

Lattice birth-and-death processes

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August 6, 2015

Abstract

Lattice birth-and-death Markov dynamics of particle systems with spins from \mathbb{Z}_+ are constructed as unique solutions to certain stochastic equations. Pathwise uniqueness, strong existence, Markov property and joint uniqueness in law are proven, and a martingale characterization of the process is given. Sufficient conditions for the existence of an invariant distribution are formulated in terms of Lyapunov functions. We apply obtained results to discrete analogs of the Bolker–Pacala–Dieckmann–Law model and an aggregation model.

1 Introduction

The evolution of a birth-and-death process admits the following description. Two functions characterize the development in time, the birth rate b and the death rate d . If the system is in state $\eta \in \mathbb{Z}_+^{\mathbb{Z}^d}$ at time t , then the probability that the number of points at a site $x \in \mathbb{Z}^d$ is increased by 1 (“birth”) over the next time interval of length Δt is

$$b(x, \eta)\Delta t + o(\Delta t),$$

the probability that the number of points at the site x is decreased by 1 (“death”) over the next time interval of length Δt is

$$d(x, \eta)\Delta t + o(\Delta t),$$

and no two changes occur at the same time. Put differently, a birth at the site x occurs at the rate $b(x, \eta)$, a death at the site x occurs at the rate $d(x, \eta)$, and no two events, births or deaths, happen simultaneously.

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The (heuristic) generator of such process is of the form

$$LF(\eta) = \sum_{x \in \mathbb{Z}^d} b(x, \eta)[F(\eta^{+x}) - F(\eta)] + \sum_{x \in \mathbb{Z}^d} d(x, \eta)[F(\eta^{-x}) - F(\eta)], \quad (1)$$

where

$$\eta^{+x}(y) = \begin{cases} \eta(y), & \text{if } y \neq x, \\ \eta(y) + 1, & \text{if } y = x, \end{cases} \quad \eta^{-x}(y) = \begin{cases} \eta(y), & \text{if } y \neq x, \text{ or if } y = x, \eta(x) = 0 \\ \eta(y) - 1, & \text{if } y = x, \eta(x) \neq 0. \end{cases}$$

Birth-and-death processes we consider here correspond to lattice interacting particle systems with a non-compact (single) spin space and general transition rates. The existence of the underlying stochastic dynamics is not obvious. The first result of this article is the construction of the corresponding Markov process. Following ideas of Garcia and Kurtz [GK06], we construct the process as a unique solution to a certain stochastic integral equation with Poisson noise.

Birth-and-death processes constructed in [GK06] are represented by a collection of points in a separable complete metric space. The scheme proposed there covers the case of \mathbb{Z}_+^d -valued processes. The existence and uniqueness theorem, [GK06, Theorem 2.13], was obtained under the assumption that the death rate is constant, which in the settings of this paper corresponds to $d(x, \eta) = \eta(x)$. Theorem 2.6 in this paper covers more general death rates.

A growing interest to the study of spatial birth-and-death processes which we have recently observed is stimulated by, among other things, an important role which these processes play in several applications. For example, in spatial plant ecology, a general approach to the so-called individual based models was developed in a series of works, see e.g. [BP97, BP99, DL05, MDL04, OFK⁺14] and references therein. These models are represented by birth-and-death Markov processes in continuous configuration space (over \mathbb{R}^d) with specific rates b and d which reflect biological notions such as competition, establishment, fecundity etc. Other examples of birth-and-death processes may be found in mathematical physics, see e.g. [KKZ06, KKM10, FKK12] and references therein. Usually, lattice models can be compared with continuous ones if we discretize the space \mathbb{R}^d by partitioning it into cubes with centers at vertices of the lattice. It is worth pointing out that in many applications a lattice version can be constructed which will stochastically dominate the original continuous model. Of course, such comparison arguments seem to be loose, because the construction of the continuous original process which is in general a very difficult problem. Nevertheless, we may hope to deduce a priori information for the continuous process from the corresponding lattice one.

There is a considerable amount of literature related to interacting particle systems in \mathbb{Z}_+^d . Systems with a non-compact discrete spin space appear as early as 1970 in the work of Spitzer [Spi70], where the invariant product measure was constructed for the zero range interaction. The

zero range process was constructed in a companion paper by Holley [Hol70] and later by Andjel [And82] under more general conditions (see also Balázs et al. [BRASS07]). Some generalizations of the class the zero range processes were considered later, one of them, the so-called misanthrope processes, were introduced by Coccozza-Thivent [CT85]. Another type of models was considered for example by Kesten and Sidoravicius [KS05] (see also [KS08]), where an interacting particle system with non-trivial interaction was constructed and studied. The system models a spread of a rumor or infection and involves infinitely many particles.

The scheme proposed by Etheridge and Kurtz [EK14] covers a wide range of interactions and applies to discrete and continuous models. Their approach is based on, among other things, assigning a certain mark ('level') to each particle and letting this mark evolve according to some law. A critical event, such as birth or death, occurs when the level hits some threshold. Penrose [Pen08] gives a general existence result for particle systems with local interaction and uniformly bounded jump rates but non-compact spin space. Sometimes the existence of microscopic stochastic dynamics is simply assumed, see for example Balázs et al. [BFKR10].

The state space of our process will be

$$\left\{ \eta \in \mathbb{Z}_+^{\mathbb{Z}^d} : \sum_{x \in \mathbb{Z}^d} w(x) \eta(x) < \infty \right\},$$

where w is a summable positive even function. Such subspaces of the product space naturally arise in the analysis of systems with unbounded transition rates as the process started from arbitrary $\eta \in \mathbb{Z}_+^{\mathbb{Z}^d}$ need not exist; compare with [And82, (1.2)].

We also mention that although our lattice is given by \mathbb{Z}^d , the approach to construction that we use should work for an arbitrary connected bounded degree graph, provided, of course, that the assumptions are appropriately modified. Indeed, in our assumptions and proofs we use only the graph distance on \mathbb{Z}^d given by

$$|x - y|_1 := \sum_{j=1}^d |x_j - y_j|$$

for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$.

The paper is organized as follows. In Section 2 we collect the main results. The first result of the paper is Theorem 2.6 which is an extension of the research done in the thesis [Bez14, Chapter 5]. A martingale characterization of the constructed process and sufficient conditions for the existence of an invariant distribution are given. The proofs of the theorems from Section 2 as well as some further comments are given in Sections 3 through 5. In Section 6 we discuss survival of the process for a model with local death rate and independent branching birth rate. We use comparison with the contact process to establish existence of a critical value of the birth

rate parameter.

2 The set-up and main results

Let $T > 0$, and let N_1, N_2 be Poisson point processes on $\mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $\# \times ds \times du$, where $\#$ is the counting measure on \mathbb{Z}^d . Consider the equation

$$\begin{aligned} \eta_t(x) &= \int_{(0,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) N_1(x, ds, du) \\ &- \int_{(0,t] \times [0,\infty)} I_{[0,d(x,\eta_{r-})]}(v) N_2(x, dr, dv) + \eta_0(x), \end{aligned} \tag{2}$$

where $(\eta_t)_{t \in [0,T]}$ is a cadlag \mathcal{X} -valued solution process, $x \in \mathbb{Z}^d$, η_0 is a (random) initial condition, b, d are birth and death rates. We require processes N_1, N_2, η_0 to be independent of each other. The first (second) term on the right hand side represents the number of births (deaths, respectively) for (η_t) at x before t . The integrals on the right-hand side are taken in the Lebesgue–Stieltjes sense: if for example $N(\{x\} \times \mathbb{R}_+ \times \mathbb{R}_+) = \sum_i \delta_{(x, s_i, u_i)}$, then

$$\int_{(0,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) N_1(x, ds, du) = \sum_{i: 0 < s_i \leq t} I_{[0,b(x,\eta_{s_i-})]}(u_i).$$

The theory of integration with respect to Poisson point processes can be found in Chapter 2 of [IW81]. Equation (2) is understood in the sense that the equality holds a.s. for all $x \in \mathbb{Z}^d$ and $t \in (0, T]$.

Assumptions on η_0, w, b and d . Let us fix the assumptions we use throughout the paper. Let w be a summable positive even function, $\sum_{x \in \mathbb{Z}^d} w(x) < \infty$. Denote by \mathcal{X} the set

$$\{\eta \in \mathbb{Z}_+^{\mathbb{Z}^d} : \sum_{x \in \mathbb{Z}^d} w(x) \eta(x) < \infty\}.$$

We equip \mathcal{X} with the topology induced by the distance

$$d_{\mathcal{X}}(\eta, \zeta) = \sum_{x \in \mathbb{Z}^d} w(x) |\eta(x) - \zeta(x)|.$$

Note that $(\mathcal{X}, d_{\mathcal{X}})$ is a complete separable metric space and that convergence in \mathcal{X} implies pointwise convergence: if $\eta_k \in \mathcal{X}$, $k = 0, 1, \dots$, $\eta \in \mathcal{X}$ and $\eta_k \rightarrow \eta$ in \mathcal{X} , then $\eta_k(x) \rightarrow \eta(x)$ for any $x \in \mathbb{Z}^d$ and, since $\eta_k(x)$ is a natural number or zero, $\eta_k(x) = \eta(x)$ for all but finitely many

k. We require that $\eta_0 \in \mathcal{X}$ a.s. and

$$E \sum_{x \in \mathbb{Z}^d} w(x) \eta_0(x) < \infty.$$

We assume throughout that the following conditions are satisfied:

$$\begin{aligned} \text{if } \xi, \eta \in \mathcal{X}, x \in \mathbb{Z}^d \text{ and } \xi(x) \geq \eta(x), \text{ then} \\ b(x, \xi) - b(x, \eta) \leq \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi(y) - \eta(y)|, \end{aligned} \quad (3)$$

$$\begin{aligned} \text{if } \xi, \eta \in \mathcal{X}, x \in \mathbb{Z}^d \text{ and } \xi(x) \geq \eta(x), \text{ then} \\ d(x, \xi) - d(x, \eta) \geq - \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi(y) - \eta(y)|, \end{aligned} \quad (4)$$

and

$$\sum_{y \in \mathbb{Z}^d} w(y) a(x-y) \leq C_{w,a} w(x), \quad x \in \mathbb{Z}^d \quad (5)$$

where $a : \mathbb{Z}^d \rightarrow \mathbb{R}_+$ is a summable even function, $C_{w,a} > 0$. Denote by $\mathbf{0}$ the “zero” element of \mathcal{X} : $\mathbf{0} \in \mathbb{Z}_+^d$, $\mathbf{0}(x) = 0$, $x \in \mathbb{Z}^d$. If there are no particles at a site then no death can occur, so d should satisfy

$$d(x, \eta) = 0, \quad \text{whenever } \eta(x) = 0.$$

We also require

$$\sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0}) < \infty. \quad (6)$$

For some possible choices of a and w satisfying these conditions and for a few examples, see Remark 2.7 below.

