Equilibrium diffusion on the cone of discrete Radon measures

Diana Conache

Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany;

e-mail: dputan@math.uni-bielefeld.de

Yuri G. Kondratiev

Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany; NPU, Kyiv, Ukraine

e-mail: kondrat@math.uni-bielefeld.de

Eugene Lytvynov

Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP, U.K.

e-mail: e.lytvynov@swansea.ac.uk

Abstract

Let $\mathbb{K}(\mathbb{R}^d)$ denote the cone of discrete Radon measures on $\mathbb{R}^d$. There is a natural differentiation on $\mathbb{K}(\mathbb{R}^d)$: for a differentiable function $F:\mathbb{K}(\mathbb{R}^d)\to\mathbb{R}$, one defines its gradient $\nabla^{\mathbb{K}}F$ as a vector field which assigns to each $\eta\in\mathbb{K}(\mathbb{R}^d)$ an element of a tangent space $T_\eta(\mathbb{K}(\mathbb{R}^d))$ to $\mathbb{K}(\mathbb{R}^d)$ at point η. Let $\phi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be a potential of pair interaction, and let μ be a corresponding Gibbs perturbation of (the distribution of) a completely random measure on $\mathbb{R}^d$. In particular, μ is a probability measure on $\mathbb{K}(\mathbb{R}^d)$ such that the set of atoms of a discrete measure $\eta\in\mathbb{K}(\mathbb{R}^d)$ is μ-a.s. dense in $\mathbb{R}^d$. We consider the corresponding Dirichlet form

$$\mathscr{E}^{\mathbb{K}}(F,G)=\int_{\mathbb{K}(\mathbb{R}^d)}\langle\nabla^{\mathbb{K}}F(\eta),\nabla^{\mathbb{K}}G(\eta)\rangle_{T_\eta(\mathbb{K})}\,d\mu(\eta).$$

Integrating by parts with respect to the measure μ, we explicitly find the generator of this Dirichlet form. By using the theory of Dirichlet forms, we prove the main result of the paper: If $d\geq 2$, there exists a conservative diffusion process on $\mathbb{K}(\mathbb{R}^d)$ which is properly associated with the Dirichlet form $\mathscr{E}^{\mathbb{K}}$.

Keywords: Completely random measure, diffusion process, discrete Radon measure, Dirichlet form, Gibbs measure

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1 Introduction

Let X denote the Euclidean space $\mathbb{R}^d$ and let $\mathscr{B}(X)$ denote the Borel σ-algebra on X. Let $\mathbb{M}(X)$ denote the space of all Radon measures on $(X, \mathscr{B}(X))$. The space $\mathbb{M}(X)$ is equipped with the vague topology, and let $\mathscr{B}(\mathbb{M}(X))$ denote the corresponding Borel σ-algebra on it. A random measure on X is a measurable mapping $\xi : \Omega \to \mathbb{M}(X)$, where $(\Omega, \mathscr{F}, P)$ is a probability space, see e.g. [8]. A random measure ξ is called completely random if, for any mutually disjoint sets $A_1, \dots, A_n \in \mathscr{B}(X)$, the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent [9].

The cone of discrete Radon measures on X is defined by

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) \,\middle|\, s_i > 0,\ x_i \in X \right\}.$$

Here δ_{x_i} denotes the Dirac measure with mass at x_i. In the above representation, the atoms x_i are assumed to be distinct and their total number is at most countable. By convention, the cone $\mathbb{K}(X)$ contains the null mass $\eta = 0$, which is represented by the sum over an empty set of indices i. As shown in [6], $\mathbb{K}(X) \in \mathscr{B}(\mathbb{M}(X))$. One endows $\mathbb{K}(X)$ with the vague topology.

A random measure ξ which takes values in $\mathbb{K}(X)$ with probability one is called a random discrete measure. It follows from Kingman's result [9] that each completely random measure ξ can be represented as $\xi = \xi' + \eta$, where ξ' is a deterministic measure on X and η is a random discrete measure. An important example of a random discrete measure is the gamma measure [19], which has many distinguished properties. It should be noted that, for a wide class of random discrete measures (including the gamma measure), the set of atoms of $\eta = \sum_i s_i \delta_{x_i}$, i.e., $\{x_i\}$, is dense in X.

In this paper, we will only use the distribution μ of a random discrete measure. So, below by a random discrete measure we will always mean a probability measure μ on $(\mathbb{K}(X), \mathscr{B}(\mathbb{K}(X)))$. (Here $\mathscr{B}(\mathbb{K}(X))$ is the Borel σ-algebra on $\mathbb{K}(X)$.)

