

Traveling waves and long-time behavior in a doubly nonlocal Fisher–KPP equation

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Abstract

We consider a Fisher–KPP-type equation, where both diffusion and nonlinear part are nonlocal, with anisotropic probability kernels. Under minimal conditions on the coefficients, we prove existence, uniqueness, and uniform space-time boundedness of the positive solution. We investigate existence, uniqueness, and asymptotic behavior of monotone traveling waves for the equation. We also describe the existence and main properties of the front of propagation.

Keywords: nonlocal diffusion, Fisher–KPP equation, traveling waves, nonlocal nonlinearity, long-time behavior, anisotropic kernels, integral equation

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1 Introduction

We will deal with the following nonlinear nonlocal evolution equation

$$\frac{\partial u}{\partial t}(x, t) = \varkappa^+(a^+ * u)(x, t) - mu(x, t) - \varkappa^-u(x, t)(a^- * u)(x, t) \quad (1.1)$$

with a bounded initial condition $u(x, 0) = u_0(x)$, $x \in \mathbb{R}^d$, $d \geq 1$. Constants m, \varkappa^\pm are assumed to be positive, and $(a^\pm * u)(x, t)$ mean the convolutions (in x) between u and nonnegative integrable probability kernels $a^\pm(x)$; namely,

$$(a^\pm * u)(x, t) = \int_{\mathbb{R}^d} a^\pm(x - y)u(y, t)dy, \quad \int_{\mathbb{R}^d} a^\pm(x) dx = 1.$$

The equation (1.1) first appeared, probably, in [27] (a ‘crabgrass model’), for $\varkappa^+a^+ = \varkappa^-a^-$, and later in [8] (a model of spatial ecology), for different kernels. The meaning of $u(x, t)$ is the (approximate) value of the local density of a system in a point $x \in \mathbb{R}^d$ at a moment of time $t \geq 0$. A short review for the history of derivation of (1.1) see in Section 6 below. This equation was considered as a spatial (nonhomogeneous) version of the classical logistic (Verhulst) equation

$$\frac{du}{dt} = (\varkappa^+ - m)u(t) - \varkappa^-(u(t))^2, \quad (1.2)$$

corresponding to $u(x, t) = u(t)$, $x \in \mathbb{R}^d$. Of course, in the original logistic model one needs to assume that $\varkappa^+ > m$; then (1.2) has two stationary solutions: unstable $u = 0$ and stable $u = \frac{\varkappa^+ - m}{\varkappa^-}$. For $\varkappa^+ \leq m$, (1.2) has the unique stationary stable solution $u = 0$.

The equation (1.1) can be rewritten as follows:

$$\frac{\partial u}{\partial t}(x, t) = (L_{a^+}u)(x, t) + F(u, a^- * u)(x, t), \quad (1.3)$$

where, for a bounded function v on \mathbb{R}^d , the operator

$$(L_{a^+}v)(x) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x - y)[v(y) - v(x)] dy \quad (1.4)$$

describes the so-called nonlocal diffusion, see e.g. [4] and references below, and F is a bilinear mapping on bounded functions, given by

$$F(v_1, v_2)(x) = \varkappa^-v_1(x)(\theta - v_2(x)), \quad \theta = \frac{\varkappa^+ - m}{\varkappa^-}. \quad (1.5)$$

For the known results about (1.3), one can refer to [33, 34, 41] in the general case; to [62, 76, 81] in the case $\theta > 0$, i.e. $\varkappa^+ > m$, see also details below; and to [72, 73] for $\varkappa^+ = m$.

The equation (1.3) may be addressed to a doubly nonlocal Fisher–KPP equation. Recall that the classical Fisher–KPP (Kolmogorov–Petrovski–Piskunov) equation in \mathbb{R}^d goes back to [40, 49] and has the form

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + f(v(x, t)), \quad (1.6)$$

see the seminal paper [6]. Here Δ is the Laplace operator on \mathbb{R}^d , and f is a nonlinear monostable function on \mathbb{R} : namely, let $\theta > 0$, cf. (1.5), then we assume that $f(0) = f(\theta) = 0$, $f'(0) > 0$, $f'(\theta) < 0$; for example,

$$f(s) = \varkappa^- s(\theta - s), \quad s \geq 0. \quad (1.7)$$

The Fisher–KPP equation may be informally obtained from (1.3) under two scaling procedures. Namely, let

$$\int_{\mathbb{R}^d} a^+(y)(|y_i| + |y_i|^2) dy < \infty, \quad \int_{\mathbb{R}^d} a^+(y)y_i dy = 0, \quad 1 \leq i \leq d,$$

and consider the scaling (for small $\varepsilon > 0$)

$$a^+(x) \mapsto a_\varepsilon^+(x) := \varepsilon^{-d} a^+(\varepsilon^{-1}x), \quad x \in \mathbb{R}^d, \quad \varkappa^+ \mapsto \varepsilon^{-2}. \quad (1.8)$$

Then, we will get

$$\begin{aligned} (L_{a_\varepsilon^+} v)(x) &= \varepsilon^{-2} \int_{\mathbb{R}^d} a^+(y)[v(\varepsilon y + x) - v(x)] dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d} a^+(y)(v''(x)y, y)_{\mathbb{R}^d} dy + o(\varepsilon) = \alpha \Delta v(x) + o(\varepsilon), \end{aligned}$$

where $(\cdot, \cdot)_{\mathbb{R}^d}$ is the scalar product on \mathbb{R}^d , and $\alpha = \frac{1}{2} \int_{\mathbb{R}^d} a^+(y)y_1^2 dy$.

As a result, we will get the following nonlocal Fisher–KPP equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u + F(u, a^- * u). \quad (1.9)$$

For the theory of such equations with different mappings F ‘similar’ to (1.5), see e.g. [1, 3, 5, 7, 21, 31, 32, 43–45, 58] and some details below.

On the other hand, the scaling

$$a^- \mapsto a_\varepsilon^-(x) = \varepsilon^{-d} a^-(\varepsilon^{-1}x) \quad (1.10)$$

yields $F(v, a_\varepsilon^- * v) \rightarrow f(v)$, $\varepsilon \rightarrow 0$, where f is given by (1.7). Hence one gets from (1.3) the another nonlocal Fisher–KPP equation

$$\frac{\partial u}{\partial t} = L_{a^+} u + f(u). \quad (1.11)$$

For a general monostable f as above, this equation was considered in e.g. [2, 10, 14–17, 19, 20, 42, 48, 54, 66, 70, 80], see also some details below.

Combination of the both considered scalings produces hence the classical Fisher–KPP equation (1.6), with f given by (1.7), from the equation (1.3). It should be stressed, however, that we do not state that solutions to (1.3) converge to solutions to (1.9), (1.11), or (1.6). Up to our knowledge, a convergence of solutions to (1.9) or (1.11) to the solutions to (1.6) was not considered rigorously in the literature as well.

Of course, there are a lot of generalisations for the equations (1.3), (1.9), (1.11): the monostable-type function f may depend on time and space variables (e.g. nonlocal reaction-diffusion equation in a periodic media), the mapping F may include a convolution in time or just a time-delay, and many others. For some recent generalisations, see e.g. [5, 18, 24, 46, 48, 54, 55, 61, 65, 67–70, 74, 79, 83].

Let us formulate the main problems traditionally addressed to the equations above.

- (P1) Existence and uniqueness of solutions in Banach spaces of functions on \mathbb{R}^d , e.g. in $L^\infty(\mathbb{R}^d)$, $C_{ub}(\mathbb{R}^d)$ (the space of uniformly continuous functions with sup-norm), or $L^1(\mathbb{R}^d)$.
- (P2) Uniform in time bounds for the norms of the solutions in the Banach spaces.
- (P3) Existence and stability of stationary solutions.
- (P4) Existence, uniqueness and properties of the traveling waves: solutions of the special form $u(x, t) = \psi(x \cdot \xi - ct)$, where ψ is a function on \mathbb{R} called the profile of the wave, ξ belongs to the unit sphere S^{d-1} in \mathbb{R}^d , $x \cdot \xi = (x, \xi)_{\mathbb{R}^d}$, and $c \in \mathbb{R}$ describes the speed of the wave. Depending on the class of functions ψ the question may be referred to decaying waves, bounded waves *etc.*
- (P5) Existence and time-behavior of the front of propagation, i.e. a set $\Gamma_t = \mathbb{R}^d \setminus (\mathcal{C}_t \cup \mathcal{O}_t)$, such that for any $x_t \in \mathcal{C}_t$, the values of $u(x_t, t)$ will converge (as $t \rightarrow \infty$) to the upper stationary solution (θ in the notations above), whereas, for any $y_t \in \mathcal{O}_t$, the values of $u(y_t, t)$ will converge to the low stationary solution (i.e. to 0).

We present now an overview of our results concerning the problems (P1)–(P5) for the equation (1.1)/(1.3), and compare them with the existing results in the literature, including some information about ‘partially local’ equations (1.9) and (1.11).

Problem (P1) We will study (1.1) in the spaces $C_{ub}(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. To get an answer on the problem (P1), one does not need any further assumptions on parameters $m, \varkappa^\pm > 0$ and probability kernels $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ (see Theorem 2.2 and Remark 2.3). We use standard fixed point arguments, which take into account, however, the negative sign before a^- in (1.1). The solution hence may be constructed on a time-interval $[\tau, \tau + \Delta\tau]$, whereas the $\Delta\tau$ depends on the supremum of the solution at τ . Since the values at the moment $\tau + \Delta\tau$ might be bigger, the next time-interval appears, in general, shorter. The mentioned usage of the negative sign allows us to show that, however, the series of the time-intervals diverges, and thus one can construct solution on an arbitrary big time-interval.

Problem (P2) In spite of the possible growth of the solution's space-supremum in time, we show (Theorem 2.7) that the solution in $C_{ub}(\mathbb{R}^d)$ remains uniformly bounded in time on $[0, \infty)$ under very weak assumptions: one needs only that a^- would be separated from zero in a neighbourhood of the origin and that a^+ would have a regular behavior at infinity, e.g. $a^+(x) \leq p(|x|)$, where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d and $p \in L^1(\mathbb{R})$ monotonically decays at $\pm\infty$. This result is an analog of [45, Theorem 1.2] for the equation (1.9) (the latter used, however, the advantages of the powerful PDE technique for the linear part, that is absent in the our case).

The rest of our results requires additional hypotheses. For the shortness, some of them are presented here in a more strict form than we really need (compare them with the real assumptions (A1)–(A9) within the paper); and surely, a particular result requires a part of the assumptions only.

(H1) $0 \leq a^\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $\varkappa^+ > m$, i.e. $\theta = \frac{\varkappa^+ - m}{\varkappa^-} > 0$.

(H2) the function

$$J_\theta := \varkappa^+ a^+ - \theta \varkappa^- a^- = \varkappa^+ a^+ - (\varkappa^+ - m) a^-$$

is almost everywhere (a.e., in the sequel) non-negative and it is separated from 0 a.e. in a neighbourhood of the origin.

(H3) There exists $\lambda > 0$, such that $\int_{\mathbb{R}^d} a^+(x) e^{\lambda|x|} dx < \infty$.

Let us compare these hypotheses with existing in the literature. First, we are working in the multi-dimensional settings, cf. [33, 34, 62]. We show (Proposition 4.4) how the problem (P4) may be reduced to a one-dimensional equation, whose kernels, however, will depend on a direction $\xi \in S^{d-1}$. Regarding to this, it should be emphasised, that we do not assume that a^+ is symmetric; we deal with the so-called anisotropic settings, cf. [2, 17, 70] for the equation (1.11).

The hypothesis (H3) is called the Mollison condition, see [56, 57]. In particular, it holds if a^+ exponentially decays as $|x| \rightarrow \infty$. The equation (1.11), under the Mollison condition (H3), was considered in [2, 10, 17, 20]. The corresponding results in [76, 81] about our equation (1.1) required, however, symmetric and quickly decaying a^+ ; the latter meant that (H3) must hold for all $\lambda > 0$. Note that [76] dealt with a system of equations for a multi-type epidemic model, which is reduced in the one-type case to (1.1) with $\varkappa^+ a^+ \equiv \varkappa^- a^-$. It is worth noting also that we do not need a continuity of a^+ as well.

The most restrictive, in some sense, hypothesis is (H2). It implies the comparison principle for the equation (1.1), cf. Theorem 3.1, Proposition 3.4. In particular, the latter states that the solution will be inside the strip $0 \leq u(x, t) \leq \theta$, for all $t > 0$, provided that the initial condition $u(x, 0)$ was inside this strip. On the other hand, we show that (H2) is, in some sense, a necessary condition to have a comparison principle at all; see the discussion at end of Subsection 3.1.

Problem (P3) In Subsection 3.1, we show also that $u \equiv \theta$ is a uniformly and asymptotically stable solution, whereas $u \equiv 0$ is an unstable one. The absence of non-constant stationary solutions is shown in Proposition 5.12, see also Problem (P5) below.

The maximum principle is considered in Subsection 3.2, cf. Theorem 3.8. In particular, we prove that the solution to (1.1) is strictly positive, even for a compactly supported initial condition $u_0(x) := u(x, 0)$, and lies strictly less than θ , for any $u_0 \not\equiv \theta$ (Proposition 3.7, Corollary 3.9).

It is worth noting that the lack of the comparison principle for the equation (1.9) leads to a non-trivial behavior: if \varkappa^- is big enough, then the upper stationary solution $u \equiv \theta$ may not be stable, moreover, a stationary inhomogeneous solution may exist, see [5, 43], and also [7, 58]; for the further results about (1.9) without the comparison principle technique, see e.g. [3, 45]. Note also that the hypothesis (H2) is not preserved in course of the scaling (1.8), which, recall, produces (1.9) from (1.3): indeed, the inequality $\varepsilon^{-2-d} a^+(\varepsilon^{-1}x) \geq \theta \varkappa^- a^-(x)$ can not hold for a.a. $x \in \mathbb{R}^d$ and any $\varepsilon > 0$ simultaneously, as this would mean that a^- is ‘localised’ arbitrary close to the origin. This is a possible reason why the comparison principle which we show for (1.1) loses for (1.9). However, if, additionally, a^- depends on ε as in the second scaling (1.10), the inequality in (H2) becomes possible, provided that $\varepsilon^{-2} > \theta \varkappa^-$, thus the comparison principle for the classical Fisher–KPP equation (1.6) may be informally derived from our results.

Problem (P4) We study monotonically non-increasing traveling waves only (i.e. the profile ψ is a non-increasing function on \mathbb{R}). To ensure the existence of the traveling wave solution to (1.1) in a direction $\xi \in S^{d-1}$ the Mollison condition (H3) can be relaxed as follows:

$$(H3_\xi) \text{ There exists } \lambda > 0, \text{ such that } \mathbf{a}_\xi(\lambda) := \int_{\mathbb{R}^d} a^+(x) e^{\lambda x \cdot \xi} dx < \infty.$$

Namely, we prove that there exists a minimal traveling wave speed $c_*(\xi) \in \mathbb{R}$, such that, for any $c \geq c_*(\xi)$, there exists a traveling wave in the direction ξ with the speed c ; and, for any $c < c_*(\xi)$, such a traveling wave does not exist (Theorem 4.9). We use here an abstract result from [80] and apply it to (1.1) similarly to how it was done in [80] for (1.11). This allow us to prove the existence of such finite $c_*(\xi)$ without an assumption about a quick decaying of a^+ in the direction ξ ; i.e. that we do not need that (H3 $_\xi$) holds, for all $\lambda > 0$, in contrast to [76, 81]. It is worth noting that the hypothesis (H2) evidently holds under the assumptions from [76], where $\varkappa^+ = \varkappa^-$, $a^+ = a^-$, as well as it holds under the assumptions from [81], where one of the considered cases may be rewritten in the form $\frac{\partial}{\partial t} u = J_\theta * u - mu + (\theta - u) \varkappa^-(a^- * u)$, which is equivalent to (1.1).

A specific feature of the equation (1.1) is that any monotonic traveling wave with a non-zero speed $c \geq c_*(\xi)$ has a smooth profile $\psi_c \in C^1(\mathbb{R})$, whereas, for the travelling wave with the zero speed (which does exist, if only $c_*(\xi) \leq 0$), one can only prove that its profile $\psi_0 \in C(\mathbb{R})$ (Proposition 4.11), in contrast to the equation (1.11), cf. [17]. This allow us to consider the equation for traveling waves point-wise:

$$c\psi'(s) + \varkappa^+(\tilde{a}^+ * \psi)(s) - m\psi(s) - \varkappa^-\psi(s)(\tilde{a}^- * \psi)(s) = 0, \quad s \in \mathbb{R}, \quad (1.12)$$

where kernels \tilde{a}^\pm are obtained by the integration of a^\pm over the orthogonal complement $\{\xi\}^\perp$, see (4.6) below. Moreover, in Corollary 4.12, we show that ψ is a strictly decaying function.

We study properties of the solutions to (1.12) using a bilateral-type Laplace transform: $(\mathcal{L}\psi)(z) = \int_{\mathbb{R}} \psi(s) e^{zs} ds$, $\operatorname{Re} z > 0$. To do this, we prove that any

solution (1.12) has a positive abscissa $\lambda_0(\psi)$ of this Laplace transform, i.e. that $(\mathfrak{L}\psi)(\lambda) < \infty$, for some $\lambda > 0$ (Proposition 4.13). Moreover, in Theorem 4.22, we prove, in particular, that $\lambda_0(\psi)$ is finite and bounded by $\lambda_0(\check{a}^+)$; note that the latter abscissa will be infinite in the case of quickly decaying kernel a^+ , i.e. when (H3 $_{\xi}$) holds, for any $\lambda > 0$. We also find in Theorem 4.22 the explicit formula for $c_*(\xi)$:

$$c_*(\xi) = \inf_{\lambda > 0} \frac{\varkappa^+ \mathfrak{a}_{\xi}(\lambda) - m}{\lambda},$$

where \mathfrak{a}_{ξ} is defined in (H3 $_{\xi}$); and we show that the abscissas $\lambda_0(\psi_c)$ of the traveling waves profiles are strictly decreasing in $c \geq c_*(\xi)$.

Thus, for ‘exponentially decaying’ a^+ (i.e. when there exists a finite supremum of λ ’s for which (H3 $_{\xi}$) does hold), it is possible the situation, when the abscissa $\lambda_* = \lambda_0(\psi_{c_*(\xi)})$ of the travelling wave with the minimal possible speed coincides with $\lambda_0(\check{a}^+)$. This case is traditionally more difficult for an analysis of profiles’ properties, cf. e.g. [2, Theorem 3, Remark 8]. We consider this special case in details and describe it in terms of the function a^+ and the parameters m, \varkappa^{\pm} , cf. Definition 4.19, Theorem 4.22.

The variety of possible situations demonstrates the following natural example, cf. Example 4.21. Let

$$a^+(x) = \frac{\alpha e^{-\mu|x|}}{1 + |x|^q}, \quad q \geq 0, \quad \mu > 0, \quad (1.13)$$

where α is a normalising constant. Then, for any $\xi \in S^{d-1}$, the abscissa $\lambda_0(\check{a}^+) = \mu$ is finite. We show that the strict inequality $\lambda_* < \mu$ always hold, for $q \in [0, 2]$. Next, there exist critical values $\mu_* > 0$ and $m_* \in (0, \varkappa^+)$, such that, for $q > 2$, one has $\lambda_* < \mu$ if $\mu \geq \mu_*$ or if $\mu \in (0, \mu_*)$ and $m \in (m_*, \varkappa^+)$. Respectively, for $q > 2$, $\mu \in (0, \mu_*)$, and $m \in (0, m_*]$, we show the equality $\lambda_* = \mu$, see Theorem 4.22.

To study the uniqueness of traveling waves, we find also the exact asymptotic at ∞ of the profiles of traveling waves with non-zero speeds. Namely, we show in Proposition 4.24, that, for a profile ψ corresponding to the speed $c \neq 0$,

$$\psi(t) \sim D e^{-\lambda_0(\psi)t}, \quad c > c_*(\xi), \quad \psi(t) \sim D t e^{-\lambda_0(\psi)t}, \quad c = c_*(\xi), \quad (1.14)$$

as $t \rightarrow \infty$. Here $D > 0$ is a constant which may be chosen equal to 1 by a shift of ψ (see Remark 4.31). To get (1.14), one needs an additional assumption in the critical case for the speed $c = c_*(\xi)$; for example, in terms of the function (1.13), this assumption does not hold for the case $q \in (2, 3]$, $\mu \in (0, \mu_*)$, $m = m_*$ only (Remark 4.26).

The asymptotic (1.14) was known for the equation (1.11), cf. e.g. [2, 12, 17]. In the two latter references, there was used a version of the Ikehara theorem which belongs to Delange [23]. However, we have met here with the following problem.

Both the classical Ikehara theorem (see e.g. [78]) and the Ikehara–Delange theorem [23] (see also [28]) dealt with functions growing on infinity to ∞ . In [12, 17], the corresponding results were postulated for functions (decreasing or increasing) which tend to 0 (on ∞ or $-\infty$, respectively). We did not find any arguments why we could apply or how one could modify the proofs of Ikehara-type theorems for such functions without proper additional assumptions. The natural assumption under which it can be realized is that the decreasing function

$\psi(s)$ (a traveling wave in our context) must become an increasing one, being multiplied on an exponent $e^{\nu s}$, for a big enough $\nu > 0$.

Under such assumption the Ikehara-type theorems might hold true, however, one needs more to cover the aforementioned case $\lambda_* = \mu$. In this case, the Laplace transform of \check{a}^+ is not analytic at its abscissa, that was a requirement for the mentioned theorems. Therefore, we used an another modification of the Ikehara theorem, the so-called Ikehara–Ingham theorem [71]. Under assumptions that a constant ν as above exists, we prove in Proposition 4.27 a version of the Ikehara–Ingham theorem for such decreasing functions. Next, using the ideas from [82], we show that, for any solution to (1.12) with $c \neq 0$, such a ν does exist.

Note also that the technique from [2] did not require the usage of Ikehara-type theorem, however, even for the local nonlinearity like in (1.11) it did not work in the critical case above.

The asymptotic (1.14) allows us to prove the uniqueness of the profiles for a traveling wave with a non-zero speed (Theorem 4.32). We follow there the technique proposed in [12].

Problem (P5) The question of the front of propagation for the equations (1.9), (1.11) was studied less intensively. For our equation (1.3), one can refer to [62] ($\varkappa^+ a^+ = \varkappa^- a^-$, see also below) and [76] ($\varkappa^+ a^+ = \varkappa^- a^-$, $d = 1$, quickly decaying kernels). Note that the generalization in [83] does not cover the equation (1.3). One of the traditional way for the study of the front of propagation for integro-differential equations is the usage of the abstract Weinberger’s results from [75] (which are going back to that for the Fisher–KPP equation (1.6) from [6]). The information we obtained for the traveling waves allow us to describe in more details the behavior of $u(xt, t)$ ‘out of front’; here u is the solution to (1.3). Namely, in Theorem 5.9, we prove that, for a proper compact convex set Υ_1 , the function $u(tx, t)$ decays exponentially in time, uniformly in $x \in \mathbb{R}^d \setminus \mathcal{O}$, for any open $\mathcal{O} \supset \Upsilon_1$, provided that the initial condition decays in space quicker than any exponent (in particular, we do not require a compactly supported initial condition).

To describe the behavior of $u(tx, t)$, for $x \in \Upsilon_1$, we start with an adaption of the results from [75] to our case. However, that abstract technique required that the initial condition should be separated from 0 on a set which can not be described explicitly (only existence of such a set was shown, cf. Lemma 5.13 and Proposition 5.17 below). To avoid this restriction, we find, in Proposition 5.18, an explicit sub-solution to (1.3), and, moreover, we prove, in Proposition 5.19, that this sub-solution indeed becomes a minorant for the solution, after a finite time. This arguments allow us to show that $u(tx, t)$ converges to θ uniformly in $x \in \mathcal{C}$, for any compact $\mathcal{C} \subset \Upsilon_1$ (Theorem 5.10, Corollary 5.11). In notations of Problem (P5), it means informally that $\Gamma_t \approx t \partial \Upsilon_1$.

As a consequence, we prove that, under additional technical assumptions, there are not other non-negative time-stationary solutions to (1.3) except constant solutions 0 and θ (Proposition 5.12).

The Mollison condition (H3 $_{\xi}$) is crucial: we show in Theorem 5.20 and Corollary 5.21 that the absence of a λ and a $\xi \in S^{d-1}$ which ensure (H3 $_{\xi}$) leads to an infinite speed of propagation (i.e. the compact set Υ_1 above may be chosen arbitrary big) and hence to the absence of traveling waves at all. The corresponding result for (1.11) was received in [42] and it is goes back to [56, 57]

mentioned above. The results of [62] cover Theorems 5.9, 5.10, and 5.20, for the equation (1.3) with $\varkappa^+ a^+ = \varkappa^- a^-$; however, a lot of details of the proofs (which used completely another technique) were omitted.

To summarize, the structure of the paper is the following. In Section 2, we study Problems (P1) and (P2); Section 3 is devoted to comparison and maximum principles, and, partially, to Problem (P3). Traveling waves, Problem (P4), are considered in Section 4. The long-time behavior, i.e. Problem (P5), and the rest of Problem (P3) are the topics of Section 5. In Section 6, we present some historical comments about the derivation of the equation (1.3).

2 Existence, uniqueness, and boundedness

Let $u = u(x, t)$ describe the local density of a system at the point $x \in \mathbb{R}^d$, $d \geq 1$, at the moment of time $t \in I$, where I is either a finite interval $[0, T]$, for some $T > 0$, or the whole $\mathbb{R}_+ := [0, \infty)$. The time evolution of u is given by the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \varkappa^+(a^+ * u)(x, t) - mu(x, t) \\ \quad - \varkappa^- u(x, t)(a^- * u)(x, t), & x \in \mathbb{R}^d, t \in I \setminus \{0\}, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (2.1)$$

which we will study in the class of bounded in x nonnegative functions.

Here $m > 0$, $\varkappa^\pm > 0$ are constants, and functions $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ are probability densities:

$$\int_{\mathbb{R}^d} a^+(y) dy = \int_{\mathbb{R}^d} a^-(y) dy = 1. \quad (2.2)$$

Here and below, for a function $u = u(y, t)$, which is bounded in $y \in \mathbb{R}^d$, and a function (a kernel) $a \in L^1(\mathbb{R}^d)$, we denote

$$(a * u)(x, t) := \int_{\mathbb{R}^d} a(x - y)u(y, t) dy. \quad (2.3)$$

We assume that u_0 is a bounded function on \mathbb{R}^d . For technical reasons, we will consider two Banach spaces of bounded real-valued functions on \mathbb{R}^d : the space $C_{ub}(\mathbb{R}^d)$ of bounded uniformly continuous functions on \mathbb{R}^d with sup-norm and the space $L^\infty(\mathbb{R}^d)$ of essentially bounded (with respect to the Lebesgue measure) functions on \mathbb{R}^d with esssup-norm. We will also use notations $C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ for the space of bounded continuous functions and the space of continuous functions with compact supports, correspondingly.

Let E be either $C_{ub}(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. Consider the equation (2.1) in E ; in particular, u must be continuously differentiable in t , for $t > 0$, in the sense of the norm in E . Moreover, we consider u as an element from the space $C_b(I \rightarrow E)$ of continuous bounded functions on I (including 0) with values in E and with the following norm

$$\|u\|_{C_b(I \rightarrow E)} = \sup_{t \in I} \|u(\cdot, t)\|_E.$$

Such solution is said to be a classical solution to (2.1); in particular, u will continuously (in the sense of the norm in E) depend on the initial condition u_0 .

there is a time-dependent linear bounded operator (acting in u) in the space $C_{ub}(\mathbb{R}^d)$ whose coefficients are continuous on $[\tau, T]$. Therefore, there exists a unique solution to (2.5) in $C_{ub}(\mathbb{R}^d)$ on $[\tau, T]$, given by $u = \Phi_\tau v$ with

$$(\Phi_\tau v)(x, t) := (Bv)(x, \tau, t)u_\tau(x) + \int_\tau^t (Bv)(x, s, t)\varkappa^+(a^+ * v)(x, s) ds, \quad (2.6)$$

for $x \in \mathbb{R}^d$, $t \in [\tau, T]$, where we set

$$(Bv)(x, s, t) := \exp\left(-\int_s^t (m + \varkappa^-(a^- * v)(x, p)) dp\right), \quad (2.7)$$

for $x \in \mathbb{R}^d$, $t, s \in [\tau, T]$. Note that, in particular, $(\Phi_\tau v)(\cdot, t), (Bv)(\cdot, s, t) \in C_{ub}(\mathbb{R}^d)$. Clearly, $(\Phi_\tau v)(x, t) \geq 0$ and, for any $\Upsilon \in (\tau, T]$,

$$\|\Phi_\tau v(\cdot, t)\| \leq \|u_\tau\| + \varkappa^+(\Upsilon - \tau)\|v\|_{\tau, \Upsilon}, \quad t \in [\tau, \Upsilon], \quad (2.8)$$

where we used (2.4). Therefore, Φ_τ maps $\mathcal{X}_{\tau, \Upsilon}^+$ into itself, $\Upsilon \in (\tau, T]$.

Let now $0 \leq \tau < \Upsilon \leq T$, and take any $v, w \in \mathcal{X}_{\tau, \Upsilon}^+$. By (2.6), one has, for any $x \in \mathbb{R}^d$, $t \in [\tau, \Upsilon]$,

$$|(\Phi_\tau v)(x, t) - (\Phi_\tau w)(x, t)| \leq J_1 + J_2, \quad (2.9)$$

where

$$\begin{aligned} J_1 &:= |(Bv)(x, \tau, t) - (Bw)(x, \tau, t)|u_\tau(x), \\ J_2 &:= \varkappa^+ \int_\tau^t |(Bv)(x, s, t)(a^+ * v)(x, s) - (Bw)(x, s, t)(a^+ * w)(x, s)| ds. \end{aligned}$$

Since $|e^{-a} - e^{-b}| \leq |a - b|$, for any constants $a, b \geq 0$, one has, by (2.7), (2.4),

$$J_1 \leq \varkappa^-(\Upsilon - \tau)\|u_\tau\|\|v - w\|_{\tau, \Upsilon}. \quad (2.10)$$

Next, for any constants $a, b, p, q \geq 0$,

$$|pe^{-a} - qe^{-b}| \leq e^{-a}|p - q| + q \max\{e^{-a}, e^{-b}\}|a - b|,$$

therefore, by (2.7), (2.4),

$$\begin{aligned} J_2 &\leq \varkappa^+ \int_\tau^t (Bv)(x, s, t)(a^+ * |v - w|)(x, s) ds \\ &\quad + \varkappa^+ \int_\tau^t \max\{(Bv)(x, s, t), (Bw)(x, s, t)\}(a^+ * w)(x, s) \\ &\quad \times \varkappa^- \int_s^t (a^- * |v - w|)(x, r) dr ds \\ &\leq \varkappa^+(\Upsilon - \tau)\|v - w\|_{\tau, \Upsilon} + \varkappa^+ \varkappa^- \|w\|_{\tau, \Upsilon} \|v - w\|_{\tau, \Upsilon} \int_\tau^t e^{-m(t-s)}(t-s) ds \\ &\leq \varkappa^+ \left(1 + \frac{\varkappa^-}{me}\|w\|_{\tau, \Upsilon}\right) (\Upsilon - \tau)\|v - w\|_{\tau, \Upsilon}, \end{aligned} \quad (2.11)$$

as $re^{-r} \leq e^{-1}$, $r \geq 0$.

For any $T_2 > T_1 \geq 0$, we define

$$\mathcal{X}_{T_1, T_2}^+ := \{v \in \mathcal{X}_{T_1, T_2}^+ \mid \|v\|_{T_1, T_2} \leq r\}, \quad r > 0.$$

Take any $\mu \geq \|u_\tau\|$. By (2.8), (2.9), (2.10), (2.11), one has, for any $v, w \in \mathcal{X}_{\tau, \Upsilon}^+(r)$, $r > 0$,

$$\begin{aligned} |(\Phi_\tau v)(x, t) - (\Phi_\tau w)(x, t)| &\leq \left(\varkappa^- \mu + \varkappa^+ + \frac{\varkappa^+ \varkappa^-}{me} r \right) (\Upsilon - \tau) \|v - w\|_{\tau, \Upsilon}, \\ |(\Phi_\tau v)(x, t)| &\leq \mu + \varkappa^+ r (\Upsilon - \tau). \end{aligned}$$

Therefore, Φ_τ will be a contraction mapping on the set $\mathcal{X}_{\tau, \Upsilon}^+(r)$ if only

$$\left(\varkappa^- \mu + \varkappa^+ + \frac{\varkappa^+ \varkappa^-}{me} r \right) (\Upsilon - \tau) < 1 \quad \text{and} \quad \mu + \varkappa^+ r (\Upsilon - \tau) \leq r. \quad (2.12)$$

Take any $\alpha \in (0, 1)$ and set

$$\begin{aligned} C &:= \varkappa^- + \frac{\varkappa^+ \varkappa^-}{me}, \quad r := \mu + \frac{\alpha \varkappa^+}{C}, \\ \Upsilon &:= \tau + \frac{\alpha}{Cr} = \tau + \frac{\alpha}{C\mu + \alpha \varkappa^+}. \end{aligned} \quad (2.13)$$

Then, the second inequality in (2.12) evidently holds (and it is just an equality), and the first one may be rewritten as follow

$$\left(C\mu + \varkappa^+ + \frac{\varkappa^+ \varkappa^-}{me} \frac{\alpha \varkappa^+}{C} \right) \frac{\alpha}{Cr} < 1,$$

or, equivalently,

$$\alpha C\mu + \alpha^2 \frac{\varkappa^+ \varkappa^-}{me} \frac{\varkappa^+}{C} < C\mu. \quad (2.14)$$

To fulfil (2.14), one should choose $\alpha \in (0, 1)$ such that

$$\frac{\alpha^2}{1 - \alpha} < \frac{C^2 \mu me}{(\varkappa^+)^2 \varkappa^-}. \quad (2.15)$$

Since function $f(\alpha) = \frac{\alpha^2}{1 - \alpha}$ is strictly increasing on $[0, 1)$ and $f(0) = 0$, one can always choose $\alpha \in (0, 1)$ that satisfies (2.15).

As a result, choosing $\mu = \mu(\tau) > \|u_\tau\|$ (to include the case $u_\tau \equiv 0$) and α that satisfies (2.15), one gets that Φ_τ will be a contraction on the set $\mathcal{X}_{\tau, \Upsilon}^+(r)$ with Υ and r given by (2.13); the latter set naturally forms a complete metric space. Therefore, there exists a unique $u \in \mathcal{X}_{\tau, \Upsilon}^+(r)$ such that $\Phi_\tau u = u$. This u will be a solution to (2.1) on $[\tau, \Upsilon]$.

To fulfil the proof of the statement, one can do the following. Set $\tau := 0$, choose any $\mu_1 > \|u_0\|$ and fix an α that satisfies (2.15) with $\mu = \mu_1$. One gets a solution u to (2.1) on $[0, \Upsilon_1]$ with $\Upsilon_1 = \frac{\alpha}{C\mu_1 + \alpha \varkappa^+}$, $\|u\|_{\Upsilon_1} \leq \mu_1 + \frac{\alpha \varkappa^+}{C}$.

Iterating this scheme, take sequentially, for each $n \in \mathbb{N}$, $\tau := \Upsilon_n$, $u_{\Upsilon_n}(x) := u(x, \Upsilon_n)$, $x \in \mathbb{R}^d$,

$$\mu_{n+1} := \mu_n + \frac{\alpha \varkappa^+}{C} \geq \|u_{\Upsilon_n}\|.$$

Since $\mu_{n+1} > \mu_n$, the same α as before will satisfy (2.15) with $\mu = \mu_{n+1}$ as well. Then, one gets a solution u to (2.1) on $[\Upsilon_n, \Upsilon_{n+1}]$ with initial condition u_{Υ_n} , where

$$\Upsilon_{n+1} := \Upsilon_n + \frac{\alpha}{C\mu_{n+1} + \alpha\chi^+},$$

and

$$\|u\|_{\Upsilon_n, \Upsilon_{n+1}} \leq \mu_{n+1} + \frac{\alpha\chi^+}{C} = \mu_{n+2}.$$

As a result, we will have a solution u to (2.1) on intervals $[0, \Upsilon_1], [\Upsilon_1, \Upsilon_2], \dots, [\Upsilon_n, \Upsilon_{n+1}]$, $n \in \mathbb{N}$, where $\mu_{n+1} = \mu_1 + n\frac{\alpha\chi^+}{C}$, and, thus,

$$\Upsilon_{n+1} := \Upsilon_n + \frac{\alpha}{C\mu_1 + (n+1)\alpha\chi^+}. \quad (2.16)$$

By Lemma 2.1, the r.h.s. of (2.1), will be continuous on each of constructed time-intervals, therefore, one has that u is continuously differentiable on $(0, \Upsilon_{n+1}]$ and solves (2.1) there. By (2.16),

$$\Upsilon_{n+1} := \sum_{j=1}^{n+1} \frac{\alpha}{C\mu_1 + j\alpha\chi^+} \rightarrow \infty, \quad n \rightarrow \infty,$$

therefore, one has a solution to (2.1) on any $[0, T]$, $T > 0$.

To prove uniqueness, suppose that $v \in \mathcal{X}_T$ is a solution to (2.1) on $[0, T]$, with $v(x, 0) \equiv u_0(x)$, $x \in \mathbb{R}^d$. Choose $\mu_1 > \|v\|_T \geq \|u_0\|$. Since $\{\mu_n\}_{n \in \mathbb{N}}$ above is an increasing sequence, v will belong to each of sets $\mathcal{X}_{\Upsilon_n, \Upsilon_{n+1}}^+(\mu_{n+1})$, $n \geq 0$, $\Upsilon_0 := 0$, considered above. And, being solution to (2.1) on each $[\Upsilon_n, \Upsilon_{n+1}]$, v will be a fixed point for Φ_{Υ_n} . By the uniqueness of such point, v coincides with u on each $[\Upsilon_n, \Upsilon_{n+1}]$ and, thus, on the whole $[0, T]$. \square

Remark 2.3. The statement of Theorem 2.2 holds true for solutions in $L^\infty(\mathbb{R}^d)$ with $u \in \tilde{X}_T$: the proof will be mainly identical. See also [33, Theorem 4.1].

Consider the following quantity

$$\theta := \frac{\chi^+ - m}{\chi^-} \in \mathbb{R}. \quad (2.17)$$

Theorem 2.2 has a simple corollary:

Corollary 2.4. *Let $t_0 \geq 0$ be such that the solution u_t to (2.1) is a constant in space at the moment of time t_0 , namely, $u_{t_0}(x) \equiv u_{t_0} \geq 0$, $x \in \mathbb{R}^d$. Then this solution will be a constant in space for all further moments of time, more precisely,*

$$u(x, t) = u(t) = \frac{u_{t_0}}{u_{t_0}g_\theta(t) + \exp(-\theta\chi^-t)} \geq 0, \quad x \in \mathbb{R}^d, t \geq t_0, \quad (2.18)$$

where

$$g_\theta(t) = \begin{cases} \frac{1 - \exp(-\theta\chi^-t)}{\theta}, & \theta \neq 0, \\ \chi^-t, & \theta = 0, \end{cases} \quad t \geq t_0.$$

In particular, $u(t) \rightarrow \max\{0, \theta\}$, $t \rightarrow \infty$.

Proof. First of all, we note that in the proof of Theorem 2.2 we proved that the problem (2.1) has a unique solution. Next, straightforward calculations show that (2.18) solves (2.1) for $\tau = t_0$, that implies the first statement. The last statement is also straightforward then. \square

Remark 2.5. Note that (2.18) solves the classical logistic equation, cf. (1.2):

$$\frac{d}{dt}u(t) = \varkappa^- u(t)(\theta - u(t)), \quad t > t_0, \quad u(t_0) = u_{t_0} \geq 0. \quad (2.19)$$

By Lemma 2.1, the mapping $A^+v = \varkappa^+ a^+ * v$ defines a linear operator on $C_{ub}(\mathbb{R}^d)$, which is evidently bounded: by (2.4) and $A^+1 = \varkappa^+$, one has $\|A^+\| = \varkappa^+$. Then a solution u to (2.1) satisfies the following equation

$$u(x, t) = e^{-tm} e^{tA^+} u_0(x) - \int_0^t e^{-(t-s)m} e^{(t-s)A^+} \varkappa^- u(x, s) (a^- * u)(x, s) ds.$$

Therefore, $u(x, t) \geq 0$ implies $u(x, t) \leq e^{-tm} e^{tA^+} u_0(x)$, $x \in \mathbb{R}^d$, $t \geq 0$; and hence, by Theorem 2.2, $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$ yields

$$\|u(\cdot, t)\| \leq e^{(\varkappa^+ - m)t} \|u_0\|, \quad t \geq 0. \quad (2.20)$$

In particular, for $m > \varkappa^+$, the solution $u(x, t)$ to (2.1) exponentially quickly in t tends to 0, uniformly in $x \in \mathbb{R}^d$.

We proceed now to show that, in fact, the solution to (2.1) is uniformly bounded in time on the whole \mathbb{R}_+ , provided that the kernel a^- does not degenerate in a neighborhood of the origin and a^+ has an integrable decay at ∞ .

Let $\mathbb{1}_A$ be the indicator function of a measurable set $A \subset \mathbb{R}^d$. Recall that the sequence $f_n \in L_{loc}^\infty(\mathbb{R}^d)$ is said to be convergent to $f \in L_{loc}^\infty(\mathbb{R}^d)$ locally uniformly if $\mathbb{1}_\Lambda f_n \rightarrow \mathbb{1}_\Lambda f$ in $L^\infty(\mathbb{R}^d)$, $n \rightarrow \infty$, for any compact $\Lambda \subset \mathbb{R}^d$. We denote this convergence by $f_n \xrightarrow{loc} f$. We will use the same notation to say that, for some $T > 0$ and $v_n, v \in L_{loc}^\infty(\mathbb{R}^d \times [0, T])$, one has $\mathbb{1}_\Lambda v_n \rightarrow \mathbb{1}_\Lambda v$ in $L^\infty(\mathbb{R}^d \times [0, T])$, for any compact $\Lambda \subset \mathbb{R}^d$.

We start with a simple statement useful for the sequel.

