Three-dimensional Navier-Stokes equations driven by space-time white noise *

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Abstract

In this paper we prove existence and uniqueness of local solutions to the three dimensional (3D) Navier-Stokes (N-S) equation driven by space-time white noise using two methods: first, the theory of regularity structures introduced by Martin Hairer in [16] and second, the paracontrolled distribution proposed by Gubinelli, Imkeller, Perkowski in [12]. We also compare the two approaches.

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1 Introduction

In this paper, we consider the three dimensional (3D) Navier-Stokes equation driven by space-time white noise: Recall that the Navier-Stokes equations describe the time evolution of an incompressible fluid and are given by

$$\begin{aligned} \partial_t u + u\cdot\nabla u =&\nu\Delta u - \nabla p + \xi \\ u(0) =&u_0, \quad divu = 0 \end{aligned} \tag{1.1}$$

where $u(t,x)\in\mathbb{R}^3$ denotes the value of the velocity field at time t and position x, $p(t,x)$ denotes the pressure, and $\xi(t,x)$ is an external force field acting on the fluid. We will consider the case when $x\in\mathbb{T}^3$, the three-dimensional torus. Our mathematical model for the driving force ξ is a Gaussian field which is white in time and space.

Random Navier-Stokes equations, especially the stochastic 2D Navier-Stokes equation driven by trace-class noise, have been studied in many articles (see e.g. [9], [17], [5], [21] and the

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reference therein). In the two dimensional case existence and uniqueness of strong solutions have been obtained if the noisy forcing term is white in time and colored in space. In the three dimensional case, existence of martingale (=probabilistic weak) solutions, which form a Markov selection, have been constructed for the stochastic 3D Navier-Stokes equation driven by trace-class noise in [10], [7], [13]. Furthermore, the ergodicity has been obtained for every Markov selection of the martingale solutions if driven by non-degenerate trace-class noise (see [10]).

This paper aims at giving a meaning to the equation (1.1) when ξ is space-time white noise and at obtaining local (in time) solution. Such a noise might not be relevant for the study of turbulence. However, in other cases, when a flow is subjected to an external forcing with a very small time and space correlation length, a space-time white noise may be appropriate to model this situation. The main difficulty in this case is that ξ and hence u are so singular that the non-linear term is not well-defined.

In the two dimensional case, the Navier-Stokes equation driven by space-time white noise has been studied in [6], where a unique global solution in the (probabilistically) strong sense has been obtained by using the Gaussian invariant measure for this equation. Thanks to the incompressibility condition, we can write $u \cdot \nabla u = \frac{1}{2}\mathrm{div}(u \otimes u)$. The authors split the unknown into the solution to the linear equation and the solution to a modified version of the Navier-Stokes equations:

$$\partial_t z = \nu\Delta z - \nabla\pi + \xi, \quad \mathrm{div} z = 0;$$

$$\partial_t v = \nu\Delta v - \nabla q - \frac{1}{2}\mathrm{div}[(v+z)\otimes(v+z)], \quad \mathrm{div} v = 0. \tag{1.2}$$

The first part z is a Gaussian process with non-smooth paths, whereas the second part v is smoother. The only term in the nonlinear part, initially not well defined, is $z \otimes z$, which, however, can be defined by using the Wick product. By a fixed point argument they obtain existence and uniqueness of local solutions in the two dimensional case. Then by using the Gaussian invariant measure for the 2D Navier-Stokes equation driven by space-time white noise, existence and uniqueness of (probabilistically) strong solutions starting from almost every initial value is obtained. (For the one-dimensional case we refer to [8]).

However, in the three dimensional case, the trick in the two dimensional case breaks down since v and z in (1.2) are so singular that not only $z \otimes z$ is not well-defined but also $v \otimes z$ and $v \otimes v$ have no meaning. Here v is the solution to the nonlinear equation (1.2) and we cannot define these terms by using the Wick product. As a result, we cannot make sense of (1.2) and obtain existence and uniqueness of local solutions as in the two dimensional case. As a way out one might try to iterate the above trick as follows: we write $v = v_2 + v_3$, where v_2, v_3 are the solutions to the following equations:

$$\partial_t v_2 = \nu\Delta v_2 - \nabla q_2 - \frac{1}{2}\mathrm{div}(z \otimes z), \quad \mathrm{div} v_2 = 0,$$

$$\partial_t v_3 = \nu\Delta v_3 - \nabla q_3 - \frac{1}{2}\mathrm{div}[(v_3+v_2)\otimes(v_3+v_2)] - \frac{1}{2}\mathrm{div}((v_3+v_2)\otimes z) - \frac{1}{2}\mathrm{div}(z\otimes(v_3+v_2)), \quad \mathrm{div} v_3 = 0. \tag{1.3}$$

Now we can make sense of the terms without v_3 in the right hand side of (1.3), hope v_3 becomes smoother such that the nonlinear terms including v_3 are well-defined and try to obtain a well-posed equation. However, this is not the case. For the unknown v_3 the nonlinear term on the right hand side of (1.3) including $v_3 \otimes z$ is still not well-defined. Indeed, in this case $z \in \mathcal{C}^{-\frac{1}{2}-\kappa}$ for every $\kappa > 0$. As a consequence, we cannot expect that the regularity of v_3 is better than $\mathcal{C}^{\frac{1}{2}-\kappa}$ for every $\kappa > 0$, which makes $v_3 \otimes z$ not well-defined. No matter how many times we modify this equation again as above, the equation always contains the multiplication for the unknown and z, which is not well-defined. Hence, this equation is ill-posed in the traditionally sense.

Thanks to the theory of regularity structures introduced by Martin Hairer in [16] and the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski in [12] we can solve this problem and obtain existence and uniqueness of local solutions to the stochastic three dimensional Navier-Stokes equations driven by space-time white noise. Recently, these two approaches have been successful in giving a meaning to a lot of ill-posed stochastic PDEs like the Kardar-Parisi-Zhang (KPZ) equation ([18], [2], [15]), the dynamical Φ^4_3 model ([16], [4]) and so on. From a philosophical perspective, the theory of regularity structures and the paracontrolled distribution are inspired by the theory of controlled rough paths [20], [11]. The main difference is that the regularity structure theory considers the problem locally, while the paracontrolled distribution method is a global approach using Fourier analysis. For a comparison of these two methods we refer to Remark 3.13.

The key idea of the theory of regularity structures is as follows: we perform an abstract Talyor expansion on both sides of the equation. Originally Talyor expansions are only for functions. Here the right objects, e.g. regularity structure that could possibly take the place of Taylor polynomials, can be constructed. The regularity structure can be endowed with a model $\iota\xi$, which is a concrete way of associating every element in the abstract regularity structure to the actual Taylor polynomial at every point. Multiplication, differentiation, the state space of solutions, and the convolution with singular kernels can be defined on this regularity structure, which is the major difficulty when trying to give a meaning to such singular stochastic partial differential equations as above. On the regularity structure, a fixed point argument can be applied to obtain local existence and uniqueness of the solution Φ to the equation lifted onto the regularity structure. Furthermore, we can go back to the real world with the help of another central tool of the theory, namely the reconstruction operator $\mathcal{R}$. If ξ is a smooth process, $\mathcal{R}\Phi$ coincides with the classic solution to the equation. Now we have the following maps

$$\xi \mapsto \iota\xi \mapsto \Phi \mapsto \mathcal{R}\Phi,$$

and one is led to the following question: Given a sequence ξ_ε of regularisations of the space-time white noise ξ, can we obtain the solution associated with ξ by taking the limit of $\mathcal{R}\Phi_\varepsilon$, as ε goes to 0, where Φ_ε is the solution associated to ξ_ε. However, the answer to this question is no. Indeed, while the last two maps are continuous with respect to suitable topologies, the above sequence $\iota\xi_\varepsilon$ of canonical models fails to converge. It may, however, still be possible to renormalize the model $\iota\xi_\varepsilon$ into some converging model $\hat{\iota}\xi_\varepsilon$, which in turn can be related to a specific renormalised equation.

With these considerations in mind, let us go back to the 3D Navier-Stokes equations driven by space-time white noise. We apply Martin Hairer's regularity structure theory to solve it. First, as in the two dimensional case we write the nonlinear term $u \cdot \nabla u = \frac{1}{2}\mathrm{div}(u \otimes u)$ and construct the associated regularity structure (Theorem 2.8). As in [16] we construct different admissible models to denote different realizations of the equations corresponding to different noises. Then for any suitable models, we obtain local existence and uniqueness of solutions by a fixed point argument. Finally, we renormalize the models associated with the approximations as mentioned above such that the solution to the equations associated with these renormalised models converge to the solution to the 3D Navier-Stokes equation driven by space-time white noise in probability, locally in time.

The theory of paracontrolled distributions combines the idea of Gubinelli's controlled rough path [11] and Bony's paraproduct [3], which is defined as follows: Let $\Delta_j f$ be the jth Littlewood-Paley block of a distribution f and define

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{j\geq -1}\sum_{i<j-1} \Delta_i f \Delta_j g, \quad \pi_0(f,g) = \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g.$$

Formally $fg = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g)$. Observing that, if f is regular, $\pi_<(f,g)$ behaves like g and is the only term in Bony's paraproduct not increasing the regularity, the authors in [12] consider a paracontrolled ansatz of type

$$u = \pi_<(u', g) + u^\sharp,$$

where $\pi_<(u',g)$ represents the "bad-term" in the solution, g is a functional of the Gaussian field and $u^\sharp$ is regular enough to allow the required multiplication. Then to make sense of the product uf we only need to define gf by using a commutator estimate (Lemma 3.3).

In the second part of this paper we apply the paracontrolled distribution method to the 3D Navier-Stokes equations driven by space-time white noise. First we split the equation into four equations and consider the approximation equations. Here as in the theory of regularity structures, we still approximate ξ by smooth functions ξ_ε and obtain the approximation equation associated with ξ_ε. By using the paracontrolled ansatz we obtain uniform estimates for the approximation equations and moreover we also get the local Lipschitz continuity of solutions with respect to initial values and some extra terms $\mathbb{Z}(\xi_\varepsilon)$, which are independent of the solutions. These extra terms $\mathbb{Z}(\xi_\varepsilon)$ play a similar role as the models associated with the "distributional-like" elements in the abstract regularity structures. If $\mathbb{Z}(\xi_\varepsilon)$ converges to some $\mathbb{Z}$ in some suitable space, then the solution u_ε associated with $\mathbb{Z}(\xi_\varepsilon)$ will converge to the desired solution. However, as in the theory of regularity structures, we have to do suitable renormalisations for these terms such that they converge in suitable spaces. Here, inspired by [16], we prove Lemma 3.10, which makes the calculations for the renormalisation easier. Moreover taking the limit of the solutions to the approximation equations we obtain local existence and uniqueness of the solutions. Indeed, by choosing a suitable solution space we can also give a meaning to the original equation (see Remark 3.9).

The main result of this article is the following theorem.

Theorem 1.1 Let $u_0 \in \mathcal{C}^\eta$ for $\eta \in (-1, \alpha+2]$ with $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Let $\xi = (\xi^1, \xi^2, \xi^3)$, with $\xi^i, i = 1,2,3$ being independent white noises on $\mathbb{R}\times\mathbb{T}^3$, which we extend periodically to $\mathbb{R}^4$. Let $\rho : \mathbb{R}^4 \to \mathbb{R}$ be a smooth compactly supported function with Lebesgue integral equal to 1, and symmetric with respect to space variable, set $\rho_\varepsilon(t,x) = \varepsilon^{-5}\rho(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ and define $\xi^i_\varepsilon = \rho_\varepsilon * \xi^i$. Consider the maximal solution u_ε to the following equation

$$\partial_t u^i_\varepsilon = \Delta u^i_\varepsilon + \sum_{i_1=1}^3 P^{ii_1}\xi^{i_1}_\varepsilon - \frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u^{i_1}_\varepsilon u^j_\varepsilon)), \quad u_\varepsilon(0) = Pu_0.$$

Then there exists $u \in C([0,\tau);\mathcal{C}^\eta)$ and a sequence of random time τ_L converging to the explosion time τ of u such that

$$\sup_{t\in[0,\tau_L]} \|u^\varepsilon - u\|_\eta \to^P 0.$$

Remark 1.2 i) From Theorem 1.1 we know that although some diverging terms appear in the intermediate stages of the analysis, no renormalisation is actually necessary in (1.1).

ii)The results obtained by using paracontrolled distribution method are expressed a little bit differently (see Theorem 3.12).

This paper is organized as follows. In Section 2, we use the regularity structure theory to obtain local existence and uniqueness of solutions to the 3D Navier-Stokes equations driven by space-time white noise. In Section 3, we apply the paracontrolled distribution method to deduce local existence and uniqueness of solutions. In Remark 3.13 we compare the two approaches.

2 N-S equation by regularity structure theory

2.1 Preliminary on regularity structure theory

In this subsection we recall some preliminaries for the theory of regularity structures from [16].

Definition 2.1 A regularity structure $\mathfrak{T} = (A, T, G)$ consists of the following elements:

(i) An index set $A \subset \mathbb{R}$ such that $0 \in A$, A is bounded from below and locally finite.

(ii) A model space T, which is a graded vector space $T = \oplus_{\alpha\in A} T_\alpha$, with each T_α a Banach space. Furthermore, T_0 is one-dimensional and has a basis vector $\mathbf{1}$. Given $\tau \in T$ we write $\|\tau\|_\alpha$ for the norm of its component in T_α.

(iii) A structure group G of (continuous) linear operators acting on T such that for every $\Gamma \in G$, every $\alpha \in A$ and every $\tau_\alpha \in T_\alpha$ one has

$$\Gamma\tau_\alpha - \tau_\alpha \in T_{<\alpha} := \bigoplus_{\beta<\alpha} T_\beta.$$

Furthermore, $\Gamma\mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

Now we have the regularity structure $\bar{T} = \bigoplus_{n\in\mathbb{N}} \bar{T}_n$ given by all polynomials in $d+1$ indeterminates, let us call them $X_0, ..., X_d$, which denote the time and space directions respectively. Denote $X^k = X_0^{k_0} \cdots X_d^{k_d}$ with k a multi-index. In this case, $A = \mathbb{N}$ and $\bar{T}_n$ denote the space of monomials that are homogeneous of degree n. The structure group can be defined by $\Gamma_h X^k = (X-h)^k$, $h \in \mathbb{R}^{d+1}$.

Given a scaling $\mathfrak{s} = (\mathfrak{s}_0, \mathfrak{s}_1, ..., \mathfrak{s}_d)$ of $\mathbb{R}^{d+1}$. We call $|\mathfrak{s}| = \mathfrak{s}_0 + \mathfrak{s}_1 + ... + \mathfrak{s}_d$ scaling dimension. We define the associate metric on $\mathbb{R}^{d+1}$ by

$$\|z - z'\|_\mathfrak{s} := \sum_{i=0}^d |z_i - z_i'|^{1/\mathfrak{s}_i}.$$

For $k = (k_0, ..., k_d)$ we define $|k|_\mathfrak{s} = \sum_{i=0}^d \mathfrak{s}_i k_i$.

Given a smooth compactly supported test function φ and a space-time coordinate $z = (t, x_1, ..., x_d) \in \mathbb{R}^{d+1}$, we denote by φ_z^λ the test function

$$\varphi_z^\lambda(s, y_1, ..., y_d) = \lambda^{-|\mathfrak{s}|}\varphi(\frac{s-t}{\lambda^{\mathfrak{s}_0}}, \frac{y_1 - x_1}{\lambda^{\mathfrak{s}_1}}, ..., \frac{y_d - x_d}{\lambda^{\mathfrak{s}_d}}).$$

Denote by $\mathcal{B}_\alpha$ the set of smooth test functions $\varphi : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ that are supported in the centred ball of radius 1 and such that their derivatives of order up to $1 + |\alpha|$ are uniformly bounded by 1. We denote by $\mathcal{S}'$ the space of all distributions on $\mathbb{R}^{d+1}$ and denote by $L(E, F)$ the set of all continuous linear maps between the topological vector spaces E and F. Now we give the definition of a model, which is a concrete way of associating every element in the abstract regularity structure to the actual Taylor polynomial at every point.

Definition 2.2 Given a regularity structure $\mathfrak{T}$, a model for $\mathfrak{T}$ consists of maps

$$\mathbb{R}^{d+1} \ni z \mapsto \Pi_z \in L(T, \mathcal{S}'), \quad \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \ni (z, z') \mapsto \Gamma_{zz'} \in G,$$

satisfying the algebraic compatibility conditions

$$\Pi_z \Gamma_{zz'} = \Pi_{z'}, \quad \Gamma_{zz'} \circ \Gamma_{z'z''} = \Gamma_{zz''},$$

as well as the analytical bounds

$$|\Pi_z \tau(\varphi_z^\lambda)| \lesssim \lambda^\alpha \|\tau\|_\alpha, \quad \|\Gamma_{zz'}\tau\|_\beta \lesssim \|z - z'\|_{\mathfrak{s}}^{\alpha-\beta} \|\tau\|_\alpha.$$

Here, the bounds are imposed uniformly over all $\tau \in T_\alpha$, all $\beta < \alpha \in A$ with $\alpha < \gamma$, $\gamma > 0$, and all test functions $\varphi \in \mathcal{B}_r$ with $r = \inf A$. They are imposed locally uniformly in z and z'.

Then for every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$ and any two models $Z = (\Pi, \Gamma)$ and $\bar{Z} = (\bar{\Pi}, \bar{\Gamma})$ we define

$$|||Z; \bar{Z}|||_{\gamma;\mathfrak{R}} := \sup_{z \in \mathfrak{R}}[\sup_{\varphi,\lambda,\alpha,\tau} \lambda^{-\alpha}|(\Pi_z\tau - \bar{\Pi}_z\tau)(\varphi_z^\lambda)| + \sup_{\|z-z'\|_{\mathfrak{s}} \le 1} \sup_{\alpha,\beta,\tau} \|z - z'\|_{\mathfrak{s}}^{\beta-\alpha} \|\Gamma_{zz'}\tau - \bar{\Gamma}_{zz'}\tau\|_\beta],$$

where the suprema are taken over the same sets as in Definition 2.2, but with $\|\tau\|_\alpha = 1$. This gives a natural topology for the space of all models for a given regularity structure.

Now we have the following definition for the spaces of distributions $\mathcal{C}_{\mathfrak{s}}^\alpha$, $\alpha < 0$, which is an extension of the definition of Hölder space to include $\alpha < 0$.

Definition 2.3 Let $\eta \in \mathcal{S}'$ and $\alpha < 0$. We say that $\eta \in \mathcal{C}_{\mathfrak{s}}^\alpha$ if the bound

$$|\eta(\varphi_z^\lambda)| \lesssim \lambda^\alpha,$$

holds uniformly over all $\lambda \in (0, 1]$, all $\varphi \in \mathcal{B}_\alpha$ and locally uniformly over $z \in \mathbb{R}^{d+1}$.

For every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$, we will denote by $\|\eta\|_{\alpha;\mathfrak{R}}$ the seminorm given by

$$\|\eta\|_{\alpha;\mathfrak{R}} := \sup_{z \in \mathfrak{R}} \sup_{\varphi \in \mathcal{B}_\alpha} \sup_{\lambda \le 1} \lambda^{-\alpha}|\eta(\varphi_z^\lambda)|.$$

We also write $\|\cdot\|_\alpha$ for the same expression with $\mathfrak{R} = \mathbb{R}^{d+1}$.

In the following we also use $\mathcal{C}^\alpha$ to denote $\mathcal{C}^\alpha_{\bar{\mathfrak{s}}}$ on $\mathbb{R}^d$ for the scaling $\bar{\mathfrak{s}} := (\mathfrak{s}_1, ..., \mathfrak{s}_d)$. On a bounded domain, $\mathcal{C}^\alpha$ coincides with the Besov space $B^\alpha_{\infty,\infty}$ defined in Section 3.

We also have the following definition of spaces of modelled distributions, which are the Hölder spaces on the regularity structure. Set $\mathfrak{P} = \{(t, x) : t = 0\}$. Given a subset $\mathfrak{K} \subset \mathbb{R}^{d+1}$ we also denote by $\mathfrak{K}_\mathfrak{P}$ the set

$$\mathfrak{K}_\mathfrak{P} = \{(z, \bar{z}) \in (\mathfrak{K} \setminus \mathfrak{P})^2 : z \neq \bar{z} \text{ and } \|z - \bar{z}\|_\mathfrak{s} \leq |t|^{\frac{1}{\mathfrak{s}_0}} \wedge |\bar{t}|^{\frac{1}{\mathfrak{s}_0}} \wedge 1\},$$

where $z = (t, x), \bar{z} = (\bar{t}, \bar{x})$.

Definition 2.4 Given a model (Π, Γ) for a regularity structure $\mathfrak{T}$ and $\mathfrak{P}$ as above. Then for any $\gamma > 0$ and $\eta \in \mathbb{R}$, the space $\mathcal{D}^{\gamma,\eta}$ consists of all functions $f : \mathbb{R}^{d+1} \setminus \mathfrak{P} \to \bigoplus_{\alpha<\gamma} T_\alpha$ such that for every compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$ one has

$$|||f|||_{\gamma,\eta;\mathfrak{K}} := \sup_{z\in\mathfrak{K}\setminus\mathfrak{P}} \sup_{l<\gamma} \frac{\|f(z)\|_l}{|t|^{\frac{\eta-l}{\mathfrak{s}_0}\wedge 0}} + \sup_{(z,\bar{z})\in\mathfrak{K}_\mathfrak{P}} \sup_{l<\gamma} \frac{\|f(z) - \Gamma_{z\bar{z}} f(\bar{z})\|_l}{\|z - \bar{z}\|_\mathfrak{s}^{\gamma-l} (|t| \wedge |\bar{t}|)^{\frac{\eta-\gamma}{\mathfrak{s}_0}}} < \infty.$$

Here we wrote $\|\tau\|_l$ for the norm of the component of τ in T_l and also used t and $\bar{t}$ as shorthands for the time components of the space-time points z and $\bar{z}$.

For $f \in \mathcal{D}^{\gamma,\eta}$ and $\bar{f} \in \bar{\mathcal{D}}^{\gamma,\eta}$ (denoting by $\bar{\mathcal{D}}^{\gamma,\eta}$ the space built over another model $(\bar{\Pi}, \bar{\Gamma})$), we also set

$$|||f; \bar{f}|||_{\gamma,\eta;\mathfrak{K}} := \sup_{z\in\mathfrak{K}\setminus\mathfrak{P}} \sup_{l<\gamma} \frac{\|f(z) - \bar{f}(z)\|_l}{|t|^{\frac{\eta-l}{\mathfrak{s}_0}\wedge 0}} + \sup_{(z,\bar{z})\in\mathfrak{K}_\mathfrak{P}} \sup_{l<\gamma} \frac{\|f(z) - \bar{f}(\bar{z}) - \Gamma_{z\bar{z}} f(\bar{z}) + \bar{\Gamma}_{z\bar{z}} \bar{f}(\bar{z})\|_l}{\|z - \bar{z}\|_\mathfrak{s}^{\gamma-l} (|t| \wedge |\bar{t}|)^{\frac{\eta-\gamma}{\mathfrak{s}_0}}},$$

which gives a natural distance between elements $f \in \mathcal{D}^{\gamma,\eta}$ and $\bar{f} \in \bar{\mathcal{D}}^{\gamma,\eta}$.

Given a regularity structure, we say that a subspace $V \subset T$ is a sector of regularity α if it is invariant under the action of the structure group G and it can be written as $V = \oplus_{\beta\in A} V_\beta$ with $V_\beta \subset T_\beta$, and $V_\beta = \{0\}$ for $\beta < \alpha$. We will use $\mathcal{D}^{\gamma,\eta}(V)$ to denote all functions in $\mathcal{D}^{\gamma,\eta}$ taking values in V.

On the regularity structure a product $\star$ is a bilinear map on T satisfying that for every $a \in T_\alpha$ and $b \in T_\beta$ one has $a \star b \in T_{\alpha+\beta}$ and $\mathbf{1} \star a = a \star \mathbf{1} = a$ for every $a \in T$. The product induces the pointwise product between modelled distribution under some conditions. For more details we refer to [16, Section 4].

Under suitable regularity assumptions, we can reconstruct from a given modelled distribution f, a distribution $\mathcal{R}f$ in the real world which "looks like $\Pi_x f(x)$ near x". This result, which defines the so-called reconstruction operator, is one of the most fundamental result in the regularity structures theory.

Theorem 2.5 (cf. [16, Proposition 6.9]) Given a regularity structure and a model (Π, Γ). Let $f \in \mathcal{D}^{\gamma,\eta}(V)$ for some sector V of regularity $\alpha \leq 0$, some $\gamma > 0$, and some $\eta \leq \gamma$. Then provided that $\alpha \wedge \eta > -\mathfrak{s}_0$, there exists a unique distribution $\mathcal{R}f \in \mathcal{C}^{\eta\wedge\alpha}_\mathfrak{s}$ such that

$$|(\mathcal{R}f - \Pi_z f(z))(\varphi_z^\lambda)| \lesssim \lambda^\gamma,$$

holds uniformly over $\lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_r$ with φ_z^λ compactly supported away from $\mathfrak{P}$ and locally uniformly over $z \in \mathbb{R}^{d+1}$.

Moreover, $(\Pi, \Gamma, f) \to \mathcal{R}f$ is jointly (locally) Lipschitz continuous with respect to the metric for (Π, Γ) and f defined in Definitions 2.2 and 2.4.

In order to define the integration against a singular kernel K, Martin Hairer in [16] introduced an abstract integration map $\mathcal{I} : T \to T$ to provide an "abstract" representation of $\mathcal{K}$ operating at the level of the regularity structure. In the regularity structure theory $\mathcal{I}$ is a linear map from T to T such that $\mathcal{I}T_\alpha \subset T_{\alpha+\beta}$ and $\mathcal{I}\bar{T} = 0$ and for every $\Gamma \in G, \tau \in T$ one has $\Gamma\mathcal{I}\tau - \mathcal{I}\Gamma\tau \in \bar{T}$.

Furthermore, we say that K is a β-regularising kernel if one can write $K = \sum_{n\geq 0} K_n$ where each $K_n : \mathbb{R}^{d+1} \to \mathbb{R}$ is smooth and compactly supported in a ball of radius 2^{-n} around the origin. Furthermore, we assume that for every multi-index k, one has a constant C such that

$$\sup_x |D^k K_n(x)| \leq C 2^{n(d+1-\beta+|k|_\mathfrak{s})},$$

holds uniformly in n. Finally, we assume that $\int K_n(x)E(x)dx = 0$ for every polynomial E of degree at most r for some sufficiently large value of r.

We say that a model Π realises K for $\mathcal{I}$ on a sector V if, for every $\alpha \in A$, every $a \in V_\alpha$ and every $x \in \mathbb{R}^d$, one has

$$\Pi_x \mathcal{I}a = \int_{\mathbb{R}^{d+1}} K(\cdot - z)(\Pi_x a)(dz) - \Pi_x \mathcal{J}(x)a,$$

where $\mathcal{J}(x)a = \sum_{|k|_\mathfrak{s}<\alpha+\beta} \frac{X^k}{k!} \int_{\mathbb{R}^{d+1}} D^k K(\cdot - z)(\Pi_x a)(dz)$.

The reason that $\Pi_x\mathcal{I}\tau \neq K * \Pi_x\tau$ is the following: Intuitively, $T_{\alpha+\beta}$ contains the elements that vanish at the order $\alpha + \beta$. Since $\mathcal{I}T_\alpha \subset T_{\alpha+\beta}$, one should subtract a suitable polynomial that forces the $\Pi_x\mathcal{I}a$ to vanish at the correct order.

Then we have the following results from [16, Proposition 6.16].

Theorem 2.6 Let $\mathfrak{T} = (A, T, G)$ be a regularity structure and (Π, Γ) be a model for $\mathfrak{T}$. Let K be a β-regularising kernel for some $\beta > 0$, let $\mathcal{I}$ be an abstract integration map acting on some sector V of regularity $\alpha \leq 0$, and let Π be a model realising K for $\mathcal{I}$. Let $\gamma > 0$, $\eta \leq \gamma$. Then provided that $\alpha \wedge \eta > -2$, $\gamma + \beta, \eta + \beta$ not in $\mathbb{N}$, there exists a continuous linear operator $\mathcal{K}_\gamma : \mathcal{D}^{\gamma,\eta}(V) \to \mathcal{D}^{\bar{\gamma},\bar{\eta}}$ with $\bar{\gamma} = \gamma + \beta$ and $\bar{\eta} = (\eta \wedge \alpha) + \beta$, such that

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R}f,$$

holds for $f \in \mathcal{D}^{\gamma,\eta}(V)$.

