

Closability of Quadratic Forms Associated to Invariant Probability Measures of SPDEs *

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Abstract

By using the integration by parts formula of a Markov operator, the closability of quadratic forms associated to the corresponding invariant probability measure is proved. The general result is applied to the study of semilinear SPDEs, infinite-dimensional stochastic Hamiltonian systems, and semilinear SPDEs with delay.

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1 Introduction

Let \mathbb{B} be a separable Banach space and μ a reference probability measure on \mathbb{B} . For any $k \in \mathbb{B}$, let ∂_k denote the directional derivative along k . According to [8], the form

$$\mathcal{E}_k(f, g) := \mu((\partial_k f)(\partial_k g)) := \int_{\mathbb{B}} (\partial_k f)(\partial_k g) d\mu, \quad f, g \in C_b^2(\mathbb{B}),$$

is closable on $L^2(\mu)$ if $\rho_s := \frac{d\mu(s k + \cdot)}{d\mu}$ exists for any s such that $s \mapsto \rho_s$ is lower semi-continuous μ -a.e.; i.e. for some fixed μ -versions of $\rho_s, s \in \mathbb{R}$,

$$\liminf_{s \rightarrow t} \rho_s(x) \geq \rho_t(x), \quad \mu\text{-a.e. } x, \quad t \in \mathbb{R}.$$

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In this paper, we aim to investigate the closability of \mathcal{E}_k for μ being the invariant probability measure of a (degenerate/delay) semilinear SPDE. Since in this case the above lower semi-continuity condition is hard to check, in this paper we make use of the integration by parts formula for the associated Markov semigroup in the line of [10] using coupling arguments.

The main motivation to study the closability of $E\mathcal{E}_k$ (respectively of ∂_k) on $L^2(\mu)$ is that it leads to a concept of weak differentiability on \mathbb{B} with respect to μ and one can define the corresponding Sobolev space on \mathbb{B} in $L^p(\mu)$, $p \in [1, \infty)$. In particular, one can analyze the generator of a Markov process (e.g. arising from a solution of an SPDE) on these Sobolev spaces when μ is its (infinitesimally) invariant measure, see e.g. [7] for details.

Before considering specific models of SPDEs, we first introduce a general result on the closability of \mathcal{E}_k using the integration by parts formula. To this end, we consider a family of \mathbb{B} -valued random variables $\{X^x\}_{x \in \mathbb{B}}$ measurable in x , and let $P(x, dy)$ be the distribution of X^x for $x \in \mathbb{B}$. Then we have the following Markov operator on $\mathcal{B}_b(\mathbb{B})$:

$$Pf(x) := \int_{\mathbb{B}} f(y)P(x, dy) = \mathbb{E}f(X^x), \quad x \in \mathbb{B}, f \in \mathcal{B}_b(\mathbb{B}).$$

A probability measure μ on \mathbb{B} is called an invariant measure of P if $\mu(Pf) = \mu(f)$ for all $f \in \mathcal{B}_b(\mathbb{B})$.

P1.1 **Proposition 1.1.** *Assume that the Markov operator P has an invariant probability measure μ . Let $k \in \mathbb{B}$. If there exists a family of real random variables $\{M_x\}_{x \in \mathbb{B}}$ measurable in x such that $M \in L^2(\mathbb{P} \times \mu)$, i.e.*

$$(1.1) \quad (\mathbb{P} \times \mu)(|M|^2) := \int_{\mathbb{B}} \mathbb{E}|M_x|^2 \mu(dx) < \infty;$$

and the integration by parts formula

$$(1.2) \quad P(\partial_k f)(x) = \mathbb{E}\{f(X^x)M_x\}, \quad f \in C_b^2(\mathbb{B}), \mu\text{-a.e. } x \in \mathbb{B}$$

holds, then $(\mathcal{E}_k, C_b^2(\mathbb{B}))$ is closable in $L^2(\mu)$.

Proof. Since μ is P -invariant, by (1.1) and (1.2) we have

$$\mu(\partial_k f) = \int_{\mathbb{B}} P(\partial_k f)(x)\mu(dx) = (\mathbb{P} \times \mu)(f(X^\cdot)M), \quad f \in C_b^2(\mathbb{B}).$$

So,

$$\begin{aligned} \mathcal{E}_k(f, g) &:= \mu((\partial_k f)(\partial_k g)) = \mu(\partial_k \{f \partial_k g\}) - \mu(f \partial_k^2 g) \\ &= (\mathbb{P} \times \mu)(\{f \partial_k g\}(X^\cdot)M) - \mu(f \partial_k^2 g), \quad f, g \in C_b^2(\mathbb{B}). \end{aligned}$$

It is standard that this implies the closability of the form $(\mathcal{E}_k, C_b^2(\mathbb{B}))$ in $L^2(\mu)$. Indeed, for $\{f_n\}_{n \geq 1} \subset C_b^2(\mathbb{B})$ with $f_n \rightarrow 0$ and $\partial_k f_n \rightarrow Z$ in $L^2(\mu)$, it suffices to prove that $Z = 0$.

Since $\mu(f_n^2) \rightarrow 0$ and $(\mathbb{P} \times \mu)(|f_n \partial_k g|^2(X \cdot)) = \mu(|f_n \partial_k g|^2)$ as μ is P -invariant, the above formula yields

$$\begin{aligned} |\mu(Zg)| &= \lim_{n \rightarrow \infty} |\mu(g \partial_k f_n)| \\ &= \lim_{n \rightarrow \infty} |(\mathbb{P} \times \mu)(\{f_n \partial_k g\}(X \cdot)M) - \mu(f_n \partial_k^2 g)| \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \sqrt{(\mathbb{P} \times \mu)(|f_n \partial_k g|^2(X \cdot)) \cdot (\mathbb{P} \times \mu)(|M|^2)} + \sqrt{\mu(f_n^2) \mu(|\partial_k^2 g|^2)} \right\} \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \|\partial_k g\|_\infty \sqrt{\mu(f_n^2) \cdot (\mathbb{P} \times \mu)(|M|^2)} + \|\partial_k^2 g\|_\infty \sqrt{\mu(f_n^2)} \right\} = 0, \quad g \in C_b^2(\mathbb{B}). \end{aligned}$$