Lemma 2.6. *Let $a \in L^1(\mathbb{R}^d)$, $\{f_n, f\} \subset L^\infty(\mathbb{R}^d)$, $\|f_n\| \leq C$, for some $C > 0$, and $f_n \xrightarrow{loc} f$. Then $a * f_n \xrightarrow{loc} a * f$.*

Proof. Let $\{a_m\} \subset C_0(\mathbb{R}^d)$ be such that $\|a_m - a\|_{L^1(\mathbb{R}^d)} \rightarrow 0$, $m \rightarrow \infty$, and denote $A_m := \text{supp } a_m$. Note that, there exists $D > 0$, such that $\|a_m\|_{L^1(\mathbb{R}^d)} \leq D$, $m \in \mathbb{N}$. Next, for any compact $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned} |\mathbb{1}_\Lambda(x)(a_m * (f_n - f))(x)| &\leq \int_{\mathbb{R}^d} \mathbb{1}_{A_m}(y) \mathbb{1}_\Lambda(x) |a_m(y)| |f_n(x - y) - f(x - y)| dy \\ &\leq \|a_m\|_{L^1(\mathbb{R}^d)} \|\mathbb{1}_{\Lambda_m}(f_n - f)\| \rightarrow 0, n \rightarrow \infty, \end{aligned}$$

for some compact $\Lambda_m \subset \mathbb{R}^d$. Next,

$$\begin{aligned} \|\mathbb{1}_\Lambda(a * (f_n - f))\| &\leq \|\mathbb{1}_\Lambda(a_m * (f_n - f))\| + \|\mathbb{1}_\Lambda((a - a_m) * (f_n - f))\| \\ &\leq \|\mathbb{1}_{\Lambda_m}(f_n - f)\| + (C + \|f\|) \|a - a_m\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

and the second term may be arbitrary small by a choice of m . \square

The next theorem is an adaptation of [45, Theorem 1.2].

Below, $|\cdot| = |\cdot|_{\mathbb{R}^d}$ denotes the Euclidean norm in \mathbb{R}^d , $B_r(x)$ is a ball in \mathbb{R}^d with the center at $x \in \mathbb{R}^d$ and the radius $r > 0$; and b_r is a volume of this ball. Consider also, for any $z \in \mathbb{Z}^d$, $q > 0$, a hypercube in \mathbb{R}^d with the center at $2qz \in \mathbb{R}^d$ and the side $2q$:

$$H_q(z) := \{y \in \mathbb{R}^d \mid 2z_i q - q \leq y_i \leq 2z_i q + q, i = 1, \dots, d\}.$$

Theorem 2.7. *Suppose that there exists $r_0 > 0$ such that*

$$\alpha := \inf_{|x| \leq r_0} a^-(x) > 0. \quad (2.21)$$

Suppose also that, for some $q \in (0, \frac{r_0}{2\sqrt{d}}]$,

$$a_q^+ := \sum_{z \in \mathbb{Z}^d} \sup_{x \in H_q(z)} a^+(x) < \infty \quad (2.22)$$

(e.g. let, for some $\varepsilon > 0$, $A > 0$, one has $a^+(x) \leq \frac{A}{1+|x|^{d+\varepsilon}}$, for a.a. $x \in \mathbb{R}^d$). Then, the solution $u \geq 0$ to (2.1), with $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$, belongs to \mathcal{X}_∞ .

Proof. If $m \geq \varkappa^+$ then the statement is trivially followed from (2.20). Suppose that $m < \varkappa^+$ and rewrite (2.1) in the form

$$\frac{\partial}{\partial t} u(x, t) = (L_{a^+} u)(x, t) + \varkappa^- u(x, t) (\theta - (a^- * u)(x, t)), \quad (2.23)$$

where $\theta = \frac{\varkappa^+ - m}{\varkappa^-} > 0$ and the operator L_{a^+} acts in x and is given by (1.4).

It is easily seen that $H_q(z) \subset B_{q\sqrt{d}}(2qz)$, $z \in \mathbb{Z}^d$, $q > 0$. Take any $q \leq \frac{r_0}{2\sqrt{d}}$ such that (2.22) holds, and set $r = q\sqrt{d} \leq \frac{r_0}{2}$. Define

$$v(x, t) := (\mathbb{1}_{B_r(0)} * u)(x, t) = \int_{B_r(x)} u(y, t) dy. \quad (2.24)$$

By Lemma 2.1, Theorem 2.2, (2.20), $0 \leq v \in \mathcal{X}_T$, $T > 0$, and

$$\|v(\cdot, t)\| \leq b_r e^{(\varkappa^+ - m)t} \|u_0\|, \quad t \geq 0.$$

Note that, by (1.4),

$$L_{a^+} v = \varkappa^+ a^+ * \mathbb{1}_{B_r(0)} * u - \varkappa^+ \mathbb{1}_{B_r(0)} * u = \mathbb{1}_{B_r(0)} * (L_{a^+} u).$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} v(x, t) - (L_{a^+} v)(x, t) &= \left(\mathbb{1}_{B_r(0)} * \frac{\partial}{\partial t} u \right)(x, t) - \left(\mathbb{1}_{B_r(0)} * (L_{a^+} u) \right)(x, t) \\ &= \varkappa^- \int_{B_r(x)} u(y, t) (\theta - (a^- * u)(y, t)) dy. \end{aligned} \quad (2.25)$$

By (2.24), one has $\|v(\cdot, 0)\| \leq b_r \|u_0\|$. Set

$$M > \max \left\{ b_r \|u_0\|, \frac{\theta}{\alpha} \right\}. \quad (2.26)$$

First, we will prove that

$$\|v(\cdot, t)\| \leq M, \quad t \geq 0. \quad (2.27)$$

On the contrary, suppose that there exists $t' > 0$ such that $\|v(\cdot, t')\| > M$. By (2.24) and Lemma 2.1, $\|v(\cdot, t)\|$ is continuous in t . Next, since $\|v(\cdot, 0)\| < M$, there exists $t_0 > 0$ such that $\|v(\cdot, t_0)\| = M$ and $\|v(\cdot, t)\| < M$, for all $t \in [0, t_0)$.

Consider the sequence $\{x_n\} \subset \mathbb{R}^d$ such that $v(x_n, t_0) \rightarrow M$, $n \rightarrow \infty$. Define the following functions:

$$u_n(x, t) := u(x + x_n, t), \quad v_n(x, t) := v(x + x_n, t) = (\mathbb{1}_{B_r(0)} * u_n)(x, t),$$

for $x \in \mathbb{R}^d$, $t \geq 0$. Take any $T > 0$. Evidently, $u \in C_{ub}(\mathbb{R}^d \times [0, T])$, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in \mathbb{R}^d$, $t, s \in [0, T]$, with $|x - y|_{\mathbb{R}^d} + |t - s| < \delta$, one has $|u_n(x, t) - u_n(y, s)| = |u(x + x_n, t) - u(y + x_n, s)| < \varepsilon$. And, by (2.20),

$$\|u_n(\cdot, t)\| \leq \|u(\cdot, t)\| \leq e^{(\varkappa^+ - m)T} \|u_0\|, \quad n \in \mathbb{N}, t \in [0, T]. \quad (2.28)$$

Hence $\{u_n\}$ is a uniformly bounded and uniformly equicontinuous sequence of functions on $\mathbb{R}^d \times [0, T]$. Thus, by a version of the Arzelà–Ascoli Theorem, see e.g. [29, Appendix C.8], there exists a subsequence $\{u_{n_k}\}$ and a continuous function u_∞ on $\mathbb{R}^d \times [0, T]$ such that $u_{n_k} \xrightarrow{\text{loc}} u_\infty$. Moreover, one can easily show that $u_\infty \in C_{ub}(\mathbb{R}^d \times [0, T])$. By (2.28) and Lemma 2.6, $v_{n_k} \xrightarrow{\text{loc}} v_\infty = \mathbb{1}_{B_r(0)} * u_\infty$, moreover, $v_\infty \in C_{ub}(\mathbb{R}^d \times [0, T])$.

It is easily seen that both parts of (2.25) belong to \mathcal{X}_T . Hence one can integrate (2.25) on $[0, t] \subset [0, T]$, namely,

$$\begin{aligned} v(x, t) &= v(x, 0) + \int_0^t (L_{a^+} v)(x, s) ds \\ &\quad + \varkappa^- \int_0^t \int_{B_r(x)} u(y, s) (\theta - (a^- * u)(y, s)) dy ds. \end{aligned} \quad (2.29)$$

Substitute $x + x_{n_k}$ instead of x into (2.29) and use twice the integration by substitution in the second integral, then one gets the same equality (2.29), but for v_{n_k} , u_{n_k} instead of v , u , respectively. Next, by Lemma 2.6 and the dominated convergence arguments, one can pass to the limit in k in the obtained equality. As a result, one get (2.29) for v_∞ , u_∞ instead of v and u , respectively. Next, since $C_{ub}(\mathbb{R}^d \times [0, T]) \subset \mathcal{X}_T$, the integrands with respect to s in the left hand side (l.h.s. in the sequel) of the modified equation (2.29) (with $u_\infty, v_\infty \in \mathcal{X}_T$) will belong to \mathcal{X}_T as well. As a result, v_∞ will be differentiable in t in the sense of the norm in $C_{ub}(\mathbb{R}^d)$. Finally, after differentiation, one get (2.25) back, but for v_∞, u_∞ , namely,

$$\begin{aligned} \frac{\partial}{\partial t} v_\infty(x, t) - (L_{a^+} v_\infty)(x, t) \\ = \varkappa^- \int_{B_r(x)} u_\infty(y, t) (\theta - (a^- * u_\infty)(y, t)) dy. \end{aligned} \quad (2.30)$$

Going back to the definition of x_n , one can see that

$$v_\infty(0, t_0) = \lim_{k \rightarrow \infty} v_{n_k}(0, t_0) = \lim_{k \rightarrow \infty} v(x_{n_k}, t_0) = M, \quad (2.31)$$

whereas, for any $x \in \mathbb{R}^d$, $t \in [0, t_0)$, $v_\infty(x, t) = \lim_{k \rightarrow \infty} v(x + x_{n_k}, t) \leq M$. Therefore, $\frac{\partial}{\partial t} v_\infty(0, t_0) \geq 0$ and, by (1.4), $(L_{a^+} v_\infty)(0, t_0) \leq 0$. Then, by (2.30),

$$\int_{B_r(0)} u_\infty(y, t_0) (\theta - (a^- * u_\infty)(y, t_0)) dy \geq 0. \quad (2.32)$$

Next, the function $u_\infty(\cdot, t_0)$, by the construction above, is nonnegative. It can not be identically equal to 0 on $B_r(0)$, since otherwise, by (2.24), $v_\infty(0, t_0) = 0$ that contradicts (2.31). Hence by (2.32), the function $\theta - (a^- * u_\infty)(\cdot, t_0)$ cannot be strictly negative on $B_r(0)$. Thus, there exists $y_0 \in B_r(0)$ such that $\theta \geq (a^- * u_\infty)(y_0, t_0)$. Since $2r \leq r_0$, one has that $\inf_{x \in B_{2r}(0)} a^-(x) \geq \alpha$, cf. (2.21).

Therefore, one can continue:

$$\begin{aligned} \theta &\geq (a^- * u_\infty)(y_0, t_0) \geq \int_{B_{2r}(0)} a^-(y) u_\infty(y_0 - y, t_0) dy \\ &\geq \alpha \int_{B_{2r}(0)} u_\infty(y_0 - y, t_0) dy = \alpha \int_{B_{2r}(y_0)} u_\infty(y, t_0) dy \\ &\geq \alpha \int_{B_r(0)} u_\infty(y, t_0) dy = \alpha v_\infty(0, t_0) = \alpha M, \end{aligned}$$

that contradicts (2.26). Therefore, our assumption was wrong, and (2.27) holds.

We proceed now to show that $\|u(\cdot, t)\|$ is uniformly bounded in time. By (2.24), (2.27), (2.22), one has, for $r = q\sqrt{d}$,

$$\begin{aligned} (a^+ * u)(x, t) &= \sum_{z \in \mathbb{Z}^d} \int_{H_q(z)} a^+(y) u(x - y, t) dy \\ &\leq \sum_{z \in \mathbb{Z}^d} \sup_{y \in H_q(z)} a^+(y) \int_{B_r(2qz)} u(x - y, t) dy \\ &= \sum_{z \in \mathbb{Z}^d} \sup_{y \in H_q(z)} a^+(y) \int_{B_r(x - 2qz)} u(y, t) dy \leq M a_q^+. \end{aligned} \quad (2.33)$$

Therefore, by (2.1), (2.33), using the same arguments as for the proof of (2.20) one gets that

$$\begin{aligned} 0 \leq u(x, t) &\leq e^{-mt} u_0(x) + \int_0^t e^{-(t-s)m} \varkappa^+ M a_q^+ ds \\ &= e^{-mt} u_0(x) + \frac{\varkappa^+ M a_q^+}{m} (1 - e^{-mt}) \\ &\leq \max \left\{ \frac{\varkappa^+ M a_q^+}{m}, \|u_0\| \right\}, \quad x \in \mathbb{R}^d, t \geq 0, \end{aligned} \quad (2.34)$$

that fulfills the proof. \square

Remark 2.8. It should be stressed that we essentially used the uniform continuity of the solution to prove Theorem 2.7.

Under conditions of Theorem 2.7, the solution u will be uniformly continuous on $\mathbb{R}^d \times \mathbb{R}_+$, namely, the following simple proposition holds true.

Proposition 2.9. *Let u be a solution to (2.1) with $u_0 \in C_{ub}(\mathbb{R}^d)$, and suppose that there exists $C > 0$, such that*

$$|u(x, t)| \leq C, \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Then $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$. Moreover, $\|u(\cdot, t)\| \in C_{ub}(\mathbb{R}_+)$.

Proof. Being solution to (2.1), u satisfies the integral equation

$$u(x, t) = u_0(x) + \int_0^t (\varkappa^+(a^+ * u)(x, s) - \varkappa^- u(x, s)(a^- * u)(x, s) - mu(x, s)) ds.$$

Hence for any $x, y \in \mathbb{R}^d$, $0 \leq \tau < t$, one has

$$\begin{aligned} |u(x, t) - u(y, \tau)| &\leq \int_\tau^t (2\varkappa^+ C + 2\varkappa^- C^2 + 2mC) ds \\ &= 2(\varkappa^+ + \varkappa^- C + m)C(t - \tau), \end{aligned}$$

that fulfils the proof of the first statement. Then, the second one follows from the inequality $\| \|u(\cdot, t)\| - \|u(\cdot, \tau)\| \| \leq \|u(\cdot, t) - u(\cdot, \tau)\|$. \square

3 Around the comparison principle

We will discuss below the comparison and maximum principles. Being standard tools for the study of parabolic and elliptic PDE, they are hold true for (1.11), however, do not hold, in general, for (1.9), see e.g. [45] and the references therein.

3.1 Comparison principle

Let $T > 0$ be fixed. Define the sets \mathcal{X}_T^1 and $\tilde{\mathcal{X}}_T^1$ of functions from \mathcal{X}_T , respectively, $\tilde{\mathcal{X}}_T$, which are continuously differentiable on $(0, T]$ in the sense of the norm in $C_{ub}(\mathbb{R}^d)$, respectively, in $L^\infty(\mathbb{R}^d)$. Here and below we consider the left derivative at $t = T$ only. For any u from \mathcal{X}_T^1 one can define the following function

$$\begin{aligned} (\mathcal{F}u)(x, t) &:= \frac{\partial u}{\partial t}(x, t) - \varkappa^+(a^+ * u)(x, t) \\ &\quad + mu(x, t) + \varkappa^- u(x, t)(a^- * u)(x, t) \end{aligned} \quad (3.1)$$

for all $t \in (0, T]$ and all $x \in \mathbb{R}^d$. Moreover, for any $u \in \tilde{\mathcal{X}}_T^1$, one can consider the function $\frac{\partial u}{\partial t}(\cdot, t) \in L^\infty(\mathbb{R}^d)$, for all $t \in (0, T]$. Then, one can also define (3.1), which will be considered a.e. in $x \in \mathbb{R}^d$ now.

Theorem 3.1. *Let there exist $c > 0$, such that*

$$\varkappa^+ a^+(x) \geq c\varkappa^- a^-(x), \quad a.a. \quad x \in \mathbb{R}^d. \quad (3.2)$$

Let $T \in (0, \infty)$ be fixed and functions $u_1, u_2 \in \mathcal{X}_T^1$ be such that, for any $(x, t) \in \mathbb{R}^d \times (0, T]$,

$$(\mathcal{F}u_1)(x, t) \leq (\mathcal{F}u_2)(x, t), \quad (3.3)$$

$$u_1(x, t) \geq 0, \quad 0 \leq u_2(x, t) \leq c, \quad u_1(x, 0) \leq u_2(x, 0). \quad (3.4)$$

Then $u_1(x, t) \leq u_2(x, t)$, for all $(x, t) \in \mathbb{R}^d \times [0, T]$. In particular, $u_1 \leq c$.

Proof. Define the following function

$$f(x, t) := (\mathcal{F}u_2)(x, t) - (\mathcal{F}u_1)(x, t) \geq 0, \quad x \in \mathbb{R}^d, t \in (0, T], \quad (3.5)$$

cf. (3.3). We set

$$K = m + \varkappa^- \|u_1\|_T, \quad (3.6)$$

and consider a linear mapping

$$\begin{aligned} F(t, w) := & Kw - mw + \varkappa^+(a^+ * w) \\ & - \varkappa^- w(a^- * u_1) - \varkappa^- u_2(a^- * w) + e^{Kt} f, \end{aligned} \quad (3.7)$$

for $w \in \mathcal{X}_T$. By (3.4), (3.5), (3.6), (3.2), (2.4), $w \geq 0$ implies

$$\begin{aligned} F(t, w) = & Kw - mw + (\varkappa^+ a^+ - c\varkappa^- a^-) * w \\ & + \varkappa^-(c - u_2)(a^- * w) - \varkappa^- w(a^- * u_1) + e^{Kt} f \\ \geq & Kw - mw + (\varkappa^+ a^+ - c\varkappa^- a^-) * w - \varkappa^- \|u_1\|_T w + e^{Kt} f \geq 0. \end{aligned} \quad (3.8)$$

Define also the function

$$v(x, t) := e^{Kt}(u_2(x, t) - u_1(x, t)), \quad x \in \mathbb{R}^d, t \in [0, T].$$

Clearly, $v \in \mathcal{X}_T^1$, and it is straightforward to check that

$$F(t, v(s, t)) = \frac{\partial}{\partial t} v(x, t), \quad (3.9)$$

for all $x \in \mathbb{R}^d$, $t \in (0, T]$. Therefore, v solves the following integral equation in $C_{ub}(\mathbb{R}^d)$:

$$\begin{cases} v(x, t) = v(x, 0) + \int_0^t F(s, v(x, s)) ds, & (x, t) \in \mathbb{R}^d \times (0, T], \\ v(x, 0) = u_2(x, 0) - u_1(x, 0), & x \in \mathbb{R}^d, \end{cases} \quad (3.10)$$

where $v(x, 0) \geq 0$, by (3.4).

Consider also another integral equation in $C_{ub}(\mathbb{R}^d)$:

$$\tilde{v}(x, t) = (\Psi \tilde{v})(x, t) \quad (3.11)$$

where

$$(\Psi w)(x, t) := v(x, 0) + \int_0^t \max\{F(s, w(x, s)), 0\} ds, \quad w \in \mathcal{X}_T. \quad (3.12)$$

It is easily seen that $w \in \mathcal{X}_T^+$ yields $\Psi w \in \mathcal{X}_T^+$. Next, for any $\tilde{T} < T$ and for any $w_1, w_2 \in \mathcal{X}_{\tilde{T}}^+$, one gets from (3.7), (3.12), that

$$\begin{aligned} \|\Psi w_1 - \Psi w_2\|_{\tilde{T}} &\leq \tilde{T}(K + m + \varkappa^+ + \varkappa^- \|u_1\|_T + c\varkappa^-) \|w_2 - w_1\|_{\tilde{T}} \\ &:= q_T \tilde{T} \|w_2 - w_1\|_{\tilde{T}}, \end{aligned} \quad (3.13)$$

where we used the elementary inequality $|\max\{a, 0\} - \max\{b, 0\}| \leq |a - b|$, $a, b \in \mathbb{R}$. Therefore, for $\tilde{T} < (q_T)^{-1}$, Ψ is a contraction on $\mathcal{X}_{\tilde{T}}^+$. Thus, there

exists a unique solution to (3.11) on $[0, \tilde{T}]$. In the same way, the solution can be extended on $[\tilde{T}, 2\tilde{T}]$, $[2\tilde{T}, 3\tilde{T}]$, \dots , and therefore, on the whole $[0, T]$. By (3.11), (3.12),

$$\tilde{v}(x, t) \geq v(x, 0) \geq 0, \quad (3.14)$$

hence, by (3.8), (3.12),

$$\tilde{v}(x, t) = v(x, 0) + \int_0^t F(s, \tilde{v}(x, s)) ds =: \Xi(\tilde{v})(x, t). \quad (3.15)$$

Since $\tilde{v} \in \mathcal{X}_T$, (3.15) implies that \tilde{v} is a solution to (3.10) as well. The same estimate as in (3.13) shows that Ξ is a contraction on $\mathcal{X}_{\tilde{T}}$, for small enough \tilde{T} . Thus $\tilde{v} = v$ on $\mathbb{R}^d \times [0, \tilde{T}]$, and one continue this consideration as before on the whole $[0, T]$. Then, by (3.14), $v(x, t) \geq 0$ on $\mathbb{R}^d \times [0, T]$, that yields the statement. \square

Remark 3.2. The previous theorem holds true in $\tilde{\mathcal{X}}_T^1$. Here and below, for the L^∞ -case, one can assume that (3.2), (3.3), (3.4) hold almost everywhere in x only.

From the proof of Theorem 3.1, one can see that we used the fact that u_1, u_2 belong to \mathcal{X}_T^1 to ensure that (3.9) implies (3.10) only. For technical reasons we will need to extend the result of Theorem 3.1 for a wider class of functions. Naturally, to get (3.10) from (3.9), it is enough to assume absolute continuity of $v(x, t)$ in t , for a fixed x . Consider the corresponding statement.

For any $T \in (0, \infty]$, define the set \mathcal{D}_T of all functions $u : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}$, such that, for all $t \in [0, T)$, $u(\cdot, t) \in C_{ub}(\mathbb{R}^d)$, and, for all $x \in \mathbb{R}^d$, the function $f(x, t)$ is absolutely continuous in t on $[0, T)$. Then, for any $u \in \mathcal{D}_T$, one can define the function (3.1), for all $x \in \mathbb{R}^d$ and a.a. $t \in [0, T)$.

Proposition 3.3. *The statement of Theorem 3.1 remains true, if we assume that $u_1, u_2 \in \mathcal{D}_T$ and, for any $x \in \mathbb{R}^d$, the inequality (3.3) holds for a.a. $t \in (0, T)$ only.*

Proof. One can literally repeat the proof of Theorem 3.1; for any $x \in \mathbb{R}^d$, the function (3.5) and the mapping (3.7) will be defined for a.a. $t \in (0, T)$ now (and it will not be a mapping on \mathcal{X}_T , of course). Similarly, (3.8) and (3.9) hold, for all x and a.a. t . However, for any $x \in \mathbb{R}^d$, one gets that (3.10) holds still for all $t \in [0, T]$. Hence, the rest of the proof remains the same, stress that $F(t)v \notin \mathcal{X}_T$, whereas $\Xi(v) \in \mathcal{X}_T$, cf. (3.15). \square

The standard way to use Theorem 3.1 is to take u_1 and u_2 which solve (2.1), thus, $\mathcal{F}u_1 = \mathcal{F}u_2 = 0$, and (3.3) holds. Then Theorem 3.1 gives a comparison between these solutions provided that there exists a comparison between the initial conditions. However, to do this, one needs to know *a priori* that $u_2(x, t) \leq c$. For example, one can demand that c is not smaller than the constant in the r.h.s. of (2.34). Another possibility is to compare the solution to (2.1) with the solution to its homogeneous version (2.19) (with $t_0 = 0$).

Namely, let (3.2) hold, $0 < v \leq c$, and, cf. (2.18),

$$\begin{aligned} \psi(t, v) &:= \frac{v}{v g_\theta(t) + \exp(-\theta \varkappa^- t)} \geq 0, \\ g_\theta(t) &:= \lim_{y \rightarrow \theta} \frac{1 - \exp(-y \varkappa^- t)}{y} \geq 0. \end{aligned}$$

It is easily seen that, for $\theta \leq 0$, $\psi(t, v)$ decreases monotonically to 0 on $t \in [0, \infty)$: exponentially fast, for $\theta < 0$, and linearly fast, for $\theta = 0$. In particular, $\psi(t, v) \leq v \leq c$, $t \geq 0$. As a result,

- if $\varkappa^+ \leq m$ and $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$ be such that $\|u_0\| \leq c$, then $\|u(\cdot, t)\| \leq \psi(t, \|u_0\|)$. In particular, u converges to 0 uniformly in space as $t \rightarrow \infty$.

Next, for $\theta > 0$, the function $\psi(t)$ increases monotonically to θ on $[0, \infty)$ if $v < \theta$, it decreases monotonically to θ if $v > \theta$, and, clearly, $\psi(t) \equiv \theta$, if $v = \theta$. Therefore, if (3.2) holds with $c > \theta$ and $\theta < \|u_0\| \leq c$ then $\psi(t, \|u_0\|) \leq \|u_0\| \leq c$, and therefore, $\|u(\cdot, t)\| \leq \psi(t, \|u_0\|) \downarrow \theta$, $t \rightarrow \infty$. Set also $\inf_{\mathbb{R}^d} u_0(x) =: \beta \geq 0$, then one can apply the comparison principle to the functions $u_1 = \psi(t, \beta)$ and $u_2 = u$. (Note that $\psi(t, 0) = 0$.) As a result,

- if $\varkappa^+ > m$ and $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$ be such that $\theta < \|u_0\| \leq c$, then $\psi(t, \beta) \leq u(x, t) \leq \psi(t, \|u_0\|)$, $x \in \mathbb{R}^d$, $t \geq 0$, where $\beta = \inf_{\mathbb{R}^d} u_0(x) \geq 0$. In particular, if $\beta > 0$ then u converges to θ exponentially fast as $t \rightarrow \infty$ and uniformly in space.

If (3.2) holds with $c \geq \theta$ whereas $\|u_0\| \leq \theta$, then one can consider $u_2 \equiv \theta$ (that is a solution to (2.1)). Therefore, $\|u(\cdot, t)\| \leq \theta = \psi(t, \theta)$. And the same low estimate holds. Of course, for this case it is enough to have (3.2) with $c = \theta$ only. The latter case constitutes the basic assumptions for the most part of our further results. Let us formulate the corresponding statement as a proposition.

Proposition 3.4. *Suppose that the following assumptions hold:*

$$\varkappa^+ > m, \quad (\text{A1})$$

$$\varkappa^+ a^+(x) \geq (\varkappa^+ - m)a^-(x), \quad \text{a.a. } x \in \mathbb{R}^d. \quad (\text{A2})$$

Let $0 \leq u_0 \in C_{ub}(\mathbb{R}^d)$ be an initial condition to (2.1) and $u \in \mathcal{X}_T$ be the corresponding solutions on any $[0, T]$, $T > 0$. Suppose that $0 \leq u_0(x) \leq \theta$, $x \in \mathbb{R}^d$. Then $u \in \mathcal{X}_\infty$, with $\|u\|_\infty \leq \theta$.

Let $v_0 \in C_{ub}(\mathbb{R}^d)$ be another initial condition to (2.1) such that $u_0(x) \leq v_0(x) \leq \theta$, $x \in \mathbb{R}^d$; and $v \in \mathcal{X}_\infty$ be the corresponding solution. Then

$$u(x, t) \leq v(x, t), \quad x \in \mathbb{R}^d, t \geq 0.$$

If, additionally, $\beta := \inf_{x \in \mathbb{R}^d} u_0(x) > 0$, then

$$\frac{\beta\theta}{\beta + (\theta - \beta)\exp(-\theta\varkappa^-t)} \leq u(x, t) \leq \theta, \quad x \in \mathbb{R}^d, t \geq 0. \quad (3.16)$$

In particular,

$$\|u(\cdot, t) - \theta\| \leq \frac{\theta - \beta}{\beta} \exp(-\theta\varkappa^-t), \quad t \geq 0.$$

Proof. The first two parts were proved above; note that $\theta\varkappa^- = \varkappa^+ - m$. The last one is followed from the definition of the function ψ above and the estimate for the difference between low and upper bounds in (3.16). \square

Remark 3.5. The same result may be formulated for $\tilde{\mathcal{X}}_T$ and $\tilde{\mathcal{X}}_\infty$. All inequalities will hold true almost everywhere only.

We did not consider all possible relations between c , $\theta > 0$, and $\|u_0\|$. In particular, the previous-type considerations will not cover the case when (3.2) holds with $c < \theta$. In such a case, the solution to (2.19) (with $t_0 = 0$) can not be considered as a function u_2 in Theorem 3.1 since that solution tends to θ as $t \rightarrow \infty$, hence, (3.4) will not hold. This situation remains open.

Another case, which is not covered by the comparison method is when $\theta > 0$, i.e. (A1) holds, and $\|u_0\| > c$. However, it may be analyzed using stability arguments in the case when $c \geq \theta$, the latter naturally implies (A2). Under assumptions (A1), (A2), we set, cf. (3.19) below,

$$J_\theta(x) := \varkappa^+ a^+(x) - (\varkappa^+ - m)a^-(x) \geq 0, \quad x \in \mathbb{R}^d. \quad (3.17)$$

Next, denote the r.h.s. of (2.1) by $G(u)$. Recall, that $G(\theta) = 0$, hence, $u^* \equiv \theta$ is a stationary solution to (2.1). Another stationary solution is $u_* \equiv 0$. Consider the stability property of these solutions. To do this, find the linear operator $G'(u)$ on $C_{ub}(\mathbb{R}^d)$: for $v \in C_{ub}(\mathbb{R}^d)$,

$$\begin{aligned} G'(u)v &= \left. \frac{d}{ds} G(u + sv) \right|_{s=0} \\ &= \varkappa^+(a^+ * v) - mv - \varkappa^-v(a^- * u) - \varkappa^-u(a^- * v). \end{aligned} \quad (3.18)$$

Therefore, by (3.17),

$$G'(\theta)v = \varkappa^+(a^+ * v) - mv - \varkappa^-v - \varkappa^-\theta(a^- * v) = J_\theta * v - \varkappa^+v.$$

By (3.17), $\int_{\mathbb{R}^d} J_\theta(x) dx = m$, thus, the spectrum $\sigma(A)$ of the operator $Av := J_\theta * v$ on $C_{ub}(\mathbb{R}^d)$ is a subset of $\{z \in \mathbb{C} \mid |z| \leq m\}$. Therefore,

$$\sigma(G'(\theta)) = \sigma(A - \varkappa^+) \subset \{z \in \mathbb{C} \mid |z + \varkappa^+| \leq m\} \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\},$$

by (A1). Hence, by e.g. [22, Chapter VII], $u^* \equiv \theta$ is uniformly and asymptotically stable solution, in the sense of Lyapunov, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any solution $u \in C_{ub}(\mathbb{R}^d)$ to (2.1) and for all $t_1 \geq 0$, the inequality $\|u(\cdot, t_1) - \theta\| < \delta$ implies that, for any $t \geq t_1$, $\|u(\cdot, t) - \theta\| < \varepsilon$; and, for some $\delta_0 > 0$, the inequality $\|u(\cdot, t_1) - \theta\| < \delta_0$ yields $\lim_{t \rightarrow \infty} \|u(\cdot, t) - \theta\| = 0$. In particular, it works if $\theta < \|u_0\| \leq \theta + \delta_0$. Moreover, it is possible to show that $u^* \equiv \theta$ is a globally asymptotically (exponentially) stable solution to (2.1), that means, in particular, that $\|u_0\| > \theta$ may be arbitrary; we expect to discuss this in a forthcoming paper.

Note also, that, by (3.18), $G'(0)v = \varkappa^+(a^+ * v) - mv$. Then, the operator $G'(0)$ has an eigenvalue $\varkappa^+ - m > 0$ whose corresponding eigenfunctions will be constants on \mathbb{R}^d . Therefore $\sigma(G'(0))$ has points in the right half-plane and since $G'''(0)$ exists, one has, again by [22, Chapter VII], that $u_* \equiv 0$ is unstable, i.e. there exists a solution u such that $\inf_{\mathbb{R}^d} |u(x, t)| \geq \varepsilon$, for some $\varepsilon > 0$, for all $x \in \mathbb{R}^d$ and for all $t \geq t_0 = t_0(\varepsilon)$. It is worth noting that, $u_* \equiv 0$ is a locally stable solution, see Subsection 3.3 below.

The natural question arises whether it is possible to characterize some properties of the solution to (2.1) without comparison between a^+ and a^- like (3.2).

For a particular answer, one can refer to [33, Theorem 4.3], namely, there was proved that if (3.2) holds on a set $\Omega \subset \mathbb{R}^d$ of positive Lebesgue measure, and if $\int_{\Omega} (\varkappa^+ a^+(x) - c \varkappa^- a^-(x)) dx < m$, then $\|u_0\| < \theta$ implies $\|u(\cdot, t)\| \leq \theta$, $t > 0$.

On the other hand, let the condition (A2) (that is, recall, (3.2) with $c = \theta$) fail in some ball $B_r(y_0)$ only, $r > 0$, $y_0 \in \mathbb{R}^d$, i.e. $J_{\theta}(x) < 0$, for a.a. $x \in B_r(y_0)$, where J_{θ} is given by (3.17). Take any $y \in B_r(y_0)$ with $\frac{r}{4} < |y - y_0| < \frac{3r}{4}$, then $y_0 \notin B_{\frac{r}{4}}(y)$ whereas $B_{\frac{r}{4}}(y) \subset B_r(y_0)$. Take $u_0 \in C_{ub}(\mathbb{R}^d)$ such that $u_0(x) = \theta$, $x \in \mathbb{R}^d \setminus B_{\frac{r}{4}}(y)$, and $u_0(x) < \theta$, $x \in B_{\frac{r}{4}}(y)$. Since $\int_{\mathbb{R}^d} J_{\theta}(x) dx = m$, one has

$$\begin{aligned} \frac{\partial u}{\partial t}(y_0, 0) &= -m\theta + \varkappa^+(a^+ * u)(y_0, \theta) - \varkappa^-\theta(a^- * u)(y_0, 0) \\ &= (J_{\theta} * u)(y_0, 0) - m\theta = (J_{\theta} * (u_0 - \theta))(y_0) \\ &= \int_{B_{\frac{r}{4}}(y)} J_{\theta}(y_0 - x)(u_0(x) - \theta) dx > 0, \end{aligned}$$

Therefore, $u(y_0, t) > u(y_0, 0) = \theta$, for small enough $t > 0$, and hence, the statement of Proposition 3.4 does not hold in this case. The similar counterexample may be considered if (3.2) fails, for $c \geq \theta$. Note that the case $c < \theta$ is again unclear.

3.2 Maximum principle

The maximum principle is a ‘standard analogue’ of the comparison principle, see e.g. [15].

We will give sufficient conditions that solutions to (2.1) will never reach at positive times the stationary values θ and 0, provided that the corresponding initial conditions were not these constants.

Through the rest of the paper we will suppose that (A1), (A2) hold and $\theta > 0$ is given by (2.17). Under these assumptions, for any $q \in (0, \theta]$, one can generalize the function (3.17) as follows

$$\begin{aligned} J_q(x) &:= \varkappa^+ a^+(x) - q \varkappa^- a^-(x), \\ &\geq \varkappa^+ a^+(x) - \theta \varkappa^- a^-(x) \geq 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.19}$$

since $\theta \varkappa^- = \varkappa^+ - m$ and (A2) holds.

Definition 3.6. For $\theta > 0$, given by (2.17), consider the following sets

$$U_{\theta} := \{f \in C_{ub}(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta, x \in \mathbb{R}^d\}, \tag{3.20}$$

$$L_{\theta} := \{f \in L^{\infty}(\mathbb{R}^d) \mid 0 \leq f(x) \leq \theta, \text{ for a.a. } x \in \mathbb{R}^d\}. \tag{3.21}$$

We introduce also the following assumption:

$$\text{there exists } \rho, \delta > 0 \text{ such that } a^+(x) \geq \rho, \text{ for a.a. } x \in B_{\delta}(0). \tag{A3}$$

Prove that then the solutions to (2.1) (or, equivalently, (2.23)) are strictly positive; this is quite common feature of linear parabolic equations, however, in general, it may fail for nonlinear ones.

Proposition 3.7. *Let (A1), (A2), (A3) hold. Let $u_0 \in U_\theta$, $u_0 \not\equiv 0$, $u_0 \not\equiv \theta$, be the initial condition to (2.1), and $u \in \mathcal{X}_\infty$ be the corresponding solution. Then*

$$u(x, t) > \inf_{\substack{y \in \mathbb{R}^d \\ s > 0}} u(y, s) \geq 0, \quad x \in \mathbb{R}^d, t > 0.$$

Proof. By Theorem 2.2 and Proposition 3.4, $0 \leq u(x, t) \leq \theta$, $x \in \mathbb{R}^d$, $t \geq 0$. Then, by (2.23),

$$\frac{\partial u}{\partial t}(x, t) - (L_{a^+}u)(x, t) \geq 0. \quad (3.22)$$

Prove that, under (3.22), u cannot attain its infimum on $\mathbb{R}^d \times (0, \infty)$ without being a constant. Indeed, suppose that, for some $x_0 \in \mathbb{R}^d$, $t_0 > 0$,

$$u(x_0, t_0) \leq u(x, t), \quad x \in \mathbb{R}^d, t > 0. \quad (3.23)$$

Then, clearly,

$$\frac{\partial u}{\partial t}(x_0, t_0) = 0, \quad (3.24)$$

and (3.22) yields $(L_{a^+}u)(x_0, t_0) \leq 0$. On the other hand, (3.23) and (1.4) imply $(L_{a^+}u)(x_0, t_0) \geq 0$. Therefore,

$$\int_{\mathbb{R}^d} a^+(x_0 - y)(u(y, t_0) - u(x_0, t_0)) dy = 0. \quad (3.25)$$

Then, by (A3), for all $y \in B_\delta(x_0)$,

$$u(y, t_0) = u(x_0, t_0). \quad (3.26)$$

By the same arguments, for an arbitrary $x_1 \in \partial B_\delta(x_0)$, we obtain (3.26), for all $y \in B_\delta(x_1)$. Hence, (3.26) holds on $B_{2\delta}(x_0)$, and so on. As a result, (3.26) holds, for all $y \in \mathbb{R}^d$, thus $u(\cdot, t_0)$ is a constant. Then, considering (2.1) at (x_0, t_0) , and taking into account (3.24), one gets $u(x_0, t_0)(\theta - u(x_0, t_0)) = 0$ with $u(x, t_0) = u(x_0, t_0)$, $x \in \mathbb{R}^d$; cf. (2.19). By (3.23), $u(x_0, t_0) = \theta \geq \sup_{y \in \mathbb{R}^d, s > 0} u(y, s)$ implies $u \equiv \theta$, that contradicts $u_0 \not\equiv \theta$. Hence $u(x, t_0) = u(x_0, t_0) = 0$, $x \in \mathbb{R}^d$. Then, by (2.18), $u(x, t) = 0$, $x \in \mathbb{R}^d$, $t \geq t_0$. And now one can consider the reverse time in (2.1) starting from $t = t_0$. Namely, we set $w(x, t) := u(x, t_0 - t)$, $t \in [0, t_0]$, $x \in \mathbb{R}^d$. Then $w(x, 0) = v(t_0) = 0$, $x \in \mathbb{R}^d$, and

$$\frac{\partial w}{\partial t}(x, t) = mw(x, t) - \varkappa^+(a^+ * w)(x, t) + \varkappa^-w(x, t)(a^- * w)(x, t). \quad (3.27)$$

The equation (3.27) has a unique classical solution in $C_{ub}(\mathbb{R}^d)$ on $[0, t_0]$. Indeed, if $w_1, w_2 \in \mathcal{X}_{t_0}$ both solve (3.27), then the difference $w_2 - w_1$ is a solution to the following linear equation

$$\begin{aligned} \frac{\partial h}{\partial t}(x, t) &= mh(x, t) - \varkappa^+(a^+ * h)(x, t) \\ &+ \varkappa^-h(x, t)(a^- * w_2)(x, t) - \varkappa^-w_1(x, t)(a^- * h)(x, t), \end{aligned} \quad (3.28)$$

with $h(x, 0) = 0$, $x \in \mathbb{R}^d$. The r.h.s. of (3.28), for any $w_1, w_2 \in \mathcal{X}_{t_0}$, is a bounded linear operator on $C_{ub}(\mathbb{R}^d)$, therefore, there exists a unique solution to (3.28), hence, $h \equiv 0$. As a result, $w_1 \equiv w_2$. Since $w \equiv 0$ satisfies (3.27) with the initial condition above, one has $u(x, t_0 - t) = 0$, $t \in [0, t_0]$, $x \in \mathbb{R}^d$. Hence, $u(\cdot, t) \equiv 0$, for all $t \geq 0$, that contradicts $u_0 \not\equiv 0$. Thus, the initial assumption was wrong, and (3.23) can not hold. \square

In contrast to the case of the infimum, the solution to (2.1) may attain its supremum but not the value θ . One can prove this under a modified version of (A3): suppose that, cf. (3.19),

$$\begin{aligned} & \text{there exists } \rho, \delta > 0, \text{ such that} \\ J_\theta(x) &= \varkappa^+ a^+(x) - (\varkappa^+ - m)a^-(x) \geq \rho, \text{ for a.a. } x \in B_\delta(0). \end{aligned} \quad (\text{A4})$$

As a matter of fact, under (A4), a much stronger statement than unattainability of θ does hold.