In the following we will only consider (1.1) with periodic boundary conditions. By the theory of regularity structures proposed in [16] we can define translation maps and use it to define the modelled distribution to be periodic. Now the fundamental domain of the translation maps is compact. We use the notations $O_T = (-\infty, T] \times \mathbb{R}^d$ and use $|||\cdot|||_{\gamma,\eta;T}$ as a short hand for $|||\cdot|||_{\gamma,\eta;O_T}$. Moreover, we have for some $\theta > 0$

$$|||\mathcal{K}_\gamma 1_{t>0} f|||_{\bar{\gamma},\bar{\eta};T} \lesssim T^\theta |||f|||_{\gamma,\eta;T}.$$

2.2 N-S equation

In this subsection we apply the regularity structure theory to the 3D Navier-Stokes equations on $\mathbb{T}^3$ driven by space-time white noise. In this case we have the scaling $\mathfrak{s} = (2, 1, 1, 1)$, so that the scaling dimension of space-time is 5. Since the kernel $G^{ij}, i, j = 1, 2, 3$, given by the heat kernel composed with the Leray projection P has the scaling property $G^{ij}(\frac{t}{\delta^2}, \frac{x}{\delta}) = \delta^3 G^{ij}(t, x)$ for $\delta > 0$, by [16, Lemma 5.5] it can be decomposed into $K^{ij} + R^{ij}, i, j = 1, 2, 3$, where K^{ij} is a 2-regularising kernel and $R^{ij} \in \mathcal{C}^\infty$. By [16] we can choose K^{ij} compactly supported and smooth away from the origin and such that it annihilates all polynomials up to some degree $r > 2$. Moreover, by [19] we have K^{ij} is of order -3, i.e. $|D^k K(z)| \leq C\|z\|_{\mathfrak{s}}^{-3-|k|_{\mathfrak{s}}}$ for every z with $\|z\|_{\mathfrak{s}} \leq 1$ and every multi-index k. We also use $D_j K$, $j = 1, 2, 3$, to represent the derivative of K with respect to the j-th space variable and $D_j K$ is also a 1-regularising kernel and of order -4.

Consider the regularity structure generated by the stochastic N-S equation with $\beta = 2, -\frac{13}{5} < \alpha < -\frac{5}{2}$. In the regularity structure we use symbol the Ξ_i to replace the driving noise ξ^i. For $i, i_1 = 1, 2, 3$, we introduce the integration map $\mathcal{I}^{ii_1}$ associated with K^{ii_1} and the integration map $\mathcal{I}_k^{ii_1}$ for a multiindex k, which represents integration against $D^k K^{ii_1}$. We recall the following notations from [16]: defining a set $\mathcal{F}$ by postulating that $\{\mathbf{1}, \Xi_i, X_j\} \subset \mathcal{F}$ and whenever $\tau, \bar{\tau} \in \mathcal{F}$, we have $\tau\bar{\tau} \in \mathcal{F}$ and $\mathcal{I}_k^{ij}(\tau) \in \mathcal{F}$; defining $\mathcal{F}_+$ as the set of all elements $\tau \in \mathcal{F}$ such that either $\tau = \mathbf{1}$ or $|\tau|_{\mathfrak{s}} > 0$ and such that, whenever τ can be written as $\tau = \tau_1\tau_2$ we have either $\tau_i = \mathbf{1}$ or $|\tau_i|_{\mathfrak{s}} > 0$; $\mathcal{H}, \mathcal{H}_+$ denote the sets of finite linear combinations of all elements in $\mathcal{F}, \mathcal{F}_+$, respectively. Here for each $\tau \in \mathcal{F}$ a weight $|\tau|_{\mathfrak{s}}$ which is obtained by setting $|\mathbf{1}|_{\mathfrak{s}} = 0$,

$$|\tau\bar{\tau}|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + |\bar{\tau}|_{\mathfrak{s}},$$

for any two formal expressions τ and $\bar{\tau}$ in $\mathcal{F}$ such that

$$|\Xi_i|_{\mathfrak{s}} = \alpha, \quad |X_i|_{\mathfrak{s}} = \mathfrak{s}_i, \quad |\mathcal{I}_k^{ii_1}(\tau)|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + 2 - |k|_{\mathfrak{s}}.$$

To apply the regularity structure theory we write the equation as follows: for $i = 1, 2, 3$

$$\begin{aligned}\partial_t v_1^i =& \nu \sum_{i_1=1}^3 P^{ii_1} \Delta v_1^{i_1} + \sum_{i_1=1}^3 P^{ii_1} \xi^{i_1}, \quad \mathrm{div} v_1 = 0, \\ \partial_t v^i =& \nu \sum_{i_1=1}^3 P^{ii_1} \Delta v^{i_1} - \sum_{i_1,j=1}^3 P^{ii_1} \frac{1}{2} D_j[(v^{i_1} + v_1^{i_1})(v^j + v_1^j)], \quad \mathrm{div} v = 0.\end{aligned} \tag{2.1}$$

Then $v_1 + v$ is the solution to the 3D Navier-Stokes equations driven by space-time white noise. Now we consider the second equation in (2.1). Define for $i, j = 1, 2, 3$,

$$\mathfrak{M}_F^{ij} = \{1, \mathcal{I}^{ii_1}(\Xi_{i_1}), \mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}), U_i, U_j, U_i U_j, \mathcal{I}^{ii_1}(\Xi_{i_1})U_j, U_i\mathcal{I}^{jj_1}(\Xi_{j_1}), i_1, j_1 = 1, 2, 3\}.$$

Then we build subsets $\{\mathcal{P}_n^i\}_{n\geq 0}$ and $\{\mathcal{W}_n\}_{n\geq 0}$ by the following algorithm: For $i, j = 1, 2, 3$, set $\mathcal{W}_0^{ij} = \mathcal{P}_0^i = \varnothing$ and

$$\mathcal{W}_n^{ij} = \mathcal{W}_{n-1}^{ij} \cup \bigcup_{\mathcal{Q}\in\mathfrak{M}_F^{ij}} \mathcal{Q}(\mathcal{P}_{n-1}^i, \mathcal{P}_{n-1}^j),$$

$$\mathcal{P}_n^i = \{X^k\} \cup \{\mathcal{I}_{i_2}^{ii_1}(\tau) : \tau \in \mathcal{W}_{n-1}^{i_1 i_2}, i_1, i_2 = 1, 2, 3\},$$

and

$$\mathcal{F}_F := \bigcup_{n\geq 0}\bigcup_{i,j=1}^{3} \mathcal{W}_n^{ij}, \quad \mathcal{F}_F^{ij} := \bigcup_{n\geq 0} \mathcal{W}_n^{ij}, i,j = 1,2,3.$$

Then $\mathcal{F}_F$ contains the elements required to describe both the solution and the terms in the equation (2.1). We denote by $\mathcal{H}_F, \mathcal{H}_F^{ij}, i,j = 1,2,3$, the set of finite linear combinations of elements in $\mathcal{F}_F$, $\mathcal{F}_F^{ij}$, respectively.

Remark 2.7 Here we construct $\mathcal{F}_F$ in a slightly different way from [16]. From (2.1) we observe that the integration map $\mathcal{I}_j^{ii_1}$ only acts on the elements belonging to $\mathcal{W}_n^{i_1 j}$. The regularity structure does not contain the elements belonging to $\mathcal{I}_j^{ii_1}(\mathcal{W}_n^{i_2 j_1})$ for $(i_1,j) \neq (i_2,j_1)$ and $(i_1,j) \neq (j_1,i_2)$, which is enough for us to describe the solution and the equations.

Now we follow [16] to construct the structure group G. Define a linear projection operator $P_+ : \mathcal{H} \to \mathcal{H}_+$ by imposing that

$$P_+\tau = \tau, \quad \tau \in \mathcal{F}_+, \quad P_+\tau = 0, \quad \tau \in \mathcal{F}\setminus\mathcal{F}_+,$$

and two linear maps $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}_+$ and $\Delta^+ : \mathcal{H}_+ \to \mathcal{H}_+\otimes\mathcal{H}_+$ by

$$\Delta\mathbf{1} = \mathbf{1}\otimes\mathbf{1}, \quad \Delta^+\mathbf{1} = \mathbf{1}\otimes\mathbf{1},$$

$$\Delta X_i = X_i\otimes\mathbf{1} + \mathbf{1}\otimes X_i, \quad \Delta^+ X_i = X_i\otimes\mathbf{1} + \mathbf{1}\otimes X_i,$$

$$\Delta\Xi^i = \Xi^i\otimes\mathbf{1},$$

and recursively by

$$\Delta(\tau\bar{\tau}) = (\Delta\tau)(\Delta\bar{\tau})$$

$$\Delta(\mathcal{I}_k^{ij}\tau) = (\mathcal{I}_k^{ij}\otimes I)\Delta\tau + \sum_{l,m}\frac{X^l}{l!}\otimes\frac{X^m}{m!}(P_+\mathcal{I}_{k+l+m}^{ij}\tau),$$

$$\Delta^+(\tau\bar{\tau}) = (\Delta^+\tau)(\Delta^+\bar{\tau})$$

$$\Delta^+(\mathcal{I}_k^{ij}\tau) = (I\otimes\mathcal{I}_k^{ij}\tau) + \sum_{l}(P_+\mathcal{I}_{k+l}^{ij}\otimes\frac{(-X)^l}{l!})\Delta\tau.$$

By using the theory of regularity structures (see [16, Section 8]) we can define a structure group G_F of linear operators acting on $\mathcal{H}_F$ satisfying Definition 2.1 as follows: For group-like elements $g \in \mathcal{H}_+^*$, the dual of $\mathcal{H}_+$, $\Gamma_g : \mathcal{H} \to \mathcal{H}, \Gamma_g\tau = (I\otimes g)\Delta\tau$. By [16, Theorem 8.24] we construct the following regularity structure.

Theorem 2.8 Let $T = \mathcal{H}_F$ with $T_\gamma = \langle\{\tau \in \mathcal{F}_F : |\tau|_{\mathfrak{s}} = \gamma\}\rangle$, $A = \{|\tau|_{\mathfrak{s}} : \tau \in \mathcal{F}_F\}$ and let G_F be as above. Then $\mathfrak{T}_F = (A, \mathcal{H}_F, G_F)$ defines a regularity structure $\mathfrak{T}$. Furthermore, for every $i, i_1 = 1,2,3$, $\mathcal{I}^{ii_1}$ is an abstract integration map of order 2.

Proof In our case, the nonlinearity is locally subcritical. (i) (ii) in Definition 2.1 can be checked easily. (iii) in Definition 2.1 and the last result for $\mathcal{I}^{ii_1}$ follow from the definitions of Δ and Γ_g. □

We also endow $\mathfrak{T}_F$ with a natural commutative product $\star$ by setting $\tau\star\tau' = \tau\tau'$ for all basis vectors τ, τ'.

Now we come to construct suitable models associated with the regularity structure above. Given any continuous approximation ξ_ε to the driving noise ξ, we set for $x, y \in \mathbb{R}^4$

$$(\Pi_x^{(\varepsilon)}\Xi_i)(y) = \xi_\varepsilon^i(y), \quad (\Pi_x^{(\varepsilon)}X^k)(y) = (y-x)^k,$$

and recursively define

$$(\Pi_x^{(\varepsilon)}\tau\bar{\tau})(y) = (\Pi_x^{(\varepsilon)}\tau)(y)(\Pi_x^{(\varepsilon)}\bar{\tau})(y),$$

and

$$(\Pi_x^{(\varepsilon)}\mathcal{I}_k^{ij}\tau)(y) = \int D_1^k K^{ij}(y-z)(\Pi_x^{(\varepsilon)}\tau)(z)dz + \sum_l \frac{(y-x)^l}{l!} f_x^{(\varepsilon)}(P_+\mathcal{I}_{k+l}^{ij}\tau). \tag{2.2}$$

Here $f_x^{(\varepsilon)}(\mathcal{I}_l^{ij}\tau)$ are defined by

$$f_x^{(\varepsilon)}(\mathcal{I}_l^{ij}\tau) = -\int D_1^l K^{ij}(x-z)(\Pi_x^{(\varepsilon)}\tau)(z)dz. \tag{2.3}$$

Furthermore, we impose $f_x^{(\varepsilon)}(X_i) = -x_i$, $f_x^{(\varepsilon)}(\tau\bar{\tau}) = f_x^{(\varepsilon)}(\tau)f_x^{(\varepsilon)}(\bar{\tau})$ and extend this to all of $\mathcal{H}_+$ by linearity. Then define

$$\Gamma_{xy}^{(\varepsilon)} = (\Gamma_{f_x^{(\varepsilon)}})^{-1} \circ \Gamma_{f_y^{(\varepsilon)}}, \tag{2.4}$$

where $\Gamma_{f_x^{(\varepsilon)}}\tau := (I \otimes f_x^{(\varepsilon)})\Delta\tau$ for $\tau \in \mathcal{H}$.

By [16, Proposition 8.27] we have

Proposition 2.9 $(\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$ is a model for the regularity structure $\mathfrak{T}_F$ constructed in Theorem 2.8.

Definition 2.10 A model (Π, Γ) for $\mathfrak{T}$ is admissible if it satisfies $(\Pi_x X^k)(y) = (y-x)^k$ as well as (2.2), (2.3) and (2.4). We denote by $\mathcal{M}_F$ the set of admissible models.

Set

$$\begin{aligned}\mathcal{F}_0 =\{&\mathbf{1}, \Xi_i, \mathcal{I}^{ii_1}(\Xi_{i_1}), \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})), \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1})), \mathcal{I}_j^{ii_1}(\mathcal{I}^{jj_1}(\Xi_{j_1})),\\ &\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}), \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}),\\ &\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})), \mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}),\\ &\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{ll_1}(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}), i, j, k, l, i_1, i_2, i_3, j_1, j_2, k_1, l_1, l_2 = 1, 2, 3\}\end{aligned}$$

and

$$\mathcal{F}_* = \{\mathcal{I}^{ik}(\Xi_k), \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}), i, k, i_1, i_2, j, j_1, k_1 = 1, 2, 3\}.$$

To make our paper more readable we use the tree notation from [16] to explain the complicated elements in $\mathcal{F}_0$. However, unlike as in the Φ_3^4 case, the solution to the stochastic N-S equation is vector valued and there are a lot of superscripts and subscripts for the elements in $\mathcal{F}_0$, which will not be noticeable in the tree notation. The tree notation only helps us to make the complicated notation clearer.

For Ξ we simply draw a dot. The integration map $\mathcal{I}$ is then represented by a downfacing line while the integration map $\mathcal{I}_j$ is then represented by a downfacing dotted line. The multiplication of symbols is obtained by joining them at the root.

$$\mathcal{F}_0 = \{\mathbf{1}, \Xi_i,$$,  ,  ,  ,  ,  ,  ,  $\},$

$\mathcal{F}_* = \{$  ,  $\}.$

We choose $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$ and the reason for $\alpha > -\frac{13}{5}$ is that this is precisely the value of α at which the homogeneity of the term $\mathcal{I}_j(\tau)\mathcal{I}(\Xi)$ vanishes for $\tau =$ .

Then $\mathcal{F}_0 \subset \mathcal{F}_F$ contains every $\tau \in \mathcal{F}_F$ with $|\tau|_\mathfrak{s} \le 0$ and for every $\tau \in \mathcal{F}_0$, $\Delta\tau \in \langle \mathcal{F}_0 \rangle \otimes \langle \mathrm{Alg}(\mathcal{F}_*) \rangle$. Here $\langle \mathcal{F}_0 \rangle$ denotes the linear span of $\mathcal{F}_0$ and $\mathrm{Alg}(\mathcal{F}_*)$ denotes the set of all elements in $\mathcal{F}_+$ of the form $X^k \prod_{i,i_1,i_2} \mathcal{I}_{l_i}^{i_1 i_2} \tau_i$ for some multiindices k and l_i such that $|\mathcal{I}_{l_i}^{ii_1}\tau_i|_\mathfrak{s} > 0$ and $\tau_i \in \mathcal{F}_*$.

As mentioned in the introduction, we should do renormalisations for the model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ built from ξ_ε such that it converges as $\xi_\varepsilon \to \xi$ in a suitable sense. In the theory of regularity structure, this has been transferred to find a sequence of M_ε belonging to the renormalisation group $\mathfrak{R}_0$ defined in [16, Definition 8.43] such that $M_\varepsilon(\Pi^\varepsilon, \Gamma^\varepsilon)$ converges to a finite limit. In the following we use the notations and definitions in [16, Section 8.3] and follow Hairer's idea to define M. We also use the tree notation as above to make it clearer.

For constants $C^1_{ii_1jj_1}, C^2_{ii_1i_2jj_1j_2kk_1ll_1}, C^3_{ii_1i_2i_3kk_1ll_1jj_1}, C^4_{ii_1i_2kk_1ll_1l_2jj_1}, i, j, k, l, i_1, i_2, i_3, j_1, k_1, l_1, l_2 = 1, 2, 3$, we define a linear map M on $\langle \mathcal{F}_0 \rangle$ by

$$M(\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1})) = \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^1_{ii_1jj_1}\mathbf{1},$$

M  $=$  $- C^1_{ii_1jj_1}\mathbf{1},$

$$M(\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))$$
$$=\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})) - C^2_{ii_1i_2jj_1j_2kk_1ll_1}\mathbf{1},$$

M  $=$  $- C^2_{ii_1i_2jj_1j_2kk_1ll_1}\mathbf{1},$

$$M(\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}))$$
$$=\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^3_{ii_1i_2i_3kk_1ll_1jj_1}\mathbf{1}, \quad (2.5)$$

M  $=$  $- C^3_{ii_1i_2i_3kk_1ll_1jj_1}\mathbf{1},$

$$M(\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{ll_1}(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}))$$
$$=\mathcal{I}_l^{ii_1}(\mathcal{I}_k^{ll_1}(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}) - C^4_{ii_1i_2kk_1ll_1l_2jj_1}\mathbf{1},$$

as well as $M(\tau) = \tau$ for the remaining basis vectors in $\mathcal{F}_0$. Here we omit the tree notation for the last one since it is the same as the one including C^3. We claim that for any $\tau \in \mathcal{F}_0$,

$$\Delta^M \tau = (M\tau) \otimes \mathbf{1}. \quad (2.6)$$

Since τ satisfies $M\tau = \tau - C\mathbf{1}$ for any $\tau \in \mathcal{F}_0$, it is easy to check that (2.6) holds. Here for the definitions of $\Delta^M, \mathcal{A}, \hat{M}, \hat{\Delta}^M$ we refer to [16, Section 8.3].

For $\tau =$ , we have

Δ^+  $=$  $\otimes \mathbf{1} + \mathbf{1} \otimes$  .

$(\mathcal{A}\hat{M}\mathcal{A}\otimes\hat{M})\Delta^{+}$  $=$  $\otimes\mathbf{1}+\mathbf{1}\otimes$  ,

It follows that

$\hat{\Delta}^{M}$  $=$  $\otimes\mathbf{1}.$

For $\tau=\mathcal{I}_l^{ii_1}(\tau_1)$, where $\tau_1=$  , $i,i_1=1,2,3$, we have

$$\Delta^{+}\mathcal{I}_l^{ii_1}(\tau_1)=\mathcal{I}_l^{ii_1}(\tau_1)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{I}_l^{ii_1}(\tau_1).$$

$$(\mathcal{A}\hat{M}\mathcal{A}\otimes\hat{M})\Delta^{+}\mathcal{I}_l^{ii_1}(\tau_1)=\mathcal{I}_l^{ii_1}(\tau_1)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{I}_l^{ii_1}(\tau_1),$$

which implies that

$$\hat{\Delta}^{M}\mathcal{I}_l^{ii_1}(\tau_1)=\mathcal{I}_l^{ii_1}(\tau_1)\otimes\mathbf{1}.$$

As a consequence of this expression, M belongs to the renormalisation group $\mathfrak{R}_0$ defined in [16, Definition 8.43]. Then by [16, Theorem 8.46] we can define (Π^M,Γ^M) and it is an admissible model for $\mathfrak{T}_F$ on $\langle\mathcal{F}_0\rangle$. Furthermore, it extends uniquely to an admissible model for all of $\mathfrak{T}_F$. By (2.6) we also have

$$\Pi_x^M\tau=\Pi_x M\tau.$$

Now we lift the equation onto the abstract regularity structure. First, we define for any $\alpha_0<0$ and compact set $\mathfrak{K}$ the norm

$$|\xi|_{\alpha_0;\mathfrak{K}}=\sup_{s\in\mathbb{R}}\|\xi 1_{t\geq s}\|_{\alpha_0;\mathfrak{K}},$$

and we denote by $\bar{\mathcal{C}}_{\mathfrak{s}}^{\alpha_0}$ the intersections of the completions of smooth functions under $|\cdot|_{\alpha_0;\mathfrak{K}}$ for all compact sets $\mathfrak{K}$.

Since $\alpha<-\frac{5}{2}$, Theorem 2.5 does not apply to $\mathbf{R}^{+}\Xi_i$ directly, where $\mathbf{R}^{+}:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is given by $\mathbf{R}^{+}(t,x)=1$ for $t>0$ and $\mathbf{R}^{+}(t,x)=0$ otherwise. To define the reconstruction operator for $\mathbf{R}^{+}\Xi_i$ by hand, we need the following results, which have been proved by [16, Proposition 9.5].

Proposition 2.11 Let $\xi=(\xi^1,\xi^2,\xi^3)$, with $\xi^i, i=1,2,3$ being independent white noises on $\mathbb{R}\times\mathbb{T}^3$, which we extend periodically to $\mathbb{R}^4$. Let $\rho:\mathbb{R}^4\to\mathbb{R}$ be a smooth compactly supported function with Lebesgue integral equal to 1, set $\rho_\varepsilon(t,x)=\varepsilon^{-5}\rho(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$ and define $\xi_\varepsilon^i=\rho_\varepsilon*\xi^i$. Then for every $i,i_1=1,2,3$, $K^{ii_1}*\xi^{i_1}\in C(\mathbb{R},\mathcal{C}^{\alpha+2}(\mathbb{R}^3))$ almost surely. Moreover, for every compact set $\mathfrak{K}\subset\mathbb{R}^4$ and every $0<\theta<-\alpha-\frac{5}{2}$ we have

$$E|\xi^i-\xi_\varepsilon^i|_{\alpha;\mathfrak{K}}\lesssim\varepsilon^{\theta}.$$

Finally for every $0<\kappa<-\alpha-\frac{5}{2}$, we have the bound

$$E\sup_{t\in[0,1]}\|K^{ii_1}*\xi^{i_1}(t,\cdot)-K^{ii_1}*\xi_\varepsilon^{i_1})(t,\cdot)\|_{\alpha+2}\lesssim\varepsilon^{\kappa}.$$

Now we reformulate the fixed point map as

$$\begin{aligned}v_1^i&=\sum_{i_1=1}^{3}(\mathcal{K}_{\bar{\gamma}}^{ii_1}+R_{\gamma}^{ii_1}\mathcal{R})\boldsymbol{R}^{+}\Xi_{i_1},\\ u^i&=-\frac{1}{2}\sum_{i_1,j=1}^{3}((\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar{\gamma}}+(D_jR^{ii_1})_{\gamma}\mathcal{R})\boldsymbol{R}^{+}(u^{i_1}\star u^j)+v_1^i+\sum_{i_1=1}^{3}\mathcal{G}^{ii_1}u_0^{i_1}.\end{aligned}\tag{2.7}$$

Here for $i, i_1, j = 1, 2, 3$, $\mathcal{K}_{\bar\gamma}^{ii_1}$ and $(\mathcal{D}_j\mathcal{K}^{ii_1})_{\bar\gamma}$ are the continuous linear operators obtained by Theorem 2.6 associated with the kernel K^{ii_1} and $D_jK^{ii_1}$ respectively,

$$R_\gamma^{ii_1} : \mathcal{C}_\mathfrak{s}^\alpha \to \mathcal{D}^{\gamma,\eta}, (R_\gamma^{ii_1}f)(z) = \sum_{|k|_\mathfrak{s}<\gamma} \frac{X^k}{k!} \int D_1^k R^{ii_1}(z-\bar z)f(\bar z)d\bar z,$$

$$(D_jR^{ii_1})_\gamma : \mathcal{C}_\mathfrak{s}^\alpha \to \mathcal{D}^{\gamma,\eta}, (D_jR^{ii_1})_\gamma f(z) = \sum_{|k|_\mathfrak{s}<\gamma} \frac{X^k}{k!} \int D_1^k (D_jR^{ii_1})(z-\bar z)f(\bar z)d\bar z,$$

$$\mathcal{G}u_0 = \sum_{|k|_\mathfrak{s}<\gamma} \frac{X^k}{k!} D^k(Gu_0)(z),$$

where $\gamma, \bar\gamma$ will be chosen below. We also use that $\int K(x-y)D_jf(y)dy = \int D_jK(x-y)f(y)dy$ and define $\mathcal{R}\boldsymbol{R}^+\Xi$ as the distribution $\xi\mathbf{1}_{t\geq 0}$.

We consider the second equation in (2.7): Define

$$V^i := \oplus_{i_1,j=1}^3 \mathcal{I}_j^{ii_1}(\mathcal{H}_F^{i_1j}) \oplus \text{span}\{\mathcal{I}^{ii_1}(\Xi_{i_1}), i_1 = 1, 2, 3\} \oplus \bar T,$$

$$V = V^1 \times V^2 \times V^3.$$

For $\gamma > 0, \eta \in \mathbb{R}$ we also define

$$\mathcal{D}^{\gamma,\eta}(V) := \mathcal{D}^{\gamma,\eta}(V^1) \times \mathcal{D}^{\gamma,\eta}(V^2) \times \mathcal{D}^{\gamma,\eta}(V^3).$$

$$(\mathcal{D}^{\gamma,\eta})^3 := \mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma,\eta}.$$

Lemma 2.12 For $\gamma > |\alpha + 2|$ and $-1 < \eta \leq \alpha + 2$, the map $u \mapsto u^iu^j$ is locally Lipschitz continuous from $\mathcal{D}^{\gamma,\eta}(V)$ into $\mathcal{D}^{\gamma+\alpha+2,2\eta}$.

Proof This is a consequence of [16, Proposition 6.12, Proposition 6.15]. □

Now for γ, η as in Lemma 2.12 and $u_0^{i_1} \in \mathcal{C}^\eta(\mathbb{R}^3), i_1 = 1, 2, 3$, periodic, we have $P^{ii_1}u_0^{i_1} \in \mathcal{C}^\eta(\mathbb{R}^3), i, i_1 = 1, 2, 3$ (see Lemma 3.6), which by [16, Lemma 7.5] implies that $\mathcal{G}^{ii_1}u_0^{i_1} \in \mathcal{D}^{\gamma,\eta}, i, i_1 = 1, 2, 3$. By Proposition 2.11 and [16, Remark 6.17] we also have that $v_1^i \in \mathcal{D}^{\gamma,\eta}$ for $i = 1, 2, 3$. Now we can apply a fixed point argument in $(\mathcal{D}^{\gamma,\eta})^3$ to obtain existence and uniqueness of local solutions to (2.7).

Proposition 2.13 Let $\mathfrak{T}_F$ be the regularity structure from Theorem 2.8 associated to the stochastic N-S equation driven by space-time white noise with $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Let $\eta \in (-1, \alpha + 2]$, $\gamma > |\alpha + 2|$, $u_0 \in \mathcal{C}^\eta(\mathbb{R}^3)$, periodic and let $Z = (\Pi, \Gamma) \in \mathcal{M}_F$ be an admissible model for $\mathfrak{T}_F$ with the additional properties that for $i, i_1 = 1, 2, 3$, $\xi^i := \mathcal{R}\Xi^i$ belongs to $\bar{\mathcal{C}}_\mathfrak{s}^\alpha$ and that $K^{ii_1} * \xi^{i_1} \in C(\mathbb{R}, \mathcal{C}^\eta)$. Then there exists a maximal solution $\mathcal{S}^L \in (\mathcal{D}^{\gamma,\eta})^3$ to the equation (2.7).

