

Lattice approximation to the dynamical Φ_3^4 model ^{*}

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Abstract

We study the lattice approximations to the dynamical Φ_3^4 model by paracontrolled distributions proposed in [GIP13]. We prove that the solutions to the lattice systems converge to the solution to the dynamical Φ_3^4 model in probability, locally uniformly in time. Since the dynamical Φ_3^4 model is not well defined in the classical sense and renormalisation has to be performed in order to define the non-linear term, a corresponding suitable drift term is added in the stochastic equations for the lattice systems.

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1 Introduction

Recall that the usual continuum Euclidean Φ_d^4 -quantum field is heuristically described by the following probability measure:

$$N^{-1} \Pi_{x \in \mathbb{T}^d} d\phi(x) \exp \left(- \int_{\mathbb{T}^d} (|\nabla \phi(x)|^2 + \phi^2(x) + \phi^4(x)) dx \right), \quad (1.1)$$

where N is a normalization constant and ϕ is the real-valued field and \mathbb{T}^d is the d -dimensional torus. There have been many approaches to the problem of giving a meaning to the above heuristic measure for $d = 2$ and $d = 3$ (see [GRS75, GIJ86] and references therein). In [PW81] Parisi and Wu proposed a program for Euclidean quantum field theory of getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study the properties of the Gibbs states. This procedure is called

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stochastic field quantization (see [JLM85]). The Φ_d^4 model is the simplest non-trivial Euclidean quantum field (see [GLJ86] and the reference therein). The issue of the stochastic quantization of the Φ_d^4 model is to solve the following equation:

$$d\Phi = (\Delta\Phi - \Phi^3)dt + dW(t) \quad \Phi(0) = \Phi_0. \quad (1.2)$$

where W is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$. The solution Φ is also called dynamical Φ_d^4 model. (1.2) is ill-posed in both two and three dimensions.

In two spatial dimensions, the dynamical Φ_2^4 model was previously treated in [AR91], [DD03] and [MW15]. In three spatial dimensions this equation (1.2) is ill-posed and the main difficulty in this case is that W and hence the solutions are so singular that the non-linear term is not well-defined in the classical sense. It was a long-standing open problem to give a meaning to equation (1.2) in the three dimensional case. A breakthrough result was achieved recently by Martin Hairer in [Hai14], where he introduced a theory of regularity structures and gave a meaning to equation (1.2) successfully. He also proved existence and uniqueness of a local (in time) solution. By using the paracontrolled distributions proposed by Gubinelli, Imkeller and Perkowski in [GIP13] existence and uniqueness of local solutions to (1.2) has also been obtained in [CC13]. Recently, these two approaches have been successful in giving a meaning to several other ill-posed stochastic PDEs like the Kardar-Parisi-Zhang (KPZ) equation ([KPZ86], [BG97], [Hai13]), the Navier-Stokes equation driven by space-time white noise ([ZZ14, ZZ15]), the dynamical sine-Gordon equation ([HS14]) and so on (see [HP14] for further interesting examples). From a philosophical perspective, the theory of regularity structures and the paracontrolled distributions are inspired by the theory of controlled rough paths [Lyo98], [Gub04]. The main difference is that the regularity structure theory considers the problem locally, while the paracontrolled distribution method is a global approach using Fourier analysis. In [Kup14] the author also uses renormalization group techniques to make sense of the dynamical Φ_3^4 model.

The lattice approximation is an important tool in constructing and studying the continuum Φ_3^4 field (see [P75, P77, ABZ04]). It also preserves Osterwalder-Schrader positivity and also the ferromagnetic nature of the measure (see [GLJ86] and the references therein). Let us set $\Lambda_\varepsilon := \{\varepsilon x \in \mathbb{T}^3, x \in \mathbb{Z}^3\}$. Heuristically, the quantities $\int |\nabla\phi(x)|^2 dx$, $\int \phi^2(x) dx$, and $\int \phi^4(x) dx$ can be approximated by $\varepsilon \sum_{|x-y|=\varepsilon, x, y \in \Lambda_\varepsilon} (\phi(x) - \phi(y))^2$, $\varepsilon^3 \sum_{x \in \Lambda_\varepsilon} \phi(x)^2$ and $\varepsilon^3 \sum_{x \in \Lambda_\varepsilon} \phi(x)^4$, respectively, as ε tends to zero. Thus heuristically (1.1) can be approximated by the following probability measure μ_ε :

$$N_\varepsilon^{-1} \Pi_{x \in \Lambda_\varepsilon} d\phi_x \exp \left(2\varepsilon \sum_{|x-y|=\varepsilon, x, y \in \Lambda_\varepsilon} \phi(x)\phi(y) - (\varepsilon^3 + 12\varepsilon) \sum_{x \in \Lambda_\varepsilon} \phi^2(x) - \varepsilon^3 \sum_{x \in \Lambda_\varepsilon} \phi^4(x) \right), \quad (1.3)$$

where N_ε is a normalization constant. (1.3) is still just a heuristic expression, but one can give a rigorous meaning to it since it is a finite dimensional Gaussian measure with a density (see [GLJ86] and the references therein). We call this the lattice Φ_3^4 -field measure. From μ_ε by deriving suitable bounds on its moments and choosing subsequences if necessary, one gets limit measures by weak convergence. These are then the continuum Φ_3^4 -field measures.

The following stochastic PDE on Λ_ε , $\varepsilon > 0$, is the stochastic quantization associated to the

lattice Φ_3^4 -field measure:

$$\begin{aligned} d\Phi^\varepsilon(t, x) &= (\Delta_\varepsilon \Phi^\varepsilon(t, x) - (\Phi^\varepsilon)^3(t, x) + (3C_0^\varepsilon - 9C_1^\varepsilon)\Phi^\varepsilon(t, x))dt \\ &\quad + \varepsilon^{-3/2} dW_\varepsilon(t, x) \\ \Phi^\varepsilon(0) &= \Phi_0^\varepsilon. \end{aligned} \tag{1.4}$$

Here $W_\varepsilon(t) = \{W(t, x)\}_{x \in \Lambda_\varepsilon}$ is a family of independent Brownian motions and C_0^ε and C_1^ε are constants defined in (6.3) and (1.10) below. For $x \in \Lambda_\varepsilon$ define

$$\Delta_\varepsilon f(x) := \varepsilon^{-2} \sum_{y \in \Lambda_\varepsilon, y \sim x} (f(y) - f(x)),$$

and the nearest neighbor relation $x \sim y$ is to be understood with periodic boundary conditions on Λ_ε .

The aim of this paper is to prove that as $\varepsilon \rightarrow 0$ the dynamical lattice approximation, i.e. the solution to (1.4), converges to the dynamical Φ_3^4 model. This problem is also related to the convergence of a rescaled discrete spin system to the solution of the dynamical Φ_3^4 model (see [MW14] for the dynamical Φ_2^4 model). We emphasize that to make sense of (1.2) we need to renormalise some ill-defined terms in (1.2). This is done by adding the renormalisation terms $C_0^\varepsilon \Phi^\varepsilon$ and $C_1^\varepsilon \Phi^\varepsilon$ in the approximating equation (1.4).

In the one dimensional case, approximations to general stochastic partial differential equations driven by space-time white noise have been very well studied (see [Gy98, Gy99, DG01, HMW14] and the reference therein). In [GP15] the authors study the Sasamoto-Spohn type discretizations of the conservative stochastic Burgers equation. In the three dimensional case we have also studied the discrete approximations to stochastic Navier-Stokes equations driven by space-time white noise (see [ZZ14]).

In this paper we use the paracontrolled distribution method to prove that the solutions to the lattice approximation equation converge to the dynamical Φ_3^4 model. The theory of paracontrolled distributions combines the idea of Gubinelli's controlled rough path [Gub04] and Bony's paraproduct [Bon84], which is defined as follows: Let $\Delta_j f$ be the j th Littlewood-Paley block of a (Scharwtz) distribution f . For its definition we refer to Section 2. Define for distributions f and g

$$\pi_{<}(f, g) = \pi_{>}(g, f) = \sum_{j \geq -1} \sum_{i < j-1} \Delta_i f \Delta_j g, \quad \pi_0(f, g) = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

Formally $fg = \pi_{<}(f, g) + \pi_0(f, g) + \pi_{>}(f, g)$. Observing that, if f is regular, $\pi_{<}(f, g)$ behaves like g and is the only term in the Bony's paraproduct not improving the regularities, the authors in [GIP13] consider a paracontrolled ansatz of the type

$$u = \pi_{<}(u', g) + u^\sharp,$$

where $\pi_{<}(u', g)$ represents the "bad-part" of the solution, u' is some suitable function and g is some functional of the Gaussian field and u^\sharp is regular enough to define the multiplication required. Then to make sense of the product of $u'f$ we only need to define gf .

Using the paracontrolled distribution method, to perform the lattice approximation of the dynamical Φ_3^4 model we shall meet the projection operators P_N , which do not commute with the

paraproduct. Here we use a random operator technique from [GP15] to handle these operators. However, for the dynamical Φ_3^4 model this technique is not enough and we have to estimate an additional error term D_N by stochastic calculations in Section 6.4 (see Remark 4.5).

Framework and main result

For $N \geq 1$, let $\Lambda^N = \{-N, -(N-1), \dots, N\}^3$. Set $\varepsilon = \frac{2}{2N+1}$. Every point $k \in \Lambda^N$ can be identified with $x = \varepsilon k \in \Lambda_\varepsilon = \{x = (x^1, x^2, x^3) \in \varepsilon\mathbb{Z}^3 : -1 < x^1, x^2, x^3 < 1\}$. We view Λ_ε as a discretisation of the continuous three-dimensional torus \mathbb{T}^3 identified with $[-1, 1]^3$. In the following for simplicity we fix a cylindrical Wiener process in (1.2) on $L^2(\mathbb{T}^3)$ given by $2^{-3/2} \sum_k \beta_k e^{i\pi k \cdot x}$ for $x \in \mathbb{T}^3$ and restrict it to $L^2(\Lambda_\varepsilon)$ as $W_N(x) = 2^{-3/2} \sum_{|k|_\infty \leq N} \beta_k e^{i\pi k \cdot x}$ for $x \in \Lambda_\varepsilon$, which is also a cylindrical Wiener process on $L^2(\Lambda_\varepsilon)$. Here $\{\beta_k\}$ is a family of complex valued Brownian motions with $\bar{\beta}_{-k}(t) = \beta_k(t)$ and $E[\beta_{k_1}(t_1)\beta_{k_2}(t_2)] = 1_{\{k_1+k_2=0\}}t_1 \wedge t_2$ and $|k|_\infty = \max(|k^1|, |k^2|, |k^3|)$. For fixed N (1.4) is a finite dimensional SDE and we can easily obtain existence and uniqueness of solutions to (1.4) by [PR07], which implies that the solution to (1.4) has the same distribution as the solution to the following equation:

$$\begin{aligned} d\Phi^\varepsilon(t, x) &= (\Delta_\varepsilon \Phi^\varepsilon(t, x) - (\Phi^\varepsilon)^3(t, x) + (3C_0^\varepsilon - 9C_1^\varepsilon)\Phi^\varepsilon(t, x))dt \\ &\quad + dW_N(t, x) \\ \Phi^\varepsilon(0) &= \Phi_0^\varepsilon. \end{aligned} \tag{1.5}$$

Following [MW14] we discuss a suitable extension of functions defined on Λ_ε onto all of the torus \mathbb{T}^3 (which we identify with the interval $[-1, 1]^3$). For any function $Y : \Lambda_\varepsilon \rightarrow \mathbb{R}$, we define the discrete Fourier transform \hat{Y} through

$$\hat{Y}(k) = \begin{cases} \sum_{x \in \Lambda_\varepsilon} \varepsilon^3 Y(x) e^{-i\pi k \cdot x}, & \text{if } k \in \{-N, \dots, N\}^3 \\ 0 & \text{if } k \in \mathbb{Z}^3 \setminus \{-N, \dots, N\}^3. \end{cases}$$

In this context Fourier inversion states

$$Y(x) = \frac{1}{8} \sum_{k \in \mathbb{Z}^3} \hat{Y}(k) e^{i\pi k \cdot x} \text{ for all } x \in \Lambda_\varepsilon. \tag{1.6}$$

It is thus natural to extend Y to all of \mathbb{T}^3 by taking (1.6) as a definition of $Y(x)$ for $x \in \mathbb{T}^3 \setminus \Lambda_\varepsilon$. More explicitly, for $Y : \Lambda_\varepsilon \rightarrow \mathbb{R}$ we define $(\text{Ext}Y) : \mathbb{T}^3 \rightarrow \mathbb{R}$ as

$$\text{Ext}Y(x) = \frac{1}{2^3} \sum_{k \in \{-N, \dots, N\}^3} \sum_{y \in \Lambda_\varepsilon} \varepsilon^3 e^{i\pi k \cdot (x-y)} Y(y).$$

By the definition of the operators Δ_ε we have

$$\widehat{e^{t\Delta_\varepsilon} v}(k) = \begin{cases} e^{-|k|^2 f(\varepsilon k)} \hat{v}(k), & \text{if } k \in \{-N, \dots, N\}^3 \\ 0 & \text{if } k \in \mathbb{Z}^3 \setminus \{-N, \dots, N\}^3. \end{cases}$$

Here for $x = (x^1, x^2, x^3)$

$$f(x) = \frac{4}{|x|^2} \left(\sin^2 \frac{x^1 \pi}{2} + \sin^2 \frac{x^2 \pi}{2} + \sin^2 \frac{x^3 \pi}{2} \right).$$

Now we extend the solutions to all of \mathbb{T}^3 . It is easy to see that

$$\text{Ext}\Phi^\varepsilon(t) = P_t^\varepsilon \text{Ext}\Phi_0^\varepsilon - \int_0^t P_{t-s}^\varepsilon Q_N [(\text{Ext}\Phi^\varepsilon)^3 - (3C_0^\varepsilon - 9C_1^\varepsilon)\text{Ext}\Phi^\varepsilon] ds + \int_0^t P_{t-s}^\varepsilon \text{Ext} dW_N. \quad (1.7)$$

Here $P_t^\varepsilon = \text{Ext}e^{t\Delta^\varepsilon}$ and $Q_N u(x) = P_N u(x) + \Pi_N u(x)$ with

$$P_N = \mathcal{F}^{-1} 1_{|k|_\infty \leq N} \mathcal{F}, \quad (1.8)$$

and Π_N is defined for u satisfying $\text{supp}\mathcal{F}u \subset \{k : |k|_\infty \leq 3N\}$ as follows

$$\begin{aligned} \Pi_N u(x) &= \sum_{i_1, i_2, i_3 \in \{-1, 0, 1\}, \sum_{j=1}^3 i_j^2 \neq 0} e_N^{i_1 i_2 i_3}(x) \mathcal{F}^{-1} 1_{k \in P^{i_1 i_2 i_3}} \mathcal{F}u(x) \\ &= \sum_{i_1, i_2, i_3 \in \{-1, 0, 1\}, \sum_{j=1}^3 i_j^2 \neq 0} P_N [e_N^{i_1 i_2 i_3} u](x), \end{aligned} \quad (1.9)$$

where $P^{i_1 i_2 i_3} = \{k = (k^1, k^2, k^3) : k^j i_j > N \text{ if } i_j = -1, 1; |k^j| \leq N, \text{ if } i_j = 0, j = 1, 2, 3\}$ is a rectangular division of $\mathbb{Z}^3 \setminus \{k \in \mathbb{Z}^3, |k|_\infty \leq N\}$, $e_N^{i_1 i_2 i_3}(x) = \prod_{j=1}^3 e^{-i\pi(2N+1)i_j x^j}$ and $|k|_\infty = \max(|k^1|, |k^2|, |k^3|)$. Here and in the following the Fourier transform and the inverse Fourier transform are denoted by \mathcal{F} and \mathcal{F}^{-1} , respectively.

Remark 1.1 When we use (1.6) to write f and g in terms of discrete Fourier transform and take the product of f and g , it is easy to see where the Π_N part comes from. When $\text{supp}\mathcal{F}(\text{Ext}f\text{Ext}g) \not\subset \{k \in \mathbb{Z}^3 : |k|_\infty \leq N\}$, we should multiply $e_N^{i_1 i_2 i_3}$ to make $\text{supp}\mathcal{F}(\text{Ext}f\text{Ext}g e_N^{i_1 i_2 i_3})$ belong to the set $\{k \in \mathbb{Z}^3 : |k|_\infty \leq N\}$.

Now choose C_0^ε as in (6.3) and

$$C_1^\varepsilon = C_{11}^\varepsilon + \sum_{i_1, i_2, i_3 \in \{-1, 0, 1\}, \sum_{j=1}^3 i_j^2 \neq 0} C_{12}^{\varepsilon, i_1 i_2 i_3}, \quad (1.10)$$

with $C_{11}^\varepsilon, C_{12}^{\varepsilon, i_1 i_2 i_3}$ defined in (6.4) and (6.5) respectively. In the following we omit the summation with respect to i_1, i_2, i_3 if there's no confusion.

The main result of this paper is the following theorem:

Theorem 1.2 Let $z \in (1/2, 2/3)$ and $\Phi_0 \in \mathcal{C}^{-z}$. Let (Φ, τ) be the unique (maximal in time) solution to (1.2) and let for $\varepsilon \in (0, 1)$ the function Φ^ε be the unique solution to (1.5) on $[0, \infty)$. If the initial data satisfies $\text{Ext}\Phi_0^\varepsilon - \Phi_0 \rightarrow 0$ in \mathcal{C}^{-z} , then there exists a sequence of random times τ_L such that $\lim_{L \rightarrow \infty} \tau_L = \tau$ and

$$\sup_{t \in [0, \tau_L]} \|\text{Ext}\Phi^\varepsilon - \Phi\|_{-z} \rightarrow 0 \quad \text{in probability, as } \varepsilon \rightarrow 0.$$

Remark 1.3 (i) Existence and uniqueness of (Φ, τ) has been obtained in [Hai14, CC13]. For the definition of \mathcal{C}^{-z} and the norm $\|\cdot\|_{-z}$ see Section 2 below.

(ii) The constant C_1^ε is the corresponding renormalization constant of order $-\log \varepsilon$ and is divided into two parts: C_{11}^ε and C_{12}^ε which come from terms with P_N and Π_N defined in (1.8) and (1.9) respectively. Moreover,

$$C_0^\varepsilon \simeq \frac{1}{\varepsilon}, \quad C_{11}^\varepsilon \simeq -\log \varepsilon, \quad C_{12}^{\varepsilon, i_1 i_2 i_3} \simeq 1.$$

(iii) After our original paper was published on arXiv, Hairer and Matetski in [HM15] also obtained similar results by using the theory of regularity structure. Moreover, by using the results in [BFS83] they obtained the existence of a global solution to the dynamical Φ_3^4 model starting from almost every point and the Φ_3^4 field is an invariant measure of the solution to (1.2) when the coupling constant is small in their Corollary 1.2. By a similar argument as in the proof of Corollary 1.2 in [HM15] we can also obtain these results. Compared to the piecewise constant extension in [HM15, Corollary 1.2], our extension is smooth and based on discrete Fourier transform and does not change the inner product from $L^2(\Lambda_\varepsilon)$ to $L^2(\mathbb{T}^3)$, which coincides with the extension considered in [GJ86]. Moreover, we use the lattice approximation and this extension to study the Dirichlet form associated with the Φ_3^4 field in our forthcoming paper.

The structure of the paper is organized as follows. In Section 2, we recall some basic notions and results for the paracontrolled distribution method. In Section 3 we prove some estimates for the approximating operators. In Section 4 we use the paracontrolled distribution method to prove uniform bounds for the lattice approximation equations. In Section 5 we give the proof of our main result Theorem 1.2. In Section 6 convergence of several stochastic terms is proved.

2 Besov spaces and paraproduct

In the following we recall the definitions and some properties of Besov spaces and paraproducts. For a general introduction we refer to [BCD11, GIP13]. First we introduce the following notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space E with norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to E , equipped with the supremum norm $\|\cdot\|_{C_T E}$. For $\alpha \in (0, 1)$ we also define $C_T^\alpha E$ as the space of α -Hölder continuous functions from $[0, T]$ to E , endowed with the seminorm $\|f\|_{C_T^\alpha E} = \sup_{s, t \in [0, T], s \neq t} \frac{\|f(s) - f(t)\|_E}{|t - s|^\alpha}$.

The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or \mathcal{D} . The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on \mathbb{R}^d , such that

- i. the support of χ is contained in a ball and the support of θ is contained in an annulus;
- ii. $\chi(z) + \sum_{j \geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.
- iii. $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\text{supp}(\theta(2^{-i}\cdot)) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $|i - j| > 1$.

We call the pair (χ, θ) a dyadic partition of unity, and refer to [BCD11, Proposition 2.10] for its existence. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot) \mathcal{F}u).$$

For $\alpha \in \mathbb{R}$, the Hölder-Besov space \mathcal{C}^α is given by $\mathcal{C}^\alpha = B_{\infty, \infty}^\alpha(\mathbb{R}^d)$, where for $p, q \in [1, \infty]$ we define

$$B_{p,q}^\alpha(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B_{p,q}^\alpha} = \left(\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p})^q \right)^{1/q} < \infty\},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. For $\alpha \in \mathbb{R}$ we write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B_{\infty, \infty}^\alpha}$ in the following for simplicity.

We point out that everything above and everything that follows can be applied to distributions on the torus (see [S85, SW71]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on \mathbb{T}^d . Therefore, Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as

$$B_{p,q}^\alpha(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B_{p,q}^\alpha} = \left(\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q \right)^{1/q} < \infty\}.$$

We will need the following Besov embedding theorem on the torus (c.f. [GIP13, Lemma 41]):

Lemma 2.1 Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B_{p_1, q_1}^\alpha(\mathbb{T}^d)$ is continuously embedded in $B_{p_2, q_2}^{\alpha - d(1/p_1 - 1/p_2)}(\mathbb{T}^d)$.

Now we recall the following paraproduct introduced by Bony (see [Bon81]). In general, the product fg of two distributions $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley blocks, the product fg of two distributions f and g can be formally decomposed as

$$fg = \sum_{j \geq -1} \sum_{i \geq -1} \Delta_i f \Delta_j g = \pi_{<}(f, g) + \pi_0(f, g) + \pi_{>}(f, g),$$

with

$$\pi_{<}(f, g) = \pi_{>}(g, f) = \sum_{j \geq -1} \sum_{i < j-1} \Delta_i f \Delta_j g, \quad \pi_0(f, g) = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

For $j \geq 0$ we also use the notations

$$S_j f = \sum_{i \leq j-1} \Delta_i f,$$

and for $k_1, k_2 \in \mathbb{Z}^3$

$$\psi_{<}(k_1, k_2) = \sum_{j \geq -1} \sum_{i < j-1} \theta_i(k_1) \theta_j(k_2), \quad \psi_0(k_1, k_2) = \sum_{|i-j| \leq 1} \theta_i(k_1) \theta_j(k_2),$$

with $\theta_i = \theta(2^{-i}\cdot)$ for $i \geq 0$ and $\theta_{-1} = \chi$. We will use without comment that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ for $\alpha \leq \beta$, that $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_\alpha$ for $\alpha > 0$, and that $\|\cdot\|_\alpha \lesssim \|\cdot\|_{L^\infty}$ for $\alpha \leq 0$. We will also use that $\|S_j u\|_{L^\infty} \lesssim 2^{-j\alpha} \|u\|_\alpha$ for $\alpha < 0, j \geq 0$ and $u \in \mathcal{C}^\alpha$, where $\|\cdot\|_\alpha$ denotes the norm in $\mathcal{C}^\alpha, \alpha \in \mathbb{R}$.

