ERGODICITY AND LOCAL LIMITS FOR STOCHASTIC LOCAL AND NONLOCAL p-LAPLACE EQUATIONS

BENJAMIN GESS

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany

JONAS M. TÖLLE

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany

Abstract. Ergodicity for local and nonlocal stochastic singular p-Laplace equations is proven, without restriction on the spatial dimension and for all $p \in [1,2)$. This generalizes previous results from [Gess, Tölle; JMPA, 2014], [Liu, Tölle; ECP, 2011], [Liu; JEE, 2009]. In particular, the results include the multivalued case of the stochastic (nonlocal) total variation flow, which solves an open problem raised in [Barbu, Da Prato, Röckner; SIAM, 2009]. Moreover, under appropriate rescaling, the convergence of the unique invariant measure for the nonlocal stochastic p-Laplace equation to the unique invariant measure of the local stochastic p-Laplace equation is proven.

1. Introduction

We consider stochastic nonlocal singular p-Laplace equations of the type

$$(1.1) \qquad \begin{aligned} dX_t &\in \left(\int_{\mathcal{O}} J(\cdot - \xi)|X_t(\xi) - X_t(\cdot)|^{p-2}(X_t(\xi) - X_t(\cdot))\, d\xi \right) dt + B dW_t \\ X_0 &= x_0, \end{aligned}$$

E-mail addresses: bgess@math.uni-bielefeld.de, jonasmtoelle@gmail.com.

Date: July 16, 2015.

2010 *Mathematics Subject Classification.* 35K55, 35K92, 60H15; 37L15, 45E10.

Key words and phrases. Stochastic variational inequality, nonlocal stochastic partial differential equations, singular-degenerate SPDE, stochastic p-Laplace equation, ergodicity.

J.M.T. gratefully acknowledges travel funds granted by the CRC 701 "Spectral Structures and Topological Methods in Mathematics" of the DFG (German Research Foundation).

and stochastic (local) singular p-Laplace equations of the type

$$(1.2)\qquad \begin{aligned} dX_t &\in \operatorname{div}\left(|\nabla X_t|^{p-2}\nabla X_t\right)\,dt + BdW_t \\ X_0 &= x_0, \end{aligned}$$

with zero Neumann boundary conditions on bounded, smooth domains $\mathcal{O} \subseteq \mathbb{R}^d$ with convex boundary $\partial\mathcal{O}$, mean zero initial conditions $x_0 \in H := L^2_{\mathrm{av}}(\mathcal{O})$ and $p \in [1,2)$. Here, W is a cylindrical Wiener process on H, $B \in L_2(H)$ is a symmetric Hilbert-Schmidt operator and $J : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, continuous, radial kernel with compact support and $J(0) > 0$. In particular, this includes the case of the stochastic total variation flow ($p = 1$) recently studied in [9,10].

Our results are twofold: First, we prove the existence and uniqueness of an invariant probability measure to (1.1) and (1.2). Second, the convergence of the respective invariant probability measures for (1.1) to the invariant probability measure for (1.2) is shown, under appropriate rescaling of the kernel J.

Uniqueness of invariant probability measures to (1.2) has been previously considered in [19,25,27]. The difficulties arising in proving uniqueness of invariant probability measures for (1.2) are due to the singular nature of the drift and the resulting low regularity properties of the solutions. More precisely, the energy space associated to (1.2) is given by $W^{1,p}_{\mathrm{av}}$, which is compactly embedded into L^2_{av} only if

$$(1.3)\qquad d < \frac{2p}{2-p}.$$

The validity of this embedding is crucial for previously established methods and thus (1.3) had to be assumed in all of the works [19,25,27], which led to stringent restrictions on the spatial dimension d, e.g. $d \leqslant 2$ for $p \approx 1$. For the case of nonlocal stochastic p-Laplace equations, the situation is even worse, since the energy associated to (1.1) is given by[1]

$$\varphi(u) = \frac{1}{2p}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)|u(\xi)-u(\zeta)|^p \, d\xi d\zeta$$

which is equivalent to the L^p norm. Hence, based on this no compactness and thus tightness for the laws of the solutions in L^2_{av} can be expected. These obstacles are overcome in the present work, by establishing a cascade of energy inequalities for L^m norms of the solutions to (1.1) and (1.2), for all $m \geqslant 2$. These new estimates are then used in order to prove concentration of mass of the solutions around zero, which in turn allows the application of results developed in [23], based on coupling techniques. In conclusion, we prove the existence and uniqueness of an invariant probability measure for (1.1) and (1.2) without any restriction on the dimension $d \in \mathbb{N}$ and for all $p \in [1,2)$. In particular, this solves the open question from [9] of uniqueness of invariant measures for the stochastic total variation flow.

In the second part of this paper, we consider the convergence of invariant probability measures under rescaling of the kernel J in (1.1). More precisely, we consider

$$(1.4)\qquad dX^\varepsilon_t = \left(\int_{\mathcal{O}} J^\varepsilon(\cdot-\xi)\,|X^\varepsilon_t(\xi)-X^\varepsilon_t(\cdot)|^{p-2}(X^\varepsilon_t(\xi)-X^\varepsilon_t(\cdot))d\xi\right)dt + BdW_t,$$

where $p \in (1,2)$ and[1]

$$J^\varepsilon(\xi) = \frac{1}{\varepsilon^{d+p}} J\left(\frac{\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^d$$

and prove that the corresponding invariant measures μ^ε converge weakly* to the invariant measure μ corresponding to (1.2). Somewhat related questions of convergence of invariant measures of (1.2) with respect to perturbations in p have

[1]For the sake of notational simplicity we drop normalization constants in the introduction.

been considered in [13, 14], under stringent restrictions on the spatial dimension, i.e. assuming (1.3). Again, such dimensional restrictions are crucial to the approach developed in [13,14], since the argument relies on tightness of the respective sequence of invariant probability measures μ^p, which in turn requires compactness of the embedding $W^{1,p} \hookrightarrow L^2$.
In the setting of local limits for (1.4) for general dimension d, this leads to two fundamental problems: First, no concentration of the invariant probability measures on some uniform, compactly embedded space can be expected. Second, as observed in [20], only weak convergence of the solutions to (1.1) to the solution to (1.2) is available, that is, $X_t^\varepsilon \rightharpoonup X_t$ in H for $\varepsilon \to 0$. Hence, we do not have the convergence of the associated Markovian semigroups $P_t^\varepsilon F$ for all $F \in \mathrm{Lip}_b(H)$, a crucial ingredient in previously developed methods such as in [13, 14]. These problems are overcome in the present work and we prove that μ^ε converges to μ in the topology of weak* convergence of measures on L^p_{av}, without any restriction on the spatial dimension d.

We note that, in general, the invariant measures μ^ε to (1.4) will only be concentrated on the domains of the corresponding energy functionals[1]

$$\varphi^\varepsilon(u) := \frac{1}{2p}\int_{\mathcal{O}}\int_{\mathcal{O}} J^\varepsilon(\xi-\zeta)\,|u(\zeta)-u(\xi)|^p\, d\zeta d\xi,$$

rather than on $W^{1,p}_{\mathrm{av}}$ as for (1.2). Roughly speaking, one has $\varphi^\varepsilon(u) \uparrow \|u\|_{W^{1,p}}$. In this sense, at least *asymptotic* concentration on $W^{1,p}_{\mathrm{av}}$ is still satisfied. This is reflected in our proof by working with *asymptotic* tightness rather than tightness. Non-compactness of $W^{1,p}_{\mathrm{av}}$ in L^2_{av} is dealt with by considering weak* convergence of measures on L^p_{av} rather than on L^2_{av}. However, this leads to the further difficulty of working with two topologies: weak* convergence of μ^ε on L^p_{av} and weak convergence of X_t^ε on L^2_{av}. These issues are resolved by a careful treatment in Section 5 below.

A detailed treatment of deterministic nonlocal p-Laplace equations may be found in [1–4] and the references therein. Relying on non-degeneracy assumptions on the noise, gradient estimates, Harnack inequalities and exponential convergence rates for stochastic p-Laplace equations and stochastic porous media equations have been obtained in [25, 37, 38] and the references therein. Ergodicity for stochastic fast diffusion equations has been considered in [8, 19, 26–28, 37, 38]. The case of stochastic degenerate p-Laplace equations, that is for $p > 2$, has been investigated in [7,24,30,32,36–38] and ergodicity for stochastic porous media equations has been shown in [6, 7, 15, 16, 21, 22, 24, 31, 32, 35, 36].

1.1. **Structure of the paper.** In Section 2 ergodicity for the stochastic nonlocal p-Laplace equation is proven. The case of the stochastic local p-Laplace equation is treated in Section 3. Convergence of the solutions of the nonlocal stochastic p-Laplace equation to its local version is shown in Section 4. The respective convergence of invariant probability measures is shown in Section 5. For notations see Appendix A.

2. Ergodicity for stochastic nonlocal p-Laplace equations

In this section we derive an SVI formulation for stochastic singular nonlocal p-Laplace equations with homogeneous Neumann boundary condition of the type

$$\begin{aligned} dX_t &\in \left(\int_{\mathcal{O}} J(\cdot-\xi)|X_t(\xi)-X_t(\cdot)|^{p-2}(X_t(\xi)-X_t(\cdot))d\xi\right)dt + BdW_t \\ X_0 &= x_0 \in L^2(\Omega,\mathcal{F}_0;L^2_{\mathrm{av}}(\mathcal{O})), \end{aligned} \tag{2.1}$$

where $p \in [1,2)$ and $\mathcal{O}$ is a bounded, smooth domain in $\mathbb{R}^d$. The kernel $J : \mathbb{R}^d \to \mathbb{R}$ is supposed to be a nonnegative, continuous, radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^d} J(z)\,dz = 1$. Furthermore, W is a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $H = L^2_{\mathrm{av}}(\mathcal{O})$. Hence,

$$W_t^B := BW_t$$

is a trace-class Wiener process in H. We further assume that there is an orthonormal basis e_k of H such that

$$\sum_{k=1}^{\infty} \|Be_k\|_\infty^2 < \infty, \tag{2.2}$$

cf. e.g. [9] where similar conditions on B have been used in the case of the stochastic total variation flow. For $u \in L^p(\mathcal{O})$ we set

$$\varphi(u) := \frac{1}{2p} \int_{\mathcal{O}} \int_{\mathcal{O}} J(\zeta - \xi)|u(\xi) - u(\zeta)|^p \, d\xi d\zeta$$

and obtain, if $p > 1$,

$$A(u) := -\partial_{L^2}\varphi(u) = \int_{\mathcal{O}} J(\cdot - \xi)|u(\xi) - u(\cdot)|^{p-2}(u(\xi) - u(\cdot))\,d\xi$$

and, if $p = 1$,

$$\begin{aligned} A(u) :=& -\partial_{L^2}\varphi(u) \\ =& \Big\{ \int_{\mathcal{O}} J(\cdot - \xi)\eta(\xi, \cdot)\,d\xi : \|\eta\|_{L^\infty} \leqslant 1,\ \eta(\xi,\zeta) = -\eta(\zeta,\xi) \text{ and} \\ & \quad J(\zeta - \xi)\eta(\xi,\zeta) \in J(\zeta - \xi)\,\mathrm{sgn}(u(\xi) - u(\zeta)) \text{ for a.e. } (\xi,\zeta) \in \mathcal{O} \times \mathcal{O} \Big\}, \end{aligned}$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. We note that A defines a continuous, monotone operator on H, satisfying

$$\|A(u)\|_H^2 \lesssim 1 + \|u\|_H^2 \quad \forall u \in H. \tag{2.3}$$

Hence, we can write (2.1) in its relaxed form

$$dX_t \in -\partial\varphi(X_t)dt + BdW_t.$$

Existence and uniqueness of an SVI solution $X = X^{x_0} \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$ to (2.1) has been proven in [20, Section 4] and

$$\mathbb{E}\|X_t^x - X_t^y\|_{L^2}^2 \lesssim \|x - y\|_{L^2}^2. \tag{2.4}$$

Since $\int_{\mathcal{O}} \eta d\zeta = 0$ for all $\eta \in A(u)$, $u \in H$, from the construction of SVI solutions presented in [20, Section 4] it easily follows that the average value is preserved, that is,

$$X_t \in L^2_{\mathrm{av}}(\mathcal{O}) \quad \forall t \geqslant 0, \mathbb{P}\text{-a.s.} \tag{2.5}$$

if $x_0 \in L^2(\Omega; L^2_{\mathrm{av}}(\mathcal{O}))$. Furthermore, analogously to [19, proof of Proposition 5.2], it follows that

$$P_t F(x) := \mathbb{E}F(X_t^x) \quad \text{for } F \in \mathcal{B}_b(H)$$

defines a Feller semigroup on $\mathcal{B}_b(H)$. As a main result in this Section we obtain

Theorem 2.1. *There is a unique invariant measure μ for P_t satisfying*[2]

$$0 \in \ supp\ (\mu) \subseteq \mathcal{T}(\mu)$$

and

$$\int_H \varphi(x) d\mu(x) \lesssim \|B\|_{L_2(H)}^2. \tag{2.6}$$

[2]Cf. Appendix A for notation.

We first need to derive suitable a-priori bounds on general L^m norms of the solutions.

Lemma 2.2. *Let $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, $m \in [2, \infty)$ and let X be the corresponding SVI solution to (2.1). Then there are $c = c(m), C = C(m) > 0$ such that*

$$\frac{1}{m}\mathbb{E}\|X_t\|_m^m + c\mathbb{E}\int_0^t \|X_r\|_{p+m-2}^{p+m-2} dr \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + tC \quad \forall t \geqslant 0. \tag{2.7}$$

If $B \equiv 0$ then we can choose $C = 0$.

