

PHASE TRANSITIONS IN CONTINUUM FERROMAGNETS WITH UNBOUNDED SPINS

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ABSTRACT. States of thermal equilibrium of an infinite system of interacting particles in \mathbb{R}^d are studied. The particles bear ‘unbounded’ spins with a given symmetric a priori distribution. The interaction between the particles is pairwise and splits into position-position and spin-spin parts. The position-position part is described by a superstable potential, and the spin-spin part is attractive and of finite range. Thermodynamic states of the system are defined as tempered Gibbs measures on the space of marked configurations. It is proved that the set of such measures contains at least two elements if the activity is big enough.

1. INTRODUCTION

1.1. Posing the problem. The mathematical theory of thermal equilibrium of infinite particle systems relies on the use of conditional probabilities, see [9, 24, 30], by means of which one defines the set of Gibbs measures that exist at given values of the model parameters. The multiplicity of such measures is then interpreted as the possibility for the system to undergo a phase transition and is one of the most fundamental aspects of the theory. Historically, the Gibbsian formalism was first developed for the Ising spin model, where each ‘particle’ was associated with a point $x \in \mathbb{Z}^d$ and can be in one of two states, cf. [7, 8]. This is the simplest model of a crystalline magnet. It took, however, eight years (since the publication of first Dobrushin’s papers) until the Gibbs states of lattice models with ‘unbounded’ spins were constructed in [22] by means of new tools developed during that time. In noncrystalline magnets, the particles are distributed over a continuous medium (e.g., \mathbb{R}^d), and their positions may not be fixed. The corresponding physical substances are e.g. magnetic gases, ferrofluids, amorphous magnets, etc., see [12] for further information on this issue. For a ferrofluid with hard core repulsion and Ising spins, the existence of spontaneous magnetization was proved in [12], which later on was extended in [28] to similar models with continuous bounded spins. The results of both these works can be interpreted as the proof of the multiplicity of the corresponding Gibbs measures provided their existence is established. In [10], the existence and multiplicity of Gibbs measures were proved for the model in

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which each particle can be in one of q states – continuum Potts model. Our aim is to elaborate the theory of phase transitions in systems of particles in continuum carrying ‘unbounded’ spins. To this end we employ the latest developments in the theory of Gibbs measures with irregular underlying sets [5, 15, 16, 19, 20] combined with contemporary methods of the analysis on configuration spaces [1, 17, 18, 21].

There are two different approaches to studying continuum systems of particles with spins: (a) the positions of the particles are taken at random from an ensemble characterized by a given probability law, and the spins are distributed according to a random ‘spin-only’ Gibbs measure; (b) the interaction between the particles contains spin-spin and position-position parts and the joint probability distribution is given by a ‘position and spin’ Gibbs measure. Phase transitions in the systems of the first type (quenched magnets with Poisson-distributed positions) have been considered in [4]. In the present paper, we study a system of the second type, with the position-position interactions satisfying the strong superstability condition, cf. [29, 26, 27, 19]. Our main technical tool is the finite volume reduction to a quenched system and the use of the percolation theory, in the spirit of [10] and [13].

1.2. The paper overview. We consider the following infinite-particle model. Each particle is characterized by position $x \in X = \mathbb{R}^d$, $d \geq 1$, and spin $\sigma \in S = \mathbb{R}$. The particles interact via a pair interaction potential of the form

$$\Psi(x \times \sigma, x' \times \sigma') = \Phi(x - x') - \phi(x - x')\sigma\sigma' \quad (1.1)$$

and are characterized by a single-particle probability measure χ on S . Here $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ are suitable functions, see Section 3 below.

The Gibbs measures of the model are defined as probability measures on the space $\Gamma(X, S) = \{\hat{\gamma} \subset X \times S : p_X(\hat{\gamma}) \subset \Gamma(X)\}$ of marked configurations, where p_X is the natural projection $X \times S \rightarrow X$ and $\Gamma(X)$ is the space of locally finite subsets of X . As is typical for systems with unbounded spins, cf. [4, 15, 20, 22], we work with the Gibbs measures that are supported on the configurations satisfying certain bounds on their density and spin growth (the *tempered* Gibbs measures). In the study of the set $\mathcal{G}^t(\Psi, \chi)$ of all such measures one typically poses the following questions:

- (E) *Existence*: is $\mathcal{G}^t(\Psi, \chi)$ not empty?
- (U) *Uniqueness*: is $\mathcal{G}^t(\Psi, \chi)$ a singleton?
- (M) *Multiplicity*: does $\mathcal{G}^t(\Psi, \chi)$ contain at least two elements?

Usually, only sufficient conditions for positive answer to these questions are obtained, which justifies distinguishing between (U) and (M). Positive answer to (M) indicates the appearance of phase transitions in the system. The comprehensive answer to all the three questions is known only for the classical Ising model where $X = \mathbb{Z}^d$ and $S = \{-1, 1\}$, see e.g., [30]. (E) is well-studied also for more general ‘crystalline’ type spin models, including

the case of X being a general graph and $S = \mathbb{R}$, see [15]. For continuum models with $X = \mathbb{R}^d$ and compact spin space S , (E) is essentially similar to the case of continuum gas models without spins, see [19] and the references therein. In the case of $S = \mathbb{R}$, questions (E) and (U) have recently been studied in [2]. In [10], question (M) was studied by exploiting a continuum version of the random cluster model and the percolation theory.

In the present work, we give an answer to question (M) in the general case of $X = \mathbb{R}^d$, $S = \mathbb{R}$ in the absence of the restrictive ‘hard core’ and ‘compactness of spins’ conditions. Instead, we assume the strong super-stability of the position-position interaction and the exponential moment bound of the single-particle measure χ , see Section 3. We exploit the fibre bundle structure of the space $\Gamma(X, S)$ studied in [4, 5], which allows us to disintegrate any element $\mu \in \mathcal{G}^t(\Psi, V)$ as $\mu(d\gamma) = \omega_\gamma(d\sigma) (p_X^* \mu)(d\gamma)$, where $\omega_\gamma(d\sigma)$ is a Gibbs measure on the product space S^γ , for a.a. $\gamma \in \Gamma(X)$. This allows for applying a suitable modification of methods developed in [4].

The structure of the paper is as follows. In Section 2, we present a number of facts from the theory of marked configuration spaces. The crucial one is a fibre bundle structure of such spaces. In Section 3, we describe the model, cf. Assumption (M), and present the main result of this paper in Theorem 3.2. In Subsection 3.3, we sketch the proof of the existence of tempered Gibbs measures of our model. The proof of Theorem 3.2 is given in Section 4 and is based on Lemma 4.1, which states that the magnetization in local states can be uniformly positive. The proof of Lemma 4.1 is in turn based on a modification of Wells’ inequality [31] and the result of [13] that relates the existence of a ferromagnetic phase of the Ising model on a general graph to the Bernoulli bond percolation thereon. The existence of such percolation in our framework is stated in Lemma 4.3 and proved in Section 5, by extending the general scheme proposed in [10]. The main idea is to pass to an auxiliary percolation model, which is dominated by the percolation in question, see Lemmas 5.1, 5.2, and 5.4.

2. MARKED CONFIGURATION SPACES

2.1. The spaces of configurations. The configuration space on $X = \mathbb{R}^d$, $d \geq 1$, is

$$\Gamma(X) = \{\gamma \subset X : N(\gamma_\Lambda) < \infty \text{ for any } \Lambda \in \mathcal{B}_0(X)\}, \quad (2.1)$$

where $\mathcal{B}_0(X)$ is the collection of all compact subsets of X , $\gamma_\Lambda := \gamma \cap \Lambda$, and $N(\cdot)$ denotes cardinality. Let $C_0(X)$ be the set of all continuous functions $f : X \rightarrow \mathbb{R}$ with compact support. The configuration space $\Gamma(X)$ is endowed with the vague topology, which is the weakest topology that makes continuous all the maps

$$\Gamma(X) \ni \gamma \mapsto \sum_{x \in \gamma} f(x), \quad f \in C_0(X).$$

This topology is metrizable in the way that makes $\Gamma(X)$ a Polish space (see, e.g., [14, Section 15.7.7] or [25, Proposition 3.17]). An explicit construction of the appropriate metric can be found in [18]. By $\mathcal{P}(\Gamma(X))$ we denote the set of all probability measures on the Borel σ -algebra $\mathcal{B}(\Gamma(X))$ of subsets of $\Gamma(X)$.