In [6] Gibbs perturbations of the gamma measure were constructed, and in [16] this result was extended to Gibbs perturbations of a general completely random discrete measure. More precisely, let $\phi : X \times X \to \mathbb{R}$ be a potential of pair interaction, which satisfies the conditions (C1), (C2) below. In particular, it is assumed that the function ϕ is symmetric, bounded, has finite range (i.e., $\phi(x, x') = 0$ if the distance between x and x' is sufficiently large), and the positive part of ϕ dominates, in a sense, its negative part. For $\eta \in \mathbb{K}(X)$, we heuristically define the energy of η (Hamiltonian) by

$$H(\eta) := \frac{1}{2} \int_{X^2 \setminus D} \phi(x, x')\, d\eta(x)\, d\eta(x'),$$

where $D = \{(x, x') \in X^2 \mid x = x'\}$. Let ν be a completely random discrete measure. The Gibbs perturbation of ν corresponding to the potential ϕ is heuristically defined as a probability measure μ on $\mathbb{K}(X)$ given by

$$d\mu(\eta) := \frac{1}{Z}\, e^{-H(\eta)}\, d\nu(\eta),$$

where Z is a normalizing factor. A rigorous definition of μ is given through the Dobrushin–Lanford–Ruelle equation. It is proven in [6] that such a Gibbs measure exists. In [16], it was shown that such a Gibbs measure is unique, provided the supremum norm of ϕ, i.e., $\|\phi\|_\infty$, and the first moment of ν are sufficiently small. In the general case, the uniqueness problem is still open.

Any Gibbs measure μ satisfies the Nguyen–Zessin identity in which the relative energy of interaction between a single atom measure $\eta = s\delta_x$ and a discrete measure $\eta' \in \mathbb{K}(X)$, with no atom at x, is given by

$$H(\eta \mid \eta') = s \int_X \phi(x, x')\, d\eta'(x').$$

In [10] (see also [7]), some elements of differential geometry on $\mathbb{K}(X)$ were introduced. In particular, for a differentiable function $F : \mathbb{K}(X) \to \mathbb{R}$, one defines its gradient $\nabla^{\mathbb{K}} F$ as a vector field which assigns to each $\eta \in \mathbb{K}(X)$ an element of a tangent space $T_\eta(\mathbb{K}(X))$ to $\mathbb{K}(X)$ at point η. It should be stressed that $\mathbb{K}(X)$ is not a flat space, in the sense that the tangent space $T_\eta(\mathbb{K})$ changes with a change of η.

So, in this paper, we consider the Dirichlet form

$$\mathscr{E}^{\mathbb{K}}(F, G) := \int_{\mathbb{K}(\mathbb{R}^d)} \left\langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta) \right\rangle_{T_\eta(\mathbb{K})} d\mu(\eta). \tag{1}$$

This bilinear form is initially defined on an appropriate set of smooth cylinder functions on $\mathbb{K}(X)$. Using the Nguyen–Zessin identity, we carry out integration by parts with respect to the Gibbs measure μ, and find the L^2-generator of the bilinear form $\mathscr{E}^{\mathbb{K}}$ (containing the potential ϕ and its gradient). This, in particular, proves the closability of the bilinear form $\mathscr{E}^{\mathbb{K}}$ on $L^2(\mathbb{K}(X), \mu)$. This result extends [10] (see also [7]), where the L^2-generator of $\mathscr{E}^{\mathbb{K}}$ (the Laplace operator) was derived in the case of no interaction, $\phi = 0$, and when the completely random measure $\mu = \nu$ is the law of a measure-valued Lévy process.

The main result of the paper is the existence of a conservative diffusion process on $\mathbb{K}(X)$ which is properly associated with the Dirichlet form $\mathscr{E}^{\mathbb{K}}$. For this, one assumes that the dimension of the underlying space X is ≥ 2. (It is intuitively clear that in the case where the dimension of X is equal to one, such a result should fail.) We note that this diffusion process has continuous sample paths in $\mathbb{K}(X)$ with respect to the vague topology. The diffusion process has μ as invariant (and even symmetrizing) measure. To prove the main result, we use the general theory of Dirichlet forms [13] as well as the theory of Dirichlet forms over configuration spaces [14, 18], see also [1, 11].

The paper is organized as follows. In Section 2, we recall how differentiation on $\mathbb{K}(X)$ is introduced [10], and how the Gibbs measure μ is constructed [6, 16]. In Section 3, we formulate the results of the paper. Finally, Section 4 contains the proofs.

2 Preliminaries

2.1 Differentiation on $\mathbb{K}(X)$

In this subsection, we follow [10]. A starting point to define differentiation on $\mathbb{K}(X)$ is the choice of a natural group $\mathfrak{G}$ of transformations of $\mathbb{K}(X)$. So let $\mathrm{Diff}_0(X)$ denote the group of C^∞ diffeomorphisms of X which are equal to the identity outside a compact set. Let $C_0(X \to \mathbb{R}_+)$ denote the multiplicative group of continuous functions on X with values in $\mathbb{R}_+ := (0, \infty)$ which are equal to one outside a compact set. The group $\mathrm{Diff}_0(X)$ naturally acts on X, hence on $C_0(X \to \mathbb{R}_+)$. So we define a group $\mathfrak{G}$ by

$$\mathfrak{G} := \mathrm{Diff}_0(X) \leftthreetimes C_0(X \to \mathbb{R}_+),$$

the semidirect product of $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$. As a set, $\mathfrak{G}$ is equal to the Cartesian product of $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$, and the product in $\mathfrak{G}$ is given by

$$g_1 g_2 = (\psi_1 \circ \psi_2,\, \theta_1(\theta_2 \circ \psi_1^{-1})) \quad \text{for } g_1 = (\psi_1, \theta_1),\, g_2 = (\psi_2, \theta_2) \in \mathfrak{G}.$$

The group $\mathfrak{G}$ naturally acts on $\mathbb{K}(X)$: for any $g = (\psi, \theta) \in \mathfrak{G}$ and any $\eta \in \mathbb{K}(X)$, we define $g\eta \in \mathbb{K}(X)$ by

$$d(g\eta)(x) := \theta(x)\, d(\psi^*\eta)(x).$$

Here $\psi^*\eta$ is the pushforward of η under ψ.