Proof. For notational convenience let

$$\begin{aligned}\psi(\xi) :=& \frac{1}{p}|\xi|^p \\ \phi(\xi) :=& \partial\psi(\xi) = |\xi|^{p-2}\xi, \quad \xi \in \mathbb{R}^d.\end{aligned}$$

Step 1: We start by proving that for $x \in L^m(\Omega, \mathcal{F}_0; L^m_{\mathrm{av}}(\mathcal{O}))$, we have $\mathbb{E}\|X_t\|^m_{L^m} < \infty$ for all $t \geqslant 0$.

We aim to apply Itô's formula for $\frac{1}{m}\|\cdot\|_m^m$. To do so, we need to consider appropriate approximations. Let

$$\iota^\alpha(r) := \frac{1}{m}\begin{cases}|r|^m & \text{if } |r| \leqslant \frac{1}{\alpha} \\ |\frac{1}{\alpha}|^m + \frac{m}{\alpha^{m-1}}(r - \frac{1}{\alpha}) + \frac{m(m-1)}{2\alpha^{m-2}}(r - \frac{1}{\alpha})^2 & \text{if } r \geqslant \frac{1}{\alpha} \\ |\frac{1}{\alpha}|^m + \frac{m}{\alpha^{m-1}}(r + \frac{1}{\alpha}) + \frac{m(m-1)}{2\alpha^{m-2}}(r + \frac{1}{\alpha})^2 & \text{if } r \leqslant -\frac{1}{\alpha}.\end{cases}$$

and observe, for α small enough,

$$\begin{aligned}(\iota^\alpha)''(r) :=& \begin{cases}(m-1)|r|^{m-2} & \text{if } |r| \leqslant \frac{1}{\alpha} \\ \frac{m-1}{|\alpha|^{m-2}} & \text{otherwise.}\end{cases} \\ \lesssim& 1 + \iota^\alpha(r).\end{aligned} \tag{2.8}$$

Let θ^β be a standard Dirac sequence on $\mathbb{R}^d$. For $v \in L^2(\mathcal{O})$ we set[3]

$$\begin{aligned}\eta^\alpha(v) :=& \int_{\mathcal{O}} \iota^\alpha(v)\, d\zeta \\ \eta^{\alpha,\beta}(v) :=& \int_{\mathcal{O}} \iota^\alpha\left(\theta^\beta * \bar{v}\right) d\zeta\end{aligned}$$

and observe that $\eta^{\alpha,\beta} \in C^2(L^2)$ with uniformly continuous derivatives on bounded sets. We recall that the SVI solution X to (2.1) has been constructed in [20, Section 4] as a limit in $L^2(\Omega; C([0,T]; H))$ of (strong) solutions X^δ corresponding to the approximating SPDE

$$dX_t^\delta = \left(\int_{\mathcal{O}} J(\cdot - \xi)\phi^\delta(X_t^\delta(\xi) - X_t^\delta(\cdot))d\xi\right) dt + BdW_t,$$

where ψ^δ is the Moreau-Yosida approximation (cf. e.g. [5]) of $\psi(\cdot) = \frac{1}{p}|\cdot|^p$ and $\phi^\delta := \partial\psi^\delta$. Hence, by Itô's formula

$$\begin{aligned}\mathbb{E}\eta^{\alpha,\beta}(X_t^\delta) =& \mathbb{E}\eta^{\alpha,\beta}(x_0) + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(\theta^\beta * X_r^\delta)(\theta^\beta * A^\delta(X_r^\delta))d\zeta dr \\ &+ \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(\theta^\beta * X_r^\delta)(\theta^\beta * Be_k)^2 d\zeta dr.\end{aligned} \tag{2.9}$$

[3]Cf. Appendix A for notations.

Using (for $\alpha > 0$ fixed)

$$(\iota^\alpha)'(r) \lesssim 1 + |r|$$
$$(\iota^\alpha)''(r) \lesssim 1 \quad \forall r \in \mathbb{R},$$

and dominated convergence, we may let $\beta \to 0$ in (2.9) to obtain that

$$\mathbb{E}\eta^\alpha(X_t^\delta) \leqslant \mathbb{E}\eta^\alpha(x_0) + \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)'(X_r^\delta) A^\delta(X_r^\delta) d\zeta dr + \sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} (\iota^\alpha)''(X_r^\delta)(Be_k)^2 d\zeta dr. \tag{2.10}$$

We note that by [4, Lemma 6.5] and monotonicity of $(\iota^\alpha)'$ we have that

$$\begin{aligned}
&\int_{\mathcal{O}} (\iota^\alpha)'(v) A^\delta(v) d\zeta \\
&= \int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(v(\xi)-v(\zeta))(\iota^\alpha)'(v(\zeta)) d\xi d\zeta \\
&= -\frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(v(\xi)-v(\zeta))((\iota^\alpha)'(v(\zeta)) - (\iota^\alpha)'(v(\zeta))) d\xi d\zeta \\
&\leqslant 0,
\end{aligned}$$

for all $v \in H$. Hence, using (2.8) we observe that

$$\mathbb{E}\eta^\alpha(X_t^\delta) \leqslant \mathbb{E}\eta^\alpha(x_0) + C\mathbb{E}\int_0^t (1 + \eta^\alpha(X_r^\delta)) dr.$$

Gronwall's Lemma then implies that

$$\begin{aligned}
\mathbb{E}\eta^\alpha(X_t^\delta) &\lesssim \mathbb{E}\eta^\alpha(x_0) + 1 \\
&\leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1.
\end{aligned}$$

Hence, taking $\alpha \to 0$ and using Fatou's Lemma we obtain that

$$\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \lesssim \frac{1}{m}\mathbb{E}\|x_0\|_m^m + 1. \tag{2.11}$$

Taking $\delta \to 0$ finishes the proof.

Step 2: We first note that it is enough to prove (2.7) for $x \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\text{av}}(\mathcal{O}))$. Due to (2.4) the case of $x \in L^m(\Omega, \mathcal{F}_0; L^m_{\text{av}}(\mathcal{O}))$ can then be concluded by approximation and Fatou's Lemma. Hence, assume $x \in L^\infty(\Omega, \mathcal{F}_0; L^\infty_{\text{av}}(\mathcal{O}))$ from now on. By step one we have $\mathbb{E}\|X_t\|_m^m < \infty$ for all $t \geqslant 0$, $m \in \mathbb{N}$.

Letting $\alpha \to 0$ in (2.10), using dominated convergence and (2.3), we obtain

$$\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + \mathbb{E}\int_0^t \int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta dr + (m-1)\sum_{k=1}^\infty \mathbb{E}\int_0^t \int_{\mathcal{O}} |X_r^\delta|^{m-2}(Be_k)^2 d\zeta dr.$$

Using [4, Lemma 6.5] we observe that

$$\begin{aligned}
&\int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta \\
&= \int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(X_r^\delta(\xi) - X_r^\delta(\zeta))(X_r^\delta)^{[m-1]}(\zeta) d\xi d\zeta \\
&= -\frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)\phi^\delta(X_r^\delta(\xi) - X_r^\delta(\zeta))((X_r^\delta)^{[m-1]}(\xi) - (X_r^\delta)^{[m-1]}(\zeta)) d\xi d\zeta.
\end{aligned}$$

From [16], for every $m \in [2,\infty)$ there is a $c>0$ such that

$$(a-b)(a^{[m-1]}-b^{[m-1]}) \geqslant c|a-b|^m \quad \forall a,b \in \mathbb{R}.$$

Since $\operatorname{sgn}(\phi^\delta(a-b)) = \operatorname{sgn}(a-b)$ and (cf. [20, Appendix A])

$$\begin{aligned}\phi^\delta(a)a &\geqslant \psi^\delta(a)\\ &\geqslant c\psi(a) - C\delta\end{aligned}$$

for some $c>0$, $C>0$ and $\delta>0$ small enough, this yields

$$\begin{aligned}\phi^\delta(a-b)(a^{[m-1]}-b^{[m-1]}) &\geqslant c\phi^\delta(a-b)(a-b)|a-b|^{m-2}\\ &\geqslant c\psi^\delta(a-b)|a-b|^{m-2}\\ &\geqslant c|a-b|^{p+m-2} - C\delta|a-b|^{m-2} \quad \forall a,b\in\mathbb{R}.\end{aligned}$$

Using this and the Poincaré type inequality [4, Proposition 6.19] in combination with (2.5) we get

$$\begin{aligned}&\mathbb{E}\int_{\mathcal{O}} (X_r^\delta)^{[m-1]} A^\delta(X_r^\delta) d\zeta\\ &\leqslant -\mathbb{E}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)(c|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{p+m-2} - C\delta|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{m-2})d\zeta d\xi\\ &\leqslant -c\mathbb{E}\|X_r^\delta\|_{p+m-2}^{p+m-2} + C\delta\mathbb{E}\int_{\mathcal{O}}\int_{\mathcal{O}} J(\zeta-\xi)|X_r^\delta(\xi)-X_r^\delta(\zeta)|^{m-2}d\zeta d\xi\\ &\leqslant -c\mathbb{E}\|X_r^\delta\|_{p+m-2}^{p+m-2} + C\delta(\mathbb{E}\|X_r^\delta\|_m^m + 1),\end{aligned}$$

for some $c>0$. Now, using (2.2) we obtain that

$$\sum_{k=1}^\infty \int_{\mathcal{O}} |X_t^\delta|^{m-2}(Be_k)^2 d\zeta \leqslant \varepsilon\|X_t^\delta\|_{p+m-2}^{p+m-2} + C_\varepsilon,$$

for all $\varepsilon>0$ and some $C_\varepsilon \geqslant 0$. Choosing $\varepsilon,\delta>0$ small enough, we conclude that

$$\frac{1}{m}\mathbb{E}\|X_t^\delta\|_m^m \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m - c\mathbb{E}\int_0^t \|X_r^\delta\|_{p+m-2}^{p+m-2}dr + C\delta\mathbb{E}\int_0^t \|X_r^\delta\|_m^m dr + tC.$$

Letting $\delta\to 0$ concludes the proof. □

We next analyze the deterministic situation, i.e. $B=0$ in (2.1). Let u be the unique SVI solution to

$$\begin{aligned}du_t &\in \left(\int_{\mathcal{O}} J(\cdot-\xi)|u_t(\xi)-u_t(\cdot)|^{p-2}(u_t(\xi)-u_t(\cdot))d\xi\right)dt\\ u_0 &= x_0 \in H.\end{aligned} \tag{2.12}$$

Lemma 2.3. *Let $m_0 = 4-p \in (2,3]$, $x_0 \in L^{m_0}_{av}(\mathcal{O}) \subseteq H$ and u be the corresponding SVI solution to* (2.12)*. Then there is a $C>0$ such that*

$$\|u_t\|_2^2 \leqslant \frac{C}{t}\|x_0\|_{m_0}^{m_0}.$$

Proof. From Lemma 2.2 we know that

$$\begin{aligned}\frac{1}{m_0}\|u_t\|_{m_0}^{m_0} &\leqslant \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - c\int_0^t \|u_r\|_{p+m_0-2}^{p+m_0-2}dr\\ &= \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - c\int_0^t \|u_r\|_2^2 dr\end{aligned}$$

In particular, $t \mapsto \|u_t\|_{m_0}^{m_0}$ is non-increasing. Using that also $t \mapsto \|u_t\|_2^2$ is non-increasing yields

$$\frac{1}{m_0}\|u_t\|_{m_0}^{m_0} \leqslant \frac{1}{m_0}\|x_0\|_{m_0}^{m_0} - ct\|u_t\|_2^2.$$

Hence,

$$\|u_t\|_2^2 \leqslant \frac{C}{t}\|x_0\|_{m_0}^{m_0}.$$

□

Next we prove concentration on bounded L^{m_0} sets for sufficiently regular initial conditions.

Lemma 2.4. *Let $\varepsilon > 0$ and $x \in L^{m_1}_{av}(\mathcal{O}) \subseteq H$ with $m_1 = m_0 + 2 - p \in (2, 4]$, $m_0 = 4 - p \in (2, 3]$. Then there is an $R = R(\varepsilon) > 0$ such that*

$$Q_T(x, B_R^{m_0}(0)) \geqslant 1 - \varepsilon$$

for all $T \geqslant 1$.

Proof. By Lemma 2.2 we have

$$\frac{1}{t}\mathbb{E}\|X_t^x\|_{m_1}^{m_1} + c\mathbb{E}\frac{1}{t}\int_0^t \|X_r^x\|_{m_0}^{m_0} dr \leqslant \frac{1}{t}\mathbb{E}\|x\|_{m_1}^{m_1} + C.$$

Thus, for $T \geqslant 1$,

$$\begin{aligned} Q_T(x, B_R^{m_0}(0)) &= \frac{1}{T}\int_0^T P_r(x, B_R^{m_0}(0))dr \\ &\geqslant \frac{1}{T}\int_0^T \left(1 - \frac{\mathbb{E}\|X_r^x\|_{m_0}^{m_0}}{R}\right) dr \\ &= 1 - \frac{1}{R}\frac{1}{T}\int_0^T \mathbb{E}\|X_r^x\|_{m_0}^{m_0} dr \\ &\geqslant 1 - \frac{C}{RT}\mathbb{E}\|x\|_{m_1}^{m_1} - \frac{C}{R}. \end{aligned}$$

Choosing R large enough yields the claim. □

Lemma 2.5. *For each $T \geqslant 0$, $\eta > 0$ we have*

$$\inf_{x \in B} \mathbb{P}\big(\sup_{t \in [0,T]} \|X_t^x - u_t^x\|_H^2 \leqslant \eta\big) > 0$$

for all bounded sets $B \subseteq H$.

