + (s-t)^{-d/2-2}|x-z|^2]\exp(-M|x-z|^2/(s-t)).$$

is finite and which, in addition, is a Lebesgue point for the functions ϱ and $b^i\varrho$ (see, e.g., [2, § 5.6]), that is, a point (z,s) for which

$$\lim_{r\to 0} r^{-d-1}\int_{U_r(z,s)} \Big[|\varrho(z,s) - \varrho(y,t)| + |b^i(z,s)\varrho(z,s) - b^i(y,t)\varrho(y,t)|\Big]\,dy\,dt = 0.$$

It will be important below that for this point (z,s) the function

$$(x,t) \mapsto \varrho(x,t)(a^{ij}(x,t) - a^{ij}(z,t))\partial_{x_i}\partial_{x_j}H(x,z)$$

is integrable in a neighborhood of (z,s), since

$$|a^{ij}(x,t) - a^{ij}(z,s)| \le \omega(|x-z|) + \omega(|s-t|), \quad |\partial_{x_i}\partial_{x_j}G(x,t,z,s)| \le C|s-t|^{-d/2-1}.$$

We can assume that $z = 0$. For notational simplicity we write

$$H(x,t) := G(x,t,0,s).$$

Let $r > 0$ be smaller than the radius of Ω divided by 4. Let us take a function $\zeta = \zeta_r \in C^\infty(\mathbb{R}^d \times \mathbb{R})$ that vanishes in the ball U_r of radius r centered at $(0,s)$, equals 1 outside the ball U_{2r} belonging with its closure to U and such that

$$0 \le \zeta \le 1,\ |D_x\zeta| \le Cr^{-1},\ |\partial_{x_i}\partial_{x_j}\zeta| \le Cr^{-2},\ |\partial_t\zeta| \le Cr^{-1},$$

where C does not depend on r; here and throughout we denote by C generic constants (possibly, different) appearing in our estimates. The index r for ζ_r will not be indicated below, but in all limits as $r \to 0$ in the integrals with ζ considered below it is meant that dependence on r is due to ζ_r.

It is clear that $\zeta - 1 \in C_0^\infty(\mathbb{R}^d \times \mathbb{R})$. Since ζ vanishes in $U_r(0,s)$, the function $\zeta\eta H$ extended by zero on the set $t \ge s$ is smooth on $\mathbb{R}^{d+1}$. Using it in (3) and taking into account that the left-hand side of (3) vanishes for it, we arrive at the equality (in this equality and below we do not indicate the arguments of the integrands in case of integration in all variables)

$$\int_0^s \int_\Omega (\partial_t + L)(\zeta\eta H)\varrho\, dx\, dt = 0. \tag{14}$$

Into this equality we substitute the identities

$$\partial_t(\zeta\eta H) = \partial_t\zeta(\eta H) + \zeta(\partial_t\eta)H + \zeta\eta\partial_t H$$

and

$$L(\zeta\eta H) = a^{ij}\zeta\partial_{x_i}\partial_{x_j}(\eta H) + a^{ij}\zeta_{ij}\eta H + 2a^{ij}\zeta_i H\eta_j + 2a^{ij}\eta\partial_{x_j}H\zeta_i + b^i\zeta_i\eta H + \zeta b^i\partial_{x_i}(\eta H) + c\zeta\eta H,$$

which is transformed to the following more compact form taking into account that the function η equals 1 on the ball U_{2r} outside of which the derivatives of ζ vanish:

$$L(\zeta\eta H) = a^{ij}\zeta\partial_{x_i}\partial_{x_j}(\eta H) + a^{ij}\zeta_{ij}H + 2a^{ij}\partial_{x_j}H\zeta_i + b^i\zeta_i H + \zeta b^i\partial_{x_i}(\eta H) + c\zeta\eta H. \tag{15}$$

As $r \to 0$ (recall that ζ depends on r, which is suppressed in our notation), the integrals with the weight ϱ of the last two terms in this identity tend to the corresponding integrals in the right-hand side of (12) by the Lebesgue dominated convergence theorem.

For estimating the integral of the fourth term in (15) with the weight ϱ we observe that $H(x,t) \le Cr^{-d/2}$ outside of $U_r(0,s)$. Hence by the local boundedness of b and the fact that $(0,s)$ is a Lebesgue point we have

$$\int_{U_{2r}} |b^i\zeta_i H\varrho|\, dx\, dt \le C' r^{-1-d/2}\int_{U_{2r}} |\varrho|\, dx\, dt \le C'' r^{-1-d/2+1+d},$$

which tends to zero as $r \to 0$.