The basic results about these bilinear operations are given by the following estimates: From these estimates we know that $\pi_{<}(f, g)$ part is the only term in the paraproduct not improving the regularity even if f is regular. $\pi_0(f, g)$ part is the only term in the paraproduct not well-defined for arbitrary distributions f, g .

Lemma 2.2 (Paraproduct estimates, [Bon 81, GIP13, Lemma 2]) For any $\beta \in \mathbb{R}$ we have

$$\|\pi_{<}(f, g)\|_{\beta} \lesssim \|f\|_{L^{\infty}} \|g\|_{\beta}, \quad f \in L^{\infty}, g \in \mathcal{C}^{\beta},$$

and for $\alpha < 0$ furthermore

$$\|\pi_{<}(f, g)\|_{\alpha+\beta} \lesssim \|f\|_{\alpha} \|g\|_{\beta}, \quad f \in \mathcal{C}^{\alpha}, g \in \mathcal{C}^{\beta}.$$

For $\alpha + \beta > 0$ we have

$$\|\pi_0(f, g)\|_{\alpha+\beta} \lesssim \|f\|_{\alpha} \|g\|_{\beta}, \quad f \in \mathcal{C}^{\alpha}, g \in \mathcal{C}^{\beta}.$$

The following basic commutator lemma is important for our use:

Lemma 2.3 ([GIP13, Lemma 5]) Assume that $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$. Then for smooth f, g, h , the trilinear operator

$$C(f, g, h) = \pi_0(\pi_{<}(f, g), h) - f\pi_0(g, h)$$

satisfies the bound

$$\|C(f, g, h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_{\alpha} \|g\|_{\beta} \|h\|_{\gamma}.$$

Thus, C can be uniquely extended to a bounded trilinear operator from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ to $\mathcal{C}^{\alpha+\beta+\gamma}$.

Now we recall the following properties of the heat semigroup $P_t := e^{t\Delta}$, which corresponds to the smoothing effect of the heat semigroup.

Lemma 2.4 ([GIP13, Lemma 47]) Let $u \in \mathcal{C}^{\alpha}$ for some $\alpha \in \mathbb{R}$. Then for every $\delta \geq 0$

$$\|P_t u\|_{\alpha+\delta} \lesssim t^{-\delta/2} \|u\|_{\alpha}.$$

Lemma 2.5 ([CC13, Lemma A.1]) Let $u \in \mathcal{C}^{\alpha}$ for some $\alpha < 1$ and $v \in \mathcal{C}^{\beta}$ for some $\beta \in \mathbb{R}$. Then for $\delta \geq \alpha + \beta$

$$\|P_t \pi_{<}(u, v) - \pi_{<}(u, P_t v)\|_{\delta} \lesssim t^{\frac{\alpha+\beta-\delta}{2}} \|u\|_{\alpha} \|v\|_{\beta}.$$

Lemma 2.6 ([CC13, Lemma 2.5]) Let $u \in \mathcal{C}^{\alpha+\delta}$ for some $\alpha \in \mathbb{R}, \delta > 0$. Then for every $t \geq 0$

$$\|(P_t - I)u\|_{\alpha} \lesssim t^{\delta/2} \|u\|_{\alpha+\delta}.$$

We also have the following result, which will be used later.

Lemma 2.7 (Bernstein type lemma) Let $u \in \mathcal{C}^{\alpha}$ for some $\alpha \in \mathbb{R}$.

1) If $\text{supp } \mathcal{F}u \subset \{k : |k| \leq CN\}$ for some $C > 0$, then for $\beta > \alpha$

$$\|u\|_{\beta} \lesssim N^{\beta-\alpha} \|u\|_{\alpha}.$$

2) If $\text{supp } \mathcal{F}u \subset \{k : |k| > CN\}$ for some $C > 0$, then for $\beta < \alpha$

$$\|u\|_{\beta} \lesssim N^{\beta-\alpha} \|u\|_{\alpha}.$$

Here all the constants we omit are independent of N .

Proof We have

$$\|u\|_{\beta} = \sup_j 2^{j\beta} \|\Delta_j u\|_{L^{\infty}} = \sup_j 2^{j(\beta-\alpha)} 2^{j\alpha} \|\Delta_j u\|_{L^{\infty}}.$$

For the first case we have that $\Delta_j u \neq 0$ iff $2^j \lesssim N$, which implies the first result. If $\text{supp } \mathcal{F}u \subset \{k : |k| > CN\}$, we have that $\Delta_j u \neq 0$ iff $2^j \gtrsim N$, which implies the second result. \square

3 Estimates for the approximating operators

In this section we prove the estimates for the approximating operators on \mathbb{T}^3 , which will be used to prove the main result. First we prove estimates for P_N and Π_N defined in (1.8) and (1.9). Compared to the estimates proved in the one dimensional case in [GP15], we prove them here in the three dimensional case. Moreover, we prove a commutator estimate for P_t^ε whereas in [GP15] a commutator estimate for Δ_N was proved.

Lemma 3.1 Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$. Then for any $\kappa > 0$ small enough we have the following estimates:

(1) (Estimates for P_N)

$$\|P_N u\|_{\alpha-\kappa} \lesssim \|u\|_\alpha, \quad \|(I - P_N)u\|_{\alpha-\kappa} \lesssim N^{-\kappa/2} \|u\|_\alpha.$$

(2) (Estimates for Π_N) If $\alpha > 5\kappa/4$, then for u satisfying $\text{supp } \mathcal{F}u \subset \{k : |k|_\infty \leq 3N\}$

$$\|\Pi_N u\|_{\alpha-\kappa} \lesssim N^{-\kappa/2} \|u\|_\alpha.$$

If $\alpha < 0$ and $\text{supp } \mathcal{F}u \subset \{k : |k|_\infty \leq N\}$, then

$$\|e_N^{i_1 i_2 i_3} u\|_{\alpha-\kappa} \lesssim N^{-\kappa/2} \|u\|_\alpha.$$

Here all the constants we omit are independent of N .

Proof We have for $p > 1$ large enough

$$\|P_N u\|_{\alpha-\kappa} \lesssim \|P_N u\|_{B_{p,\infty}^\alpha} \lesssim \|u\|_{B_{p,\infty}^\alpha} \lesssim \|u\|_\alpha,$$

where in the first inequality we used Lemma 2.1 and in the second inequality we used that $\mathcal{F}^{-1}1_{|k|_\infty \leq N}\mathcal{F}$ is an L^p -multiplier. Similarly,

$$\|(I - P_N)u\|_{\alpha-\kappa} \lesssim N^{-\kappa/2} \|(I - P_N)u\|_{\alpha-\kappa/2} \lesssim N^{-\kappa/2} \|u\|_\alpha,$$

where in the first inequality we used Lemma 2.7 and in the second inequality we used the result for P_N . For (2), we have that for $\alpha > 5\kappa/4$

$$\begin{aligned} \|\Pi_N u\|_{\alpha-\kappa} &\lesssim N^{\alpha-5\kappa/4} \|\Pi_N u\|_{\kappa/4} \lesssim N^{\alpha-\kappa} \|\mathcal{F}^{-1}1_{k \in P^{i_1 i_2 i_3}} \mathcal{F}u\|_{\kappa/4} \\ &\lesssim N^{-\kappa/2} \|\mathcal{F}^{-1}1_{k \in P^{i_1 i_2 i_3}} \mathcal{F}u\|_{\alpha-\kappa/4} \lesssim N^{-\kappa/2} \|u\|_\alpha. \end{aligned}$$

Here in the first and third inequalities we used Lemma 2.7, in the second inequality we used that $\|e_N^{i_1 i_2 i_3}\|_{\kappa/4} \lesssim N^{\kappa/4}$ and in the last inequality we used a similar argument for P_N since $\mathcal{F}^{-1}1_{k \in P^{i_1 i_2 i_3}}\mathcal{F}$ is an L^p -multiplier. Similarly for $\alpha < 0$

$$\|e_N^{i_1 i_2 i_3} u\|_{\alpha-\kappa} \lesssim N^{\alpha-3\kappa/2} \|e_N^{i_1 i_2 i_3} u\|_{\kappa/2} \lesssim N^{\alpha-\kappa} \|u\|_{\kappa/2} \lesssim N^{-\kappa/2} \|u\|_\alpha.$$

Here we used $\text{supp } \mathcal{F}(e_N^{i_1 i_2 i_3} u) \subset \{k : |k| > N\}$ and Lemma 2.7 in the first inequality as well as Lemma 2.7 in the last inequality. Thus the results in (2) follows. \square

Now we prove several properties for the approximating semigroup $P_t^\varepsilon = \text{Exte}^{t\Delta_\varepsilon}$ such as smoothing effect, commutator estimate, which are parallel to the properties of the heat semigroup in Lemmas 2.4-2.6. In fact,

$$P_t^\varepsilon = \mathcal{F}^{-1}1_{|k|_\infty \leq N} e^{-t|k|^2 f(\varepsilon k)} \mathcal{F} = \mathcal{F}^{-1}1_{|k|_\infty \leq N} e^{-t|k|^2 f(\varepsilon k)} \varphi(\varepsilon k) \mathcal{F} = P_N \tilde{P}_t^\varepsilon,$$

with

$$\tilde{P}_t^\varepsilon := \mathcal{F}^{-1} e^{-t|k|^2 f(\varepsilon k)} \varphi(\varepsilon k) \mathcal{F},$$

where φ is a smooth function and equals 1 on $\{|x|_\infty \leq 1\}$ with $\text{supp}\varphi \subset \{|x| \leq 1.8\}$. Here we introduce \tilde{P}_t^ε for the following technique calculations. Then by similar arguments as in [GIP13, Lemma 47] we have the following results.

Lemma 3.2 Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$. Then for every $\delta \geq 0, \kappa > 0, t > 0$,

$$\begin{aligned} \|P_t^\varepsilon u\|_{\alpha+\delta-\kappa} &\lesssim t^{-\delta/2} \|u\|_\alpha, \\ \|(P_t^\varepsilon - P_t)u\|_{\alpha+\delta-\kappa} &\lesssim \varepsilon^{\kappa/2} t^{-\delta/2} \|u\|_\alpha. \end{aligned}$$

Here the constants we omit are independent of N .

Proof To obtain the first result, by Lemma 3.1 it suffices to prove that for every $\delta \geq 0$

$$\|\tilde{P}_t^\varepsilon u\|_{\alpha+\delta} \lesssim t^{-\delta/2} \|u\|_\alpha. \quad (3.1)$$

In the following we prove (3.1) and have that for $j \geq 0$

$$\begin{aligned} \|\Delta_j \tilde{P}_t^\varepsilon u\|_{L^\infty} &= \|\mathcal{F}^{-1} \theta_j \phi^\varepsilon \mathcal{F} u\|_{L^\infty} = \|\mathcal{F}^{-1} \theta_j \tilde{\theta}(2^{-j}\cdot) \phi^\varepsilon \mathcal{F} u\|_{L^\infty} \\ &\leq \|\mathcal{F}^{-1}(\phi^\varepsilon \tilde{\theta}(2^{-j}\cdot))\|_{L^1} \|\Delta_j u\|_{L^\infty}. \end{aligned}$$

Here and in the following

$$\phi^\varepsilon(\xi) = e^{-t|\xi|^2 f(\varepsilon\xi)} \varphi(\varepsilon\xi), \quad \phi(\xi) = e^{-t|\xi|^2}, \quad (3.2)$$

and $\tilde{\theta}$ is a smooth function supported in an annulus such that $\tilde{\theta}\theta = \theta$. Then we get that for $\delta \geq 0$

$$\begin{aligned} \|\mathcal{F}^{-1}(\phi^\varepsilon \tilde{\theta}(2^{-j}\cdot))\|_{L^1} &= \|\mathcal{F}^{-1}(\phi^\varepsilon(2^j\cdot) \tilde{\theta})\|_{L^1} \lesssim \|(1-\Delta)^2(\phi^\varepsilon(2^j\cdot) \tilde{\theta})\|_{L^1} \\ &\lesssim \sum_{0 \leq |m| \leq 4} 2^{j|m|} \|(D_m \phi^\varepsilon)(2^j\cdot)\|_{\mathcal{C}^{\text{supp}\tilde{\theta}}} \| \tilde{\theta} \|_{L^\infty} \lesssim \sum_{0 \leq |m| \leq 4} 2^{j|m|} \frac{1}{2^{j|m|} (2^j \sqrt{t})^\delta} \lesssim (2^j \sqrt{t})^{-\delta}. \end{aligned}$$

Here in the third inequality we used that $f(\varepsilon\xi) \geq c > 0$ and $|\varepsilon\xi| \lesssim 1$ on the support of ϕ^ε , which implies that for any multiindex m satisfying $|m| \leq 4$ and every $\delta \geq 0$ we have $|D_m \phi^\varepsilon(\xi)| \lesssim \frac{1}{|\xi|^{|m|+\delta t^{1/2}}}$. For $j = -1$ we can use Bernstein's lemma to obtain the estimate. Thus (3.1) follows.

For the second result we have

$$P_t^\varepsilon - P_t = P_N(\tilde{P}_t^\varepsilon - P_t) + (I - P_N)P_t.$$

By Lemmas 2.4 and 3.1 it is sufficient to consider $\tilde{P}_t^\varepsilon - P_t$. Since $\phi^\varepsilon(\xi) - \phi(\xi) = \varphi(\varepsilon\xi)(e^{-t|\xi|^2 f(\varepsilon\xi)} - e^{-t|\xi|^2}) + (\varphi(\varepsilon\xi) - 1)e^{-t|\xi|^2}$ and $|\varphi(\varepsilon\xi) - 1| \lesssim |\varepsilon\xi|^\eta$, $|f(\varepsilon\xi) - \pi^2| \lesssim |\varepsilon\xi|^\eta$ for every $0 < \eta < 1$, we obtain that for any multiindex m satisfying $|m| \leq 4$ and every $\delta \geq 0, 0 < \eta < 1$, we have $|D_m(\phi^\varepsilon - \phi)(\xi)| \leq \frac{(\varepsilon|\xi|)^\eta}{|\xi|^{|m|+\delta}t^{\delta/2}}$. Thus the second result follows by a similar argument as in the proof of (3.1). \square

In the following we prove a commutator estimate for P_t^ε . However, P_N does not commute with paraproduct and we can only obtain the following.

Lemma 3.3 Let $u \in \mathcal{C}^\alpha$ for some $\alpha < 1$ and $v \in \mathcal{C}^\beta$ for some $\beta \in \mathbb{R}$. Then for $\delta \geq \alpha + \beta$ and any $\kappa > 0$

$$\|P_t^\varepsilon \pi_{<}(u, v) - P_N \pi_{<}(u, \tilde{P}_t^\varepsilon v)\|_{\delta-\kappa} \lesssim t^{\frac{\alpha+\beta-\delta}{2}} \|u\|_\alpha \|v\|_\beta, \quad (3.3)$$

$$\|(P_t^\varepsilon - P_t) \pi_{<}(u, v) - P_N \pi_{<}(u, \tilde{P}_t^\varepsilon v) + \pi_{<}(u, P_t v)\|_{\delta-\kappa} \lesssim \varepsilon^{\kappa/2} t^{\frac{\alpha+\beta-\delta}{2}} \|u\|_\alpha \|v\|_\beta. \quad (3.4)$$

Here the constants we omit are independent of N .

Proof We have

$$P_t^\varepsilon \pi_{<}(u, v) - P_N \pi_{<}(u, \tilde{P}_t^\varepsilon v) = P_N (\tilde{P}_t^\varepsilon \pi_{<}(u, v) - \pi_{<}(u, \tilde{P}_t^\varepsilon v)).$$

By Lemma 3.1 it suffices to prove that

$$\|\tilde{P}_t^\varepsilon \pi_{<}(u, v) - \pi_{<}(u, \tilde{P}_t^\varepsilon v)\|_\delta \lesssim t^{\frac{\alpha+\beta-\delta}{2}} \|u\|_\alpha \|v\|_\beta. \quad (3.5)$$

In fact, we have that

$$\tilde{P}_t^\varepsilon \pi_{<}(u, v) - \pi_{<}(u, \tilde{P}_t^\varepsilon v) = \sum_{j=-1}^{\infty} (\tilde{P}_t^\varepsilon (S_{j-1} u \Delta_j v) - S_{j-1} u \tilde{P}_t^\varepsilon \Delta_j v),$$

and that the Fourier transform of $\tilde{P}_t^\varepsilon (S_{j-1} u \Delta_j v) - S_{j-1} u \tilde{P}_t^\varepsilon \Delta_j v$ has its support in a suitable annulus $2^j \mathcal{A}$. Let $\psi \in \mathcal{D}(\mathbb{R}^3)$ with support in an annulus \mathcal{A} be such that $\psi = 1$ on \mathcal{A} .

Thus by the same argument as in the proof of [CC13, Lemma A.1] we obtain that

$$\begin{aligned} & \|\tilde{P}_t^\varepsilon (S_{j-1} u \Delta_j v) - S_{j-1} u \tilde{P}_t^\varepsilon \Delta_j v\|_{L^\infty} \\ & \lesssim \sum_{\eta \in \mathbb{N}^d, |\eta|=1} \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1} \|\partial^\eta S_{j-1} u\|_{L^\infty} \|\Delta_j v\|_{L^\infty}, \end{aligned}$$

where ϕ^ε is introduced in (3.2). Now we have that

$$\begin{aligned} & \|x^\eta \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1} \\ & \leq 2^{-j} \|\mathcal{F}^{-1}((\partial^\eta \psi)(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1} + \|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta \phi^\varepsilon)\|_{L^1} \\ & = 2^{-j} \|\mathcal{F}^{-1}(\partial^\eta \psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^1} + \|\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta \phi^\varepsilon(2^j\cdot))\|_{L^1} \\ & \lesssim 2^{-j} \|(1 + |\cdot|^2) \mathcal{F}^{-1}(\partial^\eta \psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^\infty} + \|(1 + |\cdot|^2) \mathcal{F}^{-1}(\psi(\cdot)\partial^\eta \phi^\varepsilon(2^j\cdot))\|_{L^\infty} \\ & = 2^{-j} \|\mathcal{F}^{-1}((1 - \Delta)^2 (\partial^\eta \psi(\cdot)\phi^\varepsilon(2^j\cdot)))\|_{L^\infty} + \|\mathcal{F}^{-1}((1 - \Delta)^2 (\psi(\cdot)\partial^\eta \phi^\varepsilon(2^j\cdot)))\|_{L^\infty} \\ & \lesssim 2^{-j} \|(1 - \Delta)^2 (\partial^\eta \psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^1} + \|(1 - \Delta)^2 (\psi(\cdot)\partial^\eta \phi^\varepsilon(2^j\cdot))\|_{L^1} \\ & \lesssim 2^{-j} \sum_{0 \leq |m| \leq 4} (2^j)^{|m|} \frac{t^{-\mu} 2^{-2j\mu}}{(2^j)^{|m|}} + \sum_{|m| \leq 5} (2^j)^{|m|} \frac{t^{-\mu} 2^{-2j\mu}}{(2^j)^{|m|+1}} \\ & \lesssim 2^{-j} t^{-\mu} 2^{-2j\mu}, \end{aligned}$$

where in the fourth inequality we used that $|D^m \phi^\varepsilon(\xi)| \lesssim |\xi|^{-|m|} t^{-\mu} |\xi|^{-2\mu}$, $\mu \geq 0$, for any multi-index m satisfying $|m| \leq 5$. Hence we get that

$$\|\tilde{P}_t^\varepsilon(S_{j-1}u\Delta_j v) - S_{j-1}u\tilde{P}_t^\varepsilon\Delta_j v\|_{L^\infty} \lesssim t^{\frac{\alpha+\beta-\delta}{2}} 2^{j(\alpha+\beta-\delta)} 2^{-j(\alpha+\beta)} \|u\|_\alpha \|v\|_\beta,$$

which yields (3.5) by [BCD11, Lemma 2.69].

Moreover, we have

$$\begin{aligned} & (P_t^\varepsilon - P_t)\pi_{<}(u, v) - P_N\pi_{<}(u, \tilde{P}_t^\varepsilon v) + \pi_{<}(u, P_t v) \\ &= P_N[(\tilde{P}_t^\varepsilon - P_t)\pi_{<}(u, v) - \pi_{<}(u, (\tilde{P}_t^\varepsilon - P_t)v)] - (I - P_N)(P_t\pi_{<}(u, v) - \pi_{<}(u, P_t v)). \end{aligned}$$

The estimate for the second term can be obtained by Lemmas 2.5 and 3.1. By a similar argument as the proof of Lemma 3.2 we obtain that for any multiindex m satisfying $|m| \leq 5$ and every $\delta \geq 0, 0 < \eta < 1$, we have $|D_m(\phi^\varepsilon - \phi)(\xi)| \leq \frac{(\varepsilon|\xi|)^\eta}{|\xi|^{|m|+\delta}t^{\delta/2}}$. Thus (3.4) follows by a similar argument as in the proof of (3.5). Here ϕ^ε and ϕ are introduced in (3.2). \square

The continuity result for P_t^ε takes as follows.

Lemma 3.4 Let $u \in \mathcal{C}^{\alpha+\delta}$ for some $\alpha \in \mathbb{R}, 0 < \delta < 1$. Then for every $\varepsilon \in (0, 1), \kappa > 0, t > s > 0$

$$\|(P_t^\varepsilon - P_s^\varepsilon)u\|_{\alpha-\kappa} \lesssim (t-s)^{\delta/2} \|u\|_{\alpha+\delta}.$$

Here the constants are independent of N .

Proof We have $(P_t^\varepsilon - P_s^\varepsilon)u = P_N(\tilde{P}_t^\varepsilon - \tilde{P}_s^\varepsilon)u$. By Lemma 3.1 it suffices to prove that

$$\|(\tilde{P}_t^\varepsilon - \tilde{P}_s^\varepsilon)u\|_\alpha \lesssim (t-s)^{\delta/2} \|u\|_{\alpha+\delta}.$$

Since $|1 - e^{-(t-s)f(\varepsilon\xi)|\xi|^2}| \leq (t-s)^{\delta/2} |\xi|^\delta$, we obtain that for any multiindex m satisfying $|m| \leq 4$ and any $\delta \geq 0$, we have $|D_m(\phi_t^\varepsilon - \phi_s^\varepsilon)(\xi)| \lesssim \frac{(t-s)^{\delta/2} |\xi|^\delta}{|\xi|^{|m|}}$, where ϕ^ε is introduced in (3.2). Thus by a similar argument as in the proof of Lemma 3.2 the result follows. \square

4 Paracontrolled analysis for the approximating equations

Now for simplicity let $u^\varepsilon = \text{Ext}\Phi^\varepsilon$. Then we have the following equation:

$$u^\varepsilon(t) = P_t^\varepsilon \text{Ext}\Phi_0^\varepsilon - \int_0^t P_{t-s}^\varepsilon Q_N[(u^\varepsilon)^3 - (3C_0^\varepsilon - 9C_1^\varepsilon)u^\varepsilon] ds + \int_0^t P_{t-s}^\varepsilon P_N dW. \quad (4.1)$$

Therefore it suffices to prove the convergence result for solutions to (4.1). In this section we give a uniform estimate for solutions to (4.1) by using paracontrolled analysis.

