Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications *

Michael Röckner[c], **Rongchan Zhu**[a,†]**Xiangchan Zhu**[b], [‡]

[a]Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China,
[b]School of Science, Beijing Jiaotong University, Beijing 100044, China
[c] Department of Mathematics, University of Bielefeld, D-33615 Bielefeld, Germany,

Abstract

In this paper we obtain restricted Markov uniqueness of the generator and uniqueness of martingale (probabilistically weak) solutions for the stochastic quantization problem in both the finite and infinite volume case by clarifying the precise relation between the solutions to the stochastic quantization problem obtained by the Dirichlet form approach and those obtained in [DD03] and in [MW15]. We prove that the solution $X-Z$, where X is obtained by the Dirichlet form approach in [AR91] and Z is the corresponding O-U process, satisfies the corresponding shifted equation (see (1.4) below). Moreover, we obtain that the infinite volume $p(\Phi)_2$ quantum field is an invariant measure for the $X_0 = Y + Z$, where Y is the unique solution to the shifted equation.

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1 Introduction

In this paper we analyze stochastic quantization equations on $\mathbb{T}^2$ and on $\mathbb{R}^2$: So, let $H = L^2(\mathbb{T}^2)$ or $L^2(\mathbb{R}^2)$ and consider

$$\begin{aligned} dX =&(AX - :p(X):)dt + dW(t), \\ X(0) =&z, \end{aligned} \tag{1.1}$$

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†Corresponding author

‡E-mail address: roeckner@math.uni-bielefeld.de(M. Röckner), zhurongchan@126.com(R. C. Zhu), zhuxiangchan@126.com(X. C. Zhu)

where $A : D(A) \subset H \to H$ is the linear operator

$$A\phi = \Delta\phi - \phi, \quad p(\phi) = \sum_{n=1}^{2N} n a_n \phi^{n-1},$$

where $a_{2N} > 0$ and $: p(\phi) :$ means the renormalization of $p(\phi)$ whose definition we will give in Section 3 and Section 4. W is a cylindrical $\mathcal{F}_t$-Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$.

This equation arises in stochastic quantization of Euclidean quantum field theory. Heuristically, (1.1) has an invariant measure ν defined as

$$\nu(d\phi) = ce^{-2\int :q(\phi):dx}\mu(d\phi),$$

where $q(\phi) = \sum_{n=0}^{2N} a_n\phi^n$, c is a normalization constant and μ is the Gaussian free field. ν is called the $p(\Phi)_2$-quantum field. There have been many approaches to the problem of giving a meaning to the above heuristic measure for the two dimensional case and the three dimensional case (see [GRS75], [GlJ86] and references therein). In [PW81] Parisi and Wu proposed a program for Euclidean quantum field theory of getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study the properties of the Gibbs states. This procedure is called stochastic field quantization (see [JLM85]). The $p(\Phi)_2$ model is the simplest non-trivial Euclidean quantum field (see [GlJ86] and the reference therein). The issue of the stochastic quantization of the $p(\Phi)_2$ model is to solve the equation (1.1).

In [AR91] weak solutions to (1.1) have been constructed by using the Dirichlet form approach in the finite and infinite volume case. However, *Markov uniqueness* for the corresponding generator (L, D) has been an open problem for many years. Here D is the "minimal" domain contained in the domain of the generator. Consider a measure ν on a Banach space E. The problem of Markov uniqueness is whether there exists exactly one negative definite self-adjoint operator L^ν on $L^2(E;\nu)$ which extends (L, D) and is a Dirichlet operator, i.e. $T_t := e^{tL^\nu}$ is sub-Markovian. The latter property is equivalent to the quadratic form given by L^ν on $L^2(E;\nu)$ being a *Dirichlet form*. Then Markov uniqueness is equivalent to the fact that there exists exactly one Dirichlet form whose generator extends (L, D). This problem is completely solved in the finite dimensional case in [RZ94] where Markov uniqueness was obtained under the most general conditions. The situation is quite different in the infinite dimensional case. We refer to [ARZ93a], [ARZ93b], [LR98], [KR07], [AKR12] for the best results in this direction known so far. In these papers Markov uniqueness has been obtained for a modified stochastic quantization equation

$$dX = (-\Delta + 1)^{-\varepsilon}(AX- : p(X) :)dt + (-\Delta + 1)^{-\frac{\varepsilon}{2}} dW(t),$$

with $\varepsilon > 0$. However, Markov uniqueness for the case that $\varepsilon = 0$ is still an open problem.

In this paper we study Markov uniqueness for the operator associated with the stochastic quantization problem in both the finite volume case and the infinite volume case and obtain the restricted Markov uniqueness of the operator, i.e. there exists exactly one *quasi-regular Dirichlet form* whose generator extends (L, D) (see Theorem 3.12, Theorem 4.10).

This problem is also related to the uniqueness of the martingale problem for (L, D), i.e. whether there exists exactly one (up to ν-equivalence defined in Section 3) strong Markov process solving the martingale problem for (L, D), and the uniqueness of probabilistically weak solution to (1.1). In this paper we also obtain that there exists exactly one (up to ν-equivalence defined in Section 3) martingale solution (probabilistically weak solution) to (1.1) (see Theorem 3.12 and Theorem 4.10).

We obtain Markov uniqueness in the restricted sense and the uniqueness of the martingale solution to (1.1) by studying the relations between the solutions to the stochastic quantization problem obtained by the Dirichlet form approach and those obtained in [DD03] and in [MW15]. In fact, (1.1) has been studied by many authors: In [MR99] the stationary solution to (1.1) has also been considered in their general theory of martinglae solutions for stochastic partial differential equaitons; In [DD03] Da Prato and Debussche define the Wick powers of solutions to the stochastic heat equation in the paths space and study a shifted equation instead of (1.1) in the finite volume case. They split the unknown X into two parts: $X = Y_1 + Z_1$, where $Z_1(t) = \int_{-\infty}^{t} e^{(t-s)A} dW(s)$. Observe that Y_1 is much smoother than X and that in the stationary case

$$: X^k := \sum_{l=0}^{k} C_k^l Y_1^l : Z_1^{k-l} :, \tag{1.2}$$

with $C_k^l = \frac{k!}{l!(k-l)!}$ and $: Z_1^{k-l} :$ being the Wick product, which motivate them to consider the following shifted equation:

$$\begin{aligned} \frac{dY_1}{dt} =& AY_1 - \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C_{k-1}^l Y_1^l : Z_1^{k-1-l} : \\ Y_1(0) =& z - Z_1(0), \end{aligned} \tag{1.3}$$

and obtain local existence and uniqueness of the solution Y_1 to (1.3) by a fixed point argument. By using the invariant measure ν they obtain a global solution to (1.1) by defining $X = Y_1 + Z$ starting from almost every starting point. In [MW15] the authors consider the following equation with $N = 2$ instead of (1.3):

$$\begin{aligned} \frac{dY}{dt} =& AY - \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C_{k-1}^l Y^l : \bar{Z}^{k-1-l} : \\ Y(0) =& 0, \end{aligned} \tag{1.4}$$

where $\bar{Z}(t) = e^{tA} z + \int_0^t e^{(t-s)A} dW(s)$ and $: \bar{Z}^{k-1-l} :$ will be defined later. We call (1.4) *the shifted equation* for short. They obtain global existence and uniqueness of the solution to (1.4) directly from every starting point both in the finite and infinite volume case. Actually, (1.3) is equivalent to (1.4). For the solution Y_1 to (1.3), defining $Y(t) := Y_1(t) + e^{tA} Z_1(0) - e^{tA} z$, we can easily check that Y is a solution to (1.4) by using the binomial formula (3.1) below.

It is natural to ask whether the unique solution obtained by the methods in [DD03] and [MW15] satisfies the original equation (1.1) and has ν as an invariant measure. Furthermore, it is a priori far from being clear what is the relation between the solutions obtained by the Dirichlet form approach and the solution obtained in [DD03] and in [MW15]. In this paper we study this problem and we prove that $X - \bar{Z}$, where X is obtained by the Dirichlet form approach in [AR91] and $\bar{Z}(t) = \int_0^t e^{(t-s)A} dW(s) + e^{tA} z$, also satisfies the shifted equation (1.4). We emphasize that it is not obvious that $X - \bar{Z}$ satisfies the shifted equation (1.4) since (1.2) only holds in the stationary case and we do not know whether the marginal distribution of the solution is absolutely continuous with respect to ν. However, by using Dirichlet form theory we can solve this problem and obtain the desired results (see Theorem 3.9 and Theorem 4.8).

Moreover, we obtain that the $p(\Phi)_2$ quantum field ν is an invariant measure for the process $X_0 = Y + \bar{Z}$, where Y is the unique solution to the shifted equation (1.4). As a consequence, we deduce uniqueness of martingale solutions (probabilistically weak solution) to (1.1) and Markov uniqueness for the corresponding generator in the restricted sense in both the finite and infinite volume case (see Theorem 3.12 and Theorem 4.10). We also emphasize that the $p(\Phi)_2$ field is not absolutely continuous with respect to Gaussian measure in the infinite volume case. This makes it more difficulty to analyze the support of ν. Here we use [GlJ86] and techniques from Dirichlet form theory to solve this problem (see Theorem 4.7).

We also want to mention that recently there has arisen a renewed interest in SPDEs related to such problems, particularly in connection with Hairer's theory of regularity structures [Hai14] and related work by Imkeller, Gubinelli, Perkowski in [GIP13]. By using these theories one can obtain local existence and uniqueness of solution to (1.1) in the three dimensional case (see [Hai14, CC13]). In a forthcoming paper we also prove ergodicity of the solution to (1.4) in the finite volume case. To the best of our knowledge, this is still an open problem in the periodic case (i.e. on the torus). In the infinte volume case this has been studied in [AKR97].

This paper is organized as follows: In Section 2 we collect some results related to Besov and weighted Besov spaces. In Section 3 we consider the finite volume case and prove that the solution obtained by Dirichlet form theory satisfies the shifted equation. Moreover, we obtain Markov uniqueness in the restricted sense and uniqueness of the martingale solutions (probabilistically weak solution) to (1.1). In Section 4 we prove that all the results also hold in the infinite volume case.

2 Preliminary

In the following we recall the definitions of Besov spaces. For a general introduction to the theory we refer to [BCD11, Tri78, Tri06]. First we introduce the following notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \backsimeq b$ if $a \lesssim b$ and $b \lesssim a$. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

i. the support of χ is contained in a ball and the support of θ is contained in an annulus;

ii. $\chi(z) + \sum_{j\geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}\theta(2^{-i}\cdot) \cap \mathrm{supp}\theta(2^{-j}\cdot) = \emptyset$ for $|i - j| > 1$.

We call such (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

Besov spaces

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{D}$ we define

$$\|u\|_{B^\alpha_{p,q}} := (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p})^q)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. The Besov space $B^\alpha_{p,q}$ consists of the completion of $\mathcal{D}$ with respect to this norm and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B^\alpha_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\}.$$

$$\mathcal{C}^\alpha(\mathbb{R}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} < \infty\}.$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [S85, SW71]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $\mathcal{D}$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q)^{1/q},$$

and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d)$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$

$$B^\alpha_{p,q}(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} < \infty\}.$$

$$\mathcal{C}^\alpha \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_\alpha < \infty\}. \tag{2.1}$$

Here we choose Besov spaces as completions of smooth functions with compact support, which ensures that the Besov spaces are separable which has a lot of advantages for our analysis below.

Weighted Besov spaces

In the following we recall the definitions and some properties of weighted Besov spaces, which are used for analyzing the regularity of the distributions in the infinite volume case. For a general introduction to these theories we refer to [Tri06].

For $\sigma \in \mathbb{R}$ we let $w(x) = (1 + |x|^2)^{-\sigma/2}$, $x \in \mathbb{R}^d$. For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, we define the weighted Besov norm for $u \in \mathcal{D}$,

$$\|u\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}} := (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(wdx)})^q)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$ and $\|f\|_{L^\infty(wdx)} = \|wf\|_{L^\infty(dx)}$. The Besov space $\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}(\mathbb{R}^d)$ consists of the completion of $\mathcal{D}$ with respect to this norm. For $p, q \in [1, \infty)$,

$$\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}} < \infty\}.$$

By [Tri83, Theorem 9.2.1] we can view functions on the torus $\mathbb{T}^d$ as periodic functions on $\mathbb{R}^d$ and have that for $\alpha \in \mathbb{R}, \sigma > 0$, $f \in \mathcal{C}^\alpha$

$$\|f\|_\alpha \backsimeq \|f\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{\infty,\infty}}. \tag{2.2}$$

Wavelet analysis

We will also use wavelet analysis to determine the regularity of a distribution in a Besov space. In the following we briefly summarize wavelet analysis below and we refer to work of Meyer [Mey92], Daubechies [Dau88] and [Tri06] for more details on wavelet analysis. For every $r > 0$, there exists a compactly supported function $\varphi \in C^r(\mathbb{R})$ such that:

1. We have $\langle \varphi(\cdot), \varphi(\cdot - k)\rangle = \delta_{k,0}$ for every $k \in \mathbb{Z}$;
2. There exist $\tilde{a}_k, k \in \mathbb{Z}$ with only finitely many non-zero values, and such that $\varphi(x) = \sum_{k\in\mathbb{Z}} \tilde{a}_k \varphi(2x - k)$ for every $x \in \mathbb{R}$;
3. For every polynomial P of degree at most r and for every $x \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} \int P(y)\varphi(y - k)dy\varphi(x - k) = P(x)$.

Given such a function φ, we define for every $x \in \mathbb{R}^d$ the recentered and rescaled function φ^n_x as follows

$$\varphi^n_x(y) := \Pi^d_{i=1} 2^{\frac{n}{2}} \varphi(2^n(y_i - x_i)).$$

Observe that this rescaling preserves the L^2-norm. We let V_n be the subspace of $L^2(\mathbb{R}^d)$ generated by $\{\varphi^n_x : x \in \Lambda_n\}$, where

$$\Lambda_n := \{(2^n k_1, ..., 2^n k_d) : k_i \in \mathbb{Z}\}.$$

An important property of wavelets is the existence of a finite set Ψ of compactly supported functions in C^r such that, for every $n \geq 0$, the orthogonal complement of V_n inside V_{n+1} is given by the linear span of all the $\psi^n_x, x \in \Lambda_n, \psi \in \Psi$. For every $n \geq 0$

$$\{\varphi^n_x, x \in \Lambda_n\} \cup \{\psi^m_x : m \geq n, \psi \in \Psi, x \in \Lambda_m\},$$

forms an orthonormal basis of $L^2(\mathbb{R}^d)$. This wavelet analysis allows one to identify a countable collection of conditions that determine the regularity of a distribution.

Setting $\Psi_\star = \Psi \cup \{\varphi\}$, by [Tri06, Theorem 6.15] we know that for $p \in (1, \infty)$, $\alpha \in \mathbb{R}$, $f \in \hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}$

$$\|f\|^p_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}} \lesssim \sum_{n=0}^{\infty} 2^{n(\alpha - d/p + 1)p} \sum_{\psi\in\Psi_\star} \sum_{x\in\Lambda_n} |\langle f, \psi^n_x\rangle|^p w(x), \tag{2.3}$$

and

$$\|f\|^p_{\hat{\mathcal{B}}^{\alpha,\sigma}_{\infty,\infty}} \lesssim \sum_{n=0}^{\infty} 2^{n(\alpha+1)p} \sum_{\psi\in\Psi_\star} \sum_{x\in\Lambda_n} |\langle f, \psi^n_x\rangle|^p w(x)^p. \tag{2.4}$$

Estimates on the torus

In this part we give estimates on the torus for later use. Set $\Lambda = (-A)^{\frac{1}{2}}$. For $s \geq 0, p \in [1, +\infty]$ we use H^s_p to denote the subspace of $L^p(\mathbb{T}^d)$, consisting of all f which can be written in the form $f = \Lambda^{-s} g, g \in L^p(\mathbb{T}^d)$ and the H^s_p norm of f is defined to be the L^p norm of g, i.e. $\|f\|_{H^s_p} := \|\Lambda^s f\|_{L^p(\mathbb{T}^d)}$.