The Lie algebra of the Lie group $\mathrm{Diff}_0(X)$ is the space $\mathrm{Vec}_0(X)$ consisting of all smooth vector fields acting from X into X which have compact support. For $v \in \mathrm{Vec}_0(X)$, let $(\psi_t^v)_{t\in\mathbb{R}}$ be the corresponding one-parameter subgroup of $\mathrm{Diff}_0(X)$, see e.g. [2]. As the Lie algebra of $C_0(X \to \mathbb{R}_+)$ we may take the space $C_0(X)$ of all real-valued continuous functions on X with compact support. For each $h \in C_0(X)$, the corresponding one-parameter subgroup of $C_0(X \to \mathbb{R}_+)$ is given by $(e^{th})_{t\in\mathbb{R}}$. Thus, $\mathfrak{g} := \mathrm{Vec}_0(X) \times C_0(X)$ can be thought of as a Lie algebra that corresponds to the Lie group $\mathfrak{G}$. For an arbitrary $(v, h) \in \mathfrak{g}$, we may consider the curve $\{(\psi_t^v, e^{th}),\, t \in \mathbb{R}\}$ in $\mathfrak{G}$. For a function $F : \mathbb{K}(X) \to \mathbb{R}$ we define its derivative in direction (v, h) by

$$\nabla^{\mathbb{K}}_{(v,h)} F(\eta) := \frac{d}{dt}\Big|_{t=0} F((\psi_t^v, e^{th})\eta), \quad \eta \in \mathbb{K}(X),$$

provided the derivative on the right hand side of this formula exists.

A tangent space to $\mathbb{K}(X)$ at $\eta \in \mathbb{K}(X)$ is defined by

$$T_\eta(\mathbb{K}(X)) := L^2(X \to X \times \mathbb{R}, \eta), \tag{2}$$

the L^2-space of $X \times \mathbb{R}$-valued vector fields on X which are square integrable with respect to the measure η. We then define a gradient of a differentiable function $F : \mathbb{K}(X) \to \mathbb{R}$ at η as the element $(\nabla^{\mathbb{K}} F)(\eta)$ of $T_\eta(\mathbb{K})$ which satisfies

$$\nabla^{\mathbb{K}}_{(v,h)} F(\eta) = \langle \nabla^{\mathbb{K}} F(\eta), (v, h)\rangle_{T_\eta(\mathbb{K})} \quad \text{for all } (v, h) \in \mathfrak{g}.$$

Remark 1. Note that, in the above definitions, one could replace $\mathbb{K}(X)$ with the wider space $\mathbb{M}(X)$. This is why, in paper [10], the gradient $\nabla^{\mathbb{K}}$ was actually denoted by $\nabla^{\mathbb{M}}$.

Let us now define a set of test functions on $\mathbb{K}(X)$. Let us denote by $\tau(\eta)$ the set of atoms of η, and for each $x \in \tau(\eta)$, let $s_x := \eta(\{x\})$. Thus, we have

$$\eta = \sum_{x\in\tau(\eta)} s_x\delta_x.$$

We define a metric on $\mathbb{R}_+$ by

$$d_{\mathbb{R}_+}(s_1, s_2) := |\log(s_1) - \log(s_2)|\,, \quad s_1, s_2 \in \mathbb{R}_+.$$

Then $\mathbb{R}_+$ becomes a locally compact Polish space, and any set of the form $[a, b]$, with $0 < a < b < \infty$, is compact. We denote $\widehat{X} := \mathbb{R}_+ \times X$, and let $C_0^\infty(\widehat{X})$ denote the space of all smooth functions on $\widehat{X}$ with compact support. For each $\varphi \in C_0^\infty(\widehat{X})$ and $\eta \in \mathbb{K}(X)$, we define

$$\langle\langle \varphi, \eta\rangle\rangle := \sum_{x\in\tau(\eta)} \varphi(s_x, x).$$

Note that the latter sum contains only finitely many nonzero terms.

We denote by $\mathscr{FC}(\mathbb{K}(X))$ the set of all functions $F : \mathbb{K}(X) \to \mathbb{R}$ of the form

$$F(\eta) = g\big(\langle\langle\varphi_1, \eta\rangle\rangle, \dots, \langle\langle\varphi_N, \eta\rangle\rangle\big), \quad \eta \in \mathbb{K}(X), \tag{3}$$

where $g \in C_b^\infty(\mathbb{R}^N)$, $\varphi_1 \dots, \varphi_N \in C_0^\infty(\widehat{X})$, and $N \in \mathbb{N}$. Here $C_b^\infty(\mathbb{R}^N)$ is the set of all infinitely differentiable functions on $\mathbb{R}^N$ which, together with all their derivatives, are bounded.