In this section we fix $\delta, \beta, \kappa, \gamma > 0$ satisfying

$$2z - 1 \geq \delta > 2\kappa, \quad \beta > \frac{\delta}{2}, \quad \beta + \frac{\delta}{2} + \kappa < \gamma, \quad 5\kappa + 2\delta + 3\gamma < (2 - 3z - \beta) \wedge \left(\frac{3}{2} - 2z\right).$$

Paracontrolled analysis of solutions to (4.1)

Now we split (4.1) into the following three equations:

$$\begin{aligned} u_1^\varepsilon(t) &= \int_{-\infty}^t P_{t-s}^\varepsilon P_N dW, \\ u_2^\varepsilon(t) &= - \int_0^t P_{t-s}^\varepsilon Q_N [(u_1^\varepsilon)^{\diamond,3}] ds, \end{aligned}$$

and

$$\begin{aligned} u_3^\varepsilon(t) &= P_t^\varepsilon (\text{Ext} \Phi_0^\varepsilon - u_1^\varepsilon(0)) - \int_0^t P_{t-s}^\varepsilon \left[Q_N [6u_1^\varepsilon \diamond u_2^\varepsilon u_3^\varepsilon + 3u_1^\varepsilon (u_3^\varepsilon)^2 + 3u_1^\varepsilon \diamond (u_2^\varepsilon)^2 + (u_2^\varepsilon + u_3^\varepsilon)^3] \right. \\ &\quad \left. + P_N [3(u_1^\varepsilon)^{\diamond,2} \diamond (u_2^\varepsilon + u_3^\varepsilon) + 3e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2} \diamond (u_2^\varepsilon + u_3^\varepsilon) - 9\varphi^\varepsilon u^\varepsilon] \right] ds. \end{aligned} \tag{4.2}$$

Here

$$\begin{aligned} (u_1^\varepsilon)^{\diamond,2} &:= (u_1^\varepsilon)^2 - C_0^\varepsilon, \\ (u_1^\varepsilon)^{\diamond,3} &:= (u_1^\varepsilon)^3 - 3C_0^\varepsilon u_1^\varepsilon, \\ u_1^\varepsilon \diamond u_2^\varepsilon &:= u_2^\varepsilon u_1^\varepsilon, \\ u_1^\varepsilon \diamond (u_2^\varepsilon)^2 &:= (u_2^\varepsilon)^2 u_1^\varepsilon, \\ (u_1^\varepsilon)^{\diamond,2} \diamond u_2^\varepsilon &:= \pi_{<}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) + \pi_{>}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) + \pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) \\ &= u_2^\varepsilon (u_1^\varepsilon)^{\diamond,2} + 3(C_{11}^\varepsilon + \varphi_1^\varepsilon) u_1^\varepsilon, \\ (u_1^\varepsilon)^{\diamond,2} \diamond u_3^\varepsilon &:= \pi_{<}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) + \pi_{>}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) + \pi_{0,\diamond}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) \\ &= u_3^\varepsilon (u_1^\varepsilon)^{\diamond,2} + 3(C_{11}^\varepsilon + \varphi_1^\varepsilon)(u_2^\varepsilon + u_3^\varepsilon), \\ e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2} \diamond u_2^\varepsilon &:= \pi_{<}(u_2^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2}) + \pi_{>}(u_2^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2}) + \pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2}) \\ &= e_N^{i_1 i_2 i_3} u_2^\varepsilon (u_1^\varepsilon)^{\diamond,2} + 3(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3}) u_1^\varepsilon, \\ e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2} \diamond u_3^\varepsilon &:= \pi_{<}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} e_N^{i_1 i_2 i_3}) + \pi_{>}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} e_N^{i_1 i_2 i_3}) + \pi_{0,\diamond}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} e_N^{i_1 i_2 i_3}) \\ &= u_3^\varepsilon (u_1^\varepsilon)^{\diamond,2} e_N^{i_1 i_2 i_3} + 3(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3})(u_2^\varepsilon + u_3^\varepsilon), \\ \varphi^\varepsilon &:= \varphi_1^\varepsilon + \varphi_2^\varepsilon = \varphi_1^\varepsilon + \sum \varphi_2^{\varepsilon, i_1 i_2 i_3}, \end{aligned}$$

where $C_0^\varepsilon \in \mathbb{R}$, $C_{1i}^\varepsilon \in \mathbb{R}$, $\varphi_i^\varepsilon \in C((0, T]; \mathbb{R})$ are defined as in Section 6 below and there exists $\varphi_1 \in C((0, T]; \mathbb{R})$ such that for every $\rho > 0$ small enough $\sup_{t \in [0, T]} t^\rho |\varphi_1^\varepsilon - \varphi_1| \rightarrow 0$ and $\sup_{t \in [0, T]} t^\rho |\varphi_2^\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

In (4.2) the most difficult term to be handled is $u_3 \diamond u_1^{\diamond,2}$, which requires us to use paracontrolled ansatz and the commutator estimates. For this we introduce the following notations:

$$K^\varepsilon(t) := \int_0^t P_{t-s}^\varepsilon (u_1^\varepsilon)^{\diamond,2} ds, \quad \tilde{K}^\varepsilon(t) := \int_0^t \tilde{P}_{t-s}^\varepsilon (u_1^\varepsilon)^{\diamond,2} ds,$$

and

$$K_1^\varepsilon(t) := \int_0^t P_{t-s}^\varepsilon [e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2}] ds, \quad \tilde{K}_1^\varepsilon(t) := \int_0^t \tilde{P}_{t-s}^\varepsilon [e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2}] ds.$$

Also define

$$\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) := \pi_0(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) - C_{11}^\varepsilon - \varphi_1^\varepsilon$$

and

$$\pi_{0,\diamond}(K_1^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2}) := \pi_0(K_1^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2}) - C_{12}^{\varepsilon, i_1 i_2 i_3} - \varphi_2^{\varepsilon, i_1 i_2 i_3}.$$

Here we introduce \tilde{K}^ε and \tilde{K}_1^ε since Lemma 3.3 about the commutator estimates only holds for \tilde{P}_t^ε , not for P_t^ε . Now we introduce the following notations: for $T > 0$

$$\begin{aligned} C_W^\varepsilon(T) := & \sup_{t \in [0, T]} [\|u_1^\varepsilon\|_{-1/2-\delta/2} + \|(u_1^\varepsilon)^{\diamond,2}\|_{-1-\delta/2} + \|u_2^\varepsilon\|_{1/2-\delta} + \|\pi_0(u_2^\varepsilon, u_1^\varepsilon)\|_{-\delta} \\ & + \|\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond,2})\|_{-1/2-\delta} + \|\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2})\|_{-\delta}] + \|u_2^\varepsilon\|_{C_T^{1/4-\delta/2-\kappa/2} C^{\kappa/2}}, \end{aligned}$$

and

$$\begin{aligned} E_W^\varepsilon(T) := & \sup_{t \in [0, T]} [\|(u_1^\varepsilon)^{\diamond,2} e_N^{i_1 i_2 i_3}\|_{-1-\delta/2} + \|\pi_0(u_2^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon)\|_{-\delta} + \|\pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})\|_{-1/2-\delta} \\ & + \|\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})\|_{-\delta} + \|\pi_0(K_1^\varepsilon, (u_1^\varepsilon)^{\diamond,2})\|_{-\delta} + \|\pi_{0,\diamond}(K_1^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})\|_{-\delta}]. \end{aligned}$$

In the following we write C_W^ε and E_W^ε for simplicity if there's no confusion. Here E_W^ε appears as an error term for the lattice approximations, which goes to 0 in probability (see Section 6.2). Lemma 3.2 and (3.1) imply that for $t \in [0, T]$

$$\|K^\varepsilon(t)\|_{1-\delta} + \|\tilde{K}^\varepsilon(t)\|_{1-\delta} \lesssim C_W^\varepsilon, \quad (4.3)$$

and

$$\|K_1^\varepsilon(t)\|_{1-\delta} + \|\tilde{K}_1^\varepsilon(t)\|_{1-\delta} \lesssim E_W^\varepsilon. \quad (4.4)$$

Now we write the paracontrolled ansatz as follows:

$$u_3^\varepsilon = -3P_N[\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, \tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon)] + u^{\varepsilon, \#}$$

with $u^{\varepsilon, \#}(t) \in \mathcal{C}^{1+\beta}$ for $t > 0$. Then Lemma 2.2 yields that for $t > 0$

$$\|u_3^\varepsilon(t)\|_{1-\delta} \lesssim \|u_2^\varepsilon(t) + u_3^\varepsilon(t)\|_\gamma (C_W^\varepsilon + E_W^\varepsilon) + \|u^{\varepsilon, \#}(t)\|_{1-\delta}. \quad (4.5)$$

Furthermore, u_3^ε solves (4.2) if and only if $u^{\varepsilon, \#}$ solves the following equation:

$$\begin{aligned} u^{\varepsilon, \#}(t) = & P_t^\varepsilon(\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)) - \int_0^t P_{t-s}^\varepsilon \left[Q_N[6u_1^\varepsilon \diamond u_2^\varepsilon u_3^\varepsilon + 3u_1^\varepsilon (u_3^\varepsilon)^2 + 3u_1^\varepsilon \diamond (u_2^\varepsilon)^2 + (u_2^\varepsilon + u_3^\varepsilon)^3] \right. \\ & \left. + 3P_N[(\pi_{>} + \pi_{0,\diamond})(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})] - 9\varphi^\varepsilon u^\varepsilon \right] ds \\ & - 3 \int_0^t P_{t-s}^\varepsilon P_N[\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})] ds + 3P_N[\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, \tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon)] \\ := & P_t^\varepsilon(\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)) + \int_0^t P_{t-s}^\varepsilon [Q_N \phi_1^{\varepsilon, \#} + P_N \phi_2^{\varepsilon, \#} + 9\varphi^\varepsilon u^\varepsilon] ds + F^\varepsilon, \end{aligned} \quad (4.6)$$

where F^ε represents the last two terms.

In the following we give estimates of terms on the right hand side of (4.6).

Estimates of $\phi_i^{\varepsilon, \#}$

First we prove an estimate for $\phi_1^{\varepsilon, \sharp}$.

Proposition 4.2 For $\phi_1^{\varepsilon, \sharp}$ defined in (4.6), the following estimate holds:

$$\|Q_N \phi_1^{\varepsilon, \sharp}\|_{-1/2-\delta/2-2\kappa} \lesssim C(C_W^\varepsilon, E_W^\varepsilon)(1 + \|u_3^\varepsilon\|_{1/2+\delta}(\|u_3^\varepsilon\|_\gamma + 1) + \|u_3^\varepsilon\|_\gamma^3).$$

Here the constant we omit is independent of N .

Proof Since

$$\Pi_N[u_3^\varepsilon u_2^\varepsilon u_1^\varepsilon] = P_N[u_3^\varepsilon e_N^{i_1 i_2 i_3} u_2^\varepsilon u_1^\varepsilon],$$

we have for $\delta > 2\kappa$

$$\begin{aligned} & \|\Pi_N[u_3^\varepsilon u_2^\varepsilon u_1^\varepsilon]\|_{-1/2-\delta/2-2\kappa} \lesssim \|u_3^\varepsilon u_2^\varepsilon u_1^\varepsilon e_N^{i_1 i_2 i_3}\|_{-1/2-\delta/2-\kappa} \\ & \lesssim (\|e_N^{i_1 i_2 i_3} u_1^\varepsilon\|_{-1/2-\delta/2-\kappa} \|u_2^\varepsilon\|_{1/2-\delta} + \|\pi_0(u_2^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon)\|_{-\delta}) \|u_3^\varepsilon\|_{1/2+\delta} \\ & \lesssim (N^{-\kappa/2} \|u_2^\varepsilon\|_{1/2-\delta} \|u_1^\varepsilon\|_{-1/2-\delta/2} + \|\pi_0(u_2^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon)\|_{-\delta}) \|u_3^\varepsilon\|_{1/2+\delta}, \end{aligned}$$

where we used Lemma 3.1 in the first and last inequalities as well as Lemma 2.2 in the second inequality.

Using the paraproduct one has

$$\begin{aligned} \Pi_N[u_1^\varepsilon (u_3^\varepsilon)^2] &= P_N[u_1^\varepsilon e_N^{i_1 i_2 i_3} (u_3^\varepsilon)^2] \\ &= P_N[\pi_<((u_3^\varepsilon)^2, e_N^{i_1 i_2 i_3} u_1^\varepsilon) + \pi_0((u_3^\varepsilon)^2, e_N^{i_1 i_2 i_3} u_1^\varepsilon) + \pi_>((u_3^\varepsilon)^2, e_N^{i_1 i_2 i_3} u_1^\varepsilon)] \\ &= P_N[\pi_<((u_3^\varepsilon)^2, e_N^{i_1 i_2 i_3} u_1^\varepsilon) + \pi_0(\pi_0(u_3^\varepsilon, u_3^\varepsilon), e_N^{i_1 i_2 i_3} u_1^\varepsilon) \\ & \quad + \pi_>((u_3^\varepsilon)^2, e_N^{i_1 i_2 i_3} u_1^\varepsilon) + 2C(u_3^\varepsilon, u_3^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon) + 2u_3^\varepsilon \pi_0(u_3^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon)]. \end{aligned}$$

Here $C(u_3^\varepsilon, u_3^\varepsilon, e_N^{i_1 i_2 i_3} u_1^\varepsilon)$ is the trilinear operator as defined in Lemma 2.3. Then by using Lemmas 2.2, 2.3 and 3.1 we obtain

$$\|\Pi_N[(u_3^\varepsilon)^2 u_1^\varepsilon]\|_{-1/2-\delta/2-2\kappa} \lesssim N^{-\kappa/2} \|u_3^\varepsilon\|_{1/2+\delta} \|u_3^\varepsilon\|_\gamma \|u_1^\varepsilon\|_{-1/2-\delta/2}. \quad (4.7)$$

Moreover, by a similar argument as for (4.7) we have

$$\begin{aligned} & \|\Pi_N[(u_2^\varepsilon)^2 \diamond u_1^\varepsilon]\|_{-1/2-\delta/2-2\kappa} \\ & \lesssim N^{-\kappa/2} \|u_2^\varepsilon\|_{1/2-\delta}^2 \|u_1^\varepsilon\|_{-1/2-\delta/2} + \|u_2^\varepsilon\|_{1/2-\delta} \|\pi_0(u_2^\varepsilon, u_1^\varepsilon e_N^{i_1 i_2 i_3})\|_{-\delta}. \end{aligned}$$

Furthermore, Lemma 3.1 implies that

$$\|Q_N[(u_2^\varepsilon + u_3^\varepsilon)^3]\|_{\gamma-\kappa} \lesssim \|u_2^\varepsilon + u_3^\varepsilon\|_\gamma^3.$$

Estimates for the terms containing P_N can be obtained similarly. Hence the result follows from the above estimates. \square

Now we consider $\phi_2^{\varepsilon, \sharp}$. To prove an estimate for $\pi_{0, \diamond}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$ we have to use the paracontrolled ansatz. However, the Fourier cutoff operator P_N does not commute with the paraproduct. Here we follow the random operator technique from [GP15, Lemma 8.16] and prove the following result.

Lemma 4.3 Let $\alpha + \beta + \gamma > 0$, $\beta + \gamma < 0$, assume that $\alpha \in (0, 1)$ and let $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, $h \in \mathcal{C}^\gamma$. Define the operators

$$A_N^1(g, h)(f) := \pi_0((I - P_N)\pi_{<}(f, P_N g), h),$$

and

$$A_N^2(g, h)(f) := \pi_0(P_N \pi_{<}(f, (P_{3N} - P_N)g), h).$$

Then for all $\eta < 0$

$$\begin{aligned} & \|\pi_0(P_N \pi_{<}(f, P_{3N}g), h) - f \pi_0(P_N g, h)\|_\eta \\ & \lesssim \|f\|_\alpha \|P_N g\|_\beta \|h\|_\gamma + (\|A_N^1(g, h)\|_{L(\mathcal{C}^\alpha, \mathcal{C}^\eta)} + \|A_N^2(g, h)\|_{L(\mathcal{C}^\alpha, \mathcal{C}^\eta)}) \|f\|_\alpha. \end{aligned}$$

Here the constant we omit is independent of N and $L(\mathcal{C}^\alpha, \mathcal{C}^\eta)$ denotes the space of bounded operators between \mathcal{C}^α and \mathcal{C}^η , equipped with the operator norm.

Proof We have that

$$\pi_0(P_N \pi_{<}(f, P_{3N}g), h) = A_N^2(g, h)(f) + \pi_0(\pi_{<}(f, P_N g), h) - A_N^1(g, h)(f).$$

Thus the result follows from Lemma 2.3. \square

By using Lemma 4.3 we have the following estimate for $\phi_2^{\varepsilon, \sharp}$.

Proposition 4.4 For $\phi_2^{\varepsilon, \sharp}$ defined in (4.6), the following estimate holds:

$$\|P_N \phi_2^{\varepsilon, \sharp}\|_{-1/2-2\delta-\kappa} \lesssim C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)(1 + \|u_3^\varepsilon\|_{1/2+\delta} + \|u^{\varepsilon, \sharp}\|_{1+\beta}).$$

with

$$\begin{aligned} A_N(T) := & \|A_N^1(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{C_T L(\mathcal{C}^{1-\delta}, \mathcal{C}^{-1/2-2\delta})} \\ & + \|A_N^2(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{C_T L(\mathcal{C}^{1-\delta}, \mathcal{C}^{-1/2-2\delta})} \end{aligned}$$

and

$$\begin{aligned} D_N(T) := & \sup_{t \in [0, T]} (\|\pi_0((I - P_N)\pi_{<}(u_2^\varepsilon, K^\varepsilon + K_1^\varepsilon), (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{-\delta} \\ & + \|\pi_0(P_N \pi_{<}(u_2^\varepsilon, (P_{3N} - P_N)(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon)), (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{-\delta}). \end{aligned}$$

Proof First we consider $\pi_0(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$. By the paracontrolled ansatz we obtain

$$\begin{aligned} & \pi_0(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}) \\ = & -3\pi_0(P_N[\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, P_{3N}(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon))], (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}) + \pi_0(u^{\varepsilon, \sharp}, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}). \end{aligned}$$

Here in the equality we used that $P_{3N}(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon) = \tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon$. Then by using Lemma 4.3 and that $P_N(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon) = K^\varepsilon + K_1^\varepsilon$, we obtain that for $\beta > \delta/2$

$$\begin{aligned} & \|\pi_{0, \diamond}(u_3^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{-1/2-2\delta} \\ \lesssim & \|u_2^\varepsilon + u_3^\varepsilon\|_{1/2-\delta} [\|K^\varepsilon + K_1^\varepsilon\|_{1-\delta} \|(u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}\|_{-1-\delta/2} + \|\pi_{0, \diamond}(K^\varepsilon + K_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} \\ & + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})\|_{-\delta}] + A_N \|u_3^\varepsilon\|_{1-\delta} + D_N + \|u^{\varepsilon, \sharp}\|_{1+\beta} \|(u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}\|_{-1-\delta/2}. \end{aligned}$$

The estimate for $\pi_{>}(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$ can be obtained by Lemma 2.2. Thus the result follows from (4.3), (4.4) and (4.5). \square

Remark 4.5 (i) Compared to the random operator technique used for the KPZ equation in [GP15], we need to introduce a new random operator part A_N^2 in the dynamical Φ_3^4 model case since $P_N(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon) \neq \tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon$.

(ii) In this case to use the random operator technique, we need that $u_3^\varepsilon \in \mathcal{C}^{1/2+\beta+\kappa}$. However, the best regularity we can obtain for u_2^ε is $\mathcal{C}^{1/2-\delta}$. Thus for the error terms including u_2^ε we have to bound them directly by stochastic calculations, which correspond to D_N (see Section 6.4).

Estimates of F^ε

We now turn to F^ε : We divide F^ε into two parts:

$$\begin{aligned} & \|F^\varepsilon(t)\|_{1+\beta} \\ & \lesssim \left\| \int_0^t P_{t-s}^\varepsilon \pi_{<}(u_2^\varepsilon(s) + u_3^\varepsilon(s) - (u_2^\varepsilon(t) + u_3^\varepsilon(t)), (u_1^\varepsilon)^{\diamond,2}(s) + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2}(s)) ds \right\|_{1+\beta} \\ & \quad + \left\| \int_0^t P_{t-s}^\varepsilon \pi_{<}(u_2^\varepsilon(t) + u_3^\varepsilon(t), (u_1^\varepsilon)^{\diamond,2}(s) + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2}(s)) ds - P_N \pi_{<}(u_2^\varepsilon(t) + u_3^\varepsilon(t), \tilde{K}^\varepsilon(t) + \tilde{K}_1^\varepsilon(t)) \right\|_{1+\beta} \\ & := I_1 + I_2. \end{aligned}$$

Estimate of I_2 can be obtained by Lemma 3.3:

$$I_2 \lesssim t^{\frac{\gamma-\beta-\frac{\delta}{2}-\kappa}{2}} \|u_2^\varepsilon(t) + u_3^\varepsilon(t)\|_\gamma (C_W^\varepsilon + E_W^\varepsilon), \quad (4.8)$$

where by the condition on β we have $\frac{\gamma-\beta-\frac{\delta}{2}-\kappa}{2} > 0$.