Remark 2.1. In a similar fashion, the configuration space $\Gamma(Y)$ can be defined for an arbitrary Riemannian manifold Y . In Subsection 4.3, we use the space $\Gamma(X^{(2)})$, where $X^{(2)}$ is the collection of two-element subsets of X , which can be identified with the symmetrization of the space $(X \times X) \setminus \{(x, x) : x \in X\}$ and thus possesses a Riemannian manifold structure.

Let us now consider the product $X \times S$, $S = \mathbb{R}$. The canonical projection $p_X : X \times S \rightarrow X$ can naturally be extended to the configuration space $\Gamma(X \times S)$. However, for a configuration $\hat{\gamma} \in \Gamma(X \times S)$, its image $p_X(\hat{\gamma})$ is a subset of X that in general admits accumulation and multiple points and hence does not belong to $\Gamma(X)$. The marked configuration space $\Gamma(X, S)$ is defined in the following way:

$$\Gamma(X, S) = \{\hat{\gamma} \in \Gamma(X \times S) : p_X(\hat{\gamma}) \in \Gamma(X)\}.$$

The space $\Gamma(X, S)$ is endowed with a metrizable topology defined as the weakest topology that makes continuous the maps

$$\Gamma(X, S) \ni \hat{\gamma} \mapsto \sum_{x \in p_X(\hat{\gamma})} f(x, \sigma_x) \quad (2.2)$$

for all bounded continuous functions $f \in X \times S \rightarrow \mathbb{R}$ such that $\text{supp} f(\cdot, \sigma) \subset \Lambda$, for some $\Lambda \in \mathcal{B}_0(X)$ and all $\sigma \in S$. This topology has been used in [1, 3, 21]. It makes $\Gamma(X, S)$ a Polish space, cf. [3, Section 2], where a concrete metric is given. We equip $\Gamma(X, S)$ with the corresponding Borel σ -algebra $\mathcal{B}(\Gamma(X, S))$.

Along with $\Gamma(X, S)$ we will also use the spaces $\Gamma(\Lambda, S)$, $\Lambda \in \mathcal{B}_0(X)$, and the space $\Gamma_0(X, S) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \Gamma(\Lambda, S)$ of finite marked configurations, endowed with the Borel σ -algebras $\mathcal{B}(\Gamma(\Lambda, S))$ and $\mathcal{B}(\Gamma_0(X, S))$ respectively, which are induced by the Euclidean structure of X . It is known that $\mathcal{B}(\Gamma_0(X, S)) = \{A \cap \Gamma_0(X, S) : A \in \mathcal{B}(\Gamma(X, S))\}$.

The spaces $\Gamma(\Lambda, S)$ and $\Gamma_0(X, S)$ can be identified with the corresponding subspaces of $\Gamma(X, S)$ via the natural embedding. Clearly, these subspaces belong to $\mathcal{B}(\Gamma(X, S))$ and σ -algebras $\mathcal{B}(\Gamma(\Lambda, S))$ and $\mathcal{B}(\Gamma_0(X, S))$ can be considered as sub-algebras of $\mathcal{B}(\Gamma(X, S))$.

On the other hand, we can introduce the algebras $\mathcal{B}_\Lambda(\Gamma(X, S))$ of sets $C_B := \{\gamma \in \Gamma(X) : \gamma_\Lambda \in B\}$, $B \in \mathcal{B}(\Gamma(\Lambda, S))$ and define the algebra of local (cylinder) sets

$$\mathcal{B}_{\text{loc}}(\Gamma(X, S)) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \mathcal{B}_\Lambda(\Gamma(X, S)). \quad (2.3)$$

In a similar way, one introduces the spaces $\Gamma(\Lambda)$, $\Gamma_0(X)$ and the corresponding algebras $\mathcal{B}(\Gamma(\Lambda))$, $\mathcal{B}(\Gamma_0(X))$ and $\mathcal{B}_{\text{loc}}(\Gamma(X))$.

It is possible to show that a given $F : \Gamma_0(X, S) \rightarrow \mathbb{R}$ is $\mathcal{B}(\Gamma_0(X, S))$ -measurable if and only if, for each $n \in \mathbb{N}$, there exists a symmetric Borel function $F_n : (X \times S)^n \rightarrow \mathbb{R}$ such that

$$F(\hat{\gamma}) = F_n((x_1, \sigma_1), \dots, (x_n, \sigma_n)), \quad \hat{\gamma} = \{(x_1, \sigma_1), \dots, (x_n, \sigma_n)\}.$$

For the single-spin measure $\chi \in \mathcal{P}(S)$ (=: the space of probability measures on S) and some $z > 0$, we introduce the Lebesgue-Poisson measure $\hat{\lambda}_z$ on $\mathcal{B}(\Gamma_0(X, S))$ by the relation

$$\begin{aligned} \int_{\Gamma_0(X, S)} F(\hat{\gamma}) \hat{\lambda}_z(d\hat{\gamma}) &= F(\emptyset) \\ &+ \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{(X \times S)^n} F_n((x_1, \sigma_1), \dots, (x_n, \sigma_n)) \chi(d\sigma_1) dx_1 \cdots \chi(d\sigma_n) dx_n, \end{aligned} \quad (2.4)$$

which has to hold for all measurable $F : \Gamma_0(X, S) \rightarrow \mathbb{R}_+$. Likewise, the Lebesgue-Poisson measure λ_z on $\mathcal{B}(\Gamma_0(X))$ is defined by

$$\int_{\Gamma_0(X)} F(\gamma) \lambda_z(d\gamma) = F(\emptyset) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{X^n} F_n(x_1, \dots, x_n) dx_1 \cdots dx_n, \quad (2.5)$$

holding for all measurable $F : \Gamma_0(X) \rightarrow \mathbb{R}_+$.

2.2. Disintegration of measures. The space $\Gamma(X, S)$ has the structure of a fibre bundle over $\Gamma(X)$, with fibres $p_X^{-1}(\gamma)$ which can be identified with the product

$$S^\gamma = \prod_{x \in \gamma} S_x, \quad S_x = S.$$

Thus, each $\hat{\gamma} \in \Gamma(X, S)$ can be represented by the pair

$$\hat{\gamma} = (\gamma, \sigma_\gamma), \quad \text{where } \gamma = p_X(\hat{\gamma}) \in \Gamma(X), \quad \sigma_\gamma = (\sigma_x)_{x \in \gamma} \in S^\gamma.$$

It follows directly from the definition of the corresponding topologies that the map $p_X : \Gamma(X, S) \rightarrow \Gamma(X)$ is continuous. For each $B \in \mathcal{B}(\Gamma(X))$, its preimage $p_X^{-1}(B)$ is in $\mathcal{B}(\Gamma(X, S))$. Likewise, $p_X^{-1}(B) \in \mathcal{B}(\Gamma_0(X, S))$ for each $B \in \mathcal{B}(\Gamma_0(X))$. In particular, $p_X^{-1}(\gamma) = p_X^{-1}(\{\gamma\}) = S^\gamma \in \mathcal{B}(\Gamma_0(X, S)) \subset \mathcal{B}(\Gamma(X, S))$. We equip each S^γ with the product topology and denote by $\mathcal{B}(S^\gamma)$ the corresponding Borel σ -algebra. By Kuratowski's theorem, see [23], it is possible to show that

$$\mathcal{B}(S^\gamma) = \{A \cap S^\gamma : A \in \mathcal{B}(\Gamma(X, S))\}.$$

Then, for each probability measure μ on $\mathcal{B}(\Gamma(X, S))$, one can define its projection $p_X^* \mu$ on $\mathcal{B}(\Gamma(X))$ by setting

$$(p_X^* \mu)(B) = \mu(p_X^{-1}(B)), \quad B \in \mathcal{B}(\Gamma(X)).$$

This in turn allows one to disintegrate

$$\mu(d\hat{\gamma}) = \omega_\gamma(d\sigma_\gamma)(p_X^*\mu)(d\gamma), \quad (2.6)$$

where ω_γ is a probability measure on $\mathcal{B}(S^\gamma)$ for $p_X^*\mu$ -almost all $\gamma \in \Gamma(X)$. Moreover, for each $B \in \mathcal{B}(S^\gamma)$, the map $\gamma \mapsto \omega_\gamma(B)$ is $\mathcal{B}(\Gamma(X))$ -measurable. A similar disintegration can be applied to measures on $\mathcal{B}(\Gamma_0(X, S))$. In particular, for the measures introduced in (2.4) and (2.5), one has

$$\hat{\lambda}_z(d\hat{\gamma}) = \chi_\gamma(d\sigma_\gamma)\lambda_z(d\gamma), \quad \chi_\gamma(d\sigma_\gamma) := \bigotimes_{x \in \gamma} \chi(d\sigma_x), \quad \gamma \in \Gamma_0(X). \quad (2.7)$$

