Fractional approximation of solutions of evolution equations

Anatoly N. Kochubei, Yuri G. Kondratiev

Abstract

We show how to approximate a solution of the first order linear evolution equation, together with its possible analytic continuation, using a solution of the time-fractional equation of order $\delta > 1$, where $\delta \to 1 + 0$.

1 Introduction

In many applications (see, for example, [2, 4]) we encounter evolution equations, which are too singular for application of the standard semigroup approach. A possible remedy is to consider the equation on a scale of Banach spaces and to use an approach known as Ovsyannikov's method; see [3] for its history and complete references.

Barkova and Zabreiko [1] extended this method to evolution equations with the Caputo-Djrbashian fractional derivative $\mathbb{D}_t^{(\alpha)}$ of order $\alpha > 0$. In particular, they showed that the Cauchy problem considered in this setting becomes less singular with the growth of α. For example, it can happen that a solution u of the first order equation $u'_t = Au$ exists only on a finite time interval, while the existence of solutions of the equation $\mathbb{D}_t^{\delta} u = Au$, $1 < \delta < 2$ can be guaranteed for all $t > 0$. Therefore such solutions u_δ are natural means of approximating solutions of the initial first order equation.

In this note we prove an even stronger result – the "fractional approximations" $u_\delta(t^{1/\delta})$ approach, as $\delta \to 1 + 0$, not only the solution u, but its

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maximal possible analytic continuation. As we will show, this follows from Hardy's theorem [5] about approximate analytic continuation obtained by summing certain divergent series.

2 Cauchy problems

Consider the Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = u_0,$$

in a scale of Banach spaces X_ω, $\omega \in [0, 1)$, $X'_\omega \subset X''_\omega$ for $\omega' < \omega''$. Here A is a linear operator, which is bounded from X'_ω to X''_ω for each couple of indices with $\omega' < \omega''$, and

$$\|A\|_{\omega' \to \omega''} \leqslant \frac{C}{\omega'' - \omega'}.$$

For each couple (ω', ω'') and any initial vector $u_0 \in X_{\omega'}$, such that $Au_0 \in X_{\omega'}$, there exists a local solution in $X_{\omega''}$ of the form

$$u(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n u_0, \quad 0 \leqslant t < T_{\omega', \omega''};$$

see [2, 7].

Let $1 < \delta < 2$. Consider also the Cauchy problem with the Caputo-Djrbashian fractional derivative

$$\mathbb{D}_t^{(\delta)} u_\delta = A u_\delta, \quad u_\delta(0) = u_0, u'_\delta(0) = 0.$$

Under the same assumptions, it has the solution (see [1])

$$u_\delta(t) = \sum_{n=0}^{\infty} \frac{t^{\delta n}}{\Gamma(\delta n + 1)} A^n u_0,$$

which exists for all $t > 0$ (the above series equivalent to the iteration process from [1] converges on any finite interval).

In fact, the solution $u(t)$ can be continued to a holomorphic function on the disk $\{t \in \mathbb{C} : \ |t| < T_{\omega', \omega''}\}$ while the function

$$u_\delta(t^{1/\delta}) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\delta n + 1)} A^n u_0$$

is extended to an entire function. Both are with values in $X_{\omega''}$. Note that

$$u_\delta(t^{1/\delta}) = \sum_{n=0}^{\infty} \lambda_n(\delta)\,(A^n u_0)\,\frac{t^n}{n!}$$

where $\lambda_n(\delta) = \dfrac{n!}{\Gamma(\delta n + 1)}$.

3 Analytic continuations

Definition (see [5]). *Let $f(z)$ be a holomorphic function on a neighborhood of the point $z = 0$ determined there by a convergent power series. The Mittag-Leffler star $G(f)$ of the function f is a domain obtained from $\mathbb{C}$ as follows: draw a ray from the origin to each singular point of the function f, and cut the plane along the part of the ray located after the singular point.*

For example, the star of the function $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$ is $\mathbb{C}\backslash[1,\infty)$.

Theorem. *Suppose that the solution $u(t)$ is continued along rays to a single-valued holomorphic function on $G(u)$. Then*

$$u_\delta(t^{1/\delta}) \longrightarrow u(t), \quad t \in G(u),$$

uniformly on any closed bounded domain inside $G(u)$.

Proof. By Theorem 135 (Section 8.10) from [5] (evidently valid also for vector-functions), it is sufficient to prove that the function

$$\varphi_\delta(z) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\delta n + 1)} z^n, \quad \delta > 1,$$

is entire and tends to $\dfrac{1}{1-z}$ uniformly, as $\delta \to 1 + 0$, in each closed bounded domain not intersecting the semi-axis $[1, \infty)$. The entireness property follows from the Stirling formula. The above convergence is obvious for $|z| < 1$. Due to the uniqueness theorem for holomorphic functions, it will be sufficient to prove that $\varphi_\delta(z)$ tends to some holomorphic function uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

The function φ_δ can be represented as a Wright function:

$$\varphi_\delta(z) = {}_2\Psi_1\left[\begin{matrix}(1,1) & (1,1)\\ (1,\delta) & \end{matrix}\Big| z\right] = \sum_{k=0}^{\infty} \frac{[\Gamma(1+k)]^2}{\Gamma(1+\delta k)} \frac{z^k}{k!}$$

with parameter $\Delta = \delta - 2$, which means that $-1 < \Delta < 1$ (for information about the Wright functions see [6], 1.11).

Let us use the integral representation for the Wright functions. In our case we have

$$\varphi_\delta(z) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)[\Gamma(1-s)]^2}{\Gamma(1-\delta s)} (-z)^{-s} ds.$$

Since $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, we find that

$$\varphi_\delta(z) = \frac{1}{2i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(1-s)}{\sin(\pi s)\Gamma(1-\delta s)} (-z)^{-s} ds, \quad |\arg(-z)| < \frac{(3-\delta)\pi}{2} \qquad (1)$$

(see [6], (1.11.21), (1.11.22)). Note that for δ close to 1 this representation becomes valid for any z outside the semi-axis $(0,\infty)$.

By a corollary of the Stirling formula ([6], (1.5.14)), for $s = \frac{1}{2} + iy$, $|y| \to \infty$, we have

$$\left|\frac{\Gamma(1-s)}{\Gamma(1-\delta s)}\right| \leqslant C|y|^{\frac{\delta-1}{2}} \exp\left\{\frac{\pi(\delta-1)|y|}{2}\right\}.$$

Noticing also that $|\sin(\pi(\frac{1}{2}+iy)| = |\cos(i\pi y)| = \frac{1}{2}|e^{\pi y} + e^{-\pi y}|$ we see the possibility to pass to the limit in (3.1), as $\delta \to 1+0$. Thus, for $z \notin [0,\infty)$, uniformly on compact sets,

$$\varphi_\delta(z) \longrightarrow \frac{1}{2i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(-z)^{-s}}{\sin(\pi s)} ds.$$

The right-hand side is a holomorphic function on $\mathbb{C}\backslash[0,\infty)$. □

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Anatoly N. Kochubei
Institute of Mathematics
National Academy of Sciences
of Ukraine
Tereshchenkivska 3
01601 Kyiv, Ukraine
kochubei@i.com.ua
kochubei@imath.kiev.ua

Yuri G. Kondratiev
Fakultät für Mathematik
Universität Bielefeld
33615 Bielefeld,Germany
kondrat@math.uni-bielefeld.de