We say that a Poisson point process N on $\mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ is *compatible* with a right-continuous complete filtration $\{\mathcal{F}_t\}$ if all random variables of the form $N(\{x\} \times [a, b] \times U)$, $x \in \mathbb{Z}^d$, $0 \leq a < b \leq t$, $U \in \mathcal{B}(\mathbb{R}_+)$, are \mathcal{F}_t -measurable, and, in addition, all random variables of the form $N(\{x\} \times [a, b] \times U)$, $x \in \mathbb{Z}^d$, $t < a < b$, $U \in \mathcal{B}(\mathbb{R}_+)$, are independent of \mathcal{F}_t .

Definition 2.1. A (*weak*) *solution* of equation (2) is a triple $((\eta_t)_{t \in [0, T]}, N_1, N_2)$, (Ω, \mathcal{F}, P) , $(\{\mathcal{F}_t\}_{t \in [0, T]})$, where

- (i) (Ω, \mathcal{F}, P) is a probability space, $\{\mathcal{F}_t\}_{t \in [0, T]}$ is an increasing, right-continuous and complete filtration of sub- σ -algebras of \mathcal{F} ,
- (ii) $(\eta_t)_{t \in [0, T]}$ is a cadlag process in \mathcal{X} , adapted to $\{\mathcal{F}_t\}_{t \in [0, T]}$, such that

$$E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} \eta_t(x) < \infty,$$

(iii) N_1, N_2 are independent Poisson point processes with measure intensity $\# \times ds \times du$, compatible with $\{\mathcal{F}_t\}_{t \in [0, T]}$,

(iv) all integrals in (2) are well-defined, and $E \int_0^T [b(x, \eta_{s-}) + d(x, \eta_{s-})] ds < \infty$ for every $x \in \mathbb{Z}^d$.

(v) equality (2) holds a.s. for all $t \in [0, T]$ and $x \in \mathbb{Z}^d$.

Remark 2.2. The definition above as well as many of the definitions and theorems below can be extended to the case of the time interval $[0, \infty)$ in an obvious manner.

Definition 2.3. A solution is called *strong* if $(\eta_t)_{t \in [0, T]}$ is adapted to the completion under P of the filtration

$$\mathcal{S}_t = \sigma\{\eta_0, N_k(\{x\} \times [0, q] \times C), x \in \mathbb{Z}^d, C \in \mathcal{B}(\mathbb{R}_+), q \in [0, t], k = 1, 2\}.$$

Definition 2.4. We say that *pathwise uniqueness* holds for equation (2) and an initial distribution ν if, whenever the triples $((\eta_t)_{t \in [0, T]}, N_1, N_2)$, (Ω, \mathcal{F}, P) , $(\{\mathcal{F}_t\}_{t \in [0, T]})$ and $((\bar{\eta}_t)_{t \in [0, T]}, N_1, N_2)$, $(\Omega, \bar{\mathcal{F}}, P)$, $(\{\bar{\mathcal{F}}_t\}_{t \in [0, T]})$ are weak solutions of (2) with $P\{\eta_0 = \bar{\eta}_0\} = 1$ and $\text{Law}(\eta_0) = \nu$, we have $P\{\eta_t = \bar{\eta}_t, t \in [0, T]\} = 1$ (that is, the processes $(\eta_t), (\bar{\eta}_t)$ are indistinguishable).

Definition 2.5. We say that *joint uniqueness in law* holds for (2) if the triple $((\eta_t)_{t \in [0, T]}, N_1, N_2)$ has the same distribution in $D_{\mathcal{X}}[0, T] \times D_{\mathbf{E}}[0, T] \times D_{\mathbf{E}}[0, T]$ for every weak solution (cf., e.g., [Kur07, Definition 2.9]).

Here \mathbf{E} is the space of locally finite simple counting measures on $\mathbb{Z}^d \times \mathbb{R}_+$ with the minimal σ -algebra such that every set of the form

$$\{\gamma \in \mathbf{E} \mid \gamma(Q) \in B\}, \quad Q \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{R}_+), B \in \mathcal{B}(\mathbb{R}_+)$$

is measurable, and endowed with the metric compatible with the vague topology (also called the *space of locally finite configurations*; see e.g. [Kal02, Appendix A2] or [KK02], and references therein).

Now we formulate the existence and uniqueness theorem, which will be proven in the next section.

Theorem 2.6. *Under the above assumptions pathwise uniqueness, strong existence and uniqueness in law hold for equation (2). The unique solution is a Markov process.*

Remark 2.7. The conditions on b, d and η_0 given in terms of functions w and a may seem somewhat indirect. Here we point out three possible choices of w and a .

(i) $w(x) = e^{-q|x|_1}$, $a(x) = ce^{-p|x|_1}$, $p > q > 0$, $c > 0$;

- (ii) $w(x) = e^{-q|x|_1}$, $a(x) = cI\{|x|_1 \leq k\}$, $q > 0$, $c > 0$, $k \in \mathbb{N}$;
- (iii) $w(x) = \frac{1}{1+|x|_1^{-q}}$, $a(x) = \frac{c}{1+|x|_1^{-p}}$, $p > q > d$, $c > 0$.

Now we give two examples where Theorem 2.6 applies, and we can obtain discrete counterparts of some continuous particle systems as unique solutions to (2).

A discrete version of the Bolker–Pacala–Dieckmann–Law model also known as spatial stochastic logistic model (Bolker and Pacala [BP97, BP99]; Dieckmann and Law et al. [DL05, MDL04]). An individual-based description of this model is as follows:

- (1) Existing individuals produce offsprings at a per capita fecundity rate.
- (2) A newly produced offspring is distributed (instantaneously) according to a dispersal kernel, and it is assumed to establish (instantaneously) as a newborn individual, which matures (instantaneously) and starts to produce offsprings.
- (3) Existing individuals may die for two reasons. Firstly, there is a constant background per capita mortality rate m , yielding an exponentially distributed lifetime. Secondly, mortality has a density dependent component (self-thinning), so that competition among the individuals may also lead to death. The density dependent component of the death rate of a focal individual is a sum of contributions from all the other individuals within the entire \mathbb{Z}^d , but the strength of the competitive effect decreases with distance.

The model is defined by

$$b(x, \eta) = b_0 + \sum_{y \in \mathbb{Z}^d} a_+(x - y)\eta(y), \quad (7)$$

and

$$d(x, \eta) = \eta(x) \left\{ m + \sum_{y \in \mathbb{Z}^d} a_-(x - y)\eta(y) \right\}. \quad (8)$$

In this model (η_t) represents an evolution of a biological population with independent branching given by the kernel a_+ , immigration at a constant rate b_0 , constant “intrinsic” mortality rate m and the competition kernel a_- . The model with rates (7) and (8) with $b_0 = 0$ can be regarded as a translation invariant discretization of the Bolker–Pacala–Dieckmann–Law model studied by Fournier and Méléard [FM04].

The stepping stone and superprocess versions of the model were considered by Etheridge [Eth04]. The process with a finite number of particles in continuum was studied in [FM04], where, among other things, it was shown that the superprocess version can be obtained as a scaling limit of continuous processes. Statistical dynamics were considered by Finkilstein et al. [FKK09, FKK13]; see also Ovaskainen et al. [OFK⁺14]. The model with rates (7) and (8) with $b_0 = 0$ can be regarded as a translation invariant discretization of the Bolker–Pacala–Dieckmann–Law model described in [FM04]. Unlike in the continuous model, in the discrete

model we allow the particles to be at the same place; otherwise the density would be bounded.

A *discrete version of an aggregation model*. Here the birth rate is either as in (7), or is given by a constant, $b(x, \eta) \equiv c > 0$. The death rate is given by

$$d(x, \eta) = e^{-c \sum_{y \in \mathbb{Z}^d} \varphi(x-y)\eta(y)},$$

or

$$d(x, \eta) = \frac{1}{1 + c \sum_{y \in \mathbb{Z}^d} \varphi(x-y)\eta(y)},$$

where $c > 0$ and $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}_+$. For a statistical dynamics corresponding to this model in continuum, see [FKKZ14] and references therein; see also [Bez14] for continuous systems with a finite number of particles.

The following two propositions establish a rigorous relation between the unique solution to (2) and L defined by (1). To formulate the first of them, let us consider the class \mathcal{C}_b of cylindrical functions $F : \mathcal{X} \rightarrow \mathbb{R}_+$ with bounded increments. We say that F has bounded increments if

$$\sup_{\eta \in \mathcal{X}, x \in \mathbb{Z}^d} (F(\eta^{+x}) - F(\eta)) < \infty.$$

We say that F is cylindrical if $F(\eta)$ depends on values of η in finitely many sites only, i.e. for some $R = R_F > 0$

$$F(\eta) = F(\zeta) \text{ whenever } \eta(x) = \zeta(x) \text{ for all } x, |x|_1 \leq R.$$

We recall that $|x|_1 = \sum_{j=1}^d |x_j|$ for $x = (x_1, \dots, x_d)$ and that the filtration $\{\mathcal{S}_t, t \geq 0\}$ appeared in Definition 2.3.

Proposition 2.8. *Let $(\eta_t)_{t \geq 0}$ be a weak solution to (2). Then for any $F \in \mathcal{C}_b$ the process*

$$F(\eta_t) - \int_0^t LF(\eta_{s-}) ds \tag{9}$$

is an $\{\mathcal{S}_t\}$ -martingale. In particular, the integral in (9) is a.s. well-defined.

The next proposition says that under some additional assumptions the converse is true.