For I_1 we will use the regularity of $u_2^\varepsilon + u_3^\varepsilon$ with respect to time to control it. Lemmas 2.2 and 3.2 yield that for $3\delta/4 + \beta/2 + 2\kappa < 1/4$

$$\begin{aligned} I_1 & \lesssim \int_0^t (t-s)^{-1-\frac{\delta/2+\beta+2\kappa}{2}} \|(u_1^\varepsilon)^{\diamond,2}(s) + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2}(s)\|_{-1-\delta/2} \|u_2^\varepsilon(t) + u_3^\varepsilon(t) - u_2^\varepsilon(s) - u_3^\varepsilon(s)\|_{\kappa/2} ds \\ & \lesssim (C_W^\varepsilon + E_W^\varepsilon) (C_W^\varepsilon + \int_0^t (t-s)^{-1-\frac{\delta/2+\beta+2\kappa}{2}} \|u_3^\varepsilon(t) - u_3^\varepsilon(s)\|_{\kappa/2} ds), \end{aligned}$$

and we note that by (4.2), Lemmas 3.2 and 3.4 that for $t > s > 0$

$$\begin{aligned} & \|u_3^\varepsilon(t) - u_3^\varepsilon(s)\|_{\kappa/2} \\ & \lesssim \|(P_{\frac{t}{2}}^\varepsilon - P_{\frac{s}{2}}^\varepsilon)(P_{\frac{t}{2}}^\varepsilon + P_{\frac{s}{2}}^\varepsilon)(\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0))\|_{\frac{\kappa}{2}} + \left\| \int_0^s (P_{t-r}^\varepsilon - P_{s-r}^\varepsilon) G^\varepsilon(r) dr \right\|_{\frac{\kappa}{2}} + \left\| \int_s^t P_{t-r}^\varepsilon G^\varepsilon(r) dr \right\|_{\frac{\kappa}{2}} \\ & \lesssim (t-s)^{b_0} s^{-\frac{z+2\kappa+2b_0}{2}} \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + (t-s)^b \int_0^s (s-r)^{-\frac{1+\delta+\kappa+2b}{2}} \|G^\varepsilon(r)\|_{-1-\delta} dr \\ & \quad + (t-s)^{b_1} \left(\int_s^t (t-r)^{-\frac{1+\delta+\kappa}{2(1-b_1)}} \|G^\varepsilon(r)\|_{-1-\delta}^{1-b_1} dr \right)^{1-b_1}, \end{aligned}$$

where in the last inequality for the third term we used Hölder's inequality. Here $\frac{\delta}{2} + \beta + 2\kappa < 2b_0 < 2 - z - 2\kappa$, $\frac{\delta}{2} + \beta + 2\kappa < 2b < 1 - \kappa - \delta$, $\frac{1}{2}(\frac{\delta}{2} + \beta + 2\kappa) < b_1 < 1 - \frac{3(\gamma+z+\kappa)}{2} < \frac{1}{2}(1 - \delta - \kappa)$ and

$$G^\varepsilon = Q_N [3u_1^\varepsilon \diamond (u_2^\varepsilon)^2 + 6u_1^\varepsilon \diamond u_2^\varepsilon u_3^\varepsilon + 3u_1^\varepsilon (u_3^\varepsilon)^2 + 3(u_1^\varepsilon)^{\diamond,2} \diamond u_2^\varepsilon + 3(u_1^\varepsilon)^{\diamond,2} \diamond u_3^\varepsilon + (u_2^\varepsilon + u_3^\varepsilon)^3] - 9\varphi^\varepsilon u^\varepsilon.$$

Moreover, by Propositions 4.2 and 4.4 and Lemma 2.2 one has the following estimate

$$\|G^\varepsilon(t)\|_{-1-\delta} \lesssim C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)U_0^\varepsilon(t) + t^{-\rho}(C_W^\varepsilon + \|u_3^\varepsilon(t)\|_\gamma). \quad (4.9)$$

Here and in the following

$$U_0^\varepsilon(t) = 1 + \|u_3^\varepsilon(t)\|_{1/2+\delta}(\|u_3^\varepsilon(t)\|_\gamma + 1) + \|u_3^\varepsilon(t)\|_\gamma^3 + \|u^{\varepsilon,\sharp}(t)\|_{1+\beta}.$$

Thus we obtain that

$$\begin{aligned} I_1 &\lesssim (C_W^\varepsilon + E_W^\varepsilon) \left(C_W^\varepsilon + t^{-\frac{\delta/2+\beta+z}{2}-2\kappa} \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \right. \\ &\quad + \int_0^t \int_r^t (t-s)^{-1-\frac{\delta/2+\beta+2\kappa}{2}+b} (s-r)^{-\frac{1+\delta+\kappa+2b}{2}} ds \|G^\varepsilon(r)\|_{-1-\delta} dr \\ &\quad + \left(\int_0^t (t-s)^{-1-\frac{\delta/2+\beta+2\kappa}{2}+b_1} ds \right)^{b_1} \left(\int_0^t \int_0^r (t-s)^{-1-\frac{\delta/2+\beta+2\kappa}{2}+b_1} (t-r)^{-\frac{1+\delta+\kappa}{2(1-b_1)}} \right. \\ &\quad \left. \|G^\varepsilon(r)\|_{-1-\delta}^{\frac{1}{1-b_1}} ds dr \right)^{1-b_1} \Big), \end{aligned}$$

where for the last term we used Hölder's inequality. Then by changing variable $s = r + (t-r)\sigma$ for the third term and using (4.9) we have

$$\begin{aligned} I_1 &\lesssim (C_W^\varepsilon + E_W^\varepsilon) t^{-\frac{\delta/2+\beta+z}{2}-2\kappa} \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N) \\ &\quad + C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N) \int_0^t (t-r)^{-\frac{1}{2}-\frac{3\delta/2+\beta+3\kappa}{2}} (U_0^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon\|_\gamma) dr \\ &\quad + C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N) \left[\int_0^t (t-r)^{-\frac{1+\delta+\kappa}{2(1-b_1)}} (U_0^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon\|_\gamma)^{\frac{1}{1-b_1}} dr \right]^{1-b_1}. \end{aligned} \quad (4.10)$$

Combining (4.8) and (4.10) we could control $\|F^\varepsilon\|_{1+\beta}$ by the right hand side of (4.8) and (4.10).

In the following we will bound $\|u_3^\varepsilon\|_{1/2+\delta}$ and $\|u_3^\varepsilon\|_\gamma$. Estimates for these two terms are much easier. We do not need to use Lemma 3.3 and can obtain the following estimates for $\int_0^t P_{t-s}^\varepsilon P_N [\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2})] ds$ by Lemmas 2.2 and 3.2 directly:

$$\begin{aligned} &\left\| \int_0^t P_{t-s}^\varepsilon P_N [\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2})] ds \right\|_{1/2+\delta} \\ &\lesssim (C_W^\varepsilon + E_W^\varepsilon) \int_0^t (t-s)^{-\frac{3+3\delta+\kappa}{4}} \|u_3^\varepsilon\|_\gamma ds + C(C_W^\varepsilon, E_W^\varepsilon), \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} &\left\| \int_0^t P_{t-s}^\varepsilon P_N [\pi_{<}(u_2^\varepsilon + u_3^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond,2})] ds \right\|_\gamma \\ &\lesssim (C_W^\varepsilon + E_W^\varepsilon) \int_0^t (t-r)^{-\frac{1+\delta/2+\kappa+\gamma}{2}} \|u_3^\varepsilon\|_\gamma dr + C(C_W^\varepsilon, E_W^\varepsilon). \end{aligned} \quad (4.12)$$

Uniform estimates of the solutions to (4.2)

Now we introduce the following random times: Define for any $L \geq 1$

$$\tau_L^\varepsilon := \inf\{t \geq 0 : \|u^\varepsilon(t)\|_{-z} \geq L\} \wedge L \quad \rho_L^\varepsilon := \inf\{t \geq 0 : C_W^\varepsilon(t) + E_W^\varepsilon(t) + A_N(t) + D_N(t) \geq L\}.$$

Proposition 4.6 For any $L, L_1 \geq 1$, we have

$$\sup_{t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]} \left(t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon, \#}(t)\|_{1+\beta} + t^{\frac{1/2+\delta+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_{1/2+\delta} + t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_\gamma \right) \lesssim C(L, L_1).$$

Moreover, before $\tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon$ one has that $u_3^\varepsilon(t)$ depends in a Lipschitz continuous way on the data $\text{Ext}\Phi_0^\varepsilon$ and terms in $(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)$. Here we consider $u_3^\varepsilon(t)$ with respect to $\|\cdot\|_{-z}$ norm and the Lipschitz constant can be chosen uniformly over $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$.

Proof It follows from Propositions 4.2, 4.4 and (4.8), (4.10) that for $\frac{3(\gamma+z+\kappa)}{2} < 1$ and $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$,

$$\begin{aligned} & t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon, \#}(t)\|_{1+\beta} \\ & \lesssim C \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + t^{\frac{3(\gamma+z+\kappa)}{2}} C \int_0^t (t-r)^{-\frac{3}{4}-\delta-\frac{\beta}{2}-\kappa} (r^{-\frac{3(\gamma+z+\kappa)}{2}} U^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon(r)\|_\gamma) dr + C \\ & \quad + Ct^{\frac{3(\gamma+z+\kappa)}{2}} \int_0^t (t-r)^{-\frac{1}{2}-\frac{3\delta/2+\beta+3\kappa}{2}} (r^{-\frac{3(\gamma+z+\kappa)}{2}} U^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon(r)\|_\gamma) dr \\ & \quad + t^{\frac{3(\gamma+z+\kappa)}{2(1-b_1)}} \int_0^t (t-r)^{-\frac{1+\delta+\kappa}{2(1-b_1)}} (r^{-\frac{3(\gamma+z+\kappa)}{2}} U^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon(r)\|_\gamma)^{\frac{1}{1-b_1}} dr + t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_\gamma. \end{aligned} \quad (4.13)$$

Here and in the following $C = C(L_1)$ and $U^\varepsilon(r) = r^{\frac{3(\gamma+z+\kappa)}{2}} U_0^\varepsilon(r)$. A similar argument as that for (4.13) and using (4.11), (4.12) one also has that for $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$ and $0 < 4\kappa < \frac{3}{2} - 2z - 2\delta - 3\gamma$

$$\begin{aligned} & t^{\frac{1/2+\delta+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_{1/2+\delta} \\ & \lesssim \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + t^{\frac{1/2+\delta+z+\kappa}{2}} C \int_0^t (t-r)^{-\frac{1+3\delta+2\kappa}{2}} (r^{-\frac{3(\gamma+z+\kappa)}{2}} U^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon(r)\|_\gamma) dr \\ & \quad + C + Ct^{\frac{1/2+\delta+z+\kappa}{2}} \int_0^t (t-r)^{-\frac{3+3\delta+\kappa}{4}} r^{-\frac{\gamma+z+\kappa}{2}} r^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(r)\|_\gamma dr, \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} & t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_\gamma \\ & \lesssim \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + t^{\frac{\gamma+z+\kappa}{2}} C \int_0^t (t-r)^{-\frac{1}{4}-\delta-\frac{\gamma}{2}-\kappa} (r^{-\frac{3(\gamma+z+\kappa)}{2}} U^\varepsilon(r) + r^{-\rho} \|u_3^\varepsilon(r)\|_\gamma) dr \\ & \quad + C + Ct^{\frac{\gamma+z+\kappa}{2}} \int_0^t (t-r)^{-\frac{1+\delta/2+\gamma+\kappa}{2}} r^{-\frac{(\gamma+z+\kappa)}{2}} r^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(r)\|_\gamma dr. \end{aligned} \quad (4.15)$$

Since $\frac{1/2+\delta+\kappa+z}{2} \leq \gamma + z + \kappa$, combining with (4.13-4.15) we get that by Hölder's inequality and Bihari's inequality there exists some T_0 (depending on L_1) such that

$$\sup_{t \in [0, T_0]} \left(t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon, \#}(t)\|_{1+\beta} + t^{\frac{1/2+\delta+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_{1/2+\delta} + t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_\gamma \right) \lesssim C(L, L_1),$$

which combined with Propositions 4.2 and 4.4 implies that

$$\sup_{t \in [0, T_0]} t^{3(\gamma+z+\kappa)/2} \|Q_N \phi_1^{\varepsilon, \sharp} + P_N \phi_2^{\varepsilon, \sharp}\|_{-1/2-2\delta-\kappa} \lesssim C(L, L_1). \quad (4.16)$$

Moreover, by (4.2) and Lemma 2.2 we obtain that for $t \in [0, T_0]$ and $5\kappa + 3\gamma + 2\delta < 3/2 - 2z$

$$\begin{aligned} \|u_3^\varepsilon(t)\|_{-z} &\lesssim C + \|\text{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + \int_0^t s^{-\rho} \|u_3^\varepsilon(s)\|_\gamma ds \\ &+ \int_0^t [(t-s)^{-\frac{1/2+2\delta+2\kappa-z}{2}} \vee 1] s^{-\frac{3(\gamma+z+\kappa)}{2}} s^{\frac{3(\gamma+z+\kappa)}{2}} \|Q_N \phi_1^{\varepsilon, \sharp} + P_N \phi_2^{\varepsilon, \sharp}\|_{-1/2-2\delta-\kappa} ds \\ &+ \int_0^t (t-s)^{-\frac{1+\delta/2+\kappa-z}{2}} s^{-\frac{\gamma+\kappa+z}{2}} ds \sup_{s \in [0, t]} s^{\frac{\gamma+\kappa+z}{2}} \|u_2^\varepsilon + u_3^\varepsilon\|_\gamma \\ &\lesssim C(L, L_1). \end{aligned}$$

Here in the last inequality we used (4.16). Moreover, similar arguments as above imply that $u_3^\varepsilon(t)$ before T_0 depends in a Lipschitz continuous way on the data $\text{Ext}\Phi_0^\varepsilon$ and terms in $(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)$. The Lipschitz constant can be chosen uniformly for $t \in [0, T_0]$. Furthermore, we can extend the time from T_0 to $\tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon$ as we did in [ZZ15]. \square

5 Proof of main result

In [CC13] it is proved that the solution to (1.1) can be obtained as a limit of solutions $\bar{\Phi}^\varepsilon$ to the following equation:

$$\begin{aligned} d\bar{\Phi}^\varepsilon &= \Delta \bar{\Phi}^\varepsilon dt + P_N dW - (\bar{\Phi}^\varepsilon)^3 dt + (3\bar{C}_0^\varepsilon - 9\bar{C}_1^\varepsilon) \bar{\Phi}^\varepsilon dt, \\ \bar{\Phi}^\varepsilon(0) &= \Phi_0. \end{aligned}$$

Here \bar{C}_0^ε and \bar{C}_1^ε are defined in Section 6.1 below. For this equation we can also divide it into three equations and define $\bar{u}_1^\varepsilon, \bar{u}_2^\varepsilon, \bar{u}_3^\varepsilon, \bar{K}^\varepsilon$ and other terms similarly as $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, K^\varepsilon$ and the associated terms, respectively. For $L \geq 0$ define $\tau_L := \inf\{t \geq 0 : \|\Phi(t)\|_{-z} \geq L\} \wedge L$. Then τ_L increases to the explosion time τ as $L \rightarrow \infty$. Moreover, define $\bar{\tau}_L^\varepsilon := \inf\{t \geq 0 : \|\bar{\Phi}^\varepsilon(t)\|_{-z} \geq L\} \wedge L$ and $\bar{\rho}_L^\varepsilon := \inf\{t \geq 0 : \bar{C}_W^\varepsilon(t) \geq L\}$. Here \bar{C}_W^ε defined similarly as C_W^ε with u_i^ε replaced by corresponding \bar{u}_i^ε . A similar argument as in the proof in [CC13] implies that for $L, L_3, L_4 \geq 1$

$$\sup_{t \in [0, \tau_L \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|\bar{\Phi}^\varepsilon(t) - \Phi(t)\|_{-z} \xrightarrow{P} 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.1)$$

Here Φ is the solution to (1.2). Define

$$\begin{aligned} \delta C_W^\varepsilon &:= \sup_{t \in [0, T]} [\|u_1^\varepsilon - \bar{u}_1^\varepsilon\|_{-1/2-\delta/2} + \|(u_1^\varepsilon)^{\diamond, 2} - (\bar{u}_1^\varepsilon)^{\diamond, 2}\|_{-1-\delta/2} + \|u_2^\varepsilon - \bar{u}_2^\varepsilon\|_{1/2-\delta} \\ &+ \|\pi_0(u_2^\varepsilon, u_1^\varepsilon) - \pi_0(\bar{u}_2^\varepsilon, \bar{u}_1^\varepsilon)\|_{-\delta} + \|\pi_{0, \diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond, 2}) - \pi_{0, \diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond, 2})\|_{-1/2-\delta} \\ &+ \|\pi_{0, \diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond, 2}) - \pi_{0, \diamond}(\bar{K}^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond, 2})\|_{-\delta}] + \|u_2^\varepsilon - \bar{u}_2^\varepsilon\|_{C_T^{1/4-\delta-\kappa/2} C^{\kappa/2}}. \end{aligned}$$

In Section 6 we will prove that $\delta C_W^\varepsilon \rightarrow^P 0$, $E_W^\varepsilon \rightarrow^P 0$, $A_N \rightarrow^P 0$ and $D_N \rightarrow^P 0$ as $\varepsilon \rightarrow 0$. Then using the estimates for $P_t^\varepsilon - P_t$ obtained in Section 3 and by similar arguments as in Section 4 we have that for $L, L_i \geq 1$ with $i = 1, 2, 3, 4$ that

$$\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon(t) - \bar{\Phi}^\varepsilon(t)\|_{-z} \rightarrow^P 0, \quad \varepsilon \rightarrow 0. \quad (5.2)$$

Here $E_W^\varepsilon, A_N, D_N$ appear as error terms for the lattice approximations. Then (5.1) and (5.2) imply that

$$\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon(t) - \Phi(t)\|_{-z} \rightarrow^P 0, \quad \varepsilon \rightarrow 0. \quad (5.3)$$

Moreover, we have the following estimate: for each $\epsilon > 0$

$$\begin{aligned} & P\left(\sup_{t \in [0, \tau_L]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\right) \\ & \leq P\left(\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\right) + P(\tau_L \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon > \tau_{L_1}^\varepsilon) \\ & \quad + P(\tau_L \wedge \bar{\rho}_{L_3}^\varepsilon > \bar{\tau}_{L_4}^\varepsilon) + P(\tau_L > \rho_{L_2}^\varepsilon) + P(\tau_L > \bar{\rho}_{L_3}^\varepsilon). \end{aligned} \quad (5.4)$$

The first term goes to zero as $\varepsilon \rightarrow 0$ by (5.3). Also for $L_1 > L + \epsilon$

$$P(\tau_L \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon > \tau_{L_1}^\varepsilon) \leq P\left(\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\right),$$

which goes to zero as $\varepsilon \rightarrow 0$ by (5.3). Furthermore, for $L_4 > L + \epsilon$ we have

$$P(\tau_L \wedge \bar{\rho}_{L_3}^\varepsilon > \bar{\tau}_{L_4}^\varepsilon) \leq P\left(\sup_{t \in [0, \tau_L \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|\bar{\Phi}^\varepsilon - \Phi\|_{-z} > \epsilon\right)$$

which goes to zero by (5.1) as $\varepsilon \rightarrow 0$. The last two terms on the right hand side of (5.4) go to zero uniformly over $\varepsilon \in (0, 1)$ as L_2, L_3 go to ∞ . Thus the result follows. \square

6 Stochastic convergence

In this section we will prove that $\delta C_W^\varepsilon \rightarrow 0$, $E_W^\varepsilon \rightarrow 0$, $A_N \rightarrow 0$, $D_N \rightarrow 0$ in probability as $\varepsilon \rightarrow 0$.

To simplify the arguments below, we assume that $\mathcal{F}W(0) = 0$ and restrict ourselves to the flow of $\int_{\mathbb{T}^3} u(x) dx = 0$. We follow the notations from [GP15, Section 9]. We represent the white noise in terms of its spatial Fourier transform. More precisely, let $E = \mathbb{Z}^3 \setminus \{0\}$ and let $W(s, k) = \langle W(s), e_k \rangle$ for $e_k(x) = 2^{-3/2} e^{i\pi x \cdot k}$, $x \in \mathbb{T}^3$, and we view $W(s, k)$ as a Gaussian process on $\mathbb{R} \times E$ with covariance given by

$$E \left[\int_{\mathbb{R} \times E} f(\eta) W(d\eta) \int_{\mathbb{R} \times E} g(\eta') W(d\eta') \right] = \int_{\mathbb{R} \times E} g(\eta_1) f(\eta_{-1}) d\eta_1,$$

where $\eta_a = (s_a, k_a)$, $s_{-a} = s_a$, $k_{-a} = -k_a$ and the measure $d\eta_a = ds_a dk_a$ is the product of the Lebesgue measure ds_a on \mathbb{R} and of the counting measure dk_a on E . Then

$$u_1^\varepsilon(t, x) = \int_{\mathbb{R} \times E} e_k(x) P_{t-s}^\varepsilon(k) W(d\eta), \quad \bar{u}_1^\varepsilon(t, x) = \int_{\mathbb{R} \times E} e_k(x) \bar{P}_{t-s}^\varepsilon(k) W(d\eta),$$

where $p_t^\varepsilon(k) = e^{-|k|^2 f(\varepsilon k)t} 1_{\{t \geq 0\}}$, $P_t^\varepsilon(k) = p_t^\varepsilon(k) 1_{\{|k|_\infty \leq N\}}$, $p_t(k) = e^{-|k|^2 \pi^2 t} 1_{\{t \geq 0\}}$, and $\bar{P}_t^\varepsilon(k) = p_t(k) 1_{\{|k|_\infty \leq N\}}$. Moreover,

$$\int P_{t-s}^\varepsilon(k) P_{\sigma-s}^\varepsilon(k) ds = \frac{e^{-|k|^2 f(\varepsilon k)|t-\sigma|} 1_{\{|k|_\infty \leq N\}}}{2|k|^2 f(\varepsilon k)} := V_{t-\sigma}^\varepsilon(k), \quad (6.1)$$

and

$$\int \bar{P}_{t-s}^\varepsilon(k) \bar{P}_{\sigma-s}^\varepsilon(k) ds = \frac{e^{-|k|^2 \pi^2 |t-\sigma|} 1_{\{|k|_\infty \leq N\}}}{2\pi^2 |k|^2} := \bar{V}_{t-\sigma}^\varepsilon(k). \quad (6.2)$$

Now we introduce the following notations: $k_{[1\dots n]} = \sum_{i=1}^n k_i$, $\eta_{1\dots n} = (\eta_1, \dots, \eta_n) \in (\mathbb{R} \times E)^n$, $d\eta_{1\dots n} = d\eta_1 \dots d\eta_n$, $dk_{1\dots n} = dk_1 \dots dk_n$, $\tilde{k}^{i_1 i_2 i_3} = (k^j - i_j(2N+1))_{j=1,2,3}$ for $i_j = 1, 0, -1$ and $\sum_{j=1}^3 i_j^2 \neq 0$. In the following we always omit the superscript of k if there is no confusion. Denote by

$$\int_{(\mathbb{R} \times E)^n} f(\eta_{1\dots n}) W(d\eta_{1\dots n})$$

a generic element of the n -th chaos of W on $\mathbb{R} \times E$. By [GP15, Section 9.2] We know that

$$E[|\int_{(\mathbb{R} \times E)^n} f(\eta_{1\dots n}) W(d\eta_{1\dots n})|^2] \leq (n!) \int_{(\mathbb{R} \times E)^n} |f(\eta_{1\dots n})|^2 d\eta_{1\dots n}.$$

Hence for bounding the variance of the chaos it is enough to bound the L^2 norm of the unsymmetrized kernels. To obtain the results we first recall the following lemma from [ZZ15] for our later use:

Lemma 6.1 ([ZZ15, Lemma 3.10]) Let $0 < l, m < d, l + m - d > 0$. Then we have

$$\sum_{k_1, k_2 \in \mathbb{Z}^d \setminus \{0\}, k_1 + k_2 = k} \frac{1}{|k_1|^l |k_2|^m} \lesssim \frac{1}{|k|^{l+m-d}}.$$

By similar arguments as in the proof of [ZZ15, Lemma 3.11] we have the following results.

Lemma 6.2 For every $0 < \kappa < 1, i \geq 0, t \geq 0, k_1, k_2 \in E$ we have

$$|e^{-|k_{[12]}|^2 \pi^2 t} \theta(2^{-i} k_{[12]}) - e^{-|k_2|^2 \pi^2 t} \theta(2^{-i} k_2)| \lesssim |k_1|^\kappa 2^{-i\kappa}.$$

Lemma 6.3 For every $0 < \kappa < 1, i \geq 0, t \geq 0$ we have that for $k_1, k_2 \in E$ with $|k_{[12]}|_\infty \leq N, |k_2|_\infty \leq N$

$$|e^{-|k_{[12]}|^2 t f(\varepsilon k_{[12]})} \theta(2^{-i} k_{[12]}) - e^{-|k_2|^2 t f(\varepsilon k_2)} \theta(2^{-i} k_2)| \lesssim |k_1|^\kappa 2^{-i\kappa}.$$

Now we prove the following estimate for the approximating operators:

Lemma 6.4 For any $0 < \kappa < 1$ and $t > 0, k \in E, \varepsilon > 0$

(i)

$$|p_t^\varepsilon(k) - p_t(k)| \lesssim e^{-|k|^2 \bar{c}_f t} |\varepsilon k|^\kappa, \quad |P_t^\varepsilon(k) - p_t(k)| \lesssim e^{-|k|^2 \bar{c}_f t} |\varepsilon k|^\kappa;$$

(ii)

$$|P_t^\varepsilon(k) - \bar{P}_t^\varepsilon(k)| \lesssim e^{-|k|^2 \bar{c}_f t} |\varepsilon k|^\kappa, \quad |V_t^\varepsilon(k) - \bar{V}_t^\varepsilon(k)| \lesssim \frac{e^{-|k|^2 \bar{c}_f t} |\varepsilon k|^\kappa}{|k|^2}.$$

Here $\bar{c}_f = c_f \wedge \pi^2 > 0$, $c_f = \min\{f(x) : |x| \leq 1.8\}$.