Proof Consider the second equation in (2.7). We have that u takes values in a sector of regularity $\zeta = \alpha + 2$ and $u^iu^j, i, j = 1, 2, 3$, takes value in a sector of regularity $\bar\zeta = 2\alpha + 4$ satisfying $\zeta < \bar\zeta + 1$. For η and γ we have $\bar\eta = 2\eta$ and $\gamma > \bar\gamma = \gamma + \alpha + 2 > 0$ and $\bar\gamma > \gamma + 1$. By Lemma 2.12 for $i, j = 1, 2, 3$, u^iu^j is locally Lipschitz continuous from $\mathcal{D}^{\gamma,\eta}(V)$ to $\mathcal{D}^{\bar\gamma,\bar\eta}$. Then $\eta < (\bar\eta \wedge \bar\zeta) + 1$ and $(\bar\eta \wedge \bar\zeta) + 2 > 0$ are satisfied by our assumptions. We consider a fixed model. Denote by $M_F^i(u)$ the right hand side of the second equation in (2.7). By [16, Theorem 7.1, Lemma 7.3] and local Lipschitz continuity of $u \mapsto u^i u^j$ we obtain that there exists $\kappa > 0$ such that for every $R > 0$

$$\sum_{i=1}^{3} |||M_F^i(u) - M_F^i(\bar{u})|||_{\gamma,\eta;T} \lesssim T^{\kappa} \sum_{i,j=1}^{3} |||u^i u^j - \bar{u}^i \bar{u}^j|||_{\bar{\gamma},\bar{\eta};T}$$
$$\lesssim T^{\kappa} \sum_{i=1}^{3} |||u^i - \bar{u}^i|||_{\gamma,\eta;T},$$

uniformly over $T \in [0,1]$ and over all $u, \bar{u}$ such that $|||u^i|||_{\gamma,\eta;T} + |||\bar{u}^i|||_{\gamma,\eta;T} \leq R$. Then we obtain local existence and uniqueness of the solutions by similar arguments as in the proof of [16, Theorem 7.8]. Here we consider vector valued solutions and the corresponding norm is the sum of the norm for each component. To extend this local map up to the first time where $\sum_{i=1}^{3} \|(\mathcal{R}u^i)(t,\cdot)\|_\eta$ blows up, we write $u = v_1 + v_2 + v_3$ with v_1 in (2.7) and

$$v_2^i = -\frac{1}{2} \sum_{i_1,j=1}^{3} ((\mathcal{D}_j \mathcal{K}^{ii_1})_{\bar{\gamma}} + (D_j R^{ii_1})_\gamma \mathcal{R}) \boldsymbol{R}^+ (v_1^{i_1} \star v_1^j),$$
$$v_3^i = -\frac{1}{2} \sum_{i_1=1}^{3} ((\mathcal{D}_j \mathcal{K}^{ii_1})_{\bar{\gamma}} + (D_j R^{ii_1})_\gamma \mathcal{R}) \boldsymbol{R}^+ [(v_3^{i_1} + v_2^{i_1}) \star (v_3^j + v_2^j)$$
$$+ (v_3^{i_1} + v_2^{i_1}) \star v_1^j + v_1^{i_1} \star (v_3^j + v_2^j)] + \sum_{i_1=1}^{3} \mathcal{G}^{ii_1} u_0^{i_1},$$

In this case v_3^i takes values in a function-like sector of regularity $3\alpha + 8$ and we can use similar arguments as in the proof of [16, Proposition 7.11] to conclude the results. $\square$

Remark 2.14 Here the lower bound for η is -1, which seems to be optimal by the theory of regularity structures. The reason for this is as follows: the nonlinear term always contains $v \star v$ and thus $\bar{\eta} \leq 2\eta$ which should be larger than -2 required by [16, Theorem 7.8]. As a result, $\eta > -1$.

Set $O := [-1, 2] \times \mathbb{R}^3$. Given a model $Z = (\Pi, \Gamma)$ for $\mathfrak{T}_F$, a periodic initial condition $u_0 \in (\mathcal{C}^\eta)^3$, and some cut-off value $L > 0$, we denote by $u = \mathcal{S}^L(u_0, Z) \in (\mathcal{D}^{\gamma,\eta})^3$ and $T = T^L(u_0, Z) \in \mathbb{R}_+ \cup \{+\infty\}$ the (unique) modelled distribution and time such that (2.7) holds on $[0, T]$, such that $\|(\mathcal{R}u)(t,\cdot)\|_\eta < L$ for $t < T$, and such that $\|(\mathcal{R}u)(t,\cdot)\|_\eta \geq L$ for $t \geq T$. Then by [16, Corollary 7.12] we obtain the following result.

Proposition 2.15 Let $L > 0$ be fixed. In the setting of Proposition 2.13, for every $\varepsilon > 0$ and $C > 0$ there exists $\delta > 0$ such that setting $T = 1 \wedge T^L(u_0, Z) \wedge T^L(\bar{u}_0, \bar{Z})$ we have

$$\|\mathcal{S}^L(u_0, Z) - \mathcal{S}^L(\bar{u}_0, \bar{Z})\|_{\gamma,\eta;T} \leq \varepsilon,$$

for all $u_0, \bar{u}_0, Z, \bar{Z}$ provided that $|||Z|||_{\gamma;O} \leq C$, $|||\bar{Z}|||_{\gamma;O} \leq C$, $\|u_0\|_\eta \leq L/2$, $\|\bar{u}_0\|_\eta \leq L/2$, $\|u_0 - \bar{u}_0\|_\eta \leq \delta$, and $|||Z; \bar{Z}|||_{\gamma;O} \leq \delta$ and

$$|\xi|_{\alpha;O} + |\bar{\xi}|_{\alpha;O} \leq C,$$

$$\sum_{i,i_1=1}^{3} \sup_{t\in[0,1]} \left[\|(K^{ii_1} * \xi^{i_1})(t,\cdot)\|_\eta + \|(K^{ii_1} * \bar{\xi}^{i_1})(t,\cdot)\|_\eta\right] \le C,$$

as well as

$$|\xi - \bar{\xi}|_{\alpha;O} \le \delta,$$

$$\sum_{i,i_1=1}^{3} \sup_{t\in[0,1]} \|(K^{ii_1} * \xi^{i_1})(t,\cdot) - (K^{ii_1} * \bar{\xi}^{i_1})(t,\cdot)\|_\eta \le \delta,$$

where $\bar{\xi}^i = \bar{\mathcal{R}}\Xi^i$ and $\bar{\mathcal{R}}$ is the reconstruction operator associated to $\bar{Z}$.

As in [16, Section 9] we now identify solutions corresponding to a model that has been renormalised by M with classical solutions to a modified equation.

Proposition 2.16 Given a continuous periodic vector $\xi_\varepsilon = (\xi_\varepsilon^1, \xi_\varepsilon^2, \xi_\varepsilon^3)$, denote by $Z_\varepsilon = (\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$ the associated canonical model realising $\mathfrak{T}_F$ given in Proposition 2.9. Let M be the renormalisation map defined in (2.5). Then for every $L > 0$ and periodic $u_0 \in C^\eta(\mathbb{R}^3;\mathbb{R}^3)$, $u_\varepsilon = \mathcal{R}S^L(u_0, Z_\varepsilon)$ satisfies the following equation on $[0, T^L(u_0, Z_\varepsilon)]$ in the mild sense:

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{2}P\mathrm{div}(u_\varepsilon \otimes u_\varepsilon) + P\xi_\varepsilon, \quad \mathrm{div} u_\varepsilon = 0, \quad u_\varepsilon(0) = Pu_0.$$

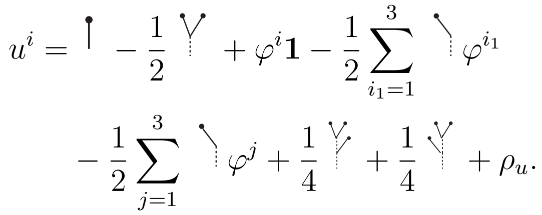
Furthermore, $u_\varepsilon^M = \mathcal{R}S^L(u_0, MZ_\varepsilon)$ also satisfies the same equation on $[0, T^L(u_0, MZ_\varepsilon)]$ in the mild sense.

Proof We follow a similar argument as in the proof of [16, Proposition 9.4].

For $i = 1,2,3$, the solution u^i to the abstract fixed point map can be expanded as

$$\begin{aligned} u^i =& \sum_{i_1=1}^{3} \mathcal{I}^{ii_1}(\Xi_{i_1}) - \frac{1}{2}\sum_{j,i_1,i_2,j_1=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1})) + \varphi^i\mathbf{1} - \frac{1}{2}\sum_{j,i_1,j_1=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{jj_1}(\Xi_{j_1}))\varphi^{i_1} \\ & - \frac{1}{2}\sum_{j,i_1,i_2=1}^{3} \mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\varphi^j + \frac{1}{4}\sum_{i_1,i_2,i_3,j,j_1,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}_j^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{jj_1}(\Xi_{j_1}))\mathcal{I}^{kk_1}(\Xi_{k_1})) \\ & + \frac{1}{4}\sum_{i_1,i_2,j,j_1,k,k_1,k_2=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}_j^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))) + \rho_u. \end{aligned}$$

i.e.

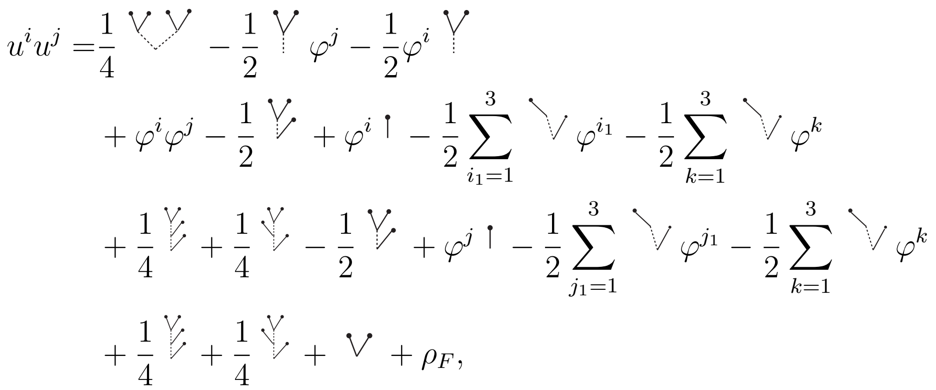


Here every component of ρ_u has homogeneity strictly greater than $3\alpha + 8$. Then we have

$$\begin{aligned} u^iu^j =& \frac{1}{4}\sum_{i_1,i_2,j_1,j_2,k,k_1,l,l_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})) \\ & - \frac{1}{2}\sum_{i_1,i_2,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^j - \frac{1}{2}\varphi^i \sum_{j_1,j_2,k,k_1=1}^{3} \mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{kk_1}(\Xi_{k_1})) \end{aligned}$$

$$
\begin{aligned}
&+\varphi^i\varphi^j - \frac{1}{2}\sum_{i_1,i_2,j_1,k,k_1=1}^{3} \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) + \varphi^i\sum_{j_1=1}^{3}\mathcal{I}^{jj_1}(\Xi_{j_1})\\
&-\frac{1}{2}\sum_{i_1,j_1,k,k_1=1}^{3}\mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^{i_1}\mathcal{I}^{jj_1}(\Xi_{j_1}) - \frac{1}{2}\sum_{i_1,i_2,j_1,k=1}^{3}\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\varphi^k\mathcal{I}^{jj_1}(\Xi_{j_1})\\
&+\frac{1}{4}\sum_{i_1,i_2,i_3,l,l_1,k,k_1,j_1=1}^{3}\mathcal{I}_k^{ii_1}(\mathcal{I}_l^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})\\
&+\frac{1}{4}\sum_{i_1,i_2,k,k_1,k_2,l,l_1,j_1=1}^{3}\mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}_l^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))\mathcal{I}^{jj_1}(\Xi_{j_1})\\
&-\frac{1}{2}\sum_{i_1,j_1,j_2,k,k_1=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ii_1}(\Xi_{i_1}) + \sum_{i_1=1}^{3}\varphi^j\mathcal{I}^{ii_1}(\Xi_{i_1})\\
&-\frac{1}{2}\sum_{i_1,j_1,k,k_1=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\varphi^{j_1}\mathcal{I}^{ii_1}(\Xi_{i_1}) - \frac{1}{2}\sum_{i_1,j_1,j_2,k=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2}))\varphi^k\mathcal{I}^{ii_1}(\Xi_{i_1})\\
&+\frac{1}{4}\sum_{i_1,j_1,j_2,j_3,l,l_1,k,k_1=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}_l^{j_1j_2}(\mathcal{I}^{j_2j_3}(\Xi_{j_3})\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ii_1}(\Xi_{i_1})\\
&+\frac{1}{4}\sum_{i_1,j_1,j_2,l,l_1,k,k_1,k_2=1}^{3}\mathcal{I}_k^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}_l^{kk_1}(\mathcal{I}^{k_1k_2}(\Xi_{k_2})\mathcal{I}^{ll_1}(\Xi_{l_1})))\mathcal{I}^{ii_1}(\Xi_{i_1})\\
&+\sum_{i_1,j_1=1}^{3}\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}) + \rho_F,
\end{aligned}
$$

i.e.



where ρ_F has strictly positive homogeneity. Moreover, we have

$$
\mathcal{R}u^i = -\frac{1}{2}\sum_{i_1,i_2,j,j_1=1}^{3} D_jK^{ii_1} * (K^{i_1i_2} * \xi_\varepsilon^{i_2} \cdot K^{jj_1} * \xi_\varepsilon^{j_1}) + \varphi^i + \sum_{i_1=1}^{3} K^{ii_1} * \xi_\varepsilon^{i_1},
$$

where $\mathcal{R}$ is the reconstruction operator associated with Z_ε. Since $\Delta^M\tau = M\tau \otimes 1$, one has the identity $(\Pi_z^{M,(\varepsilon)}\tau)(z) = (\Pi_z^{(\varepsilon)}M\tau)(z)$. It follows that for the reconstruction operator $\mathcal{R}^M$

associated with MZ_ε

$$\mathcal{R}^M(u^iu^j) =\mathcal{R}u^i\mathcal{R}u^j - \frac{1}{4}\sum_{i_1,i_2,j_1,j_2,k,k_1,l,l_1=1}^{3} C^2_{ii_1i_2jj_1j_2kk_1ll_1} - \sum_{i_1,j_1=1}^{3} C^1_{ii_1jj_1}$$
$$- \frac{1}{4}\sum_{i_1,i_2,i_3,k,k_1,l,l_1,j_1=1}^{3} C^3_{ii_1i_2i_3ll_1kk_1jj_1} - \frac{1}{4}\sum_{i_1,i_2,k,k_1,k_2,l,l_1,j_1=1}^{3} C^4_{ii_1i_2ll_1kk_1k_2jj_1}$$
$$- \frac{1}{4}\sum_{i_1,k,k_1,l,l_1,j_1,j_2,j_3=1}^{3} C^3_{jj_1j_2j_3ll_1kk_1ii_1} - \frac{1}{4}\sum_{i_1,k,k_1,k_2,l,l_1,j_1,j_2=1}^{3} C^4_{jj_1j_2ll_1kk_1k_2ii_1},$$

which together with the fact that $\int_0^t\int D_jG^{ii_1}(t-s,x-y)dyds = 0$ implies the results. $\square$

Now we follow [16, Section 10] to show that if $\xi_\varepsilon \to \xi$ with Z_ε denoting the corresponding model, then one can find a sequence $M_\varepsilon \in \mathfrak{R}_0$ such that $M_\varepsilon Z_\varepsilon \to \hat{Z}$.

Theorem 2.17 Let $\mathfrak{T}_F$ be the regularity structure associated to the stochastic N-S equation driven by space-time white noise for $\beta = 2, \alpha \in (-\frac{13}{5}, -\frac{5}{2})$, let $\xi_\varepsilon = \rho_\varepsilon * \xi$ be as in Proposition 2.11, ρ_ε symmetric in the sense that $\rho_\varepsilon(t,x) = \rho_\varepsilon(t,-x)$, and let Z_ε be the associated canonical model and M_ε be a sequence of renormalisation linear maps defined in (2.5) corresponding to $C^{1,\varepsilon}, C^{2,\varepsilon}, C^{3,\varepsilon}, C^{4,\varepsilon}$, which will be defined in the proof. Set $\hat{Z}_\varepsilon = M_\varepsilon Z_\varepsilon$. Then, there exists a random model $\hat{Z}$ independent of the choice of the mollifier ρ and $M_\varepsilon \in \mathfrak{R}_0$ such that $M_\varepsilon Z_\varepsilon \to \hat{Z}$ in probability.

More precisely, for any $\theta < -\frac{5}{2} - \alpha$, any compact set $\mathfrak{K}$ and any $\gamma < r$ we have

$$E|||M_\varepsilon Z_\varepsilon; \hat{Z}|||_{\gamma;\mathfrak{K}} \lesssim \varepsilon^\theta,$$

uniformly over $\varepsilon \in (0,1]$.

Proof By [16, Theorem 10.7] it is sufficient to prove that for $\tau \in \mathcal{F}$ with $|\tau|_\mathfrak{s} < 0$, any test function $\varphi \in \mathcal{B}_r$ and every $x \in \mathbb{R}^4$, there exist random variables $\hat{\Pi}_x\tau(\varphi)$ such that for $\kappa > 0$ small enough

$$E|(\hat{\Pi}_x\tau)(\varphi_x^\lambda)|^2 \lesssim \lambda^{2|\tau|_\mathfrak{s}+\kappa}, \tag{2.8}$$

and such that for some $0 < \theta < -\frac{5}{2} - \alpha$,

$$E|(\hat{\Pi}_x\tau - \hat{\Pi}_x^{(\varepsilon)}\tau)(\varphi_x^\lambda)|^2 \lesssim \varepsilon^{2\theta}\lambda^{2|\tau|_\mathfrak{s}+\kappa}. \tag{2.9}$$

Since the map $\varphi \mapsto (\hat{\Pi}_x\tau)(\varphi)$ is linear, we can find some functions $\hat{\mathcal{W}}^{(\varepsilon;k)}\tau$ with $(\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(x) \in L^2(\mathbb{R}\times\mathbb{T}^3)^{\otimes k}$, where $x \in \mathbb{R}^4$ and such that

$$(\hat{\Pi}_0^{(\varepsilon)}\tau)(\varphi) = \sum_{k\leq\|\tau\|} I_k\Big(\int\varphi(y)(\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(y)dy\Big),$$

where $\|\tau\|$ denotes the number of occurrences of Ξ in the expression τ and I_k is defined as in [16, Section 10.1]. To obtain (2.8) and (2.9) it is sufficient to find functions $\hat{\mathcal{W}}^{(k)}\tau \in L^2(\mathbb{R}\times\mathbb{T}^3)^{\otimes k}$, define

$$(\hat{\Pi}_x\tau)(\varphi) := \sum_{k\leq\|\tau\|} I_k\Big(\int\varphi(y)S_x^{\otimes k}(\hat{\mathcal{W}}^{(k)}\tau)(y)dy\Big),$$

and estimate the terms $|\langle(\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z),(\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(\bar{z})\rangle|$ and $|\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z),(\delta\hat{\mathcal{W}}^{(\varepsilon;k)}\tau)(\bar{z})\rangle|$, where $\{S_x\}_{x\in\mathbb{R}^4}$ is the unitary operators associated with translation invariance and $\delta\hat{\mathcal{W}}^{(\varepsilon;k)}\tau=\hat{\mathcal{W}}^{(\varepsilon;k)}\tau-\hat{\mathcal{W}}^{(k)}\tau$.

For $\tau=\Xi_i,\mathcal{I}^{ii_1}(\Xi_{i_1}),i,i_1=1,2,3$, it is easy to conclude that (2.8), (2.9) hold in this case. For $\tau=\mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,j,j_1=1,2,3$, we have

$$\hat{\Pi}_x^{(\varepsilon)}\tau(y)=\int K^{ii_1}(y-z)\xi_\varepsilon^{i_1}(z)dz\int K^{jj_1}(y-z)\xi_\varepsilon^{j_1}(z)dz-C^{1,\varepsilon}_{ii_1jj_1}.$$

If we choose $C^{1,\varepsilon}_{ii_1jj_1}:=\langle K_\varepsilon^{ii_1},K_\varepsilon^{jj_1}\rangle$ with $K_\varepsilon=\rho_\varepsilon*K$, we have

$$\hat{\Pi}_x^{(\varepsilon)}\tau(y)=\int K^{ii_1}(y-z_1)K^{jj_1}(y-z_2)\xi_\varepsilon^{i_1}(z_1)\diamond\xi_\varepsilon^{j_1}(z_2)dz_1dz_2,$$

so that $\hat{\Pi}_x^{(\varepsilon)}\tau(y)$ belongs to the homogeneous chaos of order 2 with

$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y;z_1,z_2)=K_\varepsilon^{ii_1}(y-z_1)K_\varepsilon^{jj_1}(y-z_2).$$

Since for $i,j=1,2,3$, K^{ij} is of order -3, applying [16, Lemma 10.14] we deduce that

$$|\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle|\lesssim\|y-\bar{y}\|_{\mathfrak{s}}^{-2}$$

holds uniformly over $\varepsilon\in(0,1]$, which for $4\alpha+10+\kappa<0$ implies the bound

$$|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}|\lesssim\lambda^{-10}\int_{\|y\|_{\mathfrak{s}}\le\lambda,\|\bar{y}\|_{\mathfrak{s}}\le\lambda}\|y-\bar{y}\|_{\mathfrak{s}}^{-2}dyd\bar{y}$$
$$\lesssim\lambda^{-5}\int_{\|y\|_{\mathfrak{s}}\le2\lambda}\|y\|_{\mathfrak{s}}^{-2}dy\lesssim\lambda^{-2}\lesssim\lambda^{\kappa+2(2\alpha+4)}.$$

Hence we can choose

$$(\hat{\mathcal{W}}^{(2)}\tau)(y;z_1,z_2)=K^{ii_1}(y-z_1)K^{jj_1}(y-z_2),$$

and we use it to define $(\hat{\Pi}_x\tau)(\psi)$. In the same way, it is straightforward to obtain an analogous bound on $(\hat{\mathcal{W}}^{(2)})(\tau)$, which implies that (2.8) holds in this case. So it remains to find similar bounds for $(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)=(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)-(\hat{\mathcal{W}}^{(2)}\tau)$. Similarly, by [16, Lemma 10.17] we have for $0<\kappa+\theta<-2(2\alpha+5)$

$$|\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle|\lesssim\varepsilon^\theta\|y-\bar{y}\|_{\mathfrak{s}}^{-2-\theta},$$

holds uniformly over $\varepsilon\in(0,1]$. Then we obtain the bound

$$|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}|\lesssim\varepsilon^\theta\lambda^{\kappa+2(2\alpha+4)},$$

which implies (2.9) holds in this case.

For $\tau=\mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$, $i,i_1,i_2,j,j_1=1,2,3$, we have the following identity

$$\hat{\Pi}_x^{(\varepsilon)}\tau(y)=\int D_jK^{ii_1}(y-y_1)\int K^{i_1i_2}(y_1-z)\xi_\varepsilon^{i_2}(z)dz\int K^{jj_1}(y_1-z)\xi_\varepsilon^{j_1}(z)dzdy_1$$
$$=\int D_jK^{ii_1}(y-y_1)\int\int K^{i_1i_2}(y_1-z_1)K^{jj_1}(y_1-z_2)\xi_\varepsilon^{i_2}(z_1)\diamond\xi_\varepsilon^{j_1}(z_2)dz_1dz_2dy_1,$$

so that $\hat{\Pi}_x^{(\varepsilon)}\tau(y)$ belongs to the homogeneous chaos of order 2 with

$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y;z_1,z_2)=\int D_jK^{ii_1}(y-y_1)K_\varepsilon^{i_1i_2}(y_1-z_1)K_\varepsilon^{jj_1}(y_1-z_2)dy_1.$$

Then by [16, Lemma 10.14] we obtain that for any $\delta>0$

$$|\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle|\lesssim\|y-\bar{y}\|_{\mathfrak{s}}^{-\delta},$$

holds uniformly over $\varepsilon\in(0,1]$, which implies the bound

$$|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}|\lesssim\lambda^{-10}\int_{\|y\|_{\mathfrak{s}}\leq\lambda,\|\bar{y}\|_{\mathfrak{s}}\leq\lambda}\|y-\bar{y}\|_{\mathfrak{s}}^{-\delta}dyd\bar{y}$$
$$\lesssim\lambda^{-5}\int_{\|y\|_{\mathfrak{s}}\leq2\lambda}\|y\|_{\mathfrak{s}}^{-\delta}dy\lesssim\lambda^{-\delta}\lesssim\lambda^{\kappa+2(2\alpha+5)},$$

for $0<\kappa+\delta<-2(2\alpha+5)$. Hence we can choose

$$(\hat{\mathcal{W}}^{(2)}\tau)(y;z_1,z_2)=\int D_jK^{ii_1}(y-y_1)K^{i_1i_2}(y_1-z_1)K^{jj_1}(y_1-z_2)dy_1,$$

and deduce easily that (2.8) holds for $\tau=\mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$. Similarly for $0<\kappa+\delta+\theta<-2(2\alpha+5)$ we have that the bound

$$|\int\int\psi^\lambda(y)\psi^\lambda(\bar{y})\langle(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(y),(\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{y})\rangle dyd\bar{y}|\lesssim\varepsilon^\theta\lambda^{\kappa+2(2\alpha+5)},$$

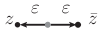
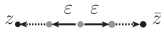
holds uniformly over $\varepsilon\in(0,1]$, which also implies that (2.9) holds for $\tau=\mathcal{I}_j^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{jj_1}(\Xi_{j_1}))$.

In the following we use to represent a factor K or K_ε and  to represent DK or DK_ε, where for simplicity we write $K^{ii_1}=K, D_jK^{ii_1}=DK$ and we do not make a difference between the graphs associated with different K^{ii_1}, since they have the same order. In the graphs below we also omit the dependence on ε if there's no confusion. We also use the convention that if a vertex is drawn in grey, then the corresponding variable is integrated out.

For $\tau=\mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i,i_1,k,k_1,j,j_1=1,2,3$ we have

$(\mathcal{W}^{(\varepsilon;2)}\tau)(z)=$  .

Defining kernels Q_ε^0, P_ε^0 by

$P_\varepsilon^0(z-\bar{z})=$  , $Q_\varepsilon^0(z-\bar{z})=$  ,

we have

$$\langle\mathcal{W}^{(\varepsilon;2)}\tau(z),\mathcal{W}^{(\varepsilon;2)}\tau(\bar{z})\rangle=P_\varepsilon^0(z-\bar{z})\delta^{(2)}Q_\varepsilon^0(z,\bar{z}),$$

where for any function Q of two variables we have set

$$\delta^{(2)}Q(z,\bar{z})=Q(z,\bar{z})-Q(z,0)-Q(0,\bar{z})+Q(0,0).$$

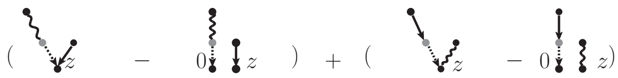
It follows from [16, Lemma 10.14, Lemma 10.17] that for every $\delta > 0$ we have

$$|Q^0_\varepsilon(z) - Q^0_\varepsilon(0)| \lesssim \|z\|_{\mathfrak{s}}^{1-\delta}, \quad |P^0_\varepsilon(z)| \lesssim \|z\|_{\mathfrak{s}}^{-1}.$$

As a consequence we have the desired a priori bounds for $\mathcal{W}^{(\varepsilon;2)}\tau$, namely for every $\delta > 0$

$$\langle (\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{z})\rangle \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta})$$

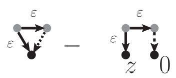
holds uniformly over $\varepsilon \in (0, 1]$. As previously, we define $\hat{W}^{(2)}\tau$ like $\hat{\mathcal{W}}^{(\varepsilon;2)}\tau$, but with K_ε replaced by K. Moreover, we use $\rightsquigarrow$ to represent the kernel $K - K_\varepsilon$ and we have

$(\delta\mathcal{W}^{(\varepsilon;2)}\tau)(z) =$  .

By a similar calculation as above we obtain the following bounds

$$\begin{aligned}\langle (\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z), (\delta\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(\bar{z})\rangle \lesssim & \varepsilon^{2\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-2\theta-\delta} + \|z\|_{\mathfrak{s}}^{1-2\theta-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-2\theta-\delta}) \\ & + \varepsilon^{2\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-1-2\theta}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta}),\end{aligned}$$

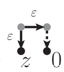
which is valid uniformly over $\varepsilon \in (0, 1]$, provided that $\theta < 1, \delta > 0$. Here we used [16, Lemma 10.17]. We come to $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$ and have

$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) =$  .

Since K is symmetric and DK is anti-symmetric with respect to the space variable, we conclude that

 $= 0,$

which implies the following

$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) = -$  .