Proposition 2.9. *Let $(\eta_t)_{t \in [0, T]}$ be a cadlag \mathcal{X} -valued process defined on a probability space (Ω, \mathcal{F}, P) , adapted to a right-continuous complete filtration $(\{\mathcal{F}_t\}_{t \in [0, T]})$ and satisfying*

$$\eta_t = \eta_0 + \sum_{0 < s \leq t} (\eta_s - \eta_{s-}), t \in [0, T],$$

and

$$\sup_{t \geq 0} \sum_{x \in \mathbb{Z}^d} |\eta_t(x) - \eta_{t-}(x)| \leq 1$$

a.s. Assume that the probability space (Ω, \mathcal{F}, P) and the filtration $(\{\mathcal{F}_t\}_{t \in [0, T]})$ are rich enough to support required randomization processes, and that

$$\eta_t(x) = \eta_0(x) + \eta_t^{(b)}(x) - \eta_t^{(d)}(x), \quad (10)$$

where $\{\eta_t^{(b)}, t \geq 0\}$ and $\{\eta_t^{(d)}, t \geq 0\}$ are cadlag non-decreasing \mathcal{X} -valued processes, $\eta_0^{(b)} = \eta_0^{(d)} = \mathbf{0}$, such that for every finite $E_1, E_2 \subset \mathbb{Z}^d$ the process

$$\sum_{x \in E_1} \eta_t^{(b)}(x) + \sum_{y \in E_2} \eta_t^{(d)}(y) - \int_0^t \sum_{x \in E_1} b(x, \eta_{s-}) ds - \int_0^t \sum_{y \in E_2} d(y, \eta_{s-}) ds$$

is an $(\{\mathcal{F}_t\}_{t \in [0, T]})$ -martingale. Then there exist independent Poisson point processes N_1 and N_2 such that the triple $(\{\eta_t\}_{t \in [0, T]}, N_1, N_2)$, (Ω, \mathcal{F}, P) , $(\{\mathcal{F}_t\}_{t \in [0, T]})$ is a weak solution to (2).

Remark 2.10. Some classic interacting particle systems, including the stochastic Ising model, the contact process and the voter model (see, e.g., [Lig85]) can be constructed using the above results. For example, the unique solution of (2) with initial condition $\beta \in \{0, 1\}^{\mathbb{Z}^d}$, the death rate $d(x, \eta) = I\{\eta(x) > 0\}$ and the birth rate

$$b(x, \eta) = b_{cont}(x, \eta) := I_{\{\eta(x)=0\}} \lambda \sum_{y: y \sim x} \eta(y) \quad (11)$$

is the contact process with parameter $\lambda > 0$ and initial state β . This follows from the uniqueness of solutions for the associated martingale problem, see [HS76, Theorem (4.12)], and Proposition 2.8.

Sometimes we will denote the solution to (2) with initial condition $\eta_0 \equiv \alpha$, $\alpha \in \mathcal{X}$, by $(\eta(\alpha, t))_{t \in [0, T]}$, emphasizing the dependence on α .

For $\alpha \in \mathcal{X}$ we denote by P_α the law of $(\eta(\alpha, t))$,

$$P_\alpha(H) = P\{(\eta(\alpha, t))_{t \geq 0} \in H\}, \quad H \in D_{\mathcal{X}}[0, \infty).$$

Note that P_α is well defined by Theorem 2.6. Also, for every $H \in D_{\mathcal{X}}[0, \infty)$, $P_\alpha(H)$ can be shown to be measurable in α .

Let $C_b(\mathcal{X})$ be the space of bounded continuous functions on \mathcal{X} equipped with the supremum norm. For $\alpha \in \mathcal{X}$ and $f \in C_b(\mathcal{X})$ we define

$$P^t f(\alpha) := E f(\eta(\alpha, t)) \quad (= E_\alpha f(\eta_t)). \quad (12)$$

The function $P^t f$ is continuous on \mathcal{X} . Indeed, by Lemma 3.5 and Gronwall's inequality

$$E \sum_{x \in \mathbb{Z}^d} w(x) |\eta(\alpha, t)(x) - \eta(\beta, t)(x)| \leq E \sum_{x \in \mathbb{Z}^d} w(x) |\alpha(x) - \beta(x)| \exp\{4C_{w,a}t\}, \quad (13)$$

and hence by Lebesgue's dominated convergence theorem $P^t f$ is continuous. Therefore P^t is a bounded operator on $C_b(\mathcal{X})$.

A probability measure π on \mathcal{X} is called invariant for equation (2) if

$$\int P^t f(\alpha) \pi(d\alpha) = \int f(\alpha) \pi(d\alpha)$$

for every $f \in C_b(\mathcal{X})$.

Consider the following additional assumption: there exists an even summable function $v : \mathbb{Z}^d \rightarrow (0, \infty)$ and a constant $C_{v,a} > 0$ satisfying

$$\frac{v(x)}{w(x)} \rightarrow \infty, \quad x \rightarrow \infty, \quad (14)$$

and

$$\sum_{y \in \mathbb{Z}^d} v(y) a(x-y) \leq C_{v,a} v(x), \quad x \in \mathbb{Z}^d. \quad (15)$$

Let $\mathcal{X}_v := \{\eta \in \mathcal{X} : \sum_{x \in \mathbb{Z}^d} v(x) \eta(x) < \infty\}$ and let $V : \mathcal{X}_v \rightarrow \mathbb{R}_+$ be given by

$$V(\eta) := \sum_{x \in \mathbb{Z}^d} v(x) \eta(x), \quad \eta \in \mathcal{X}_v.$$

Theorem 2.11. *Assume that there exists an even summable function $v : \mathbb{Z}^d \rightarrow (0, \infty)$ such that (14) and (15) hold. Also, assume that for some constants $c_1, c_2 > 0$*

$$LV(\eta) \leq c_1 - c_2 V(\eta), \quad \text{for all } \eta \in \mathcal{X}_v. \quad (16)$$

Then there exists an invariant measure for equation (2).

Note that, for $\eta \in \mathcal{X}_v$, $LV(\eta)$ may be equal to $-\infty$, in which case (16) is fulfilled. Using the theorem above, we can establish existence of an invariant measure for the model given by (7) and (8).

Proposition 2.12. *Let $w(x) = e^{-|x|_1}$. Assume that a_+ and a_- in (7) and (8) have the finite range property: there exists $R > 0$ such that $a_+(x) = a_-(x) = 0$ whenever $|x|_1 \geq R$. Then*

equation (2) has an invariant measure.

3 Proof of Theorem 2.6

The statement of Theorem 2.6 is contained in Propositions 3.6, 3.8, 3.10 and 3.11, which we prove below.

We start with the following Lemma.

Lemma 3.1. *For every $x \in \mathbb{Z}^d$ the maps*

$$\mathcal{X} \ni \xi \mapsto b(x, \xi) \in \mathbb{R}_+,$$

$$\mathcal{X} \ni \xi \mapsto d(x, \xi) \in \mathbb{R}_+$$

are continuous.

Proof. We give the proof for b only, as the proof for d can be done in the same way. Fix $x \in \mathbb{Z}^d$ and $\xi \in \mathcal{X}$. Take $\delta \in (0, w(x))$ and $\eta \in \mathcal{X}$ such that $d_{\mathcal{X}}(\xi, \eta) \leq \delta$. Then $\xi(x) = \eta(x)$. We have by (3)

$$b(x, \xi) - b(x, \eta) \leq \sum_{y \in \mathbb{Z}^d} a(x - y) |\xi(y) - \eta(y)|$$

and

$$b(x, \eta) - b(x, \xi) \leq \sum_{y \in \mathbb{Z}^d} a(x - y) |\xi(y) - \eta(y)|$$

Hence

$$|b(x, \xi) - b(x, \eta)| \leq \sum_{y \in \mathbb{Z}^d} a(x - y) |\xi(y) - \eta(y)|.$$

Now, (5) implies $w(y)a(x-y) \leq C_{w,a}w(x)$, or, after swapping x and y , $w(x)a(x-y) \leq C_{w,a}w(y)$ and $a(x-y) \leq \frac{C_{w,a}}{w(x)}w(y)$. Thus,

$$|b(x, \xi) - b(x, \eta)| \leq \frac{C_{w,a}}{w(x)} \sum_{y \in \mathbb{Z}^d} w(y) |\xi(y) - \eta(y)|.$$

□

Before treating equation (2) in a general form, let us consider the case of a “finite” initial condition. We say that the initial condition η_0 is finite if

$$\sum_x \eta_0(x) < \infty \quad \text{a.s.} \tag{17}$$

and

$$E \sum_x \eta_0(x) < \infty. \quad (18)$$

Proposition 3.2. *Pathwise uniqueness and strong existence hold for (2) with a finite initial condition. Furthermore, the unique solution $(\eta_t)_{t \in [0, T]}$ satisfies*

$$E \sum_x \sup_{t \in [0, T]} \eta_t(x) < \infty. \quad (19)$$

The proof can be done constructively, “from one jump to another”, following the proof of the existence and uniqueness theorem for a similar equation in continuous settings [Bez14, Theorem 2.1.6]. The assertion (19) follows from comparison with the Yule process. The Yule process $(Z_t)_{t \geq 0}$ is an \mathbb{N} -valued birth process such that for all $n \in \mathbb{N}$

$$P\{Z_{t+\Delta t} - Z_t = 1 \mid Z_t = n\} = \mu n + o(\Delta t).$$

for some $\mu > 0$; see e.g. [AN72, Chapter 3] or [dLF06], and references therein.

Note that (19) implies

$$E \sum_x w(x) \sup_{t \in [0, T]} \eta_t(x) < \infty, \quad t \in [0, T], \quad (20)$$

since w is summable and therefore bounded.

Consider now two solutions $(\eta_t^{(k)})$, $k = 1, 2$, to the equations

$$\begin{aligned} \eta_t(x) = & \int_{(0, t] \times [0, \infty)} I_{[0, b_k(x, \eta_{s-})]}(u) N_1(x, ds, du) \\ & - \int_{(0, t] \times [0, \infty)} I_{[0, d_k(x, \eta_{r-})]}(v) N_2(x, dr, dv) + \eta_0^{(k)}(x) \end{aligned} \quad (21)$$

with finite initial conditions.

Proposition 3.3. *Assume that almost surely $\eta_0^{(1)} \leq \eta_0^{(2)}$, and*

(i) *for any $\xi^{(1)}, \xi^{(2)} \in \mathcal{X}$ such that $\xi^{(1)} \leq \xi^{(2)}$ and $\sum_x \xi^{(2)}(x) < \infty$,*

$$b_1(x, \xi^{(1)}) \leq b_2(x, \xi^{(2)}), \quad x \in \mathbb{Z}^d, \quad (22)$$

(ii) *for any $x \in \mathbb{Z}^d$ and $\xi^{(1)}, \xi^{(2)} \in \mathcal{X}$ such that $\xi^{(1)} \leq \xi^{(2)}$, $\sum_x \xi^{(2)}(x) < \infty$ and $\xi^{(1)}(x) = \xi^{(2)}(x)$,*

$$d_1(x, \xi^{(1)}) \geq d_2(x, \xi^{(2)}). \quad (23)$$

Then

$$\eta_t^{(1)} \leq \eta_t^{(2)}, \quad t \in [0, T]. \quad (24)$$

Furthermore, the inclusion

$$\left\{ t : \sum_{x \in \mathbb{Z}^d} [(\eta_t^{(1)}(x) - \eta_{t-}^{(1)}(x)) \vee 0] > 0 \right\} \subset \left\{ t : \sum_{x \in \mathbb{Z}^d} [(\eta_t^{(2)}(x) - \eta_{t-}^{(2)}(x)) \vee 0] > 0 \right\}$$

holds a.s. In other words, every moment of birth for $(\eta_t^{(1)})$ is a moment of birth for $(\eta_t^{(2)})$ as well.