2.3. Tempered marked configurations. In the sequel, we use the following partition of X . For $k = (k^{(1)}, \dots, k^{(d)}) \in \mathbb{Z}^d$ and $l > 0$, we set

$$\Xi_k := \left\{ x \in X : x^{(i)} \in [l(k^{(i)} - 1/2), l(k^{(i)} + 1/2)] \right\}. \quad (2.8)$$

Given integer $v > 2$, we take $w \in \mathbb{N}$ such that

$$w \geq \frac{2(v-1)}{v-2}. \quad (2.9)$$

For these v and w , we then define, cf. (2.1),

$$F(\hat{\gamma}) = [N(\gamma)]^v + \sum_{x \in \gamma} |\sigma_x|^w, \quad \hat{\gamma} = (\gamma, \sigma_\gamma) \in \Gamma_0(X, S), \quad (2.10)$$

and

$$F_\alpha(\hat{\gamma}) = \sup_{k \in \mathbb{Z}^d} F(\hat{\gamma}_k) \exp(-\alpha|k|), \quad \hat{\gamma} \in \Gamma(X, S), \quad \alpha > 0, \quad (2.11)$$

where $\gamma_k := \gamma \cap \Xi_k$. By means of these functions we then set

$$\Gamma^t(X, S) = \{ \hat{\gamma} \in \Gamma(X, S) : F_\alpha(\hat{\gamma}) < \infty \text{ for each } \alpha > 0 \}, \quad (2.12)$$

which is the space of tempered marked configurations. Note that $\Gamma^t(X, S) \in \mathcal{B}(\Gamma(X, S))$ and is independent of l used in (2.8). In a similar way, we can define the space $\Gamma^t(X)$ of tempered configurations in X using the function $F_X(\gamma) := [N(\gamma)]^v$ in place of $F(\hat{\gamma})$. Observe that, for any $\gamma \in \Gamma^t(X)$ and $\sigma_\gamma = (\sigma_x)_{x \in \gamma}$ with $\sup_{x \in \gamma} |\sigma_x| < \infty$, we have $(\gamma, \sigma_\gamma) \in \Gamma^t(X, S)$.

Definition 2.2. A probability measure ν on $\mathcal{B}(\Gamma(X, S))$ is said to be tempered if $\nu(\Gamma^t(X, S)) = 1$.

3. THE MODEL AND MAIN RESULT

3.1. Description of the model. The interaction between the particles is supposed to be pair-wise and consisting of position-position and spin-spin parts described by measurable functions $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$, respectively, cf. (1.1). Another model ‘parameter’ is a single-spin measure $\chi \in \mathcal{P}(S)$. Since $\phi \geq 0$, the spin-spin interaction is of ferromagnetic type, cf. (3.1). By Φ_+ we denote the positive part of Φ , i.e.,

$\Phi_+ = \max\{\Phi; 0\}$. Thereby, for $\hat{\gamma} = (\gamma, \sigma_\gamma)$ with $\gamma \in \Gamma_0(X)$ and $\sigma_\gamma \in S^\gamma$, we define

$$H(\gamma) = \sum_{\{x,y\} \in \gamma} \Phi(x-y), \quad (3.1)$$

$$E(\sigma_\gamma) = - \sum_{\{x,y\} \in \gamma} \phi(x-y) \sigma_x \sigma_y.$$

The model parameters are supposed to satisfy the following

Assumption (M).

- (1) There exists $r > 0$ such that $\Phi_+(x) = 0$ whenever $|x| > r$.
- (2) For each $\delta > 0$, there exists $C_\delta < +\infty$ such that

$$\int_{|x| \geq \delta} \Phi_+(x) dx \leq C_\delta < \infty. \quad (3.2)$$

- (3) Φ is bounded from below and there exist $\epsilon > 0$ and positive A_Φ, B_Φ such that

$$H(\gamma) \geq A_\Phi \sum_{k \in \mathbb{Z}^d} [N(\gamma_k)]^{v+\epsilon} - B_\Phi N(\gamma), \quad \gamma_k = \gamma \cap \Xi_k, \quad (3.3)$$

for any $\gamma \in \Gamma_0(X)$, where v is as in (2.10).

- (4) $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is bounded and such that there exist $\phi_* > 0$ and $R > 0$, for which the following holds

$$\phi(x) \geq \phi_*, \quad \text{for } |x| \leq R; \quad \phi(x) = 0, \quad \text{for } |x| > R. \quad (3.4)$$

- (5) The measure $\chi \in \mathcal{P}(S)$ is symmetric with respect to $\sigma \rightarrow -\sigma$. There exist constants $\varkappa > 0$ and $u > w$, see (2.9), such that

$$\int_S \exp(\varkappa |s|^u) \chi(ds) < \infty, \quad (3.5)$$

and $\chi(\{0\}) < 1$.

- (6) The parameters r and R satisfy the relation

$$r < R/4. \quad (3.6)$$

Remark 3.1. Clearly, positive ϵ in (3.3) can be chosen in such a way that u in (3.5) also satisfies $u > 2(v + \epsilon - 1)/(v + \epsilon - 2)$, which is important for proving Proposition 3.4 below, see [2].

The property of Φ as in (3.3) is called *strong superstability* [29]. One of the best-understood examples of interaction of this type is given by the potential, which satisfies $\Phi(x) \geq c|x|^{-d(1+\epsilon)}$ in the vicinity of $x = 0$. In this case, one can take any $v > 2$. For a detailed study and historical comments see [26] and also [19, Remark 4.1.].

3.2. Main result. For $\Delta \subset X$, we write $\Delta^c = X \setminus \Delta$. Given $\Delta \in \mathcal{B}_0(X)$, for $\hat{\eta} = (\eta, \sigma_\eta) \in \Gamma(\Delta, S)$ and $\hat{\gamma} = (\gamma, \sigma_\gamma) \in \Gamma(\Delta^c, S)$, we set

$$H(\eta|\gamma) = H(\eta) + \sum_{x \in \eta} \sum_{y \in \gamma} \Phi(x - y) \quad (3.7)$$

and

$$E(\sigma_\eta|\sigma_\gamma) = E(\sigma_\eta) - \sum_{x \in \eta} \sum_{y \in \gamma} \phi(x - y) \sigma_x \sigma_y. \quad (3.8)$$

The Gibbs specification Π of the model is the family of probability kernels Π_Δ , $\Delta \in \mathcal{B}_0(X)$, defined by the integrals

$$\begin{aligned} \int_{\Gamma(X, S)} F(\hat{\eta}) \Pi_\Delta(d\hat{\eta}|\hat{\gamma}) &= [Z_\Delta(\hat{\gamma})]^{-1} \int_{\Gamma(\Delta, S)} F(\hat{\eta}_\Delta \cup \hat{\gamma}_{\Delta^c}) \\ &\times \exp\left(-H(\eta_\Delta|\gamma_{\Delta^c}) - E(\sigma_{\eta_\Delta}|\sigma_{\gamma_{\Delta^c}})\right) \hat{\lambda}_z(d\hat{\eta}_\Delta), \end{aligned} \quad (3.9)$$

which has to hold for all measurable functions $F : \Gamma(X, S) \rightarrow \mathbb{R}_+$ and all $\hat{\gamma} \in \Gamma^t(X, S)$, see (2.12). Here $\hat{\lambda}_z$ is the marked Lebesgue-Poisson measure defined in (2.4) and

$$Z_\Delta(\hat{\gamma}) = \int_{\Gamma(\Delta, S)} \exp\left(-H(\eta_\Delta|\gamma_{\Delta^c}) - E(\sigma_{\eta_\Delta}|\sigma_{\gamma_{\Delta^c}})\right) \hat{\lambda}_z(d\hat{\eta}_\Delta)$$

is the normalizing factor (partition function) making $\Pi_\Delta(\cdot|\hat{\gamma})$ a probability measure on $\Gamma(X, S)$, provided $Z_\Delta(\hat{\gamma}) \neq 0$ which is the case under Assumption (M), see [2].