By [16, Lemma 10.14, Lemma 10.17] we have that for every $\delta > 0$

$$|(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z)| \lesssim \|z\|_{\mathfrak{s}}^{-\delta},$$

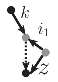
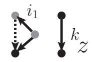
holds uniformly over $\varepsilon \in (0, 1]$. Similar bounds also hold for $(\delta\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)$. Then we can easily conclude that (2.8) (2.9) hold for $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$.

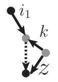
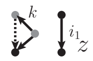
For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1})$, $i, i_1, i_2, k, j, j_1 = 1, 2, 3$, we can prove similar bounds as above, since in this case we also have

 $= 0.$

For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) =$ , $i, i_1, i_2, k, k_1, j, j_1 = 1, 2, 3$, we have the following identities

$(\hat{\mathcal{W}}^{(\varepsilon;3)}\tau)(z) =$  ,

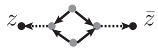
$$(\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z) = $$  $-$  ,

$$(\hat{\mathcal{W}}_2^{(\varepsilon;1)}\tau)(z) = $$  $-$  .

Then

$$\langle \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(\bar{z})\rangle = P_\varepsilon^0(z-\bar{z})Q_\varepsilon(z-\bar{z}),$$

where

$$Q_\varepsilon(z-\bar{z}) = $$  ,  $= 0.$

By [16, Lemmas 10.14 and 10.17] for every $\delta > 0$ we obtain the bound

$$|Q_\varepsilon(z-\bar{z})| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

which implies that

$$|\langle \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;3)}\tau(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-1-\delta},$$

holds uniformly over $\varepsilon \in (0,1]$. As previously, we define $\hat{W}^{(3)}\tau$ like $\hat{\mathcal{W}}^{(\varepsilon;3)}\tau$, but with K_ε replaced by K. Then $\delta\hat{\mathcal{W}}^{(\varepsilon;3)}\tau$ can be bounded in a manner similar as before. Now for $\hat{\mathcal{W}}^{(\varepsilon;1)}\tau$, we have

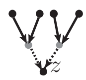
$$(\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z) = ((\mathcal{R}_1 L_\varepsilon) * K_\varepsilon^{kk_1})(z),$$

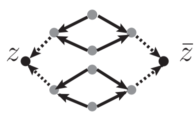
where $L_\varepsilon(z) =$  and $(\mathcal{R}_1 L_\varepsilon)(\psi) = \int L_\varepsilon(x)(\psi(x)-\psi(0))dx$ for ψ smooth with compact support. It follows from [16, Lemma 10.16] that the bound

$$|\langle (\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(z), (\hat{\mathcal{W}}_1^{(\varepsilon;1)}\tau)(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-1}$$

holds uniformly for $\varepsilon \in (0,1]$. Similarly, this bound also holds for $(\hat{\mathcal{W}}_2^{(\varepsilon;1)}\tau)(z)$. Again, $\delta\hat{\mathcal{W}}_i^{(\varepsilon;1)}\tau, i = 1,2$ can be bounded in a manner similar as before. Then we can easily conclude that (2.8), (2.9) hold for $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{jj_1}(\Xi_{j_1})$.

For $\tau = \mathcal{I}_k^{ii_1}(\mathcal{I}^{i_1i_2}(\Xi_{i_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}_l^{jj_1}(\mathcal{I}^{j_1j_2}(\Xi_{j_2})\mathcal{I}^{ll_1}(\Xi_{l_1})) =$  $, i, i_1, i_2, k, k_1, j, j_1, j_2, l, l_1 = 1,2,3$, we have the identities

$$(\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z) = $$  ,

$$\langle (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar{z})\rangle = $$  .

Then we obtain the bound for every $\delta > 0$

$$|\langle (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta}.$$

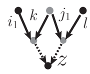
Similarly, we obtain

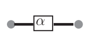
$$|\langle (\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z), (\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(\bar{z})\rangle| \lesssim \varepsilon^{2\theta}\|z-\bar{z}\|_{\mathfrak{s}}^{-2\theta-\delta}$$

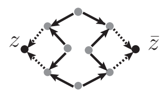
holds uniformly for $\varepsilon \in (0,1]$, provided $\theta < 1$.

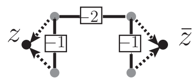
For $(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z)$, we have the identity

$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z) = \sum_{i=1}^{4} (\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau)(z).$$

$$(\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z) =$$  .

Other terms can be obtained by changing the position for i_1, k or j_1, l. Since the estimates are similar, we omit them here. We also use the notation  for $\|z-\bar{z}\|_{\mathfrak{s}}^{\alpha} 1_{\|z-\bar{z}\|_{\mathfrak{s}} \le C}$ for a constant C. We obtain that for $\delta > 0$

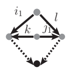
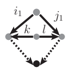
$$\langle (\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(\bar{z}) \rangle =$$ 

$\lesssim$  $\lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta}$,

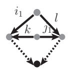
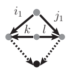
holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality in the first inequality. Similarly, we have

$$\langle (\delta\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(z), (\delta\hat{\mathcal{W}}_1^{(\varepsilon;2)}\tau)(\bar{z}) \rangle \lesssim \varepsilon^{2\theta} \|z-\bar{z}\|_{\mathfrak{s}}^{-2\theta-\delta},$$

provided $\theta < 1$. Now for $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$ we have

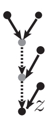
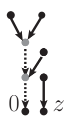
$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) =$$  $+$  $- C^{2,\varepsilon}_{ii_1i_2jj_1j_2kk_1ll_1}$.

Hence we choose

$$C^{2,\varepsilon}_{ii_1i_2jj_1j_2kk_1ll_1} =$$  $+$ 

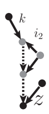
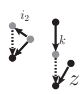
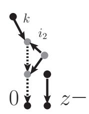
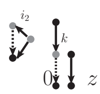
and also in this case (2.8), (2.9) follow.

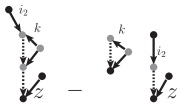
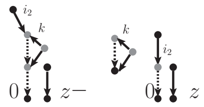
For $\tau = \mathcal{I}_l^{ii_1}(\mathcal{I}_k^{i_1i_2}(\mathcal{I}^{i_2i_3}(\Xi_{i_3})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{ll_1}(\Xi_{l_1}))\mathcal{I}^{jj_1}(\Xi_{j_1}) =$ , $i, i_1, i_2, i_3, j, j_1, k, k_1, l, l_1 = 1, 2, 3$, we have the following identities:

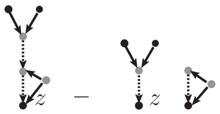
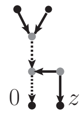
$$(\hat{\mathcal{W}}^{(\varepsilon;4)}\tau)(z) =$$  $-$  .

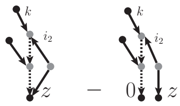
$$(\hat{\mathcal{W}}^{(\varepsilon;2)}\tau)(z) = \sum_{i=1}^{5} (\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau)(z) = \sum_{i=1}^{5} [(\hat{\mathcal{W}}_{i1}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{i2}^{(\varepsilon;2)}\tau)(z)],$$

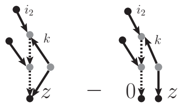
where

$(\hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau)(z) =$  $-$  , $(\hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau)(z) =$   ,

$$(\hat{\mathcal{W}}_{21}^{(\varepsilon;2)}\tau)(z) =$$  $$,\ (\hat{\mathcal{W}}_{22}^{(\varepsilon;2)}\tau)(z) =$$  ,

$$(\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z) =$$  $$,\ (\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(z) =$$  ,

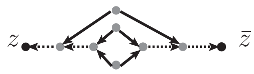
$$(\hat{\mathcal{W}}_{4}^{(\varepsilon;2)}\tau)(z) = (\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(z) =$$  ,

$$(\hat{\mathcal{W}}_{5}^{(\varepsilon;2)}\tau)(z) = (\hat{\mathcal{W}}_{51}^{(\varepsilon;2)}\tau)(z) - (\hat{\mathcal{W}}_{52}^{(\varepsilon;2)}\tau)(z) =$$  .

Now for $\hat{\mathcal{W}}^{(\varepsilon;4)}\tau$ we have

$$\langle \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(\bar{z})\rangle = P_\varepsilon^0(z-\bar{z})\delta^{(2)}Q_\varepsilon^2(z,\bar{z}),$$

where

$$Q_\varepsilon^2(z,\bar{z}) =$$  ,  $= 0.$

By [16, Lemmas 10.14, 10.16 and 10.17] for every $\delta > 0$ we have that the bound

$$|\langle \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(z), \hat{\mathcal{W}}^{(\varepsilon;4)}\tau(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta})$$

holds uniformly for $\varepsilon \in (0,1]$, and that

$$\begin{aligned}
&|\langle \hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau(z) - \hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau(z), \hat{\mathcal{W}}_{11}^{(\varepsilon;2)}\tau(\bar{z}) - \hat{\mathcal{W}}_{12}^{(\varepsilon;2)}\tau(z)\rangle| \\
\lesssim& \|z-\bar{z}\|_{\mathfrak{s}}^{-1}|\langle K * \mathcal{R}_1 L_\varepsilon^1 * DK(z-\cdot) - K * \mathcal{R}_1 L_\varepsilon^1 * DK(-\cdot), \\
&K * \mathcal{R}_1 L_\varepsilon^1 * DK(\bar{z}-\cdot) - K * \mathcal{R}_1 L_\varepsilon^1 * DK(-\cdot)\rangle| \\
\lesssim& \|z-\bar{z}\|_{\mathfrak{s}}^{-1}(\|z-\bar{z}\|_{\mathfrak{s}}^{1-\delta} + \|z\|_{\mathfrak{s}}^{1-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{1-\delta})
\end{aligned}$$

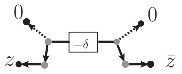
holds uniformly for $\varepsilon \in (0,1]$, where $L_\varepsilon^1(z) =$ . Then define $\hat{\mathcal{W}}^{(4)}\tau, \hat{\mathcal{W}}_i^{(2)}\tau, i = 1,2$, in a similar way as before. Similarly, these bounds also hold for $(\hat{\mathcal{W}}_2^{(\varepsilon;2)}\tau)(z)$. Again, $\delta\hat{\mathcal{W}}^{(\varepsilon;4)}\tau$, $\delta\hat{\mathcal{W}}_i^{(\varepsilon;2)}\tau, i = 1,2$ can be bounded in a manner similar as before. For $\hat{\mathcal{W}}_3^{(\varepsilon;2)}\tau$ we have

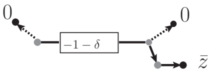
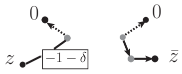
$$(\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z) = ((\mathcal{R}_1 L_\varepsilon^1) * L_\varepsilon^2)(z),$$

where $L_\varepsilon^1(z) =$ , $L_\varepsilon^2(z) =$ . It follows from [16, Lemma 10.16] that for every $\delta > 0$, the bound

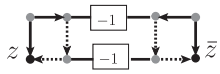
$$|\langle (\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{31}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{-\delta}$$

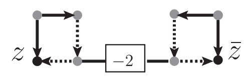
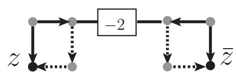
holds uniformly for $\varepsilon \in (0,1]$. Moreover, for $\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau$ we have for every $\delta \in (0,1)$

$$|\langle(\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{32}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| = |$$  $$|$$

$$\lesssim$$  $$+$$  $$\lesssim \|z\|_{\mathfrak{s}}^{-\delta}\|\bar{z}\|_{\mathfrak{s}}^{-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

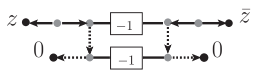
where we used Young's inequality. Again, $\delta\hat{\mathcal{W}}_{3}^{(\varepsilon;2)}\tau$, can be bounded in a manner similar as before. For $\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau$ we have that for $\delta > 0$

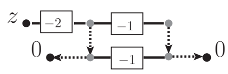
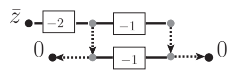
$$|\langle(\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{41}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| =$$ 

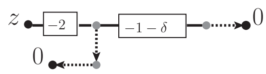
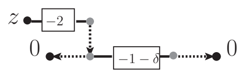
$$\lesssim$$  $$+$$ 

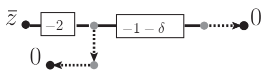
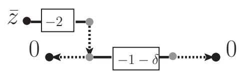
$$\lesssim \|z - \bar{z}\|_{\mathfrak{s}}^{-\delta},$$

holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality. For $\delta \in (0,1)$ we have that

$$|\langle(\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(z), (\hat{\mathcal{W}}_{42}^{(\varepsilon;2)}\tau)(\bar{z})\rangle| =$$ 

$$\lesssim$$  $$+$$ 

$$\lesssim$$  $$+$$ 

$$+$$  $$+$$ 

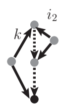
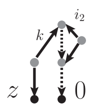
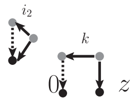
$$\lesssim \|z\|_{\mathfrak{s}}^{-\delta} + \|\bar{z}\|_{\mathfrak{s}}^{-\delta},$$

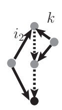
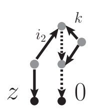
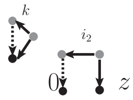
holds uniformly for $\varepsilon \in (0,1]$, where we used Young's inequality for each inequality. Similarly, these bounds also hold for $(\hat{\mathcal{W}}_{5}^{(\varepsilon;2)}\tau)(z)$. Again, defining $\hat{\mathcal{W}}_{i}^{(2)}\tau, i = 4,5$, similarly as before and $\delta\hat{\mathcal{W}}_{i}^{(\varepsilon;2)}\tau, i = 4,5$ can be bounded in a manner similar as before.

We now turn to $\hat{\mathcal{W}}^{(\varepsilon;0)}\tau$:

$$(\hat{\mathcal{W}}^{(\varepsilon;0)}\tau)(z) = \sum_{i=1}^{2}(\hat{\mathcal{W}}_{i}^{(\varepsilon;0)}\tau)(z) = \sum_{i=1}^{2}[(\hat{\mathcal{W}}_{i1}^{(\varepsilon;0)}\tau)(z) - (\hat{\mathcal{W}}_{i2}^{(\varepsilon;0)}\tau)(z)] - C^{3,\varepsilon}_{ii_1i_2i_3kk_1ll_1jj_1},$$

where

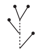
$$(\hat{\mathcal{W}}_{11}^{(\varepsilon;0)}\tau)(z) =$$  $$, (\hat{\mathcal{W}}_{12}^{(\varepsilon;0)}\tau)(z) =$$  $$-$$  $$,$$

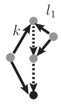
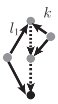
$$(\hat{\mathcal{W}}_{21}^{(\varepsilon;0)}\tau)(z) =$$  $$, (\hat{\mathcal{W}}_{22}^{(\varepsilon;0)}\tau)(z) =$$  $$-$$  $$,$$

we choose $C^{3,\varepsilon}_{ii_1i_2i_3kk_1ll_1jj_1} = (\hat{\mathcal{W}}^{(\varepsilon;0)}_{11}\tau)(z) + (\hat{\mathcal{W}}^{(\varepsilon;0)}_{21}\tau)(z)$. By [16, Lemma 10.16] we have that for every $\delta > 0$, $i = 1, 2$,

$$|(\hat{\mathcal{W}}^{(\varepsilon;0)}_{i2}\tau)(z)| \lesssim \|z\|_{\mathfrak{s}}^{-\delta}$$

holds uniformly for $\varepsilon \in (0, 1]$. Similarly as before, we obtain the bounds for $\delta\hat{\mathcal{W}}^{(\varepsilon;0)}_{i2}\tau$. Then (2.8), (2.9) also follow in this case.

For $\tau = \mathcal{I}^{ii_1}_l(\mathcal{I}^{ll_1}_k(\mathcal{I}^{l_1l_2}(\Xi_{l_2})\mathcal{I}^{kk_1}(\Xi_{k_1}))\mathcal{I}^{i_1i_2}(\Xi_{i_2}))\mathcal{I}^{jj_1}(\Xi_{j_1}) =$ , $i, i_1, i_2, l, l_1, l_2, k, k_1, j, j_1 = 1, 2, 3$, we have similar bounds as above with

$C^4_{ii_1i_2kk_1ll_1l_2jj_1} =$  $+$  .

□

Now combining Theorem 2.17 and Propositions 2.13 and 2.15, we conclude Theorem 1.1 easily.

3 N-S equation by paracontrolled distributions

3.1 Besov spaces and paraproduct

In the following we recall the definitions and some properties of Besov spaces and paraproducts. For a general introduction to these theories we refer to [1], [12]. Here the notations are different from the previous section.

First, we introduce the following notations. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. If u is a vector of n tempered distributions on $\mathbb{R}^d$, then we write $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

i. the support of χ is contained in a ball and the support of θ is contained in an annulus;

ii. $\chi(z) + \sum_{j\geq 0}\theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\mathrm{supp}(\chi)\cap\mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}(\theta(2^{-i}\cdot))\cap\mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $|i-j| > 1$.

We call such a pair (χ, θ) a dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [1, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi\mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

For $\alpha \in \mathbb{R}$, the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^n)$, where for $p, q \in [1, \infty]$ we define

$$B^\alpha_{p,q}(\mathbb{R}^d, \mathbb{R}^n) = \{u = (u^1, ..., u^n) \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n) : \|u\|_{B^\alpha_{p,q}} = \sum_{i=1}^n (\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u^i\|_{L^p})^q)^{1/q} < \infty\},$$

with the usual interpretation as the l^∞-norm in case $q = \infty$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$.

We point out that everything above and everything that follows can be applied to distributions on the torus. More precisely, let $\mathcal{D}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Therefore, Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as

$$B^\alpha_{p,q}(\mathbb{T}^d, \mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{T}^d, \mathbb{R}^n) : \|u\|_{B^\alpha_{p,q}} = \sum_{i=1}^n (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u^i\|_{L^p(\mathbb{T}^d)})^q)^{1/q} < \infty\}.$$

We will need the following Besov embedding theorem on the torus (c.f. [12, Lemma 41]):

Lemma 3.1 Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha - d(1/p_1 - 1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.

Now we recall the following paraproduct introduced by Bony (see [3]). In general, the product fg of two distributions $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley blocks, the product fg can be formally decomposed as

$$fg = \sum_{j \geq -1} \sum_{i \geq -1} \Delta_i f \Delta_j g = \pi_<(f, g) + \pi_0(f, g) + \pi_>(f, g),$$

with

$$\pi_<(f, g) = \pi_>(g, f) = \sum_{j \geq -1} \sum_{i < j-1} \Delta_i f \Delta_j g, \quad \pi_0(f, g) = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

We use the notation

$$S_j f = \sum_{i \leq j-1} \Delta_i f.$$

We will use without comment that $\| \cdot \|_\alpha \leq \| \cdot \|_\beta$ for $\alpha \leq \beta$, that $\| \cdot \|_{L^\infty} \lesssim \| \cdot \|_\alpha$ for $\alpha > 0$, and that $\| \cdot \|_\alpha \lesssim \| \cdot \|_{L^\infty}$ for $\alpha \leq 0$. We will also use that $\|S_j u\|_{L^\infty} \lesssim 2^{-j\alpha} \|u\|_\alpha$ for $\alpha < 0$ and $u \in \mathcal{C}^\alpha$.

The basic result about these bilinear operations is given by the following estimates:

Lemma 3.2 (Paraproduct estimates, [3], [12, Lemma 2]) For any $\beta \in \mathbb{R}$ we have

$$\|\pi_<(f, g)\|_\beta \lesssim \|f\|_{L^\infty} \|g\|_\beta \quad f \in L^\infty, g \in \mathcal{C}^\beta,$$

and for $\alpha < 0$ furthermore

$$\|\pi_<(f, g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \quad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

For $\alpha + \beta > 0$ we have

$$\|\pi_0(f, g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta \quad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

From this lemma we know that $\pi_<(f, g)$ and $\pi_>(f, g)$ are well defined if $f \in L^\infty$. The only term not well defined in defining fg is $\pi_0(f, g)$. Furthermore, if f is smooth, the regularity of $\pi_>(f, g)$ and $\pi_0(f, g)$ will become better than the regularity of g. $\pi_<(f, g)$ retains the same regularity as g.

The following basic commutator lemma is important for our later use:

Lemma 3.3 ([12, Lemma 5]) Assume that $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha+\beta+\gamma > 0$ and $\beta+\gamma < 0$. Then for smooth f, g, h, the trilinear operator

$$C(f,g,h) = \pi_0(\pi_<(f,g), h) - f\pi_0(g,h)$$

has the bound

$$\|C(f,g,h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma.$$

Thus, C can be uniquely extended to a bounded trilinear operator in $L^3(\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma, \mathcal{C}^{\alpha+\beta+\gamma})$.

By using this commutator estimate to make sense of the product of $\pi_<(f,g)$ and h for $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta, h \in \mathcal{C}^\gamma$, it is sufficient to define $\pi_0(g,h)$.

Now we prove the following commutator estimate for the Leray projection. We follow a similar argument as [4, Lemma A.1]. In the following we use the notation $f(D)u = \mathcal{F}^{-1} f \mathcal{F} u$.

Lemma 3.4 Let $u \in \mathcal{C}^\alpha$ for some $\alpha < 1$ and $v \in \mathcal{C}^\beta$ for some $\beta \in \mathbb{R}$. Then for every $k, l = 1, 2, 3$

$$\|P^{kl}\pi_<(u,v) - \pi_<(u, P^{kl}v)\|_{\alpha+\beta} \lesssim \|u\|_\alpha \|v\|_\beta,$$

where P is the Leray projection.

Proof We have

$$P^{kl}\pi_<(u,v) - \pi_<(u, P^{kl}v) = \sum_{j=-1}^{\infty} [P^{kl}(S_{j-1}u\Delta_j v) - S_{j-1}u\Delta_j P^{kl}v]$$

and every term of this series has a Fourier transform with support in an annulus of the form $2^j\mathcal{A}$ where $\mathcal{A}$ is an annulus. Let $\psi \in \mathcal{D}$ with support in an annulus be such that $\psi = 1$ on $\mathcal{A}$. Then

$$P^{kl}(S_{j-1}u\Delta_j v) - S_{j-1}u\Delta_j P^{kl}v = [\hat{P}^{kl}(D), S_{j-1}u]\Delta_j v = [(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]\Delta_j v.$$

Here $\hat{P}^{kl}(x) = \delta_{kl} - \frac{x_k x_l}{|x|^2}$ and

$$[(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]f = (\psi(2^{-j}\cdot)\hat{P}^{kl})(D)(S_{j-1}uf) - S_{j-1}u(\psi(2^{-j}\cdot)\hat{P}^{kl})(D)f$$

denotes the commutator. By a similar argument as in the proof of [4, Lemma A.1] we have

$$\|[(\psi(2^{-j}\cdot)\hat{P}^{kl})(D), S_{j-1}u]\Delta_j v\|_{L^\infty} \lesssim \sum_{\eta \in \mathbb{N}^d, |\eta|=1} \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1} \|\partial^\eta S_{j-1}u\|_{L^\infty} \|\Delta_j v\|_{L^\infty}.$$

Moreover, we have the following estimates

$$\begin{aligned}
&\|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1} \\
\leq& 2^{-j}\|\mathcal{F}^{-1}((\partial^\eta\psi)(2^{-j}\cdot)\hat{P}^{kl})\|_{L^1} + \|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta\hat{P}^{kl})\|_{L^1} \\
=& 2^{-j}\|\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^1} + \|\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^1} \\
\lesssim& 2^{-j}\|(1+|\cdot|^2)^d\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^\infty} + \|(1+|\cdot|^2)^d\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&=2^{-j}\|\mathcal{F}^{-1}((1-\Delta)^d(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot)))\|_{L^\infty}+\|\mathcal{F}^{-1}((1-\Delta)^d(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot)))\|_{L^\infty}\\
&\lesssim 2^{-j}\|(1-\Delta)^d(\partial^\eta\psi(\cdot)\hat{P}^{kl}(2^j\cdot))\|_{L^1}+\|(1-\Delta)^d(\psi(\cdot)\partial^\eta\hat{P}^{kl}(2^j\cdot))\|_{L^1}\\
&\lesssim 2^{-j}\sum_{0\leq|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|}}+\sum_{|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|+1}}\\
&\lesssim 2^{-j},
\end{aligned}
$$

where in the fourth inequality we used $|D^m\hat{P}^{kl}(x)|\lesssim|x|^{-|m|}$ for any multiindices m. Thus we get that

$$
\|[\psi(2^{-j}\cdot)\hat{P}^{kl}(D),S_{j-1}u]\Delta_j v\|_{L^\infty}\lesssim 2^{-j(\alpha+\beta)}\|u\|_\alpha\|v\|_\beta,
$$

which implies the result by a similar argument as in the proof of [4, Lemma A.1]. □

Now we recall the following heat semigroup estimate.

Lemma 3.5 ([12, Lemma 47]) Let $u\in\mathcal{C}^\alpha$ for some $\alpha\in\mathbb{R}$. Then for every $\delta\geq 0$

$$
\|P_t u\|_{\alpha+\delta}\lesssim t^{-\delta/2}\|u\|_\alpha,
$$

where P_t is the heat semigroup on $\mathbb{T}^d$.

For the Leray projection we have the following estimate on $\mathbb{T}^d$:

Lemma 3.6 Let $u\in\mathcal{C}^\alpha$ on $\mathbb{T}^d$ for some $\alpha\in\mathbb{R}$. Then for every $k,l=1,2,3$

$$
\|P^{kl}u\|_\alpha\lesssim\|u\|_\alpha,
$$

where P is the Leray projection.

Proof Let $\psi\in\mathcal{D}$ with support in an annulus be such that $\psi=1$ on the support of θ. We have that for $j\geq 0$

$$
\begin{aligned}
\|\Delta_j P^{kl}u\|_{L^\infty}=&\|\mathcal{F}^{-1}(\hat{P}^{kl}(\cdot)\psi(2^{-j}\cdot))\theta_j\mathcal{F}u\|_{L^\infty}\\
\lesssim&\|\mathcal{F}^{-1}(\hat{P}^{kl}(\cdot)\psi(2^{-j}\cdot))\|_{L^1}2^{-j\alpha}\|u\|_\alpha=\|\mathcal{F}^{-1}(\hat{P}^{kl}(2^j\cdot)\psi)\|_{L^1}2^{-j\alpha}\|u\|_\alpha.
\end{aligned}
$$

Here $\hat{P}^{kl}(x)=\delta_{kl}-\frac{x^kx^l}{|x|^2}$. By a similar calculaton as in the proof of Lemma 3.4 we obtain that

$$
\|\mathcal{F}^{-1}(\hat{P}^{kl}(2^j\cdot)\psi)\|_{L^1}\lesssim\|(1-\Delta)^d(\hat{P}^{kl}(2^j\cdot)\psi)\|_{L^1}\lesssim\sum_{0\leq|m|\leq 2d}(2^j)^{|m|}\frac{1}{(2^j)^{|m|}}\lesssim C.
$$

By the theory in [22] we know that the above calculations also hold on $\mathbb{T}^d$. Moreover, we have on $\mathbb{T}^d$ for $1<p<\infty$

$$
\|\Delta_{-1}P^{kl}u\|_{L^\infty(\mathbb{T}^d)}=\|\mathcal{F}^{-1}\hat{P}^{kl}\chi\mathcal{F}u\|_{L^\infty(\mathbb{T}^d)}\lesssim\|\mathcal{F}^{-1}\hat{P}^{kl}\chi\mathcal{F}u\|_{L^p(\mathbb{T}^d)}\lesssim\|\Delta_{-1}u\|_{L^p(\mathbb{T}^d)}\lesssim\|\Delta_{-1}u\|_{L^\infty(\mathbb{T}^d)},
$$

where in the first inequality we used that $\mathrm{supp}(\chi\hat{P}\mathcal{F}u)$ is contained in a ball and in the second inequality we used Mihlin's multiplier theorem. Thus the result follows. □

3.2 N-S equation

Let us focus on the equation on $\mathbb{T}^3$:

$$Lu^i = \sum_{i_1=1}^{3} P^{ii_1}\xi^{i_1} - \frac{1}{2}\sum_{i_1=1}^{3} P^{ii_1}(\sum_{j=1}^{3} D_j(u^i u^j)), \tag{3.1}$$

$$u(0) = Pu_0 \in \mathcal{C}^{-z},$$

where $\xi = (\xi^1, \xi^2, \xi^3)$, ξ^1, ξ^2, ξ^3 are the periodic independent space time white noise, $L = \partial_t - \Delta$ and $z \in (1/2, 1/2 + \delta_0)$ with $0 < \delta_0 < 1/2$. Here without loss of generality we suppose that $\nu = 1$. As we mentioned in the introduction the nonlinear term of this equation is not well defined because of the singularity of ξ. In the following we follow the idea of [12] to give the definition of the solution to the equation as a limit of solutions u^ε to the following equations:

$$Lu^{\varepsilon,i} = \sum_{i_1=1}^{3} P^{ii_1}\xi^{\varepsilon,i_1} - \frac{1}{2}\sum_{i_1=1}^{3} P^{ii_1}(\sum_{j=1}^{3} D_j(u^\varepsilon u^{\varepsilon,j})),$$

$$u(0) = Pu_0 \in \mathcal{C}^{-z}.$$

Here ξ^ε is a family of smooth approximations of ξ such that $\xi^\varepsilon \to \xi$ as $\varepsilon \to 0$. Now we prove a uniform estimate for u^ε.