To study (1.1) in the finite volume case, we will need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus first (c.f. [Tri78, Theorem 4.6.1], [GIP13, Lemma 41]):

Lemma 2.1 (i) Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$, and let $\alpha \in \mathbb{R}$. Then $B^{\alpha}_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.

(ii) Let $s \ge 0$, $1 < p < \infty$, $\epsilon > 0$. Then $H^{s+\epsilon}_p \subset B^s_{p,1}(\mathbb{T}^d) \subset B^s_{1,1}(\mathbb{T}^d)$.

(iii) Let $1 \le p_1 \le p_2 < \infty$ and let $\alpha \in \mathbb{R}$. Then $H^{\alpha}_{p_1}$ is continuously embedded in $H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$.

Here $\subset$ means that the embedding is continuous and dense.

We recall the following Schauder estimates, i.e. the smoothing effect of the heat flow, for later use.

Lemma 2.2 ([GIP13, Lemma 47]) (i) Let $u \in B^{\alpha}_{p,q}(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}, p, q \in [1,\infty]$. Then for every $\delta \ge 0$

$$\|e^{tA}u\|_{B^{\alpha+\delta}_{p,q}(\mathbb{T}^d)} \lesssim t^{-\delta/2}\|u\|_{B^{\alpha}_{p,q}(\mathbb{T}^d)}.$$

(ii) Let $\alpha \le \beta \in \mathbb{R}$. Then

$$\|(1-e^{tA})u\|_{\alpha} \lesssim t^{\frac{\beta-\alpha}{2}}\|u\|_{\beta}.$$

One can extend the multiplication on suitable Besov spaces and also have the duality properties of Besov spaces from [Tri78, Chapter 4]:

Lemma 2.3 (i) The bilinear map $(u;v) \mapsto uv$ extends to a continuous map from $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha\wedge\beta}$ if and only if $\alpha + \beta > 0$.

(ii) Let $\alpha \in (0,1)$, $p, q \in [1,\infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u;v) \mapsto \int uv dx$ extends to a continuous bilinear form on $B^{\alpha}_{p,q}(\mathbb{T}^d) \times B^{-\alpha}_{p',q'}(\mathbb{T}^d)$.

We recall the following interpolation inequality and multiplicative inequality for the elements in H^s_p, which is required for the a-priori estimate in the proof of Theorem 3.10: (cf. [Tri78, Theorem 4.3.1], [Re95, Lemma A.4], [RZZ15a, Lemma 2.1]):

Lemma 2.4 (i) Suppose that $s \in (0,1)$ and $p \in (1,\infty)$. Then for $u \in H^1_p$

$$\|u\|_{H^s_p} \lesssim \|u\|^{1-s}_{L^p(\mathbb{T}^d)}\|u\|^s_{H^1_p}.$$

(ii) Suppose that $s > 0$ and $p \in (1,\infty)$. If $u, v \in C^{\infty}(\mathbb{T}^2)$ then

$$\|\Lambda^s(uv)\|_{L^p(\mathbb{T}^d)} \lesssim \|u\|_{L^{p_1}(\mathbb{T}^d)}\|\Lambda^s v\|_{L^{p_2}(\mathbb{T}^d)} + \|v\|_{L^{p_3}(\mathbb{T}^d)}\|\Lambda^s u\|_{L^{p_4}(\mathbb{T}^d)},$$

with $p_i \in (1,\infty], i = 1, ..., 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Estimates on the whole space

We also collect some important properties for the weighted Besov spaces from [MW15] and [Tri06], which are parallel to those for Besov spaces on the torus. The Schauder estimate takes the following form:

Lemma 2.5 ([MW15, Propositions 3.11, 3.12]) (i) Let $u \in \hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}$ for some $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. Then we have for every $\delta \geq 0$

$$\|e^{tA}u\|_{\hat{\mathcal{B}}^{\alpha+\delta,\sigma}_{p,q}} \lesssim t^{-\delta/2}\|u\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}}.$$

(ii) Let $\alpha \leq \beta \in \mathbb{R}$ be such that $\beta - \alpha \leq 2$, $\sigma > 0$ and $p, q \in [1, \infty]$. Then for $u \in \hat{\mathcal{B}}^{\beta,\sigma}_{p,q}$

$$\|(1 - e^{tA})u\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}} \lesssim t^{\frac{\beta-\alpha}{2}}\|u\|_{\hat{\mathcal{B}}^{\beta,\sigma}_{p,q}}.$$

The multiplicative structure and Besov embedding theorems can be written as follows:

Lemma 2.6 ([MW15, Corollary 3.19, Corollary 3.21]) (1) For $\alpha > 0, p_1, p_2, p, q \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a continuous map from $\hat{\mathcal{B}}^{\alpha,\sigma}_{p_1,q} \times \hat{\mathcal{B}}^{\alpha,\sigma}_{p_2,q}$ to $\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}$.

(2) For $\alpha < 0, \alpha + \beta > 0, p_1, p_2, p, q \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a continuous map from $\hat{\mathcal{B}}^{\alpha,\sigma}_{p_1,q} \times \hat{\mathcal{B}}^{\beta,\sigma}_{p_2,q}$ to $\hat{\mathcal{B}}^{\alpha,\sigma}_{p,q}$.

(3) (Besov embedding [Tri06, Chapter 6]) Let $\alpha_1 \leq \alpha_2$, $1 \leq p_1 \leq p_2 \leq \infty$, and $1 \leq q_1 \leq q_2 \leq \infty$. Then

$$\hat{\mathcal{B}}^{\alpha_2,\sigma}_{p_1,q_1} \subset \hat{\mathcal{B}}^{\alpha_1,\sigma}_{p_1,q_1}; \quad \hat{\mathcal{B}}^{\alpha_1,\sigma}_{p_1,q_1} \subset \hat{\mathcal{B}}^{\alpha_1,\sigma}_{p_1,q_2}.$$

If $\sigma > d$, then

$$\hat{\mathcal{B}}^{\alpha_1,\sigma}_{p_2,q_1} \subset \hat{\mathcal{B}}^{\alpha_1,\sigma}_{p_1,q_1}.$$

Here $\subset$ means that the embedding is continuous and dense.

3 Finite volume case

In this section we consider (1.1) on the torus $\mathbb{T}^2$.

3.1 Wick power

In the following we define the Wick powers. First we define Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$ with $\mu = N(0, \frac{1}{2}(-\Delta + 1)^{-1}) := N(0, C)$.

Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$

In fact μ is a measure supported on $\mathcal{S}'(\mathbb{T}^2)$. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}'(\mathbb{T}^2), \mu) = \bigoplus_{n \geq 0} \mathcal{H}_n.$$

Now we define the Wick powers by using approximations: for $\phi \in \mathcal{S}'(\mathbb{T}^2)$ define

$$\phi_\varepsilon := \rho_\varepsilon * \phi,$$

with ρ_ε an approximate delta function on $\mathbb{R}^2$ given by

$$\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\frac{x}{\varepsilon}) \in \mathcal{D}, \int \rho = 1.$$

Here the convolution means that we view ϕ as a periodic distribution in $\mathcal{S}'(\mathbb{R}^2)$ and do convolution on $\mathbb{R}^2$. For every $n \in \mathbb{N}$ we set

$$: \phi_\varepsilon^n :_C := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}\phi_\varepsilon),$$

where $P_n, n = 0, 1, ...,$ are the Hermite polynomials defined by the formula

$$P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{(n-2j)!j!2^j} x^{n-2j},$$

and $c_\varepsilon = \int \phi_\varepsilon^2 \mu(d\phi) = \frac{1}{2}\int\int \bar{G}(x-y)\rho_\varepsilon(y)dy\rho_\varepsilon(x)dx = \|\bar{K}_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}$. Then

$$: \phi_\varepsilon^n :_C \in \mathcal{H}_n.$$

Here and in the following $\bar{G}$ is the Green function associated with $-A$ on $\mathbb{T}^2$ and $\bar{K}(t, x)$ is the heat kernel associated with A on $\mathbb{T}^2$ and $\bar{K}_\varepsilon = \bar{K} * \rho_\varepsilon$, where $*$ means convolution in space and we view $\bar{K}$ as a periodic function on $\mathbb{R}^2$.

For Hermite polynomial P_n we have that for $s, t \in \mathbb{R}$

$$P_n(s+t) = \sum_{m=0}^{n} C_n^m P_m(s) t^{n-m}, \tag{3.1}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

A direct calculation yields the following:

Lemma 3.1 Let $\alpha < 0$, $n \in \mathbb{N}$ and $p > 1$. $: \phi_\varepsilon^n :_C$ converges to some element in $L^p(\mathcal{S}'(\mathbb{T}^2), \mu; \mathcal{C}^\alpha)$. This limit is called the n-th Wick power of ϕ with respect to the covariance C and denoted by $: \phi^n :_C$.

Proof In fact, for every $p > 1, \varepsilon_1, \varepsilon_2 > 0$, $m \in \mathbb{N}$ by (2.2) and (2.4) we have that

$$\begin{aligned}
&\int \| : \phi_{\varepsilon_1}^m :_C - : \phi_{\varepsilon_2}^m :_C \|_\alpha^{2p} \mu(d\phi) \\
\lesssim &\sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} 2^{2\alpha pn+2np} \int |\langle : \phi_{\varepsilon_1}^m :_C - : \phi_{\varepsilon_2}^m :_C, \psi_x^n\rangle|^{2p} \mu(d\phi) w(x)^{2p} \\
\lesssim &\sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} 2^{2\alpha pn+2np} w(x)^{2p} (\int |\langle : \phi_{\varepsilon_1}^m :_C - : \phi_{\varepsilon_2}^m :_C, \psi_x^n\rangle|^2 \mu(d\phi))^p,
\end{aligned}$$

where $\sigma > 0$ in $w(x)$ and in the last inequality we used the hypercontractivity of the Gaussian measure. Moreover, we obtain that

$$\begin{aligned}
&\int |\langle :\phi^m_{\varepsilon_1}:_C - :\phi^m_{\varepsilon_2}:_C, \psi^n_x\rangle|^2\mu(d\phi)\\
\lesssim&\int\int |\psi^n_x(y)\psi^n_x(\bar y)|\Big|(\int \phi_{\varepsilon_1}(y)\phi_{\varepsilon_1}(\bar y)\mu(d\phi))^m - 2(\int \phi_{\varepsilon_1}(y)\phi_{\varepsilon_2}(\bar y)\mu(d\phi))^m\\
&+(\int \phi_{\varepsilon_2}(y)\phi_{\varepsilon_2}(\bar y)\mu(d\phi))^m\Big|dyd\bar y\\
\lesssim&\int\int |\psi^n_x(y)\psi^n_x(\bar y)|\Big|(\int\int \rho_{\varepsilon_1}(y-x_1)\rho_{\varepsilon_1}(\bar y-x_2)\bar G(x_1-x_2)dx_1dx_2)^m\\
&-2(\int\int \rho_{\varepsilon_1}(y-x_1)\rho_{\varepsilon_2}(\bar y-x_2)\bar G(x_1-x_2)dx_1dx_2)^m\\
&+(\int\int \rho_{\varepsilon_2}(y-x_1)\rho_{\varepsilon_2}(\bar y-x_2)\bar G(x_1-x_2)dx_1dx_2)^m\Big|dyd\bar y\\
\lesssim&(\varepsilon_1^\kappa+\varepsilon_2^\kappa)\int\int |\psi^n_x(y)\psi^n_x(\bar y)||y-\bar y|^{-\delta}dyd\bar y \lesssim (\varepsilon_1^\kappa+\varepsilon_2^\kappa)2^{-2n+n\delta},
\end{aligned}$$

where $\delta>\kappa>0, 2\alpha+\delta<0$. Here in the third inequality we have used Lemma 10.17 in [Hai14] and $|\bar G(x)|\lesssim -\log|x|$. Thus the results follow from a direct calculation. $\square$

Remark We can also use the approximations from [DD03] to define the Wick powers. We can prove that these two approximations converge to the same limit. This follows from the fact that under μ, $\phi =^d \int_{\mathbb{T}^2}\int_{-\infty}^t \bar K(t-s,\cdot-y)\xi(ds,dy)$ and for every $\varphi\in\mathcal{S}(\mathbb{T}^2)$,

$$\langle :\phi^n:,\varphi\rangle =^d \int_{[(-\infty,t]\times\mathbb{T}^2]^n}\langle\varphi,\prod_{j=1}^n \bar K(t-r_j,\cdot-y_j)\rangle\xi(dr_1,dy_1)...\xi(dr_n,dy_n)$$

Here $=^d$ means having the same distribution and ξ is space-time white noise.

Wick powers on a fixed probability space

Now we follow the idea from [DD03] and [MW15] to define the Wick powers of the solutions to the stochastic heat equation in the paths space. We fix a probability space $(\Omega,\mathcal{F},P)$ and W is a cylindrical Wiener process on $L^2(\mathbb{T}^2)$. We also have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\Omega,\mathcal{F},P)=\bigoplus_{n\geq 0}\mathcal{H}'_n.$$

In the following we set $Z(t)=\int_0^t e^{(t-s)A}dW(s)$, and we can also define Wick powers of $Z(t)$ with respect to different covariances by approximations: Let $Z_\varepsilon(t,x)=\int_0^t\langle\bar K_\varepsilon(t-s,x-\cdot),dW(s)\rangle=\rho_\varepsilon * Z$. Here $\langle\cdot,\cdot\rangle$ means inner product in $L^2(\mathbb{T}^2)$. Let $C_t:=\frac12(-A)^{-1}(I-e^{2tA})$. For every $n\in\mathbb{N}$ we set

$$:Z^n_\varepsilon(t):_{C_t}:=(c_{\varepsilon,t})^{\frac n2}P_n((c_{\varepsilon,t})^{-\frac12}Z_\varepsilon(t))\in\mathcal{H}'_n,$$

where $P_n, n=0,1,...,$ are the Hermite polynomials and $c_{\varepsilon,t}=\|1_{[0,t]}\bar K_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}$.

In the following we prove that $:Z^n_\varepsilon(t):_{C_t}$ is a Cauchy sequence in $C([0,T];\mathcal{C}^\alpha)$ and define the Wick powers of Z as the limit. To prove this we have to use the following result from [ZZ15, Lemma 4.1], the proof of which is a modification of the proof of [Hai14, Lemma 10.18].