A probability measure $\nu \in \mathcal{P}(\Gamma(X, S))$ is said to be a Gibbs measure associated with the specification Π if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equation

$$\nu(B) = \int_{\Gamma(X, S)} \Pi_\Delta(B|\hat{\gamma}) \nu(d\hat{\gamma}), \quad (3.10)$$

which has to hold for all $B \in \mathcal{B}(\Gamma(X, S))$ and $\Delta \in \mathcal{B}_0(X)$. By $\mathcal{G}^t(\Gamma(X, S))$ we denote the set of all tempered Gibbs measures, see Definition 2.2. The result of this work is given in the following

Theorem 3.2. *Let Assumption (M) hold and $d \geq 2$. Then there exists $z_c > 0$ such that*

$$N(\mathcal{G}^t(\Gamma(X, S))) \geq 2$$

for all $z > z_c$.

Observe that Theorem 3.2 contains two quite different in their nature statements: (i) $N(\mathcal{G}^t(\Gamma(X, S))) \neq \emptyset$ and (ii) $\mathcal{G}^t(\Gamma(X, S))$ contains at least two elements. In the next section, we present a sketch of the proof of (i). A complete proof of this is given in [2]. The proof of (ii) is based on the comparison with the classical Ising model on a random geometric graph and its relationship with percolation theory on this graph and will be given in Sections 4 and 5.

3.3. The existence of Gibbs measures. The main idea here is to show that, for at least some $\hat{\gamma}$, the family

$$\{\Pi_\Lambda(\cdot|\hat{\gamma})\}_{\Lambda \in \mathcal{B}_0(X)} \subset \mathcal{P}(\Gamma(X, S))$$

has accumulation points, which solve (3.10) and are tempered measures in the sense of Definition 2.2. These accumulation points are sought in the local set convergence topology (\mathfrak{L} -topology), which is defined as the weakest topology on $\mathcal{P}(\Gamma(X, S))$ that makes continuous all the evaluation maps $\mu \mapsto \mu(A)$, $A \in \mathcal{B}_{\text{loc}}(\Gamma(X, S))$, see (2.3). This topology is weaker than the usual weak topology for which the relative compactness is established by means of Prokhorov's theorem, see, e.g., [23]. Instead we can use the following instruments, cf. [9, Prop. 4.9].

Definition 3.3. A sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(\Gamma(X, S))$ is said to be locally equicontinuous (LEC) if for any $\Delta \in \mathcal{B}_0(X)$ and any $\{B_m\}_{m \in \mathbb{N}} \subset \mathcal{B}(\Gamma(\Delta, S))$, $B_m \searrow \emptyset$, $m \rightarrow \infty$, it follows that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu_n(B_m) = 0.$$

Proposition 3.4. *Each LEC sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(\Gamma(X, S))$ has accumulation points in the \mathfrak{L} -topology, which are probability measure on $\Gamma(X, S)$.*

Let $\{\Lambda_m\}_{m \in \mathbb{N}}$, be an exhausting sequence of compact subsets of X . This means that $\Lambda_m \subset \Lambda_{m+1}$ for all $m \in \mathbb{N}$ and $\Lambda_m \nearrow X$, $m \rightarrow \infty$. Set $\Pi_m = \Pi_{\Lambda_m}(\cdot|\hat{\gamma})$, $\hat{\gamma} \in \Gamma^t(X, S)$. The following fact was proved in [2].

Proposition 3.5. *For any $\hat{\gamma} \in \Gamma^t(X, S)$ and any choice of the exhausting sequence $\{\Lambda_m\}_{m \in \mathbb{N}}$, the sequence $\{\Pi_m\}_{m \in \mathbb{N}}$ is LEC.*

The next theorem states sufficient conditions for the existence and uniqueness of tempered Gibbs measures.

Theorem 3.6. [2] *Under Assumption (M) the following holds.*

- (i) *The set $\mathcal{G}^t(\Gamma(X, S))$ is nonempty; each of its elements has the property*

$$\forall \alpha > 0 \quad \sup_{k \in \mathbb{Z}^d} \int_{\Gamma(X, S)} e^{\alpha F(\hat{\gamma}_k)} \mu(d\hat{\gamma}) < \infty, \quad (3.11)$$

cf. (2.10) and (2.11).

- (ii) *There exist constants $\phi_0, z_0 > 0$ such that $N(\mathcal{G}(\Gamma^t(X, S))) = 1$ whenever $\phi(x) \leq \phi_0$, $|x| \leq R$, and $z \leq z_0$.*

In order to fix certain notations, we give a sketch of the proof of (i). It follows from Proposition 3.5 that, for any $\hat{\gamma} \in \Gamma^t(X, S)$, the sequence $\{\Pi_n\}_{n \in \mathbb{N}}$ has an accumulation point $\mu^{\hat{\gamma}} \in \mathcal{P}(\Gamma(X, S))$, so that there exists a subsequence Λ_{n_j} , $j \in \mathbb{N}$ such that

$$\mu^{\hat{\gamma}}(B) = \lim_{j \rightarrow \infty} \Pi_{\Lambda_{n_j}}(B|\hat{\gamma}), \quad (3.12)$$

holding for any $B \in \mathcal{B}_0(\Gamma(X, S))$. Standard limit transition arguments show that $\mu^{\hat{\gamma}}$ satisfies (3.10) and the estimate in (3.11).

4. PROOF OF THE MAIN RESULT

4.1. Proof of Theorem 3.2. From now on we fix the value of l in (2.8) by setting

$$l = R/2\sqrt{d}, \quad (4.1)$$

where R is as in (3.4). Then by $\mathcal{L} \subset \mathcal{B}_0(X)$ we denote the family of finite unions of the cells defined in (2.8) such that Ξ_0 is contained in each $\Lambda \in \mathcal{L}$. Next, for $n_* \in \mathbb{N}$ and $a > 0$, we define the sets $\widehat{\Gamma}_\pm(n_*, a) \subset \Gamma^t(X, S)$ as consisting of all those $\hat{\gamma} = (\gamma, \sigma_\gamma)$ that satisfy the following two conditions:

$$(a) \ \forall k \in \mathbb{Z}^d \quad N(\gamma_k) \geq n_*; \quad (b) \ \forall x \in \gamma \quad \sigma_x = \pm a. \quad (4.2)$$

In view of (2.11) and (2.12), each $\hat{\gamma} \in \widehat{\Gamma}_\pm(n_*, a)$ should have the property: for every $\alpha > 0$, there exists $N_\alpha > 0$ such that

$$\forall k \in \mathbb{Z}^d \quad N(\gamma_k) \leq N_\alpha e^{\alpha|k|},$$

i.e., γ should be in $\Gamma^t(X)$. Now we set

$$M(\hat{\gamma}) = \sum_{x \in \gamma_0} \sigma_x, \quad \hat{\gamma} \in \Gamma^t(X, S). \quad (4.3)$$

The map $\Gamma(X, S) \ni \hat{\gamma} \mapsto M(\hat{\gamma})$ is clearly measurable, cf. (2.2). The proof of Theorem 3.2 is based on the following result, which will be gradually proved in the remaining part of the paper.

Lemma 4.1. *Under the assumptions of Theorem 3.2, there exist $z_c > 0$, $n_* \in \mathbb{N}$ and positive constants a and m_c such*

$$\int_{\Gamma(X, S)} M(\hat{\gamma}) \Pi_\Lambda(d\hat{\gamma}|\hat{\xi}) \geq m_c. \quad (4.4)$$

for any $z > z_c$, $\hat{\xi} \in \widehat{\Gamma}_+(n_*, a)$ and $\Lambda \in \mathcal{L}$.

Note that $M(\hat{\gamma})$ clearly is $\Pi_\Lambda(\cdot|\hat{\xi})$ -integrable for each $\hat{\xi} \in \Gamma^t(X, S)$.