Theorem 3.8. *Let (A1), (A2), (A4) hold. Let $u_1, u_2 \in \mathcal{X}_\infty$ be two solutions to (2.1), such that $u_1(x, t) \leq u_2(x, t) \leq \theta$, $x \in \mathbb{R}^d$, $t \geq 0$. Then either $u_1(x, t) = u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$ or $u_1(x, t) < u_2(x, t)$, $x \in \mathbb{R}^d$, $t > 0$.*

Proof. Let $u_1(x, t) \leq u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$, and suppose that there exist $t_0 > 0$, $x_0 \in \mathbb{R}^d$, such that $u_1(x_0, t_0) = u_2(x_0, t_0)$. Define $w := u_2 - u_1 \in \mathcal{X}_\infty$. Then $w(x, t) \geq 0$ and $w(x_0, t_0) = 0$, hence $\frac{\partial}{\partial t} w(x_0, t_0) = 0$. Since both u_1 and u_2 solve (2.1), one easily gets that w satisfies the following linear equation

$$\begin{aligned} \frac{\partial}{\partial t} w(x, t) &= (J_\theta * w)(x, t) + \varkappa^-(\theta - u_1(x, t))(a^- * w)(x, t) \\ &\quad - w(x, t)(m + (a^- * u_2)(x, t)); \end{aligned} \quad (3.29)$$

or, at the point (x_0, t_0) , we will have

$$0 = (J_\theta * w)(x_0, t_0) + \varkappa^-(\theta - u_1(x_0, t_0))(a^- * w)(x_0, t_0). \quad (3.30)$$

Since the both summands in (3.30) are nonnegative, one has $(J_\theta * w)(x_0, t_0) = 0$. Then, by (A4), we have that $w(x, t_0) = 0$, for all $x \in B_\delta(x_0)$. Using the same arguments as in the proof of Proposition 3.7, one gets that $w(x, t_0) = 0$, $x \in \mathbb{R}^d$. Then, by Corollary 2.4, $w(x, t) = 0$, $x \in \mathbb{R}^d$, $t \geq t_0$. Finally, one can reverse the time in the linear equation (3.29) (cf. the proof of Proposition 3.7), and the uniqueness arguments imply that $w \equiv 0$, i.e. $u_1(x, t) = u_2(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$. The statement is proved. \square

By choosing $u_2 \equiv \theta$ in the Theorem 3.8, we immediately get the following

Corollary 3.9. *Let (A1), (A2), (A4) hold. Let $u_0 \in U_\theta$, $u_0 \not\equiv \theta$, be the initial condition to (2.1), and $u \in \mathcal{X}_\infty$ be the corresponding solution. Then $u(x, t) < \theta$, $x \in \mathbb{R}^d$, $t > 0$.*

3.3 Local stability

We will prove now that any solution to (2.1) is locally stable in a certain sense, provided that (3.2) holds. This stability is very ‘weak’, for example, $u_* \equiv 0$, being unstable solution (see Subsection 3.1 above), will be still locally stable.

Theorem 3.10. *Let (A1), (A2) hold. Let $T > 0$ be fixed. Consider a sequence of functions $u_n \in \mathcal{X}_T$ which are solutions to 2.1 with uniformly bounded initial conditions: $u_n(\cdot, 0) \in U_\theta$, $n \in \mathbb{N}$. Let $u \in \mathcal{X}_T$ be a solution to (2.1) with initial condition $u(\cdot, 0)$ such that $u_n(\cdot, 0) \xrightarrow{\text{loc}} u(\cdot, 0)$. Then $u_n(\cdot, t) \xrightarrow{\text{loc}} u(\cdot, t)$, uniformly in $t \in [0, T]$.*

Proof. It is easily seen that $u(\cdot, 0) \in U_\theta$. By Proposition 3.4, $u_n(\cdot, t), u(\cdot, t) \in U_\theta$, $n \in \mathbb{N}$, for any $t \geq 0$. We define, for any $n \in \mathbb{N}$, the following functions on \mathbb{R}^d :

$$\bar{u}_n(x, 0) := \max \{u_n(x, 0), u(x, 0)\}, \quad \underline{u}_n(x, 0) := \min \{u_n(x, 0), u(x, 0)\}.$$

Then, clearly, $0 \leq \underline{u}_n(x, 0) \leq u(x, 0) \leq \bar{u}_n(x, 0) \leq \theta$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. Hence the corresponding solutions $\bar{u}_n(x, t), \underline{u}_n(x, t)$ to (2.1) belongs to U_θ as well. By Theorem 3.4, one has

$$\underline{u}_n(x, t) \leq u(x, t) \leq \bar{u}_n(x, t), \quad x \in \mathbb{R}^d, t \in [0, T].$$

In the same way, one gets $\underline{u}_n(x, t) \leq u_n(x, t) \leq \bar{u}_n(x, t)$ on $\mathbb{R}^d \times [0, T]$. Therefore, it is enough to prove that \bar{u}_n and \underline{u}_n converge locally uniformly to u .

Prove that $\bar{u}_n \xrightarrow{\text{loc}} u$. For any $n \in \mathbb{N}$, the function $h_n(\cdot, t) = \bar{u}_n(\cdot, t) - u(\cdot, t) \in U_\theta$, $t \geq 0$, satisfies the equation $\frac{\partial}{\partial t} h_n = A_n h_n$ with $h_{n,0}(x) := h_n(x, 0) = \bar{u}_n(x, 0) - u(x, 0) \geq 0$, $x \in \mathbb{R}^d$, where, for any $0 \leq h \in \mathcal{X}_T$,

$$A_n h := -mh + \varkappa^+(a^+ * h) - \varkappa^- h(a^- * \bar{u}_n) - \varkappa^- u(a^- * h).$$

For any u_n and u , A_n is a bounded linear operator on $C_{ub}(\mathbb{R}^d)$, therefore, $h_n(x, t) = (e^{tA_n} h_{n,0})(x)$, $x \in \mathbb{R}^d$, $t \in [0, T]$. Since $u \geq 0$, one has that, for any $0 \leq h \in \mathcal{X}_T$, $(A_n h)(x, t) \leq (Ah)(x, t)$, $x \in \mathbb{R}^d$, $t \in [0, T]$, where a bounded linear operator A is given on $C_{ub}(\mathbb{R}^d)$ by

$$Ah := \varkappa^+(a^+ * h) - \varkappa^- u(a^- * h).$$

Next, the series expansions for e^{tA_n} and e^{tA} converge in the topology of norms of operator on the space $C_{ub}(\mathbb{R}^d)$. Then, for any $n \in \mathbb{N}$, and for $x \in \mathbb{R}^d$, $t \in [0, T]$,

$$h_n(x, t) = (e^{tA_n} h_{n,0})(x) \leq (e^{tA} h_{n,0})(x) = \sum_{m=0}^{\infty} \frac{T^m}{m!} A^m h_{n,0}, \quad (3.31)$$

and, moreover, for any $\varepsilon > 0$ one can find $M = M(\varepsilon) \in \mathbb{N}$, such that we get from (3.31) that

$$h_n(x, t) \leq \sum_{m=0}^M \frac{T^m}{m!} A^m h_{n,0}(x) + \varepsilon\theta, \quad x \in \mathbb{R}^d, t \in [0, T]. \quad (3.32)$$

as $h_{n,0} \in U_\theta$, $n \in \mathbb{N}$. Finally, the assumptions of the statement yield that $h_{n,0} \xrightarrow{\text{loc}} 0$. Then, by (3.31) and Lemma 2.6, $h_n(x, t) \xrightarrow{\text{loc}} 0$ uniformly in $t \in [0, T]$. Hence, $\bar{u}_n \xrightarrow{\text{loc}} u$ uniformly on $[0, T]$. The convergence $\underline{u}_n \xrightarrow{\text{loc}} u$ may be proved by an analogy. \square

Remark 3.11. An analogous statement holds in the space $\tilde{\mathcal{X}}_T$, $T > 0$.

In the case of measurable bounded functions, cf. Remark 3.11, we will need also a weaker form of the local stability above.

Proposition 3.12. *Let (A1), (A2) hold. Let $T > 0$ be fixed. Consider a sequence of functions $u_n \in \tilde{\mathcal{X}}_T$ which are solutions to 2.1 with uniformly bounded initial conditions: $u_n(\cdot, 0) \in L_\theta$, $n \in \mathbb{N}$. Let $u \in \tilde{\mathcal{X}}_T$ be a solution to 2.1 with initial condition $u(\cdot, 0)$ such that $u_n(x, 0) \rightarrow u(x, 0)$, for a.a. $x \in \mathbb{R}^d$. Then $u_n(x, t) \rightarrow u(x, t)$, for a.a. $x \in \mathbb{R}^d$, uniformly in $t \in [0, T]$.*

Proof. The proof will be fully analogous to that for Theorem 3.10 until the inequality (3.32), in which $M = M(\varepsilon, x)$ now. The rest of the proof is the same, taking into account that an analogue of Lemma 2.6 with both convergences almost everywhere holds true by the dominated convergence theorem. \square

3.4 Further toolkits

In the sequel, it will be sometimes useful to consider the solution to (2.1) as a nonlinear transformation of the initial condition.

Definition 3.13. For a fixed $T > 0$, define the mapping Q_T on $L_+^\infty(\mathbb{R}^d) := \{f \in L^\infty(\mathbb{R}^d) \mid f \geq 0 \text{ a.e.}\}$, as follows

$$(Q_T f)(x) := u(x, T), \quad x \in \mathbb{R}^d, \quad (3.33)$$

where $u(x, t)$ is the solution to (2.1) with the initial condition $u(x, 0) = f(x)$.

Let us collect several properties of Q_T needed below.

Proposition 3.14. *Let (A1), (A2) hold. The mapping $Q = Q_T : L_+^\infty(\mathbb{R}^d) \rightarrow L_+^\infty(\mathbb{R}^d)$ satisfies the following properties*

(Q1) $Q : L_\theta \rightarrow L_\theta$, $Q : U_\theta \rightarrow U_\theta$,

(Q2) let $T_y : L_+^\infty(\mathbb{R}^d) \rightarrow L_+^\infty(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, be a translation operator, given by

$$(T_y f)(x) = f(x - y), \quad x \in \mathbb{R}^d; \quad (3.34)$$

then

$$(QT_y f)(x) = (T_y Qf)(x), \quad x, y \in \mathbb{R}^d, \quad (3.35)$$

(Q3) $Q0 = 0$, $Q\theta = \theta$, and $Qr > r$, for any constant $r \in (0, \theta)$,

(Q4) if $f(x) \leq g(x)$, for a.a. $x \in \mathbb{R}^d$, then $(Qf)(x) \leq (Qg)(x)$, for a.a. $x \in \mathbb{R}^d$;

(Q5) if $f_n \xrightarrow{\text{loc}} f$, then $(Qf_n)(x) \rightarrow (Qf)(x)$, for a.a. $x \in \mathbb{R}^d$.

Proof. The property (Q1) follows from Remark 3.5 and Proposition 3.4. To prove (Q2) we note that, by (2.3), $T_y(a^\pm * u) = a^\pm * (T_y u)$, and then, by (2.7), $B(T_y v) = T_y(Bv)$, therefore, by (2.6), if $\tau = 0$ and $u_\tau = T_y f$, then $\Phi_\tau T_y = T_y \Phi$, where Φ is given by (2.6) with f in place of u_τ only. As a result, $\Phi_\tau^n T_y = T_y \Phi^n$ hence

$$Q_T(T_y f) = \lim_{n \rightarrow \infty} \Phi_\tau^n T_y f = \lim_{n \rightarrow \infty} T_y \Phi^n f = T_y(Q_T f);$$

and one can continue the same considerations on the next time-interval. The property (Q3) is a straightforward consequence of Corollary 2.4; indeed, (2.18) implies, for $\alpha_T := \exp(-\theta\kappa^-T) \in (0, 1)$,

$$Q_T r - r = \frac{\theta r}{r(1 - \alpha_T) + \theta \alpha_T} - r = \frac{r(\theta - r)(1 - \alpha_T)}{r(1 - \alpha_T) + \theta \alpha_T} > 0.$$

The property (Q4) holds also by Remark 3.5 and Proposition 3.4. The property (Q5) is a weaker version of Remark 3.11 and Proposition 3.12. \square

Let S^{d-1} denotes a unit sphere in \mathbb{R}^d centered at the origin:

$$S^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}; \quad (3.36)$$

in particular, $S^0 = \{-1, 1\}$.

Definition 3.15. A function $f \in L^\infty(\mathbb{R}^d)$ is said to be increasing (decreasing, constant) along the vector $\xi \in S^{d-1}$ if, for a.a. $x \in \mathbb{R}^d$, the function $f(x + s\xi) = (T_{-s\xi}f)(x)$ is increasing (decreasing, constant) in $s \in \mathbb{R}$, respectively.

Proposition 3.16. *Let (A1), (A2) hold. Let $u_0 \in L_\theta$ be the initial condition for the equation (2.1) which is increasing (decreasing, constant) along a vector $\xi \in S^{d-1}$; and $u(\cdot, t) \in L_\theta$, $t \geq 0$, be the corresponding solution (cf. Proposition 3.4 and Remark 3.5). Then, for any $t > 0$, $u(\cdot, t)$ is increasing (decreasing, constant, respectively) along the ξ .*

Proof. Let u_0 be decreasing along a $\xi \in S^{d-1}$. Take any $s_1 \leq s_2$ and consider two initial conditions to (2.1): $u_0^i(x) = u_0(x + s_i\xi) = (T_{-s_i\xi}u_0)(x)$, $i = 1, 2$. Since u_0 is decreasing, $u_0^1(x) \geq u_0^2(x)$, $x \in \mathbb{R}^d$. Then, by Proposition 3.14,

$$T_{-s_1\xi}Q_t u_0 = Q_t T_{-s_1\xi} u_0 = Q_t u_0^1 \geq Q_t u_0^2 = Q_t T_{-s_2\xi} u_0 = T_{-s_2\xi} Q_t u_0,$$

that proves the statement. The cases of a decreasing u_0 can be considered in the same way. The constant function along a vector is decreasing and decreasing simultaneously. \square

For the sequel, we need also to show that any solution to (2.1) is bounded from below by a solution to the corresponding equation with ‘truncated’ kernels a^\pm . Namely, suppose that the conditions (A1), (A2) hold. Consider a family of Borel sets $\{\Delta_R \mid R > 0\}$, such that $\Delta_R \nearrow \mathbb{R}^d$, $R \rightarrow \infty$. Define, for any $R > 0$, the following kernels:

$$a_R^\pm(x) = \mathbb{1}_{\Delta_R}(x) a^\pm(x), \quad x \in \mathbb{R}^d, \quad (3.37)$$

and the corresponding ‘truncated’ equation, cf. (2.1),

$$\begin{cases} \frac{\partial w}{\partial t}(x, t) = \kappa^+(a_R^+ * w)(x, t) - mw(x, t) \\ \quad - \kappa^-(a_R^- * w)(x, t), & x \in \mathbb{R}^d, t > 0, \\ w(x, 0) = w_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (3.38)$$

We set

$$A_R^\pm := \int_{\Delta_R} a^\pm(x) dx \nearrow 1, \quad R \rightarrow \infty, \quad (3.39)$$

by (2.2). Then the non-zero constant solution to (3.38) is equal to

$$\theta_R = \frac{\varkappa^+ A_R^+ - m}{\varkappa^- A_R^-} \rightarrow \theta, \quad R \rightarrow \infty, \quad (3.40)$$

however, the convergence θ_R to θ is, in general, not monotonic. Clearly, by (A1), $\theta_R > 0$ if only

$$A_R^+ > \frac{m}{\varkappa^+} \in (0, 1). \quad (3.41)$$

Proposition 3.17. *Let (A1), (A2) hold, and $R > 0$ be such that (3.41) holds, cf. (3.39). Let $w_0 \in C_{ub}(\mathbb{R}^d)$ be such that $0 \leq w_0(x) \leq \theta_R$, $x \in \mathbb{R}^d$. Then there exists the unique solution $w \in \mathcal{X}_\infty$ to (3.38), such that*

$$0 \leq w(x, t) \leq \theta_R, \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.42)$$

Let $u_0 \in U_\theta$ and $u \in \mathcal{X}_\infty$ be the corresponding solution to (2.1). If $w_0(x) \leq u_0(x)$, $x \in \mathbb{R}^d$, then

$$w(x, t) \leq u(x, t), \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.43)$$

Proof. Denote $\Delta_R^c := \mathbb{R}^d \setminus \Delta_R$. We have

$$\begin{aligned} \theta - \theta_R &= \frac{\varkappa^+ A_R^- - m A_R^- - \varkappa^+ A_R^+ + m}{\varkappa^- A_R^-} = \frac{\varkappa^+(1 - A_R^+) - (\varkappa^+ - m)(1 - A_R^-)}{\varkappa^- A_R^-} \\ &= \frac{1}{\varkappa^- A_R^-} \int_{\Delta_R^c} (\varkappa^+ a^+(x) - (\varkappa^+ - m)a^-(x)) dx \geq 0, \end{aligned}$$

by (A2). Therefore,

$$0 < \theta_R \leq \theta. \quad (3.44)$$

Clearly, (A2) and (3.44) yield

$$\varkappa^+ a_R^+(x) \geq \theta_R \varkappa^- a_R^-(x), \quad x \in \mathbb{R}^d. \quad (3.45)$$

Thus one can apply Proposition 3.4 to the equation (3.38) using trivial equalities $a_R^\pm(x) = A_R^\pm \tilde{a}_R^\pm(x)$, where the kernels $\tilde{a}_R^\pm(x) = (A_R^\pm)^{-1} a_R^\pm(x)$ are normalized, cf. (2.2); and the inequality (3.45) is the corresponding analog of (A2), according to (3.40). This proves the existence and uniqueness of the solution to (3.38) and the bound (3.42).

Next, for \mathcal{F} given by (3.1), one gets from (3.37) and (3.38), that the solution w to (3.38) satisfies the following equality

$$\begin{aligned} (\mathcal{F}w)(x, t) &= -\varkappa^+ \int_{\Delta_R^c} a^+(y)w(x-y, t) dy \\ &\quad + \varkappa^- w(x, t) \int_{\Delta_R^c} a^-(y)w(x-y, t) dy. \end{aligned} \quad (3.46)$$

By (3.42), (3.44), (A2), one gets from (3.46) that

$$\begin{aligned} (\mathcal{F}w)(x, t) &\leq -\varkappa^+ \int_{\Delta_R^c} a^+(y)w(x-y, t) dy + \varkappa^- \theta \int_{\Delta_R^c} a^-(y)w(x-y, t) dy \\ &\leq 0 = (\mathcal{F}u)(x, t), \end{aligned}$$

where u is the solution to (2.1). Therefore, we may apply Theorem 3.1 to get the statement. \square

Remark 3.18. The statements of Proposition 3.17 remains true for the functions from $L^\infty(\mathbb{R}^d)$ (the inequalities will hold a.e. only).

4 Traveling waves

Traveling waves were studied intensively for the original Fisher–KPP equation (1.6), see, e.g., [6]; for locally nonlinear equation with nonlocal diffusion (1.11), see, e.g., [17, 80]; and for nonlocal nonlinear equation with local diffusion (1.9), see, e.g., [3, 7, 45, 58].

Through this section we will mainly work in L^∞ -setting, see Remarks 2.3, 3.2, 3.5, 3.11 above. Recall that we will always assume that (A1) and (A2) hold, and $\theta > 0$ is given by (2.17).

4.1 Existence and properties of traveling waves

Definition 4.1. Let $\mathcal{M}_\theta(\mathbb{R})$ denote the set of all decreasing and right-continuous functions $f : \mathbb{R} \rightarrow [0, \theta]$.

Remark 4.2. There is a natural embedding of $\mathcal{M}_\theta(\mathbb{R})$ into $L^\infty(\mathbb{R})$. According to this, for a function $f \in L^\infty(\mathbb{R})$, the inclusion $f \in \mathcal{M}_\theta(\mathbb{R})$ means that there exists $g \in \mathcal{M}_\theta(\mathbb{R})$, such that $f = g$ a.s. on \mathbb{R} .

Definition 4.3. Let $\tilde{\mathcal{X}}_\infty^1 := \tilde{\mathcal{X}}_\infty \cap C^1((0, \infty) \rightarrow L^\infty(\mathbb{R}^d))$. A function $u \in \tilde{\mathcal{X}}_\infty^1$ is said to be a traveling wave solution to the equation (2.1) with a speed $c \in \mathbb{R}$ and in a direction $\xi \in S^{d-1}$ if and only if (iff, in the sequel) there exists a function $\psi \in \mathcal{M}_\theta(\mathbb{R})$, such that

$$\begin{aligned} \psi(-\infty) &= \theta, & \psi(+\infty) &= 0 \\ u(x, t) &= \psi(x \cdot \xi - ct), & t \geq 0, \text{ a.a. } x \in \mathbb{R}^d. \end{aligned} \tag{4.1}$$

Here and below S^{d-1} is defined by (3.36) and $x \cdot y = (x, y)_{\mathbb{R}^d}$ is the scalar product in \mathbb{R}^d . The function ψ is said to be the profile for the traveling wave, whereas c is its speed.

We will use some ideas and results from [80].

With necessity, to have a traveling wave solution to (2.1) one should assume that

$$u_0(x) = \psi(x \cdot \xi), \tag{4.2}$$

for some $\xi \in S^{d-1}$, $\psi \in \mathcal{M}_\theta(\mathbb{R})$. In such a case, the solution will have a special form as well.

Proposition 4.4. *Let $\xi \in S^{d-1}$, $\psi \in \mathcal{M}_\theta(\mathbb{R})$, and an initial condition to (2.1) be given by $u_0(x) = \psi(x \cdot \xi)$, a.a. $x \in \mathbb{R}^d$; let also $u \in \tilde{\mathcal{X}}_\infty^+$ be the corresponding solution. Then there exist a function $\phi : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, \theta]$, such that $\phi(\cdot, t) \in \mathcal{M}_\theta(\mathbb{R})$, for any $t \geq 0$, and*

$$u(x, t) = \phi(x \cdot \xi, t), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d. \tag{4.3}$$

Moreover, there exist functions \check{a}^\pm (depending on ξ) on \mathbb{R} with $0 \leq \check{a}^\pm \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} \check{a}^\pm(s) ds = 1$, such that ϕ is a solution to the following one-dimensional version of (2.1):

$$\begin{cases} \frac{\partial \phi}{\partial t}(s, t) = \varkappa^+(\check{a}^+ * \phi)(s, t) - m\phi(s, t) \\ \quad - \varkappa^-\phi(s, t)(\check{a}^- * \phi)(s, t), & t > 0, \text{ a.a. } s \in \mathbb{R}, \\ \phi(s, 0) = \psi(s), & \text{a.a. } s \in \mathbb{R}. \end{cases} \quad (4.4)$$

Proof. Choose any $\eta \in S^{d-1}$ which is orthogonal to the ξ . Then the initial condition u_0 is constant along η , indeed, for any $s \in \mathbb{R}$,

$$u_0(x + s\eta) = \psi((x + s\eta) \cdot \xi) = \psi(x \cdot \xi) = u_0(x), \quad \text{a.a. } x \in \mathbb{R}^d.$$

Then, by Proposition 3.16, for any fixed $t > 0$, the solution $u(\cdot, t)$ is constant along η as well. Next, for any $\tau \in \mathbb{R}$, there exists $x \in \mathbb{R}^d$ such that $x \cdot \xi = \tau$; and, clearly, if $y \cdot \xi = \tau$ then $y = x + s\eta$, for some $s \in \mathbb{R}$ and some η as above. Therefore, if we just set, for a.a. $x \in \mathbb{R}^d$, $\phi(\tau, t) := u(x, t)$, $t \geq 0$, this definition will be correct a.e. in $\tau \in \mathbb{R}$; and it will give (4.3). Next, for a.a. fixed $x \in \mathbb{R}^d$, $u_0(x + s\xi) = \psi(x \cdot \xi + s)$ is decreasing in s , therefore, u_0 is decreasing along the ξ , and by Proposition 3.16, $u(\cdot, t)$, $t \geq 0$, will be decreasing along the ξ as well. The latter means that, for any $s_1 \leq s_2$, we have, by (4.3),

$$\phi(x \cdot \xi + s_1, t) = u(x + s_1\xi, t) \geq u(x + s_2\xi, t) = \phi(x \cdot \xi + s_2, t),$$

and one can choose in the previous any x which is orthogonal to ξ to prove that ϕ is decreasing in the first coordinate.

To prove the second statement, for $d \geq 2$, choose any $\{\eta_1, \eta_2, \dots, \eta_{d-1}\} \subset S^{d-1}$ which form a complement of $\xi \in S^{d-1}$ to an orthonormal basis in \mathbb{R}^d . Then, for a.a. $x \in \mathbb{R}^d$, with $x = \sum_{j=1}^{d-1} \tau_j \eta_j + s\xi$, $\tau_1, \dots, \tau_{d-1}, s \in \mathbb{R}$, we have (using an analogous expansion of y inside the integral below and taking into account that any linear transformation of orthonormal bases preserves volumes)

$$\begin{aligned} (a^\pm * u)(x, t) &= \int_{\mathbb{R}^d} a^\pm(y) u(x - y, t) dy \\ &= \int_{\mathbb{R}^d} a^\pm \left(\sum_{j=1}^{d-1} \tau'_j \eta_j + s' \xi \right) u \left(\sum_{j=1}^{d-1} (\tau_j - \tau'_j) \eta_j + (s - s') \xi, t \right) d\tau'_1 \dots d\tau'_{d-1} ds' \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} a^\pm \left(\sum_{j=1}^{d-1} \tau'_j \eta_j + s' \xi \right) d\tau'_1 \dots d\tau'_{d-1} \right) u((s - s') \xi, t) ds', \end{aligned} \quad (4.5)$$

where we used again Proposition 3.16 to show that u is constant along the vector $\eta = \sum_{j=1}^{d-1} (\tau_j - \tau'_j) \eta_j$ which is orthogonal to the ξ .

Therefore, one can set

$$\check{a}^\pm(s) := \begin{cases} \int_{\mathbb{R}^{d-1}} a^\pm(\tau_1 \eta_1 + \dots + \tau_{d-1} \eta_{d-1} + s\xi) d\tau_1 \dots d\tau_{d-1}, & d \geq 2, \\ a^\pm(s\xi), & d = 1. \end{cases} \quad (4.6)$$

It is easily seen that $\check{a}^\pm = \check{a}_\xi^\pm$ does not depend on the choice of $\eta_1, \dots, \eta_{d-1}$, which constitute a basis in the space $H_\xi := \{x \in \mathbb{R}^d \mid x \cdot \xi = 0\} = \{\xi\}^\perp$. Note

that, clearly,

$$\int_{\mathbb{R}} \check{a}^{\pm}(s) ds = \int_{\mathbb{R}^d} a^{\pm}(y) dy = 1. \quad (4.7)$$

Next, by (4.3), $u((s-s')\xi, t) = \phi(s-s', t)$, therefore, (4.5) may be rewritten as

$$(a^{\pm} * u)(x, t) = \int_{\mathbb{R}} \check{a}^{\pm}(s') \phi(s-s', t) ds' =: (\check{a}^{\pm} * \phi)(s, t),$$

where $s = x \cdot \xi$. The rest of the proof is obvious now. \square

Remark 4.5. By Proposition 4.4, the solution u has always the form (4.3) provided that (4.2) holds, for some $\psi \in \mathcal{M}_{\theta}(\mathbb{R})$ and some $\xi \in S^{d-1}$. As a matter of fact, to prove the existence of traveling waves we will need to show that there exist functions $\psi \in \mathcal{M}_{\theta}(\mathbb{R})$ such that

$$\phi(s, t) = \psi(s - ct). \quad (4.8)$$

According to the second statement of Proposition 4.4, it means that the one-dimensional version (4.4) of the initial equation (2.1) should have a traveling wave solution with the profile ψ and the speed c given by (4.8). If such solution is found and if the initial condition to (2.1) is given by (4.2), then (4.1) holds.

Remark 4.6. One can realize all previous considerations for increasing traveling wave, increasing solution along a vector ξ etc. Indeed, it is easily seen that the function $\tilde{u}(x, t) = u(-x, t)$ with the initial condition $\tilde{u}_0(x) = u_0(-x)$ is a solution to the equation (2.1) with a^{\pm} replaced by $\tilde{a}^{\pm}(x) = a^{\pm}(-x)$; note that $(a^{\pm} * u)(-x, t) = (\tilde{a}^{\pm} * \tilde{u})(x, t)$.

Remark 4.7. It is a straightforward application of (3.35), that if $\psi \in \mathcal{M}_{\theta}(\mathbb{R})$, $c \in \mathbb{R}$ gets (4.1) then, for any $s \in \mathbb{R}$, $\psi(\cdot + s)$ is a traveling wave to (2.1) with the same c .

We will need also the following simple statement.

Proposition 4.8. *Let (A1), (A2) hold and $\xi \in S^{d-1}$ be fixed. Define, for an arbitrary $T > 0$, the mapping $\tilde{Q}_T : L_+^{\infty}(\mathbb{R}) \rightarrow L_+^{\infty}(\mathbb{R})$ as follows: $\tilde{Q}_T \psi(s) = \phi(s, T)$, $s \in \mathbb{R}$, where $\phi : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, \theta]$ solves (4.4) with $\psi \in L_+^{\infty}(\mathbb{R})$. Then such \tilde{Q}_T is well-defined, satisfies all properties of Proposition 3.14 (with $d = 1$), and, moreover, $\tilde{Q}_T(\mathcal{M}_{\theta}(\mathbb{R})) \subset \mathcal{M}_{\theta}(\mathbb{R})$.*

Proof. Consider one-dimensional equation (4.4), where \check{a}^{\pm} are given by (4.6). The latter equality together with (A2) imply that

$$\varkappa^+ \check{a}^+(s) \geq (\varkappa^+ - m) \check{a}^-(s), \quad \text{a.a. } s \in \mathbb{R}. \quad (4.9)$$

Therefore, all previous results (e.g. Theorem 2.2) hold true for the solution to (4.4) as well. In particular, all statements of Proposition 3.14 hold true, for $Q = \tilde{Q}_T$, $d = 1$. Moreover, by the proof of Theorem 2.2 (in the L^{∞} -case, cf. Remark 2.3), since the mappings B and Φ_{τ} , cf. (2.7), (2.6), map the set $\mathcal{M}_{\theta}(\mathbb{R})$ into itself, we have that \tilde{Q}_T has this property as well, cf. Remark 4.2. \square

Now we are going to prove the existence of the traveling wave solution to (2.1). Denote, for any $\lambda > 0$, $\xi \in S^{d-1}$,

$$\mathbf{a}_\xi(\lambda) := \int_{\mathbb{R}^d} a^+(x) e^{\lambda x \cdot \xi} dx \in [0, \infty]. \quad (4.10)$$

For a given $\xi \in S^{d-1}$, consider the following assumption on a^+ :

$$\text{there exists } \mu = \mu(\xi) > 0 \text{ such that } \mathbf{a}_\xi(\mu) < \infty. \quad (\text{A5})$$

Theorem 4.9. *Let (A1) and (A2) hold and $\xi \in S^{d-1}$ be fixed. Suppose also that (A5) holds. Then there exists $c_*(\xi) \in \mathbb{R}$ such that*

- 1) *for any $c \geq c_*(\xi)$, there exists a traveling wave solution, in the sense of Definition 4.3, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and the speed c ,*
- 2) *for any $c < c_*(\xi)$, such a traveling wave does not exist.*

Proof. Let $\mu > 0$ be such that (A5) holds. Then, by (4.6),

$$\begin{aligned} \int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} ds &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} a^\pm(\tau_1 \eta_1 + \dots + \tau_{d-1} \eta_{d-1} + s\xi) e^{\mu s} d\tau_1 \dots d\tau_{d-1} ds \\ &= \mathbf{a}_\xi(\mu) < \infty. \end{aligned} \quad (4.11)$$

Clearly, the integral equality in (4.11) holds true for any $\lambda \in \mathbb{R}$ as well, with $\mathbf{a}_\xi(\lambda) \in [0, \infty]$.

Let $\mu > 0$ be such that (A5) holds. Define a function from $\mathcal{M}_\theta(\mathbb{R})$ by

$$\varphi(s) := \theta \min\{e^{-\mu s}, 1\}. \quad (4.12)$$

Let us prove that there exists $c \in \mathbb{R}$ such that $\bar{\phi}(s, t) := \varphi(s - ct)$ is a supersolution to (4.4), i.e.

$$\mathcal{F}\bar{\phi}(s, t) \geq 0, \quad s \in \mathbb{R}, t \geq 0, \quad (4.13)$$

where \mathcal{F} is given by (3.1) (in the case $d = 1$). We have

$$\begin{aligned} (\mathcal{F}\bar{\phi})(s, t) &= -c\varphi'(s - ct) - \varkappa^+(\check{a}^+ * \varphi)(s - ct) + m\varphi(s - ct) \\ &\quad + \varkappa^-\varphi(s - ct)(\check{a}^- * \varphi)(s - ct), \end{aligned}$$

hence the sufficient condition for (4.13) will be

$$\mathcal{J}_c(s) := c\varphi'(s) + \varkappa^+(\check{a}^+ * \varphi)(s) - m\varphi(s) - \varkappa^-\varphi(s)(\check{a}^- * \varphi)(s) \leq 0, \quad (4.14)$$

for $s \in \mathbb{R}$. By (4.12), (4.9), for $s < 0$, we have

$$\mathcal{J}_c(s) = \varkappa^+(\check{a}^+ * \varphi)(s) - m\theta - \varkappa^-\theta(\check{a}^- * \varphi)(s) \leq ((\varkappa^+ \check{a}^- - \varkappa^- \theta \check{a}^-) * \theta)(s) - m\theta = 0.$$

Next, by (4.12),

$$(\check{a}^+ * \varphi)(s) \leq \int_{\mathbb{R}} \check{a}^+(\tau) e^{-\mu(s-\tau)} d\tau = e^{-\mu s} \mathbf{a}_\xi(\mu),$$

therefore, for $s \geq 0$, we have

$$\mathcal{J}_c(s) \leq -\mu c \theta e^{-\mu s} + \Theta e^{-\mu s} \mathbf{a}_\xi(\mu) - m\theta e^{-\mu s};$$

and to get (4.14) it is enough to demand that $\varkappa^+ \mathbf{a}_\xi(\mu) - m - \mu c \leq 0$, in particular,

$$c = \frac{\varkappa^+ \mathbf{a}_\xi(\mu) - m}{\mu}. \quad (4.15)$$

As a result, for $\bar{\phi}(s, t) = \varphi(s - ct)$ with c given by (4.15), we have

$$\mathcal{F}\bar{\phi} \geq 0 = \mathcal{F}(\tilde{Q}_t \varphi), \quad (4.16)$$

as $\tilde{Q}_t \varphi$ is a solution to (4.4). Then, by (A2) and the inequality $\bar{\phi} \leq \theta$, one can apply Proposition 3.3 and get that

$$\tilde{Q}_t \varphi(s') \leq \bar{\phi}(t, s') = \varphi(s' - ct), \quad \text{a.a. } s' \in \mathbb{R},$$

where c is given by (4.15); note that, by (4.12), for any $s \in \mathbb{R}$, the function $\bar{\phi}(s, t)$ is absolutely continuous in t . In particular, for $t = 1$, $s' = s + c$, we will have

$$\tilde{Q}_1 \varphi(s + c) \leq \varphi(s), \quad \text{a.a. } s \in \mathbb{R}. \quad (4.17)$$

And now one can apply [80, Theorem 5] which states that, if there exists a flow of abstract mappings \tilde{Q}_t , each of them maps $\mathcal{M}_\theta(\mathbb{R})$ into itself and has properties (Q1)–(Q5) of Proposition 3.14, and if, for some t (e.g. $t = 1$), for some $c \in \mathbb{R}$, and for some $\varphi \in \mathcal{M}_\theta(\mathbb{R})$, the inequality (4.17) holds, then there exists $\psi \in \mathcal{M}_\theta(\mathbb{R})$ such that, for any $t \geq 0$,

$$(\tilde{Q}_t \psi)(s + ct) = \psi(s), \quad \text{a.a. } s \in \mathbb{R}, \quad (4.18)$$

that yields the solution to (4.4) in the form (4.8), and hence, by Remark 4.5, we will get the existence of a solution to (2.1) in the form (4.1). It is worth noting that in [80] the results were obtained for increasing functions bounded by 1 whereas we are working with decreasing functions bounded by θ , however, the corresponding statements may be proved literally in the same way as in [80], see also Remark 4.6.

Next, by [80, Theorem 6], there exists $c_* = c_*(\xi) \in (-\infty, \infty]$ such that, for any $c \geq c_*$, there exists $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ such that (4.18) holds, and for any $c < c_*$ such a ψ does not exist. Since for c given by (4.15) such ψ exists, we have that $c_* \leq c < \infty$, moreover, one can take any μ in (4.15) for that (A5) holds. Therefore,

$$c_* \leq \inf_{\lambda > 0} \frac{\varkappa^+ \mathbf{a}_\xi(\lambda) - m}{\lambda}. \quad (4.19)$$

The statement is proved. \square

Remark 4.10. It can be seen from the proof above that we didn't use the special form (4.12) of the function φ after the inequality (4.16). Therefore, if a function $\varphi_1 \in \mathcal{M}_\theta(\mathbb{R})$ is such that the function $\bar{\phi}(s, t) := \varphi_1(s - ct)$, $s \in \mathbb{R}$, $t \geq 0$, is a super-solution to (4.4), for some $c \in \mathbb{R}$, i.e. if (4.13) holds, then there exists a traveling wave solution to (4.4), and hence to (2.1), with some profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and the same speed c .

Next two statements describe the properties of a traveling wave solution.

Proposition 4.11. *Let $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and $c \in \mathbb{R}$ be such that there exists a solution $u \in \tilde{\mathcal{X}}_\infty^1$ to the equation (2.1) such that (4.1) holds, for some $\xi \in S^{d-1}$. Then $\psi \in C^1(\mathbb{R} \rightarrow [0, \theta])$, for $c \neq 0$, and $\psi \in C(\mathbb{R} \rightarrow [0, \theta])$, otherwise.*

Proof. The condition (4.1) implies (4.2) for the $\xi \in S^{d-1}$. Then, by Proposition 4.4, there exists ϕ given by (4.3) which solves (4.4); moreover, by Remark 4.5, (4.8) holds.

Let $c \neq 0$. It is well-known that any monotone function is differentiable almost everywhere. Prove first that ψ is differentiable everywhere on \mathbb{R} . Fix any $s_0 \in \mathbb{R}$. It follows directly from Proposition 4.4, that $\phi \in C^1((0, \infty) \rightarrow L^\infty(\mathbb{R}))$. Therefore, for any $t_0 > 0$ and for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that, for all $t \in \mathbb{R}$ with $|ct| < \delta$ and $t_0 + t > 0$, the following inequalities hold, for a.a. $s \in \mathbb{R}$,

$$\frac{\partial \phi}{\partial t}(s, t_0) - \varepsilon < \frac{\phi(s, t_0 + t) - \phi(s, t_0)}{t} < \frac{\partial \phi}{\partial t}(s, t_0) + \varepsilon, \quad (4.20)$$

$$\frac{\partial \phi}{\partial t}(s, t_0) - \varepsilon < \frac{\partial \phi}{\partial t}(s, t_0 + t) < \frac{\partial \phi}{\partial t}(s, t_0) + \varepsilon. \quad (4.21)$$

Set, for the simplicity of notations, $x_0 = s_0 + ct_0$. Take any $0 < h < 1$ with $2h < \min\{\delta, |c|t_0\}$. Since ψ is a decreasing function, one has, for almost all $s \in (x_0, x_0 + h^2)$,

$$\begin{aligned} \frac{\psi(s_0 + h) - \psi(s_0)}{h} &\leq \frac{\psi(s - ct_0 + h - h^2) - \psi(s - ct_0)}{h} \\ &= \frac{\phi(s, t_0 + \frac{h^2 - h}{c}) - \phi(s, t_0)}{\frac{h^2 - h}{c}} \frac{h^2 - h}{ch} \leq \left(\frac{\partial \phi}{\partial t}(s, t_0) \mp \varepsilon \right) \frac{h - 1}{c}, \end{aligned} \quad (4.22)$$

by (4.20) with $t = \frac{h^2 - h}{c}$; note that then $|ct| = h - h^2 < h < \delta$, and $t_0 + t > 0$ (the latter holds, for $c < 0$, because of $t_0 + t > t_0$ then; and, for $c > 0$, it is equivalent to $ct_0 > -ct = h - h^2$, that follows from $h < ct_0$). Stress, that, in (4.22), one needs to choose $-\varepsilon$, for $c > 0$, and $+\varepsilon$, for $c < 0$, according to the left and right inequalities in (4.20), correspondingly.