Lemma 3.2 For a smooth function $K : \mathbb{R}^+ \times \mathbb{R}^2\backslash\{0\} \mapsto \mathbb{R}$ satisfying $|K(t, y)| \lesssim (t + |y|^2)^{\zeta/2}$ with $\zeta \in (-3, 0)$, then

$$|(K * \rho_\varepsilon)(t, y)| \lesssim t^{-\frac{\delta}{2}}(t + |y|^2)^{\frac{\zeta+\delta}{2}},$$

for $0 < \delta < 1, \zeta + \delta > -2$, and moreover, if K satisfies $|DK(t, y)| \lesssim (t + |y|^2)^{\frac{\zeta-1}{2}}$, then

$$|(K * \rho_\varepsilon)(t, y) - K(t, y)| \lesssim (t^{-\frac{\delta}{2}}\varepsilon^{\zeta-\bar{\zeta}}|y|^{\bar{\zeta}+\delta}) \wedge t^{\frac{\zeta}{2}},$$

for $\bar{\zeta} + \delta > -2, \zeta > \bar{\zeta}$.

Using Lemma 3.2 we obtain the Wick powers of $Z(t)$.

Lemma 3.3 For $\alpha < 0$, $n \in \mathbb{N}$, $p > 1$, $: Z^n_\varepsilon(t) :_{C_t}$ converges in $L^p(\Omega, C([0, T]; \mathcal{C}^\alpha))$. The limit is called Wick power of $Z(t)$ with respect to the covariance C_t and denoted by $: Z^n(t) :_{C_t}$.

Proof We first prove that $Z_\varepsilon \in C([0, T]; \mathcal{C}^\alpha)$ P-almost-surely. By the factorization method in [D04] we have that for $\kappa \in (0, 1)$

$$Z_\varepsilon(t) = \frac{\sin(\pi\kappa)}{\pi}\int_0^t (t - s)^{\kappa-1}\langle \bar{K}_\varepsilon(t - s, x - \cdot), U(s)\rangle ds,$$

where

$$U(s, y) = \int_0^s (s - r)^{-\kappa}\langle \bar{K}(s - r, y - \cdot), dW(r)\rangle.$$

A similar argument as in the proof of Lemma 2.7 in [D04] implies that it suffices to prove that for $p > 1/(2\kappa)$,

$$\mathbf{E}\|U\|_{L^{2p}(0,T;\mathcal{C}^\alpha)} < \infty. \tag{3.2}$$

In fact, by (2.2) and (2.4) we have that

$$\begin{aligned}\mathbf{E}\|U(s)\|^{2p}_\alpha &\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} \mathbf{E}2^{2\alpha pn+2np}|\langle U(s), \psi^n_x\rangle|^{2p}w(x)^{2p}\\ &\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} 2^{2\alpha pn+2np}(\mathbf{E}|\langle U(s), \psi^n_x\rangle|^2)^p w(x)^{2p}.\end{aligned}$$

Here $\sigma > 0$ in $w(x)$ and we used Gaussian hypercontractivity in the second inequality. Moreover we obtain that

$$\begin{aligned}\mathbf{E}|\langle U(s), \psi^n_x\rangle|^2 &\leq \int\int |\psi^n_x(y)\psi^n_x(\bar{y})| \int_0^s (s - r)^{-2\kappa}\bar{K} * \bar{K}(s - r, y - \bar{y})drdyd\bar{y}\\ &\lesssim \int\int |\psi^n_x(y)\psi^n_x(\bar{y})| \int_0^s (s - r)^{\kappa-1}|y - \bar{y}|^{-8\kappa}drdyd\bar{y}\\ &\lesssim 2^{-2n+8n\kappa}s^\kappa,\end{aligned}$$

where we used [Hai14, Lemma 10.17] to deduce that $|\bar{K}*\bar{K}(s-r, y-\bar{y})| \lesssim |s-r|^{3\kappa-1}|y-\bar{y}|^{-8\kappa}$ in the second inequality. Thus (3.2) follows by choosing κ small enough and a direct calculation.

Now we prove that for $m \in \mathbb{N}$, $: Z_\varepsilon^m :$ is a Cauchy sequence. For every $p > 1$, by (2.2) and (2.4) we have that for $t_1, t_2 \geq 0$

$$
\begin{aligned}
&\mathbf{E}\|(: Z_{\varepsilon_1}^m :_{C_{t_1}} - : Z_{\varepsilon_2}^m :_{C_{t_1}})(t_1, \cdot) - (: Z_{\varepsilon_1}^m :_{C_{t_2}} - : Z_{\varepsilon_2}^m :_{C_{t_2}})(t_2, \cdot)\|_\alpha^{2p} \\
\leq &\sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} \mathbf{E} 2^{2\alpha pn + 2np} |\langle (: Z_{\varepsilon_1}^m :_{C_{t_1}} - : Z_{\varepsilon_2}^m :_{C_{t_1}})(t_1, \cdot) - (: Z_{\varepsilon_1}^m :_{C_{t_2}} - : Z_{\varepsilon_2}^m :_{C_{t_2}})(t_2, \cdot), \psi_x^n \rangle|^{2p} w(x)^{2p} \\
\lesssim &\sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} 2^{2\alpha pn + 2np} (\mathbf{E}|\langle (: Z_{\varepsilon_1}^m :_{C_{t_1}} - : Z_{\varepsilon_2}^m :_{C_{t_1}})(t_1, \cdot) - (: Z_{\varepsilon_1}^m :_{C_{t_2}} - : Z_{\varepsilon_2}^m :_{C_{t_2}})(t_2, \cdot), \psi_x^n \rangle|^2)^p w(x)^{2p},
\end{aligned}
$$

where we used Gaussian hypercontractivity in the second inequality. For convenience we use ξ to denote space-time white noise given by $\int \phi(s, y)\xi(ds, dy) = \int_{\mathbb{R}^+} \langle \phi, dW(s) \rangle$ for $\phi \in L^2(\mathbb{R}^+ \times \mathbb{T}^2)$. Then we obtain that for $k = 1, 2$ and $j = 1, 2$

$$
: Z_{\varepsilon_k}^m(t_j) :_{C_{t_j}} = \int \Pi_{i=1}^m \bar{K}_{\varepsilon_k}(t_j - s_i, y - y_i) 1_{s_i \in [0, t_j]} \xi(d\eta_1) \ldots \xi(d\eta_m),
$$

where $\eta_a = (s_a, y_a)$, and $\int f(\eta_{1\ldots n})\xi(d\eta_1)\ldots\xi(d\eta_m)$ denotes a generic element of the n-th chaos of ξ for $\eta_{1\ldots n} = \eta_1 \ldots \eta_n$. Moreover, for $t_1 \leq t_2$ to estimate

$$
\mathbf{E}|\langle (: Z_{\varepsilon_1}^m :_{C_{t_1}} - : Z_{\varepsilon_2}^m :_{C_{t_1}})(t_1, \cdot) - (: Z_{\varepsilon_1}^m :_{C_{t_2}} - : Z_{\varepsilon_2}^m :_{C_{t_2}})(t_2, \cdot), \psi_x^n \rangle|^2,
$$

it suffices to calculate

$$
\begin{aligned}
&\int |\langle \Pi_{i=1}^m \bar{K}_{\varepsilon_1}(t_1 - s_i, \cdot - y_i) 1_{s_i \in [0, t_1]} - \Pi_{i=1}^m \bar{K}_{\varepsilon_2}(t_1 - s_i, \cdot - y_i) 1_{s_i \in [0, t_1]} \\
&- [\Pi_{i=1}^m \bar{K}_{\varepsilon_1}(t_2 - s_i, \cdot - y_i) 1_{s_i \in [0, t_2]} - \Pi_{i=1}^m \bar{K}_{\varepsilon_2}(t_2 - s_i, \cdot - y_i) 1_{s_i \in [0, t_2]}], \psi_x^n \rangle|^2 d\eta_{1\ldots m},
\end{aligned}
$$

which is bounded by

$$
\begin{aligned}
&2 \int |\langle (\Pi_{i=1}^m \bar{K}_{\varepsilon_1}(t_1 - s_i, \cdot - y_i) - \Pi_{i=1}^m \bar{K}_{\varepsilon_1}(t_2 - s_i, \cdot - y_i)) 1_{s_i \in [0, t_1]} \\
&- (\Pi_{i=1}^m \bar{K}_{\varepsilon_2}(t_1 - s_i, \cdot - y_i) - \Pi_{i=1}^m \bar{K}_{\varepsilon_2}(t_2 - s_i, \cdot - y_i)) 1_{s_i \in [0, t_1]}, \psi_x^n \rangle|^2 d\eta_{1\ldots m} \\
&+ 2 \int |\langle [\Pi_{i=1}^m \bar{K}_{\varepsilon_1}(t_2 - s_i, \cdot - y_i) 1_{s_i \in [t_1, t_2]} - \Pi_{i=1}^m \bar{K}_{\varepsilon_2}(t_2 - s_i, \cdot - y_i) 1_{s_i \in [t_1, t_2]}], \psi_x^n \rangle|^2 d\eta_{1\ldots m}.
\end{aligned}
$$

Then by Lemma 3.2 and [Hai14, Lemma 10.18] we have that for every $\delta > 0$

$$
\begin{aligned}
&|\bar{K}_{\varepsilon_1}(t_1 - s_i, y - y_i) - \bar{K}_{\varepsilon_1}(t_2 - s_i, y - y_i)| \\
\lesssim &|t_1 - t_2|^\delta (|t_2 - s_i|^{-\frac{1}{2} + \frac{\delta}{2}} + |t_1 - s_i|^{-\frac{1}{2} + \frac{\delta}{2}}) |y - y_i|^{-1 - 3\delta},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\bar{K}_{\varepsilon_1}(t_1 - s_i, y - y_i) - \bar{K}_{\varepsilon_2}(t_1 - s_i, y - y_i)| \\
\lesssim &(\varepsilon_1^{2\delta} + \varepsilon_2^{2\delta}) |t_1 - s_i|^{-\frac{1}{2} + \frac{\delta}{2}} |y - y_i|^{-1 - 3\delta},
\end{aligned}
$$

which combined with the interpolation and [Hai14, Lemma 10.14] imply that for every $\delta > 0$

$$
\begin{aligned}
&\mathbf{E}|\langle (: Z_{\varepsilon_1}^m :_{C_{t_1}} - : Z_{\varepsilon_2}^m :_{C_{t_1}})(t_1, \cdot) - (: Z_{\varepsilon_1}^m :_{C_{t_2}} - : Z_{\varepsilon_2}^m :_{C_{t_2}})(t_2, \cdot), \psi_x^n \rangle|^2 \\
\lesssim &(\varepsilon_1^{2\delta} + \varepsilon_2^{2\delta}) |t_2 - t_1|^\delta \int \int |\psi^n(y)\psi^n(\bar{y})| |y - \bar{y}|^{-8\delta} dy d\bar{y} \\
\lesssim &(\varepsilon_1^{2\delta} + \varepsilon_2^{2\delta}) |t_2 - t_1|^\delta 2^{-2n + 8n\delta}.
\end{aligned}
$$

Then the above estimates yield that

$$\begin{aligned}&\mathbf{E}\|(:Z^m_{\varepsilon_1}:_{C_{t_1}}-:Z^m_{\varepsilon_2}:_{C_{t_1}})(t_1,\cdot)-(:Z^m_{\varepsilon_1}:_{C_{t_2}}-:Z^m_{\varepsilon_2}:_{C_{t_2}})(t_2,\cdot)\|^{2p}_\alpha\\ \lesssim&\sum_{\psi\in\Psi_\star}\sum_{n\geq0}2^{2\alpha np+2np+2n}(\varepsilon_1^{2\delta}+\varepsilon_2^{2\delta})^p|t_2-t_1|^{\delta p}2^{-2np+8np\delta}.\end{aligned}$$

Thus the results follow from Kolmogorov's continuity test (in time) if we choose $\delta>0$ small enough and p sufficiently large. □

By this lemma we can define the Wick powers with respect to another covariance $:Z^n(t):_C$. For $t>0$, define

$$:Z^n_\varepsilon(t):_C:=c_\varepsilon^{\frac{n}{2}}P_n(c_\varepsilon^{-\frac12}Z_\varepsilon(t)).$$

Lemma 3.4 For $\alpha<0$, $p>1$, $n\in\mathbb{N}$, $:Z^n_\varepsilon:_C$ converges in $L^p(\Omega,C((0,T];\mathcal{C}^\alpha))$. Here $C((0,T];\mathcal{C}^\alpha)$ is equipped with the norm $\sup_{t\in[0,T]}t^{\rho/2}\|\cdot\|_\alpha$ for $\rho>0$. The limit is called Wick powers of $Z(t)$ with respect to the covariance C and denoted by $:Z^n(t):_C$. Moreover, for $t>0$

$$:Z^n(t):_C=\sum_{l=0}^{[n/2]}c_t^l\frac{n!}{(n-2l)!l!2^l}:Z^{n-2l}(t):_{C_t},$$

where $c_t:=\lim_{\varepsilon\to0}(c_{\varepsilon,t}-c_\varepsilon)$ locally uniformly for $t\in(0,T]$.

Proof By Lemma 3.3 it follows that for every $n\in\mathbb{N}$, $p>1$,

$$:Z^n_\varepsilon(t):_{C_t}\to:Z^n(t):_{C_t}\qquad\text{in } L^p(\Omega,C([0,T];\mathcal{C}^\alpha)).$$

By the definition of $c_{\varepsilon,t}$ and c_ε we also have that for every $\rho>0$, $t>0$ and $\varepsilon>0$

$$|c_{\varepsilon,t}-c_\varepsilon|\lesssim t^{-\rho/2},$$

where the constant we omit is independent of ε, and

$$c_t:=\lim_{\varepsilon\to0}(c_{\varepsilon,t}-c_\varepsilon)=-\int_t^\infty\int_{\mathbb{T}^2}\bar{K}(r,x)^2dxdr.$$

Moreover, the definition of P_n yields that

$$:Z^n_\varepsilon(t):_C=\sum_{l=0}^{[n/2]}(c_{\varepsilon,t}-c_\varepsilon)^l\frac{n!}{(n-2l)!l!2^l}:Z^{n-2l}_\varepsilon(t):_{C_t},$$

which implies the result by letting $\varepsilon\to0$. □

Now following the technique in [MW15] we combine the initial value part with the Wick powers by using (3.1). We set $V(t)=e^{tA}z, V_\varepsilon=\rho_\varepsilon*V$, $z\in\mathcal{C}^\alpha$ for $\alpha<0$ and

$$\bar{Z}(t)=Z(t)+V(t),\quad \bar{Z}_\varepsilon(t)=Z_\varepsilon(t)+V_\varepsilon(t),$$

$$:\bar{Z}^n(t):_C=\sum_{k=0}^nC_n^kV(t)^{n-k}:Z^k(t):_C,\quad :\bar{Z}^n_\varepsilon(t):_C=\sum_{k=0}^nC_n^kV_\varepsilon(t)^{n-k}:Z^k_\varepsilon(t):_C.$$

By Lemma 2.2 we know that $V \in C([0,T],\mathcal{C}^\alpha)$ and $V \in C((0,T],\mathcal{C}^\beta)$ for $\beta > \alpha$ with the norm $\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}}\|\cdot\|_\beta$. Moreover,

$$\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}}\|V(t)\|_\beta \lesssim \|z\|_\alpha, \quad \sup_{t\in[0,T]} t^{\frac{\beta-\alpha+\kappa}{2}}\|V_\varepsilon(t)-V(t)\|_\beta \lesssim \|z\|_\alpha$$

for $\beta > \alpha, \kappa > 0$. Then by Lemmas 2.3 and 3.4 we have the following results:

Lemma 3.5 Let $\alpha < 0, z \in \mathcal{C}^\alpha$, $p > 1$. Define $\bar{Z}$ and $:\bar{Z}^n:_C$ as above. Then for $n \in \mathbb{N}$ $:\bar{Z}_\varepsilon^n:_C$ converges to $:\bar{Z}^n:_C$ in $L^p(\Omega, C((0,T];\mathcal{C}^\alpha))$. Here the norm for $C((0,T];\mathcal{C}^\alpha)$ is

$$\sup_{t\in[0,T]} t^{\frac{(\beta-\alpha)n+\rho}{2}}\|\cdot\|_\alpha$$

for some $\beta > -\alpha > 0, \rho > 0$.