Therefore, $Z = 0$. □

Remark 1.1. The integration by parts formula (1.2) implies the estimate

$$\boxed{\text{DM}} \quad (1.3) \quad |\mu(\partial_k f)|^2 \leq (\mathbb{P} \times \mu)(|M|^2) \mu(f^2).$$

As the main result in [3] (Theorem 10), this type of estimate, called Fomin derivative estimate of the invariant measure, was derived as the main result for the following semilinear SPDE on $\mathbb{H} := L^2(\mathcal{O})$ for any bounded open domain $\mathcal{O} \subset \mathbb{R}^n$ for $1 \leq n \leq 3$:

$$dX(t) = [\Delta X(t) + p(X(t))]dt + (-\Delta)^{-\gamma/2} dW(t),$$

where Δ is the Dirichlet Laplacian on \mathcal{O} , p is a decreasing polynomial with odd degree, $\gamma \in (\frac{n}{2} - 1, 1)$, and $W(t)$ is the cylindrical Brownian motion on \mathbb{H} . The main point of the study is to apply the Bismut-Elworthy-Li derivative formula and the following formula for the semigroup P_t^α for the Yoshida approximation of this SPDE (see [3, Proposition 7]):

$$P_t^\alpha \partial_k f = \partial_k P_t^\alpha - \int_0^t P_{t-s} (\partial_{Ak + \partial_k p} P_s^\alpha f) ds.$$

In this paper we will establish the integration by parts formula of type (1.2) for the associated semigroup which implies the estimate (1.3). Our results apply to a general framework where the operator $(-\Delta)^{-\gamma/2}$ is replaced by a suitable linear operator σ (see Section 2) which can be degenerate (see Section 3), and the drift $p(x)$ is replaced by a general map b which may include a time delay (see Section 4). However, the price we have to pay for the generalization is that the drift b should be regular enough.

2 Semilinear SPDEs

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle, |\cdot|)$ be a real separable Hilbert space, and $(W(t))_{t \geq 0}$ a cylindrical Wiener process on \mathbb{H} with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}_{HS}(\mathbb{H})$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on H respectively. Let $\|\cdot\|$ and $\|\cdot\|_{HS}$ denote the operator norm and the Hilbert-Schmidt norm respectively.

Consider the following semilinear SPDE

$$\boxed{\text{E1}} \quad (2.1) \quad dX(t) = \{AX(t) + b(X(t))\}dt + \sigma dW(t),$$

where

(A1) $(A, \mathcal{D}(A))$ is a negatively definite self-adjoint linear operator on \mathbb{H} with compact resolvent.

(A2) Let \mathbb{H}^{-2} be the completion of \mathbb{H} under the inner product

$$\langle x, y \rangle_{\mathbb{H}^{-2}} := \langle A^{-1}x, A^{-1}y \rangle.$$

Let $b : \mathbb{H} \rightarrow \mathbb{H}^{-2}$ be such that

$$\int_0^1 |e^{tA}b(0)|dt < \infty, \quad |e^{tA}(b(x) - b(y))| \leq \gamma(t)|x - y|, \quad x, y \in \mathbb{H}, t > 0$$

holds for some positive $\gamma \in C((0, \infty))$ with $\int_0^1 \gamma(t)dt < \infty$.

(A3) $\sigma \in \mathcal{L}(\mathbb{H})$ with $\text{Ker}(\sigma\sigma^*) = \{0\}$ and $\int_0^1 \|e^{tA}\sigma\|_{HS}^2 dt < \infty$.

According to **(A1)**, the spectrum of A is discrete with negative eigenvalues. Let $0 < \lambda_0 \leq \dots \leq \lambda_n \dots$ be all eigenvalues of $-A$ counting the multiplicities, and let $\{e_i\}_{i \geq 1}$ be the corresponding unit eigen-basis. Denote $\mathbb{H}_{A,n} = \text{span}\{e_i : 1 \leq i \leq n\}$, $n \geq 1$. Then $\mathbb{H}_A := \cup_{n=1}^{\infty} \mathbb{H}_{A,n}$ is a dense subspace of \mathbb{H} . In assumption **(A2)** we have used the fact that for any $t > 0$, the operator e^{tA} extends uniquely to a bounded linear operator from \mathbb{H}^{-2} to \mathbb{H} , which is again denoted by e^{tA} .

Due to assumptions **(A1)**, **(A2)** and **(A3)**, by a standard iteration argument we conclude that for any $x \in \mathbb{H}$ the equation (2.1) has a unique mild solution $X^x(t)$ such that $X^x(0) = x$ (see [4]). Let

$$P_t f(x) = \mathbb{E}f(X^x(t)), \quad f \in \mathcal{B}_b(\mathbb{H}), x \in \mathbb{H}$$

be the associated Markov semigroup.

Let

$$\|x\|_{\sigma} = \inf \{|y| : y \in \mathbb{H}, \sqrt{\sigma\sigma^*}y = x\}, \quad x \in \mathbb{H},$$

where $\inf \emptyset := \infty$ by convention. Then $\|x\|_{\sigma} < \infty$ if and only if $x \in \text{Im}(\sigma)$.

T2.1 **Theorem 2.1.** *Assume that P_t has an invariant probability measure μ and $\mathbb{H}_A \subset \text{Im}(\sqrt{\sigma\sigma^*})$.*

(1) *For any $k \in \mathbb{H}_A$ such that*

$$\boxed{\text{**0}} \quad (2.2) \quad \sup_{x \in \mathbb{H}} \|\partial_k b(x)\|_{\sigma} := \sup_{x \in \mathbb{H}} \limsup_{\varepsilon \downarrow 0} \frac{\|b(x + \varepsilon k) - b(x)\|_{\sigma}}{\varepsilon} < \infty,$$

the form $(\mathcal{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$.

(2) If $\sigma\sigma^*$ is invertible and $b : \mathbb{H} \rightarrow \mathbb{H}$ is Lipschitz continuous, then $(\mathcal{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ for any $k \in \mathcal{D}(A)$.

Proof. Since $d\tilde{W}_t := (\sigma\sigma^*)^{-1/2}\sigma dW_t$ is also a cylindrical Brownian motion and $\sigma dW_t = \sqrt{\sigma\sigma^*}d\tilde{W}_t$, we may and do assume that σ is non-negatively definite.