Let $F : \mathbb{K}(X) \to \mathbb{R}$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$. We define

$$\nabla_x F(\eta) := \nabla_y\big|_{y=x} F(\eta - s_x\delta_x + s_x\delta_y), \tag{4}$$

$$\nabla_{s_x} F(\eta) := \frac{d}{du}\Big|_{u=s_x} F(\eta - s_x\delta_x + u\delta_x), \tag{5}$$

provided the derivatives exist. Here the variable y is from X, ∇_y denotes the gradient on X in the y variable, and the variable u is from $\mathbb{R}_+$.

An easy calculation shows that, for each function $F \in \mathscr{FC}(\mathbb{K}(X))$, the gradient $\nabla^{\mathbb{K}} F$ exists and is given by

$$(\nabla^{\mathbb{K}} F)(\eta, x) = \left(\frac{1}{s_x}\nabla_x F(\eta), \nabla_{s_x} F(\eta)\right), \quad \eta \in \mathbb{K}(X),\ x \in \tau(\eta). \tag{6}$$

2.2 The Gibbs measures

We start with defining a class of completely random measures. Let $l : \widehat{X} \to \mathbb{R}_+$ be a measurable function which satisfies the following conditions: for dx-a.a. $x \in X$

$$\int_{\mathbb{R}_+} \frac{l(s,x)}{s}\, ds = \infty \tag{7}$$

and for each $\Lambda \in \mathscr{B}_0(X)$,

$$\int_{\mathbb{R}_+\times\Lambda} l(s,x)\, ds\, dx < \infty. \tag{8}$$

Here $\mathscr{B}_0(X)$ denotes the collection of all sets from $\mathscr{B}(X)$ which have compact closure.

We define a measure σ on $\widehat{X}$ by

$$d\sigma(s,x) := \frac{l(s,x)}{s}\, ds\, dx. \tag{9}$$

Since (8) holds, we may define a completely random measure ν as a probability measure on $\mathbb{K}(X)$ which has Fourier transform

$$\int_{\mathbb{K}(X)} e^{i\langle f,\eta\rangle}\, d\nu(\eta) = \exp\left[\int_{\widehat{X}} (e^{isf(x)} - 1)\, d\sigma(s,x)\right], \quad f \in C_0(X),$$

see e.g. [3]. Here we denote $\langle f,\eta\rangle := \int_X f(x)\, d\eta(x)$. The measure ν can also be characterized through the Mecke identity: ν is the unique probability measure on $\mathbb{K}(X)$ which satisfies, for each measurable function $F : \widehat{X} \times \mathbb{K}(X) \to [0,\infty]$,

$$\int_{\mathbb{K}(X)} \sum_{x\in\tau(\eta)} F(s_x, x, \eta)\, d\nu(\eta) = \int_{\mathbb{K}(X)} d\nu(\eta) \int_{\widehat{X}} d\sigma(s,x)\, F(s,x,\eta + s\delta_x). \tag{10}$$

For example, by choosing $l(s,x) = e^{-s}$, we get the gamma measure ν [19]. More generally, we may fix measurable functions $\alpha, \beta : X \to \mathbb{R}_+$ and set

$$l(s,x) = \beta(x) e^{-s/\alpha(x)}.$$

Then conditions (7), (8) are satisfied when $\alpha(x)\beta(x) \in L^1_{\mathrm{loc}}(X, dx)$.

Let us now recall the definition of a Gibbs measure from [6,16]. Additionally to (7) and (8), we assume that, for each $\Lambda \in \mathscr{B}_0(X)$,

$$\int_{\mathbb{R}_+\times\Lambda} l(s,x) s\, ds\, dx < \infty. \tag{11}$$

Let $\phi : X \times X \to \mathbb{R}$ be a pair potential which satisfies the following two conditions:

(C1) ϕ is a symmetric, bounded, measurable function which satisfies, for some $R > 0$,

$$\phi(x, y) = 0 \quad \text{if } |x - y| > R.$$

(C2) There exists $\delta > 0$ such that

$$\inf_{x,y \in X:\ |x-y| \leq \delta} \phi(x, y) > \varepsilon \|\phi^-\|_\infty.$$

Here

$$\|\phi^-\|_\infty := \sup_{x,y \in X} (-\phi(x, y) \vee 0)$$

and $\varepsilon := 2v_d\, d^{d/2}(R/\delta + 1)$, where $v_d := \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of a unit ball in X.

Remark 2. Note that condition (C2) excludes the potential $\phi = 0$. Note also that conditions (C1) and (C2) are trivially satisfied if $\phi(x, y) = \psi(x-y)$, where $\psi \in C_0(X)$, $\psi(x) = \psi(-x)$, and $\psi(0) > v_d\, d^{d/2}\|\psi^-\|_\infty$.

For any $\eta, \xi \in \mathbb{K}(X)$ and $\Lambda \in \mathscr{B}_0(X)$, we define the relative energy (Hamiltonian)

$$H_\Lambda(\eta \mid \xi) := \frac{1}{2} \int_{\Lambda^2 \setminus D} \phi(x, y)\, d\eta(x)\, d\eta(y) + \int_{\Lambda^c} \int_\Lambda \phi(x, y)\, d\eta(x)\, d\xi(y),$$

where $\Lambda^c := X \setminus \Lambda$. Note that $H_\Lambda(\eta \mid \xi)$ is well defined and finite.