Proof By Lemma 2.3 we have

$$\|:\bar{Z}_\varepsilon^n:_C - :\bar{Z}^n:_C\|_\alpha \lesssim \sum_{k=0}^n [\|V_\varepsilon^k\|_\beta\| :Z_\varepsilon^{n-k}:_C - :Z^{n-k}:_C\|_\alpha + \|V_\varepsilon^k - V^k\|_\beta\| :Z^{n-k}:_C\|_\alpha],$$

which implies the results easily. □

Relations between two different Wick powers

First we introduce the following probability measure. Set $:q(\phi):= \sum_{n=0}^{2N} a_n :\phi^n:_C$, $:p(\phi):= \sum_{n=1}^{2N} n a_n :\phi^{n-1}:_C$ and we assume that $a_n \in \mathbb{R}$ and $a_{2N} > 0$. Let

$$\nu = c\exp(-2\int_{\mathbb{T}^2} :q(\phi):dx)\mu,$$

where c is a normalization constant. Then by [GlJ86, Sect. 8.6] for every $p \in [1,\infty)$, $\varphi(\phi) := \exp(-2\int_{\mathbb{T}^2} :q(\phi):dx) \in L^p(\mathcal{S}'(\mathbb{T}^2),\mu)$. The following result states the relations between two different Wick powers.

Lemma 3.6 Let ϕ be a measurable map from $(\Omega,\mathcal{F},P)$ to $C([0,T],B_{2,2}^{-\gamma})$ with $\gamma > 2$, $P\circ\phi(t)^{-1} = \nu$ for every $t \in [0,T]$ and let $\bar{Z}(t)$ be defined as above. Assume in addition that $y = \phi - \bar{Z} \in C([0,T];\mathcal{C}^\beta)$ P-a.s. for some $\beta > -\alpha > 0$. Then for every $t > 0$, $n \in \mathbb{N}$

$$:\phi^n(t):_C = \sum_{k=0}^n C_n^k y^{n-k}(t) :\bar{Z}^k(t):_C \quad P-a.s..$$

Proof By Lemma 3.5 it follows that for every $k \in \mathbb{N}$, $p > 1$

$$:\bar{Z}_\varepsilon^k:_C \to :\bar{Z}^k:_C \quad \text{in } L^p(\Omega, C((0,T];\mathcal{C}^\alpha)), \text{ as } \varepsilon \to 0.$$

Since $y_\varepsilon = \phi_\varepsilon - \bar{Z}_\varepsilon = \rho_\varepsilon * y$ and $y \in C([0,T];\mathcal{C}^\beta)$ P-a.s., it is obvious that $y_\varepsilon \to y$ in $C([0,T];\mathcal{C}^{\beta-\kappa})$ P-a.s. for every $\kappa > 0$ with $\beta - \kappa + \alpha > 0$, which combined with Lemma 2.3 implies that for $k \in \mathbb{N}$, $k \leq n$,

$$y_\varepsilon^{n-k} :\bar{Z}_\varepsilon^k:_C \to^P y^{n-k} :\bar{Z}^k:_C \quad \text{in } C((0,T];\mathcal{C}^\alpha), \text{ as } \varepsilon \to 0.$$

Here $\to^P$ means convergence in probability. Since $\exp(-\int_{\mathbb{T}^2} :q(\phi): dx) \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$ for every $p \geq 1$, by Hölder's inequality and Lemma 3.1 we get that for $t > 0$ and $p > 1$

$$:\phi_\varepsilon^n(t):_C \to :\phi^n(t):_C \quad \text{in } L^p(\Omega, \mathcal{C}^\alpha), \text{ as } \varepsilon \to 0.$$

Moreover, by (3.1) we have

$$\begin{aligned} :\phi_\varepsilon^n:_C =& :(y_\varepsilon + \bar{Z}_\varepsilon)^n:_C = c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}(y_\varepsilon + \bar{Z}_\varepsilon)) \\ =& \sum_{k=0}^n C_n^k c_\varepsilon^{n/2} P_k(c_\varepsilon^{-1/2}\bar{Z}_\varepsilon)(c_\varepsilon^{-1/2}y_\varepsilon)^{n-k} \\ =& \sum_{k=0}^n C_n^k :\bar{Z}_\varepsilon^k:_C y_\varepsilon^{n-k}, \end{aligned}$$

which implies the result by letting $\varepsilon \to 0$. □

In the following, we only use Wick powers $:\cdot:_C$ and we write $:\cdot:$ for simplicity.

3.2 Relations between the two solutions: starting with solutions given by Dirichlet forms

As mentioned in the introduction, weak solutions to (1.1) have been constructed in [AR91] by Dirichlet forms. In this subsection we prove that the solutions constructed in [AR91] also satisfy the shifted equation (1.4). First we recall some basic results related to Dirichlet forms from [AR91].

Solutions given by Dirichlet forms

Let $H = L^2(\mathbb{T}^2)$ and let $-\Delta + I$ be the generator of the following quadratic form on $H : (u, v) \mapsto \int_{\mathbb{T}^2} \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d} dx + \int_{\mathbb{T}^2} uv dx$ with $u, v \in \{g \in L^2(\mathbb{T}^2) | \nabla g \in L^2(\mathbb{T}^2)\}$ (where ∇ is in the sense of distributions). Let $\{e_k | k \in \mathbb{Z}^2\} \subset C^\infty(\mathbb{T}^2)$ be the (orthonormal) eigenbasis of $-\Delta + I$ in H and $\{\lambda_k | k \in \mathbb{Z}^2\} \subset (0, \infty)$ the corresponding eigenvalues. Define for $s \in \mathbb{R}$,

$$H^s := \{u \in \mathcal{S}'(\mathbb{T}^2) | \sum_{k \in \mathbb{Z}^2} \lambda_k^s {}_{\mathcal{S}'}\langle u, e_k \rangle_{\mathcal{S}}^2 < \infty\},$$

equipped with the inner product

$$\langle u, v \rangle_{H^s} := \sum_{k \in \mathbb{Z}^2} \lambda_k^s {}_{\mathcal{S}'}\langle u, e_k \rangle_{\mathcal{S}\mathcal{S}'}\langle v, e_k \rangle_{\mathcal{S}}.$$

If for $s \geq 0$ ${}_{H^s}\langle \cdot, \cdot \rangle_{H^{-s}}$ denotes the dualization between H^s and its dual space H^{-s}, then it follows that

$${}_{H^s}\langle u, v \rangle_{H^{-s}} = \langle u, v \rangle_H, u \in H^s, v \in H.$$

Let $E = H^{-1-\epsilon}, E^* = H^{1+\epsilon}$ for some $\epsilon > 0$. We denote their Borel σ-algebras by $\mathcal{B}(E), \mathcal{B}(E^*)$ respectively. Define

$$\mathcal{F}C_b^\infty = \{u : u(z) = f({}_{E^*}\langle l_1, z \rangle_E, {}_{E^*}\langle l_2, z \rangle_E, ..., {}_{E^*}\langle l_m, z \rangle_E), z \in E, l_1, l_2, ..., l_m \in E^*, m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m)\}.$$

Define for $u \in \mathcal{F}C_b^\infty$ and $l \in H$,

$$\frac{\partial u}{\partial l}(z) := \frac{d}{ds}u(z+sl)|_{s=0}, z \in E,$$

that is, by the chain rule,

$$\frac{\partial u}{\partial l}(z) = \sum_{j=1}^{m} \partial_j f({}_{E^*}\langle l_1, z\rangle_E, {}_{E^*}\langle l_2, z\rangle_E, ..., {}_{E^*}\langle l_m, z\rangle_E)\langle l_j, l\rangle_H.$$

Let Du denote the H-derivative of $u \in \mathcal{F}C_b^\infty$, i.e. the map from E to H such that

$$\langle Du(z), l\rangle = \frac{\partial u}{\partial l}(z) \text{ for all } l \in H, z \in E.$$

By [AR91] we easily deduce that the form

$$\mathcal{E}(u,v) := \frac{1}{2}\int_E \langle Du, Dv\rangle_H d\nu; u, v \in \mathcal{F}C_b^\infty$$

is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;\nu)$ in the sense of [MR92]. By [AR91, Theorem 3.6] we know that there exists a (Markov) diffusion process $M = (\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t\geq 0}, (P^z)_{z\in E})$ on E *properly associated with* $(\mathcal{E}, D(\mathcal{E}))$, i.e. for $u \in L^2(E;\nu) \cap \mathcal{B}_b(E)$, the transition semigroup $P_t u(z) := E^z[u(X(t))]$ is $\mathcal{E}$-quasi-continuous for all $t > 0$ and is a ν-version of $T_t u$, where T_t is the semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$. Here for the notion of $\mathcal{E}$-quasi-continuity we refer to [MR92, ChapterIII, Definition 3.2].

By [GlJ86, (9.1.32)] we have the following:

Theorem3.7 For each l smooth, we have that the partial log derivative β_l of ν is given by

$$\beta_l(z) = -2\sum_{n=1}^{2N} na_n : z^{n-1} : (l) + 2{}_{H^s}\langle \Delta l - l, z\rangle_{H^{-s}},$$

where $: z^n : (l)$ denotes the dualization between $: z^n :$ and l.

Now we want to extend the definition of β_l to the whole space E. By [R86, Theorem 3.1] $l \rightarrow: z^n : (l)$ can be extended to a continuous map from H to $L^2(E,\mu)$. So, by [AR91, Proposition 6.9], there exists a $\mathcal{B}(H^{-1-\epsilon})/\mathcal{B}(H^{-1-\epsilon})$ measurable map $: z^n :: H^{-1-\epsilon} \rightarrow H^{-1-\epsilon}$ such that $: z^n : (l) ={}_{H^{-1-\epsilon}} \langle : z^n :, l\rangle_{H^{1+\epsilon}}$ ν-a.e. for some $\epsilon > 0$. By [AR91, Theorem 6.10] we have the following Fukushima decomposition for $X(t)$ under P^z.

Theorem 3.8 There exist a map $W : \Omega \rightarrow C([0,\infty); E)$ and a *properly $\mathcal{E}$-exceptional set* $S \subset E$, i.e. $\nu(S) = 0$ and $P^z[X(t) \in E \setminus S, \forall t \geq 0] = 1$ for $z \in E \backslash S$, such that $\forall z \in E\backslash S$ under P^z, W is an $\mathcal{M}_t$- cylindrical Wiener process and the sample paths of the associated process $M = (\Omega, \mathcal{F}, (X(t))_{t\geq 0}, (P^z)_{z\in E})$ on E satisfy the following: for $l \in H^{2+s}$, $s > 0$

$$\begin{aligned} {}_{E^*}\langle l, X(t) - X(0)\rangle_E = \int_0^t \langle l, dW(r)\rangle + \int_0^t \Big[{}_{H^{-s-2}}\langle -\sum_{n=1}^{2N} na_n : X(r)^{n-1} :, l\rangle_{H^{2+s}} \\ + {}_{H^s}\langle \Delta l - l, X(r)\rangle_{H^{-s}} \Big] dr \quad \forall t \geq 0 \ P^z\text{–a.s..} \end{aligned} \tag{3.3}$$

Moreover, ν is an invariant measure for M in the sense that $\int P_t u d\nu = \int u d\nu$ for $u \in L^2(E;\nu) \cap \mathcal{B}_b(E)$.

Relations between the two solutions

In the following we discuss the relations between M constructed above and the shifted equation (1.4). In fact, by Lemma 2.1 we have that $\mathcal{C}^\alpha \subset E$ for $\alpha \in (-1,0)$, $\mathcal{C}^\alpha \in \mathcal{B}(E)$ and $\nu(\mathcal{C}^\alpha) = 1$. For W constructed in Theorem 3.8 define $\bar{Z}(t) := \int_0^t e^{(t-s)A} dW(s) + e^{tA}X(0)$.

Theorem 3.9 Let $\alpha \in (-\frac{1}{2N-1}, 0)$, $-\alpha < \beta < \alpha + 2$. There exists a properly $\mathcal{E}$-exceptional set $S_2 \subset E$ in the sense of Theorem 3.8 such that for every $z \in \mathcal{C}^\alpha \setminus S_2$ under P^z, $Y := X - \bar{Z} \in C([0,T];\mathcal{C}^\beta)$ is a solution to the following equation:

$$Y(t) = -\int_0^t e^{(t-s)A} \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C_{k-1}^l Y(s)^l : \bar{Z}(s)^{k-1-l} : ds. \tag{3.4}$$

Moreover,

$$P^z[X(t) \in \mathcal{C}^\alpha \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in \mathcal{C}^\alpha \setminus S_2.$$

Proof Recall that $: p(\phi) := \sum_{n=1}^{2N} n a_n : \phi^{n-1} :$. Now for $z \in E \setminus S$ under P^z we have that

$$X(t) = -\int_0^t e^{(t-\tau)A} : p(X(\tau)) : d\tau + \bar{Z}(t).$$

Since ν is an invariant measure for X, by Lemmas 2.1 and 3.1 we conclude that for every $T \geq 0$, $p > 1, \delta > 0$, with $\alpha + 2\delta < 0$, and $p_0 > 1$ large enough

$$\begin{aligned}
&\int E^z \int_0^T \| : p(X(\tau)) : \|_\alpha^p d\tau \nu(dz) \lesssim \int E^z \int_0^T \| : p(X(\tau)) : \|_{B_{p_0,p_0}^{\alpha+\delta}}^p d\tau \nu(dz) \\
=&T \int \| : p(\phi) : \|_{B_{p_0,p_0}^{\alpha+\delta}}^p \nu(d\phi) \lesssim T \int \| : p(\phi) : \|_{\alpha+2\delta}^p \nu(d\phi) < \infty,
\end{aligned}$$

which implies that there exists a properly $\mathcal{E}$-exceptional set $S_1 \supset S$ such that for $z \in E \setminus S_1$ P^z-a.s.

$$: p(X(\cdot)) :\in L^p(0,T;\mathcal{C}^\alpha), \quad E^z \int_0^T \| : p(X(\tau)) : \|_\alpha^p d\tau < \infty, \quad \forall p > 1.$$

Here we used Lemma 2.1 to deduce the first result. The second, however, does not imply the first directly because of (2.1). Lemma 2.2 implies that for $-\alpha < \beta < \alpha + 2$

$$\int_0^t e^{(t-\tau)A} : p(X(\tau)) : d\tau \in C([0,\infty);\mathcal{C}^\beta) \quad P^z - a.s..$$