Proof. We can show by induction that each moment of birth for $(\eta_t^{(1)})$ is a moment of birth for $(\eta_t^{(2)})$ as well, and that each moment τ of death for $(\eta_t^{(2)})$ is a moment of death for $(\eta_t^{(1)})$ provided $\eta_{\tau-}^{(1)}(x) = \eta_{\tau-}^{(2)}(x)$, where x is the site where the death at τ takes place. Moreover, in both cases the birth or the death occurs at the same site. Here a moment of birth is a random time at which the value of the process at one of sites is increased by 1, and a moment of death is a random time at which the value of the process at one of sites is decreased by 1. The statement formulated above implies (24).

Denote by $\{\tau_m\}_{m \in \mathbb{N}}$ the moments of jumps of $(\eta_t^{(1)})$ and $(\eta_t^{(2)})$, $0 < \tau_1 < \tau_2 < \tau_3 < \dots$. In other words, a time $t \in \{\tau_m\}_{m \in \mathbb{N}}$ if and only if at least one of the processes $(\eta_t^{(1)})$ and $(\eta_t^{(2)})$ jumps at the time t .

Here we deal only with the base case, the induction step is done in the same way. There is nothing to show if τ_1 is a moment of birth for $(\eta_t^{(2)})$ or a moment of death for $(\eta_t^{(1)})$. Assume that τ_1 is a moment of birth for $(\eta_t^{(1)})$ and let x be the place of birth:

$$\eta_{\tau_1}^{(1)}(x) - \eta_{\tau_1-}^{(1)}(x) = 1.$$

Note that

$$\eta_{\tau_1-}^{(1)} = \eta_0^{(1)} \leq \eta_0^{(2)} = \eta_{\tau_1-}^{(2)}.$$

The process $(\eta_t^{(1)})$ satisfies (21), hence

$$\begin{aligned} 1 &= \int_{(0, \tau_1] \times [0, \infty)} I_{[0, b_k(x, \eta_{s-})]}(u) N_1(x, ds, du) - \int_{(0, \tau_1) \times [0, \infty)} I_{[0, b_k(x, \eta_{s-})]}(u) N_1(x, ds, du) \\ &= \int_{\{\tau_1\} \times [0, \infty)} I_{[0, b_k(x, \eta_{s-})]}(u) N_1(x, ds, du) \end{aligned}$$

and

$$N_1(\{x\} \times \{\tau_1\} \times [0, b_1(x, \eta_0^{(1)})]) = 1 \quad \text{a.s.}$$

Since $b_2(x, \eta_0^{(2)}) \geq b_1(x, \eta_0^{(1)})$, τ_1 is a moment of birth for $(\eta_t^{(2)})$. The case when τ_1 is a moment of death for $(\eta_t^{(2)})$ at a site x and $\eta_{\tau_1-}^{(2)}(x) = \eta_{\tau_1-}^{(1)}(x)$ is analyzed similarly. \square

For $x \in \mathbb{Z}^d$, $\xi, \eta \in \mathcal{X}$ we define

$$\tilde{d}(x, \xi, \eta) = \begin{cases} d(x, \xi) & \text{if } \xi(x) > \eta(x), \\ d(x, \eta) & \text{if } \xi(x) < \eta(x), \\ d(x, \xi) \wedge d(x, \eta) & \text{if } \xi(x) = \eta(x), \end{cases} \quad (25)$$

and

$$\tilde{b}(x, \xi, \eta) = \begin{cases} b(x, \xi) & \text{if } \xi(x) > \eta(x), \\ b(x, \eta) & \text{if } \xi(x) < \eta(x), \\ b(x, \xi) \vee b(x, \eta) & \text{if } \xi(x) = \eta(x), \end{cases} \quad (26)$$

Note that $\tilde{d}(x, \xi, \eta) = \tilde{d}(x, \eta, \xi)$, $\tilde{b}(x, \xi, \eta) = \tilde{b}(x, \eta, \xi)$.

Lemma 3.4. For every $x \in \mathbb{Z}^d$, $\xi, \eta \in \mathcal{X}$,

$$\tilde{b}(x, \xi, \eta) - b(x, \eta) \leq \sum_{x \in \mathbb{Z}^d} a(x-y) |\xi(x) - \eta(x)|,$$

and

$$\tilde{d}(x, \xi, \eta) - d(x, \eta) \geq - \sum_{x \in \mathbb{Z}^d} a(x-y) |\xi(x) - \eta(x)|.$$

Proof. We have

$$\begin{aligned} \tilde{b}(x, \xi, \eta) - b(x, \eta) &= I_{\{\xi(x) > \eta(x)\}} (b(x, \xi) - b(x, \eta)) + I_{\{\xi(x) = \eta(x)\}} [(b(x, \xi) - b(x, \eta)) \vee 0] \\ &\leq \sum_{x \in \mathbb{Z}^d} a(x-y) |\xi(x) - \eta(x)|. \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{d}(x, \xi, \eta) - d(x, \eta) &= I_{\{\xi(x) > \eta(x)\}} (d(x, \xi) - d(x, \eta)) + I_{\{\xi(x) = \eta(x)\}} [(d(x, \xi) - d(x, \eta)) \wedge 0] \\ &\geq - \sum_{x \in \mathbb{Z}^d} a(x-y) |\xi(x) - \eta(x)|. \end{aligned}$$

The next lemma will play the key role in the proof of pathwise uniqueness for (2).

Lemma 3.5. Let $((\xi_t)_{t \in [0, T]}, N_1, N_2)$, (Ω, \mathcal{F}, P) , $(\{\mathcal{F}_t\}_{t \in [0, T]})$ and $((\zeta_t)_{t \in [0, T]}, N_1, N_2)$, $(\Omega, \bar{\mathcal{F}}, P)$,

$(\{\bar{\mathcal{F}}_t\}_{t \in [0, T]})$ be weak solutions to (2). Then

$$\begin{aligned} & E \sum_{x \in \mathbb{Z}^d} w(x) |\xi_t(x) - \zeta_t(x)| \\ & \leq 4C_{w,a} \int_{(0,t]} ds E \sum_{x \in \mathbb{Z}^d} w(x) |\xi_{s-}(x) - \zeta_{s-}(x)| + E \sum_{x \in \mathbb{Z}^d} w(x) |\xi_0(x) - \zeta_0(x)|. \end{aligned} \quad (27)$$

Proof. Let $(\xi \vee_t \zeta)$ be the cadlag process defined by

$$(\xi \vee_t \zeta)(x) = \xi_t(x) \vee \zeta_t(x), \quad t \in [0, T], \quad x \in \mathbb{Z}^d.$$

This process is adapted to the filtration $\{\hat{\mathcal{F}}_t\}$, where $\hat{\mathcal{F}}_t$ is the completion of the smallest σ -algebra containing \mathcal{F}_t and $\bar{\mathcal{F}}_t$.

Define also

$$\tilde{d}_t(x) := \int_{(0,t] \times [0,\infty)} I_{[0, \tilde{d}(x, \xi_{r-}, \zeta_{r-})]}(v) N_2(x, dr, dv) \quad (28)$$

and

$$\tilde{b}_t(x) := \int_{(0,t] \times [0,\infty)} I_{[0, \tilde{b}(x, \xi_{s-}, \zeta_{s-})]}(u) N_1(x, ds, du).$$

Then $\tilde{b}_t(x)$ and $\tilde{d}_t(x)$ is the number of births and deaths, respectively, for the process $(\xi \vee_t \zeta)$ at site x that occurred before time t , that is,

$$\tilde{d}_t(x) = \# \{r : r \leq t, \xi \vee_r \zeta(x) - \xi \vee_{r-} \zeta(x) = -1\},$$

and similarly for $\tilde{b}_t(x)$. Indeed, let τ be a moment of birth for $(\xi \vee_t \zeta)$, that is, $\xi \vee_\tau \zeta(x) - \xi \vee_{\tau-} \zeta(x) = 1$. Without loss of generality assume that $\xi_{\tau-}(x) \geq \zeta_{\tau-}(x)$. If $\xi_{\tau-}(x) > \zeta_{\tau-}(x)$, then τ is a moment of birth for (ξ_t) , hence $N_1(\{x\} \times \{\tau\} \times [0, b(x, \xi_{\tau-})]) = 1$ a.s. and $\tilde{b}_\tau(x) - \tilde{b}_{\tau-}(x) = 1$. If $\xi_{\tau-}(x) = \zeta_{\tau-}(x)$, then τ is a moment of birth for at least one of the processes (ξ_t) and (ζ_t) , hence

$$N_1(\{x\} \times \{\tau\} \times [0, b(x, \xi_{\tau-}) \vee b(x, \zeta_{\tau-})]) = 1 \quad \text{a.s.}$$

and again $\tilde{b}_\tau(x) - \tilde{b}_{\tau-}(x) = 1$. On the other hand, let $\tilde{b}_\tau(x) - \tilde{b}_{\tau-}(x) = 1$. Again, with no loss of generality we assume that $\xi_{\tau-}(x) \geq \zeta_{\tau-}(x)$. If $\xi_{\tau-}(x) > \zeta_{\tau-}(x)$, then

$$\tilde{b}(x, \xi_{\tau-}(x), \zeta_{\tau-}(x)) = b(x, \xi_{\tau-}),$$

hence $N_1(\{x\} \times \{\tau\} \times [0, b(x, \xi_{\tau-})]) = 1$ a.s. and τ is a moment of birth for $(\xi \vee_t \zeta)$. The remaining case $\xi_{\tau-}(x) = \zeta_{\tau-}(x)$ is similar. The proof of (28) follows the same pattern.