Similarly, for almost all $s \in (s_0 + ct_0 - h^2, s_0 + ct_0)$, one has

$$\begin{aligned} \frac{\psi(s_0 + h) - \psi(s_0)}{h} &\geq \frac{\psi(s - ct_0 + h + h^2) - \psi(s - ct_0)}{h} \\ &= \frac{\phi(s, t_0 - \frac{h^2 + h}{c}) - \phi(s, t_0)}{-\frac{h^2 + h}{c}} \frac{h^2 + h}{-ch} \geq \left(\frac{\partial \phi}{\partial t}(s, t_0) \pm \varepsilon \right) \frac{h + 1}{-c}, \end{aligned} \quad (4.23)$$

where we take again the upper sign, for $c > 0$, and the lower sign, for $c < 0$; note also that $h + h^2 < 2h < \delta$. Next, one needs to ‘shift’ values of s in (4.23) to get them the same as in (4.22). To do this note that, by (4.8),

$$\phi\left(s + h^2, t_0 + \frac{h^2}{c}\right) = \phi(s, t_0), \quad \text{a.a. } s \in \mathbb{R}^d. \quad (4.24)$$

As a result,

$$\begin{aligned} (\check{\alpha}^\pm * \phi)\left(s + h^2, t_0 + \frac{h^2}{c}\right) &= \int_{\mathbb{R}} \check{\alpha}^\pm(s') \phi\left(s - s' + h^2, t_0 + \frac{h^2}{c}\right) ds' \\ &= (\check{\alpha}^\pm * \phi)(s, t_0), \quad \text{a.a. } s \in \mathbb{R}^d. \end{aligned} \quad (4.25)$$

Then, by (4.4), (4.24), (4.25), one gets

$$\frac{\partial}{\partial t}\phi\left(s+h^2, t_0+\frac{h^2}{c}\right) = \frac{\partial}{\partial t}\phi(s, t_0), \quad \text{a.a. } s \in \mathbb{R}^d. \quad (4.26)$$

Therefore, by (4.26), one gets from (4.23) that, for almost all $s \in (x_0, x_0 + h^2)$, cf. (4.22),

$$\frac{\psi(s_0+h) - \psi(s_0)}{h} \geq \left(\frac{\partial\phi}{\partial t}\left(s, t_0+\frac{h^2}{c}\right) \pm \varepsilon\right) \frac{h+1}{-c},$$

and, since $|\frac{h^2}{c}| < \delta$, one can apply the right and left inequalities in (4.21), for $c > 0$ and $c < 0$, correspondingly, to continue the estimate

$$\geq \left(\frac{\partial\phi}{\partial t}(s, t_0) \pm 2\varepsilon\right) \frac{h+1}{-c}. \quad (4.27)$$

Combining (4.22) and (4.27), we obtain

$$\begin{aligned} \left(\operatorname{esssup}_{s \in (x_0, x_0+h^2)} \frac{\partial\phi}{\partial t}(s, t_0) \pm 2\varepsilon\right) \frac{h+1}{-c} &\leq \frac{\psi(s_0+h) - \psi(s_0)}{h} \\ &\leq \left(\operatorname{esssup}_{s \in (x_0, x_0+h^2)} \frac{\partial\phi}{\partial t}(s, t_0) \mp \varepsilon\right) \frac{h-1}{c}. \end{aligned} \quad (4.28)$$

For fixed $s_0 \in \mathbb{R}$, $t_0 > 0$ and for $x_0 = s_0 + ct_0$, the function

$$f(h) := \operatorname{esssup}_{s \in (x_0, x_0+h^2)} \frac{\partial\phi}{\partial t}(s, t_0), \quad h \in (0, 1)$$

is bounded, as $|f(h)| \leq \|\frac{\partial\phi}{\partial t}(\cdot, t_0)\|_\infty < \infty$, and monotone; hence there exists $\bar{f} = \lim_{h \rightarrow 0^+} f(h)$. As a result, for small enough h , (4.28) yields

$$(\bar{f} \pm 2\varepsilon) \frac{1}{-c} - \varepsilon \leq \frac{\psi(s_0+h) - \psi(s_0)}{h} \leq (\bar{f} \mp \varepsilon) \frac{-1}{c} + \varepsilon,$$

and, therefore, there exists $\frac{\partial\psi}{\partial s}(s_0+) = \frac{-\bar{f}}{c}$. In the same way, one can prove that there exists $\frac{\partial\psi}{\partial s}(s_0-) = \frac{-\bar{f}}{c}$, and, therefore, ψ is differentiable at s_0 . As a result, ψ is differentiable (and hence continuous) on the whole \mathbb{R} .

Next, for any $s_1, s_2, h \in \mathbb{R}$, we have

$$\begin{aligned} &\left| \frac{\psi(s_1+h) - \psi(s_1)}{h} - \frac{\psi(s_2+h) - \psi(s_2)}{h} \right| \\ &= \frac{1}{|c|} \left| \frac{\phi\left(s_1+ct_0, t_0-\frac{h}{c}\right) - \phi\left(s_1+ct_0, t_0\right)}{-\frac{h}{c}} \right. \\ &\quad \left. - \frac{\phi\left(s_1+ct_0, t_0+\frac{s_1-s_2}{c}-\frac{h}{c}\right) - \phi\left(s_1+ct_0, t_0+\frac{s_1-s_2}{c}\right)}{-\frac{h}{c}} \right|; \end{aligned}$$

and if we pass h to 0, we get

$$\begin{aligned} |\psi'(s_1) - \psi'(s_2)| &= \frac{1}{|c|} \left| \frac{\partial}{\partial t} \phi(s_1 + ct_0, t_0) - \frac{\partial}{\partial t} \phi\left(s_1 + ct_0, t_0 + \frac{s_1 - s_2}{c}\right) \right| \\ &\leq \frac{1}{|c|} \left\| \frac{\partial}{\partial t} \phi(\cdot, t_0) - \frac{\partial}{\partial t} \phi\left(\cdot, t_0 + \frac{s_1 - s_2}{c}\right) \right\|. \end{aligned} \quad (4.29)$$

And now, by the continuity of $\frac{\partial}{\partial t} \phi(\cdot, t)$ in t in the sense of the norm in $L^\infty(\mathbb{R})$, we have that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|s_1 - s_2| \leq c\delta$ implies that, by (4.29),

$$|\psi'(s_1) - \psi'(s_2)| \leq \frac{1}{|c|} \varepsilon.$$

As a result, $\psi'(s)$ is uniformly continuous on \mathbb{R} and hence continuous.

Finally, consider the case $c = 0$. Then (4.8) implies that $\phi(s, t)$ must be constant in time, i.e. $\phi(s, t) = \psi(s)$, for a.a. $s \in \mathbb{R}$. Thus one can rewrite (4.4) as follows

$$\varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \varkappa^-\psi(s)(\check{a}^- * \psi)(s) = 0, \quad (4.30)$$

or, equivalently,

$$\psi(s) = \frac{\varkappa^+(\check{a}^+ * \psi)(s)}{m + \varkappa^-(\check{a}^- * \psi)(s)}. \quad (4.31)$$

Since $\psi \in L^\infty(\mathbb{R})$, then, by Lemma 2.1, the r.h.s. of (4.31) is a continuous in s function, and hence $\psi \in C(\mathbb{R})$. \square

Corollary 4.12. *In conditions and notations of Proposition 4.11, ψ is a strictly decaying function, for any speed c .*

Proof. Let $c \in \mathbb{R}$ be the speed of a traveling wave with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ in a direction $\xi \in S^{d-1}$. By Proposition 4.11, $\psi \in C(\mathbb{R})$. Suppose that ψ is not strictly decaying, then there exists $\delta_0 > 0$ and $s_0 \in \mathbb{R}$, such that $\psi(s) = \psi(s_0)$, for all $|s - s_0| \leq \delta_0$. Take any $\delta \in (0, \frac{\delta_0}{2})$, and consider the function $\psi^\delta(s) := \psi(s + \delta)$. Clearly, $\psi^\delta(s) \leq \psi(s)$, $s \in \mathbb{R}$. By Remark 4.7, ψ^δ is a profile for a traveling wave with the same speed c . Therefore, one has two solutions to (2.1): $u(x, t) = \psi(x \cdot \xi - ct)$ and $u^\delta(x, t) = \psi^\delta(x \cdot \xi - ct)$ and hence $u(x, t) \leq u^\delta(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$. By the maximum principle, see Theorem 3.8, either $u \equiv u^\delta$, that contradicts $\delta > 0$ or $u(x, t) < u^\delta(x, t)$, $x \in \mathbb{R}^d$, $t > 0$. The latter, however, contradicts the equality $u(x, t) = u^\delta(x, t)$, which holds e.g. if $x \cdot \xi - ct = s_0$. Hence ψ is a strictly decaying function. \square

As a result, if $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and $c \in \mathbb{R}$ are such that there exists a solution $u \in \tilde{\mathcal{X}}_\infty^1$ to the equation (2.1) such that (4.1) holds, for some $\xi \in S^{d-1}$, then ϕ given by (4.8) solves (4.4) where \check{a}^\pm are given by (4.6) and depend on the $\xi \in S^{d-1}$. Next, by Proposition 4.11, for any $c \neq 0$, one can differentiate $\psi(s - ct)$ in $t \geq 0$. Thus we get

$$c\psi'(s) + \varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \varkappa^-\psi(s)(\check{a}^- * \psi)(s) = 0, \quad s \in \mathbb{R}; \quad (4.32)$$

recall that, by Lemma 2.1, the l.h.s. of (4.32) is indeed defined point-wise. For $c = 0$, one has (4.30), i.e. (4.32) holds in this case as well.

Under assumptions (A1) and (A2), define the following function, cf. (3.19),

$$\check{J}_v(s) := \varkappa^+ \check{a}^+(s) - v \varkappa^- \check{a}^-(s), \quad s \in \mathbb{R}, v \in (0, \theta]. \quad (4.33)$$

Then, by (4.9),

$$\check{J}_v(s) \geq \check{J}_\theta(s) \geq 0, \quad s \in \mathbb{R}, v \in (0, \theta).$$

Proposition 4.13. *Let (A1) and (A2) hold. Then, in the conditions and notations of Proposition 4.11, there exists $\mu = \mu(c, a^+, \varkappa^-, \theta) > 0$ such that*

$$\int_{\mathbb{R}} \psi(s) e^{\mu s} ds < \infty.$$

Proof. At first, we prove that $\psi \in L^1(\mathbb{R}_+)$. Let $v \in (0, \theta)$ and $\check{J}_v(s) > 0$, $s \in \mathbb{R}$ be given by (4.33). Since $\int_{\mathbb{R}} \check{J}_v(s) ds = \varkappa^+ - v \varkappa^- > m$, one can choose $R_0 > 0$, such that

$$\int_{-R_0}^{R_0} \check{J}_v(s) ds = m. \quad (4.34)$$

We rewrite (4.32) as follows

$$c\psi'(s) + (\check{J}_v * \psi)(s) + \varkappa^- (v - \psi(s))(\check{a}^- * \psi)(s) - m\psi(s) = 0, \quad s \in \mathbb{R}. \quad (4.35)$$

Fix arbitrary $r_0 > 0$, such that

$$\psi(r_0) < v. \quad (4.36)$$

Let $r > r_0 + R_0$. Integrate (4.35) over $[r_0, r]$; one gets

$$c(\psi(r) - \psi(r_0)) + A + B = 0, \quad (4.37)$$

where

$$\begin{aligned} A &:= \int_{r_0}^r (\check{J}_v * \psi)(s) ds - m \int_{r_0}^r \psi(s) ds, \\ B &:= \varkappa^- \int_{r_0}^r (v - \psi(s))(\check{a}^- * \psi)(s) ds. \end{aligned}$$

By (4.33), (4.34), one has

$$\begin{aligned} A &\geq \int_{r_0}^r \int_{-R_0}^{R_0} \check{J}_v(\tau) \psi(s - \tau) d\tau ds - m \int_{r_0}^r \psi(s) ds \\ &= \int_{-R_0}^{R_0} \check{J}_v(\tau) \left(\int_{r_0 - \tau}^{r - \tau} \psi(s) ds - \int_{r_0}^r \psi(s) ds \right) d\tau \\ &= \int_0^{R_0} \check{J}_v(\tau) \left(\int_{r_0 - \tau}^{r_0} \psi(s) ds - \int_{r - \tau}^r \psi(s) ds \right) d\tau \\ &\quad + \int_{-R_0}^0 \check{J}_v(\tau) \left(\int_r^{r - \tau} \psi(s) ds - \int_{r_0}^{r_0 - \tau} \psi(s) ds \right) d\tau; \quad (4.38) \end{aligned}$$

and since ψ is a decreasing function and $r - R_0 > r_0$, we have from (4.38), that

$$\begin{aligned} A &\geq (\psi(r_0) - \psi(r - R_0)) \int_0^{R_0} \tau \check{J}_v(\tau) d\tau + (\psi(r + R_0) - \psi(r_0)) \int_{-R_0}^0 (-\tau) J_v(\tau) d\tau \\ &\geq -\theta \int_{-R_0}^0 (-\tau) J_v(\tau) d\tau =: -\theta \bar{J}_{v, R_0}. \end{aligned} \quad (4.39)$$

Next, (4.36) and monotonicity of ψ imply

$$B \geq \varkappa^-(v - \psi(r_0)) \int_{r_0}^r (\check{a}^- * \psi)(s) ds. \quad (4.40)$$

Then, by (4.37), (4.39), (4.40), (4.36), one gets

$$\begin{aligned} 0 \leq \varkappa^-(v - \psi(r_0)) \int_{r_0}^r (\check{a}^- * \psi)(s) ds &\leq \theta \bar{J}_{v, R_0} + c(\psi(r_0) - \psi(r)) \\ &\rightarrow \theta \bar{J}_{v, R_0} + c\psi(r_0) < \infty, \quad r \rightarrow \infty, \end{aligned}$$

therefore, $\check{a}^- * \psi \in L^1(\mathbb{R}_+)$. Finally, (4.7) implies that there exist a measurable bounded set $\Delta \subset \mathbb{R}$, with $m(\Delta) := \int_{\Delta} ds \in (0, \infty)$, and a constant $\mu > 0$, such that $\check{a}^-(\tau) \geq \mu$, for a.a. $\tau \in \Delta$. Let $\delta = \inf \Delta \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$, one has

$$(\check{a}^- * \psi)(s) \geq \int_{\Delta} \check{a}^-(\tau) \psi(s - \tau) d\tau \geq \mu \psi(s - \delta) m(\Delta).$$

Therefore $\psi \in L^1(\mathbb{R}_+)$.

For any $N \in \mathbb{N}$, we define $\varphi_N(s) := \mathbb{1}_{(-\infty, N)}(s) + e^{-\lambda(s-N)} \mathbb{1}_{[N, \infty)}(s)$, where $\lambda > 0$. By the proved above, $\psi, \check{a}^\pm * \psi \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R})$ hence, by (4.32), $c\psi' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R})$. Therefore, all terms of (4.32) being multiplied on $e^{\lambda s} \varphi_N(s)$ are integrable over \mathbb{R} . After this integration, (4.32) will be read as follows

$$I_1 + I_2 + I_3 = 0, \quad (4.41)$$

where (recall that $\varkappa^- \theta - \varkappa^+ = -m$)

$$\begin{aligned} I_1 &:= c \int_{\mathbb{R}} \psi'(s) e^{\lambda s} \varphi_N(s) ds, \\ I_2 &:= \varkappa^+ \int_{\mathbb{R}} ((\check{a}^+ * \psi)(s) - \psi(s)) e^{\lambda s} \varphi_N(s) ds, \\ I_3 &:= \varkappa^- \int_{\mathbb{R}} \psi(s) (\theta - (\check{a}^- * \psi)(s)) e^{\lambda s} \varphi_N(s) ds \end{aligned}$$

We estimate now I_1, I_2, I_3 from below.

We start with I_2 . One can write

$$\begin{aligned} &\int_{\mathbb{R}} (\check{a}^+ * \psi)(s) e^{\lambda s} \varphi_N(s) ds = \int_{\mathbb{R}} \int_{\mathbb{R}} \check{a}^+(s - \tau) \psi(\tau) e^{\lambda s} \varphi_N(s) d\tau ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda s} \varphi_N(\tau + s) ds e^{\lambda \tau} \psi(\tau) d\tau \\ &\geq \int_{\mathbb{R}} \left(\int_{-\infty}^R \check{a}^+(s) e^{\lambda s} ds \right) \varphi_N(\tau + R) e^{\lambda \tau} \psi(\tau) d\tau, \end{aligned} \quad (4.42)$$

for any $R > 0$, as φ is nonincreasing. By (4.7), one can choose $R > 0$ such that

$$\int_{-\infty}^R \check{a}^+(\tau) d\tau > 1 - \frac{\varkappa^- \theta}{4}.$$

By continuity arguments, there exists $\nu > 0$ such that, for any $0 < \lambda < \nu$,

$$\int_{-\infty}^R \check{a}^+(\tau) e^{\lambda\tau} d\tau \geq \left(1 - \frac{\varkappa^- \theta}{4}\right) e^{\lambda R}. \quad (4.43)$$

Therefore, combining (4.42) and (4.43), we get

$$\begin{aligned} I_2 &\geq \int_{\mathbb{R}} \left(1 - \frac{\varkappa^- \theta}{4}\right) e^{\lambda R} \varphi_N(\tau + R) e^{\lambda\tau} \psi(\tau) d\tau - \int_{\mathbb{R}} \psi(s) e^{\lambda s} \varphi_N(s) ds \\ &= \int_{\mathbb{R}} \left(1 - \frac{\varkappa^- \theta}{4}\right) \varphi_N(\tau) e^{\lambda\tau} \psi(\tau - R) d\tau - \int_{\mathbb{R}} \psi(s) e^{\lambda s} \varphi_N(s) ds \\ &\geq -\frac{\varkappa^- \theta}{4} \int_{\mathbb{R}} \psi(s) e^{\lambda s} \varphi_N(s) ds, \end{aligned} \quad (4.44)$$

as $\psi(\tau - R) \geq \psi(\tau)$, $\tau \in \mathbb{R}$, $R > 0$.

Now we estimate I_3 . By (4.1), it is easily seen that the function $(\check{a}^- * \psi)(s)$ decreases monotonically to 0 as $s \rightarrow \infty$. Suppose additionally that $R > 0$ above is such that

$$(\check{a}^- * \psi)(s) < \frac{\theta}{2}, \quad s > R.$$

Then, one gets

$$\begin{aligned} I_3 &\geq \frac{\theta}{2} \int_R^\infty \psi(s) e^{\lambda s} \varphi_N(s) ds + \int_{-\infty}^R \psi(s) (\theta - (\check{a}^- * \psi)(s)) e^{\lambda s} \varphi_N(s) ds \\ &\geq \frac{\theta}{2} \int_R^\infty \psi(s) e^{\lambda s} \varphi_N(s) ds, \end{aligned}$$

as $\psi \geq 0$, $\varphi_N \geq 0$, $(\check{a}^- * \psi)(s) \leq \theta$.

It remains to estimate I_1 (in the case $c \neq 0$). Since $\lim_{s \rightarrow \pm\infty} \psi(s) e^{\lambda s} \varphi_N(s) = 0$, we have from the integration by parts formula, that

$$I_1 = -c \int_{\mathbb{R}} \psi(s) (\lambda \varphi_N(s) + \varphi'_N(s)) e^{\lambda s} ds.$$

For $c > 0$, one can use that $\varphi'_N(s) \leq 0$, $s \in \mathbb{R}$, and hence

$$I_1 \geq -c\lambda \int_{\mathbb{R}} \psi(s) \varphi_N(s) e^{\lambda s} ds.$$

For $c < 0$, we use that, by the definition of φ_N , $\lambda \varphi_N(s) + \varphi'_N(s) = 0$, $s \geq N$; therefore,

$$I_1 = -c\lambda \int_{-\infty}^N \psi(s) ds > 0. \quad (4.45)$$

Therefore, combining (4.44)–(4.45), we get from (4.41), that

$$0 \geq -\lambda \bar{c} \int_{\mathbb{R}} \psi(s) \varphi_N(s) e^{\lambda s} ds - \frac{\varkappa^{-\theta}}{4} \int_{\mathbb{R}} \psi(s) e^{\lambda s} \varphi_N(s) ds + \frac{\varkappa^{-\theta}}{2} \int_R^\infty \psi(s) e^{\lambda s} \varphi_N(s) ds,$$

where $\bar{c} = \max\{c, 0\}$.

The latter inequality can be easily rewritten as

$$\begin{aligned} \left(\frac{\varkappa^{-\theta}}{4} - \lambda \bar{c}\right) \int_R^\infty \psi(s) e^{\lambda s} \varphi_N(s) ds &\leq \left(\frac{\varkappa^{-\theta}}{4} + \lambda \bar{c}\right) \int_{-\infty}^R \psi(s) \varphi_N(s) e^{\lambda s} ds \\ &\leq \left(\frac{\varkappa^{-\theta}}{4} + \lambda \bar{c}\right) \theta \int_{-\infty}^R e^{\lambda s} ds =: I_{\lambda, R} < \infty, \end{aligned} \quad (4.46)$$

for any $0 < \lambda < \nu$.

Take now $\mu < \min\{\nu, \frac{\varkappa^{-\theta}}{4c}\}$, for $c > 0$, and $\mu < \nu$, otherwise. Then, by (4.46), for any $N > R$, one get

$$\infty > \left(\frac{\varkappa^{-\theta}}{4} - \mu \bar{c}\right)^{-1} I_{\mu, R} > \int_R^\infty \psi(s) e^{\mu s} \varphi_N(s) ds \geq \int_R^N \psi(s) e^{\mu s} ds,$$

thus,

$$\begin{aligned} \int_{\mathbb{R}} \psi(s) e^{\mu s} ds &= \int_{-\infty}^R \psi(s) e^{\mu s} ds + \int_R^\infty \psi(s) e^{\mu s} ds \\ &\leq \theta \int_{-\infty}^R e^{\mu s} ds + I_{\mu, R} \left(\frac{\varkappa^{-\theta}}{4} - \mu \bar{c}\right)^{-1} < \infty, \end{aligned}$$

that gets the statement. \square

4.2 Speed and profile of a traveling wave

Through this subsection we will suppose, additionally to (A1) and (A2), that

$$a^+ \in L^\infty(\mathbb{R}^d). \quad (A6)$$

Clearly, (A2) and (A6) imply $a^- \in L^\infty(\mathbb{R}^d)$.

Remark 4.14. All further statements remain true if we change (A6) on the condition $\check{a}^+ \in L^\infty(\mathbb{R})$, where \check{a}^+ is given by (4.6); evidently, the latter condition is, for $d \geq 2$, weaker than (A6).

Let $\xi \in S^{d-1}$ be fixed and (A5) hold. Assume also that

$$\int_{\mathbb{R}^d} |x \cdot \xi| a^+(x) dx < \infty. \quad (A7)$$

Under assumption (A7), we define

$$\mathbf{m}_\xi := \int_{\mathbb{R}^d} x \cdot \xi a^+(x) dx. \quad (4.47)$$

Suppose also, that the following modification of (A3) holds:

$$\begin{aligned} &\text{there exist } r = r(\xi) \geq 0, \rho = \rho(\xi) > 0, \delta = \delta(\xi) > 0, \text{ such that} \\ &a^+(x) \geq \rho, \text{ for a.a. } x \in B_\delta(r\xi). \end{aligned} \quad (\text{A8})$$

For an $f \in L^\infty(\mathbb{R})$, let $\mathfrak{L}f$ be a bilateral-type Laplace transform of f , cf. [78, Chapter VI]:

$$(\mathfrak{L}f)(z) = \int_{\mathbb{R}} f(s)e^{zs} ds, \quad \operatorname{Re} z > 0. \quad (4.48)$$

We collect several results about \mathfrak{L} in the following lemma.

Lemma 4.15. *Let $f \in L^\infty(\mathbb{R})$.*

(L1) *There exists $\lambda_0(f) \in [0, \infty]$ such that the integral (4.48) converges in the strip $\{0 < \operatorname{Re} z < \lambda_0(f)\}$ (provided that $\lambda_0(f) > 0$) and diverges in the half plane $\{\operatorname{Re} z > \lambda_0(f)\}$ (provided that $\lambda_0(f) < \infty$).*

(L2) *Let $\lambda_0(f) > 0$. Then $(\mathfrak{L}f)(z)$ is analytic in $\{0 < \operatorname{Re} z < \lambda_0(f)\}$, and, for any $n \in \mathbb{N}$,*

$$\frac{d^n}{dz^n}(\mathfrak{L}f)(z) = \int_{\mathbb{R}} e^{zs} s^n f(s) ds, \quad 0 < \operatorname{Re} z < \lambda_0(f).$$

(L3) *Let $f \geq 0$ a.e. and $0 < \lambda_0(f) < \infty$. Then $(\mathfrak{L}f)(z)$ has a singularity at $z = \lambda_0(f)$. In particular, $\mathfrak{L}f$ has not an analytic extension to a strip $0 < \operatorname{Re} z < \nu$, with $\nu > \lambda_0(f)$.*

(L4) *Let $f' := \frac{d}{ds}f \in L^\infty(\mathbb{R})$, $f(\infty) = 0$, and $\lambda_0(f') > 0$. Then $\lambda_0(f) \geq \lambda_0(f')$ and, for any $0 < \operatorname{Re} z < \lambda_0(f')$,*

$$(\mathfrak{L}f')(z) = -z(\mathfrak{L}f)(z). \quad (4.49)$$

(L5) *Let $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\lambda_0(f) > 0$, $\lambda_0(g) > 0$. Then $\lambda_0(f * g) \geq \min\{\lambda_0(f), \lambda_0(g)\}$ and, for any $0 < \operatorname{Re} z < \min\{\lambda_0(f), \lambda_0(g)\}$,*

$$(\mathfrak{L}(f * g))(z) = (\mathfrak{L}f)(z)(\mathfrak{L}g)(z) \quad (4.50)$$

(L6) *Let $0 \leq f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\lambda_0(f) > 0$. Then $\lim_{\lambda \rightarrow 0^+} (\mathfrak{L}f)(\lambda) = \int_{\mathbb{R}} f(s) ds$.*

(L7) *Let $f \geq 0$, $\lambda_0(f) \in (0, \infty)$ and $A := \int_{\mathbb{R}} f(s)e^{\lambda_0(f)s} ds < \infty$. Then $\lim_{\lambda \rightarrow \lambda_0(f)^-} (\mathfrak{L}f)(\lambda) = A$.*

(L8) *Let $f \geq 0$ be decreasing on \mathbb{R} , and let $\lambda_0(f) > 0$. Then, for any $0 < \lambda < \lambda_0(f)$,*

$$f(s) \leq \frac{\lambda e^\lambda}{e^\lambda - 1} (\mathfrak{L}f)(\lambda) e^{-\lambda s}, \quad s \in \mathbb{R}. \quad (4.51)$$

Moreover, for any $0 \leq g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $\lambda_0(g) > 0$,

$$\lambda_0(f(g * f)) \geq \lambda_0(f) + \min\{\lambda_0(g), \lambda_0(f)\}. \quad (4.52)$$

Proof. We can rewrite $\mathfrak{L} = \mathfrak{L}^+ + \mathfrak{L}^-$, where

$$(\mathfrak{L}^\pm f)(z) = \int_{\mathbb{R}_\pm} f(s)e^{zs} ds, \quad \operatorname{Re} z > 0,$$

$\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (\infty, 0]$. Let \mathcal{L} denote the classical (unilateral) Laplace transform:

$$(\mathcal{L}f)(z) = \int_{\mathbb{R}_+} f(s)e^{-zs} ds,$$

and $\iota_0(f)$ be its abscissa of convergence (see details, e.g., in [78, Chapter II]). Then, clearly, $(\mathfrak{L}^+ f)(z) = (\mathcal{L}f)(-z)$, $(\mathfrak{L}^- f)(z) = (\mathcal{L}f^-)(z)$, where $f^-(s) = f(-s)$, $s \in \mathbb{R}$. As a result, $\lambda_0(f) = -\iota_0(f)$.

It is easily seen that, for $f \in L^\infty(\mathbb{R})$, $\iota_0(f^-) \leq 0$, in particular, the function $(\mathfrak{L}^- f)(z)$ is analytic on $\operatorname{Re} z > 0$.

Therefore, the properties (L1)–(L3) are direct consequences of [78, Theorems II.1, II.5a, II.5b], respectively. The property (L4) may be easily derived from [78, Theorem II.2.3a, II.2.3b], taking into account that $f(\infty) = 0$. The property (L5) one gets by a straightforward computation, cf. [78, Theorem VI.16a]; note that $f * g \in L^\infty(\mathbb{R})$.

Next, $\lambda_0(f) > 0$ implies $\iota_0(f) < 0$, therefore, $\mathfrak{L}^+ f$ can be analytically continued to 0. If $\iota(f^-) < 0$, then $\mathfrak{L}^- f$ can be analytically continued to 0 as well, and (L6) will be evident. Otherwise, if $\iota(f^-) = 0$ then (L6) follows from [78, Theorem V.1]. Similar arguments prove (L7).

To prove (L8) for decreasing nonnegative f , note that, for any $0 < \lambda < \lambda_0(f)$,

$$f(s) \int_{s-1}^s e^{\lambda\tau} d\tau \leq \int_{s-1}^s f(\tau)e^{\lambda\tau} d\tau \leq (\mathfrak{L}f)(\lambda), \quad s \in \mathbb{R},$$

that implies (4.51). Next, by (L5), $\lambda_0(g * f) > 0$, and conditions on g yield that $g * f \geq 0$ is decreasing as well. Therefore, by (4.51), for any $0 < \lambda < \lambda_0(g * f)$,

$$\begin{aligned} |(\mathfrak{L}(f(g * f)))(z)| &\leq \int_{\mathbb{R}} f(s)(g * f)(s)e^{s\operatorname{Re} z} ds \\ &\leq \frac{\lambda e^\lambda}{e^\lambda - 1} (\mathfrak{L}(g * f))(\lambda) \int_{\mathbb{R}} f(s)e^{-s\lambda} e^{s\operatorname{Re} z} ds < \infty, \end{aligned}$$

provided that $\operatorname{Re} z < \lambda_0(f) + \lambda < \lambda_0(f) + \lambda_0(g * f)$. As a result, $\lambda_0(f(g * f)) \geq \lambda_0(f) + \lambda_0(g * f)$ that, by (L5), implies (4.52). \square

Fix any $\xi \in S^{d-1}$. Then, by (4.11), one has that $\lambda_0(\check{a}^\pm) > 0$. Consider, cf. (4.15), (4.19), the following complex-valued function

$$G_\xi(z) := \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z) - m}{z}, \quad \operatorname{Re} z > 0, \quad (4.53)$$

which is well-defined on $0 < \operatorname{Re} z < \lambda_0(\check{a}^+)$. Note that, by (4.11),

$$(\mathfrak{L}\check{a}^+)(\lambda) = \mathbf{a}_\xi(\lambda), \quad G_\xi(\lambda) = \frac{\varkappa^+ \mathbf{a}_\xi(\lambda) - m}{\lambda}, \quad 0 < \lambda < \lambda_0(\check{a}^+),$$

and hence, by (4.19),

$$c_*(\xi) \leq \inf_{\lambda > 0} G_\xi(\lambda), \quad (4.54)$$

where $c_*(\xi)$ is the minimal speed of traveling waves, cf. Theorem 4.9. We will show below that in fact there exists equality in (4.54), and hence in (4.19).

We start with the following notations to simplify the further statements.

Definition 4.16. Let $m > 0$, $\varkappa^\pm > 0$, $0 \leq a^- \in L^1(\mathbb{R}^d)$ be fixed, and (A1) holds. For an arbitrary $\xi \in S^{d-1}$, denote by \mathcal{U}_ξ the subset of functions $0 \leq a^+ \in L^1(\mathbb{R}^d)$ such that (A2) and (A5)–(A8) hold.

For $a^+ \in \mathcal{U}_\xi$, denote also the interval $I_\xi \subset (0, \infty)$ by

$$I_\xi := \begin{cases} (0, \infty), & \text{if } \lambda_0(\check{a}^+) = \infty, \\ (0, \lambda_0(\check{a}^+)), & \text{if } \lambda_0(\check{a}^+) < \infty \text{ and } (\mathfrak{L}\check{a}^+)(\lambda_0(\check{a}^+)) = \infty \\ (0, \lambda_0(\check{a}^+)], & \text{if } \lambda_0(\check{a}^+) < \infty \text{ and } (\mathfrak{L}\check{a}^+)(\lambda_0(\check{a}^+)) < \infty. \end{cases}$$

Proposition 4.17. Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Then there exists $\lambda_* = \lambda_*(\xi) \in I_\xi$ such that

$$\inf_{\lambda > 0} G_\xi(\lambda) = \min_{\lambda \in I_\xi} G_\xi(\lambda) = G_\xi(\lambda_*) > \varkappa^+ \mathfrak{m}_\xi. \quad (4.55)$$

Moreover, G_ξ is strictly decreasing on $(0, \lambda_*]$ and G_ξ is strictly increasing on $I_\xi \setminus (0, \lambda_*]$ (the latter interval may be empty).

Proof. First of all, by (4.6), the condition (A7) implies, cf. (4.11),

$$\mathfrak{m}_\xi = \int_{\mathbb{R}} s \check{a}^+(s) ds \in \mathbb{R}. \quad (4.56)$$

Next, to simplify notations, we set $\lambda_0 := \lambda_0(\check{a}^+) \in (0, \infty]$. Denote also

$$F_\xi(\lambda) := \varkappa^+ \mathfrak{a}_\xi(\lambda) - m = \lambda G_\xi(\lambda), \quad \lambda \in I_\xi. \quad (4.57)$$

By (L2), for any $\lambda \in (0, \lambda_0)$,

$$\mathfrak{a}_\xi''(\lambda) = \int_{\mathbb{R}} s^2 \check{a}^+(s) e^{\lambda s} ds > 0, \quad (4.58)$$

therefore, $\mathfrak{a}_\xi'(\lambda)$ is increasing on $(0, \lambda_0)$; in particular, by (4.56), we have, for any $\lambda \in (0, \lambda_0)$,

$$\int_{\mathbb{R}} s \check{a}^+(s) e^{\lambda s} ds = \mathfrak{a}_\xi'(\lambda) > \mathfrak{a}_\xi'(0) = \int_{\mathbb{R}} s \check{a}^+(s) ds = \mathfrak{m}_\xi. \quad (4.59)$$

Next, by (L6), $F_\xi(0+) = \varkappa^+ - m > 0$, hence,

$$G_\xi(0+) = \infty. \quad (4.60)$$

Finally, for $\lambda \in (0, \lambda_0)$, we have

$$G_\xi'(\lambda) = \lambda^{-2}(\lambda F_\xi'(\lambda) - F_\xi(\lambda)) = \lambda^{-1}(F_\xi'(\lambda) - G_\xi(\lambda)), \quad (4.61)$$

$$G_\xi''(\lambda) = \lambda^{-1}(F_\xi''(\lambda) - 2G_\xi'(\lambda)). \quad (4.62)$$

We will distinguish two cases.

Case 1. There exists $\mu \in (0, \lambda_0)$ with $G'_\xi(\mu) = 0$. Then, by (4.62), (4.58),

$$G''_\xi(\mu) = \mu^{-1} F''_\xi(\mu) = \mu^{-1} \varkappa^+ \mathbf{a}''_\xi(\mu) > 0.$$

Hence any stationary point of G_ξ is with necessity a point of local minimum, therefore, G_ξ has at most one such a point, thus it will be a global minimum. Moreover, by (4.61), (4.59), $G'(\mu) = 0$ implies

$$G_\xi(\mu) = F'_\xi(\mu) = \varkappa^+ \mathbf{a}'_\xi(\mu) > \varkappa^+ \mathbf{m}_\xi. \quad (4.63)$$

Therefore, in the Case 1, one can choose $\lambda_* = \mu$ (which is unique then) to fulfill the statement.

Find when the Case 1 is possible.

1. Let $\lambda_0 = \infty$. Note that (A8) implies that there exist $\delta' > 0$, $\rho' > 0$, such that $\check{a}^+(s) \geq \rho'$, for a.a. $s \in [r, r + \delta']$. Indeed, fix, for the case $d \geq 2$, a basis $\eta_1, \dots, \eta_{d-1}$ of $H_\xi = \{\xi\}^\perp$, cf. definition of (4.6), then

$$B_\delta(r\xi) \supset \left\{ (r + \sigma)\xi + \tau_1\eta_1 + \dots + \tau_{d-1}\eta_{d-1} \mid |\sigma| \leq \frac{\delta}{\sqrt{d}}, |\tau_i| \leq \frac{\delta}{\sqrt{d}} \right\}.$$

Therefore, by (4.6) and (A8),

$$\check{a}^+(s) \geq \rho \left(\frac{2\delta}{\sqrt{d}} \right)^{d-1} =: \rho', \quad s \in [r, r + \delta'], \quad \delta' := \frac{\delta}{\sqrt{d}}.$$

Hence if $\lambda_0 = \infty$, then

$$\frac{1}{\lambda} \mathbf{a}_\xi(\lambda) \geq \frac{1}{\lambda} \int_r^{r+\delta'} \check{a}^+(s) e^{\lambda s} ds \geq \rho' \frac{1}{\lambda^2} (e^{\lambda(r+\delta')} - e^{\lambda r}) \rightarrow \infty, \quad (4.64)$$

as $\lambda \rightarrow \infty$. Then, in such a case, $G_\xi(\infty) = \infty$. Therefore, by (4.60), there exists a zero of G'_ξ .

2. Let $\lambda_0 < \infty$ and $\mathbf{a}_\xi(\lambda_0) = \infty$. Then, again, (4.60) implies the existence of a zero of G'_ξ on $(0, \lambda_0)$.
3. Let $\lambda_0 < \infty$ and $\mathbf{a}_\xi(\lambda_0) < \infty$. By (4.57), (4.61),

$$\lim_{\lambda \rightarrow 0^+} \lambda^2 G'_\xi(\lambda) = -F_\xi(0+) = -(\varkappa^+ - m) < 0.$$

Therefore, the function G'_ξ has a zero on $(0, \lambda_0)$ if and only if takes a positive value at some point from $(0, \lambda_0)$.

Now, one can formulate and consider the opposite to the Case 1.

Case 2. Let $\lambda_0 < \infty$, $\mathbf{a}_\xi(\lambda_0) < \infty$, and

$$G'_\xi(\lambda) < 0, \quad \lambda \in (0, \lambda_0). \quad (4.65)$$

Therefore,

$$\inf_{\lambda > 0} G_\xi(\lambda) = \inf_{\lambda \in (0, \lambda_0]} G_\xi(\lambda) = \lim_{\lambda \rightarrow \lambda_0^-} G_\xi(\lambda) = G_\xi(\lambda_0), \quad (4.66)$$

by (L7). Hence we have the first equality in (4.55), by setting $\lambda_* := \lambda_0$. To prove the second inequality in (4.55), note that, by (4.61), the inequality (4.65) is equivalent to $F'_\xi(\lambda) < G_\xi(\lambda)$, $\lambda \in (0, \lambda_0)$. Therefore, by (4.66), (4.57), (4.59),

$$G_\xi(\lambda_0) = \inf_{\lambda \in (\frac{\lambda_0}{2}, \lambda_0)} G_\xi(\lambda) \geq \inf_{\lambda \in (\frac{\lambda_0}{2}, \lambda_0)} F'_\xi(\lambda) \geq \varkappa^+ \mathfrak{a}'_\xi\left(\frac{\lambda_0}{2}\right) > \varkappa^+ \mathfrak{m}_\xi,$$

where we used again that, by (4.58), \mathfrak{a}'_ξ and hence F'_ξ are increasing on $(0, \lambda_0)$. The statement is fully proved now. \square

The second case in the proof of Proposition 4.17 requires additional analysis. Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$, $\lambda_0 := \lambda_0(\check{a}^+)$. By (L2), one can define the following function

$$\mathfrak{t}_\xi(\lambda) := \varkappa^+ \int_{\mathbb{R}} (1 - \lambda s) \check{a}^+(s) e^{\lambda s} ds \in \mathbb{R}, \quad \lambda \in [0, \lambda_0]. \quad (4.67)$$

Note that

$$\int_{\mathbb{R}_-} |s| \check{a}^+(s) e^{\lambda_0 s} ds < \infty, \quad (4.68)$$

and $\int_{\mathbb{R}_+} s \check{a}^+(s) e^{\lambda_0 s} ds \in (0, \infty]$ is well-defined. Then, in the case $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) < \infty$, one can continue \mathfrak{t}_ξ at λ_0 , namely,

$$\mathfrak{t}_\xi(\lambda_0) := \varkappa^+ \int_{\mathbb{R}} (1 - \lambda_0 s) \check{a}^+(s) e^{\lambda_0 s} ds \in [-\infty, \varkappa^+). \quad (4.69)$$

To prove the latter inclusion, i.e. that $\mathfrak{t}_\xi(\lambda_0) < \varkappa^+$, consider the function $f_0(s) := (1 - \lambda_0 s) e^{\lambda_0 s}$, $s \in \mathbb{R}$. Then, $f'_0(s) = -\lambda_0^2 s e^{\lambda_0 s}$, and thus $f_0(s) < f_0(0) = 1$, $s \neq 0$. Moreover, the function $g_0(s) = f_0(-s) - f_0(s)$, $s \geq 0$ is such that $g'_0(s) = \lambda_0^2 s (e^{\lambda_0 s} - e^{-\lambda_0 s}) > 0$, $s > 0$. As a result, for any $\delta > 0$, $f_0(-\delta) > f_0(\delta)$, and

$$\int_{\mathbb{R}} f_0(s) \check{a}^+(s) ds \leq f_0(-\delta) \int_{\mathbb{R} \setminus [-\delta, \delta]} \check{a}^+(s) ds + \int_{[-\delta, \delta]} \check{a}^+(s) ds < \int_{\mathbb{R}} \check{a}^+(s) ds = 1.$$

Proposition 4.18. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose also that $\lambda_0 := \lambda_0(\check{a}^+) < \infty$ and $\mathfrak{a}_\xi(\lambda_0) < \infty$. Then (4.65) holds iff*

$$\mathfrak{t}_\xi(\lambda_0) \in (0, \varkappa^+), \quad (4.70)$$

$$m \leq \mathfrak{t}_\xi(\lambda_0). \quad (4.71)$$

Proof. Define the function, cf. (4.57),

$$H_\xi(\lambda) := \lambda F'_\xi(\lambda) - F_\xi(\lambda), \quad \lambda \in (0, \lambda_0). \quad (4.72)$$

By (4.61), the condition (4.65) holds iff H_ξ is negative on $(0, \lambda_0)$. By (4.72), (4.58), one has $H'_\xi(\lambda) = \lambda F''_\xi(\lambda) > 0$, $\lambda \in (0, \lambda_0)$ and, therefore, H_ξ is (strictly) increasing on $(0, \lambda_0)$. By Proposition 4.17, G'_ξ , and hence H_ξ , are negative on a right-neighborhood of 0. As a result, $H_\xi(\lambda) < 0$ on $(0, \lambda_0)$ iff

$$\lim_{\lambda \rightarrow \lambda_0^-} H_\xi(\lambda) \leq 0. \quad (4.73)$$

On the other hand, by (4.57), (4.67), one can rewrite $H_\xi(\lambda)$ as follows:

$$H_\xi(\lambda) = -\mathfrak{t}_\xi(\lambda) + m, \quad \lambda \in (0, \lambda_0). \quad (4.74)$$

By the monotone convergence theorem,

$$\lim_{\lambda \rightarrow \lambda_0^-} \int_{\mathbb{R}_+} (\lambda s - 1) \check{a}^+(s) e^{\lambda s} ds = \int_{\mathbb{R}_+} (\lambda_0 s - 1) \check{a}^+(s) e^{\lambda_0 s} ds \in (-1, \infty].$$

Therefore, by (4.68), (4.74), $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ iff $H_\xi(\lambda_0) = \lim_{\lambda \rightarrow \lambda_0^-} H_\xi(\lambda) \in \mathbb{R}$. Next, clearly, $H_\xi(\lambda_0) \in (m - \varkappa^+, 0]$ holds true iff both (4.71) and (4.70) hold.

As a result, (4.65) is equivalent to (4.73) and the latter, by (4.68), implies that $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ and hence $H_\xi(\lambda_0) \in (m - \varkappa^+, 0]$. Vice versa, (4.70) yields $\mathfrak{t}_\xi(\lambda_0) \in \mathbb{R}$ that together with (4.71) give that $H_\xi(\lambda_0) \leq 0$, i.e. that (4.65) holds. \square

According to the above, it is natural to consider two subclasses of functions from \mathcal{U}_ξ , cf. Definition 4.16.