Now we conclude that for $z \in E \setminus S_1$

$$X - \bar{Z} \in C([0,\infty);\mathcal{C}^\beta) \quad P^z - a.s..$$

Since $P^\nu \circ X(t)^{-1} = \nu$, by Lemma 3.6 we conclude that under P^ν, by Fubini's theorem $Y := X - \bar{Z}$ satisfies (3.4) and for ν-a.e. $z \in E$ under P^z, $Y := X - \bar{Z}$ satisfies (3.4). In the following we prove that the results hold under P^z for z outside a properly $\mathcal{E}$-exceptional set. First we have $\bar{Z} \in C([0,\infty);\mathcal{C}^\alpha)$ P^ν-a.s., which combined with $X - \bar{Z} \in C([0,\infty);\mathcal{C}^\beta)$ implies

$$P^\nu[X \in C([0,\infty),\mathcal{C}^\alpha)] = 1.$$

Define $Z(t) = \int_0^t e^{(t-s)A} dW(s)$ and we obtain that

$$\begin{aligned}\bar{Y}(s,t_0) :=& X(s+t_0) - Z(s+t_0) - e^{sA}(X(t_0) - Z(t_0)) \\ =& \int_{t_0}^{t_0+s} e^{(t_0+s-\tau)A} : p(X(\tau)) : d\tau \in C([0,\infty)^2;\mathcal{C}^\beta) \quad P^\nu - a.s..\end{aligned}$$

Moreover, for $s > 0, t_0 \geq 0$, define

$$: [Z(s+t_0) + e^{sA}(X(t_0) - Z(t_0))]^k ::= \sum_{l=0}^{k} C_k^l (e^{sA}(X(t_0) - Z(t_0)))^l : Z(s+t_0)^{k-l} :,$$

where we used that $e^{sA}(X(t_0) - Z(t_0)) \in \mathcal{C}^\beta$ P^ν-a.s. to make the right hand side of the above equality meaningful. Similar arguments as in the proof of Lemma 3.6 imply that $\forall s > 0, t_0 \geq 0$

$$\begin{aligned}P^\nu(: p(X(s+t_0)) :=& \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C_{k-1}^l \bar{Y}(s,t_0)^l : [Z(s+t_0) + e^{sA}(X(t_0) - Z(t_0))]^{k-1-l} :, \\ & X \in C([0,\infty),\mathcal{C}^\alpha), \bar{Y} \in C([0,\infty)^2;\mathcal{C}^\beta)) = 1,\end{aligned}$$

where we used $: [Z(s+t_0) + e^{sA}(X(t_0) - Z(t_0))]^k :\in \mathcal{C}^\alpha$. In the following we use I_{t,t_0} to denote the equality

$$\int_0^t : p(X(s+t_0)) : ds = \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} \int_0^t C_{k-1}^l \bar{Y}(s,t_0)^l : [Z(s+t_0) + e^{sA}(X(t_0) - Z(t_0))]^{k-1-l} : ds.$$

Then using Fubini's theorem we know that

$$P^\nu(I_{t,t_0} \text{ holds } \forall t \geq 0, a.e. t_0 \geq 0, X \in C([0,\infty);\mathcal{C}^\alpha), \bar{Y} \in C([0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

Here we used $X \in C([0,\infty);\mathcal{C}^\alpha)$ for $-\alpha(2N-1) < 1$ to make the right hand side of I_{t,t_0} meaningful. It is obvious that the right hand side of the first equality is continuous with respect to t_0. Since $\int_0^t : p(X(s+t_0)) : ds = \int_{t_0}^{t+t_0} : p(X(s)) : ds$ we know that $\int_0^t : p(X(s+t_0)) : ds$ is also continuous with respect to t_0 and we obtain that

$$P^\nu(I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0, X \in C([0,\infty);\mathcal{C}^\alpha), \bar{Y} \in C([0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

This implies that there exists a properly $\mathcal{E}$-exceptional set $S_2 \supset S_1$ such that for $z \in \mathcal{C}^\alpha \setminus S_2$ under P^z

$$P^z(X \in C([0,\infty);\mathcal{C}^\alpha), I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0) = 1.$$

Indeed, define

$$\Omega_0 := \{\omega : X \in C([0,\infty);\mathcal{C}^\alpha), I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0\},$$

and let $\Theta_t : \Omega \to \Omega, t > 0$, be the canonical shift, i.e. $\Theta_t(\omega) = \omega(\cdot + t), \omega \in \Omega$. Then it is easy to check that

$$\Theta_t^{-1}\Omega_0 \supset \Omega_0, \quad t \in \mathbb{R}^+,$$

and

$$\Omega_0 = \bigcap_{t>0, t\in\mathbb{Q}} \Theta_t^{-1}\Omega_0.$$

On the other hand, by the Markov property we know that

$$P^z(\Theta_t^{-1}\Omega_0) = P_t(1_{\Omega_0})(z),$$

which by [MR92, Chapter IV Theorem 3.5] is $\mathcal{E}$-quasi-continuous in the sense of [MR92, Chapter III Definition 3.2] on E. It follows that for every $t > 0$

$$P^z(\Theta_t^{-1}\Omega_0) = 1 \quad q.e. z \in E,$$

which yields that

$$P^z(\Omega_0) = 1 \quad q.e. z \in E.$$

Here q.e. means that there exists a properly $\mathcal{E}$-exceptional set such that outside this exceptional set the result follows. Now Y satisfies (3.4) P^z-a.s. for $z \in \mathcal{C}^\alpha \backslash S_2$. Moreover, for $z \in \mathcal{C}^\alpha \backslash S_2$ $Y \in C([0,\infty); \mathcal{C}^\beta), \bar{Z} \in C([0,\infty); \mathcal{C}^\alpha)$ P^z-a.s., which implies that

$$P^z[X(t) \in \mathcal{C}^\alpha \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in \mathcal{C}^\alpha \setminus S_2.$$

□

3.3 Relations between two solutions: starting with solutions to the shifted equation

Now we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ and on it a cylindrical Wiener process W in $L^2(\mathbb{T}^2)$. Define $\bar{Z}(t) = \int_0^t e^{(t-s)A} dW(s) + e^{tA} z$ as in Section 3.2 with $z \in \mathcal{C}^\alpha$ for $\alpha < 0$. Now we consider the following equation:

$$Y(t) = e^{tA} y - \int_0^t e^{(t-s)A} \sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C_{k-1}^l Y(s)^l : \bar{Z}(s)^{k-1-l} : ds. \tag{3.5}$$

When $N = 2$, global existence and uniqueness of the solutions to (3.5) have been obtained in [MW15]. Now we consider general $N \in \mathbb{N}$ and have the following existence and uniqueness results. Moreover, by using solutions given by Dirichlet form theory we also obtain that ν is an invariant measure of the solution to $\bar{X} = Y_0 + \bar{Z}$, where Y_0 is the unique solution to (3.5) with $y = 0$.

Theorem 3.10 Fix α, β such that $0 < -\alpha < \beta < \alpha + 2$ and $\beta, -\alpha$ sufficiently small. For $y \in L^p(\mathbb{T}^2)$, with p even, large enough, there exists a unique solution to (3.5) in $C((0,T]; \mathcal{C}^\beta)$ equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta}{2}+\frac{1}{p}} \|\cdot\|_\beta$.

Moreover, ν is an invariant measure of the solution to $\bar{X} = Y_0 + \bar{Z}$, where Y_0 is the unique solution to (3.5) with $y = 0$.

Proof First we prove local existence and uniqueness of solutions: for $y \in L^p(\mathbb{T}^2)$ and a.s. $\omega \in \Omega$ there exists $T^*(z,\omega)$ and a unique solution to (3.5) such that

$$Y(\omega) \in C((0, T^*(z,\omega)], \mathcal{C}^\beta).$$

In fact, we use a fixed point argument in the space

$$\mathcal{L}_T := C((0,T], \mathcal{C}^\beta)$$

equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta}{2}+\frac{1}{p}}\|\cdot\|_\beta$. By Lemma 3.5 we have that $:\bar{Z}^n_\varepsilon:$ converges to $:\bar{Z}^n:$ in $C((0,T];\mathcal{C}^\alpha)$ in probability, with the norm $\sup_{t\in[0,T]} t^{\frac{(\beta-\alpha)n+\rho}{2}}\|:\bar{Z}^n(t):\|_\alpha < \infty$ for $\beta > -\alpha > 0, \rho > 0$. For the above α, β, ρ, we introduce the following notation

$$\|\bar{Z}\|_{\mathfrak{L}} := \sum_{l=0}^{2N-1} \sup_{\tau\in[0,T]} \tau^{\frac{(\beta-\alpha)l+\rho}{2}}\|:\bar{Z}^l(\tau):\|_\alpha.$$

By Lemmas 2.2, 2.3 we obtain that for $Y \in \mathcal{L}_T$

$$\int_0^\cdot e^{(\cdot-\tau)A}\sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C^l_{k-1} Y^l(\tau):\bar{Z}^{k-1-l}(\tau):d\tau \in \mathcal{L}_T$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]} t^{\beta/2+1/p}\Big\|\int_0^t e^{(t-\tau)A}\sum_{k=1}^{2N} k a_k \sum_{l=0}^{k-1} C^l_{k-1} Y^l(\tau):\bar{Z}^{k-1-l}(\tau):d\tau\Big\|_\beta \\
\lesssim &\sup_{t\in[0,T]} t^{\beta/2+1/p}\sum_{k=1}^{2N}\int_0^t (t-\tau)^{\frac{\alpha-\beta}{2}}\sum_{l=0}^{k-1}\|Y(\tau)\|^l_\beta\|:\bar{Z}^{k-1-l}(\tau):\|_\alpha d\tau \\
\lesssim &\sup_{t\in[0,T]} t^{\beta/2+1/p}\sum_{k=1}^{2N}\int_0^t (t-\tau)^{\frac{\alpha-\beta}{2}}\Big(\sum_{l=1}^{k-1}\tau^{-(\frac{\beta}{2}+\frac{1}{p})l-\frac{(\beta-\alpha)(k-1-l)+\rho}{2}}\|Y\|^l_{\mathcal{L}_T} + \tau^{-\frac{(\beta-\alpha)(k-1)+\rho}{2}}\Big)\|Z\|_{\mathfrak{L}} d\tau \\
\lesssim &T^{(\alpha-\beta)N+1+\frac{\beta}{2}+\frac{1}{p}-\frac{\rho}{2}} \vee T^{1+\frac{\alpha-\beta-\rho}{2}-(\frac{\beta}{2}+\frac{1}{p})(2N-2)}(\|Y\|^{2N}_{\mathcal{L}_T}+1),
\end{aligned} \tag{3.6}$$

with $\beta, -\alpha > 0$ small enough and $p > 0$ large enough. Here we used Lemmas 2.2 and 2.3 in the first inequality and used Lemma 3.5 in the second inequality and used $\|\bar{Z}\|_{\mathfrak{L}} < \infty$ P-a.s. in the last inequality. Moreover, by Lemmas 2.1 and 2.2 we have

$$\|e^{tA}y\|_\beta \lesssim t^{-(\frac{\beta}{2}+\frac{1}{p})}\|y\|_{L^p(\mathbb{T}^2)}.$$

Similarly we obtain that the iteration mapping is a strict contraction in a bounded ball $\mathcal{L}_T$ with $T > 0$ small enough, which implies the local existence and uniqueness of solutions. A similar argument as (3.6) also implies that the local solution is continuous with respect to $(\bar{Z}, :\bar{Z}^2:, ..., :\bar{Z}^{2N-1}:)$.

In the following we give an a-priori estimate on the L^p norm of Y: Let Y_ε be the solution to (3.5) with $\bar{Z}$ replaced by $\bar{Z}_\varepsilon$, where $\bar{Z}_\varepsilon$ is defined in Section 3.1. Since $\bar{Z}_\varepsilon$ is smooth we know that Y_ε is smooth and we can choose Y_ε^{p-1} as a test function. Then we have

$$\begin{aligned}&\frac{1}{p}(\|Y_\varepsilon(t)\|_{L^p}^p-\|Y_\varepsilon(0)\|_{L^p}^p)\\&=\int_0^t[-(p-1)\langle\nabla Y_\varepsilon(s),Y_\varepsilon^{p-2}(s)\nabla Y_\varepsilon(s)\rangle-\|Y_\varepsilon(s)\|_{L^p}^p-\langle\sum_{k=1}^{2N}a_k\sum_{l=0}^{k-1}C_{k-1}^lY_\varepsilon^l(s):\bar{Z}_\varepsilon^{k-1-l}(s):,Y_\varepsilon(s)^{p-1}\rangle]ds.\end{aligned}$$

Without loss of generality, suppose that $a_{2N}=\frac{1}{2N}$. Then

$$\begin{aligned}&\frac{1}{p}(\|Y_\varepsilon(t)\|_{L^p}^p-\|Y_\varepsilon(0)\|_{L^p}^p)+\int_0^t[(p-1)\langle\nabla Y_\varepsilon(s),Y_\varepsilon(s)^{p-2}\nabla Y_\varepsilon(s)\rangle+\|Y_\varepsilon(s)^{p+2N-2}\|_{L^1}]ds\\&=-\int_0^t[\|Y_\varepsilon(s)\|_{L^p}^p+\langle\Psi(Y_\varepsilon(s),\bar{Z}_\varepsilon(s)),Y_\varepsilon(s)^{p-1}\rangle]ds.\end{aligned}$$

Here $\Psi(Y_\varepsilon(s),\bar{Z}_\varepsilon(s))=\sum_{k=1}^{2N}ka_k\sum_{l=0}^{k-1}C_{k-1}^lY_\varepsilon^l(s):\bar{Z}_\varepsilon^{k-1-l}(s):-Y_\varepsilon^{2N-1}(s)$. Now it is sufficient to control each term in $\langle\Psi(Y_\varepsilon(s),\bar{Z}_\varepsilon(s)),Y_\varepsilon(s)^{p-1}\rangle$ separately. We only consider $\langle Y_\varepsilon(s)^{2N-2}\bar{Z}_\varepsilon(s),Y_\varepsilon(s)^{p-1}\rangle$. The other terms can be estimated similarly. We have

$$\langle Y_\varepsilon(s)^{2N-2}\bar{Z}_\varepsilon(s),Y_\varepsilon(s)^{p-1}\rangle=\langle Y_\varepsilon(s)^{2N+p-3},\bar{Z}_\varepsilon(s)\rangle.$$

In the following we omit ε if there's no confusion. Then Lemma 2.3 implies the following duality

$$|\langle Y(s)^{2N+p-3},\bar{Z}(s)\rangle|\lesssim\|Y(s)^{2N+p-3}\|_{\mathcal{B}_{1,1}^{-\alpha}}\|\bar{Z}(s)\|_\alpha.$$

Moreover, we have

$$\|Y(s)^{2N+p-3}\|_{B_{1,1}^{-\alpha}}\lesssim\|\Lambda^{\beta_0}Y(s)^{2N+p-3}\|_{L^{p_0}}\lesssim\|\Lambda^{\beta_0}Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_1}}\|Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{q_1}},$$

with $\Lambda=(-\Delta+I)^{1/2},\beta_0>-\alpha>0,p_0>1,\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_0}$, where we used Lemma 2.1 in the first inequality and Lemma 2.4 (ii) in the second inequality. Now we estimate each term separately: Lemmas 2.1 (iii) and 2.4 (i) imply that

$$\|\Lambda^{\beta_0}Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_1}}\lesssim\|\Lambda^{\beta_1}Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}\lesssim\|\Lambda Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}^{\beta_1}\|Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}^{1-\beta_1},$$

where $\beta_1=\beta_0+\frac{2}{p_2}-\frac{2}{p_1}$, $1<p_2<p_1<2$. For $\|\Lambda Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}$ we have

$$\|\Lambda Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}\lesssim\|Y^{\frac{p}{2}+N-\frac{5}{2}}\nabla Y\|_{L^{p_2}}\lesssim\|Y^{p-2}|\nabla Y|^2\|_{L^1}^{\frac{1}{2}}\|Y^{\frac{p_2(2N-3)}{2-p_2}}\|_{L^1}^{\frac{1}{p_2}-\frac{1}{2}},$$

where we used Hölder's inequality in the last inequality. Furthermore, we have

$$\|Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{p_2}}\lesssim\|Y^{\frac{p}{2}+N-\frac{3}{2}}\|_{L^{q_1}}\lesssim\|Y^{p+2N-2}\|_{L^1}^{\frac{\frac{p}{2}+N-\frac{3}{2}}{p+2N-2}},$$

with $2<q_1<\frac{p+2N-2}{\frac{p}{2}+N-\frac{3}{2}}$. Choose p large enough (depending only on N) such that

$$\frac{4N+p-5}{2(p+2N-2)}<\frac{1}{p_2}<\frac{1}{2(N-1)}+\frac{1}{p_1}<\frac{1}{2(N-1)}+\frac{\frac{p}{2}+N-\frac{1}{2}}{p+2N-2}.\tag{3.7}$$