Proof The results follow from $|f(\varepsilon k) - \pi^2| \lesssim |\varepsilon k|^\kappa$ and

$$|e^{-|k|^2 t f(\varepsilon k)} - e^{-|k|^2 \pi^2 t}| \lesssim e^{-|k|^2 \bar{c}_f t} [1 \wedge (t^\kappa |f(\varepsilon k) - \pi^2|^\kappa |k|^{2\kappa})] \lesssim e^{-|k|^2 \bar{c}_f t} |\varepsilon k|^\kappa.$$

□

We prove the following two lemmas for the convergence of the error terms.

Lemma 6.5 For every $q \geq 0, 0 < r < 3$,

$$\int_E \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim 2^{(3-r)q} \quad \text{and} \quad \int_E \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim 2^{(3-r)q}.$$

Proof We only treat the first, the second can be obtained by a similar argument. We have

$$\int \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim \int 1_{|k| \leq 2^q} \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk + \int 1_{|k| > 2^q} \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim 2^{(3-r)q}$$

Here in the last inequality we used that the cardinality of k with $\theta(2^{-q} \tilde{k}) \neq 0$ is of order 2^{3q} . □

Lemma 6.6 For every $q \geq 0, 0 < r < 3$,

$$\int \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim \varepsilon^\kappa 2^{(3-r+\kappa)q}.$$

Here $\kappa > 0$ is small enough.

Proof We have

$$\int \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk \lesssim \int 1_{|k| \leq N} \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^r} dk + \varepsilon^\kappa \int 1_{|k| \geq N} \theta(2^{-q} \tilde{k})^2 \frac{1}{|k|^{r-\kappa}} dk \lesssim \varepsilon^\kappa 2^{(3-r+\kappa)q},$$

where in the last inequality we used that $|k| \leq N \simeq |\tilde{k}| \simeq 2^q$ and Lemma 6.5. □

6.1 Convergence of renormalization terms

In this subsection we prove $\delta C_W^\varepsilon \rightarrow 0$ in probability as $\varepsilon \rightarrow 0$. In the following we use the graph notations similar as in [Hai14] to make this paper more readable. We use $\eta \xrightarrow{\dots} \sigma$ or $s \xrightarrow{k} \sigma$ to represent a factor $P_{\sigma-s}^\varepsilon(k)$ for $\eta = (s, k)$ and use $\sigma \xrightarrow{k} t$ to represent $V_{t-\sigma}^\varepsilon(k)$ or $\bar{V}_{t-\sigma}^\varepsilon(k)$ for simplicity. We use $\eta \xrightarrow{\dots} \sigma$ to represent a factor $\bar{P}_{\sigma-s}^\varepsilon(k)$ for $\eta = (s, k)$, and use $\sigma \xrightarrow{k} t$ to represent $\bar{P}_{t-\sigma}^\varepsilon(k)$ or $p_{t-\sigma}(k)$ if there's no confusion. We also use the convention that if a vertex is drawn in grey, then the corresponding variable is integrated out. Here we

use two different graphs to denote $P_{\sigma-s}^\varepsilon(k)$ and $\bar{P}_{\sigma-s}^\varepsilon(k)$. The second one is to emphasize the appearance of k .

For the terms containing u_2^ε , there are error terms (J_t^i in the following) appears. For these terms, we use $|k_i| \simeq N$ or Lemma 6.6 to produce ε^κ .

Convergence of $u_1^\varepsilon - \bar{u}_1^\varepsilon$

In this part we prove the convergence of $u_1^\varepsilon - \bar{u}_1^\varepsilon$. We have

$$\begin{aligned} & E|\Delta_q[u_1^\varepsilon(t) - \bar{u}_1^\varepsilon(t)]|^2 \\ & \lesssim \int_{\mathbb{R} \times E} \theta(2^{-q}k)^2 |e_k(P_{t-s}^\varepsilon(k) - \bar{P}_{t-s}^\varepsilon(k))|^2 d\eta \lesssim \int \theta(2^{-q}k)^2 (\varepsilon|k|)^\kappa |k|^{-2} dk \lesssim \varepsilon^\kappa 2^{q(\kappa+1)}. \end{aligned}$$

Here $\kappa > 0$ is small enough and in the second inequality we used Lemma 6.4. Similarly by using

$$|1 - e^{-|t_2-t_1|f(\varepsilon k)|k|^2}| \lesssim |t_1 - t_2|^\kappa |k|^{2\kappa},$$

we get the desired estimates for $E|\Delta_q[(u_1^\varepsilon(t_2) - \bar{u}_1^\varepsilon(t_2)) - (u_1^\varepsilon(t_1) - \bar{u}_1^\varepsilon(t_1))]|^2$ with $t_1, t_2 \in [0, T]$, which combined with Gaussian hypercontractivity implies that for $p > 1$, $\epsilon > 0$ small enough

$$\begin{aligned} & E\|[(u_1^\varepsilon(t_2) - \bar{u}_1^\varepsilon(t_2)) - (u_1^\varepsilon(t_1) - \bar{u}_1^\varepsilon(t_1))]^p\|_{B_{p,p}^{-1/2-\kappa-\epsilon}} \\ & \lesssim \varepsilon^{p\kappa/2} |t_2 - t_1|^{\kappa p/4}, \end{aligned}$$

Then by Lemma 2.1 we obtain that for every $\delta > 0, p > 1$, $u_1^\varepsilon - \bar{u}_1^\varepsilon \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{-1/2-\delta/2})$ as $\varepsilon \rightarrow 0$.

Convergence of $u_1^\varepsilon \diamond u_1^\varepsilon - \bar{u}_1^\varepsilon \diamond \bar{u}_1^\varepsilon$

In this part we prove the convergence of $u_1^\varepsilon \diamond u_1^\varepsilon$. Recall that $u_1^\varepsilon \diamond u_1^\varepsilon = u_1^\varepsilon u_1^\varepsilon - C_0^\varepsilon$ and $\bar{u}_1^\varepsilon \diamond \bar{u}_1^\varepsilon = \bar{u}_1^\varepsilon \bar{u}_1^\varepsilon - \bar{C}_0^\varepsilon$. Now take

$$C_0^\varepsilon = 2^{-3} \int_E \frac{1_{\{|k|_\infty \leq N\}}}{2|k|^2 f(\varepsilon k)} dk, \quad \bar{C}_0^\varepsilon = 2^{-3} \int \frac{1_{\{|k|_\infty \leq N\}}}{2|k|^2 \pi^2} dk. \quad (6.3)$$

Then we have

$$\begin{aligned} & E|\Delta_q[u_1^\varepsilon \diamond u_1^\varepsilon(t) - \bar{u}_1^\varepsilon \diamond \bar{u}_1^\varepsilon(t)]|^2 \\ & \lesssim \int_{(\mathbb{R} \times E)^2} \theta(2^{-q}k_{[12]})^2 |(P_{t-s_1}^\varepsilon(k_1)P_{t-s_2}^\varepsilon(k_2) - \bar{P}_{t-s_1}^\varepsilon(k_1)\bar{P}_{t-s_2}^\varepsilon(k_2))|^2 d\eta_{12} \\ & \lesssim \varepsilon^\kappa \int \theta(2^{-q}k_{[12]})^2 \frac{|k_1|^\kappa + |k_2|^\kappa}{|k_1|^2 |k_2|^2} dk_{12} \lesssim \varepsilon^\kappa 2^{(\kappa+2)q}. \end{aligned}$$

Here $\kappa > 0$ is small enough and in the second inequality we used Lemma 6.4 and in the last inequality we used Lemma 6.1. Then by Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0, p > 1$, $u_1^\varepsilon \diamond u_1^\varepsilon - \bar{u}_1^\varepsilon \diamond \bar{u}_1^\varepsilon \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{-1-\delta})$ as $\varepsilon \rightarrow 0$.

Convergence of $u_2^\varepsilon - \bar{u}_2^\varepsilon$

In this part we consider the convergence of u_2^ε . Recall that

$$\bar{u}_2^\varepsilon(t) - u_2^\varepsilon(t) = I_t^3 - \bar{I}_t^3 + J_t^3.$$

Here

$$I_t^3 = 2^{-3} \int_{(\mathbb{R} \times E)^3} e_{k_{[123]}} \int_0^t P_{t-\sigma}^\varepsilon(k_{[123]}) P_{\sigma-s_1}^\varepsilon(k_1) P_{\sigma-s_2}^\varepsilon(k_2) P_{\sigma-s_3}^\varepsilon(k_3) d\sigma W(d\eta_{123}),$$

and \bar{I}_t^3 is defined similarly as I_t^3 with $P_{t-\sigma}^\varepsilon(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and with other P^ε replaced by \bar{P}^ε and J_t^3 is defined similarly as I_t^3 with $e_{k_{[123]}}$, $k_{[123]}$ replaced by $e_{\tilde{k}_{[123]}}$, $\tilde{k}_{[123]}$, respectively. We use a graph notation to indicate the main part in I_t^3 and \bar{I}_t^3 :

to indicate $\int_0^t P_{t-\sigma}^\varepsilon(k_{[123]}) \dots d\sigma$, for the corresponding term in \bar{I}_t^3 .

The graph for J_t^3 is the same as that for I_t^3 with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$. By Lemma 6.4 and a straightforward calculation we obtain that

$$\begin{aligned} & E|\Delta_q(I_t^3 - \bar{I}_t^3)|^2 \\ & \lesssim \int_{(\mathbb{R} \times E)^2} \theta(2^{-q}k_{[123]})^2 \left| \int_0^t \left(P_{t-\sigma}^\varepsilon(k_{[123]}) \Pi_{i=1}^3 P_{\sigma-s_i}^\varepsilon(k_i) - p_{t-\sigma}(k_{[123]}) \Pi_{i=1}^3 \bar{P}_{\sigma-s_i}^\varepsilon(k_i) \right) d\sigma \right|^2 d\eta_{123} \\ & \lesssim \int \theta(2^{-q}k_{[123]}) \frac{\varepsilon^\kappa \sum_{i=1}^3 |k_i|^\kappa + |k_{[123]}|^\kappa \varepsilon^\kappa}{|k_1|^2 |k_2|^2 |k_3|^2 [|k_1|^2 + |k_2|^2 + |k_3|^2] |k_{[123]}|^2} dk_{123} \\ & \lesssim \int_E \theta(2^{-q}k) \frac{\varepsilon^\kappa}{|k|^{4-\kappa}} dk \lesssim \varepsilon^\kappa 2^{q(-1+\kappa)}, \end{aligned}$$

where we used Lemma 6.1 in the third inequality and we used the graph notation in the second inequality. From the graph we can use Lemma 6.4 to control each $\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}$ \rightarrow $\bullet\text{---}\bullet\text{---}$, and $\varepsilon^\kappa |k_i|^\kappa$ is produced. Then for each integral w.r.t. time s_i , $|k_i|^{-2} e^{-|\sigma-\bar{\sigma}||k_i|^2}$ is produced and taking integrals w.r.t. σ and $\bar{\sigma}$ we obtain the second inequality. Similar calculations also imply that

$$\begin{aligned} E|\Delta_q J_t^3|^2 & \lesssim \int \theta(2^{-q}\tilde{k}_{[123]}) \frac{1_{\{|k_{123}| > N, 2^q \lesssim N\}}}{|k_1|^2 |k_2|^2 |k_3|^2 [|k_1|^2 + |k_2|^2 + |k_3|^2] |\tilde{k}_{[123]}|^2} dk_{123} \\ & \lesssim \int_E \theta(2^{-q}\tilde{k}) \frac{1_{\{|k| > N, 2^q \lesssim N\}}}{|k|^2 |\tilde{k}|^2} dk \lesssim \varepsilon^\kappa 2^{q(-1+\kappa)}. \end{aligned}$$

Here we use $|k| > N$ and $\frac{1}{|k|^2}$ to produce ε^κ . By a similar argument as above we also obtain that for every $\delta > 0, p > 1$, $u_2^\varepsilon - \bar{u}_2^\varepsilon \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{1/2-\delta/2})$. Similarly we obtain that $u_2^\varepsilon - \bar{u}_2^\varepsilon \rightarrow 0$ in $L^p(\Omega; C^{1/4-\delta/2-\kappa/2}([0, T], \mathcal{C}^{\kappa/2}))$.

Convergence of $\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) - \pi_{0,\diamond}(\bar{K}^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond,2})$

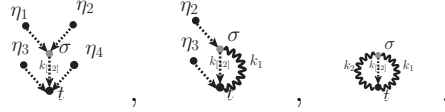
In this part we focus on $\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2})$ and prove that $\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2}) - \pi_{0,\diamond}(\bar{K}^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond,2}) \rightarrow 0$ in $C_T \mathcal{C}^{-\delta}$ for every $\delta > 0$. Now we have the following identity for $t \in [0, T]$:

$$\pi_0(K^\varepsilon, (u_1^\varepsilon)^{\diamond,2})(t) - \pi_0(\bar{K}^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond,2})(t) = I_t^1 + 4I_t^2 + 2I_t^3 - [\bar{I}_t^1 + 4\bar{I}_t^2 + 2\bar{I}_t^3],$$

where

$$\begin{aligned}
I_t^1 &= 2^{-\frac{9}{2}} \int e_{k_{[1234]}} \psi_0(k_{[12]}, k_{[34]}) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[12]}) P_{\sigma-s_1}^\varepsilon(k_1) P_{\sigma-s_2}^\varepsilon(k_2) P_{\sigma-s_3}^\varepsilon(k_3) P_{\sigma-s_4}^\varepsilon(k_4) W(d\eta_{1234}), \\
I_t^2 &= 2^{-\frac{9}{2}} \int \int e_{k_{[23]}} \psi_0(k_{[12]}, k_3 - k_1) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[12]}) P_{\sigma-s_2}^\varepsilon(k_2) P_{\sigma-s_3}^\varepsilon(k_3) V_{t-\sigma}^\varepsilon(k_1) dk_1 W(d\eta_{23}), \\
I_t^3 &= 2^{-6} \int_{E^2} \int_0^t d\sigma V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) P_{t-\sigma}^\varepsilon(k_{[12]}) dk_{12},
\end{aligned}$$

and for $i = 1, 2, 3$, \bar{I}_t^i is defined similarly with $P_{t-\sigma}^\varepsilon(k_{[12]})$ replaced by $p_{t-\sigma}(k_{[12]})$ and other $P^\varepsilon, V^\varepsilon$ replaced by $\bar{P}^\varepsilon, \bar{V}^\varepsilon$ respectively. We use a graph notation to indicate the main part in I_t^1 and I_t^2, I_t^3 :



The graphs for \bar{I}_t^1, \bar{I}_t^2 and \bar{I}_t^3 should be the same with $\bullet \cdots \bullet$ replaced by $\bullet \longrightarrow \bullet$.
In fact, choose

$$C_{11}^\varepsilon = 2^{-5} \int \int_{-\infty}^t d\sigma V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) P_{t-\sigma}^\varepsilon(k_{[12]}) dk_{12} \quad (6.4)$$

and \bar{C}_1^ε is defined with each $P^\varepsilon, V^\varepsilon$ replaced by p, \bar{V}^ε respectively. Choose $\varphi_1^\varepsilon(t) = 2I_t^3 - C_{11}^\varepsilon$ and $\bar{\varphi}_1^\varepsilon(t) = 2\bar{I}_t^3 - \bar{C}_1^\varepsilon$ and $\varphi_1(t) = -2^{-7} \int \frac{e^{-t\pi^2(|k_1|^2 + |k_2|^2 + |k_{[12]}|^2)}}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{[12]}|^2) \pi^6} dk_{12}$. Then we easily obtain that

$$\sup_{t \in [0, T]} t^\rho |\varphi_1^\varepsilon - \varphi_1| \lesssim \varepsilon^\kappa, \quad \sup_{t \in [0, T]} t^\rho |\bar{\varphi}_1^\varepsilon - \varphi_1| \lesssim \varepsilon^\kappa,$$

for every $\rho > 0, 0 < \kappa < 2\rho$.

Terms in the second chaos: Now we consider I_t^2 and by graph notations and (6.1), (6.2) we have:

$$\begin{aligned}
& E|\Delta_q(I_t^2 - \bar{I}_t^2)|^2 \\
& \lesssim \int \psi_0(k_{[12]}, k_3 - k_1) \psi_0(k_{[24]}, k_3 - k_4) \theta(2^{-q} k_{[23]})^2 \\
& \quad \frac{|\varepsilon k_{[12]}|^{\kappa/2} |\varepsilon k_{[24]}|^{\kappa/2} + |\varepsilon k_1|^{\kappa/2} |\varepsilon k_4|^{\kappa/2} + |\varepsilon k_2|^\kappa + |\varepsilon k_3|^\kappa}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2 (|k_1|^2 + |k_{[12]}|^2) (|k_4|^2 + |k_{[24]}|^2)} dk_{1234} \\
& \lesssim \varepsilon^\kappa \int \theta(2^{-q} k_{[23]})^2 \frac{2^{-2q+2\kappa}}{|k_2|^{2-\kappa} |k_3|^2} dk_{23} \\
& \lesssim \varepsilon^\kappa 2^{q3\kappa},
\end{aligned}$$

with $\kappa > 0$ small enough. Here we used that $|k_{[i2]}| \gtrsim 2^q$ on the support of $\psi_0(k_{[i2]}, k_3 - k_i) \theta(2^{-q} k_{[23]})$ for $i = 1, 4$ in the second inequality and Lemma 6.1 in the last inequality.

Terms in the fourth chaos: Now for I_t^1 by (6.1), (6.2) and graph notations we have:

$$\begin{aligned}
& E[|\Delta_q(I_t^1 - \bar{I}_t^1)|^2] \\
& \lesssim \varepsilon^\kappa \int \theta(2^{-q}k_{[1234]})^2 \frac{\psi_0(k_{[12]}, k_{[34]})}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2 |k_{[12]}|^4} (|k_{[12]}|^\kappa + \sum_{i=1}^4 |k_i|^\kappa) dk_{1234} \\
& \lesssim \int \theta(2^{-q}k_{[1234]})^2 \psi_0(k_{[12]}, k_{[34]}) \left(\frac{\varepsilon^\kappa}{|k_{[34]}| |k_{[12]}|^{5-\kappa}} + \frac{\varepsilon^\kappa}{|k_{[34]}|^{1-\kappa} |k_{[12]}|^5} \right) dk_{[12][34]} \\
& \lesssim \int \theta(2^{-q}k)^2 2^{-q(2+\kappa)} \frac{\varepsilon^\kappa}{|k|^{1-2\kappa}} dk \lesssim \varepsilon^\kappa 2^{q\kappa},
\end{aligned}$$

where we used Lemma 6.1 in the second inequality and that $|k_{[12]}| \gtrsim 2^q$ on the support of $\theta(2^{-q}k_{[1234]}) \psi_0(k_{[12]}, k_{[34]})$ in the third inequality. Now we have that for $\kappa > 0$ small enough

$$E[|\Delta_q(I_t^1 - \bar{I}_t^1)|^2] \lesssim 2^{q\kappa} \varepsilon^\kappa.$$

By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$

$$\pi_{0,\diamond}(K^\varepsilon, (u_1^\varepsilon)^{\diamond 2}) - \pi_{0,\diamond}(\bar{K}^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2}) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $\pi_0(u_2^\varepsilon, u_1^\varepsilon) - \pi_0(\bar{u}_2^\varepsilon, \bar{u}_1^\varepsilon)$

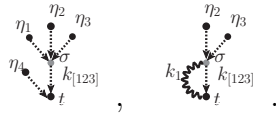
In this part we focus on $\pi_0(u_2^\varepsilon, u_1^\varepsilon)$ and prove that $\pi_0(u_2^\varepsilon, u_1^\varepsilon) - \pi_0(\bar{u}_2^\varepsilon, \bar{u}_1^\varepsilon) \rightarrow 0$ in $C_T \mathcal{C}^{-\delta}$. We have the following identity for $t \in [0, T]$:

$$\pi_0(\bar{u}_2^\varepsilon, \bar{u}_1^\varepsilon)(t) - \pi_0(u_2^\varepsilon, u_1^\varepsilon)(t) = I_t^1 + 3I_t^2 - [\bar{I}_t^1 + 3\bar{I}_t^2] + J_t^1 + 3J_t^2,$$

where

$$\begin{aligned}
I_t^1 &= 2^{-\frac{9}{2}} \int e_{k_{[1234]}} \psi_0(k_{[123]}, k_4) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[123]}) P_{\sigma-s_1}^\varepsilon(k_1) P_{\sigma-s_2}^\varepsilon(k_2) P_{\sigma-s_3}^\varepsilon(k_3) P_{t-s_4}^\varepsilon(k_4) W(d\eta_{1234}), \\
I_t^2 &= 2^{-\frac{9}{2}} \int \int e_{k_{[23]}} \psi_0(k_{[123]}, k_1) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[123]}) P_{\sigma-s_2}^\varepsilon(k_2) P_{\sigma-s_3}^\varepsilon(k_3) V_{t-\sigma}^\varepsilon(k_1) dk_1 W(d\eta_{23}),
\end{aligned}$$

and for $i = 1, 2$, \bar{I}_t^i is defined with $P_{t-\sigma}^\varepsilon(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and other $P^\varepsilon, V^\varepsilon$ replaced by $\bar{P}^\varepsilon, \bar{V}^\varepsilon$ respectively and for $i = 1, 2$, J_t^i is defined similarly as I_t^i with $k_{[123]}, e_{k_{[1234]}}, e_{k_{[23]}}$ replaced by $\tilde{k}_{[123]}, e_{\tilde{k}_{[1234]}}, e_{\tilde{k}_{[23]}}$, respectively. We also use graph notations to indicate the main part in I_t^1 and I_t^2 :



The graphs for \bar{I}_t^1, \bar{I}_t^2 should be the same with $\bullet \cdots \bullet$ replaced by $\bullet \longrightarrow \bullet$ and the graphs for J_t^1, J_t^2 should be the same as above only with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$.

Terms in the second chaos: First we consider I_t^2 and have the following calculations:

$$\begin{aligned}
& E|\Delta_q(I_t^2 - \bar{I}_t^2)|^2 \\
& \lesssim \int \psi_0(k_{[123]}, k_1)\psi_0(k_{[234]}, k_4)\theta(2^{-q}k_{[23]})^2 \\
& \quad \frac{|\varepsilon k_{[123]}|^{\kappa/2}|\varepsilon k_{[234]}|^{\kappa/2} + |\varepsilon k_1|^{\kappa/2}|\varepsilon k_4|^{\kappa/2} + |\varepsilon k_2|^\kappa + |\varepsilon k_3|^\kappa}{|k_2|^2|k_3|^2|k_1|^2(|k_1|^2 + |k_{[123]}|^2)|k_4|^2(|k_4|^2 + |k_{[234]}|^2)} dk_{1234} \\
& \lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)}\theta(2^{-q}k_{[23]})^2 \frac{1}{|k_2|^{2-\kappa}|k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{3q\kappa},
\end{aligned}$$

where $\kappa > 0$ is small enough. Here we used (6.1), (6.2) and graph notations in the first inequality and that $|k_{[123]}| \gtrsim 2^q$, $k_{[234]} \gtrsim 2^q$ in the second inequality and we used Lemma 6.1 in the last inequality. By a similar calculation as above, we see that

$$E|\Delta_q J_t^2|^2 \lesssim \int 2^{-q(2-2\kappa)}\theta(2^{-q}\tilde{k}_{[23]})^2 \frac{\varepsilon^\kappa}{|k_2|^{2-\kappa}|k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{3q\kappa}.$$

Here $\kappa > 0$ is small enough and in the first inequality we used $|k_{[123]}| \simeq N$ to deduce that $|k_i| \simeq N$ for some $i \in \{1, 2, 3\}$, which produces ε^κ , and in the last inequality we used Lemmas 6.1 and 6.5.