In the following to avoid heavy notation we omit the dependence on ε if there's no confusion and consider (3.1) for smooth ξ. We split the equation (3.1) into the following four equations:

$$Lu_1^i = \sum_{i_1=1}^{3} P^{ii_1}\xi^{i_1},$$

$$Lu_2^i = -\frac{1}{2}\sum_{i_1=1}^{3} P^{ii_1}(\sum_{j=1}^{3} D_j(u_1^{i_1} \diamond u_1^j)) \quad u_2(0) = 0,$$

$$Lu_3^i = -\frac{1}{2}\sum_{i_1=1}^{3} P^{ii_1}(\sum_{j=1}^{3} D_j(u_1^{i_1} \diamond u_2^j + u_2^{i_1} \diamond u_1^j)), \quad u_3(0) = 0,$$

and

$$\begin{aligned} Lu_4^i = &-\frac{1}{2}\sum_{i_1,j=1}^{3} P^{ii_1} D_j[u_1^{i_1} \diamond (u_3^j + u_4^j) + (u_3^{i_1} + u_4^{i_1}) \diamond u_1^j + u_2^{i_1} \diamond u_2^j \\ &+ u_2^{i_1}(u_3^j + u_4^j) + u_2^j(u_3^{i_1} + u_4^{i_1}) + (u_3^{i_1} + u_4^{i_1})(u_3^j + u_4^j)], \end{aligned} \tag{3.2}$$

$$u_4(0) = Pu_0 - u_1(0),$$

where for $i, j = 1, 2, 3$

$$u_1^i \diamond u_3^j = \pi_<(u_3^j, u_1^i) + \pi_>(u_3^j, u_1^i) + \pi_{0,\diamond}(u_3^j, u_1^i)$$

and

$$u_1^i \diamond u_4^j = \pi_<(u_4^j, u_1^i) + \pi_>(u_4^j, u_1^i) + \pi_{0,\diamond}(u_4^j, u_1^i).$$

Here for $i=1,2,3$, $u_1^i(t)=\int_{-\infty}^t\sum_{i_1=1}^3 P^{ii_1}P_{t-s}\xi^{\varepsilon,i_1}ds$ and we use $\diamond$ to replace the product of some terms, the meaning of which will be given later. In fact, the product of these terms needs to be renormalised such that they converge as $\varepsilon\to 0$. We will discuss this in Section 3.3 below. The results for the renormalised terms not including u_4 can be proved by using a similar idea as in the definition of Wick products. However, $u_4\diamond u_1$ cannot be defined by this trick since u_4 is the unknown. To deal with this term we will use the fact that u_4 has a specific structure since it satisfies (3.2). Now we do some preparations. Consider the following equations:

$$LK^i=u_1^i,\quad K^i(0)=0.$$

Then we obtain that for every $\delta>0$ small enough, if $u_1^i\in C([0,T];\mathcal{C}^{-\frac{1}{2}-\frac{\delta}{2}})$, then $K^i\in C([0,T];\mathcal{C}^{\frac{3}{2}-\delta})$ and by Lemma 3.5

$$\|K^i(t)\|_{\frac{3}{2}-\delta}\lesssim t^{\delta/4}\sup_{s\in[0,t]}\|u_1^i(s)\|_{-1/2-\delta/2}. \tag{3.3}$$

First we assume that $u_1^i\in C([0,T];\mathcal{C}^{-\frac{1}{2}-\frac{\delta}{2}})$, $u_1^i\diamond u_1^j\in C([0,T];\mathcal{C}^{-1-\delta/2})$, $u_1^i\diamond u_2^j=u_2^j\diamond u_1^i\in C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_2^i\diamond u_2^j\in C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^i,u_1^j)\in C([0,T];\mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1}D_jK^j,u_1^{j_1})$, $\pi_{0,\diamond}(P^{ii_1}D_jK^{i_1},u_1^{j_1})\in C([0,T];\mathcal{C}^{-\delta})$ for $i,j,i_1,j_1=1,2,3$, and that

$$\begin{aligned}C_\xi^\varepsilon:=\sup_{t\in[0,T]}\Big[&\sum_{i=1}^3\|u_1^{\varepsilon,i}\|_{-1/2-\delta/2}+\sum_{i,j=1}^3\|u_1^{\varepsilon,i}\diamond u_1^{\varepsilon,j}\|_{-1-\delta/2}+\sum_{i,j=1}^3\|u_1^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\|_{-1/2-\delta/2}\\&+\sum_{i,j=1}^3\|u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\|_{-\delta}+\sum_{i,j=1}^3\|\pi_{0,\diamond}(u_3^{\varepsilon,i},u_1^{\varepsilon,j})\|_{-\delta}+\sum_{i,i_1,j,j_1=1}^3\|\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j},u_1^{\varepsilon,j_1})\|_{-\delta}\\&+\sum_{i,i_1,j,j_1=1}^3\|\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1},u_1^{\varepsilon,j_1})\|_{-\delta}\Big]<\infty.\end{aligned}$$

By Lemmas 3.5 and 3.6 we easily deduce that $u_2^i\in C([0,T];\mathcal{C}^{-\delta}),u_3^i\in C([0,T];\mathcal{C}^{1/2-\delta})$ for $i=1,2,3$, and that

$$\sup_{t\in[0,T]}(\sum_{i=1}^3\|u_2^i\|_{-\delta}+\sum_{i=1}^3\|u_3^i\|_{1/2-\delta})\lesssim C_\xi. \tag{3.4}$$

In the following we will fix $\delta>0$ small enough such that

$$\delta<\delta_0\wedge\frac{1-2\delta_0}{3}\wedge\frac{1-z}{4}\wedge(2z-1).$$

By a fixed point argument it is easy to obtain local existence and uniqueness of solution to equation (3.2): More precisely, for each $\varepsilon\in(0,1)$ there exists a maximal time T_ε and $u_4\in C((0,T_\varepsilon);\mathcal{C}^{1/2-\delta_0})$ with respect to the norm $\sup_{t\in[0,T]}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0}$ such that u_4 satisfies equation (3.2) before T_ε and

$$\sup_{t\in[0,T_\varepsilon)}t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0}=\infty.$$

Indeed, since ξ_ε is smooth, by (3.2) and Lemmas 3.5 and 3.6 we have the following estimate

$$\sup_{t\in[0,T]} t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0} \lesssim C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)+T^{\frac{1/2+\delta_0-z}{2}}(\sup_{t\in[0,T]} t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0})^2,$$

where $C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)$ are constants depending on ε and we used $z<1/2+\delta_0$.

Paracontrolled ansatz: As we mentioned before, our problem lies in how to define $\pi_0(u_4^j,u_1^i)$. Observing that the worst term on the right hand side of (3.2) is $PD\pi_<(u_3+u_4,u_1)$, we write u_4 as the following paracontrolled ansatz for $i=1,2,3$:

$$u_4^i=-\frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j[\pi_<(u_3^{i_1}+u_4^{i_1},K^j)+\pi_<(u_3^j+u_4^j,K^{i_1})])+u^{\sharp,i}$$

with $u^{\sharp,i}(t)\in\mathcal{C}^{1/2+\beta}$ for some $\delta/2<\beta<(z+2\delta-1/2)<(1/2-2\delta)$ and $t\in(0,T_\varepsilon)$ (which can be done for fixed $\varepsilon>0$ since ξ_ε is smooth and by (3.2) we note that

$$t^{\frac{1/2+\beta+z}{2}}\|u_4(t)\|_{1/2+\beta}\lesssim C_\varepsilon(\|u_0\|_{-z},u_1,u_2,u_3)+t^{\frac{1/2+\delta_0-z}{2}}(\sup_{s\in[0,t]} s^{\frac{1/2-\delta_0+z}{2}}\|u_4(s)\|_{1/2-\delta_0})^2).$$

From the paracontrolled ansatz and Lemma 3.2 we easily get the following estimate for $i=1,2,3$:

$$\|u_4^i\|_{1/2-\delta}\lesssim\sum_{i_1,j=1}^3\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^j\|_{3/2-\delta}+\|u^{\sharp,i}\|_{1/2+\beta}. \tag{3.5}$$

Moreover u_4 solves (3.2) if and only if $u^\sharp$ solves the following equation:

$$\begin{aligned}
Lu^{\sharp,i}=&-\frac{1}{2}\sum_{i_1,j=1}^3 P^{ii_1}D_j\big[u_2^{i_1}\diamond u_2^j+u_2^{i_1}(u_3^j+u_4^j)+u_2^j(u_3^{i_1}+u_4^{i_1})+(u_3^{i_1}+u_4^{i_1})(u_3^j+u_4^j)\\
&-\pi_<(L(u_3^{i_1}+u_4^{i_1}),K^j)+2\sum_{l=1}^3\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)+\pi_{0,\diamond}(u_3^{i_1},u_1^j)+\pi_{0,\diamond}(u_4^{i_1},u_1^j)\\
&-\pi_<(L(u_3^j+u_4^j),K^{i_1})+2\sum_{l=1}^3\pi_<(D_l(u_3^j+u_4^j),D_lK^{i_1})+\pi_>(u_3^j+u_4^j,u_1^{i_1})+\pi_{0,\diamond}(u_3^j,u_1^{i_1})+\pi_{0,\diamond}(u_4^j,u_1^{i_1})\big]\\
:=&\,\phi^{\sharp,i}.
\end{aligned} \tag{3.6}$$

Renormalisation of $\pi_0(u_4^i,u_1^j)$: By the paracontrolled ansatz we have for $i,j=1,2,3$,

$$\begin{aligned}
\pi_0(u_4^i,u_1^j)=&-\frac{1}{2}\pi_0(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1},D_{j_1}K^{j_1}),u_1^j)-\frac{1}{2}\pi_0(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1},D_{j_1}K^{i_1}),u_1^j)\\
&-\frac{1}{2}\sum_{i_1,j_1=1}^3\pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{i_1}+u_4^{i_1}),K^{j_1}),u_1^j))-\frac{1}{2}\sum_{i_1,j_1=1}^3\pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{j_1}+u_4^{j_1}),K^{i_1}),u_1^j))\\
&+\pi_0(u^{\sharp,i},u_1^j).
\end{aligned}$$

The last three terms can be easily controlled by Lemma 3.2, and it is sufficient to consider the first two terms: For $i, i_1, j, j_1 = 1, 2, 3$,

$$
\begin{aligned}
&\pi_0(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j)\\
=&\pi_0(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j) - \pi_0(\pi_<(u_3^{i_1}+u_4^{i_1}, P^{ii_1}D_{j_1}K^{j_1}), u_1^j)\\
&+\pi_0(\pi_<(u_3^{i_1}+u_4^{i_1}, P^{ii_1}D_{j_1}K^{j_1}), u_1^j) - (u_3^{i_1}+u_4^{i_1})\pi_0(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)\\
&+(u_3^{i_1}+u_4^{i_1})\pi_0(P^{ii_1}D_{j_1}K^{j_1}, u_1^j).
\end{aligned}
$$

Applying Lemmas 3.3 and 3.4 we can control the first four terms on the right hand side of above equality. As we mentioned above for $\pi_0(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)$ we need to do renormalisation to make it convergent as $\varepsilon \to 0$, which leads to the renormalisation of $\pi_0(u_4^i, u_1^j)$. Define

$$
\begin{aligned}
&\pi_{0,\diamond}(u_4^i, u_1^j)\\
:=&-\frac{1}{2}(\pi_{0,\diamond}(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j) + \pi_{0,\diamond}(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1}, D_{j_1}K^{i_1}), u_1^j)\\
&+\sum_{i_1,j_1=1}^3 \pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{i_1}+u_4^{i_1}), K^{j_1}), u_1^j)) + \sum_{i_1,j_1=1}^3 \pi_0(P^{ii_1}\pi_<(D_{j_1}(u_3^{j_1}+u_4^{j_1}), K^{i_1}), u_1^j))\\
&+\pi_0(u^{\sharp,i}, u_1^j),
\end{aligned}
$$

where

$$
\begin{aligned}
&\pi_{0,\diamond}(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j)\\
:=&\pi_0(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j) - \pi_0(\pi_<(u_3^{i_1}+u_4^{i_1}, P^{ii_1}D_{j_1}K^{j_1}), u_1^j)\\
&+\pi_0(\pi_<(u_3^{i_1}+u_4^{i_1}, P^{ii_1}D_{j_1}K^{j_1}), u_1^j) - (u_3^{i_1}+u_4^{i_1})\pi_0(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)\\
&+(u_3^{i_1}+u_4^{i_1})\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1}, u_1^j),
\end{aligned}
$$

and $\pi_{0,\diamond}(P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1}, D_{j_1}K^{i_1}), u_1^j)$ can be defined similarly. Using Lemmas 3.2 and 3.3 we get that for $\delta \leq \delta_0 < 1/2 - 3\delta/2$

$$
\begin{aligned}
&\|\pi_{0,\diamond}(P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}), u_1^j)\|_{-\delta}\\
\lesssim&\|P^{ii_1}\pi_<(u_3^{i_1}+u_4^{i_1}, D_{j_1}K^{j_1}) - \pi_<(u_3^{i_1}+u_4^{i_1}, P^{ii_1}D_{j_1}K^{j_1})\|_{1-\delta-\delta_0}\|u_1^j\|_{-1/2-\delta/2}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|P^{ii_1}D_{j_1}K^{j_1}\|_{1/2-\delta}\|u_1^j\|_{-1/2-\delta/2}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)\|_{-\delta}\\
\lesssim&\|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}\|u_1^j\|_{-1/2-\delta/2} + \|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)\|_{-\delta}.
\end{aligned}
$$

Here in the last inequality we used Lemmas 3.4 and 3.6. Similar estimates can also be deduced for $\pi_{0,\diamond}(\sum_{i_1,j_1=1}^3 P^{ii_1}\pi_<(u_3^{j_1}+u_4^{j_1}, D_{j_1}K^{i_1}), u_1^j)$.

Hence we obtain that for $i, j = 1, 2, 3$,

$$
\begin{aligned}
\|\pi_{0,\diamond}(u_4^i, u_1^j)\|_{-\delta} \lesssim& \sum_{i_1=1}^3 \|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0} \sum_{j_1=1}^3 \|K^{j_1}\|_{3/2-\delta}\|u_1^j\|_{-1/2-\delta/2}\\
&+\sum_{i_1,j_1=1}^3 \|u_3^{i_1}+u_4^{i_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{j_1}, u_1^j)\|_{-\delta}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i_1,j_1=1}^{3}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\|\pi_{0,\diamond}(P^{ii_1}D_{j_1}K^{i_1},u_1^j)\|_{-\delta}\\
&+\|u^{\sharp,i}\|_{1/2+\beta}\|u_1^j\|_{-1/2-\delta/2}\\
\lesssim &C_\xi^3+1+\|u_4\|_{1/2-\delta_0}(C_\xi^2+1)+\|u^\sharp\|C_\xi.
\end{aligned}
$$

Estimate of $\phi^\sharp$: To obtain a uniform estimate for u_4^ε, we first prove an estimate for $\phi^\sharp$:

Lemma 3.7 For $\phi^\sharp$ defined in (3.6), the following estimate holds:

$$
\|\phi^{\sharp,i}\|_{-1-2\delta}\lesssim(1+C_\xi^4)\big[1+\|u^\sharp\|_{1/2+\beta}+\|u_4\|_{1/2-\delta_0}+\|u_4\|_\delta^2\big]. \tag{3.7}
$$

Proof First we consider $\pi_<(L(u_3^i+u_4^i),K^j), i,j=1,2,3,$: Indeed (3.2) implies that for $i=1,2,3$,

$$
\begin{aligned}
L(u_3^i+u_4^i)=&-\frac{1}{2}\sum_{i_1,j=1}^{3}P^{ii_1}D_j(u_1^{i_1}\diamond u_2^j+u_1^j\diamond u_2^{i_1}+u_1^{i_1}\diamond(u_3^j+u_4^j)+u_1^j\diamond(u_3^{i_1}+u_4^{i_1})\\
&+u_2^{i_1}\diamond u_2^j+u_2^{i_1}(u_3^j+u_4^j)+u_2^j(u_3^{i_1}+u_4^{i_1})+(u_3^{i_1}+u_4^{i_1})(u_3^j+u_4^j)),
\end{aligned}
$$

where for $i,j=1,2,3$,

$$
u_1^i\diamond(u_3^j+u_4^j)=\pi_<(u_3^j+u_4^j,u_1^i)+\pi_{0,\diamond}(u_3^j,u_1^i)+\pi_>(u_3^j+u_4^j,u_1^i)+\pi_{0,\diamond}(u_4^j,u_1^i).
$$

Using Lemmas 3.6 and 3.2 we obtain that for $i=1,2,3$,

$$
\begin{aligned}
\|L(u_3^i+u_4^i)\|_{-3/2-\delta/2}\lesssim&\sum_{i_1,j_1=1}^{3}[\|u_1^{i_1}\diamond u_2^{j_1}\|_{-1/2-\delta/2}+\|u_2^{i_1}\diamond u_2^{j_1}\|_{-\delta}+\|u_1^{i_1}\|_{-1/2-\delta/2}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|\pi_{0,\diamond}(u_3^{i_1},u_1^{j_1})\|_{-\delta}+\|u_2^{i_1}\|_{-\delta}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_\delta\|u_3^{j_1}+u_4^{j_1}\|_\delta+\|\pi_{0,\diamond}(u_4^{i_1},u_1^{j_1})\|_{-\delta}]\\
\lesssim&C_\xi^3+1+(1+C_\xi^2)\|u_4\|_{1/2-\delta_0}+C_\xi\|u^\sharp\|_{1/2+\beta}+\|u_4\|_\delta^2,
\end{aligned}
$$

where we used $\delta<\delta_0\wedge(\frac{1}{2}-\delta_0)$, which by Lemma 3.2 yields that

$$
\begin{aligned}
&\|\pi_<(L(u_3^i+u_4^i),K^j)\|_{-3\delta/2}\\
\lesssim&\|K^j\|_{3/2-\delta}[C_\xi^3+1+(1+C_\xi^2)\|u_4\|_{1/2-\delta_0}+C_\xi\|u^\sharp\|_{1/2+\beta}+\|u_4\|_\delta^2].
\end{aligned}
$$

Then we consider $\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)$ for $i_1,l,j=1,2,3$ in (3.6): Indeed Lemma 3.2 implies that

$$
\begin{aligned}
&\|\pi_<(D_l(u_3^{i_1}+u_4^{i_1}),D_lK^j)+\pi_>(u_3^{i_1}+u_4^{i_1},u_1^j)\|_{-2\delta}\\
\lesssim&(\|u_3^{i_1}\|_{1/2-\delta}+\|u_4^{i_1}\|_{1/2-\delta})(\|K^j\|_{3/2-\delta}+\|u_1^j\|_{-1/2-\delta/2})\\
\lesssim&(\|u_3^{i_1}\|_{1/2-\delta}+\sum_{i_2,j_1=1}^{3}\|u_3^{i_2}+u_4^{i_2}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta})C_\xi,
\end{aligned}
$$

where in the last inequality we used (3.5).

Combining all these estimates obtained above, by (3.6) we get that

$$\begin{aligned}
&\|\phi^{\sharp,i}\|_{-1-2\delta}\\
\lesssim&\sum_{j=1}^{3}(\|K^j\|_{3/2-\delta}+1)\sum_{i_1,j_1=1}^{3}[\|u_2^{i_1}\diamond u_2^{j_1}\|_{-\delta}+\|\pi_{0,\diamond}(u_3^{i_1},u_1^{j_1})\|_{-\delta}+\|u_2^{i_1}\|_{-\delta}\|u_3^{j_1}+u_4^{j_1}\|_{1/2-\delta_0}\\
&+\|u_3^{i_1}+u_4^{i_1}\|_{\delta}\|u_3^{j_1}+u_4^{j_1}\|_{\delta}+C_\xi^3+1+(1+C_\xi^2)\|u_4\|_{1/2-\delta_0}+C_\xi\|u^\sharp\|_{1/2+\beta}+\|u_4\|_\delta^2]\\
&+\sum_{i_1,j_1,l=1}^{3}(\|u_3^{i_1}\|_{1/2-\delta}+\sum_{i_2,j_1=1}^{3}\|u_3^{i_2}+u_4^{i_2}\|_{1/2-\delta_0}\|K^{j_1}\|_{3/2-\delta}+\|u^{\sharp,i_1}\|_{1/2+\beta})C_\xi\\
\lesssim&(1+C_\xi^4)\big[1+\|u^\sharp\|_{1/2+\beta}+\|u_4\|_{1/2-\delta_0}+\|u_4\|_\delta^2\big],
\end{aligned}$$

where we used (3.2) (3.3) and $\delta\leq\delta_0$ in the last inequality. □

Construction of the solution: In the following we will prove a uniform estimate of u_4^ε: By the paracontrolled ansatz (3.3) and Lemma 3.2 we get

$$\|u_4^i(t)\|_{1/2-\delta_0}\lesssim t^{\delta/4}C_\xi\sum_{i_1=1}^{3}\|u_3^{i_1}(t)+u_4^{i_1}(t)\|_{1/2-\delta_0}+\|u^{\sharp,i}(t)\|_{1/2-\delta_0},$$

which shows that for $t\in[0,\bar T]$ (with $\bar T>0$ only depending on C_ξ)

$$\sum_{i=1}^{3}\|u_4^i(t)\|_{1/2-\delta_0}\lesssim C_\xi^2+\sum_{i=1}^{3}\|u^{\sharp,i}(t)\|_{1/2-\delta_0}. \tag{3.8}$$

Similarly, we have for $t\in[0,\bar T]$ (with $\bar T>0$ only depending on C_ξ)

$$\sum_{i=1}^{3}\|u_4^i(t)\|_{\delta}\lesssim C_\xi^2+\sum_{i=1}^{3}\|u^{\sharp,i}(t)\|_{\delta}. \tag{3.9}$$

Moreover, Lemma 3.5 and (3.6) yield that for $\delta+z<1$

$$\begin{aligned}
&t^{\delta+z}\|u^\sharp(t)\|_{1/2+\beta}\\
\lesssim&\|Pu_0-u_1(0)\|_{-z}+t^{\delta+z}\int_0^t(t-s)^{-3/4-\delta-\beta/2}s^{-(\delta+z)}s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta}ds,
\end{aligned}\tag{3.10}$$

where we used the condition on β to deduce that $3/4+\beta/2+\delta<1$ and $\frac{1/2+\beta+z}{2}\leq\delta+z$. Similarly, we deduce that

$$\begin{aligned}
t^{\delta+z}\|u^\sharp(t)\|_\delta^2\lesssim&\|Pu_0-u_1(0)\|_{-z}^2+t^{\delta+z}(\int_0^t(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta}ds)^2\\
\lesssim&\|Pu_0-u_1(0)\|_{-z}^2+t^{(1-3\delta)/2}\int_0^t(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}(s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta})^2ds.
\end{aligned}\tag{3.11}$$

Here in the last inequality we used Hölder's inequality. Thus, by (3.7-3.11) we get that for $t \in [0, \bar{T}]$

$$\begin{aligned} t^{\delta+z}\|\phi^\sharp\|_{-1-2\delta} \lesssim & (1+C_\xi^4)\big[\|Pu_0 - u_1(0)\|_{-z}^2 + C_\xi^4 + 1 \\ & + \int_0^t t^{\delta+z}(t-s)^{-3/4-\delta-\beta/2}s^{-(\delta+z)}(s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta}) \\ & + t^{(1-3\delta)/2}(t-s)^{-\frac{1+3\delta}{2}}s^{-(\delta+z)}(s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta})^2 ds\big]. \end{aligned}$$

Then Bihari's inequality implies that for $\delta < \frac{1-z}{4}$ there exists some $0 < T_0 \le \bar{T}$ such that

$$\sup_{t\in[0,T_0]} t^{\delta+z}\|\phi^\sharp\|_{-1-2\delta} \lesssim C(T_0, C_\xi, \|u_0\|_{-z}), \tag{3.12}$$

where $C(T_0, C_\xi, \|u_0\|_{-z})$ depends on T_0, $\|u_0\|_{-z}$ and C_ξ. Here T_0 can be chosen independent of ε such that (3.12) holds for all $\varepsilon \in (0,1)$, if C_ξ^ε and $\|u_0\|_{-z}$ is uniformly bounded over $\varepsilon \in (0,1)$. Similarly as (3.10) we have

$$\begin{aligned} & t^{(1/2-\delta_0+z)/2}\|u^\sharp(t)\|_{1/2-\delta_0} \\ \lesssim & \|Pu_0 - u_1(0)\|_{-z} + t^{(1/2-\delta_0+z)/2}\int_0^t (t-s)^{-3/4-\delta+\delta_0/2}s^{-(\delta+z)}s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta}ds \\ \lesssim & \|Pu_0 - u_1(0)\|_{-z} + t^{(1-4\delta-z)/2}\sup_{s\in[0,t]} s^{\delta+z}\|\phi^\sharp(s)\|_{-1-2\delta}. \end{aligned} \tag{3.13}$$

Then by (3.8) (3.13) we obtain that

$$\sup_{t\in[0,T_0]} t^{\frac{1/2-\delta_0+z}{2}}\|u_4(t)\|_{1/2-\delta_0} \lesssim C_\xi^2 + \|u_0\|_{-z} + C(T_0, C_\xi, \|u_0\|_{-z}),$$

which implies that $T_\varepsilon \ge T_0$. Here we used $z \ge 1/2 + \delta/2$. Moreover, similarly as for (3.8) one also gets that for $t \in [0, T_0]$

$$\begin{aligned} \|u_4(t)\|_{-z} & \lesssim C_\xi^2 + \|u^\sharp(t)\|_{-z} \\ & \lesssim C_\xi^2 + \|u_0\|_{-z} + \int_0^t (t-s)^{\frac{-1-2\delta+z}{2}}s^{-(\delta+z)}s^{\delta+z}\|\phi^{\sharp,\lambda}\|_{-1-2\delta}ds, \end{aligned}$$

where in the last inequality we used Lemma 3.5. This gives us our final estimate for u^4:

$$\sup_{t\in[0,T_0]} \|u_4(t)\|_{-z} \lesssim C_\xi^2 + \|u_0\|_{-z} + C(T_0, C_\xi, \|u_0\|_{-z}).$$

We define $\mathbb{Z}(\xi^\varepsilon) := (u_1^\varepsilon, u_1^\varepsilon \diamond u_1^\varepsilon, u_1^\varepsilon \diamond u_2^\varepsilon, u_2^\varepsilon \diamond u_2^\varepsilon, \pi_{0,\diamond}(u_3^\varepsilon, u_1^\varepsilon), \pi_{0,\diamond}(PDK^\varepsilon, u_1^\varepsilon)) \in \mathbb{X} := C([0,T];\mathcal{C}^{-1/2-\delta/2}) \times C([0,T];\mathcal{C}^{-1-\delta/2}) \times C([0,T];\mathcal{C}^{-1/2-\delta/2}) \times C([0,T];\mathcal{C}^{-\delta}) \times C([0,T];\mathcal{C}^{-\delta}) \times C([0,T];\mathcal{C}^{-\delta})$. Here $\mathbb{X}$ is equipped with the product topology.