(1) Without loss of generality, we may and do assume that k is an eigenvector of A , i.e. $Ak = \lambda k$ for some $\lambda \in \mathbb{R}$. We first prove the case where b is Fréchet differentiable along the direction k . By $Ak = \lambda k$ we have

$$k(t) := \int_0^t e^{sA}k ds = \frac{e^{\lambda t} - 1}{\lambda}k, \quad t \geq 0,$$

where for $\lambda = 0$ we set $\frac{e^{\lambda t} - 1}{\lambda} = t$. Due to $\|k\|_\sigma < \infty$ and (2.2), the proof of [10, Theorem 5.1(1)] leads to the integration by parts formula

$$\boxed{\text{BO}} \quad (2.3) \quad P_T(\partial_k f)(x) = \mathbb{E}\{f(X^x(T))M_{x,T}\}, \quad f \in C_b^1(\mathbb{H}), x \in \mathbb{H}, T > 0,$$

where

$$M_{x,T} := \frac{\lambda}{e^{\lambda T} - 1} \int_0^T \left\langle \sigma^{-1} \left(k - \frac{e^{\lambda t} - 1}{\lambda} (\partial_k b)(X^x(t)) \right), dW(t) \right\rangle.$$

Since (2.2) implies

$$\boxed{\text{FH}} \quad (2.4) \quad \int_{\mathbb{B}} \mathbb{E}|M_{x,T}|^2 \mu(dx) \leq \frac{\lambda^2}{(e^{\lambda T} - 1)^2} \int_0^T \left\| \sigma^{-1} \left(k - \frac{e^{\lambda t} - 1}{\lambda} \partial_k b \right) \right\|_\infty^2 dt < \infty,$$

$(\mathcal{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ according to Proposition 1.1.

In general, for any $\varepsilon > 0$ let

$$b_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b(x + rk) \exp \left[-\frac{r^2}{2\varepsilon} \right] dr, \quad x \in \mathbb{H}.$$

Then for any $\varepsilon > 0$, b_ε is Fréchet differentiable along k and (2.2) holds uniformly in ε with b_ε replacing b . Let P_t^ε be the semigroup for the solution $X_\varepsilon(t)$ associated to equation (2.1) with b_ε replacing b . By simple calculations we have:

(i) $\lim_{\varepsilon \downarrow 0} \mathbb{E}|X_\varepsilon^x(t) - X^x(t)|^2 = 0$, $t \geq 0, x \in \mathbb{H}$.

(ii) For any $T > 0$, the family

$$M_{\cdot, T}^\varepsilon := \frac{\lambda}{e^{\lambda T} - 1} \int_0^T \left\langle \sigma^{-1} \left(k - \frac{e^{\lambda t} - 1}{\lambda} (\partial_k b_\varepsilon)(X_\varepsilon(t)) \right), dW(t) \right\rangle, \quad \varepsilon > 0$$

is bounded in $L^2(\mathbb{P} \times \mu)$; i.e. $\sup_{\varepsilon > 0} \int_{\mathbb{B}} \mathbb{E}|M_{x,T}^\varepsilon|^2 \mu(dx) < \infty$.

(iii) $P_T^\varepsilon(\partial_k f)(x) = \mathbb{E}(f(X_\varepsilon^x(T))M_{x,T}^\varepsilon)$, $f \in C_b^1(\mathbb{H}), \varepsilon > 0$.

So, there exist $M_{\cdot,T} \in L^2(\mathbb{P} \times \mu)$ and a sequence $\varepsilon_n \downarrow 0$ such that $M_{\cdot,T}^{\varepsilon_n} \rightarrow M_{\cdot,T}$ weakly in $L^2(\mathbb{P} \times \mu)$. Thus, by taking $n \rightarrow \infty$ in (iii) and using (i), we prove (2.3) for μ -a.e. $x \in \mathbb{B}$. Then the proof of the first assertion is completed as in the first case.

(2) Since σ is invertible, **(A3)** implies $\alpha := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Next, since the Lipschitz constant $\|\partial b\|_{\infty}$ of b is finite, the integration by parts formula (2.3) also implies explicit Fomin derivative estimates on the invariant probability measure, which were investigated recently in [3]. Indeed, it follows from (2.3) and (2.4) that

$$\begin{aligned} |\mu(\partial_k f)| &= \inf_{T>0} |\mu(P_T(\partial_k f))| \leq \inf_{T>0} \sqrt{\mu(P_T f^2)} \left(\int_{\mathbb{B}} \mathbb{E} |M_{x,T}|^2 \mu(dx) \right)^{\frac{1}{2}} \\ &\leq |k| \cdot \|f\|_{L^2(\mu)} \inf_{T>0} \frac{\lambda}{e^{\lambda T} - 1} \left(\int_0^T \left\| \sigma^{-1} \left(I - \frac{e^{\lambda t} - 1}{\lambda} \partial b \right) \right\|_{\infty}^2 dt \right)^{\frac{1}{2}}, \quad Ak = \lambda k. \end{aligned}$$

By taking $k = e_i$, $T = \lambda_i^{-1}$ and $\lambda = -\lambda_i$ in the above estimate, for any $k \in \mathcal{D}(A)$ we have

$$\begin{aligned} |\mu(\partial_k f)| &\leq \sum_{i=1}^{\infty} |\langle k, e_i \rangle \mu(\partial_{e_i} f)| \leq \left(\sum_{i=1}^{\infty} \lambda_i^2 \langle k, e_i \rangle^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \mu(\partial_{e_i} f)^2 \right)^{\frac{1}{2}} \\ \boxed{*Z} \quad (2.5) \quad &\leq |Ak| \left(\sum_{i=1}^{\infty} \frac{\|\sigma^{-1}\|^2}{\lambda_i (e-1)^2} \left(1 + \frac{e-1}{\lambda_i} \|\partial b\|_{\infty} \right)^2 \right)^{\frac{1}{2}} \|f\|_{L^2(\mu)} \\ &\leq C |Ak| \cdot \|f\|_{L^2(\mu)}, \end{aligned}$$