Fix $t \in [0, T]$ and $x \in \mathbb{Z}^d$. Note that

$$\begin{aligned}
& E \int_{(0,t] \times [0,\infty)} \{I_{[0,\tilde{b}(x,\xi_{s-},\zeta_{s-})]}(u) - I_{[0,b(x,\zeta_{s-})]}(u)\} N_1(x, ds, du) \\
&= \int_{(0,t]} EI \{ \xi_{s-}(x) \geq \zeta_{s-}(x) \} \{ \tilde{b}(x, \xi_{s-}, \zeta_{s-}) - b(x, \zeta_{s-}) \} ds \\
&\leq \int_{(0,t]} ds E \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi_{s-}(y) - \zeta_{s-}(y)|,
\end{aligned}$$

and

$$\begin{aligned}
& E \int_{(0,t] \times [0,\infty)} \{I_{[0,\tilde{d}(x,\xi_{s-},\zeta_{s-})]}(v) - I_{[0,d(x,\zeta_{s-})]}(v)\} N_2(x, dr, dv) \\
&= \int_{(0,t]} EI \{ \xi_{s-}(x) \geq \zeta_{s-}(x) \} \{ \tilde{d}(x, \xi_{s-}, \zeta_{s-}) - d(x, \zeta_{s-}) \} dr \\
&\geq - \int_{(0,t]} ds E \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi_{s-}(y) - \zeta_{s-}(y)|.
\end{aligned}$$

So, we can write

$$\begin{aligned}
& 0 \leq E(\xi \vee_t \zeta(x) - \zeta_t(x)) \\
&= E \int_{(0,t] \times [0,\infty)} \{I_{[0,\tilde{b}(x,\xi_{s-},\zeta_{s-})]}(u) - I_{[0,b(x,\zeta_{s-})]}(u)\} N_1(x, ds, du) \\
&- \int_{(0,t] \times [0,\infty)} \{I_{[0,\tilde{d}(x,\xi_{s-},\zeta_{s-})]}(v) - I_{[0,d(x,\zeta_{s-})]}(v)\} N_2(x, ds, dv) + E(\xi \vee_0 \zeta(x) - \zeta_0(x)) \\
&\leq 2 \int_{(0,t]} ds E \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi_{s-}(y) - \zeta_{s-}(y)| + E(\xi \vee_0 \zeta(x) - \zeta_0(x)).
\end{aligned}$$

Multiplying the last inequality by $w(x)$ and summing over x , we get

$$\begin{aligned}
& E \sum_{x \in \mathbb{Z}^d} w(x) (\xi \vee_t \zeta(x) - \zeta_t(x)) \leq 2 \int_{(0,t]} ds E \sum_{y \in \mathbb{Z}^d} |\xi_{s-}(y) - \zeta_{s-}(y)| \sum_{x \in \mathbb{Z}^d} w(x) a(x-y) \\
&+ E \sum_{x \in \mathbb{Z}^d} w(x) |\xi \vee_0 \zeta(x) - \zeta_0(x)| \leq 2 \int_{(0,t]} ds E \sum_{y \in \mathbb{Z}^d} C_{w,a} w(y) |\xi_{s-}(y) - \zeta_{s-}(y)| \quad (29) \\
&+ E \sum_{x \in \mathbb{Z}^d} w(x) |\xi \vee_0 \zeta(x) - \zeta_0(x)|.
\end{aligned}$$

Keeping in mind that $(p \vee q - p) + (p \vee q - q) = |p - q|$, we obtain (27) by swapping ξ and ζ in (29) and then adding the the obtained inequality to (29). \square

Proposition 3.6. *Pathwise uniqueness holds for equation (2).*

Proof. Let $((\xi_t)_{t \in [0, T]})$ and $((\zeta_t)_{t \in [0, T]})$ be two solutions to (2) as in Lemma 3.5. We know by item (ii) of Definition 2.1 that

$$f(t) := E \sum_{x \in \mathbb{Z}^d} w(x) |\xi_t(x) - \zeta_t(x)| < \infty.$$

Note that f is a continuous function by the dominated convergence theorem, since for a fixed $s > 0$ every solution (η_t) of (2) satisfies $\eta_{s-} = \eta_s = \eta_{s+}$ a.s. Furthermore, $f(0) = 0$, therefore Grownwall's inequality and Lemma 3.5 yield $f(t) = 0$. Since $\zeta_t(x), \xi_t(x)$ are cadlag processes, it follows that $\zeta_t(x) = \xi_t(x)$ a.s. for all $t \in (0, T]$. \square

Define $\bar{b}(x, \eta) := \sup_{\alpha \leq \eta} b(x, \alpha)$. Note that \bar{b} is non-decreasing in the sense that

$$\bar{b}(x, \eta^1) \leq \bar{b}(x, \eta^2) \quad \text{whenever } \eta^1 \leq \eta^2,$$

and that \bar{b} satisfies inequalities of the form (3). Indeed, if $\xi, \eta \in \mathcal{X}, x \in \mathbb{Z}^d, \xi(x) \geq \eta(x)$, then

$$\begin{aligned} \bar{b}(x, \xi) - \bar{b}(x, \eta) &= \sup_{\alpha: \alpha \leq \xi} \left[b(x, \alpha) - \sup_{\beta: \beta \leq \eta} b(x, \beta) \right] \\ &\leq \sup_{\alpha: \alpha \leq \xi} [b(x, \alpha) - b(x, \alpha \wedge \eta)] \leq \sup_{\alpha: \alpha \leq \xi} \sum_{y \in \mathbb{Z}^d} a(x-y) |\alpha(y) - \alpha(y) \wedge \eta(y)| \\ &\leq \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi(y) - \eta(y)|. \end{aligned}$$

Also, for every $x \in \mathbb{Z}^d$ the map

$$\mathcal{X} \ni \xi \mapsto \bar{b}(x, \xi) \in \mathbb{R}_+$$

is continuous by Lemma 3.1, since \bar{b} satisfies all conditions imposed on b .

Before proceeding to the general existence result, let us consider a pure birth equation

$$\xi_t(x) = \int_{(0, t] \times [0, \infty)} I_{[0, \bar{b}(x, \xi_{s-})]}(u) N_1(x, ds, du) + \eta_0(x). \quad (30)$$

This equation is of the form (2).

Lemma 3.7. *Equation (30) has a (unique) solution.*

Proof. Let us start with the equation (30) with a ‘truncated’ initial condition, that is, with the initial condition $\eta_0^{(n)}(x) = I_{\{|x|_1 \leq n\}} \eta_0^{(n)}(x)$, $n \in \mathbb{N}$. Then the initial condition is finite, hence there exists a unique solution by Proposition 3.2. We denote this unique solution of

$$\xi_t(x) = \int_{(0,t] \times [0,\infty)} I_{[0,\bar{b}(x,\xi_{s-})]}(u) N_1(x, ds, du) + I_{\{|x|_1 \leq n\}} \eta_0(x), \quad (31)$$

by $(\xi_t^{(n)})_{t \in [0,T]}$. By Proposition 3.3 we have $\xi_t^{(n)} \leq \xi_t^{(m)}$, $n \leq m$, and

$$\left\{ t : \sum_{x \in \mathbb{Z}^d} (\xi_t^{(n)}(x) - \xi_{t-}^{(n)}(x)) = 1 \right\} \subset \left\{ t : \sum_{x \in \mathbb{Z}^d} (\xi_t^{(m)}(x) - \xi_{t-}^{(m)}(x)) = 1 \right\}, \quad n \leq m,$$

almost surely. Therefore, the limit $\bar{\eta}_t = \lim_{n \rightarrow \infty} \xi_t^{(n)}$ exists and is cadlag (if finite). For each $n \in \mathbb{N}$

$$\begin{aligned} E \sum_{x \in \mathbb{Z}^d} w(x) \xi_t^{(n)}(x) &= E \sum_{x \in \mathbb{Z}^d} w(x) \int_{(0,t] \times [0,\infty)} I_{[0,\bar{b}(x,\xi_{s-}^{(n)})]}(u) N_1(x, ds, du) + E \sum_{x \in \mathbb{Z}^d} w(x) I_{\{|x|_1 \leq n\}} \eta_0(x) \\ &\leq E \sum_{x \in \mathbb{Z}^d} w(x) \int_{(0,t]} \bar{b}(x, \xi_{s-}^{(n)}) ds + E \sum_{x \in \mathbb{Z}^d} w(x) \eta_0(x). \end{aligned}$$

Recall that $\mathbf{0} \in \mathcal{X}$, $\mathbf{0}(x) \equiv 0$. By (3)

$$\bar{b}(x, \xi_{s-}^{(n)}) \leq \sum_{y \in \mathbb{Z}^d} a(x-y) |\xi_{s-}^{(n)}(y)| + b(x, \mathbf{0}),$$

hence

$$\begin{aligned} E \sum_{x \in \mathbb{Z}^d} w(x) \int_{(0,t]} \bar{b}(x, \xi_{s-}^{(n)}) ds &\leq E \sum_{x \in \mathbb{Z}^d} w(x) \int_{(0,t]} ds \left[\sum_{y \in \mathbb{Z}^d} a(x-y) |\xi_{s-}^{(n)}(y)| + b(x, \mathbf{0}) \right] \\ &\leq t \sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0}) + E \int_{(0,t]} ds \sum_{y \in \mathbb{Z}^d} |\xi_{s-}^{(n)}(y)| \sum_{x \in \mathbb{Z}^d} w(x) a(x-y) \\ &\leq t \sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0}) + C_{w,a} E \int_{(0,t]} ds \sum_{y \in \mathbb{Z}^d} w(y) |\xi_{s-}^{(n)}(y)|. \end{aligned}$$

Thus,

$$\begin{aligned} &E \sum_{x \in \mathbb{Z}^d} w(x) \xi_t^{(n)}(x) \\ &\leq C_{w,a} E \int_{(0,t]} ds \sum_{x \in \mathbb{Z}^d} w(x) |\xi_{s-}^{(n)}(x)| + t \sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0}) + E \sum_{x \in \mathbb{Z}^d} w(x) \eta_0(x). \end{aligned} \quad (32)$$

The expression on the left hand side is finite by Proposition 3.2 and depends continuously on t by the same argument as in the proof of Proposition 3.6, therefore Grownwall's inequality implies

$$E \sum_{x \in \mathbb{Z}^d} w(x) \xi_t^{(n)}(x) \leq e^{C_{w,a} t} [E \sum_{x \in \mathbb{Z}^d} w(x) \eta_0(x) + t \sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0})]. \quad (33)$$

Letting $n \rightarrow \infty$, we get by the monotone convergence theorem

$$E \sum_{x \in \mathbb{Z}^d} w(x) \bar{\eta}_t(x) \leq e^{C_{w,a} t} [E \sum_{x \in \mathbb{Z}^d} w(x) \eta_0(x) + t \sum_{x \in \mathbb{Z}^d} w(x) b(x, \mathbf{0})]. \quad (34)$$

It follows from the continuity of \bar{b} that $(\bar{\eta}_t)$ is a solution to (30). Uniqueness follows from Proposition 3.6. \square

Proposition 3.8. *Strong existence holds for equation (2).*

Proof. As in the proof of the previous proposition, we first consider equation (2) with the 'truncated' initial condition $\eta_0^{(n)}(x) = I_{\{|x|_1 \leq n\}} \eta_0(x)$, $n \in \mathbb{N}$. We denote the unique solution of

$$\begin{aligned} \eta_t(x) = & \int_{(0,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) N_1(x, ds, du) \\ & - \int_{(0,t] \times [0,\infty)} I_{[0,d(x,\eta_{r-})]}(v) N_2(x, dr, dv) + I_{\{|x|_1 \leq n\}} \eta_0(x), \end{aligned} \quad (35)$$

by $(\eta_t^{(n)})_{t \in [0,T]}$.