Proof of Theorem 3.2. Given $\hat{\xi} \in \widehat{\Gamma}_+(n_*, a)$, let $\hat{\xi}^- \in \widehat{\Gamma}_-(n_*, a)$ be such that $p_X(\hat{\xi}) = p_X(\hat{\xi}^-)$. By (3.9) we then get

$$\int_{\Gamma(X, S)} M(\hat{\gamma}) \Pi_\Lambda(d\hat{\gamma}|\hat{\xi}) = - \int_{\Gamma(X, S)} M(\hat{\gamma}) \Pi_\Lambda(d\hat{\gamma}|\hat{\xi}^-). \quad (4.5)$$

For $n \in \mathbb{N}$, let Λ_n be the union of all Ξ_k with $|k| \leq n$. For such $\hat{\xi}$ and $\hat{\xi}^-$, both sequences $\{\Pi_{\Lambda_n}(\cdot|\hat{\xi})\}$ and $\{\Pi_{\Lambda_n}(\cdot|\hat{\xi}^-)\}$ are relatively compact in the \mathfrak{L} -topology. Thus, one can pick the subsequence n_j , $j \in \mathbb{N}$, such that the following holds:

$$\Pi_{\Lambda_{n_j}}(\cdot|\hat{\xi}) \rightarrow \mu^{\hat{\xi}}, \quad \Pi_{\Lambda_{n_j}}(\cdot|\hat{\xi}^-) \rightarrow \mu^{\hat{\xi}^-}, \quad j \rightarrow +\infty,$$

see Propositions 3.4, 3.5 and formula (3.12). As in the proof of Theorem 3.6, his convergence yields that both $\mu^{\hat{\xi}}$ and $\mu^{\hat{\xi}^-}$ belong to $\mathcal{G}^t(\Gamma(X, S))$. At

the same time, by means of (3.11), Lemma 4.1 and standard limit transition arguments, we conclude from (4.5) that $\mu^{\hat{\xi}} \neq \mu^{\hat{\xi}^-}$, and the result follows. \square

4.2. Proof of Lemma 4.1. Given $\Lambda \in \mathcal{B}_0(X)$ and $\hat{\xi} \in \Gamma^t(X, S)$, we set $P_\Lambda^{\hat{\xi}} = p_X^* \Pi_\Lambda(\cdot | \hat{\xi})$. Then, cf. (2.6) and (2.7),

$$\Pi_\Lambda(d\hat{\gamma} | \hat{\xi}) = \pi_{\Lambda, \gamma}^{\hat{\xi}}(d\sigma_\gamma) P_\Lambda^{\hat{\xi}}(d\gamma). \quad (4.6)$$

Here

$$\begin{aligned} \pi_{\Lambda, \gamma}^{\hat{\xi}}(d\sigma_\gamma) &= \frac{1}{Q_\Lambda(\hat{\xi}_{\Lambda^c})} \exp\left(-E(\sigma_{\gamma_\Lambda} | \sigma_{\xi_{\Lambda^c}})\right) \chi_{\gamma_\Lambda}(d\sigma_{\gamma_\Lambda}) \otimes \delta_{\sigma_{\gamma_{\Lambda^c}}}(d\sigma_{\gamma_{\Lambda^c}}), \\ Q_\Lambda(\hat{\xi}_{\Lambda^c}) &= \int_{S^{\gamma_\Lambda}} \exp\left(-E(\sigma_{\gamma_\Lambda} | \sigma_{\xi_{\Lambda^c}})\right) \chi_{\gamma_\Lambda}(d\sigma_{\gamma_\Lambda}), \end{aligned} \quad (4.7)$$

and

$$P_\Lambda^{\hat{\xi}}(d\gamma) = \frac{Q_\Lambda(\hat{\xi}_{\Lambda^c})}{Z_\Lambda(\hat{\xi})} \exp\left(-H(\gamma_\Lambda | \xi_{\Lambda^c})\right) \lambda_z(d\gamma_\Lambda) \otimes \delta_{\xi_{\Lambda^c}}(d\gamma_{\Lambda^c}), \quad (4.8)$$

where δ_\cdot is the corresponding Dirac measure.

Among all those χ that satisfy (3.5) we distinguish the measure $\chi^a(d\sigma) = [\delta_{-a}(d\sigma) + \delta_a(d\sigma)]/2$, $a > 0$. This choice corresponds to the Ising model with rescaled spins. It will be used as a reference system. Let $\pi_{\Lambda, \gamma}^{a, \hat{\xi}}$ be as in (4.7) with this χ^a on the right-hand side. Next, we let $\tilde{\phi}(x) = \phi_* I_R(x)$, where I_R is the indicator of the ball $B_R = \{x \in X : |x| \leq R\}$ and ϕ_* is as in (3.4). Finally, by $\tilde{\pi}_{\Lambda, \gamma}^{a, \hat{\xi}}$ we denote the measure as in (4.7) with χ^a and with ϕ replaced in (3.1) and (3.8) by $\tilde{\phi}$.

The proof of (4.4) is based on the following statement which we prove in the next section.

Lemma 4.2. *For any $a > 0$ there exist $n_* \in \mathbb{N}$, $z_c > 0$, a constant $\theta \in (0, 1/2)$ and a family of sets $\Gamma_\Lambda(\hat{\xi}) \in \mathcal{B}(\Gamma(X))$, $\Lambda \in \mathcal{L}$, $\hat{\xi} \in \hat{\Gamma}(n_*, a)$, with the property*

$$P_\Lambda^{\hat{\xi}}(\Gamma_\Lambda(\hat{\xi})) \geq \theta \quad (4.9)$$

and such that $\gamma_0 \neq \emptyset$ and

$$\tilde{\pi}_{\Lambda, \gamma}^{a, \hat{\xi}}(\sigma_x = a) \geq \frac{1 + \theta}{2}, \quad x \in \gamma_0, \quad (4.10)$$

for all $\gamma \in \Gamma_\Lambda(\hat{\xi})$ and $z > z_c$.

Proof of Lemma 4.1. By Lemma 4.2 it follows that $\gamma_0 \neq \emptyset$ for each $\gamma \in \Gamma_\Lambda(\hat{\xi})$. For an arbitrary such γ , we have:

$$\int_{S^\gamma} \left(\sum_{x \in \gamma_0} \sigma_x \right) \pi_{\Lambda, \gamma}^{a, \hat{\xi}}(d\sigma_\gamma) \geq \int_{S^\gamma} \left(\sum_{x \in \gamma_0} \sigma_x \right) \tilde{\pi}_{\Lambda, \gamma}^{a, \hat{\xi}}(d\sigma_\gamma), \quad (4.11)$$

following by the GKS inequalities, see [30]. Now we pass to unbounded spins and take any χ , which is symmetric and satisfies (3.5). For this χ we pick $a > 0$ such that

$$\chi([a\sqrt{2}, +\infty)) \geq \chi([0, a]).$$

By Wells' inequality [31], for this a we have

$$\int_{S^\gamma} \left(\sum_{x \in \gamma_0} \sigma_x \right) \pi_{\Lambda, \gamma}^{\hat{\xi}}(d\sigma_\gamma) \geq \int_{S^\gamma} \left(\sum_{x \in \gamma_0} \sigma_x \right) \pi_{\Lambda, \gamma}^{a, \hat{\xi}}(d\sigma_\gamma) \geq a\theta, \quad (4.12)$$

see [4] for more detail. The latter estimate in (4.12) follows by (4.10) and (4.11). Now by (4.6) we integrate the left-hand side of (4.12), take into account (4.3) and (4.9), and obtain (4.4) with $m_c = a\theta^2/2$. \square