Definition 4.19. Let $\xi \in S^{d-1}$ be fixed. We denote by \mathcal{V}_ξ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that one of the following assumptions does hold:

1. $\lambda_0 := \lambda_0(\check{a}^+) = \infty$;
2. $\lambda_0 < \infty$ and $\mathfrak{a}_\xi(\lambda_0) = \infty$;
3. $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$ and $\mathfrak{t}_\xi(\lambda_0) \in [-\infty, 0]$, where $\mathfrak{t}_\xi(\lambda_0)$ is given by (4.69).

Correspondingly, we denote by \mathcal{W}_ξ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that $\lambda_0 < \infty$, $\mathfrak{a}_\xi(\lambda_0) < \infty$, and $\mathfrak{t}_\xi(\lambda_0) \in (0, \varkappa^+)$. Clearly, $\mathcal{U}_\xi = \mathcal{V}_\xi \cup \mathcal{W}_\xi$.

As a result, combining the proofs and statements of Propositions 4.17 and 4.18, one immediately gets the following corollary.

Corollary 4.20. Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, and λ_* be the same as in the Proposition 4.17. Then $\lambda_* < \lambda_0 := \lambda_0(\check{a}^+)$ iff either $a^+ \in \mathcal{V}_\xi$ or $a^+ \in \mathcal{W}_\xi$ with $m > \mathfrak{t}_\xi(\lambda_0)$. Moreover, in the both cases, $G'(\lambda_*) = 0$. Correspondingly, $\lambda_* = \lambda_0$ iff $a^+ \in \mathcal{W}_\xi$ with $m \leq \mathfrak{t}_\xi(\lambda_0)$. Moreover, in this case,

$$\lim_{\lambda \rightarrow \lambda_0^-} G'_\xi(\lambda) = \frac{m - \mathfrak{t}_\xi(\lambda_0)}{\lambda_0^2} \leq 0. \quad (4.75)$$

Example 4.21. To demonstrate the cases of Definition 4.19 on an example, consider the following family of functions

$$\check{a}^+(s) := \frac{\alpha e^{-\mu|s|^p}}{1 + |s|^q}, \quad s \in \mathbb{R}, p \geq 0, q \geq 0, \mu > 0, \quad (4.76)$$

where $\alpha > 0$ is a normalising constant to get (4.7). Clearly, the case $p \in [0, 1)$ implies $\lambda_0(\check{a}^+) = 0$, that is impossible under assumption (A5). Next, $p > 1$ leads to $\lambda_0(\check{a}^+) = \infty$, in particular, the corresponding $a^+ \in \mathcal{V}_\xi$. Let now $p = 1$, then $\lambda_0(\check{a}^+) = \mu$. The case $q \in [0, 1]$ gives $\mathfrak{a}_\xi(\lambda_0) = \infty$, i.e. $a^+ \in \mathcal{V}_\xi$ as well.

In the case $q \in (1, 2]$, we will have that $\mathbf{a}_\xi(\lambda_0) < \infty$, however, $\int_{\mathbb{R}} s\check{a}^+(s)e^{\mu s} ds = \infty$, i.e. $\mathbf{t}_\xi(\mu) = -\infty$, and again $a^+ \in \mathcal{V}_\xi$. Let $q > 2$; then, by (4.67),

$$\begin{aligned} \mathbf{t}_\xi(\mu) &= \varkappa^+ \alpha \int_{\mathbb{R}_-} \frac{1 - \mu s}{1 + |s|^q} e^{2\mu s} ds + \varkappa^+ \alpha \int_{\mathbb{R}_+} \frac{1 - \mu s}{1 + s^q} ds \\ &\geq \varkappa^+ \alpha \int_{\mathbb{R}_+} \frac{1 - \mu s}{1 + s^q} ds = \frac{\pi \varkappa^+ \alpha}{q} \left(\frac{1}{\sin \frac{\pi}{q}} - \frac{\mu}{\sin \frac{2\pi}{q}} \right) > 0, \end{aligned}$$

if only $\mu < 2 \cos \frac{\pi}{q}$; then we have $a^+ \in \mathcal{W}_\xi$. On the other hand, using the inequality $te^{-t} \leq e^{-1}$, $t \geq 0$, one gets

$$\begin{aligned} \mathbf{t}_\xi(\mu) &= \varkappa^+ \alpha \int_{\mathbb{R}_+} \frac{(1 + \mu s)e^{-2\mu s} + 1 - \mu s}{1 + s^q} ds \tag{4.77} \\ &\leq \varkappa^+ \alpha \int_{\mathbb{R}_+} \frac{1 + \frac{1}{2e} + 1 - \mu s}{1 + s^q} ds = \frac{\pi \varkappa^+ \alpha}{2eq} \left(\frac{1 + 4e}{\sin \frac{\pi}{q}} - \frac{2e\mu}{\sin \frac{2\pi}{q}} \right) < 0, \end{aligned}$$

if only $\mu > \frac{1+4e}{e} \cos \frac{\pi}{q}$; then we have $a^+ \in \mathcal{V}_\xi$. Since

$$\frac{d}{d\mu} ((1 + \mu s)e^{-2\mu s} + 1 - \mu s) = -se^{-2\mu s}(1 + 2s\mu) - s < 0, \quad s > 0, \mu > 0,$$

we have from (4.77), that $\mathbf{t}_\xi(\mu)$ is strictly decreasing and continuous in μ , therefore, there exist a critical value $\mu_* \in (2 \cos \frac{\pi}{q}, (4 + e^{-1}) \cos \frac{\pi}{q})$, such that, for all $\mu \geq \mu_*$, $a^+ \in \mathcal{V}_\xi$, whereas, for $\mu \in (0, \mu_*)$, $a^+ \in \mathcal{W}_\xi$.

Now we are ready to prove the main statement of this subsection.

Theorem 4.22. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 4.9, and let, for any $c \geq c_*(\xi)$, the function $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c . Let $\lambda_* \in I_\xi$ be the same as in the Proposition 4.17. Denote, as usual, $\lambda_0 := \lambda_0(\check{a}^+)$.*

1. *The following relations hold*

$$c_*(\xi) = \min_{\lambda > 0} \frac{\varkappa^+ \mathbf{a}_\xi(\lambda) - m}{\lambda} = \frac{\varkappa^+ \mathbf{a}_\xi(\lambda_*) - m}{\lambda_*} > \varkappa^+ \mathbf{m}_\xi, \tag{4.78}$$

$$\lambda_0(\psi) \in (0, \lambda_*], \tag{4.79}$$

$$(\mathfrak{L}\psi)(\lambda_0(\psi)) = \infty; \tag{4.80}$$

and the mapping $(0, \lambda_] \ni \lambda_0(\psi) \mapsto c \in [c_*(\xi), \infty)$ is a (strictly) monotonically decreasing bijection, given by*

$$c = \frac{\varkappa^+ \mathbf{a}_\xi(\lambda_0(\psi)) - m}{\lambda_0(\psi)}. \tag{4.81}$$

In particular, $\lambda_0(\psi_{c_(\xi)}) = \lambda_*$.*

2. *For $a^+ \in \mathcal{V}_\xi$ and for $a^+ \in \mathcal{W}_\xi$ with $m > \mathbf{t}_\xi(\lambda_0)$ we have $\lambda_* < \lambda_0$ and there exists another representation for the minimal speed than (4.81), namely,*

$$\begin{aligned} c_*(\xi) &= \varkappa^+ \int_{\mathbb{R}^d} x \cdot \xi a^+(x) e^{\lambda_* x \cdot \xi} dx \\ &= \varkappa^+ \int_{\mathbb{R}} s\check{a}^+(s) e^{\lambda_* s} ds > \varkappa^+ \mathbf{m}_\xi. \end{aligned} \tag{4.82}$$

Moreover, for all $\lambda \in (0, \lambda_*]$,

$$t_\xi(\lambda) \geq m, \quad (4.83)$$

and the equality holds for $\lambda = \lambda_*$ only.

3. For $a^+ \in \mathcal{W}_\xi$ with $m \leq t_\xi(\lambda_0)$, one has $\lambda_* = \lambda_0$. Moreover, the inequality (4.83) also holds as well as, for all $\lambda \in (0, \lambda_*]$,

$$c \geq \varkappa^+ \int_{\mathbb{R}} s \check{a}^+(s) e^{\lambda s} ds, \quad (4.84)$$

whereas the equalities in (4.83) and (4.84) hold true now for $m = t_\xi(\lambda_0)$, $\lambda = \lambda_*$, $c = c_*(\xi)$ only.

Proof. By Theorem 4.9, for any $c \geq c_*(\xi)$ there exists a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$, cf. Remark 4.7, which define a traveling wave solution (4.1) to (2.1) in the direction ξ . Then, by (4.32), we get

$$-c\psi'(s) = \varkappa^+(\check{a}^+ * \psi)(s) - m\psi(s) - \varkappa^-\psi(s)(\check{a}^- * \psi)(s), \quad s \in \mathbb{R}. \quad (4.85)$$

Step 1. By Proposition 4.13, we have that $\lambda_0(\psi) > 0$. Note also that the condition (A2) implies (4.9), therefore, $\lambda_0(\check{a}^-) \geq \lambda_0(\check{a}^+) > 0$. Take any $z \in \mathbb{C}$ with

$$0 < \operatorname{Re} z < \min\{\lambda_0(\check{a}^+), \lambda_0(\psi)\} \leq \lambda_0(\psi) < \lambda_0(\psi(\check{a}^- * \psi)), \quad (4.86)$$

where the later inequality holds by (4.52). As a result, by (L5), (L8), being multiplied on e^{zs} the l.h.s. of (4.85) will be integrable (in s) over \mathbb{R} . Hence, for any z which satisfies (4.86), $(\mathfrak{L}\psi')(z)$ converges. By (L4), it yields $\lambda_0(\psi) \geq \lambda_0(\psi') \geq \min\{\lambda_0(\check{a}^+), \lambda_0(\psi)\}$.

Therefore, by (4.49), (4.50), we get from (4.85)

$$\begin{aligned} cz(\mathfrak{L}\psi)(z) &= \varkappa^+(\mathfrak{L}\check{a}^+)(z)(\mathfrak{L}\psi)(z) \\ &\quad - m(\mathfrak{L}\psi)(z) - \varkappa^-(\mathfrak{L}(\psi(\check{a}^- * \psi)))(z), \end{aligned} \quad (4.87)$$

if only

$$0 < \operatorname{Re} z < \min\{\lambda_0(\check{a}^+), \lambda_0(\psi)\}. \quad (4.88)$$

Since $\psi \not\equiv 0$, we have that $(\mathfrak{L}\psi)(z) \neq 0$, therefore, one can rewrite (4.87) as follows

$$G_\xi(z) - c = \frac{\varkappa^-(\mathfrak{L}(\psi(\check{a}^- * \psi)))(z)}{z(\mathfrak{L}\psi)(z)}, \quad (4.89)$$

if (4.88) holds. By (4.86), both nominator and denominator in the r.h.s. of (4.89) are analytic on $0 < \operatorname{Re} z < \lambda_0(\psi)$, therefore. Suppose that $\lambda_0(\psi) > \lambda_0(\check{a}^+)$, then (4.89) holds on $0 < \operatorname{Re} z < \lambda_0(\check{a}^+)$, however, the r.h.s. of (4.89) would be analytic at $z = \lambda_0(\check{a}^+)$, whereas, by (L3), the l.h.s. of (4.89) has a singularity at this point. As a result,

$$\lambda_0(\check{a}^+) \geq \lambda_0(\psi), \quad (4.90)$$

for any traveling wave profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$. Thus one gets that (4.89) holds true on $0 < \operatorname{Re} z < \lambda_0(\psi)$.

Prove that

$$\lambda_0(\psi) < \infty. \quad (4.91)$$

Since $0 \leq \psi \leq \theta$ yields $0 \leq a^- * \psi \leq \theta$, one gets from (4.89) that, for any $0 < \lambda < \lambda_0(\psi)$,

$$c \geq G_\xi(\lambda) - \varkappa^- \frac{\theta}{\lambda} = \frac{\varkappa^+(\mathfrak{L}\tilde{a}^+)(\lambda) - \varkappa^+}{\lambda}. \quad (4.92)$$

If $\lambda_0(\tilde{a}^+) < \infty$ then (4.91) holds by (4.90). Suppose that $\lambda_0(\tilde{a}^+) = \infty$. By (4.64), the r.h.s. of (4.92) tends to ∞ as $\lambda \rightarrow \infty$, thus the latter inequality cannot hold for all $\lambda > 0$; and, as a result, (4.91) does hold.

Step 2. Recall that (4.54) holds. Suppose that $c \geq c_*(\xi)$ is such that, cf. (4.55),

$$c \geq G_\xi(\lambda_*) = \inf_{\lambda_0 \in (0, \lambda_*]} G_\xi(\lambda) = \inf_{\lambda_0 \in I_\xi} G_\xi(\lambda). \quad (4.93)$$

Then, by Proposition 4.17, the equation $G_\xi(\lambda) = c$, $\lambda \in I_\xi$, has one or two solutions. Let λ_c be the unique solution in the first case or the smaller of the solutions in the second one. Since G_ξ is decreasing on $(0, \lambda_*]$, we have $\lambda_c \leq \lambda_*$. Since the nominator in the r.h.s. of (4.89) is positive, we immediately get from (4.89) that

$$(\mathfrak{L}\psi)(\lambda_c) = \infty, \quad (4.94)$$

therefore, $\lambda_c \geq \lambda_0(\psi)$. On the other hand, one can rewrite (4.89) as follows

$$(\mathfrak{L}\psi)(z) = \frac{\varkappa^-(\mathfrak{L}(\psi(\tilde{a}^- * \psi)))(z)}{z(G_\xi(z) - c)}. \quad (4.95)$$

By (4.89), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \lambda_0(\psi) \leq \lambda_c \leq \lambda_* \leq \lambda_0(\tilde{a}^+)$. As a result, by (4.86), (L1), and (L3), $\lambda_c = \lambda_0(\psi)$, that together with (4.94) proves (4.79) and (4.80), for waves whose speeds satisfy (4.93). By (4.11), we immediately get, for such speeds, (4.81) as well. Moreover, (4.81) defines a strictly monotone function $(0, \lambda_*] \ni \lambda_0(\psi) \mapsto c \in [G_\xi(\lambda_*), \infty)$.

Next, by (4.67), (L2), (4.57), (4.61), we have that, for any $\lambda \in I_\xi$,

$$\mathfrak{t}_\xi(\lambda) = \varkappa^+ \mathfrak{a}_\xi(\lambda) - \varkappa^+ \lambda \mathfrak{a}'_\xi(\lambda) = m + F_\xi(\lambda) - \lambda F'_\xi(\lambda) = m - \lambda^2 G'_\xi(\lambda). \quad (4.96)$$

Recall that, by Proposition 4.17, the function G_ξ is strictly decreasing on $(0, \lambda_*)$. Then (4.96) implies that $\mathfrak{t}_\xi(\lambda) > m$, $\lambda \in (0, \lambda_*)$. On the other hand, by the second equality in (4.61), the inequality $G'_\xi(\lambda) < 0$, $\lambda \in (0, \lambda_*)$, yields $G_\xi(\lambda) > F'_\xi(\lambda)$, for such λ . Let $c > G_\xi(\lambda_*)$. By (4.81), (4.57), we have then $c > \varkappa^+ \mathfrak{a}'_\xi(\lambda)$, for all $\lambda \in [\lambda_0(\psi), \lambda_*)$. By (4.58), F'_ξ is increasing, hence, by (L2), the strict inequality in (4.84) does hold, for $\lambda \in (0, \lambda_*)$.

Let again $c \geq G_\xi(\lambda_*)$. Let either $a^+ \in \mathcal{V}_\xi$ or $a^+ \in \mathcal{W}_\xi$ with $m > \mathfrak{t}_\xi(\lambda_0)$. Then, by Corollary 4.20, $\lambda_* < \lambda_0(\tilde{a}^+)$ and $G'(\lambda_*) = 0$. By (4.61), the latter

equality and (4.96) give $t_\xi(\lambda_*) = m$, that fulfills the proof of (4.83), for such a^+ and m . Moreover, by (4.63),

$$G_\xi(\lambda_*) = \varkappa^+ a'_\xi(\lambda_*) = \varkappa^+ \int_{\mathbb{R}} s \check{a}^+(s) e^{\lambda_* s} ds. \quad (4.97)$$

Let $a^+ \in \mathcal{W}_\xi$ with $0 < m \leq t_\xi(\lambda_0)$, then $\lambda_* = \lambda_0(\check{a}^+)$. It means that $t_\xi(\lambda_*) = m$ if $m = t_\xi(\lambda_0)$ only, otherwise, $t_\xi(\lambda_*) > m$. Next, we get from (4.93), (4.61) (4.75),

$$c \geq G_\xi(\lambda_*) \geq \lim_{\lambda \rightarrow \lambda_*^-} F'_\xi(\lambda) = \varkappa^+ \int_{\mathbb{R}} s \check{a}^+(s) e^{\lambda_* s} ds, \quad (4.98)$$

where the latter equality may be easily verified if we rewrite, for $\lambda \in (0, \lambda_*)$,

$$F'_\xi(\lambda) = \varkappa^+ \int_{\mathbb{R}_-} s \check{a}^+(s) e^{\lambda s} ds + \varkappa^+ \int_{\mathbb{R}_+} s \check{a}^+(s) e^{\lambda s} ds, \quad (4.99)$$

and apply the dominated convergence theorem to the first integral and the monotone convergence theorem for the second one. On the other hand, (4.75) implies that the second inequality in (4.98) will be strict iff $m < t_\xi(\lambda_0)$, whereas, for $c = G_\xi(\lambda_*) = \inf_{\lambda > 0} G_\xi(\lambda)$ and $m = t_\xi(\lambda_0)$, we will get all equalities in (4.98).

Step 3. Let now $c \geq c_*(\xi)$ and suppose that $\lambda_0(\check{a}^+) > \lambda_0(\psi)$. Prove that (4.93) does hold. On the contrary, suppose that the c is such that

$$c_*(\xi) \leq c < \inf_{\lambda \in (0, \lambda_*)} G_\xi(\lambda) = \inf_{\lambda > 0} G_\xi(\lambda). \quad (4.100)$$

Again, by (4.89), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \lambda_0(\psi)$, and (4.95) holds, for such z . Since we supposed that $\lambda_0(\check{a}^+) > \lambda_0(\psi)$, one gets from (4.86), that both nominator and denominator of the r.h.s. of (4.95) are analytic on

$$\{0 < \operatorname{Re} z < \nu\} \supsetneq \{0 < \operatorname{Re} z < \lambda_0(\psi)\},$$

where $\nu = \min\{\lambda_0(\check{a}^+), \lambda_0(\psi(\check{a}^- * \psi))\}$. On the other hand, (L3) implies that $\mathfrak{L}\psi$ has a singularity at $z = \lambda_0(\psi)$. Since $\mathfrak{L}(\psi(\check{a}^- * \psi))(\lambda_0(\psi)) > 0$, the equality (4.95) would be possible if only $G_\xi(\lambda_0(\psi)) = c$, that contradicts (4.100).

Step 4. By (4.90), it remains to prove that, for $c \geq c_*(\xi)$, (4.93) does holds, in the case when we have $\lambda_0(\check{a}^+) = \lambda_0(\psi)$. Again on the contrary, suppose that (4.100) holds. For $0 < \operatorname{Re} z < \lambda_0(\psi)$, we can rewrite (4.87) as follows

$$z(\mathfrak{L}\psi)(z)(G_\xi(z) - c) = \varkappa^- (\mathfrak{L}(\psi(\check{a}^- * \psi)))(z). \quad (4.101)$$

In the notations of the proof of Lemma 4.15, the functions $\mathfrak{L}^- \psi$ and $\mathfrak{L}^- \check{a}^+$ are analytic on $\operatorname{Re} z > 0$. Moreover, $(\mathfrak{L}^+ \psi)(\lambda)$ and $(\mathfrak{L}^+ \check{a}^+)(\lambda)$ are increasing on $0 < \lambda < \lambda_0(\check{a}^+) = \lambda_0(\psi)$. Then, cf. (4.99), by the monotone convergence theorem, we will get from (4.101) and (4.86), that

$$\int_{\mathbb{R}} \psi(s) e^{\lambda_0(\psi)s} ds < \infty, \quad \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\check{a}^+)s} ds < \infty. \quad (4.102)$$

We are going to apply now Proposition 3.17, in the case $d = 1$, to the equation (4.4), where the initial condition ψ is a wave profile with the speed c

which satisfies (4.100). Namely, we set $\Delta_R := (-\infty, R) \nearrow \mathbb{R}$, $R \rightarrow \infty$, and let $\check{a}_R^\pm, \check{A}_R^\pm$ be given by (3.37), (3.39) respectively with $d = 1$ and a^\pm replaced by \check{a}^\pm . Consider a strictly monotonic sequence $\{R_n \mid n \in \mathbb{N}\}$, such that $0 < R_n \rightarrow \infty$, $n \rightarrow \infty$ and

$$\check{A}_{R_n}^+ > \frac{m}{\varepsilon^+}, \quad (4.103)$$

cf. (3.41). Let $\theta_n := \theta_{R_n}$ be given by (3.40) with A_R^\pm replaced by $\check{A}_{R_n}^\pm$. Then, by (3.44), $\theta_n \leq \theta$, $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Consider the ‘truncated’ equation (3.38) with $d = 1$, a_R^\pm replaced by $a_{R_n}^\pm$, and the initial condition $w_0(s) := \min\{\psi(s), \theta_n\} \in C_{ub}(\mathbb{R})$. By Proposition 3.17, there exists the unique solution $w^{(n)}(s, t)$ of the latter equation. Moreover, if we denote the corresponding nonlinear mapping, cf. Definition 3.13 and Proposition 4.8, by $\tilde{Q}_t^{(n)}$, we will have from (3.42) and (3.43) that

$$(\tilde{Q}_t^{(n)} w_0)(s) \leq \theta_n, \quad s \in \mathbb{R}, t \geq 0, \quad (4.104)$$

and

$$(\tilde{Q}_t^{(n)} w_0)(s) \leq \phi(s, t), \quad (4.105)$$

where ϕ solves (4.4). By (4.8), we get from (4.105) that $(\tilde{Q}_1^{(n)} w_0)(s+c) \leq \psi(s)$, $s \in \mathbb{R}$. The latter inequality together with (4.104) imply

$$(\tilde{Q}_1^{(n)} w_0)(s+c) \leq w_0(s). \quad (4.106)$$

Then, by the same arguments as in the proof of Theorem 4.9, cf. (4.17), we obtain from [80, Theorem 5] that there exists a traveling wave ψ_n for the equation (3.38) (with $d = 1$ and a_R^\pm replaced by $a_{R_n}^\pm$), whose speed will be exactly c (and c satisfies (4.100)).

Now we are going to get a contradiction, by proving that

$$\inf_{\lambda > 0} G_\varepsilon(\lambda) = \lim_{n \rightarrow \infty} \inf_{\lambda > 0} G_\varepsilon^{(n)}(\lambda), \quad (4.107)$$

where $G_\varepsilon^{(n)}$ is given by (4.53) with \check{a}^\pm replaced by $\check{a}_n^\pm := \check{a}_{R_n}^\pm$. The sequence of functions $G_\varepsilon^{(n)}$ is point-wise monotone in n and it converges to G_ε point-wise, for $0 < \lambda \leq \lambda_0(\check{a}^+)$; note we may include $\lambda_0(\check{a}^+)$ here, according to (4.102). Moreover, $G_\varepsilon^{(n)}(\lambda) \leq G_\varepsilon(\lambda)$, $0 < \lambda \leq \lambda_0(\check{a}^+)$. As a result, for any $n \in \mathbb{N}$,

$$G_\varepsilon^{(n)}(\lambda_*^{(n)}) = \inf_{\lambda > 0} G_\varepsilon^{(n)}(\lambda) \leq \inf_{\lambda > 0} G_\varepsilon(\lambda) = G_\varepsilon(\lambda_*). \quad (4.108)$$

Hence if we suppose that (4.107) does not hold, then

$$\inf_{\lambda > 0} G_\varepsilon(\lambda) - \lim_{n \rightarrow \infty} \inf_{\lambda > 0} G_\varepsilon^{(n)}(\lambda) > 0.$$

Therefore, there exist $\delta > 0$ and $N \in \mathbb{N}$, such that

$$G_\varepsilon^{(n)}(\lambda_*^{(n)}) = \inf_{\lambda > 0} G_\varepsilon^{(n)}(\lambda) \leq \inf_{\lambda > 0} G_\varepsilon(\lambda) - \delta = G_\varepsilon(\lambda_*) - \delta, \quad n \geq N. \quad (4.109)$$

Clearly, (3.37) with $\Delta_{R_n} = (-\infty, R_n)$ implies that $\lambda_0(\check{a}_n^+) = \infty$, hence $G_\xi^{(n)}$ is analytic on $\operatorname{Re} z > 0$. One can repeat all considerations of the first three steps of this proof for the equation (3.38). Let $c_*^{(n)}(\xi)$ be the corresponding minimal traveling wave speed, according to Theorem 4.9. Then the corresponding inequality (4.91) will show that the abscissa of an arbitrary traveling wave to (3.38) is less than $\lambda_0(\check{a}_n^+) = \infty$. As a result, the inequality $c_*^{(n)}(\xi) < \inf_{\lambda > 0} G_\xi^{(n)}(\lambda)$, cf. (4.100), is impossible, and hence, by the Step 3,

$$c \geq c_*^{(n)}(\xi) = \inf_{\lambda > 0} G_\xi^{(n)}(\lambda) = G_\xi^{(n)}(\lambda_*^{(n)}), \quad (4.110)$$

where $\lambda_*^{(n)}$ is the unique zero of the function $\frac{d}{d\lambda} G_\xi^{(n)}(\lambda)$. Let $\mathfrak{t}_\xi^{(n)}$ be given on $(0, \infty)$ by (4.67) with \check{a}^+ replaced by \check{a}_n^+ . Then

$$\frac{d}{d\lambda} \mathfrak{t}_\xi^{(n)}(\lambda) = -\lambda \varkappa^+ \int_{-\infty}^{R_n} \check{a}^+(s) s^2 e^{\lambda s} ds < 0, \quad \lambda > 0. \quad (4.111)$$

By (4.83), the unique point of intersection of the strictly decreasing function $y = \mathfrak{t}_\xi^{(n)}(\lambda)$ and the horizontal line $y = m$ is exactly the point $(\lambda_*^{(n)}, 0)$.

Prove that there exist $\lambda_1 > 0$, such that $\lambda_*^{(n)} > \lambda_1$, $n \geq N$, and there exists $N_1 \geq N$, such that $\mathfrak{t}_\xi^{(n)}(\lambda) \leq \mathfrak{t}_\xi^{(m)}(\lambda)$, $n > m \geq N_1$, $\lambda \geq \lambda_1$. Recall that (4.103) holds; we have

$$\begin{aligned} \lambda G_\xi^{(n)}(\lambda) &= \varkappa^+ \int_{\mathbb{R}} \check{a}_n^+(s) (e^{\lambda s} - 1) ds + \varkappa^+ \check{A}_{R_n}^+ - m \\ &\geq \varkappa^+ \int_{-\infty}^0 \check{a}_n^+(s) (e^{\lambda s} - 1) ds + \varkappa^+ \check{A}_{R_1}^+ - m, \end{aligned}$$

and the inequality $1 - e^{-s} \leq s$, $s \geq 0$ implies that

$$\left| \int_{-\infty}^0 \check{a}_n^+(s) (e^{\lambda s} - 1) ds \right| \leq \lambda \int_{-\infty}^0 \check{a}_n^+(s) |s| ds \leq \lambda \int_{\mathbb{R}} \check{a}^+(s) |s| ds < \infty,$$

by (A7). As a result, if we set

$$\lambda_1 := (\varkappa^+ \check{A}_{R_1}^+ - m) \left(\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) |s| ds + |G_\xi(\lambda_*)| \right)^{-1} > 0,$$

then, for any $\lambda \in (0, \lambda_1)$, we have

$$\lambda G_\xi^{(n)}(\lambda) \geq \varkappa^+ \check{A}_{R_1}^+ - m - \lambda_1 \varkappa^+ \int_{\mathbb{R}} \check{a}_n^+(s) |s| ds = \lambda_1 |G_\xi(\lambda_*)| \geq \lambda G_\xi(\lambda_*),$$

i.e. $G_\xi^{(n)}(\lambda) \geq G_\xi(\lambda_*) = \inf_{\lambda > 0} G_\xi(\lambda)$. Then, for any $n \geq N$, (4.109) implies that $\lambda_*^{(n)}$, being the minimum point for $G_\xi^{(n)}$, does not belong to the interval $(0, \lambda_1)$. Next, let $N_1 \geq N$ be such that $R_n \geq \frac{1}{\lambda_1}$, for all $n \geq N_1$. Then, for any $\lambda \geq \lambda_1$, and for any $n > m \geq N_1$, we have $R_n > R_m$ and

$$\begin{aligned} \mathfrak{t}_\xi^{(n)}(\lambda) - \mathfrak{t}_\xi^{(m)}(\lambda) &= \varkappa^+ \int_{R_m}^{R_n} (1 - \lambda s) \check{a}^+(s) e^{\lambda s} ds \\ &\leq \varkappa^+ \int_{R_m}^{R_n} (1 - \lambda_1 s) \check{a}^+(s) e^{\lambda s} ds \leq 0. \end{aligned}$$

As a result, the sequence $\{\lambda_*^{(n)} \mid n \geq N_1\} \subset [\lambda_1, \infty)$ is monotonically decreasing (cf. (4.111)). We set

$$\vartheta := \lim_{n \rightarrow \infty} \lambda_*^{(n)} \geq \lambda_1. \quad (4.112)$$

Next, for any $n, m \in \mathbb{N}$, $n > m \geq N_1$,

$$G_\xi^{(n)}(\lambda_*^{(n)}) \geq G_\xi^{(m)}(\lambda_*^{(n)}) \geq G_\xi^{(m)}(\lambda_*^{(m)}), \quad (4.113)$$

where we used that $G_\xi^{(n)}$ is increasing in n and $\lambda_*^{(m)}$ is the minimum point of $G_\xi^{(m)}$. Therefore, the sequence $\{G_\xi^{(n)}(\lambda_*^{(n)})\}$ is increasing and, by (4.109), is bounded. Then, there exists

$$\lim_{n \rightarrow \infty} G_\xi^{(n)}(\lambda_*^{(n)}) =: g \leq G_\xi(\lambda_*) - \delta. \quad (4.114)$$

Fix $m \geq N_1$ in (4.113) and pass n to infinity; then, by the continuity of $G_\xi^{(n)}$,

$$g \geq \lim_{\lambda \rightarrow \vartheta^+} G_\xi^{(m)}(\lambda) = G_\xi^{(m)}(\vartheta) \geq G_\xi^{(m)}(\lambda_*^{(m)}), \quad (4.115)$$

in particular, $\vartheta > 0$, as $G_\xi^{(m)}(0+) = \infty$. Next, if we pass m to ∞ in (4.115), we will get from (4.114)

$$\lim_{m \rightarrow \infty} G_\xi^{(m)}(\vartheta) = g \leq G_\xi(\lambda_*) - \delta < G_\xi(\lambda_*). \quad (4.116)$$

If $0 < \vartheta \leq \lambda_0(\check{a}^+)$ then

$$\lim_{m \rightarrow \infty} G_\xi^{(m)}(\vartheta) = G_\xi(\vartheta) \geq G_\xi(\lambda_*),$$

that contradicts (4.116). If $\vartheta > \lambda_0(\check{a}^+)$, then $\lim_{m \rightarrow \infty} G_\xi^{(m)}(\vartheta) = \infty$ (recall again that $\mathfrak{L}^-(\check{a}^+)(\lambda)$ is analytic and $\mathfrak{L}^-(\check{a}^+)(\lambda)$ is monotone in λ), that contradicts (4.116) as well.

The contradiction we obtained shows that (4.107) does hold. Then, for the chosen $c \geq c_*(\xi)$ which satisfies (4.100), one can find n big enough to ensure that, cf. (4.110),

$$c < \inf_{\lambda > 0} G_\xi^{(n)}(\lambda) = c_*^{(n)}(\xi).$$

However, as it was shown above, for this n there exists a profile ψ_n of a traveling wave to the ‘truncated’ equation (3.38) (with, recall, $d = 1$ and a_R^\pm replaced by $a_{R_n}^\pm$). The latter contradicts the statement of Theorem 4.9 applied to this equation, as $c_*^{(n)}(\xi)$ has to be a minimal possible speed for such waves.

Therefore, the strict inequality in (4.100) is impossible, hence, we have equality in (4.54). As a result, (4.11) implies (4.78), and (4.97) may be read as (4.82). The rest of the statement is evident now. \square

Remark 4.23. Clearly, the assumption $a^+(-x) = a^+(x)$, $x \in \mathbb{R}^d$, implies $\mathfrak{m}_\xi = 0$, for any $\xi \in S^{d-1}$. As a result, all speeds of traveling waves in any directions are positive, by (4.78).

4.3 Uniqueness of traveling waves

In this subsection we will prove the uniqueness (up to shifts) of a profile ψ for a traveling wave with given speed $c \geq c_*(\xi)$, $c \neq 0$. We will use the almost traditional now approach, namely, we find an *a priori* asymptotic for $\psi(t)$, $t \rightarrow \infty$, cf., e.g., [2, 12] and the references therein.

We start with the so-called characteristic function of the equation (2.1). Namely, for a given $\xi \in S^{d-1}$ and for any $c \in [c_*(\xi), \infty)$, we set

$$\mathfrak{h}_{\xi,c}(z) := \varkappa^+(\mathfrak{L}\check{a}^+)(z) - m - zc = zG_\xi(z) - zc, \quad \operatorname{Re} z \in I_\xi. \quad (4.117)$$

Proposition 4.24. *In conditions and notations of Theorem 4.22, let a traveling wave profile ψ correspond to a speed c . For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\lambda_0)$, we will assume, additionally, that*

$$\int_{\mathbb{R}} s^2 \check{a}^+(s) e^{\lambda_0 s} ds < \infty. \quad (4.118)$$

Then the function $\mathfrak{h}_{\xi,c}$ is analytic on $\{0 < \operatorname{Re} z < \lambda_0(\psi)\}$. Moreover, for any $\beta \in (0, \lambda_0(\psi))$, the function $\mathfrak{h}_{\xi,c}$ is continuous and does not equal to 0 on the closed strip $\{\beta \leq \operatorname{Re} z \leq \lambda_0(\psi)\}$, except the root at $z = \lambda_0(\psi)$, whose multiplicity j may be 1 or 2 only.

Proof. By (4.89) and the arguments around, $\mathfrak{h}_{\xi,c}(z) = z(G_\xi(z) - c)$ is analytic on $\{0 < \operatorname{Re} z < \lambda_0(\psi)\} \subset I_\xi$ and does not equal to 0 there. Then, by (4.81) and Proposition 4.17, the smallest positive root of the function $\mathfrak{h}_{\xi,c}(\lambda)$ on \mathbb{R} is exactly $\lambda_0(\psi)$. Prove that if $z_0 := \lambda_0(\psi) + i\beta$ is a root of $\mathfrak{h}_{\xi,c}$, then $\beta = 0$. Indeed, $\mathfrak{h}_{\xi,c}(z_0) = 0$ yields

$$\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\psi)s} \cos \beta s ds = m + c\lambda_0(\psi),$$

that together with (4.81) leads to

$$\varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\lambda_0(\psi)s} (\cos \beta s - 1) ds = 0,$$

and thus $\beta = 0$.

Regarding multiplicity of the root $z = \lambda_0(\psi)$, we note that, by Proposition 4.17 and Corollary 4.20, there exist two possibilities. If $a^+ \in \mathcal{V}_\xi$ or $a^+ \in \mathcal{W}_\xi$ with $m > \mathfrak{t}_\xi(\lambda_0)$ then $\lambda_0(\psi) \leq \lambda_* < \lambda_0(\check{a}^+)$ and, therefore, G_ξ is analytic at $z = \lambda_0(\psi)$. By the second equality in (4.117), the multiplicity j of this root for $\mathfrak{h}_{\xi,c}$ is the same as for the function $G_\xi(z) - c$. By Proposition 4.17, G_ξ is strictly decreasing on $(0, \lambda_*)$ and, therefore, $j = 1$ for $c > c_*(\xi)$. By Corollary 4.20, for $c = c_*(\xi)$, we have $G'_\xi(\lambda_0(\psi)) = G'_\xi(\lambda_*) = 0$ and, since $\mathfrak{h}''_{\xi,c}(\lambda_0) > 0$, one gets $j = 2$.

Let now $a^+ \in \mathcal{W}_\xi$ with $m \leq \mathfrak{t}_\xi(\lambda_0)$. Then, we recall, $\lambda_* = \lambda_0 := \lambda_0(\check{a}^+) < \infty$, $G_\xi(\lambda_0) < \infty$ and (4.75) hold. For $c > c_*(\xi)$, the arguments are the same as before, and they yield $j = 1$. Let $c = c_*(\xi)$. Then $\mathfrak{h}_{\xi,c}(\lambda_0) = 0$, and, for all $z \in \mathbb{C}$, $\operatorname{Re} z \in (0, \lambda_0)$, one has

$$\begin{aligned} \mathfrak{h}_{\xi,c}(\lambda_0 - z) &= \mathfrak{h}_{\xi,c}(\lambda_0 - z) - \mathfrak{h}_{\xi,c}(\lambda_0) = \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) (e^{(\lambda_0 - z)\tau} - e^{\lambda_0 \tau}) d\tau + cz \\ &= z \left(-\varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \int_0^\tau e^{-zs} ds d\tau + c \right). \end{aligned} \quad (4.119)$$

Let $z = \alpha + \beta i$, $\alpha \in (0, \lambda_0)$. Then $|e^{\lambda_0 \tau} e^{-zs}| = e^{\lambda_0 \tau - \alpha s}$. Next, for $\tau \geq 0$, $s \in [0, \tau]$, we have $e^{\lambda_0 \tau - \alpha s} \leq e^{\lambda_0 \tau}$; whereas, for $\tau < 0$, $s \in [\tau, 0]$, one has $e^{\lambda_0 \tau - \alpha s} = e^{\lambda_0(\tau-s)} e^{(\lambda_0 - \alpha)s} \leq 1$. As a result, $|e^{\lambda_0 \tau} e^{-zs}| \leq e^{\lambda_0 \max\{\tau, 0\}}$. Then, using that $a^+ \in \mathcal{W}_\xi$ implies $\int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \max\{\tau, 0\}} ds < \infty$, one can apply the dominated convergence theorem to the double integral in (4.119); we get then

$$\lim_{\substack{\operatorname{Re} z \rightarrow 0^+ \\ \operatorname{Im} z \rightarrow 0}} \frac{\mathfrak{h}_{\xi, c}(\lambda_0 - z)}{z} = -\varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \tau d\tau + c. \quad (4.120)$$

According to the statement 3 of Theorem 4.22, for $m < \mathfrak{t}_\xi(\lambda_0)$, the r.h.s. of (4.120) is positive, i.e. $j = 1$ in such a case. Let now $m = \mathfrak{t}_\xi(\lambda_0)$, then the r.h.s. of (4.120) is equal to 0. It is easily seen that one can rewrite then (4.119) as follows

$$\begin{aligned} \frac{\mathfrak{h}_{\xi, c}(\lambda_0 - z)}{z} &= \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \int_0^\tau (1 - e^{-zs}) ds d\tau \\ &= z \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \int_0^\tau \int_0^s e^{-zt} dt ds d\tau. \end{aligned} \quad (4.121)$$

Similarly to the above, for $\operatorname{Re} z \in (0, \lambda_0)$, one has that $|e^{\lambda_0 \tau - zt}| \leq e^{\lambda_0 \max\{\tau, 0\}}$. Then, by (4.118) and the dominated convergence theorem, we get from (4.121) that

$$\lim_{\substack{\operatorname{Re} z \rightarrow 0^+ \\ \operatorname{Im} z \rightarrow 0}} \frac{\mathfrak{h}_{\xi, c}(\lambda_0 - z)}{z^2} = \frac{\varkappa^+}{2} \int_{\mathbb{R}} \check{a}^+(\tau) e^{\lambda_0 \tau} \tau^2 d\tau \in (0, \infty).$$

Thus $j = 2$ in such a case. The statement is fully proved now. \square

Remark 4.25. Combining results of Theorem 4.22 and Proposition 4.24, we immediately get that, for the case $j = 2$, the minimal traveling wave speed $c_*(\xi)$ always satisfies (4.82).

Remark 4.26. If \check{a}^+ is given by (4.76), then, cf. Example 4.21, the case $a^+ \in \mathcal{W}_\xi$, $m = \mathfrak{t}_\xi(\lambda_0)$ together with (4.118) requires $p = 1$, $\mu < \mu_*$, $q > 3$.

We consider now the following Ikehara–Delange type Tauberian theorem, cf. [23, 47, 71].

For any $\mu > \beta > 0$, $T > 0$, we set

$$K_{\beta, \mu, T} := \{z \in \mathbb{C} \mid \beta \leq \operatorname{Re} z \leq \mu, |\operatorname{Im} z| \leq T\}.$$

Let, for any $D \subset \mathbb{C}$, $\mathcal{A}(D)$ be the class of all analytic functions on D .

Proposition 4.27. *Let $\mu > \beta > 0$ be fixed. Let $\varphi \in C^1(\mathbb{R}_+ \rightarrow \mathbb{R}_+)$ be a non-increasing function such that, for some $a > 0$, the function $\varphi(t)e^{(\mu+a)t}$ is non-decreasing, and the integral*

$$\int_0^\infty e^{zt} \varphi'(t) dt, \quad 0 < \operatorname{Re} z < \mu \quad (4.122)$$

converges. Suppose also that there exist a constant $j \in \{1, 2\}$ and complex-valued functions $H, F : \{0 < \operatorname{Re} z \leq \mu\} \rightarrow \mathbb{C}$, such that $H \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$, $F \in \mathcal{A}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \leq \mu)$, and, for any $T > 0$,

$$\lim_{\sigma \rightarrow 0^+} (\log \sigma)^{2-j} \sup_{\tau \in [-T, T]} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| = 0, \quad (4.123)$$

and also that the following representation holds

$$\int_0^{\infty} e^{zt} \varphi(t) dt = \frac{F(z)}{(z-\mu)^j} + H(z), \quad 0 < \operatorname{Re} z < \mu. \quad (4.124)$$

Then φ has the following asymptotic

$$\varphi(t) \sim F(\mu) e^{-\mu t} t^{j-1}, \quad t \rightarrow +\infty. \quad (4.125)$$

The proof of Proposition 4.27 is based on the following Tenenbaum's result.