Proof. We consider $Y_t^\delta := X_t^\delta - W_t^B$ which satisfies

$$\begin{aligned} \frac{d}{dt}Y_t^\delta &= A^\delta(Y_t^\delta + W_t^B)dt \\ Y_0^\delta &= x_0. \end{aligned}$$

Accordingly, let u^δ be the unique solution to (2.12) with $\phi(z) = |z|^{p-2}z$ replaced by ϕ^δ. Then,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|Y_t^\delta\|_H^2 &= (Y_t^\delta, A^\delta(Y_t^\delta + W_t^B))_H \\ &\leqslant \|Y_t^\delta\|_H^2 + C\|W_t^B\|_H^2. \end{aligned}$$

Thus,

$$\sup_{t \in [0,T]} \|Y_t^\delta\|_H^2 \lesssim 1 + \|x_0\|_H^2.$$

Similarly,

$$\sup_{t\in[0,T]} \|u_t^\delta\|_H^2 \lesssim 1 + \|x_0\|_H^2.$$

Moreover,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Y_t^\delta - u_t^\delta\|_H^2 &= (Y_t^\delta - u_t^\delta, A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta))_H \\ &\leqslant -(W_t^B, A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta))_H \\ &\leqslant \|W_t^B\|_H \|A^\delta(Y_t^\delta + W_t^B) - A^\delta(u_t^\delta)\|_H \\ &\leqslant C\|W_t^B\|_H(\|Y_t^\delta\|_H + \|W_t^B\|_H + \|u_t^\delta\|_H) \\ &\leqslant C\|W_t^B\|_H(\|x_0\|_H + \|W_t^B\|_H + 1).\end{aligned}$$

Since W^B is a trace class Wiener process in H, for each $\eta \in (0,1], T > 0$ we can find a subset $\Omega_\eta \subseteq \Omega$ of positive mass such that $\sup_{t\in[0,T]} \|W_t^B(\omega)\|_H < \eta$ for all $\omega \in \Omega_\eta$. For $\omega \in \Omega_\eta$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|Y_t^\delta - u_t^\delta\|_H^2 \leqslant C\eta(\|x_0\|_H + 1).$$

Choosing $\eta > 0$ small enough and letting $\delta \to 0$ yields the claim. □

Lemma 2.6. *Let $\varepsilon > 0$, $x \in L^{m_1}(\mathcal{O})$ with m_1 as before and $\delta > 0$. Then*

$$\liminf_{T\to\infty} Q_T(x, B_\delta(0)) > 0.$$

Proof. By Lemma 2.4 there is an $R > 0$ such that

$$Q_T(x, B_R^{m_0}(0)) \geqslant \frac{1}{2}$$

for all $T \geqslant 1$. Moreover, by Lemma 2.3 we have

$$\|u_t^x\|_2^2 \leqslant \frac{C}{t}\|x\|_{m_0}^{m_0} \leqslant \frac{C}{t}R$$

for all $x \in B_R^{m_0}(0)$ and thus there is a $T_0 = T_0(R,\delta)$ such that

$$\|u_t^x\|_2^2 \leqslant \frac{\delta}{2},$$

for all $t \geqslant T_0$. Using Lemma 2.5 we observe

$$P_{T_0}(x, B_\delta(0)) = P(\|X_{T_0}^x\|_H \leqslant \delta) \geqslant P(\|X_{T_0}^x - u_{T_0}^x\|_H \leqslant \frac{\delta}{2}) \geqslant \gamma > 0$$

for some $\gamma = \gamma(\delta, T_0) > 0$ and all $x \in B_R^{m_0}(0)$. Thus,

$$\begin{aligned}\liminf_{T\to\infty} Q_T(x, B_\delta(0)) &= \liminf_{T\to\infty} \frac{1}{T}\int_0^T P_s(x, B_\delta(0))ds \\ &= \liminf_{T\to\infty} \frac{1}{T}\int_0^T P_{s+T_0}(x, B_\delta(0))ds \\ &= \liminf_{T\to\infty} \frac{1}{T}\int_0^T \int_H P_s(x,dz)P_{T_0}(z, B_\delta(0))ds \\ &\geqslant \liminf_{T\to\infty} \frac{1}{T}\int_0^T \int_{B_R^{m_0}(0)} P_s(x,dz)P_{T_0}(z, B_\delta(0))ds \\ &\geqslant \gamma \liminf_{T\to\infty} Q_T(x, B_R^{m_0}(0)) \\ &\geqslant \frac{\gamma}{2} > 0.\end{aligned}$$

□

Lemma 2.7. *Let X, Y be two SVI solutions to (2.1) with initial conditions $x_0, y_0 \in L^2(\Omega, \mathcal{F}_0; L^m_{\text{av}}(\mathcal{O}))$ respectively. Then, for all $m \geqslant 1$,*

$$\|X_t - Y_t\|_{L^m(\mathcal{O})} \leqslant \|x_0 - y_0\|_{L^m(\mathcal{O})} \quad \mathbb{P}-\text{a.s.}, \forall t \geqslant 0.$$

In particular, the semigroup P_t satisfies the e-property on H.

Proof. We have that

$$d(X_t^\delta - Y_t^\delta) = (A^\delta(X_t^\delta) - A^\delta(Y_t^\delta))dt$$

and thus $t \mapsto (X_t^\delta - Y_t^\delta) \in W^{1,2}([0,T]; H)$. Let ι^α be the Moreau-Yosida approximation of $\frac{1}{m}|\cdot|^m$ and

$$\eta^\alpha(v) := \int_{\mathcal{O}} \iota^\alpha(v) d\zeta$$

for $v \in H$. By [33, Lemma IV.4.3] we obtain that

$$\frac{d}{dt}\eta^\alpha(X_t^\delta - Y_t^\delta) = (\gamma_t^\alpha, A^\delta(X_t^\delta) - A^\delta(Y_t^\delta))_H, \text{ for a.e. } t \in [0,T],$$

where $\gamma_t^\alpha := (\iota^\alpha)'(X_t^\delta - Y_t^\delta) \in L^2([0,T]; H)$. Using [4, Lemma 6.6] we conclude that, $\mathbb{P}$-a.s.,

$$\frac{d}{dt}\eta^\alpha(X_t^\delta - Y_t^\delta) \leqslant 0.$$

Letting $\alpha \to 0$, then $\delta \to 0$ concludes the proof. □

Proof of Theorem 2.1: Step 1: Existence and uniqueness of invariant measures

The proof relies on an application of [23, Theorem 1]. Let $x \in H$ and $\delta > 0$. Then we may choose $y \in L^{m_1}_{av}(\mathcal{O})$, with m_1 as in Lemma 2.4, such that

$$\|x - y\|_H^2 \leqslant \frac{\delta}{2}.$$

By Lemma 2.7 we then have

$$\|X_t^x - X_t^y\|_H^2 \leqslant \frac{\delta}{2} \quad \text{for all } t \geqslant 0. \tag{2.13}$$

Lemma 2.6 yields

$$\liminf_{T\to\infty} Q_T(y, B_{\frac{\delta}{2}}(0)) > 0.$$

Due to (2.13) we conclude

$$\liminf_{T\to\infty} Q_T(x, B_\delta(0)) \geqslant \liminf_{T\to\infty} Q_T(y, B_{\frac{\delta}{2}}(0)) > 0. \tag{2.14}$$

An application of [23, Theorem 1] implies that P_t has a unique invariant probability measure μ.

Step 2: We first note that for all $x \in H$ such that $\{Q_T(x,\cdot)\}_{T\geqslant T_0}$ is tight for some $T_0 \geqslant 0$, we have that

$$Q_T(x,\cdot) \rightharpoonup^* \mu \quad \text{for } T \to \infty$$

by uniqueness of the invariant measure μ. Hence,

$$\mathcal{T}(\mu) = \left\{x \in H : \{Q_T(x,\cdot)\}_{T\geqslant T_0} \text{ is tight for some } T_0 \geqslant 0\right\}.$$

By [23, Proposition 1] we have that

$$\text{supp}\, \mu \subseteq \mathcal{T}(\mu).$$

Moreover, using invariance of μ, Fatou's Lemma and (2.14) we note that

$$\begin{aligned}\mu(B_\delta(0)) &= \liminf_{T\to\infty} Q_T\mu(B_\delta(0))\\ &= \liminf_{T\to\infty} \int_H Q_T(x, B_\delta(0))d\mu(x)\\ &\geqslant \int_H \liminf_{T\to\infty} Q_T(x, B_\delta(0))d\mu(x)\\ &> 0\end{aligned}$$

for all $\delta > 0$. Hence,

$$0 \in \text{ supp } \mu.$$

Step 3: An application of Itô's formula yields

$$\mathbb{E}\|X_t^\delta\|_H^2 \leqslant 2\mathbb{E}\int_0^t (A^\delta(X_r^\delta), X_r^\delta)_H dr + t\|B\|_{L_2(H)}^2.$$

By [4, Lemma 6.5] we have

$$2(A^\delta(v), v)_H = -p\varphi^\delta(v)$$

and thus

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi^\delta(X_r^\delta)dr \leqslant \|B\|_{L_2(H)}^2,$$

for some $c > 0$. Since, by [20, Appendix A],

$$|\varphi^\delta(v) - \varphi(v)| \leqslant C\delta(1 + \|v\|_H^2) \quad \forall v \in H$$

we obtain that

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi(X_r^\delta)dr \leqslant \|B\|_{L_2(H)}^2 + \frac{C\delta}{t}\mathbb{E}\int_0^t (\|X_r^\delta\|_H^2 + 1)dr.$$

Letting $\delta \to 0$ yields

$$\frac{c}{t}\mathbb{E}\int_0^t \varphi(X_r)dr \leqslant \|B\|_{L_2(H)}^2.$$

Since $0 \in \mathcal{T}(\mu)$ this is easily seen to imply (2.6). □

3. Ergodicity for stochastic local p-Laplace equations

In this section we consider stochastic singular p-Laplace equations with additive noise, that is,

$$\begin{aligned} dX_t &\in \operatorname{div}\left(|\nabla X_t|^{p-2}\nabla X_t\right)\, dt + BdW_t,\\ |\nabla X_t|^{p-2}\nabla X_t \cdot \nu &\ni 0 \quad \text{on } \partial\mathcal{O},\ t > 0,\\ X_0 &= x_0 \in L^2(\Omega, \mathcal{F}_0; L^2_{\text{av}}(\mathcal{O})),\end{aligned} \tag{3.1}$$

with $p \in [1, 2)$ on a bounded, smooth domain $\mathcal{O} \subseteq \mathbb{R}^d$ with convex boundary $\partial\mathcal{O}$. In the following we set $H := L^2_{\text{av}}(\mathcal{O})$ and $S := H^1_{\text{av}}(\mathcal{O})$. Here, W is a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $B \in L_2(H, H^3_{\text{av}})$. Hence,

$$W_t^B := BW_t$$

is a trace-class Wiener process in $H^3_{\text{av}} \subseteq H$. As in Section 2, we further assume that there is an orthonormal basis e_k of H such that

$$\sum_{k=1}^\infty \|Be_k\|_\infty^2 < \infty, \tag{3.2}$$

cf. [9] where similar conditions on B have been used in the case $p = 1$. We define, for $p \in (1, 2)$,

$$\varphi(v) := \begin{cases} \frac{1}{p} \int_{\mathcal{O}} |\nabla u|^p \, d\xi & \text{if } v \in W^{1,p}(\mathcal{O}) \\ +\infty & \text{if } v \in L^p(\mathcal{O}) \setminus W^{1,p}(\mathcal{O}) \end{cases}$$

and for $p = 1$,

$$\varphi(v) := \begin{cases} \|v\|_{TV} & \text{if } v \in BV(\mathcal{O}) \\ +\infty & \text{if } v \in L^1(\mathcal{O}) \setminus BV(\mathcal{O}). \end{cases}$$

Then (3.1) can be recast in its relaxed form

$$dX_t \in -\partial_{L^2}\varphi(X_t)dt + BdW_t,$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. In [19, Section 7.2.2] the existence and uniqueness of a (limit) solution $X = X^{x_0}$ to (3.1) has been proven and

$$\|X_t^x - X_t^y\|_H^2 \leqslant \|x - y\|_H^2 \quad \forall t \geqslant 0, \ \mathbb{P}\text{-a.s.}. \tag{3.3}$$

Following [19, Appendix C] it is easy to see that X also is an SVI solution to (3.1), which by [20, Section 3] is unique. From the construction of X it is easy to see that the average value is preserved, that is,

$$X_t \in L^2_{\text{av}}(\mathcal{O}) \quad \forall t \geqslant 0, \ \mathbb{P}\text{-a.s.} \tag{3.4}$$

if $x_0 \in L^2(\Omega, \mathcal{F}_0; L^2_{\text{av}}(\mathcal{O}))$. Moreover, by [19, Proposition 5.2],

$$P_t F(x) := \mathbb{E}F(X_t^x) \quad \text{for } F \in \mathcal{B}_b(H)$$

defines a Feller semigroup on $\mathcal{B}_b(H)$. By (3.3), P_t satisfies the e-property on H. As a main result in this section we obtain

Theorem 3.1. *There is a unique invariant measure μ for P_t, which satisfies*

$$0 \in \ supp\ (\mu) \subseteq \mathcal{T}(\mu)$$

and

$$\int_H \varphi(x) d\mu(x) \lesssim \|B\|^2_{L_2(H)}.$$

The proof of Theorem 3.1 proceeds along the same principal ideas as Theorem 2.1. However, due to the local nature of (3.1) different arguments have to be used in order to deduce the cascade of L^m inequalities (cf. Lemma 3.2 below). Once, these inequalities have been shown for (3.1), the proof can be concluded essentially as in Section 2.