4.3. Proof of Lemma 4.2. The asymmetry stated in (4.10) can be established by using its relationship to the Bernoulli bond percolation in the random geometric graph $(\gamma)_R$, which we introduce now. Given a configuration $\gamma \in \Gamma_0(X)$, the vertex set of the graph is set to be γ . The edge set is then defined by setting the adjacency relation: $x \sim y$ whenever $|x - y| \leq R$. That is, $(\gamma)_R = (\gamma, \varepsilon_\gamma)$, $\varepsilon_\gamma = \{\{x, y\} \subset \gamma : |x - y| \leq R\}$. The corresponding probability distribution is introduced as follows, see [4]. Let $X^{(2)}$ be the space of two-element subsets of X and $E := \Gamma(X^{(2)})$ (cf. Remark 2.1), so that $\varepsilon_\gamma \in E$ for any $\gamma \in \Gamma(X)$. Each $\varpi \in \mathcal{P}(E)$ can be characterized by its Laplace transform

$$L_\varpi(\kappa) := \int_E \exp \left[\sum_{\{x, y\} \in \varepsilon} \log(1 + \kappa(x, y)) \right] \varpi(d\varepsilon),$$

where κ runs over the set \mathcal{K} of all measurable symmetric functions $X \times X \rightarrow (-1, 0]$. For a given $\gamma \in \Gamma(X)$, let $\varpi_\gamma \in \mathcal{P}(E)$ be the Dirac measure concentrated at ε_γ . Its Laplace transform is then

$$L_{\varpi_\gamma}(\kappa) = \exp \left[\sum_{\{x, y\} \in \varepsilon_\gamma} \log(1 + I_R(x - y)\kappa(x, y)) \right],$$

where, as above, I_R is the indicator of the ball B_R . For a given $q \in [0, 1]$, the independent q -thinning of ϖ_γ is the measure $\varpi_\gamma^q \in \mathcal{P}(E)$, cf. [6, Section 11.2], defined by the relation

$$L_{\varpi_\gamma^q}(\kappa) = L_{\varpi_\gamma}(q\kappa). \quad (4.13)$$

Note that $q\kappa \in \mathcal{K}$. The interpretation of this is that each $\{x, y\} \in \varepsilon$ is removed from the edge configuration with probability $1 - q$ and is kept with probability q . The probability distribution of such 'thinned' configurations is then ϖ_γ^q . Now let Λ and $\hat{\xi}$ be as in the statement of Lemma 4.2, and then $P_\Lambda^{\hat{\xi}}$ be as in (4.6) and (4.9). For ϖ_γ and ϖ_γ^q as in (4.13), we define

$$\zeta(d\gamma, d\varepsilon) := \varpi_\gamma(d\varepsilon)P_\Lambda^{\hat{\xi}}(d\gamma), \quad \zeta^q(d\gamma, d\varepsilon) := \varpi_\gamma^q(d\varepsilon)P_\Lambda^{\hat{\xi}}(d\gamma). \quad (4.14)$$

Let $x \leftrightarrow \infty$ denote the event that $x \in \gamma$ belongs to an infinite connected component of $(\gamma, \varepsilon_\gamma)$. The proof of Lemma 4.2 is based on the following result proved in Section 5.

Lemma 4.3. *For any $q \in (0, 1)$ and $a > 0$ there exist $z_c > 0$ and $n_* \in \mathbb{N}$ such that the bound*

$$\zeta^q(\{(\gamma, \varepsilon) : x \leftrightarrow \infty \text{ for all } x \in \gamma_0\}) \geq 2\theta \quad (4.15)$$

holds for all $z > z_c$, $\Lambda \in \mathcal{L}$, $\hat{\xi} \in \hat{\Gamma}(n_*, a)$ and some constant $\theta \in (0, 1/2)$, which depends only on the dimension of X .

Proof of Lemma 4.2. Choose q and a such that

$$\phi_* > \frac{a^2}{2} \log \frac{1+q}{1-q},$$

and let θ , z_c and n_* be as in Lemma 4.3. Fix arbitrary $\Lambda \in \mathcal{L}$ and $\hat{\xi} \in \hat{\Gamma}(n_*, a)$. Next, for a given $\gamma \in \Gamma(X)$, set

$$\Psi(\gamma) = \varpi_\gamma^q(\{\varepsilon : x \leftrightarrow \infty \text{ for all } x \in \gamma_0\}).$$

Define $\Gamma_\Lambda(\hat{\xi}) = \{\gamma \in \Gamma(X) : \Psi(\gamma) \geq \theta\}$, where θ is as in (4.15). Since $\Psi(\gamma) \leq 1$, it follows from (4.15) that $P_\Lambda^{\hat{\xi}}(\Gamma_\Lambda(\hat{\xi})) \geq \theta$, hence (4.9) holds, and

$$\varpi_\gamma^q(\{\varepsilon : x \leftrightarrow \infty \text{ for all } x \in \gamma_0\}) \geq \theta, \quad \gamma \in \Gamma_\Lambda(\hat{\xi}).$$

Then (4.10) follows by [13, Lemma 4.2]. \square

5. EXISTENCE OF THE PERCOLATION

Let $Z = (\mathbf{V}, \mathbf{E})$ be the graph with vertex set \mathbb{Z}^d and the adjacency relation: $k_1 \sim k_2$ whenever $|k_1 - k_2| = 1$. The main idea of the proof of Lemma 4.3 is to construct an auxiliary model on Z such that the percolation therein implies (4.15).

5.1. The auxiliary percolation model. In this subsection, we fix $\Lambda \in \mathcal{L}$, $n_* \in \mathbb{N}$, $a > 0$, and $\hat{\xi} \in \hat{\Gamma}(n_*, a)$.

By $\mathbf{L} \subset \mathbf{V}$ we denote the set of all those k for which $\Xi_k \subset \Lambda$. Next, we introduce two systems of random variables associated with the graph $(\gamma)_R$. Let ϑ_k take value 1 if the subgraph of $(\gamma)_R$ generated by γ_k is connected and $N(\gamma_k) \geq n_*$, and take value 0 otherwise. For $k_1 \sim k_2$, let $\varsigma_{k_1 k_2}$ take value 1 if there exist $x \in \gamma_{k_1}$ and $y \in \gamma_{k_2}$ such that $x \sim y$ in $(\gamma)_R$, and take value 0 otherwise. Clearly, the maps $(\gamma, \varepsilon) \mapsto \vartheta_k(\gamma, \varepsilon)$ and $(\gamma, \varepsilon) \mapsto \varsigma_{k_1 k_2}(\gamma, \varepsilon)$ are measurable. In view of the choice of l in (4.1), see also (2.8), the subgraph of $(\gamma)_R$ generated by each γ_k is complete; hence, the value of ϑ_k depends only on $N(\gamma_k)$. Also due to the choice of l , each vertex of γ_{k_1} is adjacent (in $(\gamma)_R$) to each vertex of γ_{k_2} whenever $k_1 \sim k_2$.

Let P be the joint probability distribution of the random fields $\{\vartheta_k\}_{k \in \mathbf{V}}$ and $\{\varsigma_{k_1 k_2}\}_{\{k_1, k_2\} \in \mathbf{E}}$ induced by the measure ζ in (4.14). By the very definition of the set $\hat{\Gamma}(n_*, a)$, see (4.2), we have that $P(\vartheta_k = 1) = 1$ for each

$k \in \mathbb{L}^c := \mathbb{V} \setminus \mathbb{L}$, and also $P(\varsigma_{k_1 k_2} = 1) = 1$ for all $k_1 \sim k_2$ such that $\vartheta_{k_1} = \vartheta_{k_2} = 1$. Let Q be the probability measure on $\{0, 1\}^{\mathbb{V}} \times \{0, 1\}^{\mathbb{E}}$ defined as follows. Its projection on $\{0, 1\}^{\mathbb{V}}$ is the product measure such that $Q(\vartheta_k = 1) = q_0$ for some $q_0 \in (0, 1)$ which will be chosen later, and $Q(\varsigma_{k_1 k_2} = 1) = 1$ for all $k_1 \sim k_2$ such that $\vartheta_{k_1} = \vartheta_{k_2} = 1$.

As in [13, Section 3.4], we introduce the usual componentwise partial order on $\{0, 1\}^{\mathbb{V}} \times \{0, 1\}^{\mathbb{E}}$, and the corresponding increasing real-valued functions on this set. Let P_1 and P_2 be probability measures on $\{0, 1\}^{\mathbb{V}} \times \{0, 1\}^{\mathbb{E}}$. We say that P_2 stochastically dominates P_1 and write $P_1 \prec P_2$ if

$$\int f dP_1 \leq \int f dP_2$$

for each increasing f .

We begin by comparing measures Q and P introduced above. Since $P(\varsigma_{k_1 k_2} = 1) = Q(\varsigma_{k_1 k_2} = 1)$ for each $k_1 \sim k_2$, we restrict our attention to the random variables ϑ_k . As in the proof of [27, Theorem 2.1], by (3.9) and (4.8) one can show that, see also (5.6) below,

$$P(\vartheta_k = 1 \ \forall k \in \mathbb{V}_1; \ \vartheta_k = 0 \ \forall k \in \mathbb{V}_2) > 0, \quad (5.1)$$

which holds for all disjoint $\mathbb{V}_1, \mathbb{V}_2 \subset \mathbb{L}$. Thus, P is irreducible in the sense of [13, Section 3.4]. Recall that P depends on the choice of z and n_* , and Q depends on the choice of $q_0 \in (0, 1)$.