The first inequality implies that $\frac{p_2(2N-3)}{2-p_2} \le p+2N-2$ and the second comes from $\beta_1 = \beta_0+\frac{2}{p_2}-\frac{2}{p_1}$ and $\beta_1-\beta_0 < \frac{1}{N-1}$ we used below and the third follows from the bound for q_1 . Thus combining the above estimates we obtain that

$$|\langle Y(s)^{2N+p-3}, \bar{Z}(s)\rangle| \lesssim \|\bar{Z}(s)\|_\alpha \|Y^{p+2N-2}\|_{L^1}^{\beta_1\frac{N-\frac{3}{2}}{p+2N-2}+(2-\beta_1)\frac{\frac{p}{2}+N-\frac{3}{2}}{p+2N-2}} \|Y^{p-2}|\nabla Y|^2\|_{L^1}^{\frac{\beta_1}{2}}.$$

By (3.7) we know that for β_0 small enough, $\beta_1(N-1)<1$, which implies that

$$\beta_1\frac{N-\frac{3}{2}}{p+2N-2}+(2-\beta_1)\frac{\frac{p}{2}+N-\frac{3}{2}}{p+2N-2}+\frac{\beta_1}{2}<1,$$

Then we have that there exists $\gamma>1$ such that

$$|\langle Y(s)^{2N+p-3}, \bar{Z}(s)\rangle| \lesssim \|\bar{Z}(s)\|_\alpha^\gamma + (\varepsilon\|Y^{p+2N-2}\|_{L^1}+\varepsilon\|Y^{p-2}|\nabla Y|^2\|_{L^1}).$$

We can do similar calculations for the other terms in $\Psi(Y(s), \bar{Z}(s))$. Then we deduce that there exist $0<\rho_1=\frac{\gamma_1(2N-1)(\beta-\alpha)+\gamma_1\rho}{2}<1, \gamma_1>1$ such that

$$\begin{aligned}&|\langle \Psi(Y(s), \bar{Z}(s)), Y(s)^{p-1}\rangle| \\ \lesssim& s^{-\rho_1}|\bar{Z}|_{\mathfrak{L}}^{\gamma_1}+\varepsilon(\|Y\|_{L^{p+2N-2}}^{p+2N-2}+\|Y^{p-2}|\nabla Y|^2\|_{L^1}).\end{aligned}$$

Here ρ_1 can be chosen less than 1 since $\beta-\alpha>0$ can be chosen small enough. Hence Gronwall's inequality yields a uniform estimate

$$\sup_{t\in[0,T]}\|Y_\varepsilon(t)\|_{L^p}^p \lesssim C_T + \|Y(0)\|_{L^p}^p.$$

Since all the constants above are independent of ε, we can extend the local solution to the unique global solution to (3.5) in $C((0,T],\mathcal{C}^\beta)$.

Moreover, consider $\bar{X}=Y_0+\bar{Z}$, where Y_0 is the unique solution to (3.5) with $y=0$. By Theorem 3.9 and the uniqueness of the solution to (3.5) we know that $\bar{X}$ has the same law as the solution X in Theorem 3.8, which combined with $\nu(\mathcal{C}^\alpha)=1$ implies that ν is an invariant measure of $\bar{X}$. □

In the following we start from the transition semigroup of $\bar{X}$: Let p_t be the transition semigroup (of sub-probability kernels) associated with $\bar{X}$. Since $\nu(\mathcal{C}^\alpha)=1$, p_t can be extended to a kernel on E by setting

$$p_t(z,dy)=\delta_z(dy),$$

for $z\in E\backslash\mathcal{C}^\alpha$, where δ_z denotes the Dirac measure in z. By Theorem 3.10 we have

$$\int p_t f\nu(dx)=\int f\nu(dx),$$

for $f\in L^2(E;\nu)$ and

$$p_t f \to^{t\to 0} f,$$

for $f\in C_b(\mathcal{C}^\alpha)$. By [MR92, Chapter II, Subsection 4a] $(p_t)_{t>0}$ uniquely determines a strongly continuous contraction semigroup $(T_t^1)_{t>0}$ of operators on $L^2(E;\nu)$. By the pathwise uniqueness of the solution to (3.5) we obtain that $p_tf(z)=P_tf(z)$ for all $f\in\mathcal{B}_b(E), t>0$ and $z\in\mathcal{C}^\alpha\backslash S_2$, which implies that p_t and hence T_t^1 is ν-symmetric. Here P_t is the transition semigroup properly associated with Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ defined in Section 3.2. Then there exists a Dirichlet form $(\mathcal{E}_1, D(\mathcal{E}_1))$ associated with T_t^1. Moreover, $(\mathcal{E}_1, D(\mathcal{E}_1))=(\mathcal{E}, D(\mathcal{E}))$, where $(\mathcal{E}, D(\mathcal{E}))$ is the Dirichlet form obtained in Section 3.2. Hence for $(\mathcal{E}_1, D(\mathcal{E}_1))$ the results in Theorem 3.8 hold.

3.4 Markov uniqueness in the restricted sense

In this subsection we prove Markov uniqueness in the restricted sense and the uniqueness of the martingale (probabilistically weak) solutions to (1.1) if the solution has ν as an invariant measure.

By [MR92, Chap. 4, Sect. 4b] it follows that there is a point separating countable $\mathbb{Q}$-vector space $D \subset \mathcal{F}C_b^\infty$ such that $D \subset D(L(\mathcal{E}))$. Let $\mathcal{E}^{\text{q.r.}}$ be the set of all quasi-regular Dirichlet forms $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ (cf. [MR92]) on $L^2(E;\nu)$ such that $D \subset D(L(\tilde{\mathcal{E}}))$ and $\tilde{\mathcal{E}} = \mathcal{E}$ on $D \times D$. Here for a Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ we denote its generator by $(L(\tilde{\mathcal{E}}), D(L(\tilde{\mathcal{E}})))$.

In the following we consider the martingale problem in the sense of [AR95] and probabilistically weak solutions to (1.1):

Definition 3.11 (i) A ν-special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (P^z))$ in the sense of [MR92, Chapter IV] with state space E is said to solve the martingale problem for $(L(\mathcal{E}), D)$ if for all $u \in D$, $u(X(t)) - u(X(0)) - \int_0^t L(\mathcal{E})u(X(s))ds$, $t \geq 0$, is an $(\mathcal{M}_t)$-martingale under P^ν.

(ii) A ν-special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (P^z))$ with state space E is called a probabilistically weak solution to (1.1) if there exists a map $W : \Omega \to C([0,\infty); E)$ such that for ν-a.e. z under P^z, W is an $\mathcal{M}_t$- cylindrical Wiener process and the sample paths of the associated process satisfy (3.3) for all $l \in H^{2+s}$, $s > 0$.

Remark If M is a probabilistically weak solution to (1.1), we can easily check that it also solves the martingale problem. Conversely, if M solves the martingale problem, then it is easy to check that

$$\beta_k(t) =: \langle e_k, X(t) - X(0)\rangle - \int_0^t \left[{}_{H^{-s-2}}\langle -\sum_{n=1}^{2N} n a_n : X(r)^{n-1} :, e_k\rangle_{H^{2+s}} + {}_{H^s}\langle \Delta e_k - e_k, X(r)\rangle_{H^{-s}} \right] dr$$

is a Brownian Motion. Here $\{e_k\}$ is the orthonormal basis defined in Section 3.2. Moreover, $W = \sum \beta_k e_k$ is a cylindrical Wiener process on $(\Omega, \mathcal{F}, P^\nu)$ and (X, W) satisfies (3.3) for $l \in H^{2+s}$, $s > 0$. That is to say, these two definitions are equivalent to each other.

To explain the uniqueness result below we also introduce the following concept:

Two strong Markov processes M and M' with state space E and transition semigroups $(p_t)_{t>0}$ and $(p'_t)_{t>0}$ are called *ν-equivalent* if there exists $S \in \mathcal{B}(E)$ such that (i) $\nu(E \backslash S) = 0$, (ii) $P^z[X(t) \in S, \forall t \geq 0] = P'^z[X'(t) \in S, \forall t \geq 0] = 1, z \in S$, (iii) $p_t f(z) = p'_t f(z)$ for all $f \in \mathcal{B}_b(E), t > 0$ and $z \in S$.

Combining Theorem 3.9 and Theorem 3.10, we obtain Markov uniqueness in the restricted sense for $(L(\mathcal{E}), D)$ (see part (iii)) and the uniqueness of martingale (probabilistically weak) solutions to (1.1) if solution has ν as an invariant measure (see part (i)):

Theorem 3.12 (i) There exists (up to ν-equivalence) exactly one probabilistically weak solution M to (1.1) satisfying $P^z(X \in C([0,\infty); E)) = 1$ for ν-a.e. and having ν as an invariant measure, i.e. for the transition semigroup $(p_t)_{t\geq 0}$, $\int p_t f d\nu = \int f d\nu$ for $f \in L^2(E;\nu)$.

(ii) There exists (up to ν-equivalence) exactly one ν-special standard process M with state space E solving the martingale problem for $(L(\mathcal{E}), D)$ and satisfying $P^z(X \in C([0,\infty); E)) = 1$ for ν-a.e. and having ν as an invariant measure.

(iii) $\sharp\mathcal{E}^{\text{q.r.}} = 1$. Moreover, there exists (up to ν-equivalence) exactly one ν-special standard process M with state space E associated with a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ solving the martingale problem for $(L(\mathcal{E}), D)$.

Proof For (i), suppose that M^1 is a probabilistically weak solution to (1.1) and let p_t^1 be the transition semigroup (of sub-probability kernels) associated with M^1. Since ν is an invariant measure and

$$p_t^1 f \to^{t\to 0} f,$$

for $f \in \mathcal{F}C_b^\infty$, by [MR92, Chapter II, Subsection 4a] $(p_t^1)_{t>0}$ uniquely determines a strongly continuous contraction semigroup $(T_t^1)_{t>0}$ of operators on $L^2(E;\nu)$. By the proof of Theorem 3.9 we know that the solution to (3.3) having ν as an invariant measure minus $\bar{Z}$ also satisfies (3.4) under P^ν. Moreover, by the pathwise uniqueness of solutions to (3.4) we obtain that $p_t^1 f(z) = P_t f(z)$ ν-a.e. for all $f \in \mathcal{B}_b(E), t > 0$, which implies that p_t^1 is associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ obtained in Section 3.2. Here P_t is the semigroup properly associated with $(\mathcal{E}, D(\mathcal{E}))$ obtained in Section 3.2. Since M^1 is a ν-special standard process and has continuous paths, by [MR92, Chapter 4, Theorem 1.15, Theorem 5.1] M^1 is properly associated with $(\mathcal{E}, D(\mathcal{E}))$. Then by [MR92, Chapter 4, Theorem 6.4] M^1 is ν-equivalent to M obtained in Section 3.2, which implies (i) easily.

(ii) follows from the first result and the above Remark.

The second result in (iii) follows from the first result and [AR95, Theorem 3.4]. We only prove the first. Since for every $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}})) \in \mathcal{E}^{\text{q.r.}}$ there exists a unique Markov process $\tilde{M}$ associated with $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ and Theorem 3.8 holds for $\tilde{M}$, by Theorems 3.9 and 3.10 we know that for the semigroup $\tilde{p}_t$ associated with $\tilde{M}$ we have $\tilde{p}_t f = P_t f$ ν-a.e. for $f \in \mathcal{B}_b(E)$, which implies that $\tilde{p}_t$ is a ν-version of the semigroup T_t associated with $(\mathcal{E}, D(\mathcal{E}))$. Then by [MR92, Chapter I] we know that $(\mathcal{E}, D(\mathcal{E})) = (\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$. Now (iii) follows. □

3.5 Stationary solution

Now we consider the stationary case. In this case, we can obtain a probabilistically strong solution to (3.3). Take two different stationary solutions X_1, X_2 to (3.3) with the same initial condition $\eta \in \mathcal{C}^\alpha$, $\alpha < 0$, $-\alpha$ small enough, having the distribution ν. We have

$$X_i(t) = e^{tA}\eta - \int_0^t e^{(t-\tau)A} : p(X_i(\tau)) : d\tau + Z(t),$$

where Z is the stochastic convolution

$$Z(t) = \int_0^t e^{(t-s)A} dW(s).$$

By a similar argument as the proof of Theorem 3.9 and using Lemma 3.1 we have that for every $p > 1$

$$E\int_0^T \| : p(X_i(\tau)) : \|_\alpha^p d\tau = T\int \| : p(\phi) : \|_\alpha^p \nu(d\phi) < \infty.$$

Then Lemma 2.2 implies that for $\alpha < 0$, $-\alpha < \beta < \alpha + 2$

$$\int_0^t e^{(t-\tau)A} : p(X_i(\tau)) : d\tau \in C([0,T];\mathcal{C}^\beta) \quad P - a.s..$$

Thus by Lemma 2.2 we conclude that

$$X_i - Z \in C((0,T];\mathcal{C}^\beta) \quad P-a.s.,$$

where $C((0,T];\mathcal{C}^\beta)$ is equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}}\|\cdot\|_\beta$. Moreover, similar arguments as in the proof of Theorem 3.9 yield that if $\alpha<0$ with $-\alpha$ small enough, $X_i - Z$ is a solution to the following equation

$$Y(t) = e^{tA}\eta + \int_0^t e^{(t-s)A}\sum_{k=1}^{2N} ka_k \sum_{l=0}^{k-1} C_{k-1}^l Y(s)^l : Z(s)^{k-1-l} : ds. \tag{3.8}$$

Here the Wick powers of Z are defined as in Lemma 3.4.

Now by similar calculations as in (3.6) we obtain local uniqueness of the solution to (3.8), which implies that

$$X_1 - Z = X_2 - Z \text{ on } [0,T] \quad P-a.s..$$

Then the pathwise uniqueness holds for the stationary solution to (3.3). Now by the existence of the stationary martingale solution (cf. [MR99]) and the Yamada-Watanabe Theorem in [Kur07] we obtain:

Theorem 3.13 For any initial condition $X(0)\in\mathcal{C}^\alpha$ with distribution ν and $\alpha<0$, $-\alpha$ small enough, there exists a unique probabilistically strong solution X to (3.3) such that X is a stationary process, i.e. for every probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$ with an $\mathcal{F}_t$-Wiener process W, there exists an $\mathcal{F}_t$-adapted stationary process $X:[0,T]\times\Omega\to E$ such that for $P-a.e.$ $\omega\in\Omega$ X satisfies (3.3). Moreover, for $0<\beta<\alpha+2$

$$X - Z \in C((0,T];\mathcal{C}^\beta) \quad P-a.s..$$

4 Infinite volume case

In this section, we analyze the stochastic quantization equations in infinite volume. The proof is similar as for the finite volume case. However, the invariant measure ν_0 defined below is more singular and the analysis becomes considerably harder. For simplicity we choose $N=2$. The general case can be proved similarly. Recall that $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered Schwartz distributions on $\mathbb{R}^2$ and $\mathcal{S}(\mathbb{R}^2)$ the associated test function space equipped with the usual topology. In this section we use weighted Besov space $\hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}$ and we fix $\sigma>2$.