Similar arguments show that for every $a > 0$ there exists a sufficiently small $T_0 > 0$ such that the map $(u_0, \mathbb{Z}(\xi_\varepsilon)) \mapsto u_4$ is Lipschitz continuous on the set

$$\max\{\|u_0\|_{-z}, C_\xi\} \le a.$$

Here we consider u_4 with respect to the norm given by

$$\sup_{t\in[0,T_0]} \|u_4(t)\|_{-z}.$$

Hence we obtain that there exists a local solution u to (3.1) with initial condition u_0, which is the limit of the solutions $u^\varepsilon, \varepsilon > 0$, to the following equation

$$Lu^{\varepsilon,i} = \sum_{i_1=1}^3 P^{ii_1}\xi^{\varepsilon,i_1} - \frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u^{\varepsilon,i_1}u^{\varepsilon,j})) \quad u^\varepsilon(0) = u_0,$$

provided that $\mathbb{Z}(\xi^\varepsilon)$ converges in $\mathbb{X}$, i.e. for $i, i_1, j, j_2 = 1, 2, 3$, there exist $v_1^i, v_2^{ij}, v_3^{ij}, v_4^{ij}, v_5^{ij}, v_6^{ii_1jj_2}, v_7^{ii_1jj_2}$ such that for any $\delta > 0$, $u_1^{\varepsilon,i} \to v_1^i$ in $C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} \to v_2^{ij}$ in $C([0,T];\mathcal{C}^{-1-\delta/2})$, $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_3^{ij}$ in $C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_4^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) \to v_5^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$, $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j}, u_1^{\varepsilon,j_2}) \to v_6^{ii_1jj_2}$ in $C([0,T];\mathcal{C}^{-\delta})$ and $\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1}, u_1^{\varepsilon,j_2}) \to v_7^{ii_1jj_2}$ in $C([0,T];\mathcal{C}^{-\delta})$. Here

$$\begin{aligned}
u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j} &:= u_1^{\varepsilon,i}u_1^{\varepsilon,j} - C_0^{\varepsilon,ij},\\
u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} &:= u_1^{\varepsilon,i}u_2^{\varepsilon,j},\\
u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} &:= u_2^{\varepsilon,i}u_2^{\varepsilon,j} - C_2^{\varepsilon,ij},\\
\pi_{0,\diamond}(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) &:= \pi_0(u_3^{\varepsilon,i}, u_1^{\varepsilon,j}) - C_1^{\varepsilon,ij},\\
\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,j}, u_1^{\varepsilon,j_2}) &:= \pi_0(P^{ii_1}D_jK^{\varepsilon,j}, u_1^{\varepsilon,j_2}),\\
\pi_{0,\diamond}(P^{ii_1}D_jK^{\varepsilon,i_1}, u_1^{\varepsilon,j_2}) &:= \pi_0(P^{ii_1}D_jK^{\varepsilon,i_1}, u_1^{\varepsilon,j_2}),
\end{aligned}$$

and $C_0^\varepsilon \in \mathbb{R}$ is defined in Section 3.3, C_1^ε is defined in Section 3.3.1 and C_2^ε is defined in Appendix 4.2. Hence we obtain the following theorem:

Theorem 3.8 Let $z \in (1/2, 1/2 + \delta_0)$ with $0 < \delta_0 < 1/2$ and assume that $(\xi^\varepsilon)_{\varepsilon>0}$ is a family of smooth functions converging to ξ as $\varepsilon \to 0$. Let for $\varepsilon > 0$ the function u^ε be the unique maximal solution to the Cauchy problem

$$Lu^{\varepsilon,i} = \sum_{i_1=1}^3 P^{ii_1}\xi^{\varepsilon,i_1} - \frac{1}{2}\sum_{i_1=1}^3 P^{ii_1}(\sum_{j=1}^3 D_j(u^{\varepsilon,i_1}u^{\varepsilon,j})) \quad u^\varepsilon(0) = Pu_0,$$

such that u_4^ε defined as above belongs to $C((0,T_\varepsilon);\mathcal{C}^{1/2-\delta_0})$, where $u_0 \in \mathcal{C}^{-z}$. Suppose that $\mathbb{Z}(\xi^\varepsilon)$ converges to $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ in $\mathbb{X}$. Then there exist $\tau = \tau(u_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7) > 0$ and $u \in C([0,\tau];\mathcal{C}^{-z})$ such that

$$\sup_{t\in[0,\tau]} \|u^\varepsilon - u\|_{-z} \to 0.$$

The limit u depends only on $(u_0, v_i), i = 1...., 7$, and not on the approximating family.

Remark 3.9 Indeed we can define the solution space as follows: $u - u_1 \in \mathcal{D}_X^L$ if

$$u - u_1 = u_2 + u_3 - \frac{1}{2}\int_0^t P_{t-s}P\sum_{j=1}^3 D_j[\pi_<(\Phi', u_1^j) + \pi_<(\Phi'^j, u_1)]ds + \Phi^\sharp$$

such that

$$\|\Phi^\sharp\|_{\star,1,L,T} := \sup_{t\in[0,T]} t^{\frac{1-\eta+z}{2}}\|\Phi_t^\sharp\|_{1-\eta} + \sup_{t\in[0,T]} t^{\frac{\gamma+z}{2}}\|\Phi_t^\sharp\|_\gamma + \sup_{s,t\in[0,T]} s^{\frac{z+a}{2}}\frac{\|\Phi_t^\sharp-\Phi_s^\sharp\|_{a-2b}}{|t-s|^b} < \infty,$$

and

$$\|\Phi'\|_{\star,2,L,T} := \sup_{t\in[0,T]} t^{\frac{2\gamma+z}{2}}\|\Phi_t'\|_{1/2-\kappa} + \sup_{s,t\in[0,T]} s^{\frac{z+a}{2}}\frac{\|\Phi_t'-\Phi_s'\|_{c-2d}}{|t-s|^d} < \infty.$$

Here $\eta,\gamma\in(0,1), a\geq 2b, 0<\kappa<1/2, c\geq 2d$. By a similar argument as in [4], if $u-u_1\in\mathcal{D}_X^L$ then the equation

$$u-u_1 = P_t(u_0-u_1(0))-\frac{1}{2}\int_0^t P_{t-s}P\sum_{j=1}^3 D_j(u_1\diamond u_1^j+(u-u_1)\diamond u_1^j+u_1\diamond(u-u_1)^j+(u-u_1)\diamond(u-u_1)^j ds$$

can be well defined and by a fixed point argument we also obtain local existence and uniqueness of solutions. The calculations for this method are more complicated and we will not go into details here.

3.3 Renormalisation

In the following we use the notation X to represent u_1, $k_{1,...,n} := \sum_{i=1}^n k_i$ and

$$\hat{f}(k) := (2\pi)^{-\frac{3}{2}}\int_{\mathbb{T}^3} f(x)e^{\imath x\cdot k}dx$$

for $k\in\mathbb{Z}^3$. To simplify the arguments below, we assume that $\hat{\xi}(0)=0$ and restrict ourselves to the flow of $\int_{\mathbb{T}^3} u(x)dx=0$. Then we know that $X_t=\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\hat{X}_t(k)e_k$ is a centered Gaussian process with covariance function given by

$$E[\hat{X}_t^i(k)\hat{X}_s^j(k')] = 1_{k+k'=0}\sum_{i_1=1}^3\frac{e^{-|k|^2|t-s|}}{2|k|^2}\hat{P}^{ii_1}(k)\hat{P}^{ji_1}(k),$$

and $\hat{X}_t(0)=0$, where $e_k(x)=(2\pi)^{-3/2}e^{\imath x\cdot k}, x\in\mathbb{T}^3$ and $\hat{P}^{ii_1}(k)=\delta_{ii_1}-\frac{k_ik_{i_1}}{|k|^2}$ for $k\in\mathbb{Z}^3\setminus\{0\}$. Let us take a smooth radial function f with compact support such that $f(0)=1$. We regularize X in the following way

$$X_t^{\varepsilon,i} = \int_{-\infty}^t\sum_{i_1=1}^3 P^{ii_1}P_{t-s}\xi^{\varepsilon,i_1}ds$$

with $\xi^\varepsilon=\sum_{k\in\mathbb{Z}^3\setminus\{0\}}f(\varepsilon k)\hat{\xi}(k)$. In this subsection we will prove that there exist $v_1,v_2,v_3,v_4,v_5,v_6,v_7$ such that $\mathbb{Z}(\xi^\varepsilon)$ converges to $(v_1,v_2,...,v_7)$ in $\mathbb{X}$.

It is easy to obtain that there exists v_1 such that $u_1^\varepsilon\to v_1$ in $L^p(\Omega,P,C([0,T];\mathcal{C}^{-1/2-\delta/2}))$ for every $p\geq 1$. The renormalisation of $u_1^{\varepsilon,i}\diamond u_1^{\varepsilon,j}, i,j=1,2,3$ and the fact that there exists $v_2\in C([0,T];\mathcal{C}^{-1-\delta})$ such that $u_1^{\varepsilon,i}\diamond u_1^{\varepsilon,j}\to v_2^{ij}$ in $L^p(\Omega,P,C([0,T];\mathcal{C}^{-1-\delta}))$ for every $p\geq 1$ can be easily obtained by using the Wick product (c.f.[4]), where

$$C_0^{\varepsilon,ij} = (2\pi)^{-3}\sum_{i_1=1}^3\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\frac{f(\varepsilon k)^2}{2|k|^2}\hat{P}^{ii_1}(k)\hat{P}^{ji_1}(k).$$

It is obvious that $C_0^{\varepsilon,ij} \to \infty$ as $\varepsilon \to 0$. Here u_1^ε and $u_1^{\varepsilon,i} \diamond u_1^{\varepsilon,j}$ correspond to $\uparrow$ and $\vee$ in Section 2 respectively. By a similar argument as in the proof of Theorem 2.16 we could conclude that $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_3^{ij}$ in $C([0,T];\mathcal{C}^{-1/2-\delta})$, $u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_4^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$. We could also use Fourier analysis to obtain it. Here for completeness of this method we calculate it in the appendix. For the terms including π_0 we cannot use a similar argument as in the proof of Theorem 2.6 to obtain the results since the definition of π_0 depends on the Fourier analysis. That is one of difference between these two approaches (see Remark 3.13).

We first prove the following two lemmas for later use, the first of which is inspired by [12, Lemma 10.14].

Lemma 3.10 Let $0 < l, m < d, l + m - d > 0$. Then

$$\sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k} \frac{1}{|k_1|^l|k_2|^m} \lesssim \frac{1}{|k|^{l+m-d}}.$$

Proof We have the following estimate:

$$\begin{aligned}\sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k} \frac{1}{|k_1|^l|k_2|^m} \lesssim & \sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k,|k_1|\leq\frac{|k|}{2}} \frac{1}{|k_1|^l|k_2|^m} \\ &+ \sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k,|k_2|\leq\frac{|k|}{2}} \frac{1}{|k_1|^l|k_2|^m} \\ &+ \sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k,|k_1|>\frac{|k|}{2},|k_2|>\frac{|k|}{2}} \frac{1}{|k_1|^l|k_2|^m}.\end{aligned}$$

Since $|k_1| \leq |k|/2$ implies that $|k_2| \geq |k| - |k_1| \geq |k|/2$, we obtain

$$\sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k,|k_1|\leq\frac{|k|}{2}} \frac{1}{|k_1|^l|k_2|^m} \lesssim \sum_{k_1\in\mathbb{Z}^d\setminus\{0\},|k_1|\leq\frac{|k|}{2}} \frac{1}{|k_1|^l|k|^m} \lesssim |k|^{-l-m+d}.$$

For the second term a similar argument also yields the desired estimate. For the third term: by $|k_2| \geq |k_1| - |k|$ and the triangle inequality, one has

$$|k_2| \geq \frac{1}{4}(|k_1| - |k|) + \frac{1-1/4}{2}|k| \geq \frac{1}{4}|k_1|,$$

which implies that

$$\sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k,|k_1|>\frac{|k|}{2},|k_2|>\frac{|k|}{2}} \frac{1}{|k_1|^l|k_2|^m} \lesssim |k|^{-l-m+d}.$$

Hence the result follows. $\square$

Lemma 3.11 For any $0 < \eta < 1$, $i, j, l = 1, 2, 3$ and for $t > 0$ the following estimate holds:

$$|e^{-|k_{12}|^2t}k_{12}^i\hat{P}^{jl}(k_{12}) - e^{-|k_2|^2t}k_2^i\hat{P}^{jl}(k_2)| \lesssim |k_1|^\eta|t|^{-(1-\eta)/2}.$$

Here $\hat{P}^{ij}(x)=\delta_{ij}-\frac{x^ix^j}{|x|^2}$.

Proof First we have the following bound:

$$|e^{-|k_{12}|^2t}k_{12}\hat{P}(k_{12})-e^{-|k_2|^2t}k_2\hat{P}(k_2)|\lesssim|t|^{-1/2}.$$

Consider the function $F(x)=e^{-|x|^2t}x\hat{P}(x)$. Then it is easy to check that $|DF|$ is bounded, which implies that

$$|e^{-|k_{12}|^2t}k_{12}\hat{P}(k_{12})-e^{-|k_2|^2t}k_2\hat{P}(k_2)|\lesssim|k_1|.$$

Thus, the result follows by the interpolation. $\square$

3.3.1 Renormalisation for $\pi_0(u_3^{\varepsilon,i_0},u_1^{\varepsilon,j_0})$

Now we consider $\pi_0(u_{31}^{\varepsilon,i_0},u_1^{\varepsilon,j_0})$. The estimates for $\pi_0(u_3^{\varepsilon,i_0}-u_{31}^{\varepsilon,i_0},u_1^{\varepsilon,j_0})$ can be obtained similarly, where $Lu_{31}^{i_0}=-\frac{1}{2}\sum_{i_1=1}^3P^{i_0i_1}\sum_{j=1}^3D_j(u_2^{i_1}\diamond u_1^j)$. We have the following identity:

$$\pi_0(u_{31}^{\varepsilon,i_0i_1},u_1^{\varepsilon,j_0})(t)=\frac{1}{4}\sum_{i=1}^7I_t^i,$$

where

$$I_t^1=(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1}\sum_{k_{1234}=k}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\int_0^tdse^{-|k_{123}|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)$$
$$\hat{X}_\sigma^{\varepsilon,i_3}(k_2)\hat{X}_s^{\varepsilon,j_1}(k_3)\hat{X}_t^{\varepsilon,j_0}(k_4):e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})e_k,$$

$$I_t^2+I_t^3=(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1}\sum_{k_{23}=k,k_1}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_1)\int_0^tdse^{-|k_{123}|^2(t-s)}$$
$$\int_0^s:\hat{X}_\sigma^{\varepsilon,i_5}(k_2)\hat{X}_s^{\varepsilon,j_1}(k_3):\frac{e^{-|k_1|^2(t-\sigma)}f(\varepsilon k_1)^2}{2|k_1|^2}\sum_{i_4=1}^3\hat{P}^{i_6i_4}(k_1)\hat{P}^{j_0i_4}(k_1)e^{-|k_{12}|^2(s-\sigma)}d\sigma$$
$$\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})(1_{i_5=i_3,i_6=i_2}+1_{i_5=i_2,i_6=i_3})e_k,$$

$$I_t^4=(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1}\sum_{k_{12}=k,k_3}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\int_0^tdse^{-|k_{123}|^2(t-s)}\int_0^s:\hat{X}_\sigma^{\varepsilon,i_2}(k_1)$$
$$\hat{X}_\sigma^{\varepsilon,i_3}(k_2):\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{2|k_3|^2}\sum_{i_4=1}^3\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_{123}^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_{123})e_k,$$

$$I_t^5+I_t^6=(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^tdse^{-|k_1|^2(t-s)}$$
$$\int_0^s:\hat{X}_\sigma^{\varepsilon,i_5}(k_1)\hat{X}_t^{\varepsilon,j_0}(k_4):\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{2|k_2|^2}\sum_{i_4=1}^3\hat{P}^{i_6i_4}(k_2)\hat{P}^{j_1i_4}(k_2)$$
$$e^{-|k_{12}|^2(s-\sigma)}d\sigma\imath k_{12}^{i_3}\imath k_1^{j_1}\hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_1)(1_{i_5=i_2,i_6=i_3}+1_{i_5=i_3,i_6=i_2})e_k,$$

$$I_t^7 = (2\pi)^{-6} \sum_{|i-j|\leq 1} \sum_{k_1,k_2} \sum_{i_1,i_2,i_3,j_1=1}^{3} \theta(2^{-i}k_2)\theta(2^{-j}k_2) \int_0^t ds e^{-|k_2|^2(t-s)} \int_0^s \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{4|k_1|^2|k_2|^2}$$

$$\sum_{i_4,i_5=1}^{3} \big(\hat{P}^{i_3i_4}(k_1)\hat{P}^{j_1i_4}(k_1)\hat{P}^{i_2i_5}(k_2)\hat{P}^{j_0i_5}(k_2) + \hat{P}^{i_2i_4}(k_1)\hat{P}^{j_1i_4}(k_1)\hat{P}^{i_3i_5}(k_2)\hat{P}^{j_0i_5}(k_2)\big)$$

$$e^{-|k_{12}|^2(s-\sigma)-|k_1|^2(s-\sigma)-|k_2|^2(t-\sigma)} d\sigma \imath k_{12}^{i_3} \imath k_2^{j_1} \hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_2)].$$

Here I_t^2, I_t^3 and I_t^5, I_t^6 correspond to the terms associated with each indicator function respectively. To make it more readable we write each term corresponding to the tree notation in Section 2. $\pi_0(u_{31}^{\varepsilon,i_0}, u_1^{\varepsilon,j_0})$ corresponds to  and $I_t^1, I_t^2, I_t^3, I_t^4, I_t^5, I_t^6, I_t^7$ correspond to the associated $\hat{\mathcal{W}}^{(\varepsilon,4)}, \hat{\mathcal{W}}_4^{(\varepsilon,2)}, \hat{\mathcal{W}}_5^{(\varepsilon,2)}, \hat{\mathcal{W}}_3^{(\varepsilon,2)}, \hat{\mathcal{W}}_1^{(\varepsilon,2)}, \hat{\mathcal{W}}_2^{(\varepsilon,2)}, \hat{\mathcal{W}}^{(\varepsilon,0)}$ in the proof of Theorem 2.16 respectively.

First we consider I_t^7: by simple calculations we have

$$I_t^7 =(2\pi)^{-6} \sum_{k_1,k_2} \sum_{i_1,i_2,i_3,j_1=1}^{3} \imath k_{12}^{i_3} \imath k_2^{j_1} \hat{P}^{i_1i_2}(k_{12})\hat{P}^{i_0i_1}(k_2) \frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2}{4|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}$$

$$\sum_{i_4,i_5=1}^{3} \big(\hat{P}^{i_3i_4}(k_1)\hat{P}^{j_1i_4}(k_1)\hat{P}^{i_2i_5}(k_2)\hat{P}^{j_0i_5}(k_2) + \hat{P}^{i_2i_4}(k_1)\hat{P}^{j_1i_4}(k_1)\hat{P}^{i_3i_5}(k_2)\hat{P}^{j_0i_5}(k_2)\big)$$

$$\Big[\frac{1-e^{-2|k_2|^2t}}{2|k_2|^2} - \int_0^t ds e^{-2|k_2|^2(t-s)} e^{-(|k_{12}|^2+|k_1|^2+|k_2|^2)s}\Big].$$

Let

$$C_{11}^{\varepsilon,i_0j_0}(t) = I_t^7$$

We could easily conclude that $C_{11}^{\varepsilon,i_0j_0}(t) \to \infty$, as $\varepsilon \to 0$.

Similarly, we can also find C_{12}^{ε} for $u_3 - u_{31}$. Define $C_1^{\varepsilon} = C_{11}^{\varepsilon} + C_{12}^{\varepsilon}$.

Terms in the second chaos: We come to I_t^2 and have the following calculations:

$$\begin{aligned}
&E|\Delta_q I_t^2|^2 \\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{23}=k,k_1,k_4} \theta(2^{-i}k_{123})\theta(2^{-j}k_1)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_4)\theta(2^{-q}k)^2 \\
&\Pi_{i=1}^4 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \int_0^t\int_0^t ds d\bar{s} e^{-|k_{123}|^2(t-s)-|k_{234}|^2(t-\bar{s})} \int_0^s\int_0^{\bar{s}} d\sigma d\bar{\sigma} e^{-|k_1|^2(t-\sigma)-|k_4|^2(t-\bar{\sigma})} \\
&e^{-(|k_{12}|^2(s-\sigma)+|k_{24}|^2(s-\bar{\sigma}))}|k_{12}k_{123}k_{24}k_{234}| \\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{23}=k,k_1,k_4} \theta(2^{-i}k_{123})\theta(2^{-j}k_1)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_4)\theta(2^{-q}k)^2 \\
&\frac{t^\eta}{|k_2|^2|k_3|^2|k_1|^{4-\eta}|k_4|^{4-\eta}} \\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{q\lesssim i} 2^{-(1-\eta-\epsilon)i} \sum_{q\lesssim i'} 2^{-(1-\eta-\epsilon)i'} \sum_{k_{23}=k} \theta(2^{-q}k)^2 \frac{t^\eta}{|k_2|^2|k_3|^2} \lesssim t^\eta 2^{2q(\eta+2\epsilon)},
\end{aligned}$$

where $\eta, \epsilon > 0$ are small enough, we used $\sup_{a\in\mathbb{R}} |a|^r \exp(-a^2) \leq C$ for $r \geq 0$ in the second inequality and Lemma 3.10 in the last inequality. Furthermore, $q \lesssim i$ follows from $|k| \leq |k_{123}| + |k_1| \lesssim 2^i$ and similarly one gets $q \lesssim i'$. Also for I_t^3 we have a similar estimate.

Now we deal with $I_t^4 = I_t^4 - \tilde{I}_t^4 + \tilde{I}_t^4 - \sum_{i_1=1}^3 u_2^{i_1}(t)C_3^{\varepsilon,i_1}(t)$ where

$$\tilde{I}_t^4 =(2\pi)^{-\frac{9}{2}}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq 1}\sum_{k_{12}=k,k_3}\sum_{i_1,i_2,i_3,j_1=1}^3\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\int_0^t :\hat{X}_\sigma^{\varepsilon,i_2}(k_1)\hat{X}_\sigma^{\varepsilon,i_3}(k_2): e^{-|k_{12}|^2(t-\sigma)}\imath k_{12}^{i_3}$$
$$\hat{P}^{i_1i_2}(k_{12})e_kd\sigma\int_0^t dse^{-|k_{123}|^2(t-s)}\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}\sum_{i_4}\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)\imath k_{123}^{j_1}\hat{P}^{i_0i_1}(k_{123}),$$

and

$$C_3^{\varepsilon,i_1}(t) =(2\pi)^{-\frac{9}{2}}\sum_{|i-j|\leq 1}\sum_{k_3}\sum_{j_1=1}^3\theta(2^{-i}k_3)\theta(2^{-j}k_3)\int_0^t ds\frac{e^{-2|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}$$
$$\sum_{i_4}\hat{P}^{j_1i_4}(k_3)\hat{P}^{j_0i_4}(k_3)\imath k_3^{j_1}\hat{P}^{i_0i_1}(k_3)=0.$$

Let $c_{k_{123},k_3}^{j_1}(t-s)=\sum_{i_1=1}^3 e^{-|k_{123}|^2(t-s)}\frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2}|k_{123}^{j_1}\hat{P}^{i_0i_1}(k_{123})|$. Then we have for $\epsilon>0$ small enough,

$$\begin{aligned}
&E|\Delta_q(I_t^4-\tilde{I}_t^4)|^2\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)\\
&\int_0^t ds\int_0^t d\bar{s}\frac{1}{|k_1|^2|k_2|^2}\sum_{j_1,j_1'=1}^3 c_{k_{123},k_3}^{j_1}(t-s)c_{k_{124},k_4}^{j_1'}(t-\bar{s})\Big[\int_0^s d\sigma\int_0^{\bar{s}}d\bar{\sigma}(e^{-|k_{12}|^2(s-\sigma)}-e^{-|k_{12}|^2(t-\sigma)})\\
&(e^{-|k_{12}|^2(\bar{s}-\bar{\sigma})}-e^{-|k_{12}|^2(t-\bar{\sigma})})|k_{12}|^2+\int_s^t d\sigma\int_{\bar{s}}^t d\bar{\sigma}e^{-|k_{12}|^2(t-\sigma)-|k_{12}|^2(t-\bar{\sigma})}|k_{12}|^2\Big]\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)\\
&\int_0^t ds\int_0^t d\bar{s}\frac{1}{|k_{12}||k_1|^2|k_2|^2}(t-s)^{1/4}(t-\bar{s})^{1/4}\sum_{j_1,j_1'=1}^3 c_{k_{123},k_3}^{j_1}(t-s)c_{k_{124},k_4}^{j_1'}(t-\bar{s})\\
\lesssim&\sum_{k\in\mathbb{Z}^3\backslash\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{12}=k,k_3,k_4}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_3)\theta(2^{-i'}k_{124})\theta(2^{-j'}k_4)\\
&\frac{t^{2\epsilon}}{|k_{12}||k_1|^2|k_2|^2|k_3|^2|k_4|^2(|k_{123}|^2+|k_3|^2)^{3/4-\epsilon}(|k_{124}|^2+|k_4|^2)^{3/4-\epsilon}}\\
\lesssim&t^{2\epsilon}\sum_{q\lesssim i}\sum_{q\lesssim i'}2^{-(i+i')(1/2-3\epsilon)}\sum_k\sum_{k_{12}=k}\theta(2^{-q}k)\frac{1}{|k_{12}||k_1|^2|k_2|^2}\\
\lesssim&t^{2\epsilon}2^{-2q(1/2-3\epsilon)}\sum_k\sum_{k_{12}=k}\theta(2^{-q}k)\frac{1}{|k_{12}||k_1|^2|k_2|^2}\lesssim t^{2\epsilon}2^{2q(3\epsilon)},
\end{aligned}$$

where in the last inequality we used Lemma 3.10 and $q\lesssim i$ follows $|k|\leq|k_{123}|+|k_3|\lesssim 2^i$ and similarly one gets $q\lesssim i'$. Moreover, by a similar argument as in the proof of Lemma 3.11 we obtain that for $\eta > \epsilon > 0$ small enough

$$
\begin{aligned}
&E[|\Delta_q(\tilde{I}^4_t - \sum_{i_1=1}^3 u_2^{\varepsilon,i_1}(t)C_3^{\varepsilon,i_1}(t))|^2] \\
\lesssim&\sum_k \sum_{k_{12}=k} \frac{1}{|k_1|^2|k_2|^2|k_{12}|^2}\theta(2^{-q}k)^2\Big[\sum_{i_1,j_1=1}^3 \sum_{|i-j|\leq 1}\sum_{k_3}\theta(2^{-j}k_3)\int_0^t \frac{e^{-|k_3|^2(t-s)}f(\varepsilon k_3)^2}{|k_3|^2} \\
&(\theta(2^{-i}k_{123})e^{-|k_{123}|^2(t-s)}k_{123}^{j_1}\hat{P}^{i_0i_1}(k_{123}) - \theta(2^{-i}k_3)e^{-|k_3|^2(t-s)}k_3^{j_1}\hat{P}^{i_0i_1}(k_3))ds\Big]^2 \\
\lesssim&\sum_k \sum_{k_{12}=k} \frac{1}{|k_1|^2|k_2|^2|k_{12}|^{2-2\eta}}\theta(2^{-q}k)^2\Big[\sum_{j=0}^\infty \sum_{k_3}\theta(2^{-j}k_3)\int_0^t \frac{e^{-|k_3|^2(t-s)}}{|k_3|^2}(t-s)^{-(1-\eta)/2}ds\Big]^2 \\
\lesssim&t^{\eta-\epsilon}2^{q(2\eta)},
\end{aligned}
$$

where in the last inequality we used Lemma 3.10.