Lemma 4.28 (“Effective” Ikehara–Ingham Theorem, cf. [71, Theorem 7.5.11]).
Let $\alpha(t)$ be a non-decreasing function such that, for some fixed $a > 0$, the following integral converges:

$$\int_0^{\infty} e^{-zt} d\alpha(t), \quad \operatorname{Re} z > a. \quad (4.126)$$

Let also there exist constants $D \geq 0$ and $j > 0$, such that for the functions

$$G(z) := \frac{1}{a+z} \int_0^{\infty} e^{-(a+z)t} d\alpha(t) - \frac{D}{z^j}, \quad \operatorname{Re} z > 0, \quad (4.127)$$

$$\eta(\sigma, T) := \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau, \quad T > 0,$$

one has that

$$\lim_{\sigma \rightarrow 0^+} \eta(\sigma, T) = 0, \quad T > 0. \quad (4.128)$$

Then

$$\alpha(t) = \left\{ \frac{D}{\Gamma(j)} + O(\rho(t)) \right\} e^{at} t^{j-1}, \quad t \geq 1, \quad (4.129)$$

where

$$\rho(t) := \inf_{T \geq 32(a+1)} \left\{ T^{-1} + \eta(t^{-1}, T) + (Tt)^{-j} \right\}. \quad (4.130)$$

Proof of Proposition 4.27. We first express $\int_0^{\infty} e^{\lambda t} \varphi(t) dt$ in the form (4.126). By the assumption, the function $\alpha(t) := e^{(\mu+a)t} \varphi(t)$ is non-decreasing. For any $0 < \operatorname{Re} z < \mu$, one has

$$\begin{aligned} \int_0^{\infty} e^{-(a+z)t} d\alpha(t) &= (\mu+a) \int_0^{\infty} e^{(\mu-z)t} \varphi(t) dt + \int_0^{\infty} e^{(\mu-z)t} \varphi'(t) dt \\ &= (\mu+a) \int_0^{\infty} e^{(\mu-z)t} \varphi(t) dt + \varphi(0) - (\mu-z) \int_0^{\infty} e^{(\mu-z)t} \varphi(t) dt \\ &= \varphi(0) + (a+z) \int_0^{\infty} e^{(\mu-z)t} \varphi(t) dt, \end{aligned} \quad (4.131)$$

where integrals converge for $0 < \operatorname{Re} z < \mu$, by (4.122) and [78, Theorem II.2.3a]. Therefore, by (4.124) and (4.131), we get

$$\frac{1}{a+z} \int_0^{\infty} e^{-(a+z)t} d\alpha(t) = \frac{F(\mu-z)}{z^j} + K(z), \quad 0 < \operatorname{Re} z < \mu,$$

where $K(z) := H(\mu-z) + \frac{\varphi(0)}{a+z}$, $0 < \operatorname{Re} z \leq \mu$.

Let now G be given by (4.127) with $\alpha(t)$ as above and $D := F(\mu)$. Check the condition (4.128); one can assume, clearly, that $0 < \sigma < 2\sigma < \mu$. Since $K \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$, one easily gets that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu)}{(2\sigma + i\tau)^j} - \frac{F(\mu - \sigma - i\tau) - F(\mu)}{(\sigma + i\tau)^j} \right| d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)}{(\sigma + i\tau)^j} \right| d\tau \\ & \quad + \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |F(\mu - 2\sigma - i\tau) - F(\mu)| \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau, \\ & =: \lim_{\sigma \rightarrow 0+} A_j(\sigma) + \lim_{\sigma \rightarrow 0+} B_j(\sigma). \end{aligned} \tag{4.132}$$

One has

$$A_j(\sigma) \leq \sup_{\tau \in [-T, T]} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| \sigma^{j-1} \int_{-T}^T \frac{1}{|\sigma + i\tau|^j} d\tau,$$

and since

$$\sigma^{j-1} \int_{-T}^T \frac{1}{|\sigma + i\tau|^j} d\tau = \sigma^{j-1} \int_{-T}^T \frac{1}{(\sigma^2 + \tau^2)^{\frac{j}{2}}} d\tau = \begin{cases} 2 \log \frac{\sqrt{T^2 + \sigma^2} + T}{\sigma}, & j = 1, \\ 2 \arctan \frac{T}{\sigma}, & j = 2, \end{cases}$$

we get, by (4.123), that $\lim_{\sigma \rightarrow 0+} A_j(\sigma) = 0$.

Next, since $F \in C(K_{\beta, \mu, T})$, there exists $C_1 > 0$ such that $|F(z)| \leq C_1$, $z \in K_{\beta, \mu, T}$. Therefore,

$$\begin{aligned} B_j(\sigma) & \leq \sigma^{j-1} \sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| \int_{|\tau| \leq \sqrt{\sigma}} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau \\ & \quad + 2C_1 \sigma^{j-1} \int_{\sqrt{\sigma} \leq |\tau| \leq T} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau. \end{aligned} \tag{4.133}$$

Note that, for any $a < b$,

$$\int_a^b \left| \frac{1}{2\sigma + i\tau} - \frac{1}{\sigma + i\tau} \right| d\tau = \int_a^b \frac{\sigma}{\sqrt{(2\sigma^2 - \tau^2)^2 + 9\sigma^2\tau^2}} d\tau = \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} h_1(x) dx;$$

$$\int_a^b \left| \frac{1}{(2\sigma + i\tau)^2} - \frac{1}{(\sigma + i\tau)^2} \right| d\tau = \sigma \int_a^b \frac{\sqrt{9\sigma^2 + 4\tau^2}}{(2\sigma^2 - \tau^2)^2 + 9\sigma^2\tau^2} d\tau = \frac{1}{\sigma} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} h_2(x) dx,$$

where

$$h_1(x) := \frac{1}{\sqrt{4 + 5x^2 + x^4}}, \quad h_2(x) := \frac{\sqrt{9 + 4x^2}}{4 + 5x^2 + x^4}.$$

Now, one can estimate terms in (4.133) separately. We have

$$\sigma^{j-1} \int_{|\tau| \leq \sqrt{\sigma}} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau = \int_{|x| \leq \frac{\sqrt{\sigma}}{\sigma}} h_j(x) dx < \int_{\mathbb{R}} h_j(x) dx < \infty.$$

Next, since F is uniformly continuous on $K_{\beta, \mu, T}$, we have that, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(\mu, \sigma, \tau) := |F(\mu - 2\sigma - i\tau) - F(\mu)| < \varepsilon$, if only $4\sigma^2 + \tau^2 < \delta$. Therefore, if $\sigma > 0$ is such that $4\sigma^2 + \sigma < \delta$ then $\sup_{|\tau| \leq \sqrt{\sigma}} f(\mu, \sigma, \tau) < \varepsilon$ hence

$$\sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| \rightarrow 0, \quad \sigma \rightarrow 0+.$$

Finally,

$$\sigma^{j-1} \int_{\sqrt{\sigma} \leq |\tau| \leq T} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| = 2 \int_{\frac{\sqrt{\sigma}}{\sigma}}^{\frac{T}{\sigma}} h_j(x) dx \rightarrow 0, \quad \sigma \rightarrow 0+,$$

as $\int_{\mathbb{R}} h_j(x) dx < \infty$. As a result, (4.133) gives $B_j(\sigma) \rightarrow 0$, as $\sigma \rightarrow 0+$. Combining this with $A_j(\sigma) \rightarrow 0$, one gets (4.128) from (4.132); and we can apply Lemma 4.28. Namely, by (4.129), there exist $C > 0$ and $t_0 \geq 1$, such that

$$De^{at}t^{j-1} \leq \varphi(t)e^{(\mu+a)t} \leq \{D + C\rho(t)\}e^{at}t^{j-1}, \quad t \geq t_0.$$

as $\Gamma(j) = 1$, for $j \in \{1, 2\}$. By (4.128), (4.130) $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$\varphi(t)e^{(\mu+a)t} \sim De^{at}t^{j-1}, \quad t \rightarrow \infty,$$

that is equivalent to (4.125) and finishes the proof. \square

To apply Proposition 4.27 to our settings, we will need the following statement, which is an adaptation of [82, Lemma 3.2, Proposition 3.7].

Proposition 4.29. *In conditions and notations of Theorem 4.22, let a traveling wave profile ψ correspond to a speed $c \neq 0$. Then there exists $\nu > 0$, such that $\psi(t)e^{\nu t}$ is a monotonically increasing function.*

Proof. We start from the case $c > 0$. Since $\psi(t) > 0$, $t \in \mathbb{R}$, it is sufficient to prove that

$$\frac{\psi'(t)}{\psi(t)} > -\nu, \quad t \in \mathbb{R}. \quad (4.134)$$

Fix any $\mu \geq \frac{\varkappa^+}{c} > 0$. Then, clearly,

$$\varkappa^-(\check{a}^- * \psi)(t) + m \leq \varkappa^- \theta + m = \varkappa^+ \leq c\mu,$$

and we will get from (4.85), that

$$0 \geq c\psi'(s) + \varkappa^+(\check{a}^+ * \psi)(s) - c\mu\psi(s), \quad s \in \mathbb{R}. \quad (4.135)$$

Multiply both parts of (4.135) on $e^{-\mu s} > 0$ and set

$$w(s) := \psi(s)e^{-\mu s} > 0, \quad s \in \mathbb{R}.$$

Then $w'(s) = \psi'(s)e^{-\mu s} - \mu w(s)$ and one can rewrite (4.135) as follows

$$\begin{aligned} 0 &\geq cw'(s) + \varkappa^+(\check{a}^+ * \psi)(s)e^{-\mu s} \\ &= cw'(s) + \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau} d\tau, \quad s \in \mathbb{R}. \end{aligned} \quad (4.136)$$

Integrate (4.136) over $s \in [t, t+1]$; one gets

$$0 \geq c(w(t+1) - w(t)) + \varkappa^+ \int_t^{t+1} \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau} d\tau ds. \quad (4.137)$$

Since $w(t)$ is a monotonically decreasing function, we have

$$\begin{aligned} &\int_t^{t+1} \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau} d\tau ds \geq \int_{\mathbb{R}} \check{a}^+(\tau)w(t+1-\tau)e^{-\mu\tau} d\tau \\ &\geq \int_2^\infty \check{a}^+(\tau)w(t+1-\tau)e^{-\mu\tau} d\tau \geq w(t-1) \int_2^\infty \check{a}^+(\tau)e^{-\mu\tau} d\tau. \end{aligned} \quad (4.138)$$

Set $C(\mu) := \frac{\varkappa^+}{c} \int_2^\infty \check{a}^+(s)e^{-\mu s} ds > 0$. Then (4.137) and (4.138) yield

$$w(t) - C(\mu)w(t-1) \geq w(t+1) > 0, \quad t \in \mathbb{R}. \quad (4.139)$$

Now we integrate (4.136) over $s \in [t-1, t]$. Similarly to above, one gets

$$\begin{aligned} 0 &\geq c(w(t) - w(t-1)) + \varkappa^+ \int_{t-1}^t \int_{\mathbb{R}} \check{a}^+(\tau)w(s-\tau)e^{-\mu\tau} d\tau ds \\ &\geq c(w(t) - w(t-1)) + \varkappa^+ \int_{\mathbb{R}} \check{a}^+(\tau)w(t-\tau)e^{-\mu\tau} d\tau. \end{aligned} \quad (4.140)$$

By (4.139) and (4.140), we have

$$\frac{1}{C(\mu)} \geq \frac{w(t-1)}{w(t)} \geq 1 + \frac{\varkappa^+}{c} \int_{\mathbb{R}} \check{a}^+(\tau) \frac{w(t-\tau)}{w(t)} e^{-\mu\tau} d\tau. \quad (4.141)$$

On the other hand, (4.85) implies that

$$-\frac{\psi'(t)}{\psi(t)} \leq \frac{\varkappa^+}{c} \frac{(\check{a}^+ * \psi)(t)}{\psi(t)} = \frac{\varkappa^+}{c} \int_{\mathbb{R}} \check{a}^+(\tau) \frac{w(t-\tau)}{w(t)} e^{-\mu\tau} d\tau, \quad t \in \mathbb{R}. \quad (4.142)$$

Finally, (4.141) and (4.142) yield (4.134) with $\nu = \frac{1}{C(\mu)}$.

Let now $c < 0$. For any $\nu \in \mathbb{R}$, one has

$$\psi'(s) = e^{-\nu s} (\psi(s)e^{\nu s})' - \nu\psi(s), \quad s \in \mathbb{R}.$$

Hence, by (4.32), (A2),

$$\begin{aligned} 0 &= ce^{-\nu s} (\psi(s)e^{\nu s})' - c\nu\psi(s) + \chi^+(\check{a}^+ * \psi)(s) - \chi^-(\check{a}^- * \psi)(s) - m\psi(s) \\ &\geq ce^{-\nu s} (\psi(s)e^{\nu s})' - c\nu\psi(s) + \chi^+(\check{a}^+ * \psi)(s) - \chi^-\theta(\check{a}^- * \psi)(s) - m\psi(s) \\ &\geq ce^{-\nu s} (\psi(s)e^{\nu s})' - c\nu\psi(s) - m\psi(s), \quad s \in \mathbb{R}. \end{aligned}$$

As a result, choosing $\nu > \frac{m}{-c}$, one gets

$$-ce^{-\nu s} (\psi(s)e^{\nu s})' \geq (-c\nu - m)\psi(s) > 0, \quad s \in \mathbb{R},$$

i.e. $\psi(s)e^{\nu s}$ is an increasing function. \square

Now, we can apply Proposition 4.27 to find the asymptotic of the profile of a traveling wave.

Proposition 4.30. *In conditions and notations of Proposition 4.24, for $c \neq 0$, there exists $D = D_j > 0$, such that*

$$\psi(t) \sim De^{-\lambda_0(\psi)t} t^{j-1}, \quad t \rightarrow \infty. \quad (4.143)$$

Proof. We set $\mu := \lambda_0(\psi)$ and

$$\begin{aligned} f(z) &:= \varkappa^-(\mathfrak{L}(\psi(\check{a}^- * \psi)))(z), \quad g_j(z) := \frac{\mathfrak{h}_{\xi,c}(z)}{(z-\mu)^j}, \\ H(z) &:= - \int_{-\infty}^0 \psi(t)e^{zt} dt, \quad F(z) := \frac{f(z)}{g_j(z)}. \end{aligned} \quad (4.144)$$

By (4.86) and Lemma 4.15, we have that $f, H \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$; in particular, for any $T > 0$, $\beta > 0$,

$$\bar{f} := \sup_{z \in K_{\beta,\mu,T}} |f(z)| < \infty. \quad (4.145)$$

By Proposition 4.24, the function g_j is continuous and does not equal to 0 on the strip $\{0 < \operatorname{Re} z \leq \mu\}$, in particular, for any $T > 0$, $\beta > 0$,

$$\bar{g}_j := \inf_{z \in K_{\beta,\mu,T}} |g_j(z)| > 0. \quad (4.146)$$

Therefore, $F \in \mathcal{A}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \leq \mu)$. As a result, one can rewrite (4.95) in the form (4.124), with $\varphi = \psi$ and with F, H as in (4.144).

Taking into account Proposition 4.29, to apply Proposition 4.27 it is enough to prove that (4.123) holds. Assume that $0 < 2\sigma < \mu$.

Let $j = 2$. Clearly, $F \in C(0 < \operatorname{Re} z \leq \mu)$ implies that F is uniformly continuous on $K_{\beta, \mu, T}$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $\tau \in [-T, T]$, the inequality

$$|\sigma| = |(\mu - 2\sigma - i\tau) - (\mu - \sigma - i\tau)| < \delta$$

implies

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| < \varepsilon,$$

and hence (4.123) holds (with $j = 2$).

Let now $j = 1$. In the case when $F \in \mathcal{A}(K_{\beta, \mu, T})$, we have, evidently, that F' is bounded on $K_{\beta, \mu, T}$, and one can apply a mean-value-type theorem for complex-valued functions, see e.g. [30], to get that F is a Lipschitz function on $K_{\beta, \mu, T}$. Therefore, for some $K > 0$,

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| < K|\sigma|,$$

for all $\tau \in [-T, T]$, that yields (4.123) (with $j = 1$). By Proposition 4.17 and Corollary 4.20, the inclusion $F \in \mathcal{A}(K_{\beta, \mu, T})$ always holds for $c > c_*$; whereas, for $c = c_*$ it does hold iff either $a^+ \in \mathcal{V}_\xi$ or $a^+ \in \mathcal{W}_\xi$ with $m > \mathfrak{t}_\xi(\lambda_0)$. Moreover, the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\lambda_0)$ and $c = c_*$ implies, by Proposition 4.24, $j = 2$ and hence it was considered above.

Therefore, it remains to prove (4.123) for the case $a^+ \in \mathcal{W}_\xi$ with $m < \mathfrak{t}_\xi(\lambda_0)$, $c = c_*$ (then $j = 1$). Denote, for simplicity,

$$z_1 := \mu - \sigma - i\tau, \quad z_2 := \mu - 2\sigma - i\tau. \quad (4.147)$$

Then, by (4.144), (4.145), (4.146), one has

$$\begin{aligned} |F(z_2) - F(z_1)| &\leq \left| \frac{f(z_2)}{g_1(z_2)} - \frac{f(z_1)}{g_1(z_2)} \right| + \left| \frac{f(z_1)}{g_1(z_2)} - \frac{f(z_1)}{g_1(z_1)} \right| \\ &\leq \frac{1}{g_1} |f(z_2) - f(z_1)| + \frac{\bar{f}}{g_1^2} |g_1(z_1) - g_1(z_2)|. \end{aligned} \quad (4.148)$$

Note that, if $0 < \phi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that $\lambda_0(\phi) > \mu$ then

$$\begin{aligned} |(\mathfrak{L}\phi)(z_2) - (\mathfrak{L}\phi)(z_1)| &\leq \int_{\mathbb{R}} \phi(s) e^{\mu s} |e^{-2\sigma s} - e^{-\sigma s}| ds \\ &\leq \sigma \int_0^\infty \phi(s) e^{(\mu - \sigma)s} s ds + \sigma \int_{-\infty}^0 \phi(s) e^{(\mu - 2\sigma)s} |s| ds = O(\sigma), \end{aligned} \quad (4.149)$$

as $\sigma \rightarrow 0+$, where we used that $\sup_{s < 0} e^{(\mu - 2\sigma)s} |s| < \infty$, $0 < 2\sigma < \mu$, and that (L2) holds. Applying (4.149) to $\phi = \psi(\check{a}^- * \psi) \leq \theta^2 \check{a}^- \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, one gets

$$\sup_{\tau \in [-T, T]} |f(z_2) - f(z_1)| = O(\sigma), \quad \sigma \rightarrow 0+.$$

Therefore, by (4.148), it remains to show that

$$\lim_{\sigma \rightarrow 0+} \log \sigma \sup_{\tau \in [-T, T]} |g_1(z_1) - g_1(z_2)| = 0. \quad (4.150)$$

Recall that, in the considered case $c = c_*$, one has $\mathfrak{h}_{\xi,c}(\mu) = 0$. Therefore, by (4.117), (4.144), (4.147), we have

$$\begin{aligned}
|g_1(z_1) - g_1(z_2)| &= \left| \frac{\mathfrak{h}_{\xi,c}(z_1) - \mathfrak{h}_{\xi,c}(\mu)}{z_1 - \mu} - \frac{\mathfrak{h}_{\xi,c}(z_2) - \mathfrak{h}_{\xi,c}(\mu)}{z_2 - \mu} \right| \\
&= \left| \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z_1) - \varkappa^+(\mathfrak{L}\check{a}^+)(\mu)}{z_1 - \mu} - \frac{\varkappa^+(\mathfrak{L}\check{a}^+)(z_2) - \varkappa^+(\mathfrak{L}\check{a}^+)(\mu)}{z_2 - \mu} \right| \\
&\leq \varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} \left| \frac{1 - e^{(-\sigma - i\tau)s}}{\sigma + i\tau} - \frac{1 - e^{(-2\sigma - i\tau)s}}{2\sigma + i\tau} \right| ds \\
&= \varkappa^+ \int_{\mathbb{R}} \check{a}^+(s) e^{\mu s} \left| \int_0^s (e^{(-\sigma - i\tau)t} - e^{(-2\sigma - i\tau)t}) dt \right| ds \\
&\leq \varkappa^+ \int_0^\infty \check{a}^+(s) e^{\mu s} \int_0^s |e^{-\sigma t} - e^{-2\sigma t}| dt ds \\
&\quad + \varkappa^+ \int_{-\infty}^0 \check{a}^+(s) e^{\mu s} \int_s^0 |e^{-\sigma t} - e^{-2\sigma t}| dt ds
\end{aligned} \tag{4.151}$$

and since, for $t \geq 0$, $|e^{-\sigma t} - e^{-2\sigma t}| \leq \sigma t$; and, for $s \leq t \leq 0$,

$$|e^{-\sigma t} - e^{-2\sigma t}| = e^{-2\sigma t} |e^{\sigma t} - 1| \leq e^{-2\sigma s} \sigma |t|,$$

one can continue (4.151)

$$\leq \frac{1}{2} \sigma \varkappa^+ \int_0^\infty \check{a}^+(s) e^{\mu s} s^2 ds + \frac{1}{2} \sigma \varkappa^+ \int_{-\infty}^0 \check{a}^+(s) e^{(\mu - 2\sigma)s} s^2 ds.$$

Since $\mu > 2\sigma$, one has $\sup_{s \leq 0} e^{(\mu - 2\sigma)s} s^2 < \infty$, therefore, by (4.118), one gets

$$\sup_{\tau \in [-T, T]} |g_1(z_1) - g_1(z_2)| \leq \text{const} \cdot \sigma,$$

that proves (4.150). The statement is fully proved now. \square

Remark 4.31. By (4.125) and (4.144), one has that the constant $D = D_j$ in (4.143) is given by

$$D = D(\psi) = \varkappa^-(\mathfrak{L}(\psi(\check{a}^- * \psi)))(\mu) \lim_{z \rightarrow \mu} \frac{(z - \mu)^j}{\mathfrak{h}_{\xi,c}(z)},$$

where $\mu = \lambda_0(\psi)$. Note that, by Proposition 4.24, the limit above is finite and does not depend on ψ . Next, by Remark 4.7, for any $q \in \mathbb{R}$, $\psi_q(s) := \psi(s + q)$, $s \in \mathbb{R}$ is a traveling wave with the same speed, and hence, by Theorem 4.22, $\lambda_0(\psi_q) = \lambda_0(\psi)$. Moreover,

$$\begin{aligned}
(\mathfrak{L}(\psi_q(\check{a}^- * \psi_q)))(\mu) &= \int_{\mathbb{R}} \psi(s + q) \int_{\mathbb{R}} \check{a}^-(t) \psi(s - t + q) dt e^{\mu s} ds \\
&= e^{-\mu q} (\mathfrak{L}(\psi(\check{a}^- * \psi)))(\mu).
\end{aligned}$$

Thus, for a traveling wave profile ψ one can always choose a $q \in \mathbb{R}$ such that, for the shifted profile ψ_q , the corresponding $D = D(\psi_q)$ will be equal to 1.

Finally, we are ready to prove the uniqueness result.

Theorem 4.32. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose, additionally, that (A4) holds. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 4.9. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathbf{t}_\xi(\lambda_0)$, we will assume, additionally, that (4.118) holds. Then, for any $c \geq c_*$, such that $c \neq 0$, there exists a unique, up to a shift, traveling wave profile ψ for (2.1).*

Proof. We will follow the sliding technique from [17]. Let $\psi_1, \psi_2 \in C^1(\mathbb{R}) \cap \mathcal{M}_\theta(\mathbb{R})$ are traveling wave profiles with a speed $c \geq c_*$, $c \neq 0$, cf. Proposition 4.11. By Proposition 4.30 and Remark 4.31, we may assume, without loss of generality, that (4.143) holds for both ψ_1 and ψ_2 with $D = 1$. By the proof of Proposition 4.24, the corresponding $j \in \{1, 2\}$ depends on a^\pm , \varkappa^\pm , m only, and does not depend on the choice of ψ_1, ψ_2 . By Theorem 4.22, $\lambda_0(\psi_1) = \lambda_0(\psi_2) =: \lambda_c \in (0, \infty)$.

Step 1. Prove that, for any $\tau > 0$, there exists $T = T(\tau) > 0$, such that

$$\psi_1^\tau(s) := \psi_1(s - \tau) > \psi_2(s), \quad s \geq T. \quad (4.152)$$

Indeed, take an arbitrary $\tau > 0$. Then (4.143) with $D = 1$ yields

$$\lim_{s \rightarrow \infty} \frac{\psi_1^\tau(s)}{(s - \tau)^{j-1} e^{-\lambda_c(s-\tau)}} = 1 = \lim_{s \rightarrow \infty} \frac{\psi_2(s)}{s^{j-1} e^{-\lambda_c s}}.$$

Then, for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > \tau$, such that, for any $s > T_1$,

$$\frac{\psi_1^\tau(s)}{(s - \tau)^{j-1} e^{-\lambda_c(s-\tau)}} - 1 > -\varepsilon, \quad \frac{\psi_2(s)}{s^{j-1} e^{-\lambda_c s}} - 1 < \varepsilon.$$

As a result, for $s > T_1 > \tau$,

$$\begin{aligned} & \psi_1^\tau(s) - \psi_2(s) > (1 - \varepsilon)(s - \tau)^{j-1} e^{-\lambda_c(s-\tau)} - (1 + \varepsilon)s^{j-1} e^{-\lambda_c s} \\ & = s^{j-1} e^{-\lambda_c s} \left(\left(1 - \frac{\tau}{s}\right)^{j-1} e^{\lambda_c \tau} - 1 - \varepsilon \left(\left(1 - \frac{\tau}{s}\right)^{j-1} e^{\lambda_c \tau} + 1 \right) \right) \\ & \geq s^{j-1} e^{-\lambda_c s} \left(\left(1 - \frac{\tau}{T_1}\right)^{j-1} e^{\lambda_c \tau} - 1 - \varepsilon(e^{\lambda_c \tau} + 1) \right) > 0 \end{aligned} \quad (4.153)$$

if only

$$0 < \varepsilon < \frac{\left(1 - \frac{\tau}{T_1}\right)^{j-1} e^{\lambda_c \tau} - 1}{e^{\lambda_c \tau} + 1} =: g(\tau, T_1). \quad (4.154)$$

For $j = 1$, the nominator in the r.h.s. of (5.64) is positive. For $j = 2$, consider $f(t) := \left(1 - \frac{t}{T_1}\right) e^{\lambda_c t} - 1$, $t \geq 0$. Then $f'(t) = \frac{1}{T_1} e^{\lambda_c t} (\lambda_c T_1 - \lambda_c t - 1) > 0$, if only $T_1 > t + \frac{1}{\lambda_c}$, that implies $f(t) > f(0) = 0$, $t > 0$.

As a result, choose $\varepsilon = \varepsilon(\tau) > 0$ with $\varepsilon < g(\tau, \tau + \frac{1}{\lambda_c})$, then, without loss of generality, suppose that $T_1 = T_1(\varepsilon) = T_1(\tau) > \tau + \frac{1}{\lambda_c} > \tau$. Therefore, $0 < \varepsilon < g(\tau, \tau + \frac{1}{\lambda_c}) \leq g(\tau, T_1)$, that fulfills (4.154), and hence (4.153) yields (4.152), with any $T > T_1$.

Step 2. Prove that there exists $\nu > 0$, such that, cf. (4.152),

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \in \mathbb{R}. \quad (4.155)$$

Let $\tau > 0$ be arbitrary and $T = T(\tau)$ be as above. Choose any $\delta \in (0, \frac{\theta}{4})$. By (4.7), (4.1), and the dominated convergence theorem,

$$\lim_{s \rightarrow -\infty} (\check{a}^- * \psi_2)(s) = \lim_{s \rightarrow -\infty} \int_{\mathbb{R}} \check{a}^-(\tau) \psi_2(s - \tau) d\tau = \theta > \delta. \quad (4.156)$$

Then, one can choose $T_2 = T_2(\delta) > T$, such that, for all $s < -T_2$,

$$\psi_1^\nu(s) > \theta - \delta, \quad (4.157)$$

$$(\check{a}^- * \psi_2)(s) > \delta. \quad (4.158)$$

Note also that (4.152) holds, for all $s \geq T_2 > T$, as well. Clearly, for any $\nu \geq \tau$,

$$\psi_1^\nu(s) = \psi_1(s - \nu) \geq \psi_1(s - \tau) > \psi_2(s), \quad s > T_2.$$

Next, $\lim_{\nu \rightarrow \infty} \psi_1^\nu(T_2) = \theta > \psi_2(-T_2)$ implies that there exists $\nu_1 = \nu_1(T_2) = \nu_1(\delta) > \tau$, such that, for all $\nu > \nu_1$,

$$\psi_1^\nu(s) \geq \psi_1^\nu(T_2) > \psi_2(-T_2) \geq \psi_2(s), \quad s \in [-T_2, T_2].$$

Let such a $\nu > \nu_1$ be chosen and fixed. As a result,

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \geq -T_2, \quad (4.159)$$

and, by (4.157),

$$\psi_1^\nu(s) + \delta > \theta > \psi_2(s), \quad s < -T_2. \quad (4.160)$$

For the $\nu > \nu_1$ chosen above, define

$$\varphi_\nu(s) := \psi_1^\nu(s) - \psi_2(s), \quad s \in \mathbb{R}. \quad (4.161)$$

To prove (4.155), it is enough to show that $\varphi_\nu(s) \geq 0$, $s \in \mathbb{R}$.

On the contrary, suppose that φ_ν takes negative values. By (4.159), (4.160),

$$\varphi_\nu(s) \geq -\delta, \quad s < -T_2; \quad \varphi_\nu(s) \geq 0, \quad s \geq -T_2. \quad (4.162)$$

Since $\lim_{s \rightarrow -\infty} \varphi_\nu(s) = 0$ and $\varphi_\nu \in C^1(\mathbb{R})$, our assumption implies that there exists $s_0 < -T_2$, such that

$$\varphi_\nu(s_0) = \min_{s \in \mathbb{R}} \varphi_\nu(s) \in [-\delta, 0). \quad (4.163)$$

We set also

$$\delta_* := -\varphi_\nu(s_0) = \psi_2(s_0) - \psi_1^\nu(s_0) \in (0, \delta]. \quad (4.164)$$

Next, both ψ_1^ν and ψ_2 solve (4.32). Let \check{J}_θ be given by (4.33). Then, recall, $\int_{\mathbb{R}} \check{J}_\theta(s) ds = m$. Denote, cf. (1.4), $\check{L}_\theta \varphi := \check{J}_\theta * \varphi - m\varphi$. Then one can rewrite (4.32), cf. (4.35),

$$c\psi'(s) + (\check{L}_\theta \psi)(s) + \kappa^-(\theta - \psi(s))(\check{a}^- * \psi)(s) = 0.$$

Writing the latter equation for ψ_1^ν and ψ_2 and subtracting the results, one gets

$$\begin{aligned} c\varphi'_\nu(s) + (\check{L}_\theta\varphi_\nu)(s) + \varkappa^- A(s) &= 0, \\ A(s) := (\theta - \psi_1^\nu(s))(\check{a}^- * \psi_1^\nu(s)) - (\theta - \psi_2(s))(\check{a}^- * \psi_2(s)). \end{aligned} \quad (4.165)$$

Consider (4.165) at the point s_0 . By (4.163),

$$\varphi'_\nu(s_0) = 0, \quad (\check{L}_\theta\varphi_\nu)(s_0) \geq 0. \quad (4.166)$$

Next, (4.164) yields

$$\begin{aligned} A(s_0) &= (\theta - \psi_1^\nu(s_0))(\check{a}^- * \psi_1^\nu(s_0)) + (\delta_* - (\theta - \psi_1^\nu(s_0)))(\check{a}^- * \psi_2(s_0)) \\ &= (\theta - \psi_1^\nu(s_0))(\check{a}^- * \varphi_\nu(s_0)) + \delta_*(\check{a}^- * \psi_2(s_0)) \\ &= (\theta - \psi_1^\nu(s_0))(\check{a}^- * (\varphi_\nu + \delta_*))(s_0) + \delta_*((\check{a}^- * \psi_2(s_0)) - (\theta - \psi_1^\nu(s_0))) \\ &> 0, \end{aligned} \quad (4.167)$$

because of (4.163), (4.157), and (4.158). The strict inequality in (4.167) together with (4.166) contradict to (4.165). Therefore, (4.155) holds, for any $\nu > \nu_1$.

Step 3. Prove that, cf. (4.155),

$$\vartheta_* := \inf\{\vartheta > 0 \mid \psi_1^\vartheta(s) \geq \psi_2(s), s \in \mathbb{R}\} = 0. \quad (4.168)$$

On the contrary, suppose that $\vartheta_* > 0$. Let $\varphi_* := \varphi_{\vartheta_*}$ be given by (4.161). By the continuity of the profiles, $\varphi_* \geq 0$.

First, assume that $\varphi_*(s_0) = 0$, for some $s_0 \in \mathbb{R}$, i.e. φ_* attains its minimum at s_0 . Then (4.166) holds with ϑ replaced by ϑ_* , and, moreover, cf. (4.165),

$$A(s_0) = (\theta - \psi_1^{\vartheta_*}(s_0))(\check{a}^- * \varphi_*)(s_0) \geq 0.$$

Therefore, (4.165) implies

$$(\check{L}_\theta\varphi_*)(s_0) = 0. \quad (4.169)$$

By the same arguments as in the proof of Proposition 4.17, one can show that (A4) implies that the function \check{J}_θ also satisfies (A4), for $d = 1$, and with some another constants. Then, arguing in the same way as in the proof of Proposition 3.7 (with $d = 1$ and a^+ replaced by \check{J}_θ), one gets that (4.169) implies that φ_* is a constant, and thus $\varphi_* \equiv 0$, i.e. $\psi_1^{\vartheta_*} \equiv \psi_2$. The latter contradicts (4.152).

Therefore, $\varphi_*(s) > 0$, i.e. $\psi_1^{\vartheta_*}(s) > \psi_2(s)$, $s \in \mathbb{R}$. By (4.152) and (4.156), there exists $T_3 = T_3(\vartheta_*) > 0$, such that $\psi_1^{\frac{\vartheta_*}{2}}(s) > \psi_2(s)$, $s > T_3$, and also, for any $s < -T_3$, (4.158) holds and (4.160) holds with ϑ replaced by $\frac{\vartheta_*}{2}$ (for some fixed $\delta \in (0, \frac{\vartheta_*}{4})$). We set

$$\alpha := \inf_{t \in [-T_3, T_3]} (\psi_1^{\vartheta_*}(s) - \psi_2(s)) > 0.$$

For any $\varepsilon \in (0, \frac{\vartheta_*}{2})$, $\psi_1^{\vartheta_* - \varepsilon} \geq \psi_1^{\frac{\vartheta_*}{2}}$, therefore,

$$\psi_1^{\vartheta_* - \varepsilon}(s) > \psi_2(s), \quad s > T_3,$$

and also (4.160) holds with ϑ replaced by $\vartheta_* - \varepsilon$, for $s < -T_3$. Since the family $\{\psi_1^{\vartheta_* - \varepsilon} \mid \varepsilon \in (0, \frac{\vartheta_*}{2})\}$ is monotone in ε , and $\lim_{\varepsilon \rightarrow 0} \psi_1^{\vartheta_* - \varepsilon}(t) = \psi_1^{\vartheta_*}(t)$, $t \in \mathbb{R}$, we

have, by Dini's theorem, that the latter convergence is uniform on $[-T_3, T_3]$. As a result, there exists $\varepsilon = \varepsilon(\alpha) \in (0, \frac{\vartheta_*}{2})$, such that

$$\psi_1^{\vartheta_*}(s) \geq \psi_1^{\vartheta_* - \varepsilon}(s) \geq \psi_2(s), \quad s \in [-T_3, T_3].$$

Then, the same arguments as in the Step 2 prove that $\psi_1^{\vartheta_* - \varepsilon}(s) \geq \psi_2(s)$, for all $s \in \mathbb{R}$, that contradicts the definition (4.168) of ϑ_* .

As a result, $\vartheta_* = 0$, and by the continuity of profiles, $\psi_1 \geq \psi_2$. By the same arguments, $\psi_2 \geq \psi_1$, that fulfills the statement. \square

5 Long-time behavior of the solution

5.1 Long-time behavior along a direction

We will follow the abstract scheme proposed in [75]. Note that all statements there were considered in the space $C_b(\mathbb{R}^d)$, however, it can be checked straightforward that they remain true in the space $C_{ub}(\mathbb{R}^d)$. We will assume that (A1) and (A2) hold. Recall that θ , U_θ , L_θ are given by (2.17), (3.20), and (3.21), respectively.

Consider the set N_θ of all nonincreasing functions $\varphi \in C(\mathbb{R})$, such that $\varphi(s) = 0$, $s \geq 0$, and

$$\varphi(-\infty) := \lim_{s \rightarrow -\infty} \varphi(s) \in (0, \theta).$$

It is easily seen that $N_\theta \subset U_\theta$.

For arbitrary $s \in \mathbb{R}$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, define the following mapping $V_{s,c,\xi} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}^d)$

$$(V_{s,c,\xi}f)(x) = f(x \cdot \xi + s + c), \quad x \in \mathbb{R}^d. \quad (5.1)$$

Fix an arbitrary $\varphi \in N_\theta$. For $T > 0$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, consider the mapping $R_{T,c,\xi} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$, given by

$$(R_{T,c,\xi}f)(s) = \max\{\varphi(s), (Q_T(V_{s,c,\xi}f))(0)\}, \quad s \in \mathbb{R}, \quad (5.2)$$

where Q_T is given by (3.33), cf. Proposition 3.14. Consider now the following sequence of functions

$$f_{n+1}(s) = (R_{T,c,\xi}f_n)(s), \quad f_0(s) = \varphi(s), \quad s \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}. \quad (5.3)$$

By Proposition 3.14 and [75, Lemma 5.1], $\varphi \in U_\theta$ implies $f_n \in U_\theta$ and $f_{n+1}(s) \geq f_n(s)$, $s \in \mathbb{R}$, $n \in \mathbb{N}$; hence one can define the following limit

$$f_{T,c,\xi}(s) := \lim_{n \rightarrow \infty} f_n(s), \quad s \in \mathbb{R}. \quad (5.4)$$

Also, by [75, Lemma 5.1], for fixed $\xi \in S^{d-1}$, $T > 0$, $n \in \mathbb{N}$, the functions $f_n(s)$ and $f_{T,c,\xi}(s)$ are nonincreasing in s and in c ; moreover, $f_{T,c,\xi}(s)$ is a lower semicontinuous function of s, c, ξ , as a result, this function is continuous from the right in s and in c . Note also, that $0 \leq f_{T,c,\xi} \leq \theta$. Then, for any c, ξ , one can define the limiting value

$$f_{T,c,\xi}(\infty) := \lim_{s \rightarrow \infty} f_{T,c,\xi}(s).$$

Next, for any $T > 0$, $\xi \in S^{d-1}$, we define

$$c_T^*(\xi) = \sup\{c \mid f_{T,c,\xi}(\infty) = \theta\} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where, as usual, $\sup \emptyset := -\infty$. By [75, Propositions 5.1, 5.2], one has

$$f_{T,c,\xi}(\infty) = \begin{cases} \theta, & c < c_T^*(\xi), \\ 0, & c \geq c_T^*(\xi), \end{cases} \quad (5.5)$$

cf. also [75, Lemma 5.5]; moreover, $c_T^*(\xi)$ is a lower semicontinuous function of ξ . It is crucial that, by [75, Lemma 5.4], neither $f_{T,c,\xi}(\infty)$ nor $c_T^*(\xi)$ depends on the choice of $\varphi \in N_\theta$. Note that the monotonicity of $f_{T,c,\xi}(s)$ in s and (5.5) imply that, for $c < c_T^*(\xi)$, $f_{T,c,\xi}(s) = \theta$, $s \in \mathbb{R}$.

Proposition 5.1. *Let $\xi \in S^{d-1}$ and suppose that (A1), (A2), and (A5) hold. Let $c_*(\xi)$ be as in Theorem 4.9. Then*

$$c_T^*(\xi) = Tc_*(\xi), \quad T > 0. \quad (5.6)$$

Proof. Take any $c > 0$ with $cT \geq c_T^*(\xi)$. Then, by (5.5), $f_{T,cT,\xi} \not\equiv \theta$. By (5.2), (5.3), one has

$$f_{n+1}(s) \geq (Q_T(V_{s,cT,\xi}f_n))(0), \quad s \in \mathbb{R}. \quad (5.7)$$

Since $f_n(s)$ is nonincreasing in s , one gets, by (5.1), that, for a fixed $x \in \mathbb{R}^d$, the function $(V_{s,cT,\xi}f_n)(x)$ is also nonincreasing in s . Next, by (5.1), (5.4) and Propositions 3.12,

$$(Q_T(V_{s,cT,\xi}f_n))(x) \rightarrow (Q_T(V_{s,cT,\xi}f_{T,cT,\xi}))(x), \quad \text{a.a. } x \in \mathbb{R}^d. \quad (5.8)$$

Note that, by (5.1) and Proposition 4.4,

$$(Q_T(V_{s,cT,\xi}f_{T,cT,\xi}))(x) = \phi(x \cdot \xi, T), \quad (5.9)$$

where $\phi(\tau, t)$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}_+$ solves (4.4) with $\psi(\tau) = f_{T,cT,\xi}(\tau + s + cT)$ (note that s is a parameter now, cf. (4.4)). On the other hand, the evident equality $(V_{s,cT,\xi}f_{T,cT,\xi})(x + \tau\xi) = f_{T,cT,\xi}(x \cdot \xi + \tau + s + cT)$, $\tau \in \mathbb{R}$ shows that the function $V_{s,cT,\xi}f_{T,cT,\xi}$ is a decreasing function on \mathbb{R}^d along the $\xi \in S^{d-1}$, cf. Definition 3.15, as $f_{T,cT,\xi}$ is a decreasing function on \mathbb{R} . Then, by Proposition 3.16 and (5.9), the function $\mathbb{R}^d \ni x \mapsto \phi(x \cdot \xi, T) \in [0, \theta]$ is decreasing along the ξ as well, i.e. $\phi(x \cdot \xi + \tau, T) = \phi((x + \tau\xi) \cdot \xi, T) \leq \phi(x \cdot \xi, T)$, $\tau \geq 0$. As a result, the function $\phi(s, T)$ is monotone (almost everywhere) in s . Since $f_{T,cT,\xi}(s)$ was continuous from the right in s , one gets from (5.7), (5.8), that

$$f_{T,cT,\xi}(s) \geq (\tilde{Q}_T f_{T,cT,\xi})(s + cT),$$

where \tilde{Q}_T is given as in Proposition 4.8. Since $f_{T,cT,\xi} \not\equiv \theta$, one has that, by [80, Theorem 5] (cf. the proof of Theorem 4.9), there exists a traveling wave profile with speed c . By Theorem 4.9, we have that $c \geq c_*(\xi)$, and hence $c_T^*(\xi) \geq Tc_*(\xi)$.