4.1 Wick powers

Let μ_0 be the mean zero Gaussian measure on $(\mathcal{S}'(\mathbb{R}^2),\mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ with covariance

$$\int {}_{\mathcal{S}}\langle k_1,z\rangle_{\mathcal{S}'}\,{}_{\mathcal{S}}\langle k_2,z\rangle_{\mathcal{S}'}\mu_0(dz) = \int\int \frac{1}{2}G(x-y)k_1(x)k_2(y)dxdy =: \langle k_1,k_2\rangle_{H_1},$$

where G denotes the Green function of the operator $-A$ on $\mathbb{R}^2$.

Wick powers on $L^2(\mathcal{S}'(\mathbb{R}^2),\mu_0)$

Let H_1 be the real Hilbert space obtained by completing $S(\mathbb{R}^2)$ w.r.t, the norm associated with the inner product $\langle\cdot,\cdot\rangle_{H_1}$. Now for $n\in\mathbb{N}$, let $\mathcal{S}_{-n}$ denote the Hilbert subspace of $\mathcal{S}'(\mathbb{R}^2)$ which is the dual of $\mathcal{S}_n$ defined as the completion of $\mathcal{S}(\mathbb{R}^2)$ w.r.t the norm

$$\|k\|_{\mathcal{S}_n} := [\sum_{|m|\le n}\int_{\mathbb{R}^2}(1+|x|^2)^n|(\frac{\partial^{m_1}}{\partial x_1^{m_1}},\frac{\partial^{m_2}}{\partial x_2^{m_2}})k(x)|^2dx]^{1/2}.$$

For $h\in H_1$ we define $X_h\in L^2(\mathcal{S}'(\mathbb{R}^2),\mu_0)$ by $X_h := \lim_{n\to\infty}{}_{\mathcal{S}}\langle k_n,\cdot\rangle_{\mathcal{S}'}$ in $L^2(\mathcal{S}'(\mathbb{R}^2),\mu_0)$ where k_n is any sequence in $\mathcal{S}(\mathbb{R}^2)$ such that $k_n\to h$ in H_1. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}'(\mathbb{R}^2),\mu_0)=\bigoplus_{n\ge 0}\mathcal{H}_n.$$

For $h\in L^2(\mathbb{R}^2,dx)$ and $n\in\mathbb{N}$, define $:z^n:(h)$ to be the unique element in $\mathcal{H}_n$ such that

$$\int :z^n:(h):\prod_{j=1}^n X_{k_j}:_n d\mu_0 = n!\int_{\mathbb{R}^2}\prod_{j=1}^n(\int_{\mathbb{R}^2}G(x-y_j)k_j(y_j)dy_j)h(x)dx$$

where $k_1,...,k_n\in\mathcal{S}(\mathbb{R}^2)$ and $: :_n$ means orthogonal projection onto $\mathcal{H}_n$ (see [S74, V.1] for existence of $:z^n:(h)$).

From now on we define for $h\in L^2(\mathbb{R}^2,dx)$

$$:P(z):(h) := \frac{1}{2}:z^4:(h).$$

We have that $\exp(-:P(z):(h))\in L^p(\mathcal{S}'(\mathbb{R}^2),\mu_0)$ for all $p\in[1,\infty)$ if $h\ge 0$ (cf. [AR91, Section 7]), hence the following probability measures (called space-time cut-off quantum fields) are well-defined for $\Lambda\in\mathcal{B}(\mathbb{R}^2)$, Λ bounded,

$$\nu_\Lambda := \frac{\exp(-:P(z):(1_\Lambda))}{\int\exp(-:P(z):(1_\Lambda))d\mu_0}\mu_0.$$

It has been proven that the weak limit

$$\lim_{\Lambda\to\mathbb{R}^2}\nu_\Lambda =: \nu_0$$

exists as a probability measure on $(\mathcal{S}'(\mathbb{R}^2),\mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ having moments of all orders (see [GlJ86] and also [AR91, Section 7]). In particular, it follows by [AR89, Proposition 3.7] that $\nu_0(\mathcal{S}_{-n})=1$ for $n\in\mathbb{N}$ large enough. We emphasize that ν_0 is not absolutely continuous with respect to μ_0 (c.f.[AR91]). By [AR91, Section 7], if n is large enough, there exists a $\mathcal{B}(\mathcal{S}_{-n})/\mathcal{B}(\mathcal{S}_{-n})$-measurable map $:\phi^3::\mathcal{S}_{-n}\to\mathcal{S}_{-n}$ such that ${}_{\mathcal{S}_{-n}}\langle:\phi^3:,l\rangle_{\mathcal{S}_n} =: \phi^3:(l)$ ν_0-a.e. for each l with compact support and $\int\|:\phi^3:\|^2_{\mathcal{S}_{-n}}d\nu_0<\infty$.

For $\phi\in\mathcal{S}'(\mathbb{R}^2)$ define

$$\phi_\varepsilon := \rho_\varepsilon * \phi$$

with ρ_ε an approximate delta function,

$$\rho_\varepsilon(x)=\varepsilon^{-2}\rho(\frac{x}{\varepsilon})\in\mathcal{D}, \int\rho=1,$$

and for every $n \in \mathbb{N}$ we set

$$: \phi_\varepsilon^n :_C = c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}\phi_\varepsilon),$$

with $c_\varepsilon = \int \phi_\varepsilon^2 \mu_0(d\phi) = \frac{1}{2}\int\int G(x-y)\rho_\varepsilon(y)dy\rho_\varepsilon(x)dx = \|K_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{R}^2)}$. Here and in the following $K(t, x-y)$ is the heat kernel associated with A on $\mathbb{R}^2$ and $K_\varepsilon = K * \rho_\varepsilon$, where $*$ means convolution in space. By [GlJ86] we know that for every smooth funcion g with compact support, $\langle : \phi_\varepsilon^3 :_C, g\rangle$ converges to $\langle : \phi^3 :, g\rangle$ in $L^2(\mathcal{S}'(\mathbb{R}^2), \nu_0)$.

Now we give estimates on the measure ν_0 for later use.

Lemma 4.1 Let $\alpha_0 < -\frac{3}{2}$, $\sigma > 2$, $p > 1, p \in \mathbb{N}$, then

$$\int \| : \phi^3 : \|^{2p}_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{2p,2p}} \nu_0(d\phi) < \infty.$$

Proof By (2.3) we have

$$\begin{aligned}
&\int \| : \phi^3 : \|^{2p}_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{2p,2p}} \nu_0(d\phi) \\
\lesssim &\int \Big(\sum_{n=0}^\infty 2^{2n(\alpha_0-1/p+1)p} \sum_{\psi\in\Psi_\star}\sum_{x\in\Lambda_n} |\langle : \phi^3 :, \psi_x^n\rangle|^{2p} w(x)\Big)\nu_0(d\phi) \\
\lesssim &\sum_{n=0}^\infty 2^{2n(\alpha_0-1/p+1)p} \sum_{\psi\in\Psi_\star}\sum_{x\in\Lambda_n} \int \langle : \phi^3 :, \psi_x^n\rangle^{2p} w(x)\nu_0(d\phi) \\
\leq &C(p)\sum_{n=0}^\infty 2^{2n(\alpha_0-1/p+1)p} \sum_{\psi\in\Psi_\star}\sum_{x\in\Lambda_n} \|\psi_x^n\|^{2p}_{L^4} w(x).
\end{aligned}$$

Here we used [GlJ86, Corollary 12.2.4] in the last inequality. Recall that the L^4-norm of ψ_x^n is of order $2^{n/2}$ and that Ψ is a finite set. Thus we obtain that the last term is of order

$$\sum_{n=0}^\infty 2^{2n(\alpha_0+\frac{3}{2})p} \int_{\mathbb{R}^2} w(x)dx.$$

Hence the sums over n and x converge for $\alpha_0 < -\frac{3}{2}$. □

Wick powers on a fixed probability space

Now we fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ and on it a cylindrical Wiener process W in $L^2(\mathbb{R}^2)$. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n\geq 0} \mathcal{H}'_n.$$

Now for $Z(t) = \int_0^t e^{(t-s)A}dW(s)$, we can also define Wick powers with respect to different covariances by approximations: Let $Z_\varepsilon(t,y) = \int\int_0^t \langle K_\varepsilon(t-s, y-x), dW(s)\rangle = \rho_\varepsilon * Z$. Here $\langle\cdot,\cdot\rangle$ means inner product in $L^2(\mathbb{R}^2)$. Let $C_t := \frac{1}{2}(-A)^{-1}(I - e^{2tA})$. For every $n \in \mathbb{N}$ we set

$$: Z_\varepsilon^n(t) :_{C_t} = (c_{\varepsilon,t})^{\frac{n}{2}} P_n((c_{\varepsilon,t})^{-\frac{1}{2}} Z_\varepsilon(t)) \in \mathcal{H}'_n,$$

where $P_n, n = 0, 1, ...,$ are the Hermite polynomials and $c_{\varepsilon,t} = \|K_\varepsilon 1_{[0,t]}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^2)}$.

By similar arguments as in the proof of Lemma 3.3 and using (2.3) we have:

Lemma 4.2 For every $\alpha < 0$ and every $p > 1$, $n = 2, 3$, $: Z_\varepsilon^n :_{C_t}$ converges in $L^p(\Omega, C([0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{2p,2p}))$. This limit is called Wick power of $Z(t)$ with respect to the covariance C_t and denoted by $: Z^n(t) :_{C_t}$.

By this lemma and a similar argument as in the proof of Lemma 3.4 we can also define $: Z^n(t) :_C$.

Lemma 4.3 For every $\alpha < 0$ and every $p > 1$, $n = 2, 3$, $: Z_\varepsilon^n :_C = (c_\varepsilon)^{\frac{n}{2}} P_n((c_\varepsilon)^{-\frac{1}{2}} Z_\varepsilon)$ converges in $L^p(\Omega, C((0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}))$. Here $C((0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{p,p})$ is equipped with the norm $\sup_{t\in[0,T]} t^\rho \|\cdot\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}}$ for $\rho > 0$. This limit is called Wick power of $Z(t)$ with respect to the covariance C and denoted by $: Z^n(t) :_C$. Moreover, for $t > 0$

$$: Z^n(t) :_C = \sum_{l=0}^{[n/2]} c_t^l \frac{n!}{(n-2l)!l!2^l} : Z^{n-2l}(t) :_{C_t},$$

where $c_t := \lim_{\varepsilon\to 0}(c_{\varepsilon,t} - c_\varepsilon) = -\int_t^\infty \int K(r,x)^2 dx dr$, $c_\varepsilon = \|K_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{R}^2)}$.

Now we combine the initial value part with the Wick power by using (3.1). In the following we fix $p_0 > 3$. For $z \in \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty}$ with $\alpha < 0, \sigma > 2$ we set $V(t) = e^{At} z$, $V_\varepsilon = \rho_\varepsilon * V$, and

$$\bar{Z}(t) = Z(t) + V(t), \quad \bar{Z}_\varepsilon(t) = Z_\varepsilon(t) + V_\varepsilon(t),$$

$$: \bar{Z}^2(t) :_C =: Z(t)^2 :_C + V(t)^2 + 2Z(t)V(t),$$

$$: \bar{Z}^3(t) :_C =: Z(t)^3 :_C + V(t)^3 + 3Z(t)V^2(t) + 3 : Z(t)^2 :_C V(t).$$

$: \bar{Z}_\varepsilon^n :_C$ is defined as $: \bar{Z}^n :_C$ with Z, V replaced by $Z_\varepsilon, V_\varepsilon$, respectively. By Lemma 2.5 we know that $V \in C([0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty})$ and $V \in C((0,T]; \hat{\mathcal{B}}^{\beta,\sigma}_{3p_0,\infty})$ for $\beta > \alpha$ equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}} \|\cdot\|_{\hat{\mathcal{B}}^{\beta,\sigma}_{3p_0,\infty}}$. Moreover,

$$\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}} \|V(t)\|_{\hat{\mathcal{B}}^{\beta,\sigma}_{3p_0,\infty}} \lesssim \|z\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty}}.$$

By Lemma 2.6 we obtain that for $\alpha < 0$, $n = 1, 2, 3$, $\sigma > 2$, $p > 1$, $: \bar{Z}^n :_C \in L^p(\Omega, C((0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{p_0,\infty}))$ and that $: \bar{Z}_\varepsilon^n :_C$ converges to $: \bar{Z}^n :_C$ in $L^p(\Omega, C((0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{p_0,\infty}))$. Here $C((0,T]; \hat{\mathcal{B}}^{\alpha,\sigma}_{p_0,\infty})$ is equipped with the norm $\sup_{t\in[0,T]} t^{\frac{(\beta-\alpha)n+\rho}{2}} \|\cdot\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p_0,\infty}}$ with $\beta > -\alpha > 0, \rho > 0$.

Relations between two different Wick powers

Lemma 4.4 Let ϕ be a measurable map from $(\Omega, \mathcal{F}, P)$ to $C([0,T], \mathcal{S}^{-n})$ for some $n > 0$ large enough, $P \circ \phi(t)^{-1} = \nu_0$ for every $t \in [0,T]$ and let $\bar{Z}$ be defined as above. Assume in addition that $y = \phi - \bar{Z} \in C([0,T]; \hat{\mathcal{B}}^{\beta,\sigma}_{p_0,\infty})$ P-a.s. for some β with $\beta > -\alpha > 0$. Then for every $t > 0$

$$: \phi^3(t) := \sum_{k=0}^{3} C_3^k y^{3-k}(t) : \bar{Z}^k(t) :_C \quad P-a.s..$$

Proof By [GlJ86, Theorem 12.2.1] it follows that for every compactly supported smooth function g and $t \geq 0$

$$\langle : \phi_\varepsilon(t)^3 :_C, g\rangle \to \langle : \phi(t)^3 :, g\rangle \quad \text{in } L^2(\Omega, P).$$

Since $y_\varepsilon = \phi_\varepsilon - \bar{Z}_\varepsilon = \rho_\varepsilon * y$, it is obvious that $y_\varepsilon(t) \to y(t)$ in $\hat{\mathcal{B}}^{\beta-\kappa,\sigma}_{p_0,\infty}$ P-a.s. for $\kappa > 0, \beta - \kappa > -\alpha > 0$, which combined with Lemmas 2.6 and 4.3 implies that for $k \in \mathbb{N}$, $k \leq 3$

$$\langle y_\varepsilon^{3-k}(t) : \bar{Z}_\varepsilon^k :_C, g\rangle \to \langle y^{3-k}(t) : \bar{Z}^k :_C, g\rangle \quad \text{in probability} .$$

Moreover, by (3.1) and similar arguments as the proof of Lemma 3.6 we have

$$: \phi_\varepsilon(t)^3 :_C = \sum_{k=0}^{3} C_3^k : \bar{Z}_\varepsilon^k(t) :_C y_\varepsilon^{3-k}(t),$$

which implies

$$\langle : \phi(t)^3 :_C, g\rangle = \sum_{k=0}^{3} C_3^k \langle y^{3-k}(t) : \bar{Z}^k(t) :_C, g\rangle \quad P-a.s..$$

by letting $\varepsilon \to 0$. Now the results follow because the test function space is separable. $\square$

In the following, we only use Wick powers $: \cdot :_C$ and we write $: \cdot :$ for simplicity.