Using arguments analogous to those in the proof of Lemma 3.5, we can write

$$\begin{aligned} & E \sup_{t \in [0,T]} \left[\int_{(0,t] \times [0,\infty)} \{I_{[0,\bar{b}(x,\eta_{s-}^{(n)}, \eta_{s-}^{(m)})]}(u) - I_{[0,b(x,\eta_{s-}^{(m)})]}(u)\} N_1(x, ds, du) \right] \\ & \leq E \sup_{t \in [0,T]} \left[\int_{(0,t] \times [0,\infty)} I_{[b(x,\eta_{s-}^{(m)}), b(x,\eta_{s-}^{(m)}) \vee b(x,\eta_{s-}^{(n)})]}(u) N_1(x, ds, du) \right] \\ & = E \int_{(0,T] \times [0,\infty)} I_{[b(x,\eta_{s-}^{(m)}), b(x,\eta_{s-}^{(m)}) \vee b(x,\eta_{s-}^{(n)})]}(u) N_1(x, ds, du) \\ & = E \int_{(0,T]} \{b(x, \eta_{s-}^{(m)}) \vee b(x, \eta_{s-}^{(n)}) - b(x, \eta_{s-}^{(m)})\} ds \\ & \leq E \int_{(0,T]} \sum_{y \in \mathbb{Z}^d} a(x-y) E |\eta_{s-}^{(n)}(y) - \eta_{s-}^{(m)}(y)| ds \end{aligned}$$

and similarly

$$\begin{aligned}
& E \inf_{t \in [0, T]} \left[\int_{(0, t] \times [0, \infty)} \{I_{[0, \tilde{d}(x, \eta_{r_-}^{(n)}, \eta_{r_-}^{(m)})]}(v) - I_{[0, d(x, \eta_{r_-}^{(m)})]}(v)\} N_2(x, ds, dv) \right] \\
& \geq -E \int_{(0, T]} \sum_{y \in \mathbb{Z}^d} a(x-y) |\eta_{r_-}^{(n)}(y) - \eta_{r_-}^{(m)}(y)| ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E \sup_{t \in [0, T]} (\eta^{(n)} \vee_t \eta^{(m)}(x) - \eta_t^{(m)}(x)) \\
& = E \sup_{t \in [0, T]} \left[\int_{(0, t] \times [0, \infty)} \{I_{[0, \tilde{b}(x, \eta_{s_-}^{(n)}, \eta_{s_-}^{(m)})]}(u) - I_{[0, b(x, \eta_{s_-}^{(m)})]}(u)\} N_1(x, ds, du) \right. \\
& \quad \left. - \int_{(0, t] \times [0, \infty)} \{I_{[0, \tilde{d}(x, \eta_{r_-}^{(n)}, \eta_{r_-}^{(m)})]}(v) - I_{[0, d(x, \eta_{r_-}^{(m)})]}(v)\} N_2(x, dr, dv) + (\eta^{(n)} \vee_0 \eta^{(m)}(x) - \eta_0^{(m)}(x)) \right] \\
& \leq 2 \int_{(0, T] \times [0, \infty)} ds E \sum_{y \in \mathbb{Z}^d} a(x-y) |\eta_{s_-}^{(m)}(y) - \eta_{s_-}^{(n)}(y)| + E (\eta^{(n)} \vee_0 \eta^{(m)}(x) - \eta_0^{(m)}(x)).
\end{aligned}$$

Multiplying by $w(x)$ and taking the sum over x , we obtain

$$\begin{aligned}
& E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} (\eta^{(n)} \vee_t \eta^{(m)}(x) - \eta_t^{(m)}(x)) \\
& \leq 2 \int_{(0, T] \times [0, \infty)} ds E \sum_{x \in \mathbb{Z}^d} w(x) \sum_{y \in \mathbb{Z}^d} a(x-y) |\eta_{s_-}^{(m)}(y) - \eta_{s_-}^{(n)}(y)| + E \sum_{x \in \mathbb{Z}^d} w(x) (\eta^{(n)} \vee_0 \eta^{(m)}(x) - \eta_0^{(m)}(x)). \\
& \leq 2 \int_{(0, T] \times [0, \infty)} ds E \sum_{y \in \mathbb{Z}^d} w(y) C_{w,a} |\eta_{s_-}^{(m)}(y) - \eta_{s_-}^{(n)}(y)| + E \sum_{x \in \mathbb{Z}^d} w(x) (\eta^{(n)} \vee_0 \eta^{(m)}(x) - \eta_0^{(m)}(x)).
\end{aligned}$$

Using the above inequality, we get

$$\begin{aligned}
& \Delta_{m,n} := E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} |\eta_t^{(n)}(x) - \eta_t^{(m)}(x)| \\
& \leq E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} (\eta^{(n)} \vee_t \eta^{(m)}(x) - \eta_t^{(m)}(x)) + E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} (\eta^{(n)} \vee_t \eta^{(m)}(x) - \eta_t^{(n)}(x)) \\
& \leq 4C_{w,a} \int_{(0, T] \times [0, \infty)} ds E \sum_{x \in \mathbb{Z}^d} w(x) |\eta_{s_-}^{(n)}(x) - \eta_{s_-}^{(m)}(x)| + E \sum_{x \in \mathbb{Z}^d} w(x) |\eta_0^{(n)}(x) - \eta_0^{(m)}(x)|,
\end{aligned}$$

and consequently

$$\Delta_{m,n} \leq E \sum_{x \in \mathbb{Z}^d} w(x) |\eta_0^{(n)} - \eta_0^{(m)}(x)| \exp\{4C_{w,a}T\} \quad (36)$$

by Lemma 3.5 and Gronwall's inequality. As $m, n \rightarrow \infty$, $E \sum_{x \in \mathbb{Z}^d} w(x) |\eta_0^{(n)} - \eta_0^{(m)}(x)| \rightarrow 0$. Hence

$$\Delta_{m,n} \rightarrow 0, \quad m, n \rightarrow \infty. \quad (37)$$

Since $w(x) > 0$ for all x , (37) implies that

$$\begin{aligned} P\left\{ \sup_{t \in [0, T]} |\eta_t^{(n)}(x) - \eta_t^{(m)}(x)| > 0 \right\} &= P\left\{ \sup_{t \in [0, T]} |\eta_t^{(n)}(x) - \eta_t^{(m)}(x)| \geq 1 \right\} \\ &\leq \frac{\Delta_{m,n}}{w(x)} \rightarrow 0, \quad m, n \rightarrow \infty. \end{aligned}$$

Cauchy convergence in probability implies existence of a subsequence along which almost sure convergence takes place; moreover, using the diagonal argument, we can find a subsequence $\{n_m\} \subset \mathbb{N}$ such that for each x there exists $(\eta_t(x))_{t \in [0, T]}$ satisfying

$$P\left\{ \sup_{t \in [0, T]} |\eta_t^{(n_k)}(x) - \eta_t(x)| \rightarrow 0 \right\} = 1. \quad k \rightarrow \infty \quad (38)$$

Furthermore, $\eta_t^{(n)} \leq \bar{\eta}_t$, $t \in [0, T]$, where $(\bar{\eta}_t)$ is the unique solution of (30). Thus, since $\eta_t \leq \bar{\eta}_t$, $t \in [0, T]$, by the dominated convergence theorem

$$P\left\{ \sup_{t \in [0, T]} \sum_{x \in \mathbb{Z}^d} w(x) |\eta_t^{(n_k)}(x) - \eta_t(x)| \rightarrow 0 \right\} = 1. \quad (39)$$

Since b, d are continuous and $\eta_t^{(n_k)} \rightarrow \eta_t$ a.s. in \mathcal{X} , $(\eta_t)_{t \in [0, T]}$ is a strong solution to (2) if we can show that $E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} \eta_t(x) < \infty$, integrals on the right hand side of (2) are well defined and

$$E \int_0^T [b(x, \eta_{s-}) + d(x, \eta_{s-})] ds < \infty. \quad (40)$$

The inequality $E \sum_{x \in \mathbb{Z}^d} w(x) \sup_{t \in [0, T]} \eta_t(x) < \infty$ follows from the inequalities

$$\eta_t^{(n)} \leq \bar{\eta}_t, \quad n \in \mathbb{N},$$

and the fact that $(\bar{\eta}_t)$ is a solution to (30). The integrals on the right hand side of (2) are well defined as pointwise limits of the corresponding integrals for $(\eta_t^{(n_k)})$.