Take now any $c > c_*(\xi)$ and consider, by Theorem 4.9, a traveling wave in a direction $\xi \in S^{d-1}$, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and a speed c . Then, by (5.1) and (4.1),

$$(Q_T(V_{s,cT,\xi}\psi))(x) = \psi((x \cdot \xi - cT) + s + cT) = \psi(x \cdot \xi + s).$$

Choose $\varphi \in N_\theta$ such that $\varphi(s) \leq \psi(s)$, $s \in \mathbb{R}$ (recall that all constructions are independent on the choice of φ). Then, one gets from (5.2), that $(R_{T,cT,\xi}\psi)(s) = \psi(s)$, $s \in \mathbb{R}$. Then, by (5.3) and (5.4), $f_{T,cT,\xi}(s) = \psi(s)$, $s \in \mathbb{R}$, and thus (5.5) implies $cT \geq c_T^*(\xi)$; as a result, $Tc_*(\xi) \geq c_T^*(\xi)$, that fulfills the statement. \square

We describe now how the solution to (2.1) behaves, for big times, along a direction $\xi \in S^{d-1}$. We start with a result about an exponential decaying along such a direction. It is worth noting that we do not need to assume either (A1) or (A2) to prove Proposition 5.2 below.

For any $\xi \in S^{d-1}$ and $\lambda > 0$, consider the following set of bounded functions on \mathbb{R}^d :

$$E_{\lambda,\xi}(\mathbb{R}^d) := \{f \in L^\infty(\mathbb{R}^d) \mid \|f\|_{\lambda,\xi} := \operatorname{esssup}_{x \in \mathbb{R}^d} |f(x)|e^{\lambda x \cdot \xi} < \infty\}. \quad (5.10)$$

Evidently, for $f \in L^\infty(\mathbb{R}^d)$,

$$\operatorname{esssup}_{x \in \mathbb{R}^d} |f(x)|e^{\lambda x \cdot \xi} < \infty \quad \text{if and only if} \quad \operatorname{esssup}_{x \cdot \xi \geq 0} |f(x)|e^{\lambda x \cdot \xi} < \infty,$$

therefore,

$$E_{\lambda,\xi}(\mathbb{R}^d) \subset E_{\lambda',\xi}(\mathbb{R}^d), \quad \lambda > \lambda' > 0, \xi \in S^{d-1}.$$

Proposition 5.2. *Let $\xi \in S^{d-1}$ and $\lambda > 0$ be fixed and suppose that (A5) holds with $\mu = \lambda$. Let $0 \leq u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$ and let $u = u(x, t)$ be a solution to (2.1). Then*

$$\|u(\cdot, t)\|_{\lambda,\xi} \leq \|u_0\|_{\lambda,\xi} e^{pt}, \quad t \geq 0, \quad (5.11)$$

where

$$p = p(\xi, \lambda) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x) e^{\lambda x \cdot \xi} dx - m \in \mathbb{R}. \quad (5.12)$$

Proof. First, we note that, for any $a \in L^1(\mathbb{R}^d)$,

$$\begin{aligned} |(a * f)(x) e^{\lambda x \cdot \xi}| &\leq \int_{\mathbb{R}^d} |a(x-y)| e^{\lambda(x-y) \cdot \xi} |f(y)| e^{\lambda y \cdot \xi} dy \\ &\leq \|f\|_{\lambda,\xi} \int_{\mathbb{R}^d} |a(y)| e^{\lambda y \cdot \xi} dy. \end{aligned} \quad (5.13)$$

We will follow the notations from the proof of Theorem 2.2, cf. Remark 2.3. Let p is given by (5.12) and suppose that, for some $\tau \in [0, T)$, $\|u_\tau\|_{\lambda,\xi} \leq \|u_0\|_{\lambda,\xi} e^{p\tau}$. Take any $v \in \mathcal{X}_{\tau,\Upsilon}^+(r)$ with Υ, r given by (2.13), (2.15), such that

$$\|v(\cdot, t)\|_{\lambda,\xi} \leq \|u_0\|_{\lambda,\xi} e^{pt}, \quad t \in [\tau, \Upsilon]. \quad (5.14)$$

Then, by (2.6), (2.7), one gets, for any $t \in [\tau, \Upsilon]$,

$$\begin{aligned} 0 &\leq (\Phi_\tau v)(x, t) e^{\lambda x \cdot \xi} \\ &\leq e^{-(t-\tau)m} u_\tau(x) e^{\lambda x \cdot \xi} + \int_\tau^t e^{-m(t-s)} \varkappa^+ (a^+ * v)(x, s) e^{\lambda x \cdot \xi} ds \\ &\leq \|u_0\|_{\lambda,\xi} e^{-m(t-\tau)} e^{p\tau} + \|u_0\|_{\lambda,\xi} \varkappa^+ \int_{\mathbb{R}^d} a^+(y) e^{\lambda y \cdot \xi} dy \int_\tau^t e^{-m(t-s)} e^{ps} ds, \\ &= \|u_0\|_{\lambda,\xi} e^{-mt} e^{(p+m)\tau} + \|u_0\|_{\lambda,\xi} (p+m) e^{-mt} \int_\tau^t e^{(p+m)s} ds \\ &= \|u_0\|_{\lambda,\xi} e^{pt}, \end{aligned}$$

where we used (5.13) and (5.14). Therefore, $\|(\Phi_\tau v)(\cdot, t)\|_{\lambda, \xi} \leq \|u_0\|_{\lambda, \xi} e^{pt}$, $t \in [\tau, \Upsilon]$. As a result,

$$\|(\Phi_\tau^n v)(\cdot, t)\|_{\lambda, \xi} \leq \|u_0\|_{\lambda, \xi} e^{pn}, \quad n \in \mathbb{N}, t \in [\tau, \Upsilon].$$

Then $\|u(\cdot, t)\|_{\lambda, \xi}$ satisfies the same inequality on $[\tau, \Upsilon]$; and, by the proof of Theorem 2.2, we have the statement. \square

Remark 5.3. It follows from (L1) and the considerations thereafter, that the statement of Proposition 5.2 remains true if (A5) holds for some $\mu > \lambda$, provided that we assume, additionally, (A6).

Define now the following set

$$\Upsilon_{T, \xi} = \{x \in \mathbb{R}^d \mid x \cdot \xi \leq c_T^*(\xi)\}, \quad \xi \in S^{d-1}, T > 0. \quad (5.15)$$

Clearly, the set $\Upsilon_{T, \xi}$ is convex and closed. Moreover, by (5.6),

$$\Upsilon_{T, \xi} = T\Upsilon_{1, \xi} \quad (5.16)$$

Here and below, for any measurable $A \subset \mathbb{R}^d$, we define $tA := \{tx \mid x \in A\} \subset \mathbb{R}^d$. We are going to explain now how a solution $u(x, t)$ to (2.1) behaves outside of the set $t\Upsilon_{1, \xi} = \Upsilon_{t, \xi}$, $t > 0$.

Theorem 5.4. *Let $\xi \in S^{d-1}$ and $a^+ \in \mathcal{U}_\xi$; i.e. all conditions of Definition 4.16 hold. Let $\lambda_* = \lambda_*(\xi) \in I_\xi$ be the same as in the Proposition 4.17. Suppose that $u_0 \in E_{\lambda_*, \xi}(\mathbb{R}^d) \cap L_\theta$ and let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Let $\mathcal{O}_\xi \subset \mathbb{R}^d$ be an open set, such that $\Upsilon_{1, \xi} \subset \mathcal{O}_\xi$ and $\delta := \text{dist}(\Upsilon_{1, \xi}, \mathbb{R}^d \setminus \mathcal{O}_\xi) > 0$. Then the following estimate holds*

$$\text{esssup}_{x \notin t\mathcal{O}_\xi} u(x, t) \leq \|u_0\|_{\lambda_*, \xi} e^{-\lambda_* \delta t}, \quad t > 0. \quad (5.17)$$

Proof. Let $p_* := p(\xi, \lambda_*)$ be given by (5.12). By (5.11), (5.10), one has

$$0 \leq u(x, t) \leq \|u_0\|_{\lambda_*, \xi} \exp\{p_* t - \lambda_* x \cdot \xi\}, \quad \text{a.a. } x \in \mathbb{R}^d. \quad (5.18)$$

Next, by (5.15) and Proposition 5.1, for any $t > 0$ and for all $x \in \mathbb{R}^d \setminus t\mathcal{O}_\xi$, one has $x \cdot \xi \geq tc_1^*(\xi) + t\delta = tc_*(\xi) + t\delta$. Then, by (4.81),

$$\begin{aligned} \inf_{x \notin t\mathcal{O}_\xi} (\lambda_* x \cdot \xi) &\geq t\lambda_* c_*(\xi) + t\lambda_* \delta \\ &= t \left(\mathcal{Z}^+ \int_{\mathbb{R}^d} a^+(x) e^{\lambda_* x \cdot \xi} dx - m \right) + t\lambda_* \delta = tp_* + t\lambda_* \delta. \end{aligned}$$

Therefore, (5.18) implies the statement. \square

Remark 5.5. The assumption $u_0 \in E_{\lambda_*, \xi}(\mathbb{R}^d)$ is close, in some sense, to the weakest possible assumption on an initial condition $u_0 \in L_\theta$ for the equation (2.1) to have

$$\lim_{t \rightarrow \infty} \text{esssup}_{x \notin t\mathcal{O}_\xi} u(x, t) = 0, \quad (5.19)$$

for an arbitrary open set $\mathcal{O}_\xi \supset \Upsilon_{1,\xi}$, where $\Upsilon_{1,\xi}$ is defined by (5.15). Indeed, take any λ_1, λ with $0 < \lambda_1 < \lambda < \lambda_* = \lambda_*(\xi)$. By Theorem 4.22, there exists a traveling wave solution to (2.1) with a profile $\psi_1 \in \mathcal{M}_\theta(\mathbb{R})$ such that $\lambda_0(\psi_1) = \lambda_1$. By Proposition 4.30 (with $j = 1$ as $\lambda_1 < \lambda_*$) we have that $\psi_1(t) \sim De^{-\lambda_1 t}$, $t \rightarrow \infty$. It is easily seen that one can choose a function $\varphi \in \mathcal{M}_\theta(\mathbb{R}) \cap C(\mathbb{R})$ such that there exist $p > 0$, $T > 0$, such that $\varphi(t) \geq \psi_1(t)$, $t \in \mathbb{R}$ and $\varphi(t) = pe^{-\lambda t}$, $t > T$. Take now $u_0(x) = \varphi(x \cdot \xi)$, $x \in \mathbb{R}^d$. We have $u_0 \in E_{\lambda,\xi}(\mathbb{R}^d) \setminus E_{\lambda_*,\xi}(\mathbb{R}^d)$. Then, by Proposition 4.4, the corresponding solution has the form $u(x, t) = \phi(x \cdot \xi, t)$. By Proposition 3.4 applied to the equation (4.4), $\phi(s, t) \geq \psi_1(s - c_1 t)$, $s \in \mathbb{R}$, $t \geq 0$, where $c_1 = G_\xi(\lambda_1) > c_*(\xi)$, cf. (4.53) and (4.81). Take $c \in (c_*(\xi), c_1)$ and consider an open set $\mathcal{O}_\xi := \{x \in \mathbb{R}^d \mid x \cdot \xi < c\}$, then $\Upsilon_{1,\xi} \subset \mathcal{O}_\xi \subset \{x \in \mathbb{R}^d \mid x \cdot \xi \leq c_1\} =: A_1$. One has

$$\begin{aligned} \sup_{x \notin t\mathcal{O}_\xi} u(x, t) &\geq \sup_{x \in tA_1 \setminus t\mathcal{O}_\xi} \phi(x \cdot \xi, t) \\ &\geq \sup_{ct < s \leq c_1 t} \psi_1(s - c_1 t) = \psi_1(ct - c_1 t) > \psi_1(0), \end{aligned}$$

as $c < c_1$ and ψ_1 is decreasing. As a result, (5.19) does not hold.

On the other hand, if $\psi_* \in \mathcal{M}_\theta(\mathbb{R})$ is a profile with the minimal speed $c_*(\xi) \neq 0$ and if $j = 2$, cf. Proposition 4.24, then $u_0(x) := \psi_*(x \cdot \xi)$ does not belong to the space $E_{\lambda_*,\xi}(\mathbb{R}^d)$, however, the arguments above do not contradict (5.19) anymore. In the next remark, we consider this case in more details.

Remark 5.6. In connection with the previous remark, it is worth noting also that one can easily generalize Theorem 5.4 in the following way. Let $u_0 \in E_{\lambda,\xi}(\mathbb{R}^d) \cap L_\theta$, for some $\lambda \in (0, \lambda_*]$, and let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Consider the set $A_{c,\xi} := \{x \in \mathbb{R}^d \mid x \cdot \xi \leq c\}$, where $c = \lambda^{-1}(\varkappa^+ \mathbf{a}_\xi(\lambda) - m)$ cf. (4.81). Then, for any open set $B_{c,\xi} \supset A_{c,\xi}$ with $\delta_c := \text{dist}(A_{c,\xi}, \mathbb{R}^d \setminus B_{c,\xi}) > 0$, one gets

$$\text{esssup}_{x \notin tB_{c,\xi}} u(x, t) \leq \|u_0\|_{\lambda,\xi} e^{-\lambda \delta_c t}. \quad (5.20)$$

Therefore, if $u_0(x) = \psi_*(x \cdot \xi)$, where ψ_* is as in Remark 5.5 above, then, evidently, $u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$, for any $\lambda \in (0, \lambda_*)$. Then, for any open $\mathcal{O}_\xi \supset \Upsilon_{1,\xi}$ with $\delta := \text{dist}(\Upsilon_{1,\xi}, \mathbb{R}^d \setminus \mathcal{O}_\xi) > 0$ one can choose, for any $\varepsilon \in (0, 1)$, $c_1 = c_*(\xi) + \delta\varepsilon$. By Theorem 4.22, there exists a unique $\lambda_1 = \lambda_1(\varepsilon) \in (0, \lambda_*)$ such that $c_1 = \lambda_1^{-1}(\varkappa^+ \mathbf{a}_\xi(\lambda_1) - m)$. Then $u_0 \in E_{\lambda_1,\xi}(\mathbb{R}^d)$ and $A_{c_1,\xi} \subset \mathcal{O}_\xi$, i.e. \mathcal{O}_ξ may be considered as a set $B_{c_1,\xi}$, cf. above. As a result, (5.20) gives (5.17), with the constant $\|u_0\|_{\lambda_1,\xi} < \|u_0\|_{\lambda_*,\xi}$, and with $\lambda_* \delta$ replaced by $\lambda_1 \delta(1 - \varepsilon)$. Note that, clearly, $\|u_0\|_{\lambda_1,\xi} \nearrow \|u_0\|_{\lambda_*,\xi}$, $\lambda_1 \nearrow \lambda_*$, $\varepsilon \rightarrow 0$.

5.2 Global long-time behavior

We are going to consider now the global long-time behavior along all possible directions $\xi \in S^{d-1}$ simultaneously. Define, cf. (5.15),

$$\Upsilon_T = \{x \in \mathbb{R}^d \mid x \cdot \xi \leq c_T^*(\xi), \xi \in S^{d-1}\}, \quad T > 0. \quad (5.21)$$

By (5.15), (5.6), (5.16),

$$\Upsilon_T = \bigcap_{\xi \in S^{d-1}} \Upsilon_{T,\xi} = \bigcap_{\xi \in S^{d-1}} T\Upsilon_{1,\xi} = T\Upsilon_1, \quad T > 0. \quad (5.22)$$

Clearly, the set Υ_T , $T > 0$ is convex and closed. To have an analog of Theorem 5.4 for the set Υ_T , one needs to have $a^+ \in \mathcal{U}_\xi$, for all $\xi \in S^{d-1}$, cf. Definition 4.16.

Since $\int_{x \cdot \xi \leq 0} a^+(x) e^{\lambda x \cdot \xi} dx \in [0, 1]$, $\xi \in S^{d-1}$, $\lambda > 0$, we have the following observation. If, for some $\xi \in S^{d-1}$, there exist $\mu^\pm > 0$, such that, cf. (4.10), $\mathbf{a}_{\pm\xi}(\mu^\pm) < \infty$, i.e. if (A5) holds for both ξ and $-\xi$, then, for $\mu = \min\{\mu^+, \mu^-\}$,

$$\begin{aligned} \int_{\mathbb{R}^d} a^+(x) e^{\mu|x \cdot \xi|} dx &= \int_{x \cdot \xi \geq 0} a^+(x) e^{\mu x \cdot \xi} dx + \int_{x \cdot \xi < 0} a^+(x) e^{-\mu x \cdot \xi} dx \\ &\leq \int_{x \cdot \xi \geq 0} a^+(x) e^{\mu^+ x \cdot \xi} dx + \int_{x \cdot (-\xi) > 0} a^+(x) e^{\mu^- x \cdot (-\xi)} dx < \infty. \end{aligned} \quad (5.23)$$

Let now $\{e_i \mid 1 \leq i \leq d\}$ be an orthonormal basis in \mathbb{R}^d . Let (A5) holds for $2d$ directions $\{\pm e_i \mid 1 \leq i \leq d\} \subset S^{d-1}$ and let $\mu_i = \min\{\mu(e_i), \mu(-e_i)\}$, $1 \leq i \leq d$, cf. (5.23). Set $\mu = \frac{1}{d} \min\{\mu_i \mid 1 \leq i \leq d\}$. Then, by the triangle and Jensen's inequalities and (5.23), one has

$$\begin{aligned} \int_{\mathbb{R}^d} a^+(x) e^{\mu|x|} dx &\leq \int_{\mathbb{R}^d} a^+(x) \exp\left(\sum_{i=1}^d \frac{1}{d} \mu_i |x \cdot e_i|\right) dx \\ &\leq \sum_{i=1}^d \frac{1}{d} \int_{\mathbb{R}^d} a^+(x) e^{\mu_i |x \cdot e_i|} dx < \infty. \end{aligned}$$

As a result, the assumption that (A5) holds, for all $\xi \in S^{d-1}$, is equivalent to the following one

$$\text{there exists } \mu_d > 0, \text{ such that } \int_{\mathbb{R}^d} a^+(x) e^{\mu_d |x|} dx < \infty. \quad (\text{A9})$$

Clearly, (A9) implies

$$\int_{\mathbb{R}^d} |x| a^+(x) dx < \infty, \quad (5.24)$$

and thus (A7) holds, for any $\xi \in S^{d-1}$. Then, one can define the (global) first moment vector of a^+ , cf. (4.47),

$$\mathbf{m} := \int_{\mathbb{R}^d} x a^+(x) dx \in \mathbb{R}^d. \quad (5.25)$$

The most 'anisotropic' assumption is (A8). We will assume, for simplicity, that (A3) holds; then (A8) holds with $r(\xi) = 0$, for all $\xi \in S^{d-1}$.

Proposition 5.7. *Let (A1), (A2), (A3), (A6), (A9) hold. Then, for any $T > 0$, $T\mathcal{X}^+\mathbf{m} \in \mathbb{R}^d$ is an interior point of Υ_T , and Υ_T is a bounded set.*

Proof. By (5.22), it is enough to prove the statement, for $T = 1$. By (4.47), for any orthonormal basis $\{e_i \mid 1 \leq i \leq d\} \subset S^{d-1}$, $\mathbf{m} = \sum_{i=1}^d \mathbf{m}_{e_i}$. As it was shown above, the assumptions of the statement imply that Theorem 4.22 holds, for any $\xi \in S^{d-1}$. Therefore, by (4.78) and Proposition 5.1,

$$(\mathcal{X}^+\mathbf{m}) \cdot \xi = \mathcal{X}^+ \int_{\mathbb{R}^d} x \cdot \xi a^+(x) dx = \mathcal{X}^+ \mathbf{m}_\xi < c_*(\xi) = c_1^*(\xi), \quad (5.26)$$

for all $\xi \in S^{d-1}$; thus $\varkappa^+ \mathbf{m} \in \Upsilon_1$. Since the inequality in (5.26) is strict, the point $\varkappa^+ \mathbf{m}$ is an interior point of Υ_1 .

Next, by Proposition 5.1, $x \in \Upsilon_1$ implies that, for any fixed $\xi \in S^{d-1}$, $x \cdot \xi \leq c_*(\xi)$ and $x \cdot (-\xi) \leq c_*(-\xi)$, i.e.

$$-c_*(-\xi) \leq x \cdot \xi \leq c_*(\xi), \quad x \in \Upsilon_1, \xi \in S^{d-1}. \quad (5.27)$$

Then (5.27) implies

$$|x \cdot \xi| \leq \max\{|c_*(\xi)|, |c_*(-\xi)|\}, \quad x \in \Upsilon_1, \xi \in S^{d-1};$$

in particular, for an orthonormal basis $\{e_i \mid 1 \leq i \leq d\}$ of \mathbb{R}^d , one gets

$$|x| \leq \sum_{i=1}^d |x \cdot e_i| \leq \sum_{i=1}^d \max\{|c_*(e_i)|, |c_*(-e_i)|\} =: R < \infty, \quad x \in \Upsilon_1,$$

that fulfills the statement. \square

Remark 5.8. It is worth noting that, by (4.78), (4.47), the following inequality holds, cf. (5.27),

$$c_*(\xi) + c_*(-\xi) > \varkappa^+(\mathbf{m}_\xi + \mathbf{m}_{-\xi}) = 0.$$

For any $T > 0$, consider the set $\mathcal{M}(T)$ of all subsets from \mathbb{R}^d of the following form:

$$M_T = M_{T, \varepsilon, K, \xi_1, \dots, \xi_K} = \{x \in \mathbb{R}^d \mid x \cdot \xi_i \leq c_T^*(\xi_i) + \varepsilon, \quad i = 1, \dots, K\}, \quad (5.28)$$

for some $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1, \dots, \xi_K \in S^{d-1}$.

We are ready now to prove a result about the long-time behavior at infinity in space.

Theorem 5.9. *Let the conditions (A1), (A2), (A3), (A6), (A9) hold. Let $u_0 \in L_\theta$ be such that*

$$\|u_0\| := \sup_{\lambda > 0} \operatorname{esssup}_{x \in \mathbb{R}^d} u_0(x) e^{\lambda|x|} < \infty \quad (5.29)$$

and let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Then, for any open set $\mathcal{O} \supset \Upsilon_1$, there exists $\nu = \nu(\mathcal{O}) > 0$, such that

$$\operatorname{esssup}_{x \notin t\mathcal{O}} u(x, t) \leq \|u_0\| e^{-\nu t}, \quad t > 0.$$

Proof. By Proposition 5.7, the set Υ_1 is bounded and nonempty. Then, by [75, Lemma 7.2], there exists $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1, \dots, \xi_K \in S^{d-1}$ and a set $M \in \mathcal{M}(1)$ of the form (5.28), with $T = 1$, such that

$$\Upsilon_1 \subset M \subset \mathcal{O}. \quad (5.30)$$

Choose now

$$\mathcal{O}_{\xi_i} = \left\{ x \in \mathbb{R}^d \mid x \cdot \xi_i < c_1^*(\xi_i) + \frac{\varepsilon}{2} \right\} \supset \Upsilon_{1, \xi_i}, \quad 1 \leq i \leq K.$$

Then, by (5.30),

$$\Upsilon_1 = \bigcap_{\xi \in S^{d-1}} \Upsilon_{1,\xi} \subset \bigcap_{i=1}^K \Upsilon_{1,\xi_i} \subset \bigcap_{i=1}^K \mathcal{O}_{\xi_i} \subset M \subset \mathcal{O},$$

and, therefore,

$$\mathbb{R}^d \setminus \mathcal{O} \subset \bigcup_{i=1}^K (\mathbb{R}^d \setminus \mathcal{O}_{\xi_i}). \quad (5.31)$$

By (5.10), the assumption (5.29) implies,

$$\begin{aligned} \|u_0\|_{\lambda,\xi} &\leq \max \left\{ \operatorname{esssup}_{x \cdot \xi \geq 0} |u_0(x)| e^{\lambda x \cdot \xi}, \operatorname{esssup}_{x \cdot \xi < 0} |u_0(x)| \right\} \\ &\leq \max \left\{ \operatorname{esssup}_{x \cdot \xi \geq 0} |u_0(x)| e^{\lambda |x|}, \operatorname{esssup}_{x \cdot \xi < 0} |u_0(x)| \right\} \leq \operatorname{esssup}_{x \in \mathbb{R}^d} |u_0(x)| e^{\lambda |x|} \leq \|u_0\|, \end{aligned}$$

for any $\lambda > 0$, $\xi \in S^{d-1}$. Denote

$$\nu_i := \lambda_*(\xi_i) \operatorname{dist}(\Upsilon_{1,\xi_i}, \mathbb{R}^d \setminus \mathcal{O}_{\xi_i}) = \lambda_*(\xi_i) \frac{\varepsilon}{2}, \quad 1 \leq i \leq K.$$

Then, by Theorem 5.4 and (5.31), one gets, for any $t > 0$,

$$\operatorname{esssup}_{x \notin t\mathcal{O}} u(x, t) \leq \max_{1 \leq i \leq K} \operatorname{esssup}_{x \notin t\mathcal{O}_{\xi_i}} u(x, t) \leq \|u_0\|_{\lambda_*(\xi_i), \xi_i} e^{-\nu_i t} \leq \|u_0\| e^{-\nu t},$$

with $\nu := \min\{\nu_i \mid 1 \leq i \leq K\}$. \square

Our second main result about the long-time behavior states that the solution $u \in \mathcal{X}_\infty$ uniformly converges to θ inside the set $t\Upsilon_1 = \Upsilon_t$. The proof of this result is quite technical. For the convenience of the reader, we present here the statement of Theorem 5.10 only, and explain the proof in the next subsection.

For a closed set $A \subset \mathbb{R}^d$, we denote by $\operatorname{int}(A)$ the interior of A .

Theorem 5.10. *Let the conditions (A1), (A2), (A4), (A6), (A9) hold. Let $u_0 \in U_\theta$, $u_0 \not\equiv 0$, and $u \in \mathcal{X}_\infty$ be the corresponding solution to (2.1). Then, for any compact set $\mathcal{C} \subset \operatorname{int}(\Upsilon_1)$,*

$$\lim_{t \rightarrow \infty} \min_{x \in t\mathcal{C}} u(x, t) = \theta. \quad (5.32)$$

Corollary 5.11. *Let the conditions (A1), (A2), (A4), (A6), (A9) hold. Let $u_0 \in L_\theta$ be such that there exist $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, with $u_0 \geq \eta$, for a.a. $x \in B_r(x_0)$. Let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Then, for any compact set $\mathcal{C} \subset \operatorname{int}(\Upsilon_1)$,*

$$\lim_{t \rightarrow \infty} \operatorname{essinf}_{x \in t\mathcal{C}} u(x, t) = \theta.$$

Proof. The assumption on u_0 implies that there exists a function $v_0 \in U_\theta \subset L_\theta$, $v_0 \not\equiv 0$, such that $u_0(x) \geq v_0(x)$, for a.a. $x \in \mathbb{R}^d$. Then, by Remark 3.5, $u(x, t) \geq v(x, t)$, for a.a. $x \in \mathbb{R}^d$, and for all $t \geq 0$, where $v \in \mathcal{X}_\infty$ is the corresponding to v_0 solution to (2.1). By Proposition 3.4, $v \in \mathcal{X}_\infty$, and one has (5.32) for v , with the same Υ_1 , cf. (Q1). The statement follows then from the evident inequality

$$\min_{x \in t\mathcal{C}} v(x, t) = \operatorname{essinf}_{x \in t\mathcal{C}} v(x, t) \leq \operatorname{essinf}_{x \in t\mathcal{C}} u(x, t) \leq \theta. \quad \square$$

As an important application of Theorem 5.10, we will prove that there are not stationary solutions $u \geq 0$ to (2.1) (i.e. solutions with $\frac{\partial}{\partial t}u = 0$), except $u \equiv 0$ and $u \equiv \theta$, provided that the origin belongs to $\text{int}(\Upsilon_1)$.

Proposition 5.12. *Let the conditions (A1), (A2), (A4), (A6), (A9), and (2.21) hold. Let the origin belongs to $\text{int}(\Upsilon_1)$. Then there exist only two non-negative stationary solutions to (2.1) in $L^\infty(\mathbb{R}^d)$, namely, $u = 0$ and $u = \theta$.*

Proof. Since $\frac{\partial}{\partial t}u = 0$, one gets from (2.1) that

$$u(x) = \frac{\varkappa^+(a^+ * u)(x)}{m + \varkappa^-(a^- * u)(x)}, \quad x \in \mathbb{R}^d. \quad (5.33)$$

Then, by Lemma 2.1, one easily gets that $u \in C_{ub}(\mathbb{R}^d)$.

Denote $M := \|u\| = \sup_{x \in \mathbb{R}^d} u(x)$. We are going to prove now that $M \leq \theta$. On the contrary, suppose that $M > \theta$. One can rewrite (5.33) as follows:

$$mu(x) + \varkappa^-(a^- * u)(x)(u(x) - \theta) = (J_\theta * u)(x) \leq Mm, \quad (5.34)$$

where $J_\theta \geq 0$ is given by (3.19) and hence $\int_{\mathbb{R}^d} J_\theta(x) dx = m$.

Choose a sequence $x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that $u(x_n) \rightarrow M$, $n \rightarrow \infty$. Substitute x_n to the inequality (5.34) and pass $n \rightarrow \infty$. Since $M > \theta$ and $u \geq 0$, one gets then that $(a^- * u)(x_n) \rightarrow 0$, $n \rightarrow \infty$. Passing to a subsequence of $\{x_n\}$ and keeping the same notation, for simplicity, one gets that

$$(a^- * u)(x_n) \leq \frac{1}{n}, \quad n \geq 1.$$

For all $n \geq r_0^{-2d}$, set $r_n := n^{-\frac{1}{2d}} < r_0$; then the inequality (2.21) holds, for any $x \in B_{r_n}(0)$, and hence

$$\frac{1}{n} \geq (a^- * u)(x_n) \geq \alpha(\mathbb{1}_{B_{r_n}(0)} * u)(x_n) \geq \alpha V_d(r_n) \min_{x \in B_{r_n}(x_n)} u(x), \quad (5.35)$$

where $V_d(R)$ is a volume of a sphere with the radius $R > 0$ in \mathbb{R}^d . Since $V(r_n) = r_n^d V_d(1) = n^{-\frac{1}{2}} V_d(1)$, we have from (5.35), that, for any $n \geq r_0^{-2d}$, there exists $y_n \in B_{r_n}(x_n)$, such that

$$u(y_n) \leq \frac{1}{\alpha \sqrt{n} V_d(1)}.$$

Thus $u(y_n) \rightarrow 0$, $n \rightarrow \infty$. Recall that $u(x_n) \rightarrow M > 0$, $n \rightarrow \infty$, however, $|x_n - y_n| \leq r_n = n^{-\frac{1}{2d}}$, that may be arbitrary small. This contradicts the fact that $u \in C_{ub}(\mathbb{R}^d)$.

As a result, $0 \leq u(x) \leq \theta = M$, $x \in \mathbb{R}^d$. Let $u \not\equiv 0$. By Theorem 5.10, for any compact set $\mathcal{C} \subset \text{int}(\Upsilon_1)$, $\min_{x \in \mathcal{C}} u(x) \rightarrow \theta$, $t \rightarrow \infty$, as $u(x, t) = u(x)$ now. Since $0 \in \text{int}(\Upsilon_1)$, the latter convergence is obviously possible for $u \equiv \theta$ only. \square

5.3 Proof of Theorem 5.10

We will do as follows. At first, in Proposition 5.17, we apply results of [75] for discrete time, to prove (5.32) for continuous time, provided that u_0 is separated from 0 on a big enough set. Next, in Proposition 5.18, we show that there exists a proper subsolution to (2.1), which will reach (as we explain thereafter) any needed level after a finite time. Finally, we properly use in Proposition 5.19 the results of [11], to prove that the solution to (2.1) will dominate the subsolution after a finite time.

We start with the following Weinberger's result (rephrased in our settings). Note that (A4) implies (A3), hence, under conditions of Theorem 5.10, we have by Proposition 5.7, that $\Upsilon_T \neq \emptyset$, $T > 0$.

Lemma 5.13 (cf. [75, Theorem 6.2]). *Let (A1), (A2), (A4), (A6), (A9) hold. Let $u_0 \in U_\theta$ and $T > 0$ be arbitrary, and Q_T be given by (3.33). Define*

$$u_{n+1}(x) := (Q_T u_n)(x), \quad n \geq 0. \quad (5.36)$$

Then, for any compact set $\mathcal{C}_T \subset \text{int}(\Upsilon_T)$ and for any $\sigma \in (0, \theta)$, one can choose a radius $r_\sigma = r_\sigma(Q_T, \mathcal{C}_T)$, such that

$$u_0(x) \geq \sigma, \quad x \in B_{r_\sigma}(0), \quad (5.37)$$

implies

$$\lim_{n \rightarrow \infty} \min_{x \in n\mathcal{C}_T} u_n(x) = \theta. \quad (5.38)$$

Remark 5.14. By the proof of [75, Theorem 6.2], the radius $r_\sigma(Q_T, \mathcal{C}_T)$ is not defined uniquely. In the sequel, $r_\sigma(Q_T, \mathcal{C}_T)$ means just a radius which fulfills the assertion of Lemma 5.13 for the chosen Q_T and \mathcal{C}_T , rather than a function of Q_T and \mathcal{C}_T .

Remark 5.15. It is worth noting, that, by (3.33) and the uniqueness of the solution to (2.1), the iteration (5.36) is just given by

$$u_n(x) = u(x, nT), \quad x \in \mathbb{R}^d, n \in \mathbb{N} \cup \{0\}. \quad (5.39)$$

Therefore, (5.38) with $T = 1$ yields (5.32), for $\mathbb{N} \ni t \rightarrow \infty$, namely,

$$\lim_{n \rightarrow \infty} \min_{x \in n\mathcal{C}} u(x, n) = \theta, \quad (5.40)$$

provided that (5.37) holds with $r_\sigma = r_\sigma(Q_1, \mathcal{C})$, $\mathcal{C} \subset \text{int}(\Upsilon_1)$.

Lemma 5.16. *Let the conditions of Theorem 5.10 hold. Fix a $\sigma \in (0, \theta)$ and a compact set $\mathcal{C} \subset \text{int}(\Upsilon_1)$. Let $u_0 \in U_\theta$ be such that $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1, \mathcal{C})}(0)$. Then, for any $k \in \mathbb{N}$,*

$$\lim_{n \rightarrow \infty} \min_{x \in \frac{n}{k}\mathcal{C}} u\left(x, \frac{n}{k}\right) = \theta. \quad (5.41)$$

Proof. Since $\mathcal{C} \subset \text{int}(\Upsilon_1)$, one can choose a compact set $\tilde{\mathcal{C}} \subset \text{int}(\Upsilon_1)$ such that

$$\mathcal{C} \subset \text{int}(\tilde{\mathcal{C}}). \quad (5.42)$$

By (5.39) and Lemma 5.13 (with $T = 1$), the assumption $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1, \mathcal{C})}(0)$ implies (5.40). Fix $k \in \mathbb{N}$, take $p = \frac{1}{k}$; then choose and fix the radius $r_\sigma(Q_p, p\tilde{\mathcal{C}})$. By (5.40), there exists an $N = N(k) \in \mathbb{N}$, such that

$$\begin{aligned} u(x, N) &\geq \sigma, \quad x \in N\mathcal{C}, \\ B_{r_\sigma(Q_p, p\tilde{\mathcal{C}})}(0) &\subset N\mathcal{C}. \end{aligned}$$

Apply now Lemma 5.13, with $u_0(x) = u(x, N)$, $x \in \mathbb{R}^d$, $T = p$, and

$$\mathcal{C}_T = \mathcal{C}_p := p\tilde{\mathcal{C}} \subset p \operatorname{int}(\Upsilon_1) = \operatorname{int}(\Upsilon_p),$$

as, by (5.22), $p\Upsilon_1 = \Upsilon_p$. We will get then

$$\lim_{n \rightarrow \infty} \min_{x \in np\tilde{\mathcal{C}}} u(x, N + np) = \theta. \quad (5.43)$$

By (5.42), there exists $M \in \mathbb{N}$ such that one has

$$\left(\frac{N}{n} + p\right)\mathcal{C} \subset p\tilde{\mathcal{C}}, \quad n \geq M. \quad (5.44)$$

Therefore, by (5.44), one gets, for $n \geq M$,

$$\begin{aligned} \min_{x \in np\tilde{\mathcal{C}}} u(x, N + np) &\leq \min_{x \in n\left(\frac{N}{n} + p\right)\mathcal{C}} u(x, N + np) \\ &= \min_{x \in (Nk + n)\frac{1}{k}\mathcal{C}} u\left(x, (Nk + n)\frac{1}{k}\right) \leq \theta. \end{aligned} \quad (5.45)$$

By (5.43) and (5.45), one gets the statement. \square

Now, one can prove Theorem 5.10, under assumption on the initial condition.

Proposition 5.17. *Let the conditions of Theorem 5.10 hold. Fix a $\sigma \in (0, \theta)$ and a compact set $\mathcal{C} \subset \operatorname{int}(\Upsilon_1)$. Let $u_0 \in U_\theta$ be such that $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1, \mathcal{C})}(0)$, and $u \in \mathcal{X}_\infty$ be the corresponding solution to (2.1). Then*

$$\lim_{t \rightarrow \infty} \min_{x \in t\mathcal{C}} u(x, t) = \theta. \quad (5.46)$$

Proof. Suppose (5.46) were false. Then, there exist $\varepsilon > 0$ and a sequence $t_N \rightarrow \infty$, such that $\min_{x \in t_N\mathcal{C}} u(x, t_N) \leq \theta - \varepsilon$, $n \in \mathbb{N}$. Since $t_N\mathcal{C}$ is a compact set and $u(\cdot, t) \in U_\theta$, $t \geq 0$, there exists $x_N \in t_N\mathcal{C}$, such that

$$u(x_N, t_N) < \theta - \varepsilon, \quad n \in \mathbb{N}. \quad (5.47)$$

Next, by Proposition 2.9, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x', x'' \in \mathbb{R}^d$ and for all $t', t'' > 0$, with $|x' - x''| + |t' - t''| < \delta$, one has

$$|u(x', t') - u(x'', t'')| < \frac{\varepsilon}{2}. \quad (5.48)$$

Since \mathcal{C} is a compact, $p(\mathcal{C}) := \sup_{x \in \mathcal{C}} \|x\| < \infty$. Choose $k \in \mathbb{N}$, such that $\frac{1}{k} < \frac{\delta}{1+p(\mathcal{C})}$. By (5.41), there exists $M(k) \in \mathbb{N}$, such that, for all $n > M(k)$,

$$u\left(x, \frac{n}{k}\right) > \theta - \frac{\varepsilon}{2}, \quad x \in \frac{n}{k}\mathcal{C}. \quad (5.49)$$

Choose $N > N_0$ big enough to ensure $t_N > \frac{M(k)}{k}$. Then, there exists $n > M(k)$, such that $t_N \in [\frac{n}{k}, \frac{n+1}{k})$. Hence

$$\left| t_N - \frac{n}{k} \right| < \frac{1}{k} < \frac{\delta}{1 + p(\mathcal{C})}. \quad (5.50)$$

Then, for the chosen N , there exists $y_N \in \mathcal{C}$, such that $x_N = t_N y_N$. Set $t' = t_N$, $t'' = \frac{n}{k}$, $x' = x_N = t_N y_N$, and $x'' = \frac{n}{k} y_N$. Then, by (5.50),

$$|t' - t''| + |x' - x''| = \left| t_N - \frac{n}{k} \right| (1 + |y_N|) < \delta.$$

Therefore, one can apply (5.48). Combining this with (5.47), one gets

$$u\left(\frac{n}{k} y_N, \frac{n}{k}\right) = u\left(\frac{n}{k} y_N, \frac{n}{k}\right) - u(t_N y_N, t_N) + u(x_N, t_N) < \frac{\varepsilon}{2} + \theta - \varepsilon = \theta - \frac{\varepsilon}{2},$$

that contradicts (5.49), as $\frac{n}{k} y_N \in \frac{n}{k} \mathcal{C}$. Hence the statement is proved. \square

Next two statements will allow us to get rid the restriction on u_0 in Proposition 5.17.

Proposition 5.18. *Let (A1), (A2), and (A4) hold; assume also that (5.24) holds, and \mathbf{m} is given by (5.25). Then there exists $\alpha_0 > 0$, such that, for all $\alpha \in (0, \alpha_0)$, there exists $q_0 = q_0(\alpha) \in (0, \theta)$, such that there exists $T = T(\alpha, q_0) > 0$, such that, for all $q \in (0, q_0)$, the function*

$$w(x, t) = q \exp\left(-\frac{|x - t\mathbf{m}|^2}{\alpha t}\right), \quad x \in \mathbb{R}^d, t > T, \quad (5.51)$$

is a subsolution to (2.1) on $t > T$; i.e. $\mathcal{F}w(x, t) \leq 0$, $x \in \mathbb{R}^d$, $t > T$, where \mathcal{F} is given by (3.1).