To prove Lemma 4.3 we need the following result which will be proved in the next section.

Lemma 5.1. *For each $n_* \in \mathbb{N}$ and $q_0 \in (0, 1)$ there exists $z_c > 0$ such that $Q \prec P$ for any $z > z_c$.*

For a given $q \in (0, 1)$ and $n \in \mathbb{N}$, consider an n -element set and connect any two elements of it by an edge with probability q , independently of other edges. Denote by $\varphi(n, q)$ the probability that the resulting graph is connected. It is known that

$$\varphi(n, q) \geq 1 - (n-1)(1-q^2)^{n-2}, \quad n \geq 3, \quad (5.2)$$

and hence $\varphi(n, q) \rightarrow 1$ as $n \rightarrow +\infty$, see [10, Lemma 3.4]. By (5.2) one gets

$$\varrho(n, q) := \inf_{m \geq n} \varphi(m, q) \rightarrow 1 \quad \text{as } n \rightarrow +\infty. \quad (5.3)$$

Likewise, for two sets A and B consisting of n_1 and n_2 elements respectively, connect any $a \in A$ and $b \in B$ with each other by an edge with probability q , independently of other edges. Let $\psi(n_1, n_2, q)$ be the probability that there is at least one edge connecting A and B . Obviously,

$$\psi(n_1, n_2, q) = 1 - (1-q)^{n_1 n_2}. \quad (5.4)$$

Set

$$h(n, q) = \varrho(n, q)\psi(n, n, q). \quad (5.5)$$

Proof of Lemma 4.3. For given $q_1, q_2 \in (0, 1)$, let Q_{q_1, q_2} be the measure on $\{0, 1\}^{\mathbb{V}} \times \{0, 1\}^{\mathbb{E}}$ such that its projection on $\{0, 1\}^{\mathbb{V}}$ is the product measure

for which $Q_{q_1, q_2}(\vartheta_k = 1) = q_0 q_1$, and $Q_{q_1, q_2}(\varsigma_{k_1 k_2} = 1) = q_2$ for all $k_1 \sim k_2$ such that $\vartheta_{k_1} = \vartheta_{k_2} = 1$. That is, Q_{q_1, q_2} is the corresponding thinning of the measure Q .

For a finite $V' \subset V$, let $G' := (V', E')$ be a subgraph of Z . By $|V'|$ and $|E'|$ we denote the cardinalities of the corresponding sets. Consider the event $A_{G'} = \{\vartheta_k = 1, k \in V', \text{ and } \varsigma_{k_1 k_2} = 1, \{k_1, k_2\} \in E'\}$. By Lemma 5.1, for the corresponding values of the parameters n_*, q, z and q_0 we have

$$Q_{q_1, q_2}(A_{G'}) = (q_0 q_1)^{|V'|} q_2^{|E'|} = q_1^{|V'|} q_2^{|E'|} Q(A_{G'}) \leq q_1^{|V'|} q_2^{|E'|} P(A_{G'}). \quad (5.6)$$

The right-hand side can be estimated in terms of the measure ζ^q defined in (4.13) and (4.14). To this end, we set

$$q_1 = \varphi(n_*, q), \quad q_2 = \psi(n_*, n_*, q), \quad (5.7)$$

where φ is as in (5.2), (5.3). We then have

$$\begin{aligned} q_1^{|V'|} q_2^{|E'|} P(A_{G'}) &\leq \int_{\Gamma(X) \times E} \left(\prod_{k \in V'} \vartheta(\gamma, \varepsilon) \varphi(N(\gamma_k), q) \right) \quad (5.8) \\ &\times \left(\prod_{\{k_1, k_2\} \in E'} \varsigma_{k_1 k_2}(\gamma, \varepsilon) \psi(N(\gamma_{k_1}), N(\gamma_{k_2}), q) \right) \zeta(d\gamma, d\varepsilon) \\ &= \int_{\Gamma(X) \times E} \left(\prod_{k \in V'} \vartheta(\gamma, \varepsilon) \prod_{\{k_1, k_2\} \in E'} \varsigma_{k_1 k_2}(\gamma, \varepsilon) \right) \zeta^q(d\gamma, d\varepsilon) \\ &= P^q(A_{G'}), \end{aligned}$$

where P^q is the joint probability distribution of $\{\vartheta_k\}_{k \in V}$ and $\{\varsigma_{k_1 k_2}\}_{\{k_1, k_2\} \in E}$ induced by the measure ζ^q in (4.14). Combining (5.6) and (5.8) we then get $Q_{q_1, q_2} \prec P^q$.

Let $0 \leftrightarrow \infty$ denote the event that $0 \in Z$ belongs to an infinite connected component of the graph. Then by (5.6) and (5.8), for q_1 and q_2 as in (5.7) we have

$$\begin{aligned} Q_{q_1, q_2}(0 \leftrightarrow \infty) &\leq P^q(0 \leftrightarrow \infty) \quad (5.9) \\ &= \zeta^q(\{(\gamma, \varepsilon) : x \leftrightarrow \infty \text{ for all } x \in \gamma_0\}). \end{aligned}$$

cf. (4.15). To estimate the left-hand side of (5.9) we proceed as follows. For a given subgraph $G \subseteq Z$, let $\theta^{\text{site}}(p; G)$ (resp. $\theta^{\text{bond}}(p; G)$), $p \in (0, 1)$, be the probability of the event $0 \leftrightarrow \infty$ in the Bernoulli site (resp. bond) percolation model on G with site (resp. bond) probability p . It is known that, see [11],

$$\theta^{\text{site}}(p; G) \leq p \theta^{\text{bond}}(p; G) \leq \theta^{\text{bond}}(p; G). \quad (5.10)$$

Let G_p be the random graph obtained from Z by independent deleting sites with probability $1 - p$. By construction of the measure Q_{q_1, q_2} and in view

of (5.10) we have the estimate

$$\begin{aligned} Q_{q_1, q_2}(0 \leftrightarrow \infty) &= \theta^{\text{bond}}(q_2; \mathbf{G}_{q_0 q_1}) \theta^{\text{site}}(q_0 q_1; \mathbf{Z}) \\ &\geq \theta^{\text{site}}(q_2; \mathbf{G}_{q_0 q_1}) \theta^{\text{site}}(q_0 q_1; \mathbf{Z}) = \theta^{\text{site}}(q_0 q_1 q_2; \mathbf{Z}) > 0. \end{aligned} \quad (5.11)$$

For $d \geq 2$, the latter estimate holds whenever

$$q_0 q_1 q_2 > p^{\text{site}}(d), \quad (5.12)$$

where $p^{\text{site}}(d)$ is the threshold probability for the Bernoulli site percolation on \mathbf{Z} . Thus (5.11) turns into the following condition, see (5.3), (5.5), (5.4), and (5.7):

$$q_0 h(n_*, q) > p^{\text{site}}(d). \quad (5.13)$$

Now we can finalize the proof of Lemma 4.3. Fix an arbitrary $q \in (0, 1)$, pick n_* such that $h(n_*, q) > p^{\text{site}}(d)$ and choose any $q_0 < 1$ satisfying (5.13). For these n_* and q_0 let z_c be as in Lemma 5.1. Then for any $z > z_c$ we have $Q \prec P$, which yields (5.9). Bound (4.15) follows now by (5.11) with $\theta = p^{\text{site}}(d)/2$. Note that estimates (5.9) and (5.11) are uniform in Λ and a , which completes the proof. \square

5.2. Proof of Lemma 5.1. We start with the following technical estimate. Recall that the parameters r and R satisfy (3.6). Let Ξ be any of the cells (2.8), (4.1) and $\Delta \subset \Xi$ be such that $|x - y| > r$ for each $x \in \Delta$ and $y \in \Xi^c$. That is, $\Delta = \Xi \setminus \{\text{boundary layer of thickness } r\}$. Thus, there is no repulsion between the particles located at $x \in \Delta$ and $y \in \Xi^c$. Observe that the Euclidean volume $\text{Vol}(\Delta)$ is positive in view of (3.6). Then, for $x \in \Delta$ and $\hat{\gamma} \in \Gamma(X, S)$, we set

$$g(\hat{\gamma}) = \int_X \exp\left(-\sum_{y \in \gamma} \Phi(x - y)\right) G(x, \hat{\gamma}) dx, \quad (5.14)$$

where

$$G(x, \hat{\gamma}) = \int_S \exp\left(s \sum_{y \in \gamma} \phi(x - y) \sigma_y\right) \chi(ds). \quad (5.15)$$