Lemma 3.2. *Let $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\text{av}}(\mathcal{O}))$, $m \in [2, \infty)$ and let X be the corresponding SVI solution to (2.1). Then there is a constant $c = c(p, m) > 0$ such that*

$$\frac{1}{m}\mathbb{E}\|X_t\|_m^m + c\mathbb{E}\int_0^t \|X_r\|_{p+m-2}^{p+m-2} dr \leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + tC \quad \forall t \geqslant 0. \tag{3.5}$$

Proof. Step 1: We start by proving that for $x_0 \in L^m(\Omega, \mathcal{F}_0; L^m_{\text{av}}(\mathcal{O}))$ we have $\mathbb{E}\|X_t\|_{L^m}^m < \infty$ for all $t \geqslant 0$.

In the following let $\psi(\cdot) = \frac{1}{p}|\cdot|^p$, $\phi := \partial\psi$, ψ^δ be the Moreau-Yosida approximation of ψ and $\phi^\delta := \partial\psi^\delta$. Recall that the unique SVI solution X to (2.1) has been constructed in [20, Theorem 4.1] as a limit of approximating solutions $X^{\varepsilon,\delta,n}$ to

$$\begin{aligned} dX_t^{\varepsilon,\delta,n} &= \varepsilon\Delta X_t^{\varepsilon,\delta,n}\, dt + \operatorname{div} \phi^\delta \left(\nabla X_t^{\varepsilon,\delta,n}\right) dt + BdW_t, \\ X_0^{\varepsilon,\delta,n} &= x_0^n, \end{aligned} \tag{3.6}$$

with zero Neumann boundary conditions, where $x_0^n \to x_0$ in $L^2(\Omega;H)$ with $x_0^n \in L^2(\Omega,\mathcal{F}_0;H^1_{\mathrm{av}})$ and $\varepsilon>0$. From [20, Equation (3.6)] we recall the bound

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^{\varepsilon,\delta,n}\|_{H^1}^2+2\varepsilon\mathbb{E}\int_0^T\|\Delta X_r^{\varepsilon,\delta,n}\|_2^2dr\leqslant C(\mathbb{E}\|x_0^n\|_{H^1}^2+1),\tag{3.7}$$

with a constant $C>0$ independent of ε, δ and n. In [20, proof of Theorem 3.1] the following subsequent convergence has been shown in $L^2(\Omega;C([0,T];H))$

$$\begin{aligned}X^{\varepsilon,\delta,n}&\to X^{\varepsilon,n}\quad\text{for }\delta\to0\\ X^{\varepsilon,n}&\to X^{n}\quad\text{for }\varepsilon\to0\\ X^{n}&\to X\quad\text{for }n\to\infty.\end{aligned}\tag{3.8}$$

Let $\iota^\alpha,\theta^\beta$ and $\eta^{\alpha,\beta}$ be as in the proof of Lemma 2.2. Then, since $X^{\varepsilon,\delta,n}$ is a strong solution to (3.6) and by Itô's formula

$$\begin{aligned}\mathbb{E}\eta^{\alpha,\beta}(X_t^{\varepsilon,\delta,n})&=\mathbb{E}\eta^{\alpha,\beta}(x_0)\\&+\mathbb{E}\int_0^t\int_{\mathcal{O}}(\iota^\alpha)'(\theta^\beta*X_r^{\varepsilon,\delta,n})\left(\theta^\beta*(\varepsilon\Delta X_r^{\varepsilon,\delta,n}+\operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right))\right)d\xi dr\\&+\sum_{k=1}^\infty\mathbb{E}\int_0^t\int_{\mathcal{O}}(\iota^\alpha)''(\theta^\beta*X_r^{\varepsilon,\delta,n})(\theta^\beta*Be_k)^2d\xi dr\end{aligned}\tag{3.9}$$

Using (3.2), (3.7) and dominated convergence, taking $\beta\to0$ yields

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(X_t^{\varepsilon,\delta,n})d\xi&=\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(x_0)d\xi\\&+\mathbb{E}\int_0^t\int_{\mathcal{O}}(\iota^\alpha)'(X_r^{\varepsilon,\delta,n})(\varepsilon\Delta X_r^{\varepsilon,\delta,n}+\operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right))d\xi dr\\&+\sum_{k=1}^\infty\mathbb{E}\int_0^t\int_{\mathcal{O}}(\iota^\alpha)''(X_r^{\varepsilon,\delta,n})(Be_k)^2d\xi dr\end{aligned}\tag{3.10}$$

Since $\int_{\mathcal{O}}(\iota^\alpha)'(X_r^{\varepsilon,\delta,n})(\varepsilon\Delta X_r^{\varepsilon,\delta,n}+\operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right))d\xi\leqslant0$, using (2.8), this implies

$$\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(X_t^{\varepsilon,\delta,n})d\xi\leqslant\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(x_0)d\xi+C\mathbb{E}\int_0^t\int_{\mathcal{O}}1+\iota^\alpha(X_r^{\varepsilon,\delta,n})dxdr.$$

Hence, by Gronwall's Lemma

$$\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(X_t^{\varepsilon,\delta,n})d\xi\lesssim\mathbb{E}\int_{\mathcal{O}}\iota^\alpha(x_0)d\xi+1\lesssim\frac{1}{m}\mathbb{E}\|x_0\|_m^m+1.$$

Taking the limit $\alpha\to0$ yields, by Fatou's Lemma and using dominated convergence

$$\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,\delta,n}\|_m^m\lesssim\frac{1}{m}\mathbb{E}\|x_0\|_m^m+1.\tag{3.11}$$

Taking the limits $\delta\to0$, $\varepsilon\to0$, $n\to\infty$ subsequently as in the proof of [20, Theorem 3.1] finishes the proof of this step by Fatou's Lemma.

Step 2: As in the proof of Lemma 2.2 it is enough to prove (3.5) for $x_0\in L^\infty(\Omega,\mathcal{F}_0;L^\infty_{\mathrm{av}}(\mathcal{O}))$. Hence, assume $x_0\in L^\infty(\Omega,\mathcal{F}_0;L^\infty_{\mathrm{av}}(\mathcal{O}))$ from now on. By step one we have $\mathbb{E}\|X_t\|_m^m<\infty$ for all $t\geqslant0$, $m\in\mathbb{N}$.

Taking the limit $\alpha \to 0$ in (3.10) yields, using dominated convergence and (2.8), (3.11),

$$\begin{aligned}
&\frac{1}{m}\mathbb{E}\int_{\mathcal{O}} \|X_t^{\varepsilon,\delta,n}\|_m^m d\xi \\
&\leqslant \frac{1}{m}\mathbb{E}\int_{\mathcal{O}} \|x_0\|_m^m + \mathbb{E}\int_0^t\int_{\mathcal{O}} (X_r^{\varepsilon,\delta,n})^{[m-1]}(\varepsilon\Delta X_r^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right))\, d\xi dr \\
&+ C\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_m^m dr + Ct.
\end{aligned}$$

We observe that, for $v \in H^2$ with $\nabla v \cdot \nu = 0$ on $\partial\mathcal{O}$,

$$\int_{\mathcal{O}} v^{[m-1]} \operatorname{div}\phi^\delta\left(\nabla v\right) d\zeta = -(m-1)\int_{\mathcal{O}} |v|^{m-2}\nabla v \cdot \phi^\delta(\nabla v) d\zeta.$$

Further, note that (since $\psi^\delta(0) = 0$)

$$\begin{aligned}
a \cdot \phi^\delta(a) &= a \cdot \partial\psi^\delta(a) \\
&\geqslant \psi^\delta(a) \quad \forall a \in \mathbb{R}^d
\end{aligned}$$

and (cf. [20, Appendix A])

$$|\psi^\delta(a) - \psi(a)| \lesssim \delta(1 + \psi(a)) \quad \forall a \in \mathbb{R}^d.$$

Hence, for $\delta > 0$ small enough, we obtain that

$$\begin{aligned}
&\int_{\mathcal{O}} (X_r^{\varepsilon,\delta,n})^{[m-1]} \operatorname{div}\phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right) d\zeta \\
&= -(m-1)\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\nabla X_r^{\varepsilon,\delta,n} \cdot \phi^\delta\left(\nabla X_r^{\varepsilon,\delta,n}\right) d\zeta \\
&\leqslant -c(m-1)\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\psi(\nabla X_r^{\varepsilon,\delta,n}) d\zeta + C(m-1)\delta\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2} d\zeta.
\end{aligned}$$

For $u \in W_{\mathrm{av}}^{1,\infty}$ we observe that

$$\begin{aligned}
\int_{\mathcal{O}} |u|^{m-2}\psi(\nabla u) d\zeta &= \frac{1}{p}\int_{\mathcal{O}} |u|^{m-2}|\nabla u|^p d\zeta \\
&= \frac{1}{p}\int_{\mathcal{O}} ||u|^{\frac{m-2}{p}}\nabla u|^p d\zeta \\
&= c\int_{\mathcal{O}} |\nabla u^{[\frac{m-2+p}{p}]}|^p d\zeta,
\end{aligned}$$

for some generic constant $c = c(m,p) > 0$. By Poincaré's inequality we obtain that

$$\int_{\mathcal{O}} |u|^{m-2}\psi(\nabla u) d\zeta \geqslant c\int_{\mathcal{O}} |u|^{p+m-2} d\zeta.$$

By smooth approximation, this inequality remains true for all $u \in H_{\mathrm{av}}^1$ with $u \in \bigcap_{m\geqslant 1} L^m$. Hence, using step one and (3.4) we conclude

$$\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2}\psi(\nabla X_r^{\varepsilon,\delta,n}) d\zeta \geqslant c\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{p+m-2} d\zeta.$$

Using this above yields that

$$\begin{aligned}
&\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,\delta,n}\|_m^m + c\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_{p+m-2}^{p+m-2} dr \\
&\leqslant \frac{1}{m}\mathbb{E}\|x_0\|_m^m + C\mathbb{E}\int_0^t \|X_r^{\varepsilon,\delta,n}\|_m^m dr + Ct + C(m-1)\delta\mathbb{E}\int_{\mathcal{O}} |X_r^{\varepsilon,\delta,n}|^{m-2} d\zeta.
\end{aligned}$$

By step one

$$\delta\mathbb{E}\int_{\mathcal{O}}|X_r^{\varepsilon,\delta,n}|^{m-2}d\zeta\to 0$$

for $\delta\to 0$. In conclusion, by Fatou's Lemma and (3.8),

$$\frac{1}{m}\mathbb{E}\|X_t^{\varepsilon,n}\|_m^m+c\int_0^t\mathbb{E}\|X_r^{\varepsilon,n}\|_{p+m-2}^{p+m-2}dr\leqslant\frac{1}{m}\mathbb{E}\|x_0\|_m^m+C\mathbb{E}\int_0^t\|X_r^{\varepsilon,n}\|_m^m dr+Ct.$$

An application of Gronwall's Lemma and then letting $\varepsilon\to 0$, $n\to\infty$ concludes the proof. □

The proof may now be concluded as in Section 2. For the readers convenience we give some details. Let u be the unique solution to

$$\begin{aligned}du_t&\in\operatorname{div}\left(|\nabla u_t|^{p-2}\nabla u_t\right)\,dt,\\|\nabla u_t|^{p-2}\nabla u_t\cdot\nu&\ni 0\quad\text{on }\partial\mathcal{O},\ t>0,\\u_0&=x_0\in H,\end{aligned}\tag{3.12}$$

Lemma 3.3. *Let $x_0\in H$ and u be the corresponding solution to* (3.12)*. Then there is a $C>0$ such that*

$$\|u_t\|_2^2\leqslant\frac{C}{t}\|x_0\|_{m_0}^{m_0},$$

where $m_0=4-p\in(2,3]$.

Proof. Using Lemma 3.2, the proof is analogous to Lemma 2.3. □

Lemma 3.4. *Let $\varepsilon>0$ and $x\in L^{m_1}(\Omega,\mathcal{F}_0;L^{m_1}(\mathcal{O}))$ with $m_1=m_0+2-p\in(2,4]$, $m_0=4-p\in(2,3]$. Then there is an $R=R(\varepsilon)>0$ such that*

$$Q_T(x,B_R^{m_0}(0))\geqslant 1-\varepsilon$$

for all $T\geqslant 1$.

Proof. Using Lemma 3.2, the proof is analogous to Lemma 2.4. □

Lemma 3.5. *For each $T\geqslant 0$, $\delta>0$ we have*

$$\inf_{x\in B}\mathbb{P}\left(\sup_{t\in[0,T]}\|X_t^x-u_t^x\|_H^2\leqslant\delta\right)>0$$

for all bounded sets $B\subseteq H$.