where $C := \frac{\|\sigma^{-1}\| \sqrt{\alpha}}{e-1} \left(1 + \frac{e-1}{\lambda_1} \|\partial b\|_{\infty} \right)$. This implies the closability of $(\mathcal{E}_k, C_b^2(\mathbb{H}))$ as explained in the proof of Proposition 1.1. Indeed, if $\{f_n\}_{n \geq 1} \subset C_b^2(\mathbb{B})$ satisfies $f_n \rightarrow 0$ and $\partial_k f_n \rightarrow Z$ in $L^2(\mu)$, then (2.5) implies

$$\begin{aligned} |\mu(gZ)| &= \lim_{n \rightarrow \infty} |\mu(g \partial_k f_n)| = \lim_{n \rightarrow \infty} |\mu(\partial_k(f_n g) - \mu(f_n \partial_k g))| \\ &\leq C |Ak| \lim_{n \rightarrow \infty} \sqrt{\mu((f_n g)^2)} = 0, \quad g \in C_b^2(\mathbb{B}), \end{aligned}$$

so that $Z = 0$. □

To conclude this section, let us recall a result concerning existence and stability of the invariant probability measure. Let $W_a(t) = \int_0^t e^{A(t-s)} \sigma dW(s)$, $t \geq 0$. Assume that b is Lipschitz continuous and $\int_0^{\infty} \|e^{tA} \sigma\|_{HS}^2 dt < \infty$. We have

$$\sup_{t \geq 0} \mathbb{E} (\|W_A(t)\|^2 + |b(W_A(t))|^2) < \infty.$$

Therefore, by [5, Theorem 2.3], if there exist $c_1 > 0$, $c_2 \in \mathbb{R}$ with $c_1 + c_2 > 0$ such that

$$\langle A(x - y), x - y \rangle \leq -c_1 |x - y|^2, \quad \langle b(x) - b(y), x - y \rangle \leq -c_2 |x - y|^2, \quad x, y \in \mathbb{H},$$

then P_t has a unique invariant probability measure such that $\lim_{t \rightarrow \infty} P_t f = \mu(f)$ holds for $f \in C_b(\mathbb{H})$.

3 Stochastic Hamiltonian systems on Hilbert spaces

Let $\tilde{\mathbb{H}}$ and \mathbb{H} be two separable Hilbert spaces. Consider the following stochastic differential equation for $Z(t) := (X(t), Y(t))$ on $\tilde{\mathbb{H}} \times \mathbb{H}$:

$$\boxed{3.1} \quad (3.1) \quad \begin{cases} dX(t) = BY(t)dt, \\ dY(t) = \{AY(t) + b(t, X(t), Y(t))\}dt + \sigma dW(t), \end{cases}$$

where $B \in \mathcal{L}(\mathbb{H} \rightarrow \tilde{\mathbb{H}})$, $(A, \mathcal{D}(A))$ satisfies **(A1)**, σ satisfies **(A3)**, $W(t)$ is the cylindrical Brownian motion on \mathbb{H} , and $b : [0, \infty) \times \tilde{\mathbb{H}} \times \mathbb{H} \rightarrow \mathbb{H}^{-2}$ satisfies: for any $T > 0$ there exists $\gamma \in C((0, T])$ with $\int_0^T \gamma(t)dt < \infty$ such that

$$\boxed{BB} \quad (3.2) \quad \begin{aligned} & \sup_{s \in [0, T]} \int_0^T |e^{tA}b(s, 0)|dt < 1, \\ & \sup_{s \in [0, T]} |e^{tA}(b(s, z) - b(s, z'))| \leq \gamma(t)|z - z'|, \quad t \in [0, T], z, z' \in \tilde{\mathbb{H}} \times \mathbb{H}. \end{aligned}$$

Obviously, for any initial data $z := (x, y) \in \mathbb{H}$, the equation has a unique mild solution $Z^z(t)$. Let P_t be the associated Markov semigroup.

When $\tilde{\mathbb{H}}$ and \mathbb{H} are finite-dimensional, the integration by parts formula of P_t has been established in [10, Theorem 3.1]. Here, we extend this result to the present infinite-dimensional setting.

P3.1 **Proposition 3.1.** *Assume that $BB^* \in \mathcal{L}(\tilde{\mathbb{H}})$ with $\text{Ker}(BB^*) = \{0\}$. Let $T > 0$ and $k := (k_1, k_2) \in \text{Im}(BB^*) \times \mathbb{H}$ be such that*

$$\boxed{EG} \quad (3.3) \quad Ak_2 = \theta_2 k_2, \quad AB^*(BB^*)^{-1}k_1 = \theta_1 B^*(BB^*)^{-1}k_1$$

for some constants $\theta_1, \theta_2 \in \mathbb{R}$. For any $\phi, \psi \in C^1([0, T])$ such that

$$\boxed{PH} \quad (3.4) \quad \phi(0) = \phi(T) = \psi(0) = \psi(T) - 1 = \int_0^T e^{\theta_2 t} \psi(t) dt = 0, \quad \int_0^T \phi(t) e^{\theta_1 t} dt = e^{\theta_1 T},$$

let

$$\begin{aligned} h(t) &= B^*(BB^*)^{-1}k_1 \int_0^t \phi'(s) e^{\theta_1(s-T)} ds + k_2 \int_0^t \psi'(s) e^{\theta_2(s-T)} ds, \\ \tilde{h}(t) &= \phi(t) e^{\theta_1(t-T)} B^*(BB^*)^{-1}k_1 + \psi(t) e^{\theta_2(t-T)} k_2, \\ \Theta(t) &= \left(\int_0^t B \tilde{h}(s) ds, \tilde{h}(t) \right), \quad t \in [0, T]. \end{aligned}$$

If for any $t \in [0, T]$, $b(s, \cdot)$ is Fréchet differentiable along $\Theta(t)$ such that

$$\boxed{*A} \quad (3.5) \quad \int_0^T \sup_{z \in \tilde{\mathbb{H}} \times \mathbb{H}} \|h'(t) - (\partial_{\Theta(t)} b(t, \cdot))(z)\|_{\sigma}^2 dt < \infty,$$

then for any $f \in C_b^1(\tilde{\mathbb{H}} \times \mathbb{H})$,

$$P_T(\partial_k f) = \mathbb{E} \left\{ f(Z(T)) \int_0^T \left\langle (\sigma \sigma^*)^{-1/2} \{h'(t) - (\partial_{\Theta(t)} b(t, \cdot))(Z(t))\}, dW(t) \right\rangle \right\}.$$