For each $\Lambda \in \mathscr{B}(X)$, we denote $\mathbb{K}(\Lambda) := \{\eta \in \mathbb{K}(X) \mid \tau(\eta) \subset \Lambda\}$. Note that $\mathbb{K}(\Lambda) \in \mathscr{B}(\mathbb{K}(X))$. Let ν_Λ denote the pushforward of the completely random measure ν under the canonical projection

$$\mathbb{K}(X) \ni \eta \mapsto \eta_\Lambda := \sum_{x \in \tau(\eta) \cap \Lambda} s_x \delta_x \in \mathbb{K}(\Lambda).$$

The measure ν_Λ has Fourier transform

$$\int_{\mathbb{K}(\Lambda)} e^{i\langle f, \eta \rangle}\, d\nu_\Lambda(\eta) = \exp\left[\int_{\mathbb{R}_+ \times \Lambda} (e^{isf(x)} - 1)\, d\sigma(s, x)\right], \quad f \in C_0(X).$$

Proposition 3 ([6, 16])**.** *Let* (7)–(9), (11) *hold and let conditions* (C1) *and* (C2) *be satisfied. Then, for any* $\Lambda \in \mathscr{B}_0(X)$ *and* $\xi \in \mathbb{K}(X)$,

$$0 < Z_\Lambda(\xi) := \int_{\mathbb{K}(\Lambda)} e^{-H(\eta \mid \xi)}\, d\nu_\Lambda(\eta) < \infty.$$

For each $\Lambda \in \mathscr{B}_0(X)$ with $\int_\Lambda dx > 0$, the local Gibbs state with boundary condition $\xi \in \mathbb{K}(X)$ is defined as a probability measure on $\mathbb{K}(\Lambda)$ given by

$$d\mu_\Lambda(\eta \mid \xi) := \frac{1}{Z_\Lambda(\xi)}\, e^{-H(\eta \mid \xi)}\, d\nu_\Lambda(\eta).$$

For each $B \in \mathscr{B}(\mathbb{K}(X))$, $\Lambda \in \mathscr{B}_0(X)$, and $\xi \in \mathbb{K}(X)$, we define

$$B_{\Lambda,\xi} := \{\eta \in \mathbb{K}(\Lambda) \mid \eta + \xi_{\Lambda^c} \in B\} \in \mathscr{B}(\mathbb{K}(\Lambda))$$

and hence we can define the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathscr{B}_0(X)}$ on $\mathbb{K}(X)$ as the family of stochastic kernels

$$\mathscr{B}(\mathbb{K}(X)) \times \mathbb{K}(X) \ni (B, \xi) \mapsto \pi_\Lambda(B \mid \xi) \in [0,1]$$

given by $\pi_\Lambda(B \mid \xi) := \mu_\Lambda(B_{\Lambda,\xi})$.

Definition 4. A Gibbs perturbation of a completely random measure ν corresponding to a pair potential ϕ is defined as a probability measure μ on $(\mathbb{K}(X), \mathscr{B}(\mathbb{K}(X)))$ which satisfies the following Dobrushin–Lanford–Ruelle (DLR) equation:

$$\int_{\mathbb{K}(X)} \pi_\Lambda(B \mid \xi)\, d\mu(\xi) = \mu(B), \tag{12}$$

for any $B \in \mathscr{B}(\mathbb{K}(X))$ and $\Lambda \in \mathscr{B}_0(X)$. We denote by $G(\nu, \phi)$ the set of all such probability measures μ.

Theorem 5 ([6,16])**.** *Let the conditions of Proposition 3 be satisfied. Then the set $G(\nu, \phi)$ is non-empty. Furthermore, each measure $\mu \in G(\nu, \phi)$ has finite moments: for each $\Lambda \in \mathscr{B}_0(X)$ and $n \in \mathbb{N}$,*

$$\int_{\mathbb{K}(X)} \eta(\Lambda)^n\, d\mu(\eta) < \infty. \tag{13}$$

Since (7) holds, for each $\Lambda \in \mathscr{B}_0(X)$ with $\int_\Lambda dx > 0$, for ν-a.a. $\eta \in \mathbb{K}(X)$, the set $\tau(\eta) \cap \Lambda$ is infinite. Using the DLR equation, we therefore obtain the following result.

Proposition 6. *Let the conditions of Proposition 3 be satisfied, and let $\mu \in G(\nu, \phi)$. Let $\Lambda \in \mathscr{B}_0(X)$ with $\int_\Lambda dx > 0$. Then, for μ-a.a. $\eta \in \mathbb{K}(X)$, the set $\tau(\eta) \cap \Lambda$ is infinite. In particular, the set $\tau(\eta)$ is μ-a.s. dense in X.*

By analogy with [15], the Gibbs measures have the following property.