To prove (40), we denote the number of births and deaths at x before t by $b_t(x)$ and $d_t(x)$

respectively, i.e.

$$b_t(x) = \#\{s : \eta_s(x) - \eta_{s-}(x) = 1\} = \int_{(0,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) N_1(x, ds, du).$$

and similarly for $d_t(x)$. Note that $\eta_t(x) = b_t(x) - d_t(x) + \eta_0(x)$. Let (τ_n) be the moments of jumps of $c_t(x) := b_t(x) + d_t(x)$, $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$. If $\tau_k < T$ and $c_t(x) = c_{\tau_k}(x)$ for all $t \in [\tau_k, T]$, we set $\tau_{k+j} = T$ for all $j \in \mathbb{N}$. Note that τ_n is a stopping time with respect to the filtration $\{\mathcal{S}_t\}$. We have

$$c_t(x) = \sum_{n \in \mathbb{N}} I\{\tau_n \leq t\}$$

a.s. for all $t \in [0, T)$. Define for $n \in \mathbb{N}$

$$\begin{aligned} c_t^{(n)}(x) &:= \int_{(0,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-}) \wedge n]}(u) N_1(x, ds, du) \\ &+ \int_{(0,t] \times [0,\infty)} I_{[0,d(x,\eta_{r-}) \wedge n]}(v) N_2(x, dr, dv). \end{aligned}$$

Then

$$M_t^{(n)}(x) = c_t^{(n)}(x) - \int_0^t (b(x, \eta_{s-}) \wedge n) ds - \int_0^t (d(x, \eta_{s-}) \wedge n) ds$$

is a martingale with respect to $\{\mathcal{S}_t\}$. By the optional stopping theorem $EM_{\tau_1}^{(n)}(x) = 0$, hence

$$E \int_0^{\tau_1} (b(x, \eta_{s-}) \wedge n + d(x, \eta_{s-}) \wedge n) ds \leq 1.$$

Similarly,

$$E \int_{\tau_m}^{\tau_{m+1}} (b(x, \eta_{s-}) \wedge n + d(x, \eta_{s-}) \wedge n) ds \leq P\{\tau_m < T\}.$$

Consequently

$$\begin{aligned} E \int_0^T (b(x, \eta_{s-}) \wedge n + d(x, \eta_{s-}) \wedge n) ds &\leq \sum_{m=0}^{\infty} E \int_{\tau_m}^{\tau_{m+1}} (b(x, \eta_{s-}) \wedge n + d(x, \eta_{s-}) \wedge n) ds \\ &\leq \sum_{m=0}^{\infty} P\{\tau_m < T\} \stackrel{*}{=} \sum_{m=0}^{\infty} P\{c_T(x) \geq m\} = Ec_T(x) + 1, \end{aligned}$$

where the transition marked by the asterisk is possible in particular since

$$c_T(x) = c_{T-}(x) \quad \text{a.s.}$$

Letting $n \rightarrow \infty$, we get by the monotone convergence theorem

$$E \int_0^T (b(x, \eta_{s-}) + d(x, \eta_{s-})) ds \leq E c_T(x) + 1 \quad (41)$$

Finally, since only existing particles may disappear, the number of deaths $d_t(x)$ satisfies for every $t \in [0, T]$

$$d_t(x) \leq b_t(x) + \eta_0(x).$$

Thus,

$$E c_T(x) \leq 2E b_T(x) + E \eta_0(x) \leq 2E \bar{\eta}_T(x) + E \eta_0(x) < \infty. \quad (42)$$

□

Remark 3.9. In fact, (41) and (42) yield even stronger inequality

$$\sum_{x \in \mathbb{Z}^d} w(x) E \int_0^T (b(x, \eta_{s-}) + d(x, \eta_{s-})) ds < \infty.$$

The following statement is a consequence of Proposition 3.6 and [Kur07, Theorem 3.14]

Proposition 3.10. *Joint uniqueness in law holds for (2).*

Proposition 3.11. *The unique solution to (2) is a Markov process: for all $\mathcal{D} \in D_{\mathcal{X}}[0, \infty)$ and $q \geq 0$,*

$$P[(\eta_{q+}) \in \mathcal{D} \mid \mathcal{S}_q] = P[(\eta_{q+}) \in \mathcal{D} \mid \eta_q]. \quad (43)$$

Proof. For $t \geq q$ we have

$$\begin{aligned} \eta_t(x) &= \int_{(q,t] \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) N_1(x, ds, du) \\ &- \int_{(q,t] \times [0,\infty)} I_{[0,d(x,\eta_{r-})]}(v) N_2(x, dr, dv) + \eta_q(x), \end{aligned} \quad (44)$$

therefore (η_{q+}) is $\sigma\{\eta_q, N_k(\{x\} \times [q, q+r] \times C), x \in \mathbb{Z}^d, C \in \mathcal{B}(\mathbb{R}_+), r \geq 0, k = 1, 2\}$ -measurable by Propositions 3.6 and 3.8; the fact that we start from the time q instead of 0 does not cause

problems. Since Poisson processes have independent increments,

$$\sigma\{N_k(\{x\} \times [q, q+r] \times C), x \in \mathbb{Z}^d, C \in \mathcal{B}(\mathbb{R}_+), r \geq 0, k = 1, 2\}$$

is independent of \mathcal{S}_q and (43) follows. \square

4 Proof of Propositions 2.8 and 2.9

Proof of Proposition 2.8. For $R > 0$ we define $\mathbf{B}_R := \{x \in \mathbb{Z}^d \mid |x|_1 \leq R\}$. By Ito's formula

$$\begin{aligned} F(\eta_t) = F(\eta_0) &+ \int_{\mathbf{B}_{R_F} \times (0,t] \times [0,\infty)} \{F(\eta_{t-}^{+x}) - F(\eta_{t-})\} I_{[0,b(x,\eta_{s-})]}(u) N_1(dx, ds, du) \\ &+ \int_{\mathbf{B}_{R_F} \times (0,t] \times [0,\infty)} \{F(\eta_{t-}^{-x}) - F(\eta_{t-})\} I_{[0,d(x,\eta_{r-})]}(v) N_2(dx, dr, dv). \end{aligned} \quad (45)$$

We can write

$$\begin{aligned} &\int_{\mathbf{B}_{R_F} \times (0,t] \times [0,\infty)} \{F(\eta_{t-}^{+x}) - F(\eta_{t-})\} I_{[0,b(x,\eta_{s-})]}(u) N_1(dx, ds, du) \\ &= \int_{(0,t]} \sum_{x \in \mathbf{B}_{R_F}} \{F(\eta_{t-}^{+x}) - F(\eta_{t-})\} b(x, \eta_{s-}) ds \\ &+ \int_{\mathbf{B}_{R_F} \times (0,t] \times [0,\infty)} \{F(\eta_{t-}^{+x}) - F(\eta_{t-})\} I_{[0,b(x,\eta_{s-})]}(u) \tilde{N}_1(dx, ds, du) \end{aligned}$$

where $\tilde{N}_1 = N_1 - \#(dx)dsdu$. Since $F(\eta^{+x}) - F(\eta)$ is bounded uniformly in x and η , the last integral with respect to \tilde{N}_1 is a martingale by item (iv) of Definition 2.1, see e.g. [IW81, Section 3 of Chapter 2]. Similarly,

$$\int_{\mathbf{B}_{R_F} \times (0,t] \times [0,\infty)} \{F(\eta_{t-}^{-x}) - F(\eta_{t-})\} I_{[0,d(x,\eta_{r-})]}(v) N_2(dx, dr, dv)$$

can be represented as a sum of

$$\int_{(0,t]} \sum_{x \in \mathbf{B}_{R_F}} \{F(\eta_{t-}^{-x}) - F(\eta_{t-})\} d(x, \eta_{r-}) dr$$

and a martingale. The assertion of the proposition now follows from (45) and (1).

To prove Proposition 2.9 we will need the following form of the martingale representation

theorem, which is a corollary of [IW81, Theorem 7.4, Chapter 2].

Theorem. Let (α_t) be a cadlag \mathcal{X} -valued process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ with the (predictable) compensator $\#(dx) \times \phi(x, t)dt$ such that

$$E\phi(x, t) < \infty$$

for each $t \geq 0$ and $x \in \mathbb{Z}^d$. Then on an extended filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t, t \geq 0\}, \tilde{P})$ there exists an adapted to $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ Poisson point process \tilde{N} on $\mathbb{Z}^d \times \mathbb{R}_+$ such that

$$\alpha_t(x) = \int_{(0, t] \times \mathbb{R}_+} I_{[0, \phi(x, s)]}(u) \tilde{N}(x, ds, du). \quad (46)$$

To see that this theorem follows from [IW81, Theorem 7.4, Chapter 2], we should take there $\mathbf{X} = \mathbb{Z}^d$, $\mathbf{Z} = \mathbb{Z}^d \times \mathbb{R}_+$, $m = \# \times du$, $q(t, E) = \sum_{x \in E} \phi(x, t)$, $\theta(t, (x, u)) = xI\{u \leq \phi(x, t)\} + \Delta I\{u > \phi(x, t)\}$.

Proof of Proposition 2.9. Define a $(\mathbb{Z}_+)^{\mathbb{Z}^d \times \{-1, 1\}}$ -valued process $\{\alpha_t, t \geq 0\}$ by

$$\alpha_t((x, 1)) = \eta_t^{(b)}(x), \quad \alpha_t((x, -1)) = \eta_t^{(d)}(x).$$

Conditions of the previous theorem are satisfied, so we get

$$\alpha_t((x, 1)) = \int_{(0, t] \times \mathbb{R}_+} I_{[0, b(x, \eta_{s-})]}(u) N((x, 1), ds, du),$$

and

$$\alpha_t((x, -1)) = \int_{(0, t] \times \mathbb{R}_+} I_{[0, d(x, \eta_{s-})]}(u) N((x, -1), ds, du),$$

a.s. for all $t \in [0, T]$, where N is a Poisson point process on $(\mathbb{Z}^d \times \{-1, 1\}) \times \mathbb{R}_+ \times \mathbb{R}_+$. Define N_1 and N_2 by

$$N_1(x \times [0, t] \times U) = N((x, 1) \times [0, t] \times U),$$

$$N_2(x \times [0, t] \times U) = N((x, -1) \times [0, t] \times U).$$

Then a.s. for all $t \in [0, T]$

$$\begin{aligned} \eta_t(x) &= \eta_0(x) + \eta_t^{(b)}(x) - \eta_t^{(d)}(x) \\ &= \int_{(0, t] \times [0, \infty)} I_{[0, b(x, \eta_{s-})]}(u) N_1(x, ds, du) - \int_{(0, t] \times [0, \infty)} I_{[0, d(x, \eta_{s-})]}(v) N_2(x, ds, dv). \end{aligned}$$

□

5 Proof of Theorem 2.11 and Proposition 2.12

Let us recall that $(\eta(\alpha, t))_{t \in [0, T]}$ is the unique solution to (2) with initial condition $\eta_0 \equiv \alpha$, $\alpha \in \mathcal{X}$.

Lemma 5.1. *The process*

$$M_t := V(\eta(\mathbf{0}, t)) - \int_0^t LV(\eta(\mathbf{0}, s-)) ds,$$

is well defined and an $\{\mathcal{F}_t\}$ -martingale.