Proof. As explained in the proof of Theorem 2.1, we simply assume that $\sigma = \sqrt{\sigma\sigma^*}$. Let $(X^0(t), Y^0(t)) = (X(t), Y(t))$ solve (3.1) with initial data (x, y) , and for $\varepsilon \in (0, 1]$ let $(X^\varepsilon(t), Y^\varepsilon(t))$ solve the equation

$$\boxed{\text{CCCO}} \quad (3.6) \quad \begin{cases} dX^\varepsilon(t) = BY^\varepsilon(t)dt, & X^\varepsilon(0) = x, \\ dY^\varepsilon(t) = \sigma dW(t) + \{b(t, X(t), Y(t)) + AY^\varepsilon(t) + \varepsilon h'(t)\}dt, & Y^\varepsilon(0) = y. \end{cases}$$

Then it is easy to see from (3.3) and (3.4) that

$$\begin{aligned} Y^\varepsilon(t) - Y(t) &= \varepsilon \int_0^t e^{(t-s)A} h'(s) ds \\ &= \varepsilon B^*(BB^*)^{-1} k_1 \int_0^t \phi'(s) e^{\theta_1(s-T)} e^{\theta_1(t-s)} ds + \varepsilon k_2 \int_0^t \psi'(s) e^{\theta_2(s-T)} e^{\theta_2(t-s)} ds \\ &= \varepsilon (\phi(t) e^{\theta_1(t-T)} B^*(BB^*)^{-1} k_1 + \psi(t) e^{\theta_2(t-T)} k_2) = \varepsilon \tilde{h}(t), \end{aligned}$$

and hence,

$$\begin{aligned} X^\varepsilon(t) - X(t) &= \varepsilon \int_0^t B \tilde{h}(s) ds \\ &= \varepsilon \left(k_1 \int_0^t \phi(r) e^{\theta_1(r-T)} dr + (Bk_2) \int_0^t \psi(r) e^{\theta_2(r-T)} dr \right). \end{aligned}$$

So,

$$\boxed{\text{X3}} \quad (3.7) \quad X^\varepsilon(t) - X(t) = \varepsilon \Theta(t), \quad t \in [0, T],$$

and in particular

$$\boxed{\text{CP}} \quad (3.8) \quad (X^\varepsilon(T), Y^\varepsilon(T)) = (X(T), Y(T)) + \varepsilon k$$

due to (3.4). Next,

$$\boxed{\text{XI}} \quad (3.9) \quad \xi_\varepsilon(s) = \varepsilon h'(s) + b(s, X(s), Y(s)) - b(s, X^\varepsilon(s), Y^\varepsilon(s))$$

and

$$R_\varepsilon = \exp \left[- \int_0^T \langle \sigma^{-1} \xi_\varepsilon(s), dW(s) \rangle - \frac{1}{2} \int_0^T |\sigma^{-1} \xi_\varepsilon(s)|^2 ds \right].$$

We reformulate (3.6) as

$$\boxed{\text{CCC}} \quad (3.10) \quad \begin{cases} dX^\varepsilon(t) = BY^\varepsilon(t)dt, & X^\varepsilon(0) = x, \\ dY^\varepsilon(t) = \sigma dW^\varepsilon(t) + \{b(t, X^\varepsilon(t), Y^\varepsilon(t)) + AY^\varepsilon(t)\}dt, & Y^\varepsilon(0) = y, \end{cases}$$

where by (3.5) and (3.7),

$$W^\varepsilon(t) := W(t) + \int_0^t \sigma^{-1} \xi_\varepsilon(s) ds, \quad t \in [0, T]$$

is a cylindrical Brownian motion under the weighted probability measure $\mathbb{Q}_\varepsilon := R_\varepsilon \mathbb{P}$. Since $|\xi_\varepsilon|$ is uniformly bounded on $[0, T]$, by the dominated convergence theorem and (3.7), for any $f \in C_b^1(\tilde{\mathbb{H}} \times \mathbb{H})$ we obtain

$$\begin{aligned} P_T(\partial_k f) &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \frac{f((X(T), Y(T)) + \varepsilon k) - f((X(t), Y(t)))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \frac{f((X^\varepsilon(T), Y^\varepsilon(T))) - R_\varepsilon f((X^\varepsilon(T), Y^\varepsilon(T)))}{\varepsilon} \\ &= \mathbb{E} \left(f(Z(T)) \lim_{\varepsilon \rightarrow 0} \frac{1 - R_\varepsilon}{\varepsilon} \right) \\ &= \mathbb{E} \left(f(Z(T)) \int_0^T \left\langle \sigma^{-1} \{h'(t) - (\partial_{\Theta(t)} b)(Z(t))\}, dW(t) \right\rangle \right). \end{aligned}$$

□

To apply this result, we present here a specific choice of (ϕ, ψ) such that (3.4) holds:

$$\phi(t) = \frac{e^{\theta_1 T} t(T-t)}{\int_0^T s(T-s)e^{\theta_1 s} ds}, \quad \psi(t) = \frac{e^{\theta_2(T-t)}}{T} \left(\frac{3t^2}{T} - 2t \right), \quad t \in [0, T].$$

T3.1 **Theorem 3.2.** *Let $\tilde{\mathbb{H}} = \mathbb{H} = \mathbb{H}$ and $\text{Ker}(B) = \{0\}$. Let $b(t, \cdot) = b$ do not dependent on t such that P_t has an invariant probability measure μ . If*

$$\text{*A2} \quad (3.11) \quad \sup_{(x,y) \in \mathbb{H} \times \mathbb{H}} \lim_{r \downarrow 0} \frac{\|b(x + rB^{-1}\tilde{k}, y + rk) - b(x, y)\|_\sigma}{r} < \infty, \quad (\tilde{k}, k) \in (B\mathbb{H}_A) \times \mathbb{H}_A,$$

Then for any $(k_1, k_2) \in (B\mathbb{H}_A) \times \mathbb{H}_A$, the form $(\mathcal{E}_k, C_b^2(\mathbb{H} \times \mathbb{H}))$ is closable in $L^2(\mu)$.