Terms in the fourth chaos: Now for I_t^1 we have:

$$\begin{aligned}
& E[|\Delta_q(I_t^1 - \bar{I}_t^1)|^2] \\
& \lesssim \varepsilon^\kappa \int \frac{\theta(2^{-q}k_{[1234]})^2 \psi_0(k_{[123]}, k_4)(|k_{[123]}|^\kappa + \sum_{i=1}^4 |k_i|^\kappa)}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2[|k_1|^2 + |k_2|^2 + |k_3|^2]|k_{[123]}|^2} dk_{1234} \\
& \lesssim \int 2^{-q(2-\kappa)}\theta(2^{-q}k)^2 \frac{\varepsilon^\kappa}{|k|} dk \lesssim \varepsilon^\kappa 2^{q\kappa},
\end{aligned}$$

where we used (6.1), (6.2) and graph notations in the first inequality, Lemma 6.1 and that $|k_{[123]}| \gtrsim 2^q$ in the second inequality. For J_t^1 , using Lemma 6.5 and by a similar argument we also obtain that

$$E|\Delta_q J_t^1|^2 \lesssim \varepsilon^\kappa 2^{\kappa q}.$$

Now by a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we have that for every $\delta > 0$, $p > 1$

$$\pi_{0,\diamond}(u_2^\varepsilon, u_1^\varepsilon) - \pi_{0,\diamond}(\bar{u}_2^\varepsilon, \bar{u}_1^\varepsilon) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2}) - \pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2})$

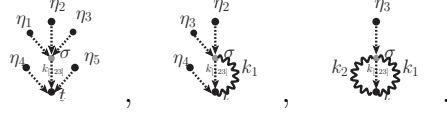
In this part we focus on $\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2})$ and prove that $\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2}) - \pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2}) \rightarrow 0$ in $C_T \mathcal{C}^{-1/2-\delta/2}$. We have the following identity for $t \in [0, T]$:

$$\pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2})(t) - \pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2})(t) = I_t^1 + 6I_t^2 + 6I_t^3 - [\bar{I}_t^1 + 6\bar{I}_t^2 + 6\bar{I}_t^3] + J_t^1 + 6J_t^2 + 6J_t^3,$$

where

$$\begin{aligned}
I_t^1 &= 2^{-6} \int e_{k_{[12345]}} \psi_0(k_{[123]}, k_{[45]}) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[123]}) \Pi_{i=1}^3 P_{\sigma-s_i}^\varepsilon(k_i) \Pi_{i=4}^5 P_{t-s_i}^\varepsilon(k_i) W(d\eta_{12345}), \\
I_t^2 &= 2^{-6} \int \int e_{k_{[234]}} \psi_0(k_{[123]}, k_4 - k_1) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(k_{[123]}) \Pi_{i=2}^3 P_{\sigma-s_i}^\varepsilon(k_i) P_{t-s_4}^\varepsilon(k_4) V_{t-\sigma}^\varepsilon(k_1) dk_1 W(d\eta_{234}), \\
I_t^3 &= 2^{-6} \int \int e_{k_3} \psi_0(k_{[123]}, k_{[12]}) \int_0^t d\sigma P_{\sigma-s_3}^\varepsilon(k_3) V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) P_{t-\sigma}^\varepsilon(k_{[123]}) dk_{12} W(d\eta_3),
\end{aligned}$$

and for $i = 1, 2, 3$, \bar{I}_t^i is defined similarly with $P_{t-\sigma}^\varepsilon(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and other $P^\varepsilon, V^\varepsilon$ replaced by $\bar{P}^\varepsilon, \bar{V}^\varepsilon$ respectively and for $i = 1, 2, 3$, J_t^i is defined similarly as I_t^i with each $k_{[123]}, e_{k_{[12345]}}, e_{k_{[234]}}, e_{k_3}$ replaced by $\tilde{k}_{[123]}, e_{\tilde{k}_{[12345]}}, e_{\tilde{k}_{[234]}}, e_{\tilde{k}_3}$, respectively. We use graph notations to indicate the main parts in I_t^1 and I_t^2, I_t^3 :

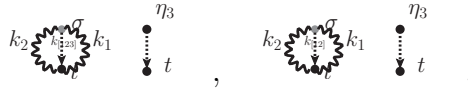


The graph for $\bar{I}_t^1, \bar{I}_t^2, \bar{I}_t^3$ should be the same with $\bullet \cdots \bullet$ replaced by $\bullet \longrightarrow \bullet$ and the graph for J_t^1, J_t^2, J_t^3 should be the same as above only with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$.

We consider the following term first:

$$I_t^3 - \bar{I}_t^3 - [\tilde{I}_t^3 - \tilde{\bar{I}}_t^3] + \tilde{I}_t^3 - \tilde{\bar{I}}_t^3 - C^\varepsilon(t)u_1^\varepsilon(t) + \bar{C}^\varepsilon(t)\bar{u}_1^\varepsilon(t),$$

where $\tilde{I}_t^3, \tilde{\bar{I}}_t^3$ are defined similarly as I_t^3, \bar{I}_t^3 with $P_{\sigma-s_3}^\varepsilon(k_3), \bar{P}_{\sigma-s_3}^\varepsilon(k_3)$ replaced by $P_{t-s_3}^\varepsilon(k_3), \bar{P}_{t-s_3}^\varepsilon(k_3)$, respectively and $C^\varepsilon(t) = \frac{1}{2}[C_{11}^\varepsilon + \varphi_1^\varepsilon(t)], \bar{C}^\varepsilon(t) = \frac{1}{2}[\bar{C}_{11}^\varepsilon + \bar{\varphi}_1^\varepsilon(t)]$. We also use graph notations to indicate the main parts in \tilde{I}_t^3 and $C^\varepsilon(t)u_1^\varepsilon(t)$:



Since for $\kappa > 0$ small enough $\int |P_{t-s_3}^\varepsilon(k_3) - P_{\sigma-s_3}^\varepsilon(k_3)|^2 ds_3 \lesssim \frac{(t-\sigma)^{\kappa/2}}{|k_3|^{2-\kappa}}$ and

$$\int |P_{t-s_3}^\varepsilon(k_3) - P_{\sigma-s_3}^\varepsilon(k_3) - [\bar{P}_{t-s_3}^\varepsilon(k_3) - \bar{P}_{\sigma-s_3}^\varepsilon(k_3)]|^2 ds_3 \lesssim \frac{(t-\sigma)^{\kappa/2} \wedge \varepsilon^\kappa}{|k_3|^{2-\kappa}},$$

by (6.1), (6.2) and graph notations we obtain that for $\kappa > 0$ small enough

$$\begin{aligned}
& E[|\Delta_q(I_t^3 - \bar{I}_t^3 - [\tilde{I}_t^3 - \tilde{\bar{I}}_t^3])|^2] \\
& \lesssim \int \theta(2^{-q}k_3)^2 \left[\frac{1}{|k_3|^{2-2\kappa}} \left(\int_0^t \int \varepsilon^{\kappa/2} (|k_{[123]}|^{\kappa/2} + |k_2|^{\kappa/2} + |k_1|^{\kappa/2}) \frac{e^{-(|k_{[123]}|^2 + |k_1|^2 + |k_2|^2)\bar{c}_f(t-\sigma)}}{|k_1|^2 |k_2|^2} \right. \right. \\
& \quad \left. \left. (t-\sigma)^{\kappa/2} dk_{12} d\sigma \right)^2 + \frac{\varepsilon^\kappa}{|k_3|^{2-2\kappa}} \left(\int_0^t \int \frac{e^{-(|k_{[123]}|^2 + |k_1|^2 + |k_2|^2)(t-\sigma)}}{|k_1|^2 |k_2|^2} (t-\sigma)^{\kappa/4} dk_{12} d\sigma \right)^2 \right] dk_3 \\
& \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}$$

Here in the last inequality we used that $\sup_{a \geq 0} a^r e^{-a} \leq C$ for $r \geq 0$ and Lemma 6.1. Moreover, by Lemmas 6.2 and 6.3 and graph notations we obtain that

$$\begin{aligned}
& E[|\Delta_q(\tilde{I}_t^3 - \bar{I}_t^3 - u_1^\varepsilon(t)C^\varepsilon(t) + \bar{u}_1^\varepsilon(t)\bar{C}^\varepsilon(t))|^2] \\
& \lesssim \int \frac{1}{|k_3|^2} \theta(2^{-q}k_3) \left(\int \int_0^t |k_{[12]}|^{-\kappa} |k_3|^\kappa \right. \\
& \quad \left. (\varepsilon^{\kappa/2}|k_2|^{\kappa/2} + \varepsilon^{\kappa/2}|k_1|^{\kappa/2} + \varepsilon^{\kappa/2}|k_3|^{\kappa/2}) \frac{e^{-|k_1|^2(t-\sigma)\bar{c}_f - |k_2|^2(t-\sigma)\bar{c}_f}}{|k_1|^2|k_2|^2} dk_{12} d\sigma \right)^2 dk_3 \\
& \quad + \int \frac{\varepsilon^\kappa |k_3|^\kappa}{|k|^2} \theta(2^{-q}k_3)^2 \left(\int \int_0^t \frac{e^{-|k_2|^2(t-\sigma) - |k_1|^2(t-\sigma)}}{|k_1|^2|k_2|^2} |k_3|^\kappa |k_{[12]}|^{-\kappa} dk_{12} d\sigma \right)^2 dk_3 \\
& \lesssim \varepsilon^\kappa \int \theta(2^{-q}k_3) \frac{1}{|k_3|^{2-3\kappa}} dk_3 \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}$$

For J_t^3 we have

$$E[|\Delta_q J_t^3|^2] \lesssim \int \frac{1}{|k_3|^2} \theta(2^{-q}\tilde{k}_3) \left(\int \frac{1_{|k_1| \leq N, |k_2| \leq N, |k_3| \leq N}}{|k_1|^2|k_2|^2(|k_1|^2 + |k_2|^2 + |\tilde{k}_{[123]}|^2)} dk_{12} \right)^2 dk_3 \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.$$

Here we used that $2^q \simeq N \simeq |\tilde{k}_3|$ and Lemma 6.6 in the last inequality.

Terms in the third chaos: Now we focus on the bounds for I_t^2 . We obtain the following inequalities:

$$\begin{aligned}
& E|\Delta_q(I_t^2 - \bar{I}_t^2)|^2 \\
& \lesssim \int \theta(2^{-q}k_{[234]}) \psi_0(k_{[123]}, k_4 - k_1) \psi_0(k_{[235]}, k_4 - k_5) \\
& \quad \prod_{i=1}^5 \frac{1}{|k_i|^2} \frac{|k_{[123]}|^{\kappa/2} |k_{[235]}|^{\kappa/2} \varepsilon^\kappa + \sum_{i=1}^4 (\varepsilon |k_i|)^\kappa}{(|k_1|^2 + |k_{[123]}|^2 + |k_2|^2)(|k_5|^2 + |k_{[235]}|^2)} dk_{12345} \\
& \lesssim \int 2^{-q(1-\kappa)} \frac{\varepsilon^\kappa \theta(2^{-q}k_{[234]})}{|k_2|^{3-2\kappa} |k_3|^2 |k_4|^2} dk_{234} \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)},
\end{aligned}$$

where we used graph notations in the first inequality and Lemma 6.1 in the last inequality. For J_t^2 by a similar calculation as above we know that

$$E|\Delta_q J_t^2|^2 \lesssim \int 2^{-q(1-\kappa)} \theta(2^{-q}\tilde{k}_{[234]})^2 \frac{1}{|k_2|^{3-\kappa} |k_3|^2 |k_4|^2} dk_{234} \lesssim \varepsilon^\kappa 2^{(1+3\kappa)q}.$$

Here $\kappa > 0$ is small enough and in the last inequality we used Lemmas 6.1 and 6.6.

Terms in the fifth chaos: Now we focus on the bounds for I_t^1 . By graph notations we obtain the following inequalities:

$$\begin{aligned}
& E|\Delta_q(I_t^1 - \bar{I}_t^1)|^2 \\
& \lesssim \int \theta(2^{-q}k_{[12345]})^2 \psi_0(k_{[123]}, k_{[45]})^2 \prod_{i=1}^5 \frac{1}{|k_i|^2} \frac{(\sum_{i=1}^5 |\varepsilon k_i|^\kappa + |\varepsilon k_{[123]}|^\kappa)}{|k_{[123]}|^2 (|k_1|^2 + |k_2|^2 + |k_{[123]}|^2)} dk_{12345} \\
& \lesssim \varepsilon^\kappa 2^{q(1+2\kappa)}.
\end{aligned}$$

For J_t^1 by similar calculations as for I_t^1 and using the fact that $|k_{[123]}| \simeq N \gtrsim |\tilde{k}_{[123]}|$ we obtain that

$$E|\Delta_q J_t^1|^2 \lesssim \varepsilon^\kappa 2^{q(1+2\kappa)}.$$

By a similar calculation as above we also obtain that there exist $\kappa, \epsilon, \gamma > 0$ small enough such that for any $t_1, t_2 \in [0, T]$

$$\begin{aligned} & E[|\Delta_q(\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2})(t_1) - \pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2})(t_2) - \pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2})(t_1) + \pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2})(t_2))|^2] \\ & \lesssim \varepsilon^\gamma |t_1 - t_2|^\kappa 2^{q(1+\epsilon)}, \end{aligned}$$

which by Gaussian hypercontractivity and Lemma 2.1 implies that for every $\delta > 0, p > 1$, $\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2}) - \pi_{0,\diamond}(\bar{u}_2^\varepsilon, (\bar{u}_1^\varepsilon)^{\diamond 2}) \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{-1/2-\delta/2})$.

6.2 Convergence of error terms

In this subsection we prove that $E_W^\varepsilon \rightarrow^P 0$ as $\varepsilon \rightarrow 0$. For the estimate we use Lemma 6.6 or the fact that there exists some $|k_i| \simeq N$ to produce ε^κ . Due to this reason most error terms converge to zero. However, for $\pi_{0,\diamond}(K_1^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})$ and $\pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})$ we still need to do renormalization such that it converges to zero. This is where C_{12}^ε comes from.

Convergence of $\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})$

We have the following identity for $t \in [0, T]$:

$$\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})(t) = I_t^1 + 4I_t^2 + 2I_t^3,$$

where

$$\begin{aligned} I_t^1 &= 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(k_{[12]}, \tilde{k}_{[34]}) \begin{array}{c} \eta_1 \quad \eta_2 \\ \eta_3 \quad \eta_4 \\ \sigma \\ k_{[12]} \\ \tilde{k}_{[34]} \\ t \end{array} W(d\eta_{1234}), \\ I_t^2 &= 2^{-\frac{9}{2}} \int \int e_{\tilde{k}_{[23]}} \psi_0(k_{[12]}, \tilde{k}_3 - k_1) \begin{array}{c} \eta_2 \\ \eta_3 \\ \sigma \\ k_{[12]} \\ \tilde{k}_3 \\ k_1 \\ t \end{array} dk_1 W(d\eta_{23}) \\ I_t^3 &= 2^{-6} \int e_N^{i_1 i_2 i_3} \psi_0(k_{[12]}, \widetilde{-k_{[12]}}) \begin{array}{c} \sigma \\ k_{[12]} \\ k_{[12]} \\ t \end{array} dk_{12}. \end{aligned}$$

Term in the 0-th chaos: We have

$$E[|\Delta_q I_t^3|^2] \lesssim \left(\int \frac{1_{|k_{[12]}| \simeq N \simeq 2^q} \psi_0(k_{[12]}, \tilde{k}_{[12]})}{|k_{[12]}|^3} dk_{[12]} \right)^2 \lesssim \varepsilon^\kappa 2^{q(3\kappa)}.$$

Term in the second chaos: Now we consider I_t^2 . We have:

$$\begin{aligned} E|\Delta_q I_t^2|^2 & \lesssim \int \psi_0(k_{[12]}, \tilde{k}_3 - k_1) \psi_0(k_{[24]}, \tilde{k}_3 - k_4) \theta(2^{-q} \tilde{k}_{[23]})^2 \\ & \quad \frac{1}{|k_2|^2 |k_3|^2 |k_1|^2 (|k_1|^2 + |k_{12}|^2) |k_4|^2 (|k_4|^2 + |k_{24}|^2)} dk_{1234} \\ & \lesssim \int 2^{(-2+\kappa)q} \theta(2^{-q} \tilde{k}_{[23]})^2 \frac{1}{|k_2|^2 |k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{q2\kappa}, \end{aligned}$$

where $\kappa > 0$ is small enough. Here we used that $|k_{[i2]}| \gtrsim 2^q$ for $i = 1, 4$ in the second inequality and used Lemmas 6.1 and 6.6 in the third inequality.

Term in the fourth chaos: Now for I_t^1 we have:

$$\begin{aligned} E[|\Delta_q I_t^1|^2] &\lesssim \int \theta(2^{-q} \tilde{k}_{[1234]})^2 \psi_0(k_{[12]}, \tilde{k}_{[34]}) \frac{1}{|k_{[34]}| |k_{[12]}|^{5-\kappa}} dk_{[12][34]} \\ &\lesssim 2^{-2q} \int \theta(2^{-q} \tilde{k}_{[1234]})^2 \frac{1}{|k_{[34]}| |k_{[12]}|^{3-\kappa}} dk_{[12][34]} \lesssim \varepsilon^\kappa 2^{q\kappa}, \end{aligned}$$

where we used Lemmas 6.1 and 6.6 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$

$$\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $\pi_0(K_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2})$

Now we have the following identity for $t \in [0, T]$:

$$\pi_0(K_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2})(t) = I_t^1 + 4I_t^2 + 2I_t^3,$$

where

$$\begin{aligned} I_t^1 &= 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(\tilde{k}_{[12]}, k_{[34]}) \begin{array}{c} \eta_1 \quad \eta_2 \\ \diagdown \quad \diagup \\ \sigma \\ \diagup \quad \diagdown \\ \eta_3 \quad \eta_4 \\ \downarrow \\ t \end{array} W(d\eta_{1234}), \\ I_t^2 &= 2^{-\frac{9}{2}} \int \int e_{\tilde{k}_{[23]}} \psi_0(\tilde{k}_{[12]}, k_3 - k_1) \begin{array}{c} \eta_2 \quad \sigma \\ \diagdown \quad \diagup \\ \eta_3 \quad k_1 \\ \downarrow \\ t \end{array} dk_1 W(d\eta_{23}) \\ I_t^3 &= 2^{-6} \int e_N^{i_1 i_2 i_3} \psi_0(\tilde{k}_{[12]}, -k_{[12]}) \begin{array}{c} \sigma \\ \downarrow \\ t \end{array} dk_{12}. \end{aligned}$$

I_t^3, I_t^2 can be estimated similarly as for the case of $\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$ and we only consider

Terms in the fourth chaos: Now for I_t^1 we have:

$$\begin{aligned} E|\Delta_q I_t^1|^2 &\lesssim \int \psi_0(\tilde{k}_{[12]}, k_{[34]}) \theta(2^{-q} \tilde{k}_{[1234]})^2 \frac{1}{|k_2|^2 |k_3|^2 |k_1|^2 (|k_1|^2 + |k_2|^2) |k_4|^2 |\tilde{k}_{[12]}|^2} dk_{1234} \\ &\lesssim \int 2^{-2q} \theta(2^{-q} \tilde{k}_{[1234]})^2 \frac{1}{|k_{[12]}|^{3-\kappa} |k_{[34]}|} dk_{[12][34]} \lesssim \varepsilon^\kappa 2^{q2\kappa}, \end{aligned}$$

where we used Lemmas 6.1 and 6.6 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for $\delta > 0$, $p > 1$

$$\pi_0(K_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2}) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $\pi_{0, \diamond}(K_1^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$

We have

$$\pi_0(K_1^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})(t) = I_t^1 + 4I_t^2 + 2I_t^3.$$

Here $I_t^i, i = 1, 2$ is defined similarly as for the case of $\pi_0(K^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$ with $k_{[12]}, e_{\tilde{k}_{[1234]}}$ and $e_{\tilde{k}_{[23]}}$ replaced by $\tilde{k}_{[12]}, e_{\tilde{k}_{[1234]}}$ and $e_{\tilde{k}_{[23]}}$, respectively and

$$I_t^3 = 2^{-6} \int e_N^{i_1 i_2 i_3} e_N^{i'_1 i'_2 i'_3} \psi_0(\tilde{k}_{[12]}^{i'_1 i'_2 i'_3}, \widetilde{-k_{[12]}^{i_1 i_2 i_3}}) \int_0^t d\sigma P_{t-\sigma}^\varepsilon(\tilde{k}_{[12]}^{i'_1 i'_2 i'_3}) V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) dk_{12},$$

for $i_j, i'_j \in \{-1, 0, 1\}$ for $j = 1, 2, 3$ with $\sum_j i_j^2 \neq 0, \sum_j (i'_j)^2 \neq 0$. Choosing

$$C_{12}^{\varepsilon, i_1 i_2 i_3} = 2^{-5} \int \int_{-\infty}^t d\sigma P_{t-\sigma}^\varepsilon(\widetilde{-k_{[12]}^{i_1 i_2 i_3}}) V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) dk_{12}, \quad (6.5)$$

and $\varphi_2^{\varepsilon, i_1 i_2 i_3}(t) = -2^{-5} \int \int_{-\infty}^0 d\sigma P_{t-\sigma}^\varepsilon(\widetilde{-k_{[12]}^{i_1 i_2 i_3}}) V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) dk_{12}$, we easily obtain that

$$|C_{12}^{\varepsilon, i_1 i_2 i_3}| \simeq 1, \quad \sup_{t \in [0, T]} t^\rho |\varphi_2^{\varepsilon, i_1 i_2 i_3}(t)| \lesssim \varepsilon^\kappa,$$

for every $\rho > \kappa/2 > 0$. For the terms in $2I_t^3 - C_{12}^{\varepsilon, i_1 i_2 i_3} - \varphi_2^{\varepsilon, i_1 i_2 i_3}$, we know that $e_N^{i_1 i_2 i_3} e_N^{i'_1 i'_2 i'_3} \neq 1$ and we easily obtain that

$$E[|\Delta_q(2I_t^3 - C_{12}^\varepsilon - \varphi_2^\varepsilon)|^2] \lesssim \varepsilon^\kappa 2^{q(3\kappa)}.$$

Term in the second chaos: Now we consider I_t^2 . We have:

$$\begin{aligned} E|\Delta_q I_t^2|^2 &\lesssim \int \psi_0(\tilde{k}_{[12]}, \tilde{k}_3 - k_1) \psi_0(\tilde{k}_{[24]}, \tilde{k}_3 - k_4) \theta(2^{-q} \tilde{k}_{[23]})^2 \mathbf{1}_{|k_{[12]}| > N, |k_{[24]}| > N} \\ &\quad \frac{1}{|k_2|^2 |k_3|^2 |k_1|^2 (|k_1|^2 + |\tilde{k}_{[12]}|^2) |k_4|^2 (|k_4|^2 + |\tilde{k}_{[24]}|^2)} dk_{1234} \\ &\lesssim \varepsilon^\kappa \int 2^{(-2+2\kappa)q} \theta(2^{-q} \tilde{k}_{[23]})^2 \frac{1}{|k_2|^{2-\kappa} |k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{3q\kappa}, \end{aligned}$$

where $\kappa > 0$ is small enough and we used that $|k_i| \simeq N$ for some $i \in \{1, 2, 4\}$ in the second inequality and Lemmas 6.1 and 6.5 in the third inequality.