Lemma 5.2. *For an arbitrary $n_* \in \mathbb{N}$, there exists $g_* > 0$ such that*

$$g(\hat{\gamma}) \geq g_* \quad (5.16)$$

for all $\hat{\gamma} \in \Gamma(X, S)$ with $N(\gamma_\Xi) < n_$.*

Proof. Fix $n_* \in \mathbb{N}$ and $\hat{\gamma} = (\gamma, \sigma_\gamma)$ such that $N(\gamma_\Xi) < n_*$. Choose δ such that $\text{Vol} \Delta - (n_* - 1) \text{Vol}(B_\delta) > 0$, where B_δ is the ball of radius δ centered at the origin in X . Define the set Δ_γ by removing from Δ the balls of radius δ with centers at the elements of $\gamma \in \Gamma(X)$, that is,

$$\Delta_\gamma := \{x \in \Delta : |x - y| \geq \delta, y \in \gamma\}.$$

Then $\text{Vol}(\Delta_\gamma) \geq \text{Vol}(\Delta) - N(\gamma_\Xi)\text{Vol}(B_\delta) \geq \text{Vol}(\Delta) - (n_* - 1)\text{Vol}(B_\delta) =: v_*$. For a given $c > 0$, introduce the sets

$$\Delta_{\gamma,c} := \left\{ x \in \Delta_\gamma : \sum_{y \in \gamma} \Phi(x-y) \geq c \right\}$$

and

$$S_{x,\hat{\gamma}} := \left\{ s \in S : s \sum_{y \in \gamma} \phi(x-y)\sigma_y \geq 0 \right\}.$$

For each x and $\hat{\gamma}$, we have either $S_{x,\hat{\gamma}} = \mathbb{R}_\pm$ or $S_{x,\hat{\gamma}} = \mathbb{R}$, which together with the symmetry of χ implies that $\chi(S_{x,\hat{\gamma}}) \geq \frac{1}{2}$, and hence, see (5.15)

$$G(x, \hat{\gamma}) \geq \frac{1}{2}, \quad x \in X, \quad \hat{\gamma} \in \Gamma(X, S).$$

Now we take this into account in (5.14) and obtain

$$g(\hat{\gamma}) \geq \frac{1}{2} \int_\Delta \exp\left(-\sum_{y \in \gamma} \Phi_+(x-y)\right) dy \geq \frac{e^{-c}}{2} \text{Vol}(\Delta_\gamma \setminus \Delta_{\gamma,c}).$$

To estimate the latter quantity we use Markov's inequality

$$\text{Vol}(\Delta_{\gamma,c}) \leq \frac{1}{c} \int_{\Delta_\gamma} \sum_{y \in \gamma_\Xi} \Phi_+(x-y) dx \leq \frac{1}{c} N(\gamma_\Xi) \int_{|x|>\delta} \Phi_+(x) dx,$$

which yields, see (3.2),

$$\text{Vol}(\Delta_\gamma \setminus \Delta_{\gamma,c}) \geq v_* - \frac{n_* - 1}{c} C_\delta,$$

so that

$$g(\hat{\gamma}) \geq \frac{1}{2} e^{-c} \left(v_* - \frac{n_* - 1}{c} C_\delta \right).$$

It is clear that the right-hand side is positive for sufficiently large c . The (5.16) follows with $g_* = \sup_{c>0} \frac{1}{2} e^{-c} \left(v_* - \frac{n_* - 1}{c} C_\delta \right)$. \square

By (5.1) and (5.6) we know that P is irreducible. Hence, we can apply here Holley's theorem, see [13, Theorem 3.7], and obtain the following statement.

Proposition 5.3. *Assume that the inequality*

$$P(\vartheta_k = 1 | \vartheta_{k'} = \beta_{k'}, k' \in \mathbb{L} \setminus \{k\}) \geq Q(\vartheta_k = 1) \quad (5.17)$$

holds for each $k \in \mathbb{L}$ and $\beta \in \{0, 1\}^{\mathbb{L} \setminus \{k\}}$. Then $Q \prec P$.

Recall that P is determined by $P_\Lambda^{\hat{\xi}}$ with a fixed $\hat{\xi} \in \hat{\Gamma}(n_*, a)$. For this $\hat{\xi}$, and k and β as in (5.17), we pick $\hat{\eta} \in \Gamma^t(X, S)$ such that: (a) $\hat{\eta}_{\Lambda^c} = \hat{\xi}_{\Lambda^c}$; (b) $\vartheta_{k'}(\hat{\eta}) = \beta_{k'}$ for each $k' \in \mathbb{L} \setminus \{k\}$. Then the conditional measure $P_\Lambda^{\hat{\xi}}(\cdot | \hat{\eta})$, such that

$$P(\vartheta_k = 1 | \vartheta_{k'} = \beta_{k'}, k' \in \mathbb{L} \setminus \{k\}) = P_\Lambda^{\hat{\xi}}(N(\gamma_k) \geq n_* | \hat{\eta}), \quad (5.18)$$

can be obtained in the form

$$P_{\Lambda}^{\hat{\xi}}(d\gamma|\hat{\eta}) = \int_{S^{\eta_{\Lambda}\setminus\Xi_k}} P_{\Xi_k}^{\hat{\eta}}(d\gamma)\chi_{\eta_{\Lambda}\setminus\Xi_k}(d\sigma_{\eta_{\Lambda}\setminus\Xi_k}), \quad (5.19)$$

see (4.6).

Lemma 5.4. *Let k and $\hat{\eta}$ be as in (5.17), (5.18), (5.19). Then for any $n_* \in \mathbb{N}$ and $q_0 \in (0, 1)$ there exists $z_c > 0$ such that*

$$P_{\Xi_k}^{\hat{\eta}}(N(\gamma_k) \geq n_*) \geq q_0 \quad (5.20)$$

for any $z > z_c$.

Proof. Let I_n be the indicator function of the set $\{\gamma : N(\gamma_k) = n\}$, $n \in \mathbb{N}$. Set also $\omega_n = P_{\Xi_k}^{\hat{\eta}}(N(\gamma_k) = n)$. By (2.4), (3.7), (4.8), and (5.14) for $n < n_*$ we get

$$\begin{aligned} \omega_{n+1} &= \frac{1}{n+1} \int_{\Gamma(X,S)} \left(\sum_{x \in \gamma_k} I_{n+1}(\gamma) \right) \Pi_{\Xi_k}(d\hat{\gamma}|\hat{\eta}) \\ &= \frac{z}{n+1} \int_{\Gamma(X,S)} I_n(\gamma) g(\hat{\gamma}) \Pi_{\Xi_k}(d\hat{\gamma}|\hat{\eta}) \\ &\geq \frac{zg_*}{n+1} \omega_n \geq zt_* \omega_n, \quad t_* := g_*/n_*, \end{aligned}$$

where we have taken into account that $n+1 \leq n_*$ and used (5.16). The latter estimate readily yields

$$\sum_{n=0}^{n_*-1} \omega_n \leq \frac{\omega_{n_*}}{zt_* - 1} \leq \frac{1}{zt_* - 1} \sum_{n \geq n_*} \omega_n.$$

Taking into account that $\sum_{n \geq 0} \omega_n = 1$ we obtain that

$$P_{\Xi_k}^{\hat{\eta}}(N(\gamma_k) \geq n_*) \geq 1 - \frac{1}{zt_*}.$$

Now we can set

$$z_c = (t_*(1 - q_0))^{-1}, \quad (5.21)$$

and (5.20) follows. \square

Proof of Lemma 5.1. For $z > z_c$ given in (5.21), we have $P_{\Xi_k}^{\hat{\eta}}(N(\gamma_k) \geq n_*) \geq q_0$, which by (5.19) and (5.18) yields (5.17) and hence $Q \prec P$ by Proposition 5.3. \square

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