Proof. It suffices to prove for $k = (k_1, k_2)$ such that $B^{-1}k_1$ and k_2 are eigenvectors of A , i.e. $AB^{-1}k_1 = \theta_1 B^{-1}k_1$ and $Ak_2 = \theta_2 k_2$ hold for some $\theta_1, \theta_2 \in \mathbb{R}$. As explained above there exists $T > 0$ such that (3.4) holds for some $\phi, \psi \in C^\infty([0, T])$. Moreover, as explained in the proof of Theorem 2.1, by taking

$$b_\varepsilon(s, x, y) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b((x, y) + r\Theta(s)) \exp \left[-\frac{r^2}{2\varepsilon} \right] dr, \quad s \in [0, T], (x, y) \in \mathbb{H} \times \mathbb{H}$$

for $\varepsilon > 0$, such that (3.11) holds uniformly in $\varepsilon > 0$ and $s \in [0, T]$ with $b_\varepsilon(s, \cdot)$ replacing b , we may and do assume that $b(s, \cdot)$ is Fréchet differentiable along $\Theta(s)$. Then the integration by parts formula in Proposition 3.1 holds, and due to (3.11) we have

$$M_{\cdot, T} := \int_0^T \left\langle (\sigma\sigma^*)^{-1/2} \{h'(t) - (\partial_{\Theta(t)} b(t, \cdot))(Z(t))\}, dW(t) \right\rangle \in L^2(\mathbb{P} \times \mu).$$

Therefore, by Proposition 1.1, the form $(\mathcal{E}_k, C_b^2(\mathbb{H} \times \mathbb{H}))$ is closable on $L^2(\mu)$. □

Below are typical examples of the stochastic Hamiltonian system with invariant probability measure such that Theorem 3.2 applies.

Example 3.1. Let $\tilde{\mathbb{H}} = \mathbb{H} = \mathbb{H}$.

(1) Let $\mathbb{H} = \mathbb{R}^d$ for some $d \geq 1$. When $\sigma = B = I$, $A \leq -\lambda I$ for some $\lambda > 0$ is a negatively definite $d \times d$ -matrix, and $b(x, y) = A^{-1}\nabla V(x)$ for some $V \in C^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} e^{-V(x)} dx < \infty$. Then the unique invariant probability measure of P_t is

$$\mu(dx, dy) = C e^{-V(x) + \frac{\lambda}{2}\langle Ay, y \rangle} dx dy,$$

where $C > 0$ is the normalization. See [2, 6, 9] for the study of hypercoercivity of the associated semigroup P_t with respect to μ , as well as [12] for the stronger property of hypercontractivity.

(2) In the infinite-dimensional setting, let $\sigma = B = I$ and A be negatively definite such that A^{-1} is of trace class. Take $b(x, y) = A^{-1}Qx$ for some positively definite self-adjoint operator Q on \mathbb{H} such that Q^{-1} is of trace class and

$$\int_0^1 \|e^{tA} A^{-1}Q\| dt < 1.$$

Then it is easy to see that

$$\mu(dx, dy) = N_{Q^{-1}}(dx) N_{-A^{-1}}(dy)$$

is an invariant probability measure.

(3) More generally, let $\sigma = B = I$ and

$$b(x, y) = \tilde{b}(x) := A^{-1}\nabla V(x), \quad (x, y) \in \mathbb{H} \times \mathbb{H}_A$$

for some Fréchet differentiable $V : \mathbb{H}_A \rightarrow \mathbb{R}$ such that (3.11) holds. For any $n \geq 1$, let

$$V_n(r) = V \circ \varphi_n(r), \quad \varphi_n(r) = \sum_{i=1}^n r_i e_i, \quad r = (r_1, \dots, r_n) \in \mathbb{R}^n.$$

If $\int_{\mathbb{R}^n} e^{-V_n(r)} dr < \infty$ and when $n \rightarrow \infty$ the probability measure

$$\nu_n(D) := \frac{1}{\int_{\mathbb{R}^n} e^{-V_n(r)} dr} \int_{\varphi_n^{-1}(D)} e^{-V_n(r)} dr, \quad D \in \mathcal{B}(\mathbb{H})$$

converges weakly to some probability measure ν , then $\mu := \nu \times N_{-A^{-1}}$ is an invariant probability measure of P_t . This can be confirmed by (1) and a finite-dimensional approximation argument. Indeed, let $\pi_n : \mathbb{H} \rightarrow \mathbb{H}_{A,n}$ be the orthogonal projection, and let $A_n = \pi_n A$, $W_n = \pi_n W$ and $b_n(x, y) = \pi_n \nabla V(x)$. Let $X_n(t)$ solve the finite-dimensional equation

$$\begin{cases} dX_n(t) = Y_n(t) dt, \\ dY_n(t) = \{A_n Y_n(t) + b_n(X_n(t))\} dt + dW_n(t) \end{cases}$$

with $(X_n(0), Y_n(0)) = (\pi_n X(0), \pi_n Y(0))$. Then the proof of [11, Theorem 2.1] yields that for every $t \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n(t) - X(t)|^2 + |Y_n(t) - Y(t)|^2) = 0$$

uniformly in the initial data $(X(0), Y(0)) \in \mathbb{H} \times \mathbb{H}$. Thus, letting $P_t^{(n)}$ be the semigroup for $(X_n(t), Y_n(t))$, we have

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathbb{H} \times \mathbb{H}} |P_t^{(n)} f(\pi_n x, \pi_n y) - P_t f(x, y)| = 0, \quad f \in C_b^1(\mathbb{H} \times \mathbb{H}).$$

Combining this with the assertion in (1) and noting that $\nu_n \times (N_{-A^{-1}} \circ \pi_n^{-1}) \rightarrow \mu$ weakly as $n \rightarrow \infty$, we conclude that μ is an invariant probability measure of P_t .