Next we analyze the second term in (15). Similarly to the estimates above, we have

$$\int_{U_{2r}} |(a^{ij}(x,t) - a^{ij}(0,s))\zeta_{ij}(x,t)H(x,t)\varrho(x,t)|\,dx\,dt$$
$$\le C\omega(r)r^{-2}r^{-d/2}\int_{U_{2r}} |\varrho(x,t)|\,dx\,dt \le C'\omega(r)r^{-1+d/2}. \quad (16)$$

On the other hand, from (11) with the function $\varphi = \zeta - 1$ we obtain

$$-\eta(0,s)\varrho(0,s) = \int_0^s \int_\Omega [\partial_t\zeta(x,t) - a^{ij}(0,s)\zeta_{ij}(x,t)]\,H(x,t)\eta(0,s)\varrho(0,s)\,dx\,dt.$$

The right-hand side differs from

$$\int_0^s \int_\Omega [\partial_t\zeta(x,t) - a^{ij}(0,s)\zeta_{ij}(x,t)]\,H(x,t)\eta(x,t)\varrho(x,t)\,dx\,dt$$

by a quantity tending to zero as $r \to 0$, since

$$|\partial_t\zeta| \le Cr^{-1}, \quad |\zeta_{ij}| \le Cr^{-2},$$

$$H(x,t) \le Cr^{-d/2} \quad \text{if } (x,t) \in U_{2r}\backslash U_r,$$

and the integral of $|\varrho(x,t) - \varrho(0,s)|$ over U_r is estimated by $o(r)r^{d+1}$ (since $(0,s)$ is a Lebesgue point). Thus, along with (16) this gives the equality

$$-\eta(0,s)\varrho(0,s) = \lim_{r\to 0}\int_0^s \int_\Omega [\partial_t\zeta(x,t) - a^{ij}(0,s)\zeta_{ij}(x,t)]\,H(x,t)\eta(x,t)\varrho(x,t)\,dx\,dt$$
$$= \lim_{r\to 0}\int_0^s \int_\Omega [\partial_t\zeta(x,t) - a^{ij}(x,t)\zeta_{ij}(x,t)]\,H(x,t)\eta(x,t)\varrho(x,t)\,dx\,dt.$$

Therefore, for the integral with the weight ϱ of the second term in (15) with the added $-\partial_t\zeta\eta H$ we have

$$\int_0^s \int_\Omega [-\partial_t\zeta + a^{ij}\zeta_{ij}]H\varrho\,dx\,dt \to \eta(0,s)\varrho(0,s)$$

as $r \to 0$. Note that now we have $2\partial_t\zeta\eta H$ at our disposal in place of the original term $\partial_t\zeta\eta H$, which will be important below.

Let us now consider the third term in (15) and verify that Therefore, for the integral with the weight ϱ of the third term in (15) we have

$$\lim_{r\to 0}\int_0^s \int_\Omega [2\partial_t\zeta H + 2a^{ij}\partial_{x_j}H\zeta_i]\varrho\,dx\,dt \to -2\eta(0,s)\varrho(0,s). \quad (17)$$

By the bounds $|\zeta_i(x,t)| \le Cr^{-1}$, $\partial_{x_j}H(x,t) \le Cr^{-d/2-1/2}$ we have

$$\int_0^s \int_\Omega |\partial_{x_j}H(x,t)\zeta_i(x,t)|\,|\varrho(x,t) - \varrho(0,s)|\,dx\,dt$$
$$\le Cr^{-(d+3)/2}\int_0^s \int_\Omega |\varrho(x,t) - \varrho(0,s)|\,dx\,dt \le o(r)r^{(d-1)/2} \to 0$$

as $r \to 0$. Hence in (17) we can replace $\varrho(x,t)$ with $\varrho(0,s)$, next we can replace $a^{ij}(x,t)$ with $a^{ij}(0,s)$ by the continuity of a^{ij}. Let us now observe that by virtue of (11) we have

$$\int_0^s \int_\Omega [\partial_t\zeta(x,t)H(x,t) + a^{ij}(0,s)\partial_{x_j}H(x,t)\zeta_i(x,t)]\,dx\,dt$$
$$= \int_0^s \int_\Omega [\partial_t(\zeta-1)(x,t)H(x,t) - a^{ij}(0,s)\partial_{x_j}\partial_{x_i}(\zeta-1)(x,t)]H(x,t)\,dx\,dt = -1,$$

hence (17) is established.