Proof. Follows from [20, Lemma 6.6]. □

Lemma 3.6. *Let $\varepsilon>0$, $x\in L^{m_1}(\mathcal{O})$ with m_1 as before and $\delta>0$. Then*

$$\liminf_{T\to\infty}Q_T(x,B_\delta(0))>0.$$

Proof. Same as Lemma 2.6. □

Proof of Theorem 3.1. Same as Theorem 2.1. □

4. Convergence of solutions: Non-local to local

In this section we investigate the convergence of the solutions to the stochastic nonlocal p-Laplace equation to solutions of the stochastic (local) p-Laplace equation, under appropriate rescaling of the kernel J. The convergence of the associated unique invariant measures will be considered in Section 5 below.

In the following let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded, smooth domain with convex boundary $\partial\mathcal{O}$ and let $J : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, $J(0) > 0$, $\int_{\mathbb{R}^d} J(z)\,dz = 1$ and $J(x) \geqslant J(y)$ for all $|x| \leqslant |y|$. Further, let W be a cylindrical Wiener process on H and $B \in L_2(H)$ symmetric with $B \in L_2(H, H^3_{\mathrm{av}})$. As above, let $H := L^2_{\mathrm{av}}(\mathcal{O})$ and $S = H^1_{\mathrm{av}}(\mathcal{O})$.

For $p \in (1,2)$, $\varepsilon > 0$, we consider the rescaled stochastic nonlocal p-Laplace equations of the type

$$\begin{aligned} dX^\varepsilon_t &= \left(\int_{\mathcal{O}} J^\varepsilon(\cdot - \xi)|X^\varepsilon_t(\xi) - X^\varepsilon_t(\cdot)|^{p-2}(X^\varepsilon_t(\xi) - X^\varepsilon_t(\cdot))d\xi\right) dt + BdW_t \\ X^\varepsilon_0 &= x_0 \in L^2(\Omega, \mathcal{F}_0; H) \end{aligned} \tag{4.1}$$

with

$$J^\varepsilon(\xi) := \frac{C_{J,p}}{\varepsilon^{p+d}} J\left(\frac{\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^d$$

and corresponding energy

$$\varphi^\varepsilon(u) := \frac{1}{2p}\int_{\mathcal{O}}\int_{\mathcal{O}} J^\varepsilon(\xi - \zeta)\left|\frac{u(\zeta) - u(\xi)}{\varepsilon}\right|^p d\zeta d\xi,$$

for $u \in L^p(\mathcal{O})$, where

$$C^{-1}_{J,p} := \frac{1}{2}\int_{\mathbb{R}^d} J(z)|z_d|^p\,dz.$$

Furthermore, we set

$$\varphi(u) := \begin{cases} \frac{1}{p}\int_{\mathcal{O}}|\nabla u|^p\,d\xi, & \text{if } u \in W^{1,p}(\mathcal{O}), \\ +\infty, & \text{if } u \in L^p(\mathcal{O}) \setminus W^{1,p}(\mathcal{O}). \end{cases}$$

By [20, Theorem 4.1], for each $\varepsilon > 0$, there is a unique SVI solution X^ε to the stochastic nonlocal p-Laplace equation

$$\begin{aligned} dX^\varepsilon_t &= -\partial_{L^2}\varphi^\varepsilon(X^\varepsilon_t)\,dt + BdW_t, \\ X^\varepsilon_0 &= x_0 \end{aligned} \tag{4.2}$$

and, by [20, Theorem 3.1], there is a unique SVI solution to the stochastic (local) p-Laplace equation

$$\begin{aligned} dX_t &= -\partial_{L^2}\varphi(X_t)\,dt + BdW_t, \\ X_0 &= x_0, \end{aligned} \tag{4.3}$$

where $\partial_{L^2}\varphi$ denotes the L^2 subgradient of φ restricted to L^2. In [20, Section 5] weak convergence of $X^\varepsilon \rightharpoonup X$ in $L^2([0,T] \times \Omega; H)$ has been shown. The aim of this section is to strengthen this to pointwise in time weak convergence, that is,

$$X^\varepsilon_t \rightharpoonup X_t \quad \text{weakly in } H \text{ for all } t \geqslant 0, \mathbb{P}\text{-a.s.}. \tag{4.4}$$

This will be crucial in order to obtain the convergence of the associated semigroups $P^\varepsilon_t F(x) = \mathbb{E}F(X^{\varepsilon,x}_t)$ to $P_t F(x) = \mathbb{E}F(X^x_t)$ for cylindrical functions $F \in \mathcal{F}C^1_b(H)$.

The strategy to prove the pointwise weak convergence (4.4) is based on considering the transformed equations (cf. Remark 4.5 below)

$$\frac{d}{dt}Y_t = -\partial_{L^2}\varphi(Y_t + W^B_t(\omega)) = A(Y_t + W^B_t(\omega)) \tag{4.5}$$

and

$$\frac{d}{dt}Y_t^\varepsilon = -\partial_{L^2}\varphi^\varepsilon(Y_t^\varepsilon + W_t^B(\omega)) = A^\varepsilon(Y_t^\varepsilon + W_t^B(\omega)) \tag{4.6}$$

and to prove the weak convergence $Y_t^\varepsilon \rightharpoonup Y_t$ in H for all $t \geqslant 0$ and a.a. $\omega \in \Omega$. The advantage of considering the transformed, random PDE (4.5) and (4.6) is that Y^ε, Y enjoy better time regularity properties than X^ε, X which may be used to deduce stronger convergence results.

Theorem 4.1. *Let $x_0 \in L^2(\Omega, \mathcal{F}_0; H)$ and X^ε, X be the unique solutions to* (4.2), (4.3) *respectively. Then, for each sequence $\varepsilon_n \to 0$,*

$$X_t^{\varepsilon_n} \rightharpoonup X_t \quad \text{for } n \to \infty$$

weakly in H for all $t \in [0,T], \mathbb{P}$-a.s.. In particular,

$$P_t^\varepsilon F(x) \to P_t F(x) \quad \text{for } \varepsilon \to 0$$

for all $F \in \mathcal{F}C_b^1(H)$, $t \in [0,T]$, $x \in H$.

Motivated by [20], the general strategy of the proof of Theorem 4.1 is based on the SVI framework. Hence, we first briefly sketch well-posedness of SVI solutions to (4.5) and then proceed with the proof of Theorem 4.1.

Remark 4.2. In this section, we restrict to the case $p \in (1,2)$ for simplicity only. The interested reader will notice that the same arguments can be applied in the case $p = 1$ with only minor changes. The only difference is the treatment of the nonlocal, transformed random PDE (4.6). In the case $p = 1$ well-posedness of SVI solutions to (4.6) has to be shown as a first step. This can be done following the same arguments as in [20, Section 3].

4.1. SVI approach to the transformed equation (4.5).

Let $H := L^2_{\text{av}}(\mathcal{O})$, $S = H^1_{\text{av}}(\mathcal{O})$. Without loss of generality we may assume $W^B(\omega) \in C([0,T]; H^3_{\text{av}})$ for all $\omega \in \Omega$. In analogy to [20, Definition 2.1] we define

Definition 4.3. Let $x_0 \in H$, $T > 0$. A map $Y \in L^2([0,T]; H)$ is said to be an *SVI solution* to (4.5) if

(i) [Regularity]

$$\operatorname{ess\,sup}_{t\in[0,T]} \|Y_t\|_H^2 + \int_0^T \varphi(Y_r + W_r^B)dr \leqslant \|x_0\|_H^2 + C, \tag{4.7}$$

for some constants $C > 0$.

(ii) [Variational inequality] for each map $Z \in W^{1,2}([0,T]; S)$ we have

$$\begin{aligned} &\|Y_t - Z_t\|_H^2 + 2\int_0^t \varphi(Y_r + W_r^B)\,dr \\ &\leqslant \mathbb{E}\|x_0 - Z_0\|_H^2 + 2\int_0^t \varphi(Z_r + W_r^B)\,dr - 2\int_0^t (G_r, Y_r - Z_r)_H\,dr, \end{aligned} \tag{4.8}$$

for a.e. $t \in [0,T]$, where $G := \frac{d}{dt}Z$.

If, in addition, $Y \in C([0,T]; H)$ then Y is said to be a (time-) continuous SVI solution to (4.5).

Proposition 4.4. *For each $x_0 \in H$, $\omega \in \Omega$ there is a unique time-continuous SVI solution $Y = Y(\omega)$ to* (4.5). *The process $(t,\omega) \mapsto Y_t(\omega)$ is $\mathcal{F}_t$-progressively measurable and the constant $C = C(\omega)$ in* (4.7) *satisfies $C \in L^2(\Omega)$.*

Proof. The proof follows the same line of arguments as the proof of [20, Theorem 3.1]. Hence, in the following we shall restrict to giving some details on required modifications of the proof. For notational convenience we set

$$\psi(z) := \frac{1}{p}|z|^p,$$
$$\phi(z) := \partial\psi(z)$$

and let ψ^δ be the Moreau-Yosida approximation of ψ, $\phi^\delta := \partial\psi^\delta$. In analogy to (3.6) we consider the three step approximation

$$\frac{d}{dt}Y_t^{\varepsilon,\delta,n} = \varepsilon\Delta Y_t^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right)$$
$$Y_0^{\varepsilon,\delta,n} = x_n \in H^1_{\mathrm{av}}. \tag{4.9}$$

Existence and uniqueness of a variational solution $Y^{\varepsilon,\delta,n}$ to (4.9) follows easily from [32, Theorem 4.2.4]. Progressive measurability of $(t,\omega) \mapsto Y_t^{\varepsilon,\delta,n}(\omega)$ follows as in [18, proof of Theorem 1.1].

Step 1: The first step consists in proving the existence of a strong solution to (4.9) and corresponding (uniform) energy bounds as in (3.7). We restrict to an informal derivation of these estimates, the rigorous justification proceeds as in [20, Theorem 3.1]. We set $\|v\|^2_{\dot H^1} := \|\nabla v\|_2^2$ for $v \in H^1$. Informally, we compute

$$\begin{aligned}
&\frac{d}{dt}\|Y_t^{\varepsilon,\delta,n}\|^2_{\dot H^1}\\
&= -2(\varepsilon\Delta Y_t^{\varepsilon,\delta,n} + \operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta Y_t^{\varepsilon,\delta,n})_H\\
&= -2\varepsilon\|\Delta Y_t^{\varepsilon,\delta,n}\|_H^2 - 2(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega)))_H\\
&+ 2(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta W_t^B(\omega))_H.
\end{aligned}$$

For $v \in H^2$ with $\nabla v\cdot\nu = 0$ on $\partial\mathcal{O}$, arguing as in [20, Example 7.11], we obtain that

$$\begin{aligned}
-(v, \operatorname{div}\phi^\delta(\nabla v))_{\dot H^1} &= -(-\Delta v, \operatorname{div}\phi^\delta(\nabla v))_2\\
&= -\lim_{n\to\infty}(T_n v, \operatorname{div}\phi^\delta(\nabla v))_2\\
&= -\lim_{n\to\infty}(nu - nJ_n u, \operatorname{div}\phi^\delta(\nabla v))_2\\
&\leqslant \lim_{n\to\infty} n\left(\int_{\mathcal{O}}\psi^\delta(\nabla J_n u)d\xi - \int_{\mathcal{O}}\psi^\delta(\nabla u)d\xi\right)\\
&\leqslant 0,
\end{aligned} \tag{4.10}$$

where T_n is the Yosida-approximation and J_n the resolvent of the Neumann Laplacian $-\Delta$ on L^2. We next observe that

$$\begin{aligned}
&2(\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right), \Delta W_t^B(\omega))_H\\
&= 2\int_{\mathcal{O}}\operatorname{div}\phi^\delta\left(\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))\right)\Delta W_t^B(\omega)d\xi\\
&\leqslant C\int_{\mathcal{O}}(1 + |\nabla(Y_t^{\varepsilon,\delta,n} + W_t^B(\omega))|)|\nabla\Delta W_t^B(\omega)|d\xi\\
&\leqslant C\left(\|\nabla Y_t^{\varepsilon,\delta,n}\|_H^2 + 1 + \|W_t^B(\omega)\|^2_{H^3}\right).
\end{aligned}$$

Hence,

$$\frac{d}{dt}\|Y_t^{\varepsilon,\delta,n}\|^2_{\dot H^1} \leqslant -2\varepsilon\|\Delta Y_t^{\varepsilon,\delta,n}\|_H^2 + C(\|Y_t^{\varepsilon,\delta,n}\|^2_{\dot H^1} + 1 + \|W_t^B(\omega)\|_{H^3}).$$

By Gronwall's lemma this implies

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|Y_t^{\varepsilon,\delta,n}\|_{\dot H^1}^2 + 2\varepsilon \int_0^T \|\Delta Y_r^{\varepsilon,\delta,n}\|_H^2 dr \\
&\leqslant C\left(\|x_0\|_{\dot H^1}^2 + \int_0^T \|W_r^B(\omega)\|_{H^3} dr + 1\right) \qquad (4.11)\\
&\leqslant C(\|x_0\|_{\dot H^1}^2 + 1),
\end{aligned}
$$

which finishes the proof of the required energy bound.