Theorem 7. *Let the conditions of Proposition 3 be satisfied, and let $\mu \in G(\nu, \phi)$. Then μ satisfies the following Nguyen–Zessin identity: for each measurable function $F : \widehat{X} \times \mathbb{K}(X) \to [0, \infty]$,*

$$\int_{\mathbb{K}(X)} \sum_{x\in\tau(\eta)} F(s_x, x, \eta)\, d\mu(\eta)$$
$$= \int_{\mathbb{K}(X)} \int_{\widehat{X}} \exp\left[-s\int_X \phi(x,x')\, d\eta(x')\right] F(s,x,\eta+s\delta_x)\, d\sigma(s,x) d\mu(\eta). \quad (14)$$

Proof. By the same arguments as in the proof of [6, Theorem 6.3], it is enough to show that, for each $\Lambda \in \mathscr{B}_0(X)$, equality (14) holds for all functions F of the form $F(s,x,\eta) = f(s,x)g(\eta_\Lambda)$, where $f \in C_0(\widehat{X})$, $f \geq 0$, the support of f is a subset of $\mathbb{R}_+ \times \Lambda$ and $g : \mathbb{K}(\Lambda) \to [0,\infty)$ is bounded and measurable. By the DLR equation (12) and the Mecke identity (10), we have

$$\int_{\mathbb{K}(X)} \sum_{x\in\tau(\eta)} F(s_x,x,\eta)\, d\mu(\eta) = \int_{\mathbb{K}(X)} \int_{\mathbb{K}(X)} \sum_{x\in\tau(\eta)\cap\Lambda} f(s_x,x)g(\eta)\, \pi_\Lambda(d\eta \mid \xi)\, d\mu(\xi)$$
$$= \int_{\mathbb{K}(X)} \int_{\mathbb{K}(\Lambda)} \sum_{x\in\tau(\eta)} f(s_x,x)g(\eta)\, \frac{1}{Z_\Lambda(\xi)} e^{-H_\Lambda(\eta|\xi_{\Lambda^c})} d\nu_\Lambda(\eta)\, d\mu(\xi)$$
$$= \int_{\mathbb{K}(X)} \int_{\mathbb{K}(\Lambda)} \int_{\mathbb{R}_+\times\Lambda} f(s,x)g(\eta+s\delta_x)\, \frac{1}{Z_\Lambda(\xi)} e^{-H_\Lambda(\eta+s\delta_x|\xi_{\Lambda^c})}\, d\sigma(s,x) d\nu_\Lambda(\eta)\, d\mu(\xi)$$
$$= \int_{\widehat{X}} \int_{\mathbb{K}(X)} \int_{\mathbb{K}(X)} F(s,x,\eta+s\delta_x) \exp\left[-s\int_{X\setminus\{x\}} \phi(x,x')\, d\eta(x')\right] \pi_\Lambda(d\eta \mid \xi)\, d\mu(\xi)\, d\sigma(s,x)$$
$$= \int_{\mathbb{K}(X)} \int_{\widehat{X}} \exp\left[-s\int_{X\setminus\{x\}} \phi(x,x')\, d\eta(x')\right] F(s,x,\eta+s\delta_x)\, d\sigma(s,x) d\mu(\eta), \quad (15)$$

where the last line is obtained by applying the DLR equation (12) again. Note that, for a fixed $\eta \in \mathbb{K}(X)$, since the set $\tau(\eta)$ is countable, we have $\sigma(\tau(\eta)\times\mathbb{R}_+) = 0$. Hence, in formula (15), instead of the integral $\int_{X\setminus\{x\}} \phi(x,x')\, d\eta(x')$, we may write $\int_X \phi(x,x')\, d\eta(x')$. □

3 The results

In this section, we will introduce the Dirichlet form $\mathscr{E}^{\mathbb{K}}$ and formulate the results. We postpone the proofs to Section 4.

Let the conditions of Proposition 3 be satisfied and let us fix any Gibbs measure $\mu \in G(\nu,\phi)$. For any $F, G \in \mathscr{F}\mathscr{C}(\mathbb{K}(X))$, we define $\mathscr{E}^{\mathbb{K}}(F,G)$ by formula (1). Note that, by (6) and (13), we indeed have

$$\int_{\mathbb{K}(X)} \left|\langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta)\rangle_{T_\eta(\mathbb{K})}\right| d\mu(\eta) < \infty.$$

Lemma 8. *Let $F, G \in \mathscr{F}\mathscr{C}(\mathbb{K}(X))$ and let $F = 0$ μ-a.e. Then $\mathscr{E}^{\mathbb{K}}(F,G) = 0$.*

Thus, we may consider $\mathscr{E}^{\mathbb{K}}$ as a symmetric bilinear form on $L^2(\mathbb{K}(X),\mu)$ with domain $\mathscr{FC}(\mathbb{K}(X))$. Note that $\mathscr{FC}(\mathbb{K}(X))$ is dense in $L^2(\mathbb{K}(X),\mu)$. Let us now find the L^2-generator of this form. Analogously to (4), (5), we define, for each function $F\in\mathscr{FC}(\mathbb{K}(X))$, $\eta\in\mathbb{K}(X)$, and $x\in\tau(\eta)$,

$$\begin{aligned}\Delta_x F(\eta) &:= \Delta_y\Big|_{y=x} F(\eta - s_x\delta_x + s_x\delta_y),\\ \Delta_{s_x} F(\eta) &:= \frac{d^2}{du^2}\Big|_{u=s_x} F(\eta - s_x\delta_x + u\delta_x),\end{aligned}$$

where Δ_y is the Laplace operator on X acting in the y variable.