Proof. Let J_q , $q \in (0, \theta)$ be given by (3.19), and consider the function (5.51). Since $w(x, t) \leq q$, we have from (3.1), that

$$\begin{aligned} (\mathcal{F}w)(x, t) &= w(x, t) \left(\frac{|x|^2}{\alpha t^2} - \frac{|\mathbf{m}|^2}{\alpha} \right) - (a^+ * w)(x, t) \\ &\quad + \varkappa^- w(x, t) (a^- * w)(x, t) + mw(x, t) \\ &\leq w(x, t) \left(\frac{|x|^2}{\alpha t^2} - \frac{|\mathbf{m}|^2}{\alpha} \right) - (J_q * w)(x, t) + mw(x, t). \end{aligned} \quad (5.52)$$

Since, for any $q_0 \in (0, \theta)$ and for any $q \in (0, q_0)$, $J_q(x) \geq J_{q_0}(x)$, $x \in \mathbb{R}^d$, one gets from (5.52), that, to have $\mathcal{F}w \leq 0$, it is enough to claim that, for all $x \in \mathbb{R}^d$,

$$m + \frac{|x|^2}{\alpha t^2} - \frac{|\mathbf{m}|^2}{\alpha} \leq \exp\left(\frac{|x - t\mathbf{m}|^2}{\alpha t}\right) \int_{\mathbb{R}^d} J_{q_0}(y) \exp\left(-\frac{|x - y - t\mathbf{m}|^2}{\alpha t}\right) dy.$$

By changing x onto $x + t\mathbf{m}$ and a simplification, one gets an equivalent inequality

$$m + \frac{|x|^2}{\alpha t^2} + \frac{2x \cdot \mathbf{m}}{\alpha t} \leq \int_{\mathbb{R}^d} J_{q_0}(y) \exp\left(\frac{2x \cdot y}{\alpha t}\right) \exp\left(-\frac{|y|^2}{\alpha t}\right) dy =: I(t). \quad (5.53)$$

One can rewrite $I(t) = I_0(t) + I^+(t) + I^-(t)$, where

$$\begin{aligned} I_0(t) &:= \int_{\mathbb{R}^d} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} dy; & I^+(t) &:= \int_{x \cdot y \geq 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} \left(e^{\frac{2x \cdot y}{\alpha t}} - 1 \right) dy; \\ I^-(t) &:= \int_{x \cdot y < 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} \left(e^{\frac{2x \cdot y}{\alpha t}} - 1 \right) dy. \end{aligned}$$

Using that $e^s - 1 \geq s$, for all $s \in \mathbb{R}$, and $e^s - 1 \geq s + \frac{s^2}{2}$, for all $s \geq 0$, one gets the following estimates

$$I^+(t) \geq \frac{2}{\alpha t} \int_{x \cdot y \geq 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y) dy + \frac{2}{\alpha^2 t^2} \int_{x \cdot y \geq 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y)^2 dy$$

and

$$I^-(t) \geq \frac{2}{\alpha t} \int_{x \cdot y < 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y) dy.$$

Therefore,

$$\begin{aligned} I(t) &\geq I_0(t) + \frac{2}{\alpha t} \left(x \cdot \int_{\mathbb{R}^d} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} y dy \right) \\ &\quad + \frac{2}{\alpha^2 t^2} \int_{x \cdot y \geq 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y)^2 dy. \end{aligned} \quad (5.54)$$

By the dominated convergence theorem,

$$I_0(t) \nearrow \int_{\mathbb{R}^d} J_{q_0}(x) dx = \varkappa^+ - q_0 \varkappa^- > m, \quad t \rightarrow \infty, \quad (5.55)$$

for any $q_0 \in (0, \theta)$. Set also

$$I_1(t) := \int_{\mathbb{R}^d} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} y dy.$$

By (5.24) and (A2), one has $\int_{\mathbb{R}^d} a^-(x) |x| dx < \infty$ and hence $\int_{\mathbb{R}^d} J_{q_0}(x) |x| dx < \infty$. Then, by the dominated convergence theorem,

$$I_1(t) \nearrow \int_{\mathbb{R}^d} J_{q_0}(y) y dy =: \mu(q_0) \in \mathbb{R}^d, \quad t \rightarrow \infty. \quad (5.56)$$

Since $0 \leq J_{q_0}(x) \leq \varkappa^+ a^+(x)$, $x \in \mathbb{R}^d$, we have, by (5.25) and the dominated convergence theorem, that $m(q_0) \rightarrow m$, $q_0 \rightarrow 0$.

For any $\varepsilon > 0$ with $m + 2\varepsilon < \varkappa^+$, one can choose $q_0 = q_0(\varepsilon) \in (0, \theta)$, such that

$$\varkappa^+ > \varkappa^+ - q_0 \varkappa^- > m + 2\varepsilon, \quad |m - \mu(q_0)| < \frac{\varepsilon}{2}. \quad (5.57)$$

By (5.55), (5.56), there exists $T_1 = T_1(\varepsilon, q_0) > 0$, such that, for all $\alpha > 0$ and $t > 0$ with $\alpha t > T_1$, one has, cf. (5.57),

$$\varkappa^+ \geq I_0(t) > m + \varepsilon, \quad |I_1(t) - \mu(q_0)| < \frac{\varepsilon}{2}. \quad (5.58)$$

Let $T > \frac{T_1}{\alpha}$ be chosen later. The function

$$I_2(t) := \int_{x \cdot y \geq 0} J_{q_0}(y) e^{-\frac{|y|^2}{\alpha t}} (x \cdot y)^2 dy$$

is also increasing in $t > 0$. Clearly, from (5.57) and (5.58), one has $|I_1(t) - \mathbf{m}| < \varepsilon$. Therefore, by (5.54) and (5.58), one gets, for $t > T > \frac{T_1}{\alpha}$,

$$\begin{aligned} I(t) &> m + \varepsilon + \frac{2}{\alpha t} x \cdot (I_1(t) - \mathbf{m}) + \frac{2}{\alpha t} x \cdot \mathbf{m} + \frac{2}{\alpha^2 t^2} I_2(t) \\ &\geq m + \varepsilon - \frac{2\varepsilon}{\alpha t} |x| + \frac{2}{\alpha t} x \cdot \mathbf{m} + \frac{2}{\alpha^2 t^2} I_2(T). \end{aligned} \quad (5.59)$$

Next, by (A2), (A4), and (3.19), $J_{q_0}(y) \geq \rho$, for a.a. $y \in B_\delta(0)$. For an arbitrary $x \in \mathbb{R}^d$, consider the set

$$B_x = \left\{ y \in \mathbb{R}^d \mid |y| \leq \delta, \frac{1}{2} \leq \frac{x \cdot y}{|x||y|} \leq 1 \right\}.$$

Then

$$I_2(T) \geq \frac{\rho}{4} |x|^2 \int_{B_x} |y|^2 e^{-\frac{|y|^2}{\alpha T}} dy. \quad (5.60)$$

The set B_x is a cone inside the ball $B_\delta(0)$, with the apex at the origin, the height which lies along x , and the apex angle $2\pi/3$. Since the function inside the integral in the r.h.s. of (5.60) is radially symmetric, the integral does not depend on x . Fix an arbitrary $\bar{x} \in \mathbb{R}^d$ and denote

$$A(\tau) = A(\tau, \delta) = \int_{B_{\bar{x}}} |y|^2 e^{-\frac{|y|^2}{\tau}} dy \nearrow \int_{B_{\bar{x}}} |y|^2 dy =: \bar{B}_\delta, \quad \tau \rightarrow \infty. \quad (5.61)$$

Then, by (5.59) and (5.60), one has, for $t > T$,

$$I(t) > m + \varepsilon - \frac{2\varepsilon}{\alpha t} |x| + \frac{2}{\alpha t} x \cdot \mathbf{m} + \frac{\rho A(\alpha T)}{2\alpha^2 t^2} |x|^2. \quad (5.62)$$

By (5.62), to prove (5.53), it is enough to show that

$$\varepsilon - \frac{2\varepsilon}{\alpha t} |x| + \frac{\rho A(\alpha T)}{2\alpha^2 t^2} |x|^2 \geq \frac{|x|^2}{\alpha t^2}, \quad t > T, \quad x \in \mathbb{R}^d,$$

or, equivalently, for $2\alpha < \rho A(\alpha T)$,

$$\begin{aligned} &\left(\sqrt{\frac{\rho A(\alpha T) - 2\alpha}{2}} \frac{|x|}{\alpha t} - \varepsilon \sqrt{\frac{2}{\rho A(\alpha T) - 2\alpha}} \right)^2 \\ &\quad + \varepsilon - \varepsilon^2 \frac{2}{\rho A(\alpha T) - 2\alpha} \geq 0. \end{aligned} \quad (5.63)$$

To get (5.63), we proceed as follows. For a given $\rho > 0$, $\delta > 0$ which provide (A4), we set $\alpha_0 := \frac{1}{2} \rho \bar{B}_\delta$, cf. (5.61). Then, for any $\alpha \in (0, \alpha_0)$, there exists $T_2 = T_2(\alpha) > 0$, such that

$$2\alpha < \rho A(\alpha T_2) < \rho \bar{B}_\delta.$$

Choose now $\varepsilon = \varepsilon(\alpha) > 0$, such that $m + 2\varepsilon < \varkappa^+$ and

$$\varepsilon < \frac{1}{2}(\rho A(\alpha T_2) - 2\alpha) < \frac{1}{2}(\rho A(\alpha T) - 2\alpha), \quad T > T_2. \quad (5.64)$$

For the chosen ε , find $q_0 = q_0(\alpha) \in (0, \theta)$ which ensures (5.57). Then, find $T_1 = T_1(\alpha, q_0) > 0$ which gives (5.58); and, finally, take $T = T(\alpha, q_1) > T_2$ such that $\alpha T > T_1$. As a result, for $t > T$, one has $\alpha t > \alpha T > T_1$, thus (5.58) holds, whereas (5.64) yields (5.63). The latter inequality gives (5.53), and hence, for all $q \in (0, q_0)$, $\mathcal{F}w \leq 0$, for w given by (5.51). The statement is proved. \square

Proposition 5.19. *Let (A1), (A2), and (A4) hold. Then, there exists $t_1 > 0$, such that, for any $t > t_1$ and for any $\tau > 0$, there exists $q_1 = q_1(t, \tau) > 0$, such that the following holds. If $u_0 \in L_\theta$ is such that there exist $\eta > 0$, $r > 0$, $x_0 \in \mathbb{R}^d$ with $u_0(x) \geq \eta$, $x \in B_r(x_0)$ and $u \in \tilde{\mathcal{X}}_\infty$ is the corresponding solution to (2.1), then*

$$u(x, t) \geq q_1 e^{-\frac{|x-x_0|^2}{\tau}}, \quad x \in \mathbb{R}^d. \quad (5.65)$$

Proof. At first, we note that (5.65) may be rewritten as follows:

$$q_1 e^{-\frac{|x|^2}{\tau}} \leq u(x + x_0, t_0) = T_{-x_0} Q_{t_0} u_0(x) = Q_{t_0} T_{-x_0} u_0(x),$$

cf. (3.33), (3.34), (3.35), and one has

$$T_{-x_0} u_0(x) = u_0(x + x_0) \geq \eta, \quad |(x + x_0) - x_0| = |x| \leq r.$$

Therefore, it is enough to prove the statement for $x_0 = 0$.

Consider now arbitrary functions $b, v_0 \in C^\infty(\mathbb{R}^d)$, such that

$$\begin{aligned} \text{supp } b &= B_\delta(0), & 0 < b(x) = b(|x|) \leq \rho, & \quad x \in B_\delta(0); \\ \text{supp } v_0 &= B_r(0), & 0 < v_0(x) \leq \eta, & \quad x \in B_r(0); \\ \exists 0 < p < \min\{r, 1\}, & 0 < \nu < \eta, & \text{ such that } v_0(x) \geq \nu, & \quad x \in B_p(0). \end{aligned}$$

Set $\langle b \rangle := \int_{\mathbb{R}^d} b(x) dx > 0$. Define two bounded operators in the space $L^\infty(\mathbb{R}^d)$, cf. (1.4): $Bu = b * u$, $L_b u = Bu - \langle b \rangle u$. One can rewrite (2.1) as follows

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= (J_\theta * u)(x, t) - mu(x, t) + \varkappa^-(\theta - u(x, t))(a^- * u)(x, t) \\ &= (b * u)(x, t) - mu(x, t) + f(x, t), \end{aligned}$$

where, for any $x \in \mathbb{R}^d$, $t \geq 0$,

$$f(x, t) := ((J_\theta - b) * u)(x, t) + \varkappa^-(\theta - u(x, t))(a^- * u)(x, t).$$

By (A4) and the choice of b , $J_\theta(x) \geq b(x)$, $x \in \mathbb{R}^d$. In particular, $m = \int_{\mathbb{R}^d} J_\theta(x) dx \geq \langle b \rangle$, and $f(x, t) \geq 0$, $x \in \mathbb{R}^d$, $t \geq 0$. Next, for any $t \geq 0$, $\|f(\cdot, t)\|_\infty \leq \theta(m - \langle b \rangle) + \varkappa^- \theta^2 < \infty$. Since $b \geq 0$ and $Bu = b * u$ defines a bounded operator on $L^\infty(\mathbb{R}^d)$, one has that $e^{tB} f(x, s) \geq 0$, for all $t, s \geq 0$, $x \in \mathbb{R}^d$. By the same argument, $u_0(x) \geq \eta \mathbb{1}_{B_r(0)}(x) \geq v_0(x) \geq 0$ implies $(e^{tB} u_0)(x) \geq (e^{tB} v_0)(x)$. Therefore,

$$\begin{aligned} u(x, t) &= e^{-tm} (e^{tB} u_0)(x) + \int_0^t e^{-(t-s)m} (e^{(t-s)B} f)(x, s) ds \\ &\geq e^{-tm} (e^{tB} u_0)(x) \geq e^{-(m-\langle b \rangle)t} (e^{tL_b} v_0)(x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (5.66)$$

We are going to apply now the results of [11]. To do this, set $\beta := \langle b \rangle^{-1}$. Then

$$(e^{tL_b}v_0)(x) = (e^{\langle b \rangle t(\beta L_b)}v_0)(x) = v(x, \langle b \rangle t), \quad (5.67)$$

where v solves the differential equation $\frac{d}{dt}v = \beta L_b$. Since $\int_{\mathbb{R}^d} \beta b(x) dx = 1$, then, by [13, Theorem 2.1, Lemma 2.2],

$$v(x, t) = e^{-t}v_0(x) + (w * v_0)(x, t), \quad (5.68)$$

where $w(x, t)$ is a smooth function. Moreover, by [11, Proposition 5.1], for any $\omega \in (0, \delta)$ there exist $c_1 = c_1(\omega) > 0$ and $c_2 = c_2(\omega) \in \mathbb{R}$, such that

$$\begin{aligned} w(x, t) &\geq h(x, t), \quad x \in \mathbb{R}^d, t \geq 0, \\ h(x, t) &:= c_1 t \exp\left(-t - \frac{1}{\omega}|x| \log|x| + (\log t - c_2) \left\lceil \frac{|x|}{\omega} \right\rceil\right). \end{aligned} \quad (5.69)$$

Here $\lceil \alpha \rceil$ means the entire part of an $\alpha \in \mathbb{R}$, and $0 \log 0 := 1$, $\log 0 := -\infty$.

Set $t_1 = e^{c_2} > 0$. Since $\lceil \alpha \rceil > \alpha - 1$, $\alpha \in \mathbb{R}$, one has, for $t > t_1$,

$$h(x, t) \geq c_1 e^{c_2} \exp\left(-t - \frac{1}{\omega}|x| \log|x| + (\log t - c_2) \frac{|x|}{\omega}\right) \geq c_3 g(x, t),$$

where $c_3 = c_1 e^{c_2} > 0$ and

$$g(x, t) := \exp\left(-t - \frac{1}{\omega}|x| \log|x|\right), \quad x \in \mathbb{R}^d, t > t_1.$$

Since $v_0 \geq \nu \mathbb{1}_{B_p(0)}$, one gets from (5.68) and (5.69), that

$$v(x, t) \geq \nu e^{-t} \mathbb{1}_{B_p(0)}(x) + \nu c_3 \int_{B_p(x)} g(y, t) dy \quad (5.70)$$

Set $V_p := \int_{B_p(0)} dx$. For any fixed $t > t_1$, since $g(\cdot, t) \in C(B_p(x))$, there exists $y_*, y^* \in B_p(x)$, such that $g(y, t)$ attains its minimal and maximal values on $B_p(x)$ at these points, respectively. Since $B_p(x)$ is a convex set, one gets that, for any $\gamma \in [0, 1]$, $y_\gamma := \gamma y^* + (1 - \gamma)y_* \in B_p(x)$. Then

$$V_p g(y_0, t) \leq \int_{B_p(x)} g(y_\gamma, t) dy \leq V_p g(y_1, t).$$

Therefore, by the intermediate value theorem there exists, $\tilde{y}_t = \tilde{y}(x, t) \in B_p(x)$, $t > t_1$, $x \in \mathbb{R}^d$, such that $\int_{B_p(x)} g(y, t) dy = V_p g(\tilde{y}_t, t)$. Hence one gets from (5.66), (5.67), (5.70), that

$$u(x, t) \geq c_4 e^{-(m-\langle b \rangle)t} g(\tilde{y}_t, \langle b \rangle t) = c_4 \exp\left(-mt - \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t|\right), \quad (5.71)$$

for $\tilde{y}_t = \tilde{y}(x, t) \in B_p(x)$, $t > t_1$; here $c_4 = c_3 \nu V_p > 0$.

As a result, to get the statement, it is enough to show that, for any $t > t_1$ and for any $\tau > 0$, there exists $q_1 = q_1(t, \tau) > 0$, such that the r.h.s. of (5.71) is estimated from below by $q_1 e^{-\frac{|x|^2}{\tau}}$, i.e. that

$$mt + \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t| - \log c_4 \leq \frac{|x|^2}{\tau} - \log q_1, \quad x \in \mathbb{R}^d, \quad (5.72)$$

Note that $\tilde{y}_t \in B_p(x)$ implies $|\tilde{y}_t| \leq p + |x|$, $x \in \mathbb{R}^d$.

Let $p + |x| \leq 1$. Then $\log |\tilde{y}_t| \leq 0$, and the l.h.s. of (5.72) is majorized by $mt - \log c_4$. Therefore, to get (5.72), it is enough to have $q_1 < c_4 e^{-mt}$, regardless of τ .

Let now $|x| + p > 1$. Recall that we chose $p < 1$. The function $s \log s$ is increasing on $s > 1$. Hence to get (5.72), we claim

$$(|x| + 1) \log(|x| + 1) \leq \frac{\omega}{\tau} |x|^2 - \omega m t + \omega \log c_4 - \omega \log q_1. \quad (5.73)$$

Consider now the function $f(s) = as^2 - (s+1) \log(s+1)$, $s \geq 0$, $a = \frac{\omega}{\tau} > 0$. Then $f(0) = 0$, $f'(s) = 2as - \log(s+1) - 1$, $f'(0) = -1$, $f''(s) = 2a - \frac{1}{s+1}$. Since $f''(s) \nearrow 2a > 0$, $s \rightarrow \infty$, there exists $s_0 > 0$, such that $f''(s) > 0$, for all $s > s_0$, i.e. $f'(s)$ increases on $s > s_0$. Since $f'(s) \rightarrow \infty$, $s \rightarrow \infty$, there exists $s_1 > s_0$, such that $f'(s) > 0$, for all $s > s_1$, i.e. f is increasing on $s > s_1$. Finally, for any $t > t_1$, one can choose $q_1 = q_1(t, \tau) > 0$ small enough, to get

$$\min_{s \in [0, s_1]} f(s) - \omega m t + \omega \log c_4 - \omega \log q_1 > 0$$

and too fulfill (5.73), for all $x \in \mathbb{R}^d$. The statement is proved. \square

Now, we are ready to prove the main Theorem 5.10.

Proof of Theorem 5.10. For $u_0 \equiv \theta$, the statement is trivial. Hence let $u_0 \not\equiv \theta$, $u_0 \not\equiv 0$. Next, recall that, (A4) implies (A3) and (A9) implies (5.24). Therefore, one may use the statements of Propositions 5.7, 5.18, 5.19.

According to Proposition 5.18, choose any $\alpha \in (0, \alpha_0)$ and take the corresponding $q_0 = q_0(\alpha) \in (0, \theta)$ and $T = T(\alpha, q_0) > 0$. Choose then arbitrary $t_2 > T$. Let \mathbf{m} be given by (5.25). Set $x_0 = t_2 \mathbf{m} \in \mathbb{R}^d$. By Proposition 3.7, there exist $\eta = \eta(t_2) > 0$ and $r = r(t_2) > 0$, such that $u(x, t_2) \geq \eta$, $|x - x_0| = |x - t_2 \mathbf{m}| \leq r$. Apply now Proposition 5.19, with $u_0(x) = u(x, t_2)$; let t_1 be the moment of time stated there. Take, for the α chosen above, $\tau = \alpha t_2 > 0$. Take any $t_3 > \max\{t_1, t_2\}$ and the corresponding $q_1 = q_1(t_3, \tau) > 0$. We will get then, by (5.65), that

$$u(x, t_3 + t_2) \geq q_1 \exp\left(-\frac{|x - t_2 \mathbf{m}|^2}{\alpha t_2}\right), \quad x \in \mathbb{R}^d. \quad (5.74)$$

Of course, one can assume that $q_1 < q_0$ (otherwise, we just pass to a weaker inequality in (5.74)).

We are going to apply now Theorem 3.1, with $c = \theta$ and, for $t \geq 0$,

$$\begin{aligned} u_1(x, t) &= q_1 \exp\left(-\frac{|x - (t + t_2) \mathbf{m}|^2}{\alpha(t + t_2)}\right) \geq 0, \\ u_2(x, t) &= u(x, t + t_3 + t_2) \in [0, \theta]. \end{aligned}$$

By (5.74), $u_1(x, 0) \leq u_2(x, 0)$, $x \in \mathbb{R}^d$. Since u solve (2.1), $\mathcal{F}u_2 \equiv 0$. Next, by Proposition 5.18, if we set $q = q_1$, we will have $\mathcal{F}u_1 \leq 0$, as $t + t_2 \geq t_2 > T$. Therefore, by Theorem 3.1,

$$u(x, t + t_3 + t_2) \geq q_1 \exp\left(-\frac{|x - (t + t_2) \mathbf{m}|^2}{\alpha(t + t_2)}\right), \quad t \geq 0, x \in \mathbb{R}^d,$$

or, equivalently,

$$u(x + (t + t_2)\mathbf{m}, t + t_3 + t_2) \geq q_1 \exp\left(-\frac{|x|^2}{\alpha(t + t_2)}\right), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

Let now $\mathcal{K} \subset \text{int}(\Upsilon_1)$ be a compact set. Choose any $\sigma \in (0, q_1)$ and consider a radius $r_\sigma = r_\sigma(Q_1, \mathcal{K})$ which fulfills Proposition 5.17, cf. Remark 5.14. Then $|x| \leq r_\sigma$ implies that there exists $t_4 = t_4(\sigma, \mathcal{K}) > 0$, such that, for all $t \geq t_4$,

$$q_1 \exp\left(-\frac{|x|^2}{\alpha(t + t_2)}\right) \geq q_1 \exp\left(-\frac{r_\sigma^2}{\alpha(t + t_2)}\right) > \sigma.$$

Then, one can apply Proposition 5.17 with $u_0(x) = u(x + (t_4 + t_2)\mathbf{m}, t_4 + t_3 + t_2)$, $x \in \mathbb{R}^d$; by (5.46), we have

$$\lim_{t \rightarrow \infty} \min_{x \in t\mathcal{K}} u(x + (t_2 + t_4)\mathbf{m}, t + t_2 + t_3 + t_4) = \theta. \quad (5.75)$$

Let, finally, $\mathcal{C} \subset \text{int}(\Upsilon_1)$ be an arbitrary compact set from the statement of Theorem 5.10. It is well-known, that the distance between a compact and a closed set is positive; in particular, one can consider the compact \mathcal{C} and the closure of $\mathbb{R}^d \setminus \Upsilon_1$. Therefore, there exists a compact set $\mathcal{K} \subset \text{int}(\Upsilon_1)$, such that $\mathcal{C} \subset \text{int}(\mathcal{K})$. Let $\delta_0 > 0$ be the distance between \mathcal{C} and the closure of $\mathbb{R}^d \setminus \mathcal{K}$. One has then that (5.75) does hold with $t_4 = t_4(\sigma, \mathcal{K}) > 0$.

By (5.75), for any $\varepsilon > 0$, there exists $t_5 > 0$ such that, for all $t > t_2 + t_3 + t_4 + t_5 =: t_6 > 0$ and for all $y \in \mathcal{K}$,

$$u((t - t_2 - t_3 - t_4)y + (t_2 + t_4)\mathbf{m}, t) > \theta - \varepsilon \quad (5.76)$$

Without loss of generality we can assume that t_5 is big enough to ensure

$$(t_2 + t_3 + t_4) \max_{x \in \mathcal{C}} |x| + (t_2 + t_4)|\mathbf{m}| < \delta_0 t_5. \quad (5.77)$$

Then, for any $x \in \mathcal{C}$ and for any $t > t_6$, the vector

$$y(x, t) := \frac{tx - (t_2 + t_4)\mathbf{m}}{t - t_2 - t_3 - t_4}$$

is such that

$$|y(x, t) - x| = \frac{|(t_2 + t_3 + t_4)x - (t_2 + t_4)\mathbf{m}|}{t - t_2 - t_3 - t_4} < \delta_0,$$

where we used (5.77). Therefore, $y(x, t) \in \mathcal{K}$, for all $x \in \mathcal{C}$ and $t > t_6$, and hence (5.76) applied for any such $y(x, t)$ yields $u(tx, t) > \theta - \varepsilon$, $x \in \mathcal{C}$, $t > t_6$, that fulfills the proof. \square

5.4 Fast propagation for slow decaying dispersal kernels

All result above about traveling waves and long-time behavior of the solutions were obtained under exponential integrability assumptions, cf. (A5) or (A9). In [42], it was proved, for the equation (1.11) on \mathbb{R} with local nonlinear term, that the case with a^+ which does not satisfy such conditions leads to ‘accelerating’ solutions, i.e. in this case the equality like (5.32) holds for arbitrary big compact

$\mathcal{C} \subset \mathbb{R}^d$. The aim of this Subsection is to show an analogous result for the equation (2.1). The detailed analysis of the propagation for the slow decaying a^+ will be done in a forthcoming paper.

We will prove an analog of the first statement in [42, Theorem 1].

Theorem 5.20. *Let the conditions (A1), (A2), (A4), (A6), and (5.24) hold. Suppose also there exists a function $0 \leq b \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, such that $a^+(x) \geq b(|x|)$, for a.a. $x \in \mathbb{R}^d$, and that, cf. (A9), for any $\lambda > 0$ and for any $\xi \in S^{d-1}$,*

$$\int_{\mathbb{R}^d} b(|x|)e^{\lambda x \cdot \xi} dx = \infty. \quad (5.78)$$

Let $u_0 \in L_\theta$ be such that there exist $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, with $u_0 \geq \eta$, for a.a. $x \in B_r(x_0)$. Let $u \in \tilde{\mathcal{X}}_\infty$ be the corresponding solution to (2.1). Then, for any compact set $\mathcal{K} \subset \mathbb{R}^d$,

$$\lim_{t \rightarrow \infty} \operatorname{ess\,inf}_{x \in t\mathcal{K}} u(x, t) = \theta. \quad (5.79)$$

Proof. By the same arguments as in the proof of Corollary 5.11, there exists $v_0 \in U_\theta$, $v_0 \not\equiv 0$, such that $u_0(x) \geq v_0(x)$, for a.a. $x \in \mathbb{R}^d$, and $u(x, t) \geq v(x, t)$, for a.a. $x \in \mathbb{R}^d$ and for all $t \geq 0$, where $v \in \tilde{\mathcal{X}}_\infty$ is the corresponding to v_0 solution to (2.1), moreover, $v \in \mathcal{X}_\infty$.

Let $\bar{\theta} \in (0, \theta)$ be chosen and fixed. We are going to apply now Proposition 3.17 to (3.37)–(3.39) with $\Delta_R := B_R(0) \nearrow \mathbb{R}^d$, $R \rightarrow \infty$. Consider an increasing sequence $\{R_n \mid n \in \mathbb{N}\}$, such that

- (i) $\delta < R_n \rightarrow \infty$, $n \rightarrow \infty$, where δ is the same as in (A4);
- (ii) $A_{R_n}^+ > \frac{m}{\varkappa^+}$, $n \in \mathbb{N}$, cf. (3.41);
- (iii) $\bar{\theta} < \theta_{R_n} \leq \theta$, cf. (3.40), (3.44).

Let $w_0 \in C_{ub}(\mathbb{R}^d)$, $w_0 \not\equiv 0$ be such that $0 \leq w_0(x) \leq v_0(x)$, $x \in \mathbb{R}^d$ and $\|w_0\| \leq \bar{\theta}$. Let, for any $n \in \mathbb{N}$, $w^{(n)} \in \mathcal{X}_\infty$ be the corresponding solution to the equation (3.38) with R replaced by R_n . Then, by (3.43), $w^{(n)}(x, t) \leq v(x, t)$, for all $x \in \mathbb{R}^d$, $t \geq 0$, $n \in \mathbb{N}$. As a result,

$$w^{(n)}(x, t) \leq v(x, t) \leq \theta, \quad \text{a.a. } x \in \mathbb{R}^d, \quad t \geq 0, \quad n \in \mathbb{N}. \quad (5.80)$$

For an arbitrary $\xi \in S^{d-1}$, consider the corresponding $\tilde{a}_{R_n}^+$, cf. (4.6). Clearly, $\lambda_0(\tilde{a}_{R_n}^+) = \infty$, i.e. $a_{R_n}^+ \in \mathcal{V}_\xi$, $n \in \mathbb{N}$, cf. Definition 4.19. Let $\mathfrak{a}_\xi^{(n)}(\lambda)$, $n \in \mathbb{N}$, $\lambda > 0$ be defined by (4.10), with a^+ replaced by $a_{R_n}^+$. Finally, let $c_*^{(n)}(\xi)$ be the corresponding minimal traveling wave's speed for the equation (3.38) (with R replaced by R_n). Prove that

$$\lim_{n \rightarrow \infty} \inf_{\xi \in S^{d-1}} c_*^{(n)}(\xi) = \infty. \quad (5.81)$$

By (4.78), it is enough to show that, cf. (5.24), for any

$$C > \varkappa^+ \int_{\mathbb{R}^d} a^+(x)|x| dx, \quad (5.82)$$

there exists $N = N(C) \in \mathbb{N}$, such that, for all $\lambda > 0$,

$$\frac{1}{\lambda} (\varkappa^+ \mathfrak{a}_\xi^{(n)}(\lambda) - m) \geq C, \quad \xi \in S^{d-1}, \quad n \geq N. \quad (5.83)$$

Denote $\Xi_\xi^\pm := \{x \in \mathbb{R}^d \mid \pm x \cdot \xi \geq 0\}$; i.e. $\Xi_\xi^+ \cup \Xi_\xi^- = \mathbb{R}^d$. Then, by (ii) above,

$$\begin{aligned} \varkappa^+ \mathbf{a}_\xi^{(n)}(\lambda) - m &= \varkappa^+ \int_{\mathbb{R}^d} a_{R_n}^+(x)(e^{\lambda x \cdot \xi} - 1) dx + \varkappa^+ A_{R_n}^+ - m \\ &\geq \varkappa^+ \int_{\Xi_\xi^-} a_{R_n}^+(x)(e^{\lambda x \cdot \xi} - 1) dx + \varkappa^+ A_{R_1}^+ - m, \end{aligned}$$

as $\int_{\Xi_\xi^+} a_{R_n}^+(x)(e^{\lambda x \cdot \xi} - 1) dx \geq 0$. By the inequality $1 - e^{-s} \leq s$, $s \geq 0$, one has that

$$\left| \int_{\Xi_\xi^-} a_{R_n}^+(x)(e^{\lambda x \cdot \xi} - 1) dx \right| \leq \lambda \int_{\Xi_\xi^-} a_{R_n}^+(x) |x \cdot \xi| dx \leq \lambda \int_{\mathbb{R}^d} a^+(x) |x| dx.$$

Hence, cf. (ii), (5.24), and (5.82), if we set

$$\lambda_1 := \frac{\varkappa^+ A_{R_1}^+ - m}{2C} > 0,$$

then, for any $\lambda \in (0, \lambda_1)$ and for any $\xi \in S^{d-1}$,

$$\varkappa^+ \mathbf{a}_\xi^{(n)}(\lambda) - m \geq \varkappa^+ A_{R_1}^+ - m - \lambda_1 \varkappa^+ \int_{\mathbb{R}^d} a^+(x) |x| dx \geq \frac{\varkappa^+ A_{R_1}^+ - m}{2} > C\lambda,$$

i.e. (5.83) holds.

On the other hand, (A4) and the condition (i) imply that, for any $n \in \mathbb{N}$, the assumption (A8) holds with a^+ replaced by $a_{R_n}^+$, where $r = 0$ and ρ, δ are the same as in (A4), and thus are independent on n . Hence, by (4.64),

$$\frac{1}{\lambda} \mathbf{a}_\xi^{(n)}(\lambda) \geq \rho' \frac{1}{\lambda^2} (e^{\lambda \delta'} - 1) \rightarrow \infty,$$

for all $n \in \mathbb{N}$, and here ρ', δ' are independent on n and on ξ . Therefore, there exists $\lambda_2 > 0$, such that, for all $\lambda > \lambda_2$, $\xi \in S^{d-1}$, $n \in \mathbb{N}$, (5.83) holds.

Let, finally, $\lambda \in [\lambda_1, \lambda_2]$. Since $a_{R_n}^+$ are compactly supported, one has

$$\frac{d}{d\lambda} \mathbf{a}_\xi^{(n)}(\lambda) = \int_{\Xi_\xi^+} a_{R_n}^+(x)(x \cdot \xi) e^{\lambda x \cdot \xi} dx + \int_{\Xi_\xi^-} a_{R_n}^+(x)(x \cdot \xi) e^{\lambda x \cdot \xi} dx. \quad (5.84)$$

The inequality $se^{-s} \leq \frac{1}{e}$, $s \geq 0$ implies

$$\left| \int_{\Xi_\xi^-} a_{R_n}^+(x)(x \cdot \xi) e^{\lambda x \cdot \xi} dx \right| \leq \frac{1}{e} \int_{\Xi_\xi^-} a_{R_n}^+(x) dx \leq \frac{1}{e}. \quad (5.85)$$

Since

$$\int_{x \cdot \xi \leq 1} b(|x|) e^{\lambda x \cdot \xi} dx \leq e^\lambda < \infty, \quad \lambda > 0,$$

one has, by (5.78), that

$$\int_{x \cdot \xi \geq 1} b(|x|) e^{\lambda x \cdot \xi} dx = \infty, \quad \lambda > 0. \quad (5.86)$$

Then, by (5.84), (5.85), (5.86), for all $\lambda \geq \lambda_1$,

$$\begin{aligned} \frac{d}{d\lambda} \int_{\mathbb{R}^d} a_{R_n}^+(x) e^{\lambda x \cdot \xi} dx &\geq \int_{\Xi_\xi^+} a_{R_n}^+(x) (x \cdot \xi) e^{\lambda x \cdot \xi} dx - \frac{1}{e} \\ &\geq \int_{x \cdot \xi \geq 1} a_{R_n}^+(x) e^{\lambda_1 x \cdot \xi} dx - \frac{1}{e} \\ &\geq \int_{x \cdot \xi \geq 1} b(|x|) \mathbb{1}_{B_{R_n}(0)}(x) e^{\lambda_1 x \cdot \xi} dx - \frac{1}{e} \rightarrow \infty, \quad n \rightarrow \infty, \end{aligned}$$

and the latter integral, evidently, does not depend on $\xi \in S^{d-1}$. Therefore, there exists $N_1 = N_1(\lambda_1) \in \mathbb{N}$, such that, for all $n \geq N_1$ and for all $\xi \in S^{d-1}$, the function $\mathfrak{a}_\xi^{(n)}(\lambda)$ is increasing on $[\lambda_1, \lambda_2]$. As a result, for $\lambda \in [\lambda_1, \lambda_2]$, $n \geq N_1$, $\xi \in S^{d-1}$,

$$\begin{aligned} \frac{1}{\lambda} (\mathfrak{z}^+ \mathfrak{a}_\xi^{(n)}(\lambda) - m) &\geq \frac{1}{\lambda_2} (\mathfrak{z}^+ \mathfrak{a}_\xi^{(n)}(\lambda_1) - m) \\ &\geq \frac{1}{\lambda_2} \left(\mathfrak{z}^+ \int_{\mathbb{R}^d} b(|x|) \mathbb{1}_{B_{R_n}(0)}(x) e^{\lambda_1 x \cdot \xi} dx - m \right) \rightarrow \infty, \quad n \rightarrow \infty, \end{aligned}$$

and, again, the latter expression does not depend on $\xi \in S^{d-1}$, thus the convergence is uniform in ξ . Therefore, one gets (5.83), for a big enough $N > N_1$ and all $\lambda \in [\lambda_1, \lambda_2]$, $\xi \in S^{d-1}$.

As a result, we have (5.81). Take an arbitrary compact $\mathcal{K} \in \mathbb{R}^d$. Choose $n \in \mathbb{N}$ big enough to ensure that

$$\max_{x \in \mathcal{K}, \xi \in S^{d-1}} x \cdot \xi < \min_{\xi \in S^{d-1}} c_*^{(n)}(\xi).$$

As a result, $\mathcal{K} \in \text{int}(\Upsilon_1^{(n)})$, where $\Upsilon_1^{(n)}$ is defined according to (5.21), but for the kernels $a_{R_n}^\pm$. Then (5.32), with $\mathcal{C} = \mathcal{K}$, yields $\min_{x \in t\mathcal{K}} w^{(n)}(x, t) = \theta$, $t \rightarrow \infty$.

Hence the inequality (5.80) fulfills the statement. \square

Corollary 5.21. *Let conditions of Theorem 5.20 hold. Then there does not exist a travelling wave solution, in the sense of Definition 4.3, to the equation (2.1).*

Proof. Suppose that, for some $\xi \in S^{d-1}$, $c \in \mathbb{R}$, and $\psi \in \mathcal{M}_\theta(\mathbb{R})$, (4.1) holds. Then $u_0(x) = \psi(x \cdot \xi)$ satisfies the assumptions of Theorem 5.20. Take a compact set $\mathcal{K} \subset \mathbb{R}^d$, such that $c_1 := \max_{y \in \mathcal{K}} y \cdot \xi > c$. Then (5.79) implies

$$\begin{aligned} \theta &= \lim_{t \rightarrow \infty} \text{essinf}_{x \in t\mathcal{K}} \psi(x \cdot \xi - ct) = \lim_{t \rightarrow \infty} \text{essinf}_{y \in \mathcal{K}} \psi(t(y \cdot \xi - c)) \\ &= \lim_{t \rightarrow \infty} \psi(t(c_1 - c)) = 0, \end{aligned}$$

where we used that ψ is decreasing. One gets a contradiction which proves the statement. \square

6 Historical comments

The solution $u = u(x, t)$ to (1.1) describes approximately a density (at the moment of time t and at the position x of the space \mathbb{R}^d) for a particle system evolving in the continuum. In course of the evolution, particles might reproduce themselves, die, and compete (say, for resources). Namely, a particle located at a point $y \in \mathbb{R}^d$ may produce a ‘child’ at a point $x \in \mathbb{R}^d$ with the intensity \varkappa^+ and according to the dispersion kernel $a^+(x - y)$. Next, any particle may die with the constant intensity m . And additionally, a particle located at x may die according to the competition with the rest of particles; the intensity of the death because of a competitive particle located at y is equal to \varkappa^- and the distribution of the competition is described by $a^-(x - y)$.

This model was originally proposed in mathematical ecology, see [8], and subsequent papers [9, 25, 53, 59]; for further biological references see e.g. [60] and the recent review [63]. Rigorous mathematical constructions were done in [34], see also [36, 38]. The mathematical approach was realized using the theory of Markov statistical dynamics on the so-called configuration spaces expressed in terms of evolution of time-dependent correlation functions of the system, see e.g. [39, 50, 51].

The particle density of such a system can not be described by a single evolution equation in a closed form (except the case $\varkappa^- = 0$, which is out of our considerations, see for it [52]). Namely, the evolutional equation for the density includes time-dependent correlations between pairs of particles, whereas the evolutional equation for that pair correlations includes correlations between triples of particles, and so on. This situation is quite common in statistical physics, see e.g. [39] and the references therein; in particular, cf. BBGKY-hierarchy for the Hamiltonian dynamics [26].

On the other hand, for purposes of applications, one needs to find though approximate numeric values of the density. To do this, the so-called moment closure procedure, which goes back at least to [77], was realized in [8, 9, 25, 53]. The idea was to rewrite higher order correlations as (nonlinear) combinations of low-order ones that yields a closed system of (nonlinear) evolutional equations. Unfortunately, this procedure had not any rigorous mathematical background and may be considered informally only. Another problem was that different ‘closings’ gave different answers (even numerically). On the contrary, in [59], a mesoscopic-type scaling, cf. [64], for the considered model was proposed. It was realized there (heuristically) for the case of homogeneous in space initial density, that gave the homogeneous version (1.2) of (1.1), for $u = u(t)$.

The nonhomogeneous equation (1.1) was rigorously derived in [36, 37] from the dynamics of infinite particle systems described above, using the so-called Vlasov scaling technique for continuous particle systems developed in [35]. The infiniteness of the systems reflected in the fact that solutions to (1.1) should be bounded but, in general, non integrable on the whole \mathbb{R}^d . The derivation was realized for all times, however, under conditions $m \gg \varkappa^+$ and $Ca^-(x) \geq a^+(x)$, $x \in \mathbb{R}^d$, with some $C > 0$; and under the assumption that the functions a^\pm are symmetric: $a^\pm(-x) = a^\pm(x)$. In the recent paper [33], the condition on m was dropped, however, the equation (1.1) was derived on a finite time-interval only. A rigorous derivation of (1.1) from an infinite particle system under the opposite assumption $a^+(x) \geq Ca^-(x)$, $x \in \mathbb{R}^d$, which is crucial for the present paper (see, in particular, the discussion in Subsection 3.1), is still an open problem. Note

also that the same question for finite systems, that leads to an integrable in space function u , should not require any comparison between kernels a^+ and a^- , cf. [41].

The relations between the particle density of the considered system and the solutions to (1.2) and (1.1) were studied in [59] and [60], correspondingly.

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