Step 2: We next derive the variational inequality (4.8) and regularity estimate (4.7) for $Y^{\varepsilon,\delta,n}$. By the chain-rule we have that

$$
\begin{aligned}
&\left\|Y_t^{\varepsilon,\delta,n} - Z_t\right\|_H^2 \\
&= \|x_0^n - Z_0\|_H^2 + 2\int_0^t (\operatorname{div}\phi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n} + W_r^B(\omega))\right) - G_r, Y_r^{\varepsilon,\delta,n} - Z_r)_H dr \\
&= \|x_0^n - Z_0\|_H^2 + 2\int_0^t \int_{\mathcal O} \phi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n} + W_r^B(\omega))\right)\cdot(\nabla Z_r - \nabla Y_r^{\varepsilon,\delta,n}) d\xi dr \\
&- 2\int_0^t (G_r, Y_r^{\varepsilon,\delta,n} - Z_r)_H dr \\
&\leqslant \|x_0^n - Z_0\|_H^2 - 2\int_0^t \int_{\mathcal O} \psi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n} + W_r^B(\omega))\right) d\xi \\
&+ 2\int_0^t \int_{\mathcal O} \psi^\delta\left(\nabla(Z_r + W_r^B(\omega))\right) d\xi dr - 2\int_0^t (G_r, Y_r^{\varepsilon,\delta,n} - Z_r)_H dr,
\end{aligned}
$$

for all $Z \in W^{1,2}([0,T];H)$ and $G := \frac{d}{dt}Z$. In particular, choosing $Z \equiv 0$ we obtain that

$$
\begin{aligned}
&\left\|Y_t^{\varepsilon,\delta,n}\right\|_H^2 + 2\int_0^t \int_{\mathcal O} \psi^\delta\left(\nabla(Y_r^{\varepsilon,\delta,n} + W_r^B(\omega))\right) d\xi\, dr \\
&\leqslant \|x_0\|_H^2 + 2\int_0^t \int_{\mathcal O} \psi^\delta\left(\nabla W_r^B(\omega)\right)\, dr \\
&\leqslant \|x_0\|_H^2 + C(\omega),
\end{aligned}
$$

with $C \in L^2(\Omega)$.

Step 3: The next step is to take the limit $\delta \to 0$, i.e. we estimate

$$
\begin{aligned}
&\frac{d}{dt}\|Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\|_H^2 \\
&= 2\varepsilon(\Delta Y_t^{\varepsilon,\delta_1,n} - \Delta Y_t^{\varepsilon,\delta_2,n}, Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n})_H \\
&+ 2(\operatorname{div}\phi^{\delta_1}\left(\nabla(Y_t^{\varepsilon,\delta_1,n} + W_t^B(\omega))\right) - \operatorname{div}\phi^{\delta_2}\left(\nabla(Y_t^{\varepsilon,\delta_2,n} + W_t^B(\omega))\right), Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n})_H \\
&\leqslant 2(\operatorname{div}\phi^{\delta_1}\left(\nabla(Y_t^{\varepsilon,\delta_1,n} + W_t^B(\omega))\right) - \operatorname{div}\phi^{\delta_2}\left(\nabla(Y_t^{\varepsilon,\delta_2,n} + W_t^B(\omega))\right), Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n})_H.
\end{aligned}
$$

Using [20, Appendix A, (A.6)] we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\|Y_t^{\varepsilon,\delta_1,n} - Y_t^{\varepsilon,\delta_2,n}\|_H^2 \\
&\leqslant C(\delta_1 + \delta_2)(1 + \|Y_t^{\varepsilon,\delta_1,n}\|_{\dot H^1}^2 + \|Y_t^{\varepsilon,\delta_2,n}\|_{\dot H^1}^2 + \|W_t^B(\omega)\|_{\dot H^1}^2).
\end{aligned}
$$

Hence,

$$\|Y_t^{\varepsilon,\delta_1,n}-Y_t^{\varepsilon,\delta_2,n}\|_H^2 \leqslant C(\delta_1+\delta_2)(1+\int_0^t\|Y_r^{\varepsilon,\delta_1,n}\|_{\dot H^1}^2dr+\int_0^t\|Y_r^{\varepsilon,\delta_2,n}\|_{\dot H^1}^2dr+\int_0^t\|W_r^B(\omega)\|_{\dot H^1}^2dr)$$

which, using (4.11), implies convergence of $Y^{\varepsilon,\delta,n}$ in $C([0,T];H)$ for $\delta\to 0$.
Step 4: The limits $\varepsilon\to 0$, $n\to\infty$ can be justified precisely as in the proof of [20, Theorem 3.1] and the proof can be concluded as in [20, Theorem 3.1]. Progressive measurability of $(t,\omega)\mapsto Y_t(\omega)$ follows from the respective property of $Y^{\varepsilon,\delta,n}$. □

Remark 4.5.

(i) Let X be the unique SVI solution to (3.1) and Y be the unique SVI solution to (4.5). Then $X=Y+W^B$, $\mathbb{P}$-a.s..
(ii) Let X^ε be the unique SVI solution to (4.1) and Y^ε be the unique variational solution to (4.5) given by [32, Theorem 4.2.4]. Then $X^\varepsilon=Y^\varepsilon+W^B$, $\mathbb{P}$-a.s..

Proof. (i): If Y is the unique SVI solution to (4.5), then $\bar X:=Y+W^B$ is progressively measurable and taking the expectation in (4.7) yields $\bar X\in L^2([0,T]\times\Omega;H)$. It is then easy to see that $\bar X$ is an SVI solution to (3.1). Thus, uniqueness of SVI solutions implies $X=\bar X$, $\mathbb{P}$-a.s.. *(ii):* Follows by the same proof as (i). In order to prove that $\bar X^\varepsilon$ is an SVI solution to (4.1) see the proof of Theorem 4.1 below, in particular (4.14), (4.15). □

4.2. **Convergence of solutions.**

Proof of Theorem 4.1. Let $\varepsilon_n\to 0$ and set $Y^n:=Y^{\varepsilon_n}$, $J^n=J^{\varepsilon_n}$.
Step 1: Weak convergence in $L^2([0,T];H)$
By [32, Theorem 4.2.4] there is a unique variational solution $Y^n\in C([0,T];H)$ to (4.6) with respect to the trivial Gelfand triple $V=H=L^2$. Indeed, it is easy to see that A^n satisfies the required hemicontinuity and monotonicity on L^2. Concerning coercivity, using [4, Lemma 6.5], we note that

$$\begin{aligned}
&{}_{V^*}\langle A^n(u),u\rangle_V\\
&=-\frac12\int_{\mathcal O}\int_{\mathcal O}J^n(\zeta-\xi)\phi(u(\xi)-u(\zeta))(u(\xi)-u(\zeta))\,d\xi\,d\zeta \qquad (4.12)\\
&=-\frac12\int_{\mathcal O}\int_{\mathcal O}J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^p\,d\xi\,d\zeta\\
&=-\frac12\|u\|_{V_n}^p\\
&\leqslant 0,
\end{aligned}$$

for all $u\in L^2$, where we have set $V_n:=L^p_{J^n}\cap L^p_{\mathrm{av}}$. Moreover, using [4, Lemma 6.5] and Hölder's inequality, we note that

$$\begin{aligned}
(A^n(u),v)_H&=\int_{\mathcal O}\int_{\mathcal O}J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^{p-2}(u(\xi)-u(\zeta))(v(\xi)-v(\zeta))\,d\xi\,d\zeta\\
&\leqslant\left(\int_{\mathcal O}\int_{\mathcal O}J^n(\zeta-\xi)|u(\xi)-u(\zeta)|^p\,d\xi\,d\zeta\right)^{\frac{p-1}{p}} \qquad (4.13)\\
&\quad\left(\int_{\mathcal O}\int_{\mathcal O}J^n(\zeta-\xi)|v(\xi)-v(\zeta)|^p\,d\xi\,d\zeta\right)^{\frac1p}\\
&\leqslant\|u\|_{V_n}^{p-1}\|v\|_{V_n}\quad\forall u,v\in L^2.
\end{aligned}$$

It is easy to see that Y^n is an SVI solution to (4.6): Indeed, by the chain-rule we have that

(4.14)
$$\|Y_t^n - Z_t\|_H^2 \leqslant \|x_0^n - Z_0\|_H^2 + 2\int_0^t \varphi^n(Z_r + W_r^B)\,dr - 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr - \int_0^t (G_r, Y_r^n - Z_r)dr,$$

for all $Z \in W^{1,2}([0,T];H)$ and $G := \frac{d}{dt}Z$. In particular, choosing $Z \equiv 0$ we obtain that

$$\|Y_t^n\|_H^2 + 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr \leqslant \|x_0\|_H^2 + 2\int_0^t \varphi^n(W_r^B)\,dr.$$

By [20, p. 24, first equation] we have

$$\varphi^n(v) \leqslant C\|v\|_{\dot W^{1,p}} \leqslant C(1 + \|v\|_{\dot H^1}^2) \quad \forall v \in H^1_{\mathrm{av}}.$$

Hence,

$$\|Y_t^n\|_H^2 + 2\int_0^t \varphi^n(Y_r^n + W_r^B)\,dr \lesssim \|x_0\|_H^2 + \int_0^t \|W_r^B\|_{\dot H^1}^2\,dr + 1. \tag{4.15}$$

and we may extract a subsequence (again denoted by Y^n) such that

$$Y^n \rightharpoonup Y \quad \text{in } L^2([0,T];H).$$

Using the Mosco convergence of $\varphi^n \to \varphi$ and $\limsup_{n\to\infty} \varphi^n(u) \leqslant \varphi(u)$ (cf. [20, Proposition 5.2]) and Mosco convergence of integral functionals (cf. [20, Appendix B]), it is easy to see that Y is an SVI solution to (4.5). Since SVI solutions to (4.5) are unique by Proposition 4.4 this implies weak convergence of the whole sequence Y^n to Y in $L^2([0,T];H)$.

Step 2: By the chain-rule, (4.12) and (4.13) we have that

$$\begin{aligned}
\|Y_t^n\|_H^2 &= \|x_0\|_H^2 + 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), Y_r^n)_H\,dr \\
&= \|x_0\|_H^2 + 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), Y_r^n + W_r^B(\omega))_H\,dr \\
&\quad - 2\int_0^t (A^n(Y_r^n + W_r^B(\omega)), W_r^B(\omega))_H\,dr \\
&\leqslant \|x_0\|_H^2 - 2\int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^p\,dr \\
&\quad + 2\int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^{p-1}\|W_r^B(\omega)\|_{V_n}\,dr \\
&\leqslant \|x_0\|_H^2 - \int_0^t \|Y_r^n + W_r^B(\omega)\|_{V_n}^p\,dr + C\int_0^t \|W_r^B(\omega)\|_{V_n}^p\,dr.
\end{aligned}$$

Hence, using (A.1),

$$\sup_{t\in[0,T]} \|Y_t^n\|_H^2 + \int_0^T \|Y_r^n + W_r^B(\omega)\|_{V_n}^p\,dr \leqslant \|x_0\|_H^2 + C\int_0^t \|W_r^B(\omega)\|_{H^1}^p\,dr. \tag{4.16}$$

We continue with an argument from [17]: Consider the set

$$\mathcal{K} := \{(Y^n, v)_H \;:\; v \in W^{1,p} \cap H,\ \|v\|_H \vee \|v\|_{W^{1,p}} \leqslant 1,\ n \in \mathbb{N}\} \subset C([0,T]).$$

By (4.16), $\mathcal{K}$ is bounded in $C([0,T])$. Moreover, by (4.13) and (4.16),

$$\begin{aligned}(Y^n_{t+s}-Y^n_t,v)_H &= \int_t^{t+s}((Y^n)'_r,v)_H\,dr\\ &= \int_t^{t+s}(A^n(Y^n_r+W^B_r(\omega)),v)_H\,dr\\ &\leqslant \int_t^{t+s}\|Y^n_r+W^B_r(\omega)\|^{p-1}_{V_n}\|v\|_{V_n}\,dr\\ &\leqslant Cs^{1/p}\int_0^T\|Y^n_r+W^B_r(\omega)\|^p_{V_n}dr\\ &\leqslant Cs^{1/p},\end{aligned}$$

where we used (A.1). Hence, $\mathcal{K}$ is a set of equibounded, equicontinuous functions and thus is relatively compact in $C([0,T])$ by the Arzelà-Ascoli theorem. Thus, for every $v\in W^{1,p}\cap H$, $\|v\|_H\vee\|v\|_{W^{1,p}}\leqslant 1$, there is a subsequence (again denoted by Y^n) such that $(Y^n,v)_H\to g$ in $C([0,T])$. Since also $(Y^n,v)_H\to(Y,v)_H$ in $L^2([0,T])$ by step one, we have $g=(Y,v)_H$. Hence, for each $v\in W^{1,p}\cap H$, the whole sequence $(Y^n,v)_H$ converges to $(Y,v)_H$ in $C([0,T])$.

For $h\in H$, $\varepsilon>0$ we can choose $v^\varepsilon\in W^{1,p}\cap H$ such that $\|h-v^\varepsilon\|\leqslant\varepsilon$. Then

$$\begin{aligned}(Y^n_t-Y_t,h)_H &= (Y^n_t-Y_t,h-v^\varepsilon)_H+(Y^n_t-Y_t,v^\varepsilon)_H\\ &\leqslant \|Y^n_t-Y_t\|_H\|h-v^\varepsilon\|_H+(Y^n_t-Y_t,v^\varepsilon)_H\\ &\leqslant C\varepsilon+(Y^n_t-Y_t,v^\varepsilon)_H.\end{aligned}$$

Hence, choosing n large enough implies

$$(Y^n_t-Y_t,h)_H\leqslant C\varepsilon\quad\forall n\geqslant n_0(\varepsilon),$$

that is $Y^n_t\rightharpoonup Y_t$ weakly in H for $n\to\infty$.

Since $Y^n=X^n-W^B$, $Y=X-W^B$, $\mathbb{P}$-a.s., this implies weak convergence of X^n_t to X_t $\mathbb{P}$-a.s..