The following proposition gives, in particular, the explicit form of the L^2-generator of the bilinear form $(\mathscr{E}^{\mathbb{K}},\mathscr{FC}(\mathbb{K}(X)))$.

Proposition 9. *Assume that $l\in C^1(\widehat{X})$ and $\phi\in C^1(X\times X)$. For each $F\in\mathscr{FC}(\mathbb{K}(X))$, we define a function $L^{\mathbb{K}}F\in L^2(\mathbb{K}(X),\mu)$ by*

$$\begin{aligned}L^{\mathbb{K}}F(\eta) = \sum_{x\in\tau(\eta)} \bigg[&\frac{1}{s_x}\Delta_x F(\eta) + \frac{1}{s_x}\langle \nabla_x \log l(s,x), \nabla_x F(\eta)\rangle_X\\ &- \int_X d(\eta - s_x\delta_x)(x')\langle\nabla_x\phi(x,x'),\nabla_x F(\eta)\rangle_X\\ &+ s_x\Delta_{s_x}F(\eta) + s_x\big(\nabla_{s_x}\log l(s_x,x)\big)\big(\nabla_{s_x}F(\eta)\big)\\ &- \bigg(\int_X d(\eta - s_x\delta_x)(x')\phi(x,x')\bigg)\, s_x\nabla_{s_x}F(\eta)\bigg]. \end{aligned} \tag{16}$$

Here $\langle\cdot,\cdot\rangle_X$ denotes the scalar product in X. Then, for any $F,G\in\mathscr{FC}(\mathbb{K}(X))$,

$$\mathscr{E}^{\mathbb{K}}(F,G) = (-L^{\mathbb{K}}F, G)_{L^2(\mathbb{K}(X),\mu)}. \tag{17}$$

The bilinear form $(\mathscr{E}^{\mathbb{K}},\mathscr{FC}(\mathbb{K}(X)))$ is closable on $L^2(\mathbb{K}(X),\mu)$, and its closure, denoted by $(\mathscr{E}^{\mathbb{K}},D(\mathscr{E}^{\mathbb{K}}))$ is a Dirichlet form. The operator $(-L^{\mathbb{K}},\mathscr{FC}(\mathbb{K}(X)))$ has Friedrichs' extension, which we denote by $(-L^{\mathbb{K}},D(L^{\mathbb{K}}))$.

Remark 10. Note that, in the case where μ is the Gibbs perturbation of the gamma measure, i.e., when $l(s,x)=e^{-s}$, formula (16) becomes

$$\begin{aligned}L^{\mathbb{K}}F(\eta) = \sum_{x\in\tau(\eta)} \bigg[&\frac{1}{s_x}\Delta_x F(\eta) - \int_X d(\eta - s_x\delta_x)(x')\langle\nabla_x\phi(x,x'),\nabla_x F(\eta)\rangle_X\\ &+ s_x\big(\Delta_{s_x}F(\eta) - \nabla_{s_x}F(\eta)\big) - \bigg(\int_X d(\eta - s_x\delta_x)(x')\phi(x,x')\bigg)\, s_x\nabla_{s_x}F(\eta)\bigg].\end{aligned}$$

We are now ready to formulate the main result of the paper.

Theorem 11. *Assume that the conditions of Propositions 3 and 9 be satisfied. Further assume that the dimension d of the space X is ≥ 2. Then there exists a conservative diffusion process on $\mathbb{K}(X)$ (i.e., a conservative strong Markov process with continuous sample paths in $\mathbb{K}(X)$),*

$$M^{\mathbb{K}} = (\Omega^{\mathbb{K}}, \mathscr{F}^{\mathbb{K}}, (\mathscr{F}_t^{\mathbb{K}})_{t\geq 0}, (\Theta_t^{\mathbb{K}})_{t\geq 0}, (\mathfrak{X}^{\mathbb{K}}(t))_{t\geq 0}, (\mathbb{P}_\eta^{\mathbb{K}})_{\eta\in\mathbb{K}(X)}),$$

(cf. [4]) which is properly associated with the Dirichlet form $(\mathscr{E}^{\mathbb{K}}, D(\mathscr{E}^{\mathbb{K}}))$, i.e., for all ($\mu$-versions of) $F \in L^2(\mathbb{K}(X), \mu)$ and all $t > 0$ the function

$$\mathbb{K}(X) \ni \eta \mapsto (p_t^{\mathbb{K}} F)(\eta) := \int_\Omega F(\mathfrak{X}(t))\, d\mathbb{P}_\eta^{\mathbb{K}}$$

is an $\mathscr{E}^{\mathbb{K}}$-quasi-continuous version of $\exp(tL^{\mathbb{K}})F$ (cf. [13, Chap. 1, Sect. 2]). Here $\Omega^{\mathbb{K}} = C([0,\infty) \to \mathbb{K}(X))$, $\mathfrak{X}^{\mathbb{K}}(t)(\omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega^{\mathbb{K}}$, $(\mathscr{F}_t^{\mathbb{K}})_{t\geq 0}$ together with $\mathscr{F}^{\mathbb{K}}$ is the corresponding minimum completed admissible family (cf. [5, Section 4.1]) and $\Theta_t^{\mathbb{K}}$, $t \geq 0$, are the corresponding natural time shifts.