4 Semilinear SPDEs with delay

For fixed $\tau > 0$, let $\mathcal{C}_\tau = C([- \tau, 0]; \mathbb{H})$ be equipped with the uniform norm $\|\eta\|_\infty := \sup_{\theta \in [- \tau, 0]} |\eta(\theta)|$. For any $\xi \in C([- \tau, \infty); \mathbb{H})$, we define $\xi \in C([0, \infty); \mathcal{C}_\tau)$ by letting

$$\xi_t(\theta) = \xi(t + \theta), \quad \theta \in [- \tau, 0], t \geq 0.$$

Consider the following stochastic differential equation with delay:

$$\boxed{4.1} \quad (4.1) \quad dX(t) = \{AX(t) + b(X_t)\}dt + \sigma dW(t), \quad X_0 \in \mathcal{C}_\tau,$$

where $(A, \mathcal{D}(A))$ satisfies **(A1)**, σ satisfies **(A3)**, and $b : \mathcal{C}_\tau \rightarrow \mathbb{H}$ satisfies: for any $T > 0$ there exists $\gamma \in C((0, T])$ with $\int_0^T \gamma(t)dt < \infty$ such that

$$\boxed{*4} \quad (4.2) \quad \int_0^T \sup_{s \in [0, T]} |e^{tA} b(s, 0)|^2 dt < \infty, \quad |e^{tA}(b(s, \xi) - b(s, \eta))|^2 \leq \gamma(t) \|\xi - \eta\|_\infty^2, \quad t, s \in [0, T].$$

Then for any initial datum $\xi \in \mathcal{C}_\tau$, the equation has a unique mild solution $X^\xi(t)$ with $X_0 = \xi$. Let P_t be the Markov semigroup for the segment solution X_t .

Let

$$\mathcal{C}_\tau^1 = \left\{ \eta \in \mathcal{C}_\tau : \eta(\theta) \in \mathcal{D}(A) \text{ for } \theta \in [- \tau, 0], \int_{- \tau}^0 (|A\eta(\theta)|^2 + |\eta'(\theta)|^2) d\theta < \infty \right\}.$$

The following result is an extension of [10, Theorem 4.1(1)] to the infinite-dimensional setting.

P4.1 **Proposition 4.1.** *For any $\eta \in \mathcal{C}_\tau^1$ and $T > \tau$, let*

$$\Gamma(t) := \begin{cases} \frac{1}{T-\tau} e^{(s+\tau-T)A} \eta(-\tau), & \text{if } s \in [0, T-\tau], \\ \eta'(s-T) - A\eta(s-T), & \text{if } s \in (T-\tau, T], \end{cases}$$

and

$$\Theta(t) := \int_0^{t \vee 0} \Gamma(s) ds, \quad t \in [- \tau, T].$$

If $b(t, \cdot)$ is Fréchet differentiable along Θ_t for $t \in [0, T]$ such that

$$\boxed{SS} \quad (4.3) \quad \sup_{\xi \in \mathcal{C}_\tau} \int_0^T \|\Gamma(t) - (\nabla_{\Theta_t} b(T, \cdot))(\xi)\|_\sigma^2 dt < \infty,$$

then

PP (4.4)

$$P_T(\partial_\eta f) = \mathbb{E} \left(f(X_T) \int_0^T \left\langle (\sigma\sigma^*)^{-1/2} (\Gamma(t) - (\nabla_{\Theta_t} b(t, \cdot))(X_t)), dW(t) \right\rangle \right), \quad f \in C_b^1(\mathcal{C}_\tau).$$

Proof. Simply let $\sigma = \sqrt{\sigma\sigma^*}$ as in the proof of Theorem 2.1. For any $\varepsilon \in (0, 1)$, let $X^\varepsilon(t)$ solve the equation

E' (4.5)
$$dX^\varepsilon(t) = \{AX^\varepsilon(t) + b(t, X_t) + \varepsilon\Gamma(t)\}dt + \sigma dW(t), \quad X_0^\varepsilon = X_0.$$

We have

AP (4.6)
$$\begin{aligned} X^\varepsilon(t) - X(t) &= \varepsilon \int_0^{t^+} e^{(t-s)A} \Gamma(s) ds \\ &= \frac{\varepsilon t^+}{T - \tau} e^{(\tau-T)A} \eta(-\tau) 1_{[-\tau, T-\tau)}(t) + \varepsilon \eta(t - T) 1_{[T-\tau, T]}(t), \quad t \in [-\tau, T]. \end{aligned}$$

In particular, we have $X_T^\varepsilon - X_T = \varepsilon\eta$. To formulate P_T using X_T^ε , rewrite (4.5) by

$$dX^\varepsilon(t) = \{AX^\varepsilon(t) + b(t, X_t^\varepsilon)\}dt + \sigma dW_\varepsilon(t), \quad X_0^\varepsilon = X_0,$$

where

$$W_\varepsilon(t) := W(t) + \int_0^t \xi_\varepsilon(s) ds, \quad \xi_\varepsilon(s) := b(s, X_s) - b(s, X_s^\varepsilon) + \varepsilon\Gamma(s).$$

By (4.3) and the Girsanov theorem, we see that $\{W_\varepsilon(t)\}_{t \in [0, T]}$ is a cylindrical Brownian motion on \mathbb{H} under the probability measure $d\mathbb{Q}_\varepsilon := R_\varepsilon d\mathbb{P}$, where

$$R_\varepsilon := \exp \left[\int_0^T \left\langle \sigma^{-1} (b(t, X_t^\varepsilon) - b(t, X_t) - \varepsilon\Gamma(t)), dW(t) \right\rangle \right].$$

Then

$$\mathbb{E}(f(X_T)) = P_T f = \mathbb{E}(R_\varepsilon f(X_T^\varepsilon)).$$

Combining this with $X_T^\varepsilon = X_T + \varepsilon\eta$ and using (4.6), we arrive at

$$\begin{aligned} P_T(\partial_\eta f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\{f(X_T + \varepsilon\eta) - f(X_T)\} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\{f(X_T^\varepsilon) - R_\varepsilon f(X_T^\varepsilon)\} \\ &= \mathbb{E} \left(f(X_T) \lim_{\varepsilon \downarrow 0} \frac{1 - R_\varepsilon}{\varepsilon} \right) = \mathbb{E} \left\{ f(X_T) \int_0^T \left\langle \sigma^{-1} (\Gamma(t) - (\nabla_{\Theta_t} b(t, \cdot))(X_t)), dW(t) \right\rangle \right\}. \end{aligned}$$