Step 3: We next prove the convergence of the associated semigroups P^ε_t,P_t. Let $F\in\mathcal{F}C^1_b(E)$ with $F=f(l_1,\dots,l_k)$ and let $t\geqslant 0$, $x\in H$. Further let $\varepsilon_n\to 0$ and set $P^n_t:=P^{\varepsilon_n}_t$. Then

$$\begin{aligned}P^n_tF(x) &= \mathbb{E}F(X^{n,x}_t)\\ &= \mathbb{E}f(l_1(X^{n,x}_t),\dots,l_k(X^{x,n}_t))\\ &\to \mathbb{E}f(l_1(X^x_t),\dots,l_k(X^x_t))\\ &= \mathbb{E}F(X^x_t)\\ &= P_tF(x),\end{aligned}$$

as $n\to\infty$, by Lebesgue's dominated convergence theorem. □

5. Convergence of invariant measures: Non-local to local

By Theorem 2.1, for each $\varepsilon>0$, there exists a unique invariant measure μ^ε to the stochastic nonlocal p-Laplace equation (4.2) and by Theorem 3.1 there is a unique invariant measure μ to the (local) stochastic p-Laplace equation (4.3). In this section we prove weak* convergence of μ^ε to μ in a suitable topology, for $p\in(1,2)$.

Several difficulties appear, due to the nonlocal and singular-degenerate nature of the SPDE (4.2). First, we expect tightness of μ^ε on $H=L^2_{\mathrm{av}}$ only under stringent dimensional restrictions. Indeed, for the (expected) limit μ we only know $\mu(W^{1,p}_{\mathrm{av}})=1$ which, roughly speaking, would lead to assuming that the embedding $W^{1,p}_{\mathrm{av}}\hookrightarrow L^2_{\mathrm{av}}$ is compact and thus restrict to one spatial dimension, i.e. $d=1$, in general. Second, we only have weak convergence $X^\varepsilon \rightharpoonup X$ in H. Therefore, the convergence of the associated semigroups $P_t^n F$ for general $F \in \mathrm{Lip}_b(H)$ is unclear; a crucial ingredient in previously available methods.

The first problem is overcome in this section by considering weak convergence of μ^ε on $E = L^p_{\mathrm{av}}$ rather than on L^2_{av}. Again, for the limit μ we know $\mu(W^{1,p}_{\mathrm{av}}) = 1$. Hence, by compactness of the embedding $W^{1,p}_{\mathrm{av}} \hookrightarrow L^p_{\mathrm{av}}$, μ is concentrated on compact sets in L^p_{av} which suggests that tightness of μ^ε on L^p_{av} should hold without restrictions on the dimension. Indeed, this is established in Lemma 5.6 below. The resulting difficulty of working with two topologies, weak* convergence of μ^ε on L^p_{av} versus weak convergence of X^ε on L^2_{av} is solved in Lemma 5.3. The second problem is overcome by first considering cylindrical functions on H. For a cylindrical function F, weak convergence $X_t^n \rightharpoonup X_t$ in H is enough to deduce $P_t^n F \to P_t F$. It turns out that this is sufficient to deduce the weak* convergence $\mu^\varepsilon \rightharpoonup^* \mu$ by means of a monotone class argument (cf. proof of Theorem 5.1). The main result of this section is

Theorem 5.1. *Let μ^ε be the unique invariant measure to* (4.2) *and μ be the unique invariant measure to* (4.3). *Then $\mu^\varepsilon \rightharpoonup^* \mu$ for $\varepsilon \to 0$ weakly* in the set of probability measures on $L^p_{\mathrm{av}}(\mathcal{O})$, that is, for each bounded, Lipschitz continuous function F on $L^p_{\mathrm{av}}(\mathcal{O})$ we have $\mu^\varepsilon(F) \to \mu(F)$ for $\varepsilon \to 0$.*

5.1. **Asymptotic invariance.**

In this section we provide a general result on the convergence of invariant measures for convergent semigroups. Compared to previous results [14] the main novelty here is to work with two distinct topologies corresponding to the convergence of the invariant measures on the one hand and to the convergence of the semigroups on the other hand.

Definition 5.2. Let E be a Banach space, $\mathcal{G} \subseteq \mathcal{B}_b(E)$ be a set of bounded, measurable functions on E and P_t be a semigroup on E. Then, a probability measure μ on E is said to be $\mathcal{G}$-invariant if

$$\int_E P_t G \, d\mu = \int_E G \, d\mu \quad \forall G \in \mathcal{G}.$$

Lemma 5.3. *Let E, H be Banach spaces with $H \hookrightarrow E$ dense. Further let $\mathcal{G} \subset \mathrm{Lip}_b(E)$, P_t^n, P_t be Feller semigroups on H and μ^n be $\mathcal{G}$-invariant probability measures for P_t^n, for all $n \in \mathbb{N}$. Suppose that $\mu_n \rightharpoonup^* \mu$ as probability measures on E, the semigroups P_t^n satisfy a uniform e-property, that is, there exists a $C > 0$ such that for all $F \in \mathrm{Lip}_b(E)$, $x, y \in E$*

$$|P_t^n F(x) - P_t^n F(y)| \leqslant C \,\mathrm{Lip}(F) \, \|x - y\|_E \quad \forall n \in \mathbb{N}, t \geqslant 0$$

and that for every $G \in \mathcal{G}$, $t \geqslant 0$, $x \in H$,

$$\lim_{n \to \infty} P_t^n G(x) = P_t G(x).$$

Then μ is $\mathcal{G}$-invariant, i.e.

$$\int_E P_t G \, d\mu = \int_E G \, d\mu \quad \textit{for all } G \in \mathcal{G}, t \geqslant 0.$$

Proof. For two (Borel) probability measures ν_1, ν_2 on $(E, \mathcal{B}(E))$, denote by $\beta_E(\nu_1, \nu_2)$ the bounded Lipschitz distance between them, that is

$$\beta_E(\nu_1, \nu_2) := \sup \left\{ \left| \int_E F \, d(\nu_1 - \nu_2) \right| \; : \; F \in \mathrm{Lip}_b(E), \; \|F\|_{E,\infty} + \mathrm{Lip}_E(F) \leqslant 1 \right\}.$$

We have $\mathrm{Lip}_b(E) \subseteq \mathrm{Lip}_b(H)$ and by continuous extension we can identify

$$\{F \in \mathrm{Lip}_b(H) \, : \, \exists C > 0 \text{ s.t. } \|F(x) - F(y)\|_E \leqslant C \|x - y\|_E, \, \forall x, y \in E\} = \mathrm{Lip}_b(E).$$

Accordingly, due to the e-property, $P_t^n : \mathrm{Lip}_b(E) \to \mathrm{Lip}_b(E)$. Let $G \in \mathcal{G}$, $t \geqslant 0$. We have that

$$\begin{aligned}
&\left|\int_E G\,d\mu - \int_E P_t G\,d\mu\right| \\
&\leqslant \left|\int_E G\,d\mu - \int_E P_t^n G\,d\mu_n\right| + \left|\int_E P_t^n G\,d\mu_n - \int_E P_t^n G\,d\mu\right| \\
&+ \left|\int_E P_t^n G\,d\mu - \int_E P_t G\,d\mu\right|.
\end{aligned}$$

By the property of being $\mathcal{G}$-invariant measures, the first term equals $\int_E G\,d(\mu-\mu_n)$ and hence tends to zero as $n \to \infty$. By the e-property, the second term can be bounded as follows (with $\|F\|_{E,\infty} := \sup_{x\in E}|F(x)|$)

$$\beta_E(\mu_n,\mu)\left[\|P_t^n G\|_{E,\infty} + \mathrm{Lip}_E(P_t^n G)\right] \leqslant \beta_E(\mu_n,\mu_0)\left[\|G\|_{E,\infty} + C\,\mathrm{Lip}_E(G)\right]$$

hence in turn tends to zero as $n \to \infty$ by weak convergence of μ_n to μ and Lebesgue's dominated convergence, since (in Polish spaces) the bounded Lipschitz metric generates the weak topology, see e.g. [34, 1.12, pp. 73/74]. Since $\mu^n(H) = 1$ and $\mu_n \rightharpoonup^* \mu$ we have $\mu(H) = 1$. Thus, the third term converges to zero by convergence of semigroups and Lebesgue's dominated convergence theorem. ☐

5.2. **Tightness.**

Definition 5.4. A sequence of probability measures μ_n on a Polish space E is called asymptotically tight, if for each $\eta > 0$ there exists a compact set K_η such that for each $\delta > 0$ it holds that

$$\limsup_{n\to\infty} \mu_n((K_\eta^\delta)^c) < \eta,$$

where $K_\eta^\delta \supset K_\eta$ is the open δ-enlargement of K_η.

The next result can be found in [34, Theorem 1.3.9].

Lemma 5.5. *If μ_n is asymptotically tight, then it is weakly relatively compact.*

Let μ^ε, $\varepsilon > 0$ be the unique invariant measure associated to (4.2).

Proposition 5.6. *Let $\varepsilon_n \searrow 0$ as $n \to \infty$ and set $\mu_n := \mu_{\varepsilon_n}$. Then μ_n is asymptotically tight on $E := L^p_{\mathrm{av}}(\mathcal{O})$.*

Proof. Let $\eta > 0$ and $C := \|B\|^2_{L_2(H)}$. Recall $\varphi_\varepsilon \leqslant K\varphi$ for all $\varepsilon \in (0,1]$ for some constant $K > 0$. Then, by Poincaré's inequality,

$$K_\eta := \left\{x \in L^p : \varphi(x) \leqslant \frac{2C}{\eta K}\right\}$$

is a bounded set in $W^{1,p}_{\mathrm{av}}(\mathcal{O})$ and hence compact in $L^p_{\mathrm{av}}(\mathcal{O})$. For $\delta > 0$, let K_η^δ be the open δ-enlargement of K_η in E. Let

$$G_n := \left\{x \in L^p : \varphi^{\varepsilon_n}(x) \leqslant \frac{2C}{\eta}\right\}$$

for some $\varepsilon_n \searrow 0$, $n \to \infty$. Since $\varphi_\varepsilon \leqslant K\varphi$ for all $\varepsilon \in (0,1]$, it holds that $G_n \supset K_\eta$ for $n \in \mathbb{N}$.

We claim that for each $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that $G_n \subset K_\eta^\delta$ for all $n \geqslant n_0$. We argue by contradiction. If there exists $\delta_0 > 0$, such that for all $n \in \mathbb{N}$ it holds that $G_n \not\subset K_\eta^{\delta_0}$, then we can find a sequence $x_n \in G_n \setminus K_\eta^{\delta_0}$ such that $\mathrm{dist}_E(x_n, K_\eta) \geqslant \delta_0$ for every n. By the definition of G_n and [4, Theorem 6.11 (2.)], $\{x_n\}$ is relatively compact in $L^p_{\mathrm{av}}(\mathcal{O})$. Hence, there exists a subsequence (denoted by $\{x_n\}$) such that $x_n \to \bar{x}$ in $L^p_{\mathrm{av}}(\mathcal{O})$ and $\bar{x} \in W^{1,p}_{\mathrm{av}}(\mathcal{O})$. By the Mosco convergence of $\varphi_\varepsilon \to \varphi$ on L^p (cf. [20, Proposition 5.2]) we obtain that

$$\varphi(\bar{x}) \leqslant \liminf_{n\to\infty} \varphi^{\varepsilon_n}(x_n) \leqslant \frac{2C}{\eta},$$

and thus $\bar{x} \in K_\eta$. Hence,

$$\delta_0 \leqslant \mathrm{dist}_E(x_n, K_\eta) \leqslant \|\bar{x} - x_n\|_{L^p_{\mathrm{av}}(\mathcal{O})} \underset{n\to\infty}{\longrightarrow} 0,$$

the desired contradiction.

Now, by Theorem 2.1, for each $n \geqslant n_0(\delta)$,

$$\begin{aligned}\mu_n((K^\delta_\eta)^c) \leqslant \mu_n(G^c_n) &\leqslant \frac{\eta}{2C}\int \varphi^{\varepsilon_n}(z)\,\mu_n(dz)\\ &\leqslant \frac{\eta}{2} < \eta.\end{aligned}$$

The proof is completed by taking the limsup as $n \to \infty$. □

5.3. **Proof of Theorem 5.1.** We aim to apply Lemma 5.3 with $E = L^p_{\mathrm{av}}(\mathcal{O})$, $H = L^2_{\mathrm{av}}(\mathcal{O})$. Since $p \in (1,2)$, we have that $H \subseteq E$. Let $\mathcal{G}$ be the space of cylindrical functions on E, that is,

$$\mathcal{G} = \mathcal{F}C^1_b(E).$$

Let $\varepsilon_n \to 0$, set $\mu_n := \mu_{\varepsilon_n}$ and let $t \geqslant 0$ be arbitrary, fixed. By Proposition 5.6 μ_n is asymptotically tight and thus has a weakly* convergent subsequence (again denoted by μ_n) such that $\mu_n \rightharpoonup \nu$. The uniform e-property for $P^n_t := P^{\varepsilon_n}_t$ on E has been verified in Lemma 2.7 and by Theorem 4.1 we have

$$P^n_t F(x) \to P_t F(x) \quad \text{for } n \to \infty$$

for all $F \in \mathcal{G}$, $t \geqslant 0$, $x \in H$. An application of Lemma 5.3 thus yields that ν is $\mathcal{G}$-invariant.