4.2 Relations between the two solutions

Solutions given by Dirichlet forms

Now choose $H = L^2(\mathbb{R}^2)$ and $E = \mathcal{S}_{-n}$ for some n large enough, which can be chosen from Theorem 4.5 below. We define the Dirichlet form as in [AR91]. Define

$$\mathcal{F}C_b^\infty = \{u : u(z) = f({}_{E^*}\langle l_1, z\rangle_E, {}_{E^*}\langle l_2, z\rangle_E, ..., {}_{E^*}\langle l_m, z\rangle_E), z \in E, l_1, l_2, ..., l_m \in E^*, m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m)\},$$

where E^* denotes the dual space of E. Define for $u \in \mathcal{F}C_b^\infty$ and $l \in H$,

$$\frac{\partial u}{\partial l}(z) := \frac{d}{ds}u(z + sl)|_{s=0}, z \in E,$$

that is, by the chain rule,

$$\frac{\partial u}{\partial l}(z) = \sum_{j=1}^{m} \partial_j f({}_{E^*}\langle l_1, z\rangle_E, {}_{E^*}\langle l_2, z\rangle_E, ..., {}_{E^*}\langle l_m, z\rangle_E)\langle l_j, l\rangle_H.$$

Let Du denote the H-derivative of $u \in \mathcal{F}C_b^\infty$, i.e. the map from E to H such that

$$\langle Du(z), l\rangle = \frac{\partial u}{\partial l}(z) \text{ for all } l \in H, z \in E.$$

By [AR91] we easily deduce that the form

$$\mathcal{E}(u, v) := \frac{1}{2}\int_E \langle Du, Dv\rangle_H d\nu_0; u, v \in \mathcal{F}C_b^\infty$$

is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;\nu_0)$ in the sense of [MR92]. By [AR91, Theorem 3.6] we know that there exists a (Markov) diffusion process $M = (\Omega, \mathcal{F}, (X(t))_{t\geq 0}, (P^z)_{z\in E})$ on E properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

By [AR91, Theorem 7.11] we have the following:

Theorem 4.5 For each l smooth and compactly supported, the partial log derivative of ν_0 is given by

$$\beta_l(z) = -2\langle : z^3 :, l\rangle + 2_{\mathcal{S}_n}\langle \Delta l - l, z\rangle_{\mathcal{S}_{-n}}.$$

If n is large enough there exists a $\mathcal{B}(\mathcal{S}_{-n})/\mathcal{B}(\mathcal{S}_{-n})$-measurable map $\beta : \mathcal{S}_{-n} \to \mathcal{S}_{-n}$ such that ${}_{\mathcal{S}_{-n}}\langle \beta, l\rangle_{\mathcal{S}_n} = \beta_l$ ν_0-a.e. for each l with compact support and $\int \|\beta\|^2_{\mathcal{S}_{-n}} d\nu_0 < \infty$.

Moreover, by [AR91, Theorem 6.1] we obtain the following results:

Theorem 4.6 There exist a map $W : \Omega \to C([0,\infty); E)$ and a *properly $\mathcal{E}$-exceptional set* $S \subset E$, i.e. $\nu_0(S) = 0$ and $P^z[X(t) \in E\setminus S, \forall t \geq 0] = 1$ for $z \in E\setminus S$, such that $\forall z \in E\setminus S$ under P^z, W is an $\mathcal{M}_t$- cylindrical Wiener process and the sample paths of the associated process $M = (\Omega, \mathcal{F}, (X(t))_{t\geq 0}, (P^z)_{z\in E})$ on E satisfy the following: for $l \in \mathcal{S}_n$ with compact support

$$\begin{aligned} {}_{E^*}\langle l, X(t) - X(0)\rangle_E = \int_0^t \langle l, dW(r)\rangle + \int_0^t (-{}_{\mathcal{S}_{-n}}\langle : X(r)^3 :, l\rangle_{\mathcal{S}_n} \\ + {}_{\mathcal{S}_n}\langle \Delta l - l, X(r)\rangle_{\mathcal{S}_{-n}}) dr \quad \forall t \geq 0 \qquad P^z\text{-a.s.}. \end{aligned} \tag{4.1}$$

Moreover, ν_0 is an invariant measure for X in the sense that $\int p_t u d\nu = \int u d\nu$ for $u \in L^2(E;\nu) \cap \mathcal{B}_b(E)$, where p_t is the transition semigroup for M.

Relations between the two solutions

In the following we discuss the relations between M constructed above and the shifted equation. For W constructed in Theorem 4.6, define $\bar{Z}(t) := \int_0^t e^{(t-s)A} dW(s) + e^{tA}X(0)$. First we prove the following property for ν_0 by using Theorem 4.6. .

Theorem 4.7 For every $\alpha < 0, \sigma > 2, p \geq 1$ we have

$$\nu_0(\hat{\mathcal{B}}^{\alpha,\sigma}_{p,\infty}) = 1.$$

Proof By using [GlJ86, Corollary 12.2.4], we have that for $\alpha_1 < -1/2, p > 1$

$$\nu_0(\hat{\mathcal{B}}^{\alpha_1,\sigma}_{2p,2p}) = 1.$$

Indeed, by (2.3), [GlJ86, Corollary 12.2.4] and similar calculation as the proof of Lemma 4.1 we have

$$\int \|\phi\|^{2p}_{\hat{\mathcal{B}}^{\alpha_1,\sigma}_{2p,2p}} \nu_0(d\phi) \leq C(p) \sum_{n=0}^{\infty} 2^{2n(\alpha_1 - 1/p+1)p} \sum_{\psi\in\Psi_\star} \sum_{x\in\Lambda_n} \|\psi^n_x\|^{2p}_{L^{4/3}} w(x).$$

Recall that the $L^{4/3}$-norm of ψ^n_x is of order $2^{-n/2}$ and that Ψ is a finite set. Thus we obtain that the sums over n and x converge for $\alpha_1 < -\frac{1}{2}$.

Then by Theorem 4.6 we have that for $z \in \hat{\mathcal{B}}^{\alpha_1,\sigma}_{p,\infty} \cap (E \setminus S)$ with $p > 1$ under P^z

$$X(t) = -\int_0^t e^{(t-\tau)A} : X(\tau)^3 : d\tau + \bar{Z}(t).$$

By a similar calculation as in the proof of Lemma 3.3 and using (2.3), it follows that for every $\alpha < 0, p > 1, t > 0$,

$$\int_0^t e^{(t-s)A} dW(s) \in \hat{\mathcal{B}}^{\alpha,\sigma}_{p,\infty} \quad P^{\nu_0} - a.s..$$

Moreover, by Lemma 2.5 for $t > 0$

$$\|e^{tA} z\|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,\infty}} \lesssim t^{-\frac{(\alpha-\alpha_1)}{2}} \|z\|_{\hat{\mathcal{B}}^{\alpha_1,\sigma}_{p,\infty}},$$

which implies that for every $t > 0, \alpha < 0$,

$$\bar{Z}(t) \in \hat{\mathcal{B}}^{\alpha,\sigma}_{p,\infty} \quad P^{\nu_0} - a.s..$$

Since ν_0 is an invariant measure for M, by Lemma 4.1 we conclude that for every $\alpha_0 < -\frac{3}{2}, T > 0$, $p > 1$,

$$E^{\nu_0} \int_0^T \| : X(\tau)^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} d\tau = T \int \| : \phi^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} \nu_0(d\phi) < \infty.$$

Then by Lemma 2.5 we have that for $p > \frac{2}{2-(\alpha-\alpha_0)}$

$$\begin{aligned} &E^{\nu_0} \sup_{t\in[0,T]} \| \int_0^t e^{(t-\tau)A} : X(\tau)^3 : d\tau \|_{\hat{\mathcal{B}}^{\alpha,\sigma}_{p,p}} \lesssim E^{\nu_0} \sup_{t\in[0,T]} \int_0^t (t-\tau)^{-\frac{\alpha-\alpha_0}{2}} \| : X(\tau)^3 : \|_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} d\tau \\ \lesssim &\Big(E^{\nu_0} \int_0^T \| : X(\tau)^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} d\tau \Big)^{\frac{1}{p}} < \infty \end{aligned}.$$

Here in the last inequality we used Hölder's inequality. Thus, by Lemma 2.6 for every $t > 0$ $X(t) \in \hat{\mathcal{B}}^{\alpha,\sigma}_{p,\infty}$ P^{ν_0}-a.s., which implies the result since $P^{\nu_0} \circ X(t)^{-1} = \nu_0$. $\square$

Now we prove that $X - \bar{Z}$ satisfies the shifted equation.

Theorem 4.8 Let $\alpha \in (-\frac{1}{3}, 0)$ and $p_0 > 3$. There exists a properly $\mathcal{E}$-exceptional set $S_2 \supset S$ in the sense of Theorem 4.6 such that for $z \in \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty} \cap (E \setminus S_2)$, $Y := X - \bar{Z} \in C([0,\infty); \hat{\mathcal{B}}^{\beta,\sigma}_{p,p})$ P^z-a.s. for every $\beta \in (0, \frac{1}{2}), p > 1$, is a solution to the following equation:

$$Y(t) = -\int_0^t e^{(t-s)A} \sum_{l=0}^3 C_3^l Y(s)^l : \bar{Z}^{3-l}(s) : ds. \tag{4.2}$$

Moreover,

$$P^z[X(t) \in \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty} \cap (E \setminus S_2), \forall t \geq 0] = 1 \text{ for } z \in \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty} \cap (E \setminus S_2). \tag{4.3}$$

Proof By Theorem 4.6 we have that for $z \in E \setminus S$

$$X(t) = -\int_0^t e^{(t-\tau)A} : X(\tau)^3 : d\tau + \bar{Z}(t) \quad P^z - a.s..$$

Since ν_0 is an invariant measure for X, by Lemma 4.1 we conclude that for every $\alpha_0 < -\frac{3}{2}$, $p > 1$,

$$\int E^z \int_0^T \| : X(\tau)^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} d\tau \nu_0(dz) = T \int \| : \phi^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} \nu_0(d\phi) < \infty,$$

which implies that there exists a properly $\mathcal{E}$-exceptional set $S_1 \supset S$ such that for $z \in E \setminus S_1$

$$E^z \int_0^T \| : X(\tau)^3 : \|^p_{\hat{\mathcal{B}}^{\alpha_0,\sigma}_{p,p}} d\tau < \infty.$$

By Lemma 2.5 we know that for $0 < \beta < \alpha_0 + 2$ and for $z \in E \backslash S_1$, $p > 1$,

$$\int_0^\cdot e^{(\cdot-\tau)A} : X(\tau)^3 : d\tau \in C([0, \infty); \hat{\mathcal{B}}^{\beta,\sigma}_{p,p}) \quad P^z - a.s..$$

Then we conclude that for every $z \in E \setminus S_1$, $p > 1$

$$X - \bar{Z} \in C([0, \infty); \hat{\mathcal{B}}^{\beta,\sigma}_{p,p}) \quad P^z - a.s..$$

By Theorem 4.7 and the fact that $\int_0^\cdot e^{(\cdot-s)A} dW(s) \in C([0, \infty); \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty})$ P^{ν_0}-a.s. for $\alpha \in (-\frac{1}{3}, 0)$, we obtain that

$$\bar{Z} \in C([0, \infty); \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty}) \quad P^{\nu_0} - a.s..$$

Thus, Lemma 4.4 and similar arguments as in the proof of Theorem 3.9 imply that

$$P^{\nu_0}[X \in C([0, \infty), \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty}), X - \bar{Z} \in C([0, \infty), \hat{\mathcal{B}}^{\beta,\sigma}_{p,p}),$$
$$\int_0^t : X(s)^3 : ds = \int_0^t \sum_{l=0}^3 C_3^l (X(s) - \bar{Z}(s))^l : \bar{Z}(s)^{3-l} : ds, \forall t \geq 0] = 1,$$

which combined with similar arguments as in the proof of Theorem 3.9 implies that there exists a properly $\mathcal{E}$-exceptional set $S_2 \supset S$ such that for $z \in \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty} \cap (E \setminus S_2)$

$$P^z[X \in C([0, \infty), \hat{\mathcal{B}}^{\alpha,\sigma}_{3p_0,\infty}), X - \bar{Z} \in C([0, \infty), \hat{\mathcal{B}}^{\beta,\sigma}_{p,p}),$$
$$\int_0^t : X(s)^3 : ds = \int_0^t \sum_{l=0}^3 C_3^l (X(s) - \bar{Z}(s))^l : \bar{Z}(s)^{3-l} : ds, \forall t \geq 0] = 1.$$

Now we can conclude the first result. (4.3) follows from the above equality and S_2 is a properly $\mathcal{E}$-exceptional set. □

Now we deduce the uniqueness of the solution to (4.2) and that ν_0 is an invariant measure of the solution $\bar{X} = Y_0 + \bar{Z}$, where Y_0 is the unique solution to (4.2).

Theorem 4.9 For $0 < \beta < \frac{1}{2}$, $\sigma > 2$, p sufficiently large, there exists a unique solution to (4.2) in $C([0, T]; \hat{\mathcal{B}}^{\beta,\sigma}_{p,\infty})$.

Moreover, ν_0 is an invariant measure of the solution $\bar{X} = Y_0 + \bar{Z}$, where Y_0 is the unique solution to (4.2).

Proof The first result follows from [MW15, Theorem 9.5] and the second follows from Theorem 4.7 and similar arguments as in the proof of Theorem 3.10. □

Similarly as in Section 3.3 we start from the transition semigroup of $\bar{X}$ and can prove that the Dirichlet form associated with this transition semigroup is $(\mathcal{E}, D(\mathcal{E}))$ defined in Section 4.2.

4.3 Markov uniqueness in the restricted sense

All the definitions introduced in Section 3.4 can be transferred here. Combining Theorem 4.8 and Theorem 4.9, we obtain uniqueness of martingale problem for $(L(\mathcal{E}), D)$ in the infinite volume case and the uniqueness of probabilistically weak solutions to (1.1) if solution has ν_0 as an invariant measure:

Theorem 4.10 (i) There exists (up to ν_0-equivalence) exactly one ν_0-special standard process M with state space E which satisfies (4.1) P^z-a.s. and $P^z(X \in C([0,\infty); E)) = 1$ for ν_0-a.e. $z \in E$ and has ν_0 as an invariant measure, i.e. for the transition semigroup $(p_t)_{t\geq 0}$, $\int p_t f d\nu_0 = \int f d\nu_0$ for $f \in L^2(E;\nu_0)$.

(ii) There exists (up to ν_0-equivalence) exactly one ν_0-special standard process M with state space E solving the martingale problem for $(L(\mathcal{E}), D)$ and satisfying $P^z(X \in C([0,\infty); E)) = 1$ for ν_0-a.e. and having ν_0 as an invariant measure.

(iii) $\sharp\mathcal{E}^{\text{q.r.}} = 1$. Moreover, there exists (up to ν_0-equivalence) exactly one ν-special standard process M with state space E associated with a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ solving the martingale problem for $(L(\mathcal{E}), D)$.

Proof It follows essentially from the same argument as the proof of Theorem 3.12 and (4.3). □

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