□

T4.1 **Theorem 4.2.** *Let $b(t, \cdot) = b$ be independent of t such that P_t has an invariant probability measure μ . If $\text{Im}(\sigma) \supset \mathbb{H}_A$ and*

CD (4.7)
$$\sup_{\xi \in \mathcal{C}_\tau} \limsup_{\varepsilon \downarrow 0} \frac{\|b(\xi + \varepsilon\eta) - b(\xi)\|_\sigma}{\varepsilon} < \infty, \quad \eta \in \mathcal{C}_\tau^1 \cap \left(\bigcup_{n \geq 1} C([- \tau, 0]; \mathbb{H}_{A, n}) \right),$$

then for any $\eta \in \mathcal{C}_\tau^1 \cap (\cup_{n \geq 1} C([- \tau, 0]; \mathbb{H}_{A,n}))$, which is dense in \mathcal{C}_τ , the form

$$\mathcal{E}_\eta(f, g) := \int_{\mathcal{C}_\tau} (\partial_\eta f)(\partial_\eta g) d\mu, \quad f, g \in C_b^2(\mathcal{C}_\tau)$$

is closable in $L^2(\mu)$.

Proof. For any $\varepsilon \in (0, 1)$ let

$$b_\varepsilon(t, \xi) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b(\xi + r\Theta_t) \exp\left[-\frac{r^2}{2\varepsilon}\right] dr, \quad \xi \in \mathcal{C}_\tau.$$

Then $b_\varepsilon(t, \cdot)$ is F chet differentiable along Θ_t and (4.7) holds uniformly in ε with $b_\varepsilon(t, \cdot)$ replacing b . Moreover, $\eta \in \mathcal{C}_\tau^1 \cap (\cup_{n \geq 1} C([- \tau, 0]; \mathbb{H}_n))$ implies that $\Theta_t \in \mathcal{C}_\tau^1 \cap (\cup_{n \geq 1} C([- \tau, 0]; \mathbb{H}_n))$ and (4.7) holds uniformly in $t \in [0, T]$ and $\varepsilon \in (0, 1)$ with Θ_t and $b_\varepsilon(t, \cdot)$ replacing η and b respectively. Combining this with $\text{Im}(\sigma) \supset \mathbb{H}_A$, we conclude that (4.3) holds uniformly in ε with b_ε replacing b . Therefore, as explained in the proof of Theorem 2.1, we may assume that b is Fr chet differentiable along Θ_t , $t \in [0, T]$, and by Proposition 4.1 the integration by parts formula (4.4) holds. Moreover, (4.7) implies

$$M_{\cdot, T} := \int_0^T \left\langle (\sigma\sigma^*)^{-1/2} (\Gamma(t) - (\nabla_{\Theta_t} b(t, \cdot))(X_t)), dW(t) \right\rangle \in L^2(\mathbb{P} \times \mu).$$

Then the proof is finished by Proposition 1.1. □

Finally, we introduce the following example to illustrate Theorem 4.2.

Example 4.1. Let $b(\xi) = F(\xi(-\tau))$, $\xi \in \mathcal{C}_\tau$, for some $F \in C_b^1(\mathbb{H})$. If σ is Hilbert-Schmidt and

$$\langle x, Ax + F(y) - F(y') \rangle \leq -\lambda_1 |x|^2 + \lambda_2 |y - y'|^2, \quad x, y \in \mathbb{H},$$

for some constants $\lambda_1 > \lambda_2 \geq 0$, then according to [1, Theorem 4.9] P_t has a unique invariant probability measure μ . If moreover $\text{Im}(\sigma) \supset \mathbb{H}_A$ and for any $y \in \mathbb{H}_A$ there exists a constant

$$\limsup_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{H}} \frac{\|F(x + \varepsilon y) - F(x)\|_\sigma}{\varepsilon} < \infty,$$

then by Theorem 4.2, for any $\eta \in \mathcal{C}_\tau^1 \cap (\cup_{n \geq 1} C([- \tau, 0]; \mathbb{H}_{A,n}))$ the form $(\mathcal{E}_\eta, C_b^2(\mathcal{C}_\tau))$ is closable on $L^2(\mu)$.

References

BTY

- [1] J. Bao, A. Truman, C. Yuan, *Stability in distribution of mild solutions to stochastic partial differential delay equations with jumps*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465 (2009), 2111–2134.

- Baudoin** [2] F. Baudoin, *Bakry-Emery meet Villani*, arXiv:1308.4938.
- DD** [3] G. Da Prato, A. Debussche, *Existence of the Fomin derivative of the invariant measure of a stochastic reaction-diffusion equation*, arXiv: 1502.07490v1.
- DZ** [4] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
- DZ2** [5] G. Da Prato, J. Zabczyk, *Convergence to equilibrium for classical and quantum spin systems*, Probab. Theory Related Fields 103 (1995), 529–552.
- GR** [6] M. Grothaus, P. Stilgenbauer, *Hypocoercivity for Kolmogorov backward evolution equations and applications*, J. Funct. Anal. 267 (2014), 3515–3556.
- 5'** [7] M. Röckner, *L^p -analysis of finite and infinite dimensional diffusion operators*, in “SPDEs and Kolmogorov’s equations in infinite dimensions (G. Da Prato, ed.) ” Lecture Notes Math. vol. 1715, Springer, Berlin, 1999, pp. 65–116.
- AR** [8] M. Röckner, N. Wielens, *Dirichlet forms, losability and change of speed measure*, Infinite-dimensional analysis and stochastic processes (Bielefeld, 1983), 119–144, Res. Notes in Math., 124, Pitman, Boston, MA, 1985.
- V** [9] C. Villani, *Hypocoercivity*, Mem. Amer. Math. Soc. 202 (2009), no. 950.
- W12** [10] F.-Y. Wang, *Integration by parts formula and shift Harnack inequality for stochastic equations*, Ann. Probab. 42 (2014), 994–1019.
- WZ** [11] F.-Y. Wang, Tusheng Zhang, *Log-Harnack inequality for mild solutions of SPDEs with multiplicative noise*, Stochastic Process. Appl. 124 (2014), 1261–1274.
- W15** [12] F.-Y. Wang, *Hypercontractivity for stochastic Hamiltonian systems*, arXiv:1409.1995v2.