We show next that this implies that ν is an invariant measure for P_t. First note that $\mathcal{G}$ is an algebra (w.r.t. pointwise multiplication) of bounded real-valued functions on E that contains the constant functions. By [29, II.3 a), p.54], $\mathcal{G}$ separates points of E, which by [11, Theorem 6.8.9] implies that $\mathcal{G}$ generates the Borel σ-algebra $\mathcal{B}(E)$. Set

$$\mathcal{H} := \mathcal{H}(\nu, t) := \left\{ F \in \mathcal{B}_b(E) \ :\ \int_E P_t F\, d\nu = \int_E F\, d\nu \right\}.$$

Clearly, $1 \in \mathcal{H}$ and $\mathcal{H}$ is closed under monotone convergence by the Markov property and Beppo-Levi's monotone convergence lemma. Further, $\mathcal{H}$ is closed under uniform convergence by Lebesgue's dominated convergence theorem and the Markov property. We have already shown $\mathcal{G} \subset \mathcal{H}$. Hence, by the monotone class theorem [11, Theorem 2.12.9 (ii)], $\mathcal{B}_b(E) \subset \mathcal{H}$ and therefore $\mathcal{B}_b(E) = \mathcal{H}$. Since $\nu(H) = 1$ this implies that ν is an invariant measure for P_t. By Theorem 3.1 there is a unique invariant measure μ for P_t. Thus $\mu = \nu$ and by uniqueness, the whole sequence μ_n converges weakly* to μ.

Appendix A. Notation

We work with generic constants $C \geqslant 0$, $c > 0$ that are allowed to change value from line to line and we write

$$A \lesssim B$$

if there is a constant $C \geqslant 0$ such that $A \leqslant CB$. For a metric space (E, d), $R > 0$, $x \in E$ we let $B_R(x)$ denote the open ball of radius R in E centered at x. Moreover, we let $\mathcal{B}(E)$ denote the Borel sigma algebra and $\mathcal{B}_b(E)$ the space of bounded, Borel-measurable functions on E. The $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ is denoted by S^{d-1}. For notational convenience we set

$$a^{[m]} := |a|^{m-1}a \quad \text{for } a \in \mathbb{R}, m \geqslant 0$$

and

$$|\xi|^{-1}\xi := \begin{cases} |\xi|^{-1}\xi & \text{if } \xi \in \mathbb{R}^d \setminus \{0\} \\ \bar{B}_1(0) & \text{if } \xi = 0. \end{cases}$$

For $m \geqslant 1$ we set $L^m(\mathcal{O})$ to be the usual Lebesgue spaces with norm $\|\cdot\|_{L^m}$ and we shall often use the shorthand notation $L^m := L^m(\mathcal{O}), \|\cdot\|_m := \|\cdot\|_{L^m(\mathcal{O})}$. We let $B^m_R(x)$ be the open ball in L^m of radius $R > 0$ centered at $x \in L^m$. We further define $L^m_{\text{av}} := L^m_{\text{av}}(\mathcal{O})$ to be the space of all functions in L^m with zero average, that is,

$$L^m_{\text{av}}(\mathcal{O}) := \{v \in L^m(\mathcal{O}) : \int_{\mathcal{O}} v d\xi = 0\}$$

and $H^k_{\text{av}} := H^k \cap L^2_{\text{av}}$, where H^k are the usual Sobolev spaces. For a function $v \in L^m(\mathcal{O})$ we define its extension to all of $\mathbb{R}^d$ by

$$\bar{v}(\xi) = \begin{cases} v(\xi) & \text{if } \xi \in \mathcal{O} \\ 0 & \text{otherwise.} \end{cases}$$

Let $J : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, continuous, radial function with compact support, $J(0) > 0$, $\int_{\mathbb{R}^d} J(z)\, dz = 1$. We then consider the following nonlocal averaged Sobolev-type spaces: For $\varepsilon > 0$, $m \geqslant 1$, let $V_\varepsilon := L^m_{J^\varepsilon}(\mathcal{O})$ be equal to $L^m_{\text{av}}(\mathcal{O})$ with the topology coming from the norm

$$\begin{aligned} \|v\|^m_{J^\varepsilon} &:= \frac{C_{J,m}}{2m\varepsilon^d} \int_{\mathcal{O}} \int_{\mathcal{O}} J\left(\frac{\xi - \zeta}{\varepsilon}\right) \left|\frac{v(\zeta) - v(\xi)}{\varepsilon}\right|^m d\zeta\, d\xi \\ &= \frac{C_{J,m}}{2m} \int_{\mathcal{O}} \int_{\mathbb{R}^d} J(z) 1_{\mathcal{O}}(\xi + \varepsilon z) \left|\frac{\bar{v}(\xi + \varepsilon z) - v(\xi)}{\varepsilon}\right|^m dz\, d\xi, \end{aligned}$$

where $C_{J,m}$ is a normalization constant given by

$$C^{-1}_{J,m} := \frac{1}{2} \int_{\mathbb{R}^d} J(z) |z_d|^m\, dz.$$

For notational convenience we set

$$J^\varepsilon(\xi) := \frac{C_{J,m}}{\varepsilon^{d+m}} J\left(\frac{\xi}{\varepsilon}\right) \quad \forall \xi \in \mathbb{R}^d.$$

By [4, Proposition 6.25] the norm $\|v\|_{J^\varepsilon}$ is equivalent to $\|v\|_m$. In particular, $L^m_{J^\varepsilon}(\mathcal{O})$ is a reflexive Banach space for all $m \in (1, \infty)$. Moreover, by [12],

$$\|\cdot\|_{V_\varepsilon} = \|\cdot\|_{J^\varepsilon} \leqslant C \|\cdot\|_{W^{1,m}}, \tag{A.1}$$

for some constant $C > 0$ independent of $\varepsilon > 0$.

We say that a function $X \in L^1([0,T] \times \Omega; H)$ is $\mathcal{F}_t$-progressively measurable if $X 1_{[0,t]} \in L^1([0,t]) \otimes \mathcal{F}_t; H)$ for all $t \geqslant 0$.

Let E be a Banach space. For a Feller semigroup P_t on $\mathcal{B}_b(E)$ we define the dual semigroup on the space of probability measures $\mathcal{M}_1(E)$ on E by

$$P^*_t \mu(B) := \int_E P_t 1_B(x) d\mu(x)$$

and the time averages

$$Q_T(x, B) := \frac{1}{T} \int_0^T P_t 1_B(x) dt, \quad \forall B \in \mathcal{B}(E).$$

We further set

$$Q_T\mu(B) := \int_E Q_T(x,B)d\mu(x).$$

We say that a probability measure μ on E is invariant for P_t if $P_t^*\mu = \mu$ for all $t \geqslant 0$. For an invariant probability measure μ we define its basin of attraction by

$$\mathcal{T}(\mu) := \{x \in E : Q_T(x,\cdot) = \int_0^T P_t(x,\cdot)dt \rightharpoonup^* \mu \text{ for } T \to \infty\} \subset E.$$

We say that P_t satisfies the e-property if, for some constant $C > 0$,

$$\|P_tF(x) - P_tF(y)\|_E \leqslant C\,\mathrm{Lip}(F)\|x-y\|_E \quad \forall x,y \in \mathbb{E},\, F \in \mathrm{Lip}(E).$$

For a Banach space E we define the space of cylindrical functions on E by

$$\mathcal{F}C_b^1(E) := \{f(l_1,\dots,l_k) \,:\, k \in \mathbb{N},\, l_1,\dots,l_k \in E^*,\, f \in C_b^1(\mathbb{R}^k)\}.$$

References

[1] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. A nonlocal p-Laplacian evolution equation with Neumann boundary conditions. *J. Math. Pures Appl. (9)*, 90(2):201–227, 2008.

[2] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions. *SIAM J. Math. Anal.*, 40(5):1815–1851, 2008/09.

[3] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. Local and nonlocal weighted p-Laplacian evolution equations with Neumann boundary conditions. *Publ. Mat.*, 55(1):27–66, 2011.

[4] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[5] V. Barbu. *Analysis and control of nonlinear infinite-dimensional systems*, volume 190 of *Mathematics in Science and Engineering*. Academic Press Inc., Boston, MA, 1993.

[6] V. Barbu, V. I. Bogachev, G. Da Prato, and M. Röckner. Weak solutions to the stochastic porous media equation via Kolmogorov equations: The degenerate case. *J. Funct. Anal.*, 237:54–75, 2006.

[7] V. Barbu and G. Da Prato. Ergodicity for nonlinear stochastic equations in variational formulation. *Appl. Math. Optim.*, 53(2):121–139, 2006.

[8] V. Barbu and G. Da Prato. Invariant measures and the Kolmogorov equation for the stochastic fast diffusion equation. *Stoch. Processes Appl.*, 120:1247–1266, 2010.

[9] V. Barbu, G. Da Prato, and M. Röckner. Stochastic nonlinear diffusion equations with singular diffusivity. *SIAM J. Math. Anal.*, 41(3):1106–1120, 2009.

[10] V. Barbu and M. Röckner. Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise. *Arch. Ration. Mech. Anal.*, 209(3):797–834, 2013.

[11] V. Bogachev. *Measure Theory*, volume 1. Springer, 2007.

[12] H. Brézis. How to recognize constant functions. Connections with Sobolev spaces. *Uspekhi Mat. Nauk, translated in Russian Math. Surveys*, 57(4(346)):59–74, 2002. Volume in honor of M. Vishik.

[13] I. Ciotir and J. M. Tölle. Convergence of invariant measures for singular stochastic diffusion equations. *Stochastic Process. Appl.*, 122(4):1998–2017, 2012.

[14] I. Ciotir and J. M. Tölle. Corrigendum to 'Convergence of invariant measures for singular stochastic diffusion equations' [Stoch. Proc. Appl. 122 (2012) 1998–2017]. *Stoch. Proc. Appl.*, 123(3):1178–1181, 2013.

[15] G. Da Prato and M. Röckner. Invariant measures for a stochastic porous medium equation. In *Stochastic analysis and related topics in Kyoto*, volume 41 of *Adv. Stud. Pure Math.*, pages 13–29. Math. Soc. Japan, Tokyo, 2004.

[16] G. Da Prato, M. Röckner, B. L. Rozovskii, and F.-Y. Wang. Strong solutions of stochastic generalized porous media equations: existence, uniqueness, and ergodicity. *Comm. Partial Differential Equations*, 31(1-3):277–291, 2006.

[17] B. Gess. Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise. *Ann. Probab.*, 42(2):818–864, 2014.

[18] B. Gess, W. Liu, and M. Röckner. Random attractors for a class of stochastic partial differential equations driven by general additive noise. *J. Differential Equations*, 251(4-5):1225–1253, 2011.

[19] B. Gess and J. M. Tölle. Multi-valued, singular stochastic evolution inclusions. *J. Math. Pures Appl.*, 101(6):789–827, 2014.

[20] B. Gess and J. M. Tölle. Stability of solutions to stochastic partial differential equations. *arXiv:1506.01230*, pages 1–39, 2015.

[21] B.-l. Guo and G.-l. Zhou. Exponential stability of stochastic generalized porous media equations with jump. *Appl. Math. Mech. (Engl. Ed.)*, 35(8):1067–1078, 2014.

[22] J. U. Kim. On the stochastic porous medium equation. *J. Differential Equations*, 220(1):163–194, 2006.

[23] T. Komorowski, S. Peszat, and T. Szarek. On ergodicity of some Markov processes. *Ann. Probab.*, 38(4):1401–1443, 2010.

[24] W. Liu. Harnack inequality and applications for stochastic evolution equations with monotone drifts. *J. Evol. Equ.*, 9(4):747–770, 2009.

[25] W. Liu. On the stochastic p-Laplace equation. *J. Math. Anal. Appl.*, 360(2):737–751, 2009.

[26] W. Liu. Ergodicity of transition semigroups for stochastic fast diffusion equations. *Front. Math. China*, 6(3):449–472, 2011.

[27] W. Liu and J. M. Tölle. Existence and uniqueness of invariant measures for stochastic evolution equations with weakly dissipative drifts. *Electron. Commun. Probab.*, 16:447–457, 2011.

[28] W. Liu and F.-Y. Wang. Harnack inequality and strong Feller property for stochastic fast-diffusion equations. *J. Math. Anal. Appl.*, 342(1):651–662, 2008.

[29] Z.-M. Ma and M. Röckner. *Introduction to the theory of (non-symmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin–Heidelberg–New York, 1992.

[30] C. Marinelli and G. Ziglio. Ergodicity for nonlinear stochastic evolution equations with multiplicative Poisson noise. *Dynamics of PDE*, 7(1):1–23, 2010.

[31] S.-X. Ouyang. Harnack inequalities and applications for multivalued stochastic evolution equations. *IDAQP*, 14(2):261–278, 2011.

[32] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[33] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[34] A. van der Vaart and J. Wellner. *Weak convergence and empirical processes*. Springer Series in Statistics. Springer, 1996.

[35] F.-Y. Wang. Harnack inequalities and applications for multivalued stochastic evolution equations. *The Annals of Probability*, 35(4):1333–1350, 2007.

[36] F.-Y. Wang. *Harnack inequalities for stochastic partial differential equations*. Springer Briefs in Mathematics. Springer, New York, 2013.

[37] F.-Y. Wang. Asymptotic couplings by reflection and applications for nonlinear monotone SPDEs. *Nonlinear Anal.*, 117:169–188, 2015.

[38] F.-Y. Wang. Exponential convergence of non-linear monotone SPDEs. *Discrete and Continuous Dynamical Systems*, 35(11):5239–5253, 2015.