It remains to consider the first term in (15) and study the behavior of the integral

$$\int_0^s \int_\Omega a^{ij}\zeta\partial_{x_i}\partial_{x_j}(\eta H)\varrho\,dx\,dt.$$

We observe that the forth term in (12) equals

$$\int_0^s \int_\Omega \varrho(x,t)(a^{ij}(x,t) - a^{ij}(0,s))\partial_{x_i}\partial_{x_j}(\eta(x,t)H(x,t))\,dx\,dt$$
$$= \lim_{r\to 0}\int_0^s \int_\Omega \zeta(x,t)\varrho(x,t)(a^{ij}(x,t) - a^{ij}(0,s))\partial_{x_i}\partial_{x_j}(\eta(x,t)H(x,t))\,dx\,dt$$

by the Lebesgue dominated convergence theorem. The integral of the first term is the one we are studying. Hence we have to consider the integral with the weight ϱ of the function

$$a^{ij}(0,s)\zeta(x,t)\partial_{x_i}\partial_{x_j}(\eta(x,t)H(x,t)) = a^{ij}(0,s)\zeta(x,t)H(x,t)\eta_{ij}(x,t)$$
$$+ a^{ij}(0,s)\zeta(x,t)\eta(x,t)\partial_{x_i}\partial_{x_j}H(x,t) + 2\zeta(x,t)a^{ij}(0,s)\partial_{x_i}H(x,t)\eta_j(x,t).$$

In this expression, the integrals of the first and last terms with the weight ϱ tend, respectively, to the second and third terms in the right-hand side of (12) by the Lebesgue dominated convergence theorem.

It remains to handle the function $a^{ij}(0,s)\zeta(x,t)\eta(x,t)\partial_{x_i}\partial_{x_j}H(x,t)$ in the expression above. However, adding to this function the term $\zeta\eta\partial_t H$ from the derivative $\partial_t(\zeta\eta H)$ (which has been left aside so far) we obtain the expression that vanishes identically since on the ball U_r the function ζ equals zero and outside this ball, whenever $t < s$, we have the equality

$$\partial_t H(x,t) - a^{ij}(0,s)\partial_{x_i}\partial_{x_j}H(x,t) = 0.$$

Finally, the integral of $\zeta(\partial_h\eta)H\varrho$ tends to the first term in the right-hand side of (12). Thus, the desired formula is completely justified. □

Corollary 4. *Under the stated assumptions, there is a continuous version of ϱ.*

Proof. It suffices to verify the continuity of the right-hand side of our representation. We show that the right-hand side in (12) is bounded and then our claim will follow by the known properties of convolutions (this will actually yield the continuity of $\eta\varrho$, but since η can be chosen arbitrarily, the continuity of ϱ follows). To this end we apply the following useful assertion from [12]: if nonnegative measurable functions u, K and f on $\mathbb{R}^n$ are such that $u, K \in L^1$, $f \in L^p + L^\infty$, $1 < p \le +\infty$, i.e., $f = f_1 + f_2$, where $f_1 \in L^p, f_2 \in L^\infty$, and almost everywhere

$$u \le K * u + f,$$

then $u \in L^p + L^\infty$. □

As an application of the obtained representation we give a sufficient condition for the global boundedness of solution densities in case of b sufficiently integrable with respect to the solution. For a Sobolev class matrix A such conditions are known (see [4]), but their proof employs membership in Sobolev classes and embedding theorems. Under our assumptions, ϱ can fail to belong to a Sobolev class.

Theorem 5. *Suppose that a bounded measure μ on $\mathbb{R}^d\times(0,1)$ satisfies equation* (1), *where $b^i, c \in L^p(|\mu|)$ with some $p > d+2$, the matrices A and A^{-1} are uniformly bounded and the functions a^{ij} have a modulus of continuity satisfying Dini's condition. Then the solution density ϱ is uniformly bounded on $\mathbb{R}^d \times (0,1)$.*

Proof. We use again the lemma from [12] cited above. To this end we have to estimate the integral

$$\int_{\mathbb{R}^d\times(0,s)} \Bigl[|b^i(x,t)|\partial_{x_i}(\eta(x,t)G(x,t,z,s)) + |c(x,t)|\eta(x,t)G(x,t,z,s)\Bigr]\, |\varrho(x,t)|\, dx\, dt.$$

It suffices to consider the integral

$$\int_{\mathbb{R}^d\times(0,s)} (s-t)^{-1}|b(x,t)|\,|x-z|G(x,t,z,s)|\varrho(x,t)|\, dx\, dt.$$

By Hölder's inequality it is dominated by

$$\|b\|_{L^p(|\mu|)}\biggl(\int_{\mathbb{R}^d\times(0,s)} (s-t)^{-p'}|x-z|^{p'}G(x,t,z,s)^{p'}|\varrho(x,t)|\, dx\, dt\biggr)^{1/p'} \le C + CK * |\varrho|,$$

where the function $K(x,t) = (s-t)^{-p'}|x-z|^{p'}G(x,t,z,s)^{p'}$ is integrable, since $p' < (d+2)' = (d+2)/(d+1)$. □

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