Now we consider $I^5_t = I^5_t - \tilde{I}^5_t + \tilde{I}^5_t - \bar{I}^5_t$, where

$$
\begin{aligned}
\tilde{I}^5_t =&(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3 \theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^t :\hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4): e^{-|k_1|^2(t-s)} \\
&\imath k_1^{j_1}\hat{P}^{i_0i_1}(k_1)e_k ds\int_0^s d\sigma e^{-|k_{12}|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}\imath k_{12}^{i_3}\hat{P}^{i_1i_2}(k_{12})\sum_{i_4=1}^3 \hat{P}^{i_3i_4}(k_2)\hat{P}^{j_1i_4}(k_2),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{I}^5_t =&(2\pi)^{-9/2}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1}\sum_{k_{14}=k,k_2}\sum_{i_1,i_2,i_3,j_1=1}^3 \theta(2^{-i}k_1)\theta(2^{-j}k_4)\int_0^t :\hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4): e^{-|k_1|^2(t-s)} \\
&\imath k_1^{j_1}\hat{P}^{i_0i_1}(k_1)e_k ds\int_0^s d\sigma e^{-|k_2|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}\imath k_2^{i_3}\hat{P}^{i_1i_2}(k_2)\sum_{i_4=1}^3 \hat{P}^{i_3i_4}(k_2)\hat{P}^{j_1i_4}(k_2) = 0.
\end{aligned}
$$

Let $d_{k_{12},k_2}(s-\sigma) = \sum_{i_2,i_3=1}^3 e^{-|k_{12}|^2(s-\sigma)}\frac{e^{-|k_2|^2(s-\sigma)}f(\varepsilon k_2)^2}{|k_2|^2}|k_{12}^{i_3}\hat{P}^{i_1i_2}(k_{12})|$. Since by Hölder's inequality we obtain

$$
\begin{aligned}
&E(:\hat{X}^{\varepsilon,i_2}_s(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4): - :\hat{X}^{\varepsilon,i_2}_\sigma(k_1)\hat{X}^{\varepsilon,j_0}_t(k_4):)\overline{(:\hat{X}^{\varepsilon,i_2}_{\bar{s}}(k_1')\hat{X}^{\varepsilon,j_0}_t(k_4'): - :\hat{X}^{\varepsilon,i_2}_{\bar{\sigma}}(k_1')\hat{X}^{\varepsilon,j_0}_t(k_4'):)} \\
\lesssim&(1_{k_1=k_1'}1_{k_4=k_4'} + 1_{k_1=k_4'}1_{k_4=k_1'})(\frac{1-e^{-|k_1|^2|s-\sigma|}}{|k_1|^2|k_4|^2})^{1/2}(\frac{1-e^{-|k_1'|^2|\bar{s}-\bar{\sigma}|}}{|k_1'|^2|k_4'|^2})^{1/2} \\
\lesssim&(1_{k_1=k_1'}1_{k_4=k_4'} + 1_{k_1=k_4'}1_{k_4=k_1'})\frac{|k_1|^\eta|k_1'|^\eta}{|k_1||k_1'||k_4||k_4'|}|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2},
\end{aligned}
$$

it follows that for $\eta, \epsilon > 0$ small enough

$$
E|\Delta_q(I^5_t - \tilde{I}^5_t)|^2 \lesssim \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)
$$

$$\int_0^t ds\int_0^t d\bar{s}\int_0^s d\sigma\int_0^{\bar{s}} d\bar{\sigma}e^{-|k_1|^2(t-s)}e^{-|k_1|^2(t-\bar{s})}|k_1|^2\frac{1}{|k_1|^{2-2\eta}|k_4|^2}$$
$$|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2}d_{k_{12},k_2}(s-\sigma)d_{k_{13},k_3}(\bar{s}-\bar{\sigma})$$
$$+\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_4)\theta(2^{-j'}k_1)$$
$$\int_0^t ds\int_0^t d\bar{s}\int_0^s d\sigma\int_0^{\bar{s}} d\bar{\sigma}e^{-|k_1|^2(t-s)}e^{-|k_4|^2(t-\bar{s})}|k_1||k_4|\frac{1}{|k_1|^{2-\eta}|k_4|^{2-\eta}}$$
$$|s-\sigma|^{\eta/2}|\bar{s}-\bar{\sigma}|^{\eta/2}d_{k_{12},k_2}(s-\sigma)d_{k_{34},k_3}(\bar{s}-\bar{\sigma})$$
$$\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)$$
$$(\frac{t^\epsilon}{|k_1|^{4-2\eta-2\epsilon}|k_4|^2}+\frac{t^\epsilon}{|k_1|^{3-\eta-\epsilon}|k_4|^{3-\eta-\epsilon}})$$
$$\lesssim t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim i}2^{-i}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^2}$$
$$+t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim j}2^{-j\epsilon}\frac{1}{|k_1|^{3-\eta-2\epsilon}|k_4|^{3-\eta-\epsilon}}$$
$$\lesssim t^\epsilon 2^{q(2\epsilon+2\eta)},$$

where in the last inequality we used Lemma 3.10 and $q\lesssim i$ follows from $|k|\leq|k_1|+|k_4|\lesssim 2^i$.

Moreover, it follows by Lemma 3.11 that for $\eta,\epsilon>0$ small enough

$$E[|\Delta_q(\tilde{I}^5_t-\bar{I}^5_t)|^2]\lesssim\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_1)\theta(2^{-j'}k_4)$$
$$\int_0^t\int_0^t|k_1|^{2+2\eta}e^{-|k_1|^2(t-s+t-\bar{s}+|s-\bar{s}|)}\frac{1}{|k_1|^2|k_4|^2}\int_0^s\frac{e^{-|k_2|^2(s-\sigma)}}{|k_2|^2}(s-\sigma)^{-(1-\eta)/2}$$
$$\int_0^{\bar{s}}\frac{e^{-|k_3|^2(\bar{s}-\bar{\sigma})}}{|k_3|^2}(\bar{s}-\bar{\sigma})^{-(1-\eta)/2}dsd\bar{s}d\sigma d\bar{\sigma}$$
$$+\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq1,|i'-j'|\leq1}\sum_{k_{14}=k,k_3,k_2}\theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_4)\theta(2^{-i'}k_4)\theta(2^{-j'}k_1)$$
$$\int_0^t\int_0^t|k_1|^{1+2\eta}|k_4|e^{-2|k_1|^2(t-s)-2|k_4|^2(t-\bar{s})}\frac{1}{|k_1|^2|k_4|^2}\int_0^s\frac{e^{-|k_2|^2(s-\sigma)}}{|k_2|^2}(s-\sigma)^{-(1-\eta)/2}$$
$$\int_0^{\bar{s}}\frac{e^{-|k_3|^2(\bar{s}-\bar{\sigma})}}{|k_3|^2}(\bar{s}-\bar{\sigma})^{-(1-\eta)/2}dsd\bar{s}d\sigma d\bar{\sigma}$$
$$\lesssim t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim i}2^{-i}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^2}$$
$$+t^\epsilon\sum_k\sum_{k_{14}=k}\theta(2^{-q}k)\sum_{q\lesssim j}2^{-j\epsilon}\frac{1}{|k_1|^{3-2\eta-2\epsilon}|k_4|^{3-\epsilon}}$$
$$\lesssim t^\epsilon 2^{q(2\epsilon+2\eta)},$$

where in the last inequality we used Lemma 3.10 and $q\lesssim i$ follows from $|k|\leq|k_1|+|k_4|\lesssim 2^i$.

Similar estimates can also be obtained for I_t^6.

Terms in the fourth chaos: Now for I_t^1 we have the following calculations:

$$\begin{aligned}
&E[|\Delta_q I_t^1|^2]\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{1234}=k,k'_{1234}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k'_{123})\theta(2^{-j'}k'_4)\\
&(1_{k_1=k'_1,k_2=k'_2,k_3=k'_3,k_4=k'_4}+1_{k_1=k'_4,k_2=k'_2,k_3=k'_3,k_4=k'_1}+1_{k_1=k'_1,k_2=k'_2,k_3=k'_4,k_4=k'_3}+1_{k_1=k'_3,k_2=k'_4,k_3=k'_1,k_4=k'_2}\\
&+1_{k_1=k'_1,k_2=k'_3,k_3=k'_2,k_4=k'_4}+1_{k_1=k'_3,k_2=k'_2,k_3=k'_4,k_4=k'_1}+1_{k_1=k'_4,k_2=k'_2,k_3=k'_1,k_4=k'_3})\\
&\int_0^t ds\int_0^t d\bar{s}e^{-|k_{123}|^2(t-s)-|k'_{123}|^2(t-\bar{s})}\int_0^s\int_0^{\bar{s}}\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}e^{-|k_{12}|^2(s-\sigma)-|k'_{12}|^2(\bar{s}-\bar{\sigma})}d\sigma d\bar{\sigma}|k_{12}k_{123}k'_{12}k'_{123}|\\
=&E_t^1+E_t^2+E_t^3+E_t^4+E_t^5+E_t^6+E_t^7.
\end{aligned}$$

Here each E_t^i corresponds to the term associated with each indicator function.

For $\epsilon,\eta>0$ small enough by Lemma 3.10 we have

$$\begin{aligned}
E_t^1\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^2|k_{123}|^{2-2\eta}}\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{1234}=k}\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{123})\theta(2^{-j'}k_4)\frac{t^\eta}{|k_4|^2|k_{123}|^{4-2\eta-\epsilon}}\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{q\lesssim i}2^{-(2-2\eta-\epsilon)i}\theta(2^{-q}k)^2\frac{t^\eta}{|k|}\lesssim 2^{q(2\eta+\epsilon)}t^\eta,
\end{aligned}$$

and

$$\begin{aligned}
E_t^2\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{|i-j|\leq 1,|i'-j'|\leq 1}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2\theta(2^{-i}k_{123})\theta(2^{-j}k_4)\theta(2^{-i'}k_{234})\theta(2^{-j'}k_1)t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}||k_{24}||k_{123}|^{1-\eta}|k_{234}|^{1-\eta}}\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta 2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{12}||k_{24}||k_{123}|^{1-\eta}|k_{234}|^{1-\eta}}\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\Big(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta 2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{12}|^2|k_{123}|^{2-2\eta}}\Big)^{1/2}\\
&\Big(\sum_{k_{1234}=k}\frac{\theta(2^{-q}k)^2t^\eta 2^{-q(2-2\eta)}}{|k_1|^{1+\eta}|k_2|^2|k_3|^2|k_4|^{1+\eta}|k_{24}|^2|k_{234}|^{2-2\eta}}\Big)^{1/2}\\
\lesssim &\sum_{k\in\mathbb{Z}^3\setminus\{0\}}2^{-(2-2\eta)q}\frac{t^\eta}{|k|}\lesssim 2^{q(2\eta)}t^\eta.
\end{aligned}$$

By a similar argument we can also obtain the same bounds for E_t^3, E_t^4, E_t^5, E_t^6 and E_t^7, which implies that for $\epsilon,\eta>0$ small enough

$$E[|\Delta_q I_t^1|^2]\lesssim 2^{q(2\eta+\epsilon)}t^\eta.$$

By a similar calculation as above we get that for $\eta, \epsilon, \gamma > 0$ small enough

$$
\begin{aligned}
&E[|\Delta_q(\pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_1) - \pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_2) - \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_1) \\
&+ \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_2))|^2] \\
\lesssim&(\varepsilon_1^{2\gamma} + \varepsilon_2^{2\gamma})|t_1 - t_2|^{\eta} 2^{q(\epsilon+2\eta)},
\end{aligned}
$$

which by Gaussian hypercontractivity and Lemma 3.1 implies that

$$
\begin{aligned}
&E[\|\pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_1) - \pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_2) - \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_1) \\
&+ \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_2)\|^p_{\mathcal{C}^{-\eta-\epsilon-3/p}}] \\
\lesssim&E[\|\pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_1) - \pi_{0,\diamond}(u_3^{\varepsilon_1,i_0}, u_1^{\varepsilon_1,j_0})(t_2) - \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_1) \qquad (3.14)\\
&+ \pi_{0,\diamond}(u_3^{\varepsilon_2,i_0}, u_1^{\varepsilon_2,j_0})(t_2)\|^p_{B^{-\eta-\epsilon}_{p,p}}] \\
\lesssim&(\varepsilon_1^{p\gamma} + \varepsilon_2^{p\gamma})|t_1 - t_2|^{p(\eta-\epsilon)/2},
\end{aligned}
$$

(see the proof of (4.2), (4.3)). Thus, for every $i_0, j_0 = 1, 2, 3$ we choose p large enough and deduce that there exist $v_5^{i_0j_0} \in C([0,T], \mathcal{C}^{-\delta}), i_0, j_0 = 1, 2, 3$, such that for $p > 1$

$$\pi_{0,\diamond}(u_3^{\varepsilon,i_0}, u_1^{\varepsilon,j_0}) \to v_5^{i_0j_0} \text{ in } L^p(\Omega, P, C([0,T], \mathcal{C}^{-\delta})).$$

Here $\delta > 0$ depending on η, ϵ, p can be chosen small enough.

3.3.2 Renormalisation for $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0}, u_1^{\varepsilon,j_1})$ and $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2}, u_1^{\varepsilon,j_1})$

In this subsection we consider $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0}, u_1^{\varepsilon,j_1})$ and $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2}, u_1^{\varepsilon,j_1})$ for $i_1, i_2, j_0, j_1 = 1, 2, 3$ and have the following identity:

$$
\begin{aligned}
&\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0}, u_1^{\varepsilon,j_1})(t) \\
=&(2\pi)^{-\frac{3}{2}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1} \sum_{k_{12}=k} \theta(2^{-i}k_1)\theta(2^{-j}k_2) \int_0^t e^{-(t-s)|k_1|^2} \imath k_1^{j_0} : \hat{X}_s^{\varepsilon,j_0}(k_1)\hat{X}_t^{\varepsilon,j_1}(k_2) : dse_k \hat{P}^{i_1i_2}(k_1) \\
&+ (2\pi)^{-3} \sum_{|i-j|\leq 1} \sum_{k_1} \theta(2^{-i}k_1)\theta(2^{-j}k_1) \int_0^t e^{-2(t-s)|k_1|^2} \imath k_1^{j_0} \frac{f(\varepsilon k_1)^2}{2|k_1|^2} ds \hat{P}^{i_1i_2}(k_1) \sum_{j_2=1}^3 \hat{P}^{j_0j_2}(k_1)\hat{P}^{j_1j_2}(k_1).
\end{aligned}
$$

Here $\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0}, u_1^{\varepsilon,j_1})$ corresponds to  and the first term and the second term on the right hand side of the above equality correspond to the associated $\hat{\mathcal{W}}^{(\varepsilon,2)}, \hat{\mathcal{W}}^{(\varepsilon,0)}$ in the proof of Theorem 2.16 respectively. It is easy to get that the second term on the right hand side of the above equality equals zero. It is straightforward to calculate for $\epsilon > 0$ small enough:

$$
\begin{aligned}
&E|\Delta_q\pi_0(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0}, u_1^{\varepsilon,j_1})|^2 \\
\lesssim& \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{|i-j|\leq 1, |i'-j'|\leq 1} \sum_{k_{12}=k} \theta(2^{-q}k)^2\theta(2^{-i}k_1)\theta(2^{-j}k_2)\theta(2^{-i'}k_1)\theta(2^{-j'}k_2) \\
&\Big[\int_0^t\int_0^t e^{-(t-s+t-\bar{s})|k_1|^2}|k_1|^2 \frac{e^{-|k_1|^2|s-\bar{s}|}}{|k_1|^2|k_2|^2} ds d\bar{s} \\
&+ \int_0^t\int_0^t e^{-2(t-s)|k_1|^2-2(t-\bar{s})|k_2|^2}|k_1||k_2| \frac{1}{|k_1|^2|k_2|^2} ds d\bar{s}\Big]
\end{aligned}
$$

$$\lesssim t^{\epsilon}\sum_{k}\sum_{q\lesssim i}\sum_{k_{12}=k}\theta(2^{-q}k)\theta(2^{-i}k_1)\frac{1}{|k_1|^{4-2\epsilon}|k_2|^2}$$
$$+t^{\epsilon}\sum_{k}\sum_{q\lesssim i}\sum_{k_{12}=k}\theta(2^{-q}k)\theta(2^{-j}k_2)\frac{1}{|k_1|^{3-2\epsilon}|k_2|^3}$$
$$\lesssim t^{\epsilon}2^{2q\epsilon},$$

where in the last inequality we used Lemma 3.10. By a similar calculation we also get that for $\epsilon,\eta>0$, $\gamma>0$ small enough

$$E[|\Delta_q(\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})(t_1)-\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})(t_2)$$
$$-\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})(t_1)+\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})(t_2))|^2]$$
$$\lesssim(\varepsilon_1^{2\gamma}+\varepsilon_2^{2\gamma})|t_1-t_2|^{\eta}2^{q(\epsilon+2\eta)},$$

which by Gaussian hypercontractivity, Lemma 3.1 and similar arguments as for (3.14) implies that there exists $v_6^{i_1i_2j_0j_1}\in C([0,T];\mathcal{C}^{-\delta})$ for $i_1,i_2,j_0,j_1=1,2,3$ such that for $p>1$

$$\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,j_0},u_1^{\varepsilon,j_1})\to v_6^{i_1i_2j_0j_1}\text{ in }L^p(\Omega,P,C([0,T];\mathcal{C}^{-\delta})).$$

Here $\delta>0$ depending on η,ϵ,p can be chosen small enough. By a similar argument we also obtain that there exists $v_7^{i_1i_2j_0j_1}\in C([0,T];\mathcal{C}^{-\delta})$ for $i_1,i_2,j_0,j_1=1,2,3$ such that

$$\pi_{0,\diamond}(P^{i_1i_2}D_{j_0}K^{\varepsilon,i_2},u_1^{\varepsilon,j_1})\to v_7^{i_1i_2j_0j_1}\text{ in }L^p(\Omega,P,C([0,T];\mathcal{C}^{-\delta})).$$

Combining all the convergence results we obtained above and Theorem 3.8 we obtain local existence and uniqueness of the solutions to the 3D Navier-Stokes equation driven by space-time white noise.

Theorem 3.12 Let $z\in(1/2,1/2+\delta_0)$ with $0<\delta_0<1/2$ and $u_0\in\mathcal{C}^{-z}$. Then there exists a unique local solution to

$$Lu^i=\sum_{i_1=1}^3P^{ii_1}\xi-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u^{i_1}u^j))\quad u(0)=Pu_0,$$

in the following sense: For $\xi^\varepsilon=\sum_k f(\varepsilon k)\hat{\xi}(k)e_k$ with f a smooth radial function with compact support satisfying $f(0)=1$ and for $\varepsilon>0$ consider the maximal unique solution u^ε to the following equation, such that u_4^ε defined above belongs to $C((0,T^\varepsilon);\mathcal{C}^{1/2-\delta_0})$,

$$Lu^{\varepsilon,i}=\sum_{i_1=1}^3P^{ii_1}\xi^\varepsilon-\frac{1}{2}\sum_{i_1=1}^3P^{ii_1}(\sum_{j=1}^3D_j(u^{\varepsilon,i_1}u^{\varepsilon,j})),\quad u^\varepsilon(0)=Pu_0.$$

Then there exists $u\in C([0,\tau);\mathcal{C}^{-z})$ and a sequence of random time τ_L converging to the explosion time τ of u such that

$$\sup_{t\in[0,\tau_L]}\|u^\varepsilon-u\|_{-z}\to^P 0,.$$

Proof By a similar argument as above we have that there exists some $\gamma > 0$ and $u_1 \in C([0,T];\mathcal{C}^{-1/2-\delta/2})$, $u_2 \in C([0,T];\mathcal{C}^{-\delta})$, $u_3 \in C([0,T];\mathcal{C}^{\frac{1}{2}-\delta})$ such that for every $p > 0$

$$E\|u_1^\varepsilon - u_1\|^p_{C([0,T];\mathcal{C}^{-1/2-\delta/2})} \lesssim \varepsilon^{\gamma p},$$

$$E\|u_2^\varepsilon - u_2\|^p_{C([0,T];\mathcal{C}^{-\delta})} \lesssim \varepsilon^{\gamma p}.$$

$$E\|u_3^\varepsilon - u_3\|^p_{C([0,T];\mathcal{C}^{1/2-\delta})} \lesssim \varepsilon^{\gamma p}.$$

Then for $\varepsilon_k = 2^{-k} \to 0$ and $\epsilon > 0$

$$\sum_{k=1}^\infty P(\|u_1^{\varepsilon_k} - u_1\|_{C([0,T];\mathcal{C}^{-1/2-\delta/2})} > \epsilon) \lesssim \sum_{k=1}^\infty 2^{-k\gamma}/\epsilon < \infty,$$

which by the Borel-Cantelli Lemma implies that $u_1^{\varepsilon_k,i} - u_1^i \to 0$ in $C([0,T];\mathcal{C}^{-1/2-\delta/2})$ a.s., as $k \to \infty$. The results for the other terms are similar. Thus we obtain that $\sup_{\varepsilon_k=2^{-k},k\in\mathbb{N}} \bar{C}_\xi^{\varepsilon_k} < \infty$ a.s., T_0 independent of ε, $u_4 := \lim_{k\to\infty} u_4^{\varepsilon_k}$ on $[0,T_0]$, $u = u_1 + u_2 + u_3 + u_4$ as the solution to (3.1) on $[0,T_0]$ and

$$\sup_{t\in[0,T_0]} \|u^{\varepsilon_k} - u\|_{-z} \to 0 \quad a.s..$$

Now we can extend the solution to the maximal solution such that

$$\sup_{t\in[0,\tau)} \|u\|_{-z} = \infty.$$

Indeed, a similar argument as in the proof in Section 3.2 implies that there exists some $T_1(C(T_0))$ (for simplicity we assume $T_1 \le T_0$) such that for every $t^* \in [0,T_0]$

$$\sup_{t\in[t^*,t^*+T_1]} \left[(t-t^*)^{\delta+z+\kappa}\|\bar{u}^{\varepsilon,\sharp}\|_{1/2+\beta} + (t-t^*)^{\frac{\delta+z+\kappa}{2}}\|\bar{u}^{\varepsilon,\sharp}(t)\|_\delta\right] \lesssim C(T_1, C_\xi^\varepsilon, C(T_0), \|u(t^*)\|_{-z}),$$

where $\bar{u}^\varepsilon$ denotes the solution starting at t^* with initial condition $\bar{u}^\varepsilon(t^*) = u(t^*)$ and we can also define $\bar{u}^{\varepsilon,\sharp}$. Here the only difference is that $\bar{K}^{\varepsilon,i}$ satisfies the following equation

$$d\bar{K}^{\varepsilon,i} = (\Delta \bar{K}^{\varepsilon,i} + u_1^{\varepsilon,i})dt, \quad \bar{K}^{\varepsilon,i}(t^*) = 0,$$

and by a similar argument as above we obtain that there exists some $\gamma > 0$ such that for every $p > 1$

$$E\sup_{r\in[0,T]} \|\pi_0(PD\int_r^\cdot P_{\cdot-s}u_1^\varepsilon ds, u_1^\varepsilon(\cdot)) - \pi_0(PD\int_r^\cdot P_{\cdot-s}u_1 ds, u_1(\cdot))\|^p_{C([0,T];\mathcal{C}^{-\delta})} \lesssim \varepsilon^{p\gamma},$$

which implies that a similar convergence also holds for $\pi_0(PD\bar{K}^\varepsilon, u_1^\varepsilon)$ in this case. Here we omit superscripts for simplicity.

Therefore for $t^* = T_0 - \frac{T_1(C(T_0))}{2}$ we obtain the following estimate

$$\begin{aligned}
&\sup_{t\in[T_0,T_0+\frac{T_1}{2}]} (t^{\delta+z+\kappa}\|\bar{u}^{\varepsilon,\sharp}\|_{1/2+\beta} + t^{\frac{\delta+z+\kappa}{2}}\|\bar{u}^{\varepsilon,\sharp}(t)\|_\delta) \\
\lesssim &\sup_{t\in[T_0,T_0+\frac{T_1}{2}]} ((t-t^*)^{\delta+z+\kappa}\|\bar{u}^{\varepsilon,\sharp}(t)\|_{1/2+\beta} + (t-t^*)^{\frac{\delta+z+\kappa}{2}}\|\bar{u}^{\varepsilon,\sharp}(t)\|_\delta) \\
\lesssim &C(T_1, C_\xi^\varepsilon, C(T_0), \|u_0\|_{-z}).
\end{aligned}$$

Hence by a similar argument as above we obtain the solution $u = \lim_{k\to\infty} \bar{u}^{\varepsilon_k}$ on $[T_0, T_0 + \frac{T_1}{2}]$. Iterating the above arguments we get that there exist the explosion time $\tau > 0$ and the maximal solution u on $[0, \tau)$ such that

$$\sup_{t\in[0,\tau)} \|u(t)\|_{-z} = \infty.$$

In the following we prove u^ε converges to u before some random time. For $L \geq 0$ define $\tau_L := \inf\{t : \|u(t)\|_{-z} \geq L\} \wedge L$. Then τ_L increases to τ. Also define $\tau_L^\varepsilon := \inf\{t : \|u^\varepsilon(t)\|_{-z} \geq L\} \wedge L$ and $\rho_L^\varepsilon := \inf\{t : C_\xi^\varepsilon(t) \geq L\}$. Then by the proof in Section 3.2 we obtain for any $L, L_1, L_2 > 0$,

$$\sup_{t\in[0,\rho^\varepsilon_{L_1}\wedge\tau_L\wedge\tau^\varepsilon_{L_2}]} \|u^\varepsilon - u\|_{-z} \to 0 \quad a.s..$$

Now we have for any $\epsilon > 0$

$$P(\sup_{t\in[0,\tau_L]} \|u^\varepsilon - u\|_{-z} > \epsilon) \leq P(\sup_{t\in[0,\tau_L\wedge\rho^\varepsilon_{L_1}\wedge\tau^\varepsilon_{L_2}]} \|u^\varepsilon - u\|_{-z} > \epsilon) + P(\tau_L > \rho^\varepsilon_{L_1}) + P(\tau_L \wedge \rho^\varepsilon_{L_1} > \tau^\varepsilon_{L_2}).$$

Here the first term goes to zero by the above result, the second term goes to zero as L_1 goes to infinity and for $L_2 > L + \epsilon$

$$P(\tau_L \wedge \rho^\varepsilon_{L_1} > \tau^\varepsilon_{L_2}) \leq P(\sup_{t\in[0,\tau_L\wedge\rho^\varepsilon_{L_1}\wedge\tau^\varepsilon_{L_2}]} \|u^\varepsilon - u\|_{-z} > \epsilon),$$

which goes to zero as $\varepsilon \to 0$ by the above result. Thus the result follows. $\square$

Remark 3.13 We used two different approaches and obtained the same results in Theorem 2.18 and Theorem 3.12. As we mentioned in the introduction from a philosophical perspective, the theory of regularity structures and the paracontrolled distribution are inspired by the theory of controlled rough paths [20], [11]. The main difficulty for this problem lies in how to define multiplication for the unknowns. In the regularity structure theory we used an extension of the Taylor expansion and split the unknown into elements of different orders of homogeneity (i.e. regularity structure). Then it suffices to define the multiplications for these elements of different orders of homogeneity. In the paracontrolled distribution method using Bony's paraproduct we split the unknown into good terms and bad terms ($\pi_<(\cdot,\cdot)$), where the singularity of the bad term is the same as the singularity of some functional of the Gaussian field. Then by using the commutator estimate it suffices to define the multiplication of some functionals of the Gaussian field.

From the proof we see that the terms required to be renormalized in the two methods are similar: The terms not including the terms with $|\cdot|_\mathfrak{s} > 0$ in the theory of the regularity sturctures are the same as the associated terms in the paracontrolled distribution, while the terms including the terms with $|\cdot|_\mathfrak{s} > 0$ (like $\mathcal{I}_l(\mathcal{I}_k(\mathcal{I}(\Xi)\mathcal{I}(\Xi))\mathcal{I}(\Xi))\mathcal{I}(\Xi)$ and $\mathcal{I}_k(\mathcal{I}(\Xi))\mathcal{I}(\Xi)$) are different from the terms in the paracontrolled distributions ($\pi_0(u_3, u_1)$ and $\pi_0(PDK, u_1)$). In the theory of regularity structures a distribution is divided into the elements of different orders of homogeneity. For example, the terms of good regularity (e.g. u_3) are split into constants, polynomials and some other terms with positive order (e.g. $\mathcal{I}_l(\mathcal{I}_k(\mathcal{I}(\Xi)\mathcal{I}(\Xi))\mathcal{I}(\Xi))$). In the paracontrolled distribution method using Bony's paraproduct for these terms it is sufficient to define $\pi_0(\cdot,\cdot)$, which plays a similar role as the term of positive order in the regularity structure theory.