Proof. Denote

$$\mathcal{D}(L) := \left\{ \eta \in \mathcal{X} : \sum_{x \in \mathbb{Z}^d} v(x) [b(x, \eta) + d(x, \eta)] < \infty \right\}.$$

For $\eta \in \mathcal{D}(L)$ the expression $LV(\eta)$ in (1) is well defined. Since v satisfies the same assumptions as w does, Theorem 2.6, Proposition 2.8 and all the other results proven in Sections 3 and 4 are still valid if we replace in their formulations w by v and \mathcal{X} by \mathcal{X}_v . Remark 3.9 implies that a.s.

$$E \int_0^T LV(\eta(\mathbf{0}, s-)) ds < \infty,$$

in particular, $\eta(\mathbf{0}, s) \in \mathcal{D}(L)$ a.s. since $(\eta(\mathbf{0}, t))$ is cadlag. By Proposition 2.8,

$$\sum_{x \in \mathbb{Z}^d} v(x) \eta(\mathbf{0}, t)(x) I\{|x|_1 \leq n\} - \int_0^t \sum_{x \in \mathbb{Z}^d} v(x) \{b(x, \eta(\mathbf{0}, s-)) - d(x, \eta(\mathbf{0}, s-))\} I\{|x|_1 \leq n\} ds$$

is an $\{\mathcal{F}_t\}$ -martingale. By the dominated convergence theorem,

$$\sum_{x \in \mathbb{Z}^d} v(x) \eta(\mathbf{0}, t)(x) I\{|x|_1 \leq n\} \xrightarrow{L^1} \sum_{x \in \mathbb{Z}^d} v(x) \eta(\mathbf{0}, t)(x).$$

Furthermore,

$$\begin{aligned} & \int_0^t \sum_{x \in \mathbb{Z}^d} v(x) \{b(x, \eta(\mathbf{0}, s-)) - d(x, \eta(\mathbf{0}, s-))\} I\{|x|_1 \leq n\} ds \\ & \xrightarrow{L^1} \int_0^t \sum_{x \in \mathbb{Z}^d} v(x) \{b(x, \eta(\mathbf{0}, s-)) - d(x, \eta(\mathbf{0}, s-))\} ds \end{aligned}$$

since the difference goes to zero in L^1 by Remark 3.9. Therefore,

$$M_t = V(\eta(\mathbf{0}, t)) - \int_0^t LV(\eta(\mathbf{0}, s-))ds,$$

is an $\{\mathcal{F}_t\}$ -martingale. □

Proof of Theorem 2.11. For $\alpha \in \mathcal{X}$ let us define

$$P^t(\alpha, B) := P\{\eta(\alpha, t) \in B\}, \quad B \in \mathcal{B}(\mathcal{X}), t \geq 0$$

and let

$$\mu_n(B) := \frac{1}{n} \int_0^n P^s\{\mathbf{0}, B\}ds, \quad B \in \mathcal{B}(\mathcal{X}).$$

Denote also $K_r := \{\eta \in \mathcal{X}_v : V(\eta) \leq r\}$, $r > 0$. Imitating the proof of Lemma 9.7 of Chapter 4 [EK86], we obtain by Lemma 5.1

$$\begin{aligned} 0 &\leq EV(\eta_n) = EV(\eta_0) + E \int_0^n LV(\eta_{s-})ds \\ &= EV(\eta_0) + E \int_0^n LV(\eta_{s-})I\{\eta_{s-} \in K_r\}ds + E \int_0^n LV(\eta_{s-})I\{\eta_{s-} \notin K_r\}ds \\ &\leq EV(\eta_0) + Ec_1 \int_0^n I\{\eta_{s-} \in K_r\}ds + (c_1 - c_2r)E \int_0^n I\{\eta_{s-} \notin K_r\}ds \\ &= EV(\eta_0) + nc_1\mu_n(K_r) + n(c_1 - c_2r)[1 - \mu_n(K_r)], \end{aligned}$$

hence

$$\mu_n(K_r) \geq 1 - \frac{c_1}{c_2r} - \frac{EV(\eta_0)}{nc_2r}.$$

We see that $\mu_n(K_r) \rightarrow 1$ as $r \rightarrow \infty$ uniformly in $n \in \mathbb{N}$. It follows from (14) that for every $r > 0$ the set K_r is precompact in \mathcal{X} , therefore the family $\{\mu_n, n \in \mathbb{N}\}$ is tight. By Prohorov's theorem there exists a measure μ on \mathcal{X} and a sequence $\{n_k\}$ such that $\mu_{n_k} \Rightarrow \mu$. Without loss of generality we assume that $\mu_n \Rightarrow \mu$. Let us show that μ is an invariant measure. Take $f \in C_b(\mathcal{X})$, then

$$\begin{aligned} \int P^t f(\eta)\mu(d\eta) &= \lim_n \int P^t f(\eta)\mu_n(d\eta) = \lim_n \frac{1}{n} \int_0^n ds \int P^t f(\eta)P^s(\mathbf{0}, d\eta) = \\ &= \lim_n \frac{1}{n} \int_0^n ds P^{t+s} f(\mathbf{0}) = \lim_n \frac{1}{n} \int_t^{n+t} ds P^s f(\mathbf{0}) \\ &= \lim_n \left[\frac{1}{n} \int_0^n + \frac{1}{n} \int_n^{n+t} - \frac{1}{n} \int_0^t \right] = \lim_n \int f(\eta)\mu_n(d\eta) = \int f(\eta)\mu(d\eta). \end{aligned}$$

□

Proof of Proposition 2.12. Let us take $v(x) = \frac{1}{1+|x|_1^{d+1}}$, and let \mathbf{o}_d be the origin in \mathbb{Z}^d . In the computations below we set $C_1 = \sum_{x \in \mathbb{Z}^d} \frac{b_0}{1+|x|_1^{d+1}}$. We have for all $\eta \in \mathcal{X}_v$

$$\begin{aligned}
LV(\eta) &= \sum_{x \in \mathbb{Z}^d} v(x) [b_0 + \sum_{y \in \mathbb{Z}^d} a_+(x-y)\eta(y)] \\
&\quad - \sum_{x \in \mathbb{Z}^d} v(x)\eta(x) \left\{ m + \sum_{y \in \mathbb{Z}^d} a_-(x-y)\eta(y) \right\} \\
&\leq C_1 + \sum_{y \in \mathbb{Z}^d} \eta(y) \sum_{x \in \mathbb{Z}^d} v(x) a_+(x-y) - a_-(\mathbf{o}_d) \sum_{x \in \mathbb{Z}^d} v(x)\eta^2(x) \\
&= C_1 + \sum_{y \in \mathbb{Z}^d} C_{v,a} v(y)\eta(y) - a_-(\mathbf{o}_d) \sum_{y \in \mathbb{Z}^d} v(y)\eta^2(y) \\
&\leq C_1 + \sum_{y \in \mathbb{Z}^d} v(y) [C_{v,a}\eta(y) - a_-(\mathbf{o}_d)\eta^2(y)]
\end{aligned}$$

There exist positive constants c_1 and c_2 such that

$$C_1 + \sum_{x \in \mathbb{Z}^d} v(x) [C_{v,a}\eta(x) - a_-(\mathbf{o}_d)\eta^2(x)] \leq c_1 - c_2 \sum_{x \in \mathbb{Z}^d} v(x)\eta(x),$$

thus the desired statement follows from Theorem 2.11.

6 Extinction and critical value for a model with independent branching birth rate and local death rate

In this section we consider the birth and death rates given by

$$b_\lambda(x, \eta) = \lambda \sum_{y: y \sim x} \eta(y), \quad d(x, \eta) = g(\eta(x)), \quad (47)$$

where $\lambda > 0$, $y \sim x$ means $|x - y| \leq 1$ and $g: \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing function such that $g(0) = 0$, $g(1) = 1$ and $g(n) \geq n$. For $g(n) = n^2$ the evolution of the process can be described as follows. Each particle is deleted from the system at a rate which is equal to the number of particles at the same site. Each particle gives birth to a new particle at a constant rate. The descendant appears at a site chosen uniformly among those neighboring to the predecessor sites and the site of the predecessor. We denote the unique solution of (2) by $(\eta_t^\lambda)_{t \in [0, \infty)}$, or simply (η_t) .

Let us consider equation

$$\begin{aligned} \xi_t(x) &= \int_{(0,t] \times [0,\infty)} I_{[0,b_{cont}(x,\xi_{s-})]}(u) N_1(x, ds, du) \\ &- \int_{(0,t] \times [0,\infty)} I_{[0,d(x,\xi_{r-})]}(v) N_2(x, dr, dv) + \xi_0(x), \end{aligned} \quad (48)$$

where $\xi_0(x) = \eta_0(x) \wedge 1$ and b_{cont} is given in (11). Equation (48) is of the form (2). The unique solution $(\xi_t^\lambda)_{t \in [0,T]}$ of (48) is in fact the contact process, see Remark 2.10.

Proposition 6.1. *Let $\lambda < \bar{\lambda}$. Then*

- (i) $\xi_t^\lambda \leq \eta_t^\lambda$ a.s. for all $t \geq 0$,
- (ii) $\eta_t^\lambda \leq \eta_t^{\bar{\lambda}}$ a.s. for all $t \geq 0$.

Proof. We saw in the proof of Proposition 3.8 that every solution is an a.s. limit of solutions with finite initial conditions. Therefore, this statement is a consequence of Proposition 3.3. \square

We recall that \mathbf{o}_d stands for the origin in \mathbb{Z}^d . Let $\eta_0(x) = I_{\{x=\mathbf{o}_d\}}$, and define

$$p_s(\lambda) = P\{\eta_t^\lambda \neq \mathbf{0} \text{ for all } t \geq 0\}.$$

From Proposition 6.1 it follows that p_s is a non-decreasing function of λ . A standard comparison with a subcritical branching process shows that $p_s(\lambda) = 0$ for sufficiently small λ , for example for $\lambda < \frac{1}{2d+1}$. On the other hand, comparison with the contact process demonstrates that $p_s(\lambda) > 0$ for $\lambda > \lambda_c^{cont}$, where λ_c^{cont} is a critical value of the contact process. Therefore, there exists a critical value:

$$\lambda_c = \inf\{\lambda > 0 : p_s(\lambda) > 0\}.$$

We summarize the above discussion in the following proposition.

Proposition 6.2. *Consider the unique solution to (2) with the birth and death rates (47) and the initial condition $\eta_0(x) = I_{\{x=\mathbf{o}_d\}}$. Then there exists $\lambda_c > 0$ such that*

- (i) *the process goes extinct if $\lambda < \lambda_c$:*

$$P\{\eta_t = \mathbf{0} \text{ for some } t \geq 0\} = 1,$$

- (ii) *the process survives with positive probability if $\lambda > \lambda_c$:*

$$P\{\eta_t \neq \mathbf{0} \text{ for all } t \geq 0\} > 0.$$

Acknowledgement

The authors acknowledge the financial support of the DFG through the SFB 701 “Spektrale Strukturen und Topologische Methoden in der Mathematik” (Bielefeld University). Viktor Bezborodov is also thankful the the support of the DFG through the IRTG (IGK) 1132 “Stochastics and Real World Models”. Viktor Bezborodov and Yuri Kondratiev would like to thank Errico Presutti for the fruitful discussions during their visit to L’Aquila.

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