Term in the fourth chaos: Now for I_t^1 we have:

$$\begin{aligned} E[|\Delta_q I_t^1|^2] &\lesssim \varepsilon^\kappa \int \theta(2^{-q} \tilde{k}_{[1234]})^2 \psi_0(\tilde{k}_{[12]}, \tilde{k}_{[34]}) \frac{1}{|k_{[34]}| |\tilde{k}_{[12]}|^{5-\kappa}} dk_{[12][34]} \\ &\lesssim \varepsilon^\kappa 2^{-2q} \int \theta(2^{-q} \tilde{k}_{[1234]})^2 \frac{1}{|k_{[34]}| |\tilde{k}_{[12]}|^{3-\kappa}} dk_{[12][34]} \lesssim \varepsilon^\kappa 2^{q\kappa}, \end{aligned}$$

where we used that $|k_{[12]}| \simeq N \gtrsim |\tilde{k}_{[12]}|$ in the first inequality as well as Lemmas 6.1 and 6.5 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0, p > 1$

$$\pi_{0, \diamond}(K_1^\varepsilon, e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2}) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $(u_1^\varepsilon)^{\diamond, 2} e_N^{i_1 i_2 i_3}$

By a similar calculation as that for $(u_1^\varepsilon)^{\diamond,2}$ in Section 6.1 we know that

$$E|\Delta_q[(u_1^\varepsilon)^{\diamond,2}e_N^{i_1i_2i_3}]|^2 \lesssim \int \theta(2^{-q}\tilde{k}_{[123]})^2 \frac{1}{|k_1|^2|k_2|^2} dk_{12} \lesssim \varepsilon^\kappa 2^{(\kappa+2)q}.$$

Here $\kappa > 0$ is small enough and in the last inequality we used Lemmas 6.1, 6.6. Then by Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0, p > 1$, $(u_1^\varepsilon)^{\diamond,2}e_N^{i_1i_2i_3} \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{-1-\delta})$.

Convergence of $\pi_0(u_2^\varepsilon, e_N^{i_1i_2i_3}u_1^\varepsilon)$

Now we have the following identity for $t \in [0, T]$:

$$\pi_0(u_2^\varepsilon, e_N^{i_1i_2i_3}u_1^\varepsilon)(t) = -[I_t^1 + 3I_t^2 + J_t^1 + 3J_t^2],$$

where

$$I_t^1 = 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(k_{[123]}, \tilde{k}_4) \begin{array}{c} \eta_1 \quad \eta_2 \quad \eta_3 \\ \vdots \\ \sigma \\ \vdots \\ \eta_4 \end{array} W(d\eta_{1234}),$$

$$I_t^2 = 2^{-\frac{9}{2}} \int \int e_{\tilde{k}_{[23]}} \psi_0(k_{[123]}, \tilde{k}_1) \begin{array}{c} \eta_2 \quad \eta_3 \\ \vdots \\ \sigma \\ \vdots \\ k_1 \end{array} dk_1 W(d\eta_{23}),$$

and for $i = 1, 2$, J_t^i is defined similarly as I_t^i with each $k_{[123]}$, $e_{\tilde{k}_{[1234]}}$, $e_{\tilde{k}_{[23]}}$ replaced by $\tilde{k}_{[123]}$, $e_{\tilde{k}_{[1234]}}$, $e_{\tilde{k}_{[23]}}$, respectively.

Terms in the second chaos: First we consider I_t^2 and by similar calculations as that for $\pi_0(u_2^\varepsilon, u_1^\varepsilon)$, we obtain

$$E|\Delta_q I_t^2|^2 \lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)} \theta(2^{-q}\tilde{k}_{[23]})^2 \frac{1}{|k_2|^2|k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{2q\kappa},$$

where $\kappa > 0$ is small enough and we used that $|k_{[123]}| \simeq |\tilde{k}_1| \simeq N$ in the first inequality and we used Lemmas 6.1 and 6.5 in the last inequality. By a similar calculation as above, we see that

$$E|\Delta_q J_t^2|^2 \lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)} \theta(2^{-q}\tilde{k}_{[23]})^2 \frac{1}{|k_2|^2|k_3|^2} dk_{23} \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

Here $\kappa > 0$ is small enough and we used that $|\tilde{k}_{[123]}| \simeq |\tilde{k}_1| \simeq N$ in the first inequality, we used Lemmas 6.1 and 6.5 in the last inequality.

Terms in the fourth chaos: Now for I_t^1, J_t^1 we similarly get that:

$$E[|\Delta_q I_t^1|^2 + |\Delta_q J_t^1|^2] \lesssim \varepsilon^\kappa 2^{\kappa q}.$$

Here for I_t^1 we used that $|k_{[123]}| \simeq |\tilde{k}_4| \simeq N$ and for J_t^1 we used that $|k_{[123]}| \simeq N \gtrsim |\tilde{k}_{[123]}|$. Now by a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0, p > 1$

$$\pi_0(u_2^\varepsilon, e_N^{i_1i_2i_3}u_1^\varepsilon) \rightarrow 0 \text{ in } L^p(\Omega; C_T \mathcal{C}^{-\delta}).$$

Convergence of $\pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})$

Now we have the following identity for $t \in [0, T]$:

$$\pi_0(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})(t) = -[I_t^1 + 6I_t^2 + 6I_t^3 + J_t^1 + 6J_t^2 + 6J_t^3],$$

where

$$I_t^1 = 2^{-6} \int e_{\tilde{k}_{[12345]}} \psi_0(k_{[123]}, \tilde{k}_{[45]}) \begin{array}{c} \eta_2 \\ \vdots \\ \eta_1 \quad \sigma \quad \eta_5 \\ \vdots \\ \eta_4 \end{array} W(d\eta_{12345}),$$

$$I_t^2 = 2^{-6} \int \int e_{\tilde{k}_{[234]}} \psi_0(k_{[123]}, \tilde{k}_4 - k_1) \begin{array}{c} \eta_2 \\ \vdots \\ \eta_3 \quad \sigma \quad k_1 \\ \vdots \\ \eta_4 \end{array} dk_1 W(d\eta_{234}),$$

$$I_t^3 = 2^{-6} \int \int e_{\tilde{k}_3} \psi_0(k_{[123]}, \tilde{k}_{[12]}) \begin{array}{c} \eta_3 \\ \vdots \\ k_2 \quad \sigma \quad k_1 \end{array} dk_{12} W(d\eta_3),$$

and for $i = 1, 2, 3$, J_t^i is defined similarly as I_t^i with each $k_{[123]}$, $e_{\tilde{k}_{[12345]}}$, $e_{\tilde{k}_{[234]}}$, $e_{\tilde{k}_3}$ replaced by $\tilde{k}_{[123]}$, $e_{\tilde{k}_{[12345]}}$, $e_{\tilde{k}_{[234]}}$, $e_{\tilde{k}_3}$, respectively.

Terms in the first chaos We consider J_t^3 . I_t^3 can be estimated similarly. We decompose $J_t^3 = J_t^{31} + J_t^{32}$, with J_t^{31}, J_t^{32} associated with the terms that $\tilde{k}_3 \neq k_3$ and $\tilde{k}_3 = k_3$, respectively. For J_t^{31} we have

$$\begin{aligned} & E[|\Delta_q J_t^{31}|^2] \\ & \lesssim \int \frac{1_{|k_3|_\infty \leq N}}{|k_3|^2} \theta(2^{-q} \tilde{k}_3) \left(\int \frac{1_{|k_1| \lesssim N, |k_2| \lesssim N}}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |\tilde{k}_{[123]}|^2)} dk_{12} \right)^2 dk_3 \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}. \end{aligned}$$

Here we used that $|\tilde{k}_3| \simeq 2^q \simeq N$ in the last inequality. For J_t^{32} we consider

$$J_t^{32} - \tilde{J}_t^{32} + \tilde{J}_t^{32} - C_2^\varepsilon(t) u_1^\varepsilon(t),$$

where \tilde{J}_t^{32} is defined as J_t^{32} with $P_{\sigma-s_3}^\varepsilon(k_3)$ replaced by $P_{t-s_3}^\varepsilon(k_3)$ and $C_2^\varepsilon(t) = \frac{1}{2}(C_{12}^\varepsilon + \varphi_2^\varepsilon(t))$.

Since $\int |P_{t-s_3}^\varepsilon(k_3) - P_{\sigma-s_3}^\varepsilon(k_3)|^2 ds_3 \leq C \frac{(t-\sigma)^{\kappa/2}}{|k_3|^{2-\kappa}}$, by a straightforward calculation we obtain that for $\kappa > 0$ small enough

$$\begin{aligned} & E[|\Delta_q (J_t^{32} - \tilde{J}_t^{32})|^2] \\ & \lesssim \int \theta(2^{-q} k_3)^2 \frac{1}{|k_3|^{2-4\kappa}} \left(\int_0^t \int \frac{e^{-(|\tilde{k}_{[123]}|^2 + |k_1|^2 + |k_2|^2) \bar{c}_f(t-\sigma)}}{|k_1|^2 |k_2|^2} (t-\sigma)^\kappa 1_{|k_{[123]}| \simeq N} dk_{12} d\sigma \right)^2 dk_3 \\ & \lesssim \varepsilon^\kappa 2^{q(1+5\kappa)}. \end{aligned}$$

Here in the last inequality we used that $|k_{123}| \simeq N$ implies that $|k_i| \simeq N$ for some $i \in \{1, 2, 3\}$ and that $\sup_{a \geq 0} a^r e^{-a} \leq C$ for $r \geq 0$ and Lemma 6.1. Moreover, by Lemmas 6.2 and 6.3 we

obtain that

$$\begin{aligned}
& E[|\Delta_q(\tilde{J}_t^{32} - u_1^\varepsilon(t)C_2^\varepsilon(t))|^2] \\
& \lesssim \int \frac{1}{|k_3|^2} \theta(2^{-q}k_3) \left(\int \int_0^t |\tilde{k}_{[12]}|^{-\kappa} |k_3|^\kappa \frac{e^{-|k_1|^2(t-\sigma)\bar{c}_f - |k_2|^2(t-\sigma)\bar{c}_f}}{|k_1|^2|k_2|^2} dk_{12} d\sigma \right)^2 dk_3 \\
& \lesssim \int \theta(2^{-q}k_3) \frac{1}{|k_3|^{2-2\kappa}} dk_3 \left[\int_{|k_{[12]}| \leq N} \frac{1}{|\tilde{k}_{[12]}|^\kappa |k_{[12]}|^3} dk_{[12]} + \varepsilon^{\kappa/2} \int_{|k_{[12]}| > N} \frac{1}{|\tilde{k}_{[12]}|^{3+\kappa/2}} dk_{[12]} \right]^2 \lesssim \varepsilon^\kappa 2^{q(1+2\kappa)},
\end{aligned}$$

where in the last inequality we used that if $|k_{[12]}| \leq N$, then $|\tilde{k}_{[12]}| \simeq N$.

Terms in the third and fifth chaos can be estimated similarly as done for the case of $\pi_{0,\diamond}(u_2^\varepsilon, (u_1^\varepsilon)^{\diamond 2})$ and we also obtain that there exist $\kappa, \epsilon, \gamma > 0$ small enough such that for any $t_1, t_2 \in [0, T]$

$$\begin{aligned}
& E[|\Delta_q(\pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})(t_1) - \pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2})(t_2))|^2] \\
& \lesssim \varepsilon^\gamma |t_1 - t_2|^\kappa 2^{q(1+\epsilon)},
\end{aligned}$$

which by Gaussian hypercontractivity and Lemma 2.1 implies that for every $\delta > 0, p > 1$, $\pi_{0,\diamond}(u_2^\varepsilon, e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond 2}) \rightarrow 0$ in $L^p(\Omega; C_T \mathcal{C}^{-1/2-\delta/2})$.

6.3 Convergence of random operators

The purpose of this subsection is to prove that A_N defined in Proposition 4.3 converges to zero in probability. Here we follow essentially the same arguments as in [GP15, Section 10.2].

Theorem 6.7 For every $T \geq 0, 0 < \eta < \delta, r \geq 1$ we have

$$E[(A_N)^r]^{1/r} \lesssim N^{-\delta+\eta}.$$

Here δ is fixed in Section 4.

To prove Theorem 6.7 we use similar arguments as in [GP15, Section 10.2] and obtain the following two lemmas.

Lemma 6.8 We have

$$\begin{aligned}
& A_N^1(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})(f) + A_N^2(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})(f) \\
& = \sum_{p,q} \int_{\mathbb{T}^3} g_{p,q}^N(t, x, y) \Delta_p f(y) dy
\end{aligned}$$

with

$$\mathcal{F}g_{p,q}^N(t, x, \cdot)(k) = \sum_{k_1, k_2} \Gamma_{p,q}^N(x, k, k_1, k_2) \mathcal{F}(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon)(t, k_1) \mathcal{F}((u_1^\varepsilon)^{\diamond,2} + e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2})(t, k_2).$$

Here

$$\begin{aligned}
\Gamma_{p,q}^N(x, k, k_1, k_2) & = 2^{-9/2} e^{i(k_1+k_2-k)\pi x} \theta_q(k_1 + k_2 - k) \tilde{\theta}_p(k) \psi_{<}(k, k_1) \psi_0(k_1 - k, k_2) \\
& \quad (\mathbf{1}_{|k_1-k|_\infty > N} \mathbf{1}_{|k_1|_\infty \leq N} + \mathbf{1}_{|k_1-k|_\infty \leq N} \mathbf{1}_{N < |k_1|_\infty \leq 3N}),
\end{aligned}$$

with $\tilde{\theta}_p$ being a smooth function supported in an annulus $2^p \mathcal{A}$ such that $\tilde{\theta}_p \theta_p = \theta_p$.

Lemma 6.9 For all $r \geq 1, \kappa > 0, \delta > 0$ we have for $A_N^1(t) := A_N^1(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$, $A_N^2(t) := A_N^2(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$

$$\begin{aligned} & E[\|A_N^1(t) + A_N^2(t) - (A_N^1(s) + A_N^2(s))\|_{L(\mathcal{C}^{1-\delta}, B_{r,r}^{-1/2-2\delta+\kappa})}^r] \\ & \lesssim \sum_{p,q} 2^{qr(-1/2-2\delta+\kappa)} 2^{-pr(1-\delta)} \left(\sup_{x \in \mathbb{T}^3} \sum_k E[|(\mathcal{F}g_{p,q}^N(t, x, \cdot) - \mathcal{F}g_{p,q}^N(s, x, \cdot))(k)|^2] \right)^{r/2}. \end{aligned}$$

Lemma 6.10 For all $p, q \geq -1$, all $0 \leq t_1 < t_2$, and all $\lambda, \kappa \in (0, 1]$ we have

$$\sum_k E[|(\mathcal{F}g_{p,q}^N(t_2, x, \cdot) - \mathcal{F}g_{p,q}^N(t_1, x, \cdot))(k)|^2] \lesssim 1_{2^p, 2^q \lesssim N} (2^{3p} 2^{2q} + 2^{2p} 2^{3q}) N^{-2+2\lambda+\kappa} |t_1 - t_2|^\lambda.$$

Proof We only prove the estimate for $\sum_k E[|\mathcal{F}g_{p,q}^N(t, x, \cdot)(k)|^2]$. The above assertion can be obtained by essentially the same arguments. First we consider the term in A_N containing \tilde{K}^ε and $(u_1^\varepsilon)^{\diamond, 2}$. We have the following chaos decomposition:

$$\mathcal{F}\tilde{K}^\varepsilon(t, l_1) \mathcal{F}(u_1^\varepsilon)^{\diamond, 2}(t, l_2) = I_t^1 + 4I_t^2 + 2I_t^3.$$

Here

$$\begin{aligned} I_t^1 &= 2^{-3} \int 1_{k_{[12]}=l_1, k_{[34]}=l_2} \int_0^t d\sigma p_{t-\sigma}^\varepsilon(k_{[12]}) \varphi(\varepsilon k_{[12]}) P_{\sigma-s_1}^\varepsilon(k_1) P_{\sigma-s_2}^\varepsilon(k_2) P_{t-s_3}^\varepsilon(k_3) P_{t-s_4}^\varepsilon(k_4) W(d\eta_{1234}), \\ I_t^2 &= 2^{-3} \int 1_{k_{[12]}=l_1, k_3-k_1=l_2} \int_0^t d\sigma p_{t-\sigma}^\varepsilon(k_{[12]}) \varphi(\varepsilon k_{[12]}) P_{\sigma-s_2}^\varepsilon(k_2) P_{t-s_3}^\varepsilon(k_3) V_{t-\sigma}^\varepsilon(k_1) dk_1 W(d\eta_{23}), \\ I_t^3 &= 2^{-3} \int \int_0^t d\sigma 1_{k_{[12]}=l_1, -k_{[12]}=l_2} V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) p_{t-\sigma}^\varepsilon(k_{[12]}) \varphi(\varepsilon k_{[12]}) dk_{12}. \end{aligned}$$

The graph for $I_t^i, i = 1, 2, 3$ is similar as that of $\pi_0(K_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2})$ and we omit them here.

Term in the chaos of order 0 By a similar calculation as in Section 6.1 we have

$$\begin{aligned} & \sum_k \left| \sum_{k_1, k_2} \Gamma_{p,q}^N(x, k, k_1, k_2) 1_{k_1+k_2=0} I_t^3 \right|^2 \\ & \lesssim \sum_k \left| \sum_{k_1} \Gamma_{p,q}^N(x, k, k_1, -k_1) \frac{1}{|k_1|^3} \right|^2 \\ & \lesssim \sum_k \tilde{\theta}_p(k)^2 \theta_q(-k)^2 \left| \sum_{k_1} (1_{|k_1-k|_\infty > N} 1_{|k_1|_\infty \leq N} + 1_{|k_1-k|_\infty \leq N} 1_{N < |k_1|_\infty \leq 3N}) \psi_\lt(k, k_1) \psi_0(k_1 - k, k_1) \frac{1}{|k_1|^3} \right|^2. \end{aligned}$$

In the first case without loss of generality we may assume that $|k_1^i - k^i| > N$ for some i . Then there are at most $|k^i|$ values of k_1^i with $|k_1^i| \leq N$ and $|k_1^i - k^i| > N$. In the second case for $1_{|k_1-k|_\infty \leq N} 1_{N < |k_1|_\infty \leq 3N}$ without loss of generality we may assume that $|k_1^i| > N$ for some i . Then there are at most $|k^i|$ values of k_1^i with $|k_1^i| > N$ and $|k_1^i - k^i| \leq N$. Moreover, observe that $|k_1| \simeq N$ on the support of $(1_{|k_1-k|_\infty > N} 1_{|k_1|_\infty \leq N} + 1_{|k_1-k|_\infty \leq N} 1_{|k_1|_\infty > N}) \psi_0(k - k_1, k_1)$ and that $|k| \lesssim N$ whenever $1_{|k_1|_\infty \leq 3N} \psi_\lt(k, k_1) \neq 0$, which implies that the above term is bounded by

$$\sum_k \tilde{\theta}_p(k)^2 \theta_q(-k)^2 |k|^2 1_{|k| \lesssim N} N^{-2} \lesssim 1_{2^p, 2^q \lesssim N} 2^{3p} 2^{2q} N^{-2}.$$

Term in the second chaos By a similar calculation as in Section 6.1 we have

$$\begin{aligned}
& \sum_k E \left| \sum_{l_1, l_2} \Gamma_{p,q}^N(x, k, l_1, l_2) I_t^2 \right|^2 \\
& \lesssim \sum_k 1_{2^p, 2^q \lesssim N} \tilde{\theta}_p(k)^2 \int \theta_q(k_{[23]} - k)^2 \Pi_{i=2}^3 \frac{1}{|k_i|^2} \left[\int \psi_{<}(k, k_{[12]}) \frac{1}{(|k_{[12]}|^2 + |k_1|^2) |k_1|^2} \right. \\
& \quad \left. (1_{|k_{[12]}-k|_\infty > N, |k_{[12]}|_\infty \leq N} + 1_{|k_{[12]}-k|_\infty \leq N, N < |k_{[12]}|_\infty \leq 3N}) dk_1 \right]^2 dk_{23} \\
& \lesssim \sum_k 1_{2^p, 2^q \lesssim N} \tilde{\theta}_p(k)^2 \int \theta_q(k_{[23]} - k)^2 \frac{1}{|k_{[23]}|} N^{-2+\kappa} dk_{[23]} \\
& \lesssim \sum_k 1_{2^p, 2^q \lesssim N} \tilde{\theta}_p(k)^2 N^{-2+\kappa} 2^{2q} \lesssim 1_{2^p, 2^q \lesssim N} 2^{3p} 2^{2q} N^{-2+\kappa}.
\end{aligned}$$

Here in the second inequality we used that $|k_{[12]}| \simeq N$ on the support of $1_{|k_{[12]}-k|_\infty > N, |k_{[12]}|_\infty \leq N} \psi_{<}(k, k_{[12]})$ and in the third inequality we used Lemma 6.5.

Term in the fourth chaos We have

$$\begin{aligned}
& \sum_k E \left| \sum_{l_1, l_2} \Gamma_{p,q}^N(x, k, l_1, l_2) I_t^1 \right|^2 \\
& \lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(k_{[1234]} - k)^2 \psi_{<}(k, k_{[12]})^2 \psi_0(k_{[12]} - k, k_{[34]})^2 \frac{1_{2^p, 2^q \lesssim N}}{|k_2|^2 |k_3|^2 |k_1|^2 |k_4|^2 |k_{[12]}|^4} \\
& \quad (1_{|k_{[12]}-k|_\infty > N, |k_{[12]}|_\infty \leq N} + 1_{|k_{[12]}-k|_\infty \leq N, N < |k_{[12]}|_\infty \leq 3N}) dk_{1234} \\
& \lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(k_{[1234]} - k)^2 \frac{1_{2^p, 2^q \lesssim N}}{|k_{[1234]}|^{1-\kappa}} dk_{[1234]} N^{-2-\kappa} \\
& \lesssim 1_{2^p, 2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 N^{-2-\kappa} 2^{(2+\kappa)q} \\
& \lesssim 1_{2^p, 2^q \lesssim N} 2^{3p} 2^{2q} N^{-2},
\end{aligned}$$

where we used Lemma 6.1 and that $|k_{[12]}| \simeq N$ in the second inequality.

Moreover, we consider

$$\mathcal{F} \tilde{K}_1^\varepsilon(t, l_1) \mathcal{F}(u_1^\varepsilon)^{\diamond, 2}(t, l_2) = J_t^1 + 4J_t^2 + 2J_t^3.$$

Here $J_t^i, i = 1, 2$, is defined similarly as $I_t^i, i = 1, 2$, with $k_{[12]}$ replaced by $\tilde{k}_{[12]}$ and

$$J_t^3 = 2^{-3} \int \int_0^t d\sigma 1_{\tilde{k}_{[12]}=l_1, -k_{[12]}=l_2} V_{t-\sigma}^\varepsilon(k_1) V_{t-\sigma}^\varepsilon(k_2) p_{t-\sigma}^\varepsilon(\tilde{k}_{[12]}) \varphi(\varepsilon \tilde{k}_{[12]}) dk_{12}.$$

Terms in the chaos of order 0 We have

$$\begin{aligned}
& \sum_k \left| \sum_{k_1, k_2} \Gamma_{p,q}^N(x, k, \tilde{k}_1, k_2) 1_{k_1+k_2=0} J_t^3 \right|^2 \\
& \lesssim \sum_k \left| \sum_{k_1} \Gamma_{p,q}^N(x, k, \tilde{k}_1, -k_1) 1_{|k_1|_\infty \lesssim N} \frac{1}{|k_1| |\tilde{k}_1|^2} \right|^2 \\
& \lesssim \sum_k \tilde{\theta}_p(k)^2 \theta_q(\tilde{k})^2 \left| \sum_{k_1} (1_{|\tilde{k}_1-k|_\infty > N} 1_{|\tilde{k}_1|_\infty \leq N} + 1_{|\tilde{k}_1-k|_\infty \leq N} 1_{N < |\tilde{k}_1|_\infty \leq 3N}) \psi_{<}(k, \tilde{k}_1) 1_{|k_1|_\infty \lesssim N} \frac{1}{|k_1| |\tilde{k}_1|^2} \right|^2.
\end{aligned}$$

Similarly as above we obtain that there are at most $|k^i|$ values of \tilde{k}_1^i with $|\tilde{k}_1^i| > N$ and $|\tilde{k}_1^i - k^i| \leq N$ or $|\tilde{k}_1^i| \leq N$ and $|\tilde{k}_1^i - k^i| > N$. Moreover, observe that $|\tilde{k}_1| \simeq N$ on the support of $1_{|\tilde{k}_1 - k|_\infty > N} 1_{|\tilde{k}_1|_\infty \leq N} \psi_\lt(k, \tilde{k}_1)$ and that $|k| \lesssim N$ whenever $1_{|k_1|_\infty < 3N} \psi_\lt(k, k_1) \neq 0$, which implies that the above term is bounded by

$$\sum_k \tilde{\theta}_p(k)^2 \theta_q(\tilde{k})^2 |k|^{21} 1_{|k| \lesssim N} N^{-2} \lesssim 1_{2^p, 2^q \lesssim N} 2^{2p} 2^{3q} N^{-2}.$$

For the terms in the second chaos by a similar calculation as above we obtain the desired estimates.