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4 Appendix

4.1 Renormalisation for $u_1^\varepsilon u_2^\varepsilon$

In this subsection we focus on $u_1^\varepsilon u_2^\varepsilon$ and prove that $u_1^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_3^{ij}$ in $C([0,T];\mathcal{C}^{-1/2-\delta})$ for $i,j = 1,2,3$. Now we have the following identity: for $t \in [0,T]$, $i,j = 1,2,3$

$$
\begin{aligned}
u_1^{\varepsilon,j} u_2^{\varepsilon,i}(t) =& \frac{(2\pi)^{-3}}{2} \sum_{i_1,i_2=1}^{3} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{k_{123}=k} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} : \hat{X}_s^{\varepsilon,i_1}(k_1)\hat{X}_s^{\varepsilon,i_2}(k_2)\hat{X}_t^{\varepsilon,j}(k_3) : ds \hat{P}^{ii_1}(k_{12}) e_k \\
&+ \frac{(2\pi)^{-3}}{2} \sum_{i_1,i_2,i_3=1}^{3} \sum_{k_1,k_2\in\mathbb{Z}^3\setminus\{0\}} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} \hat{X}_s^{\varepsilon,i_1}(k_1) \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{2|k_2|^2} ds \\
&\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2) e_{k_1} \\
&+ \frac{(2\pi)^{-3}}{2} \sum_{i_1,i_2,i_3=1}^{3} \sum_{k_1,k_2\in\mathbb{Z}^3\setminus\{0\}} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} \hat{X}_s^{\varepsilon,i_2}(k_2) \frac{e^{-|k_1|^2(t-s)} f(\varepsilon k_1)^2}{2|k_1|^2} ds \\
&\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_1)\hat{P}^{ji_3}(k_2) e_{k_2} \\
=& I_t^1 + I_t^2 + I_t^3.
\end{aligned}
$$

To make it more readable we write each term corresponding to the tree notation in Section 2: $u_1^{\varepsilon,j} u_2^{\varepsilon,i}$ corresponds to  and I_t^1, I_t^2, I_t^3 correspond to the associated $\hat{\mathcal{W}}^{(\varepsilon,3)}, \hat{\mathcal{W}}_2^{(\varepsilon,1)}, \hat{\mathcal{W}}_1^{(\varepsilon,1)}$ in the proof of Theorem 2.16 respectively.

Term in the first chaos: First, we consider I_t^2. We have

$$
I_t^2 = I_t^2 - \tilde{I}_t^2 + \tilde{I}_t^2 - \sum_{i_1=1}^{3} X_t^{\varepsilon,i_1} C_t^{\varepsilon,i_1},
$$

where

$$
\begin{aligned}
\tilde{I}_t^2 =& \frac{(2\pi)^{-3}}{2} \sum_{i_1,i_2,i_3=1}^{3} \sum_{k_1,k_2\in\mathbb{Z}^3\setminus\{0\}} \hat{X}_t^{\varepsilon,i_1}(k_1) e_{k_1} \int_0^t e^{-|k_{12}|^2(t-s)} \imath k_{12}^{i_2} \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{2|k_2|^2} ds \\
&\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2),
\end{aligned}
$$

and

$$
C_t^{\varepsilon,i_1} = \frac{(2\pi)^{-3}}{2} \sum_{i_2,i_3=1}^{3} \sum_{k_2\in\mathbb{Z}^3\setminus\{0\}} \int_0^t e^{-2|k_2|^2(t-s)} \imath k_2^{i_2} \frac{f(\varepsilon k_2)^2}{2|k_2|^2} \hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2) ds = 0.
$$

A straightforward calculation yields that for $\eta > 0$ small enough

$$\begin{aligned}E[|\Delta_q(I_t^2-\tilde{I}_t^2)|^2] \lesssim & E\Big[\Big|\sum_{i_1,i_2,i_3=1}^3\int_0^t\sum_{k_1}\theta(2^{-q}k_1)e_{k_1}a_{k_1}^{i_1i_2i_3}(t-s)(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))ds\Big|^2\Big]\\ \lesssim & \sum_{i_1,i_2,i_3=1}^3\sum_{i_1',i_2',i_3'=1}^3\int_0^t\int_0^t dsd\bar{s}\sum_{k_1,k_1'}\theta(2^{-q}k_1)\theta(2^{-q}k_1')|a_{k_1}^{i_1i_2i_3}(t-s)a_{k_1'}^{i_1'i_2'i_3'}(t-\bar{s})|\\ & E\big|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}\big|\\ \lesssim & \sum_{k_1}\theta(2^{-q}k_1)^2\frac{f(\varepsilon k_1)^2}{|k_1|^{2(1-\eta)}}\Big(\int_0^t|t-s|^{\eta/2}|a_{k_1}^{i_1i_2i_3}(t-s)|ds\Big)^2.\end{aligned}$$

Here

$$a_{k_1}^{i_1i_2i_3}(t-s)=\sum_{k_2}e^{-|k_{12}|^2(t-s)}k_{12}^{i_2}\frac{e^{-|k_2|^2(t-s)}f(\varepsilon k_2)^2}{|k_2|^2}\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_{12})\hat{P}^{ji_3}(k_{12}),$$

and in the third inequality we used that for $\eta > 0$ small enough

$$\begin{aligned}& E|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}|\\ \leq & 1_{k_1=k_1'}(E|(\hat{X}_s^{\varepsilon,i_1}(k_1)-\hat{X}_t^{\varepsilon,i_1}(k_1))|^2)^{1/2}(E|\overline{(\hat{X}_{\bar{s}}^{\varepsilon,i_1'}(k_1')-\hat{X}_t^{\varepsilon,i_1'}(k_1'))}|^2)^{1/2}\\ \lesssim & 1_{k_1=k_1'}(\frac{f(\varepsilon k_1)^2}{|k_1|^2}(1-e^{-|k_1|^2(t-s)}))^{1/2}(\frac{f(\varepsilon k_1')^2}{|k_1'|^2}(1-e^{-|k_1'|^2(t-\bar{s})}))^{1/2}\\ \lesssim & \frac{f(\varepsilon k_1)^2}{|k_1|^2}|k_1|^{2\eta}|t-s|^{\eta/2}|t-\bar{s}|^{\eta/2}.\end{aligned}$$

Since $\sup_{a\in\mathbb{R}}|a|^r\exp(-a^2)\leq C$ for $r\geq 0$ implies that for $\eta>\epsilon>0$, ϵ small enough $|a_{k_1}^{i_1i_2i_3}(t-s)|\lesssim|t-s|^{-1-\epsilon/2}\sum_{k_2}\frac{1}{|k_2|^{3+\epsilon}}$, it follows that

$$\int_0^t|t-s|^{\eta/2}|a_{k_1}^{i_1i_2i_3}(t-s)|ds\lesssim\int_0^t|t-s|^{\eta/2-1-\epsilon/2}ds\sum_{k_2}\frac{1}{|k_2|^{3+\epsilon}}\lesssim t^{(\eta-\epsilon)/2},$$

which implies that

$$E[|\Delta_q(I_t^2-\tilde{I}_t^2)|^2]\lesssim 2^{q(1+2\eta)}t^{\eta-\epsilon}.$$

Moreover, by Lemma 3.11 we deduce that for $\epsilon>0$ small enough

$$\begin{aligned}& E[|\Delta_q(\tilde{I}_t^2-\sum_{i_1=1}^3X_t^{\varepsilon,i_1}C_t^{\varepsilon,i_1})|^2]\\ \lesssim & \sum_{k_1}\frac{f(\varepsilon k_1)^2}{2|k_1|^2}\theta(2^{-q}k_1)^2\Big[\sum_{i_1,i_2,i_3=1}^3\sum_{k_2}\int_0^t\frac{e^{-|k_2|^2(t-s)}f(\varepsilon k_2)^2}{|k_2|^2}\\ & \big(e^{-|k_{12}|^2(t-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)-e^{-|k_2|^2(t-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)\big)ds\Big]^2\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{k_1} \frac{f(\varepsilon k_1)^2}{|k_1|^{2-2\eta}} \theta(2^{-q}k_1)^2 \Big(\sum_{k_2} \int_0^t \frac{e^{-|k_2|^2(t-s)} f(\varepsilon k_2)^2}{|k_2|^2} (t-s)^{-(1-\eta)/2} ds \Big)^2 \\ &\lesssim t^{\eta-\epsilon} 2^{q(1+2\eta)},\end{aligned} \tag{4.1}$$

holds uniformly over $\varepsilon \in (0,1)$, which is the desired bound for I_t^2. Here in the third inequality we also used $\sup_{a\in\mathbb{R}} |a|^r \exp(-a^2) \leq C$ for $r \geq 0$.

Similarly, we obtain that

$$E[|\Delta_q I_t^3|^2] \lesssim t^{\eta-\epsilon} 2^{q(1+2\eta)}.$$

Term in the third chaos: Now we focus on the bounds for I_t^1. Let $b_{k_{12}}^{i_1,i_2}(t-s) = e^{-|k_{12}|^2(t-s)} k_{12}^{i_2} \hat{P}^{ii_1}(k_{12})$. We obtain the following inequalities:

$$\begin{aligned}&E|\Delta_q I_t^1|^2 \\ \lesssim & 2 \sum_{i_1,i_2=1}^3 \sum_{i_1',i_2'=1}^3 \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \int_0^t \int_0^t e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|} |b_{k_{12}}^{i_1,i_2}(t-s) b_{k_{12}}^{i_1',i_2'}(t-\bar{s})| ds d\bar{s} \\ &+ 2 \sum_{i_1,i_2=1}^3 \sum_{i_1',i_2'=1}^3 \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{f(\varepsilon k_i)^2}{|k_i|^2} \int_0^t \int_0^t e^{-|k_2|^2|s-\bar{s}|-|k_1|^2(t-s)-|k_3|^2(t-\bar{s})} \\ &|b_{k_{12}}^{i_1,i_2}(t-s) b_{k_{32}}^{i_1',i_2'}(t-\bar{s})| ds d\bar{s} \\ :=& J_t^1 + J_t^2.\end{aligned}$$

Since $|b_{k_{12}}^{i_1,i_2}(t-s)| \lesssim \frac{1}{|k_{12}|^{1-\eta}(t-s)^{1-\eta/2}}$ it follows by Lemma 3.10 that for $\eta > 0$ small enough

$$\begin{aligned}J_t^1 &\lesssim \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \Pi_{i=1}^3 \frac{1}{|k_i|^2} \frac{t^\eta}{|k_{12}|^{2-2\eta}} \\ &\lesssim \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \frac{t^\eta}{|k_3|^2 |k_{12}|^{3-2\eta}} \\ &\lesssim t^\eta 2^{q(1+2\eta)},\end{aligned}$$

and

$$\begin{aligned}J_t^2 &\lesssim \sum_k \theta(2^{-q}k) \sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_{12}|^{1-\eta}|k_{32}|^{1-\eta}} \\ &\lesssim \sum_k \theta(2^{-q}k) (\sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_{12}|^{2-2\eta}})^{1/2} (\sum_{k_{123}=k} \frac{t^\eta}{|k_1|^2|k_2|^2|k_3|^2|k_{32}|^{2-2\eta}})^{1/2} \\ &\lesssim t^\eta 2^{q(1+2\eta)},\end{aligned}$$

which yield the desired estimate for I_t^1. By a similar calculation we also obtain that for $\eta > \epsilon > 0$, $\gamma > 0$ small enough,

$$E[|\Delta_q(u_2^{\varepsilon_1,i} u_1^{\varepsilon_1,j}(t_1) - u_2^{\varepsilon_1,i} u_1^{\varepsilon_1,j}(t_2) - u_2^{\varepsilon_2,i} u_1^{\varepsilon_1,j}(t_1) + u_2^{\varepsilon_2,i} u_1^{\varepsilon_1,j}(t_2))|^2] \lesssim (\varepsilon_1^{2\gamma} + \varepsilon_2^{2\gamma})|t_1 - t_2|^{\eta-\epsilon} 2^{q(1+2\eta)}, \tag{4.2}$$

which by Gaussian hypercontractivity and Lemma 3.1 implies that

$$\begin{aligned}
&E[\|(u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_1)-u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_2)-u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_1)+u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_2))\|^p_{\mathcal{C}^{-1/2-\eta-\epsilon-3/p}}]\\
\lesssim&E[\|(u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_1)-u_2^{\varepsilon_1,i}u_1^{\varepsilon_1,j}(t_2)-u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_1)+u_2^{\varepsilon_2,i}u_1^{\varepsilon_1,j}(t_2))\|^p_{B^{-1/2-\eta-\epsilon}_{p,p}}]\\
\lesssim&(\varepsilon_1^{p\gamma}+\varepsilon_2^{p\gamma})|t_1-t_2|^{p(\eta-\epsilon)/2}.
\end{aligned}\tag{4.3}$$

Thus, for every $i,j=1,2,3$ we choose p large enough and deduce that there exists $v_3^{ij}\in C([0,T];\mathcal{C}^{-1/2-\delta/2})$ such that

$$u_2^{\varepsilon,i}\diamond u_1^{\varepsilon,j}\to v_3^{ij} \text{ in } L^p(\Omega,P,C([0,T];\mathcal{C}^{-1/2-\delta/2})).$$

Here $\delta>0$ depending on η,ϵ,p can be chosen small enough. For the proof of (4.2) we only calculate the corresponding term as in (4.1) and the other terms can be obtained similarly. It is straightforward to calculate that for $0\leq t_1<t_2\leq T$

$$\begin{aligned}
&E[|\Delta_q(\tilde{I}^2_{t_1}-\sum_{i_1=1}^3X^{\varepsilon,i_1}_{t_1}C^{\varepsilon,i_1}_{t_1}-\tilde{I}^2_{t_2}+\sum_{i_1=1}^3X^{\varepsilon,i_1}_{t_2}C^{\varepsilon,i_1}_{t_2})|^2]\\
\lesssim&E\bigg|\sum_{i_1,i_2,i_3=1}^3\sum_{k_1}\hat{X}^{\varepsilon,i_1}_{t_1}(k_1)\theta(2^{-q}k_1)e_{k_1}\bigg[\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}f(\varepsilon k_2)^2}{|k_2|^2}\Big(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})\\
&\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)\Big)ds-\sum_{k_2}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}\\
&\Big(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)\Big)ds\bigg]\bigg|^2\\
&+E\bigg|\sum_{i_1,i_2,i_3=1}^3\sum_{k_1}(\hat{X}^{\varepsilon,i_1}_{t_1}(k_1)-\hat{X}^{\varepsilon,i_1}_{t_2}(k_1))\theta(2^{-q}k_1)e_{k_1}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}\\
&\Big(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\hat{P}^{i_2i_3}(k_2)\hat{P}^{ji_3}(k_2)\Big)ds\bigg|^2\\
\lesssim&L^1_t+L^2_t+L^3_t+L^4_t,
\end{aligned}$$

where

$$\begin{aligned}
L^1_t=&\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2\bigg[\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}(1-e^{-|k_2|^2(t_2-t_1)})f(\varepsilon k_2)^2}{|k_2|^2}\\
&\big(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\big)ds\bigg]^2\\
L^2_t=&\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2\bigg[\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}\Big(e^{-|k_{12}|^2(t_1-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})\\
&-e^{-|k_2|^2(t_1-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)-e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})+e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\Big)ds\bigg]^2
\end{aligned}$$

$$L_t^3 =\sum_{k_1}\sum_{i_1,i_2=1}^3\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2\Big[\sum_{k_2}\int_{t_1}^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}\Big(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})$$
$$- e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\Big)ds\Big]^2$$
$$L_t^4 =E\Big|\sum_{k_1}\sum_{i_1,i_2=1}^3(\hat{X}_{t_1}^{\varepsilon,i_1}(k_1)-\hat{X}_{t_2}^{\varepsilon,i_1}(k_1))\sum_{k_2}\theta(2^{-q}k_1)e_{k_1}\int_0^{t_2}\frac{e^{-|k_2|^2(t_2-s)}f(\varepsilon k_2)^2}{|k_2|^2}$$
$$\big(e^{-|k_{12}|^2(t_2-s)}k_{12}^{i_2}\hat{P}^{ii_1}(k_{12})-e^{-|k_2|^2(t_2-s)}k_2^{i_2}\hat{P}^{ii_1}(k_2)\big)ds\Big|^2.$$

It is easy to deduce the desired estimates for L_t^1, L_t^3, L_t^4 as for (4.1) and it is sufficient to consider L_t^2: for some $0<\beta_0<1/2, \eta>0$ small enough, by Lemma 3.11 and interpolation we have

$$L_t^2 \lesssim\sum_{k_1}\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}}{|k_2|^2}[|k_1|^\eta\wedge|t_2-t_1|^{\frac{\eta}{2}}(|k_{12}|^{2\eta}+|k_2|^{2\eta})](t_1-s)^{-\frac{1-\eta}{2}}ds)^2$$
$$\lesssim\sum_{k_1}\frac{1}{|k_1|^2}\theta(2^{-q}k_1)^2(\sum_{k_2}\int_0^{t_1}\frac{e^{-|k_2|^2(t_1-s)}}{|k_2|^2}|k_1|^{\eta(1-\beta_0)}|t_2-t_1|^{\frac{\eta\beta_0}{2}}(|k_{12}|^{2\eta\beta_0}+|k_2|^{2\eta\beta_0})(t_1-s)^{-\frac{1-\eta}{2}}ds)^2$$
$$\lesssim|t_1-t_2|^{\eta\beta_0/2}2^{q(1+2\eta(1+\beta_0))},$$

which is the required estimate for L_t^2.

4.2 Renormalisation for $u_2^{\varepsilon,i}u_2^{\varepsilon,j}$

In this subsection we deal with $u_2^{\varepsilon,i}u_2^{\varepsilon,j}, i,j=1,2,3,$ and prove that $u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}\to v_4^{ij}$ in $C([0,T];\mathcal{C}^{-\delta})$. Recall that for $i,j=1,2,3$

$$u_2^{\varepsilon,i}\diamond u_2^{\varepsilon,j}:=u_2^{\varepsilon,i}u_2^{\varepsilon,j}-C_2^{\varepsilon,ij},$$

We have the following identities:

$$u_2^{\varepsilon,i}u_2^{\varepsilon,j}:=L^1+L^2+L^3,$$

where

$$L_t^1 =(2\pi)^{-\frac{9}{2}}\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_{1234}=k}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_{34}|^2(t-\bar{s})}:\hat{X}_s^{\varepsilon,i_1}(k_1)\hat{X}_s^{\varepsilon,i_2}(k_2)\hat{X}_{\bar{s}}^{\varepsilon,j_1}(k_3)\hat{X}_{\bar{s}}^{\varepsilon,j_2}(k_4):dsd\bar{s}e_k$$
$$\hat{P}^{ii_1}(k_{12})\imath k_{12}^{i_2}\hat{P}^{jj_1}(k_{34})\imath k_{34}^{j_2}$$

$$L_t^2 =\sum_{i=1}^4 I_t^i$$
$$=(2\pi)^{-\frac{9}{2}}\sum_{i_1,i_2,j_1,j_2=1}^3\sum_{k_{24}=k,k_1}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_4-k_1|^2(t-\bar{s})}\frac{f(\varepsilon k_1)^2e^{-|k_1|^2|s-\bar{s}|}}{2|k_1|^2}$$
$$:\hat{X}_s^{\varepsilon,i_3}(k_2)\hat{X}_{\bar{s}}^{\varepsilon,j_3}(k_4):dsd\bar{s}e_k\hat{P}^{ii_1}(k_{12})\imath k_{12}^{i_2}\hat{P}^{jj_1}(k_4-k_1)\imath(k_4^{j_2}-k_1^{j_2})\sum_{j_5=1}^3\hat{P}^{i_4j_5}(k_1)\hat{P}^{j_4j_5}(k_1)$$
$$(1_{i_3=i_2,i_4=i_1,j_3=j_2,j_4=j_1}+1_{i_3=i_2,i_4=i_1,j_3=j_1,j_4=j_2}+1_{i_3=i_1,i_4=i_2,j_3=j_2,j_4=j_1}+1_{i_3=i_1,i_4=i_2,j_3=j_1,j_4=j_2}),$$

and

$$L_t^3 =(2\pi)^{-6}\sum_{i_1,i_2,j_1,j_2=1}^{3}\sum_{k_1,k_2}\int_0^t\int_0^t e^{-|k_{12}|^2(t-s+t-\bar{s})}\frac{f(\varepsilon k_1)^2 f(\varepsilon k_2)^2 e^{-(|k_1|^2+|k_2|^2)|s-\bar{s}|}}{4|k_1|^2|k_2|^2}dsd\bar{s}\hat{P}^{ii_1}(k_{12})\hat{P}^{jj_1}(k_{12})$$

$$\imath k_{12}^{i_2}(-\imath k_{12}^{j_2})\sum_{j_3,j_4=1}^{3}(\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\hat{P}^{i_2j_4}(k_2)\hat{P}^{j_2j_4}(k_2)+\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_2j_3}(k_1)\hat{P}^{i_2j_4}(k_2)\hat{P}^{j_1j_4}(k_2)).$$

Here each I_t^i corresponds to the term associated with each indicator function respectively. To make it more readable we write each term corresponding to the tree notation in Section 2. $u_2^{\varepsilon,i}u_2^{\varepsilon,j}$ corresponds to  and $L_t^1, L_t^2, I_t^1, L_t^3$ correspond to the associated $\hat{\mathcal{W}}^{(\varepsilon,4)}, \hat{\mathcal{W}}^{(\varepsilon,2)}, \hat{\mathcal{W}}_1^{(\varepsilon,2)}, \hat{\mathcal{W}}^{(\varepsilon,0)}$ in the proof of Theorem 2.16 respectively.

By an easy computation we obtain that

$$L_t^3 =(2\pi)^{-6}\sum_{i_1,i_2,j_1,j_2=1}^{3}\sum_{k_1,k_2}f(\varepsilon k_1)^2 f(\varepsilon k_2)^2\hat{P}^{ii_1}(k_{12})\hat{P}^{jj_1}(k_{12})k_{12}^{i_2}k_{12}^{j_2}\sum_{j_3,j_4=1}^{3}(\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_1j_3}(k_1)\hat{P}^{i_2j_4}(k_2)$$

$$\hat{P}^{j_2j_4}(k_2)+\hat{P}^{i_1j_3}(k_1)\hat{P}^{j_2j_3}(k_1)\hat{P}^{i_2j_4}(k_2)\hat{P}^{j_1j_4}(k_2))\frac{1}{2|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{12}|^2)}$$

$$\left[\frac{1-e^{-2|k_{12}|^2t}}{2|k_{12}|^2}-\int_0^t e^{-2|k_{12}|^2(t-s)-(|k_1|^2+|k_2|^2+|k_{12}|^2)s}ds\right].$$

Let

$$C_2^{\varepsilon,ij}(t) =L_t^3.$$

Terms in the second chaos: Now we come to L_t^2: it is sufficient to consider I_t^1 and the desired estimates for the other terms can be obtained similarly. For $\epsilon>0$ small enough we have the following inequality

$$\begin{aligned}&E|\Delta_q I_t^1|^2\\ \lesssim&\sum_k\sum_{k_{24}=k,k_{24}'=k,k_1,k_1'}\theta(2^{-q}k)^2\int_0^t\int_0^t\int_0^t\int_0^t e^{-|k_{12}|^2(t-\sigma)-|k_4-k_1|^2(t-\bar{\sigma})}|k_{12}(k_4-k_1)|\\ &e^{-|k_{12}'|^2(t-s)-|k_4'-k_1'|^2(t-\bar{s})}|k_{12}'(k_4'-k_1')|\frac{1}{|k_1|^2|k_1'|^2|k_2|^2|k_4|^2}1_{\{k_2=k_2',k_4=k_4'\}}+1_{\{k_2=k_4',k_4=k_2'\}}dsd\bar{s}d\sigma d\bar{\sigma},\end{aligned}$$

Now in the following we only estimate the term corresponding to the first characteristic function on the right hand side of the inequality. The second term can be estimated similarly:

$$\begin{aligned}E|\Delta_q I_t^1|^2\lesssim&\sum_k\sum_{k_{24}=k,k_1,k_3}\theta(2^{-q}k)^2\int_0^t\int_0^t\int_0^t\int_0^t e^{-|k_{12}|^2(t-s)-|k_4-k_1|^2(t-\bar{s})-|k_{23}|^2(t-\sigma)-|k_4-k_3|^2(t-\bar{\sigma})}\\ &\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}dsd\bar{s}|k_{12}(k_4-k_1)k_{23}(k_4-k_3)|\\ \lesssim&t^\epsilon\sum_k\sum_{k_{24}=k,k_1,k_3}\frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_1-k_4|^{1-\epsilon}|k_4-k_3||k_{12}|^{1-\epsilon}|k_{23}|}\end{aligned}$$

$$
\begin{aligned}
&\lesssim t^\epsilon \sum_k \sum_{k_{24}=k} \frac{\theta(2^{-q}k)^2}{|k_2|^2|k_4|^2} \sum_{k_1} \frac{1}{|k_1-k_4||k_1|^2|k_{12}|} \sum_{k_3} \frac{1}{|k_3-k_4||k_3|^2|k_{23}|}\\
&\lesssim t^\epsilon \sum_k \sum_{k_{24}=k} \frac{\theta(2^{-q}k)^2}{|k_2|^2|k_4|^2} (\sum_{k_1} \frac{1}{|k_1-k_4|^{2-2\epsilon}|k_1|^2})^{1/2} (\sum_{k_1} \frac{1}{|k_{12}|^{2-2\epsilon}|k_1|^2})^{1/2}\\
&\quad (\sum_{k_3} \frac{1}{|k_3-k_4|^2|k_3|^2})^{1/2} (\sum_{k_3} \frac{1}{|k_{23}|^2|k_3|^2})^{1/2}\\
&\lesssim t^\epsilon \sum_k \sum_{k_{24}=k} \frac{\theta(2^{-q}k)^2}{|k_2|^{3-\epsilon}|k_4|^{3-\epsilon}} \lesssim t^\epsilon 2^{2q\epsilon},
\end{aligned}
$$

where in the last two inequalities we used Lemma 3.10.

Terms in the fourth chaos:

Now we consider L^1_t. For $\epsilon > 0$ small enough we have the following calculations:

$$
\begin{aligned}
&E|\Delta_q L^1_t|^2\\
\lesssim &\sum_k \sum_{k_{1234}=k} \theta(2^{-q}k)^2 \Big[\int_0^t \int_0^t \int_0^t \int_0^t e^{-|k_{12}|^2(t-s+t-\sigma)-|k_{34}|^2(t-\bar{s}+t-\bar{\sigma})} \frac{e^{-(|k_1|^2+|k_2|^2)|s-\sigma|-(|k_3|^2+|k_4|^2)|\bar{s}-\bar{\sigma}|}}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}\\
&ds d\bar{s} d\sigma d\bar{\sigma} |k_{12}k_{34}|^2 + \int_0^t \int_0^t \int_0^t \int_0^t e^{-|k_{12}|^2(t-s)-|k_{23}|^2(t-\sigma)-|k_{34}|^2(t-\bar{s})-|k_{14}|^2(t-\bar{\sigma})} \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2}\\
&ds d\bar{s} d\sigma d\bar{\sigma} |k_{12}k_{34}k_{14}k_{23}| \Big]\\
\lesssim &t^\epsilon \sum_k \sum_{k_{1234}=k} \theta(2^{-q}k)^2 \Big(\frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}}\\
&+ \frac{1}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{1-\epsilon/2}|k_{34}|^{1-\epsilon/2}|k_{14}|^{1-\epsilon/2}|k_{23}|^{1-\epsilon/2}} \Big)\\
\lesssim &t^\epsilon \Big[2^{2q\epsilon} + \Big(\sum_{k_{1234}=k} \frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{12}|^{2-\epsilon}|k_{34}|^{2-\epsilon}} \Big)^{1/2} \Big(\sum_{k_{1234}=k} \frac{\theta(2^{-q}k)^2}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{14}|^{2-\epsilon}|k_{23}|^{2-\epsilon}} \Big)^{1/2} \Big]\\
\lesssim &t^\epsilon 2^{2q\epsilon},
\end{aligned}
$$

where we used Lemma 3.10 in the last inequality. By a similar calculation we also get that for $\epsilon, \eta > 0$, $\gamma > 0$ small enough

$$
\begin{aligned}
&E[|\Delta_q(u_2^{\varepsilon_1,i} \diamond u_2^{\varepsilon_1,j}(t_1) - u_2^{\varepsilon_1,i} \diamond u_2^{\varepsilon_1,j}(t_2) - u_2^{\varepsilon_2,i} \diamond u_2^{\varepsilon_2,j}(t_1) + u_2^{\varepsilon_2,i} \diamond u_2^{\varepsilon_2,j})(t_2))|^2]\\
\lesssim &(\varepsilon_1^{2\gamma} + \varepsilon_2^{2\gamma})|t_1 - t_2|^\eta 2^{q(\epsilon+2\eta)},
\end{aligned}
$$

which together with Gaussian hypercontractivity, Lemma 3.1 and similar arguments as for (4.3) implies that there exist $v_4^{ij} \in C([0,T];\mathcal{C}^{-\delta}), i, j = 1, 2, 3$ such that for $p > 1$

$$
u_2^{\varepsilon,i} \diamond u_2^{\varepsilon,j} \to v_4^{ij} \text{ in } L^p(\Omega, P, C([0,T];\mathcal{C}^{-\delta})).
$$

Here $\delta > 0$ depending on η, ϵ, p can be chosen small enough.

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