Terms in the fourth chaos We have

$$\begin{aligned} & \sum_k E \left| \sum_{l_1, l_2} \Gamma_{p,q}^N(x, k, l_1, l_2) J_t^1 \right|^2 \\ & \lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k)^2 \psi_\lt(k, \tilde{k}_{[12]})^2 \psi_0(\tilde{k}_{[12]}, k_{[34]})^2 \frac{1_{2^p, 2^q \lesssim N}}{|k_2|^2 |k_3|^2 |k_1|^2 |k_4|^2 |\tilde{k}_{[12]}|^4} 1_{|k_{[12]}| \lesssim N, |k_{[34]}| \lesssim N} dk_{1234} \\ & \quad (1_{|\tilde{k}_{12} - k|_\infty > N, |\tilde{k}_{12}|_\infty \leq N} + 1_{|\tilde{k}_{12} - k|_\infty \leq N, N < |\tilde{k}_{12}|_\infty \leq 3N}) \\ & \lesssim 1_{2^p, 2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k)^2 \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} N^{-4} 1_{|k_{[12]}| \lesssim N, |k_{[34]}| \lesssim N} dk_{1234} \\ & \lesssim 1_{2^p, 2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k) \frac{1}{|k_{[12]}|^2 |k_{[34]}|^2} dk_{[12][34]} N^{-2} \\ & \lesssim 1_{2^p, 2^q \lesssim N} 2^{3p} 2^{2q} N^{-2}. \end{aligned}$$

Here in the second inequality we used that $|\tilde{k}_{[12]}| \simeq N$ and in the third inequality we used that $N^{-1} \lesssim |k_{[12]}|^{-1}$, $N^{-1} \lesssim |k_{[34]}|^{-1}$ and in the last inequality we used Lemma 6.5.

Furthermore, for the terms associated with $\tilde{K}^\varepsilon(u_1^\varepsilon)^{\diamond, 2} e_N^{i_1 i_2 i_3}$ and $\tilde{K}_1^\varepsilon(u_1^\varepsilon)^{\diamond, 2} e_N^{i_1 i_2 i_3}$ we obtain the desired estimates by similar arguments. Thus the result follows. \square

Proof of Theorem 6.7 For $t, s \geq 0, r > 0$ large enough we have for $A_N^1(t) := A_N^1(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$, $A_N^2(t) := A_N^2(\tilde{K}^\varepsilon + \tilde{K}_1^\varepsilon, (u_1^\varepsilon)^{\diamond, 2} + e_N^{i_1 i_2 i_3} (u_1^\varepsilon)^{\diamond, 2})$

$$\begin{aligned} & E[\| (A_N^1(t) + A_N^2(t) - (A_N^1(s) + A_N^2(s))) \|_{L(C^{1-\delta}, C^{-1/2-2\delta})}^r] \\ & \lesssim E[\| (A_N^1(t) + A_N^2(t) - (A_N^1(s) + A_N^2(s))) \|_{L(C^{1-\delta}, B_{r,r}^{-1/2-2\delta+\kappa})}^r] \\ & \lesssim \sum_{p,q} 2^{qr(-1/2-2\delta+\kappa)} 2^{-pr(1-\delta)} 1_{2^p, 2^q \lesssim N} [(2^{3p} 2^{2q} + 2^{2p} 2^{3q}) |t - s|^\lambda N^{-2+2\lambda+\kappa}]^{r/2} \\ & \lesssim |t - s|^{r\lambda/2} N^{(-\delta+2\kappa+\lambda)r}. \end{aligned}$$

Here $\delta > 2\kappa + \lambda > 0$. Thus the result follows by using Kolmogorov's continuity criterion. \square

6.4 Convergence of D^N

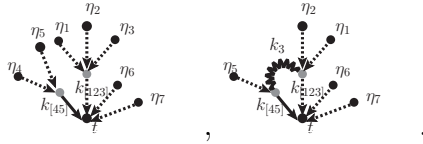
In this subsection we prove that $D^N \xrightarrow{P} 0$ as $\varepsilon \rightarrow 0$. Here we use the fact that there exists some $|k_j| \simeq N$ to produce ε^κ . We have the following identity for $t \in [0, T]$:

$$\pi_0((I - P_N)\pi_\lt(u_2^\varepsilon, K^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t) + \pi_0(P_N \pi_\lt(u_2^\varepsilon, (P_{3N} - P_N)\tilde{K}^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t) = \sum_{i=1}^4 (I_t^i + J_t^i),$$

where

$$\begin{aligned}
I_t^1 &= 2^{-9} \int e_{k_{[1234567]}} \psi_0(k_{[12345]}, k_{[67]}) \psi_{<}(k_{[123]}, k_{[45]}) (1_{|k_{[12345]}|_\infty > N} 1_{|k_{[45]}|_\infty \leq N} + 1_{|k_{[12345]}|_\infty \leq N} 1_{N < |k_{[45]}|_\infty \leq 3N}) \\
&\quad \int_0^t \int_0^t d\sigma d\bar{\sigma} P_{t-\sigma}^\varepsilon(k_{[123]}) \prod_{i=1}^3 P_{\sigma-s_i}^\varepsilon(k_i) p_{t-\bar{\sigma}}^\varepsilon(k_{[45]}) \varphi(\varepsilon k_{[45]}) \prod_{i=4}^5 P_{\bar{\sigma}-s_i}^\varepsilon(k_i) \prod_{i=6}^7 P_{t-s_i}^\varepsilon(k_i) W(d\eta_{1234567}) \\
&:= \int G(t, x, \eta_{1234567}) W(d\eta_{1234567}), \\
I_t^2 &= \sum_{i=1}^3 I_t^{2i}, I_t^{21} = 6 \int \int G(t, x, \eta_{123(-3)567}) d\eta_3 W(d\eta_{12567}), \\
I_t^{22} &= 6 \int G(t, x, \eta_{12345(-3)7}) d\eta_3 W(d\eta_{12457}), I_t^{23} = 4 \int G(t, x, \eta_{12345(-5)7}) d\eta_5 W(d\eta_{12347}), \\
I_t^3 &= \sum_{i=1}^6 I_t^{3i}, I_t^{31} = 6 \int \int G(t, x, \eta_{123(-3)(-2)67}) d\eta_{23} W(d\eta_{167}), \\
I_t^{32} &= 24 \int \int G(t, x, \eta_{123(-3)5(-2)7}) d\eta_{23} W(d\eta_{157}), I_t^{33} = 12 \int \int G(t, x, \eta_{123(-3)5(-5)7}) d\eta_{35} W(d\eta_{127}) \\
I_t^{34} &= 6 \int \int G(t, x, \eta_{12345(-2)(-3)}) d\eta_{23} W(d\eta_{145}), I_t^{35} = 12 \int \int G(t, x, \eta_{12345(-3)(-4)}) d\eta_{34} W(d\eta_{125}) \\
I_t^{36} &= 2 \int \int G(t, x, \eta_{12345(-4)(-5)}) d\eta_{45} W(d\eta_{123}), \\
I_t^4 &= \sum_{i=1}^3 I_t^{4i}, I_t^{41} = 12 \int \int G(t, x, \eta_{123(-1)(-2)(-3)7}) d\eta_{123} W(d\eta_7), \\
I_t^{42} &= 12 \int \int G(t, x, \eta_{123(-3)5(-2)(-1)}) d\eta_{123} W(d\eta_5), I_t^{43} = 24 \int \int G(t, x, \eta_{123(-3)5(-5)-2}) d\eta_{235} W(d\eta_1),
\end{aligned}$$

and J_t^1 is defined similarly as I_t^1 with $k_{[123]}, k_{[12345]}, e_{k_{[1234567]}}$ replaced by $\tilde{k}_{[123]}, \tilde{k}_{[12345]}, e_{\tilde{k}_{[1234567]}}$ respectively and $J_t^i, i = 2, 3, 4$ is defined similarly as I_t^i with the G replaced by that associated with J^1 . For the readers's convenience, we use the graph notations to denote $G(t, x, \eta_{1234567})$ and the term for I_t^{21} . The graphs for other terms are similar as the graph for I_t^{21} with the corresponding lines connected. Here we omit $(I - P_N), P_N, \pi_0, \pi_{<}$ in the graph for simplicity.



Terms in the seventh chaos We have

$$\begin{aligned}
E|\Delta_q I_t^1|^2 &\lesssim \int \theta(2^{-q} k_{[1234567]}) \psi_0(k_{[12345]}, k_{[67]}) \psi_{<}(k_{[123]}, k_{[45]}) (1_{|k_{[12345]}|_\infty > N, |k_{[45]}|_\infty \leq N} + 1_{|k_{[12345]}|_\infty \leq N, |k_{[45]}|_\infty > N}) \\
&\quad 1_{|k_{[1234567]}| \lesssim N} \prod_{i=1}^7 \frac{1}{|k_i|^2} \frac{1}{|k_{[123]}|^2 |k_{[45]}|^2 (|k_{[123]}|^2 + \sum_{i=1}^3 |k_i|^2) (|k_{[45]}|^2 + \sum_{i=4}^5 |k_i|^2)} dk_{1234567}.
\end{aligned}$$

Observe that $|k_{45}|_\infty \simeq N$ on the support of $\psi_{<}(k_{[123]}, k_{[45]})1_{|k_{[12345]}|_\infty > N}$, which combined with Lemma 6.1 implies that the above term is bounded by

$$\begin{aligned} & \int \theta(2^{-q}k_{[1234567]})1_{|k_{[45]}|_\infty \simeq N, 2^q \lesssim N} \frac{1}{|k_{[123]}|^4 |k_{[45]}|^5 |k_{[67]}|} dk_{[123][45][67]} \\ & \lesssim \int 1_{2^q \lesssim N} \theta(2^{-q}k_{[1234567]}) \frac{N^{-2-\kappa}}{|k_{[12345]}|^{3-\kappa}} \frac{1}{|k_{[67]}|} dk_{[12345][67]} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Terms in the fifth chaos: Consider I_t^{21} first: by the formula we know that $|k_5 - k_3| \simeq N$, hence

$$\begin{aligned} & E|\Delta_q I_t^{21}|^2 \\ & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[12567]}) \prod_{i=5}^7 \frac{1}{|k_i|^2} \frac{1}{|k_1|^2 |k_2|^2} \left[\left(\int \frac{1}{|k_3|^2 (|k_5 - k_3|^2 + |k_3|^2) (|k_{[123]}|^2 + |k_5 - k_3|^2)} dk_3 \right)^2 \right. \\ & \quad \left. + \left(\int \frac{1}{|k_3|^2 (|k_5 - k_3|^2 + |k_{[123]}|^2) (|k_{[123]}|^2 + |k_3|^2)} dk_3 \right)^2 \right] 1_{\{|k_5 - k_3| \simeq N, |k_5| \lesssim N, |k_{[12]}| \lesssim N\}} dk_{12567} \\ & \lesssim \int \theta(2^{-q}k_{[12567]}) 1_{2^q \lesssim N} \frac{N^{-4+2\kappa}}{|k_{[12]}|^{3-\kappa} |k_5|^{2+2\kappa} |k_{[67]}|} dk_{[12]5[67]} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Here in the first inequality we consider $\sigma > \bar{\sigma}$ and $\sigma \leq \bar{\sigma}$ separately and we used that $|k_{[123]}|^2 + |k_3|^2 \gtrsim |k_{[12]}|^2$ in the second inequality.

Now we consider I_t^{22} and in this case we have that $|k_{[45]}| \simeq N$, which implies that

$$\begin{aligned} E|\Delta_q I_t^{22}|^2 & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[12457]}) 1_{|k_{45}| \simeq N} \frac{1}{|k_1|^2 |k_2|^2 |k_{[45]}|^5 |k_7|^2} \left(\int \frac{1}{(|k_{[123]}|^2 + |k_3|^2) |k_3|^2} dk_3 \right)^2 dk_{12[45]7} \\ & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[12457]}) \frac{N^{-2-\kappa}}{|k_{[12]}|^{3-\kappa} |k_{[45]}|^{3-\kappa} |k_7|^2} dk_{[12][45]7} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Here in the second inequality we used that $|k_{[123]}|^2 + |k_3|^2 \gtrsim |k_{[12]}|^2$.

For I_t^{23} we have that $|k_{[45]}| \simeq N$, hence

$$\begin{aligned} E|\Delta_q I_t^{23}|^2 & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[12347]}) 1_{\{|k_{[45]}| \simeq N\}} \prod_{i=1}^4 \frac{1}{|k_i|^2} \frac{1}{|k_7|^2} \\ & \quad \frac{1}{(|k_{[123]}|^2 + \sum_{i=1}^3 |k_i|^2) |k_{[123]}|^2} \left(\int \frac{1}{(|k_{[45]}|^2 + |k_5|^2) |k_5|^2} dk_5 \right)^2 dk_{12347} \\ & \lesssim \int 1_{2^q \lesssim N} \theta(2^{-q}k_{[12347]}) \frac{1}{|k_{[1234]}|^{2-\kappa}} \frac{N^{-2+\kappa}}{|k_7|^2} dk_{[1234]7} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Terms in the third chaos: For I_t^{31} , we have that

$$\begin{aligned} E|\Delta_q I_t^{31}|^2 & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[167]}) \psi_0(k_1, k_{[67]}) \psi_{<}(k_{[123]}, k_{[23]}) (1_{|k_1|_\infty > N, |k_{[23]}|_\infty \leq N} + 1_{|k_1|_\infty \leq N, N < |k_{[23]}|_\infty \leq 3N}) \\ & \quad \frac{1}{|k_1|^2} \frac{1}{|k_6|^2 |k_7|^2} \left(\int \frac{1}{(|k_{[123]}|^2 + |k_{[23]}|^2) |k_{[23]}|^2 |k_2|^2 |k_3|^2} dk_{23} \right)^2 1_{|k_1| \lesssim N} dk_{167} \\ & \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[167]}) \frac{N^{-3-\kappa}}{|k_1|^{3-\kappa} |k_{[67]}|} dk_{1[67]} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Here we used that $|k_{[23]}|^2 \lesssim |k_2|^2 + |k_3|^2$ in the first inequality and in the second inequality we used that $|k_{[23]}| \asymp N$.

For I_t^{32} we obtain that

$$\begin{aligned} E|\Delta_q I_t^{32}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[157]}) 1_{\{|k_5 - k_3|_\infty \asymp N\}} \frac{1}{|k_1|^2 |k_5|^2 |k_7|^2} 1_{|k_5| \leq N} \\ &\quad \left[\left(\int \frac{1}{(|k_5 - k_3|^2 + |k_3|^2)(|k_{[123]}|^2 + |k_2|^2 + |k_5 - k_3|^2) |k_2|^2 |k_3|^2} dk_{23} \right)^2 \right. \\ &\quad \left. + \left(\int \frac{1}{(|k_{[123]}|^2 + |k_2|^2 + |k_3|^2)(|k_{[123]}|^2 + |k_2|^2 + |k_5 - k_3|^2) |k_2|^2 |k_3|^2} dk_{23} \right)^2 \right] dk_{157} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[157]}) N^{-3} \frac{1}{|k_1|^2 |k_5|^{3-\kappa} |k_7|^2} dk_{157} \lesssim \varepsilon^\kappa 2^{2q\kappa}, \end{aligned}$$

Here in the first inequality we consider $\sigma > \bar{\sigma}$ and $\sigma \leq \bar{\sigma}$ separately. For I_t^{33} we have that

$$\begin{aligned} E|\Delta_q I_t^{33}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[127]}) 1_{|k_5 - k_3| \asymp N, |k_{[12]}| \lesssim N} \frac{1}{|k_1|^2 |k_2|^2 |k_7|^2} \\ &\quad \left[\left(\int \frac{1}{(|k_3|^2 + |k_5 - k_3|^2 + |k_5|^2)(|k_{[123]}|^2 + |k_5 - k_3|^2 + |k_5|^2) |k_3|^2 |k_5|^2} dk_{35} \right)^2 \right. \\ &\quad \left. + \left(\int \frac{1}{(|k_3|^2 + |k_{[123]}|^2)(|k_{[123]}|^2 + |k_5 - k_3|^2 + |k_5|^2) |k_3|^2 |k_5|^2} dk_{35} \right)^2 \right] dk_{127} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[127]}) \frac{N^{-2}}{|k_{[12]}|^{3-\kappa} |k_7|^2} dk_{[12]7} \lesssim \varepsilon^\kappa 2^{q\kappa}. \end{aligned}$$

Here in the first inequality we consider $\sigma > \bar{\sigma}$ and $\sigma \leq \bar{\sigma}$ separately. For I_t^{34} we get that

$$\begin{aligned} E|\Delta_q I_t^{34}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[145]}) 1_{|k_{[45]}| \asymp N} \frac{1}{|k_1|^2 |k_{[45]}|^5} \\ &\quad \left(\int \frac{1_{|k_{[23]}| \lesssim N}}{(|k_{[123]}|^2 + \sum_{i=2}^3 |k_i|^2) |k_2|^2 |k_3|^2} dk_{23} \right)^2 dk_{145} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[145]}) 1_{|k_{[45]}| \asymp N} \frac{N^\kappa}{|k_{[45]}|^5 |k_1|^2} dk_{1[45]} \lesssim \varepsilon^\kappa 2^{q\kappa}. \end{aligned}$$

For I_t^{35} we have

$$\begin{aligned} E|\Delta_q I_t^{35}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[125]}) 1_{|k_{[45]}| \asymp N} \frac{1}{|k_1|^2 |k_2|^2 |k_5|^2} \\ &\quad \left(\int \frac{1}{(|k_{[45]}|^2 + |k_4|^2)(|k_{[123]}|^2 + |k_3|^2) |k_3|^2 |k_4|^2} dk_{34} \right)^2 dk_{125} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[125]}) \frac{N^{-2+\kappa}}{|k_5|^2 |k_{[12]}|^{3-\kappa}} dk_{[12]5} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Here in the second inequality we used $|k_{[123]}|^2 + |k_3|^2 \gtrsim |k_{[12]}|^2$. For I_t^{36} we obtain that

$$E|\Delta_q I_t^{36}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[123]}) \frac{1}{|k_{[123]}|^4} \left(\int \frac{(1_{|k_{[12345]}|_\infty > N} 1_{|k_{[45]}|_\infty \leq N} + 1_{|k_{[12345]}|_\infty \leq N} 1_{N < |k_{[45]}|_\infty \leq 3N})}{(|k_{[45]}|^2 + \sum_{i=4}^5 |k_i|^2) |k_4|^2 |k_5|^2} dk_{45} \right)^2 dk_{123},$$

Now we use similar argument as in Section 6.3. For the case that $1_{|k_{[12345]}|_\infty > N} 1_{|k_{[45]}|_\infty \leq N}$ without loss of generality we assume that $|k_{[123]}^i + k_{[45]}^i| > N$ for some i . Then there are at most $|k_{[123]}^i|$ values of $k_{[45]}^i$ with $|k_{[12345]}| > N$ and $|k_{[45]}^i| \leq N$. For the case that $1_{|k_{[12345]}|_\infty \leq N} 1_{N < |k_{[45]}|_\infty \leq 3N}$ similarly we obtain that there are at most $|k_{[123]}^i|$ values of $k_{[45]}^i$ with $|k_{[45]}^i| > N$ and $|k_{[12345]}| \leq N$. Thus we obtain

$$E|\Delta_q I_t^{36}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[123]}) N^{-2+\kappa} \frac{1}{|k_{[123]}|^2} dk_{[123]} \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

Terms in the first chaos: For I^{41} we obtain that

$$E|\Delta_q I_t^{41}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_7) \frac{1}{|k_7|^2} \left[\int \frac{1_{|k_{[12]}| \leq N}}{|k_2|^2 |k_3|^2 |k_1|^2 (|k_{[123]}|^2 + |k_3|^2) |k_{[12]}|^2} dk_{123} \right]^2 dk_7 \\ \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_7) \frac{N^{-2+\kappa}}{|k_7|^2} dk_7 \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

For I^{42} we have that

$$E|\Delta_q I_t^{42}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_5) \left(\int 1_{|k_5 - k_3| \leq N} \prod_{i=1}^3 \frac{1_{\{|k_i|_\infty \leq N, i=1,2,3\}}}{|k_i|^2 (|k_{[123]}|^2 + \sum_{i=1}^2 |k_i|^2) |k_5 - k_3|^2} dk_{123} \right)^2 \frac{1}{|k_5|^2} dk_5 \\ \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_5) \frac{N^{-2+\kappa}}{|k_5|^2} dk_5 \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

For I^{43} we get that

$$E|\Delta_q I_t^{43}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_1) \frac{1}{|k_1|^2} \left(\int \frac{1_{|k_5 - k_3| \leq N} 1_{\{|k_i|_\infty \leq N, i=2,3,5\}}}{|k_2|^2 |k_3|^2 |k_5|^2 (|k_{[123]}|^2 + |k_2|^2 + |k_3|^2) (|k_5 - k_3|^2 + |k_5|^2)} dk_{235} \right)^2 dk_1 \\ \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_1) \frac{N^{-2+\kappa}}{|k_1|^2} dk_1 \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

Similar arguments imply the same estimate for J_t^i . By a similar calculation as above we also obtain that there exist $\kappa, \varepsilon, \gamma > 0$ small enough such that for any $t_1, t_2 \in [0, T]$

$$E[|\Delta_q(\pi_0((I - P_N)\pi_{<}(u_2^\varepsilon, K^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t_1) + \pi_0(P_N\pi_{<}(u_2^\varepsilon, (P_{3N} - P_N)\tilde{K}^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t_1) \\ - \pi_0((I - P_N)\pi_{<}(u_2^\varepsilon, K^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t_2) - \pi_0(P_N\pi_{<}(u_2^\varepsilon, (P_{3N} - P_N)\tilde{K}^\varepsilon), (u_1^\varepsilon)^{\diamond, 2})(t_2))|^2] \\ \lesssim \varepsilon^\gamma |t_1 - t_2|^\kappa 2^{q\varepsilon}.$$

Moreover, for the other terms in D^N we can use similar calculations and Lemma 6.5 to obtain the same estimates. Then by using Gaussian hypercontractivity, Lemma 2.1 and Kolomogorov continuity criterion we obtain that $D_N \xrightarrow{P} 0$ as $\varepsilon \rightarrow 0$.

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