In particular, $M^{\mathbb{K}}$ is μ-symmetric (i.e., $\int G\, p_t^{\mathbb{K}} F\, d\mu = \int F\, p_t^{\mathbb{K}} G\, d\mu$ for all $F, G : \mathbb{K}(X) \to [0,\infty)$, $\mathscr{B}(\mathbb{K}(X))$-measurable) and has μ as an invariant measure.

$M^{\mathbb{K}}$ is up to μ-equivalence unique (cf. [13, Chap. IV, Sect. 6]).

Remark 12. In addition to (7)–(11), let us assume that the function $l(s,x)$ satisfies, for each $\Lambda \in \mathscr{B}_0(X)$,

$$\int_{\mathbb{R}_+\times\Lambda} l(s,x) s^i\, ds\, dx < \infty, \quad i = 2, 3.$$

This implies that the completely random measure ν satisfies, for each $\Lambda \in \mathscr{B}_0(X)$,

$$\int_{\mathbb{K}(X)} \eta(\Lambda)^n\, d\nu(\eta) < \infty \quad \text{for } n = 1, 2, 3, 4.$$

Then it easily follows from the proofs of Proposition 9 and Theorem 11 that these statements remain true when $l \in C^1(\widehat{X})$ and the pair potential ϕ is equal to zero, i.e., when $\mu = \nu$.

We note that, in paper [10], for a different choice of a tangent space $T_\eta(\mathbb{K})$ and in the case where $l(s,x) = l(s)$ is independent of x and $\mu = \nu$, the corresponding diffusion process on $\mathbb{K}(X)$ was constructed explicitly. However, for the choice of the tangent space $T_\eta(\mathbb{K})$ as in this paper, even in the case where $\mu = \nu$, an explicit construction of the diffusion process is an open problem, see Subsec. 5.2 in [10].

4 The proofs

4.1 Proofs of Lemma 8 and Proposition 9

We start with the following

Lemma 13. *For any* $F, G \in \mathscr{F}\mathscr{C}(\mathbb{K}(X))$,

$$\mathscr{E}^{\mathbb{K}}(F,G) = \int_{\mathbb{K}(X)} d\mu(\eta) \int_{\widehat{X}} ds\, dx\, l(s,x)\, \exp\left[-s \int_X \phi(x,x')\, d\eta(x')\right] \times \left[\frac{1}{s^2}\langle \nabla_x F(\eta + s\delta_x), \nabla_x G(\eta + s\delta_x)\rangle_X + \left(\frac{d}{ds}F(\eta+s\delta_x)\right)\left(\frac{d}{ds}G(\eta+s\delta_x)\right)\right]. \quad (18)$$

Proof. Formula (18) follows directly from (1), (2), (4)–(6), and (14). □

Proof of Lemma 8. By (C1) and (13), for a fixed $x \in X$, we get

$$\int_{\mathbb{K}(X)} \int_X |\phi(x,x')|\, d\eta(x')\, d\mu(\eta) < \infty.$$

Hence, for μ-a.a. $\eta \in \mathbb{K}(X)$, we have $\int_X |\phi(x,x')|\, d\eta(x') < \infty$. Therefore, on $\widehat{X} \times \mathbb{K}(X)$, the measures

$$l(s,x) \exp\left[-s \int_X \phi(x,x')\, d\eta(x')\right] ds\, dx\, d\mu(\eta)$$

and $ds\, dx\, d\mu(\eta)$ are equivalent.

Let $F \in \mathscr{F}\mathscr{C}(\mathbb{K}(X))$ be such that $F = 0$ μ-a.e. Then, for any $\Lambda \in \mathscr{B}_0(X)$, we get by (14)

$$\int_{\mathbb{K}(X)} d\mu(\eta) \int_{\widehat{X}} ds\, dx\, l(s,x) \exp\left[-s \int_X \phi(x,x')\, d\eta(x')\right] |F(\eta + s\delta_x)| \chi_\Lambda(x) = \int_{\mathbb{K}(X)} |F(\eta)|\, \eta(\Lambda)\, d\mu(\eta) = 0.$$

Here χ_Λ denotes the indicator function of the set Λ. Hence, $F(\eta + s\delta_x) = 0$ for $ds\, dx\, d\mu(\eta)$-a.a. $(s,x,\eta) \in \widehat{X} \times \mathbb{K}(X)$. For each fixed $\eta \in \mathbb{K}(X)$, the function $(s,x) \mapsto F(\eta + s\delta_x)$ is continuous. Therefore, for μ-a.a. $\eta \in \mathbb{K}(X)$, $F(\eta + s\delta_x) = 0$ for all $(s,x) \in \widehat{X}$. Hence, by Lemma 13, for each $G \in \mathscr{F}\mathscr{C}(\mathbb{K}(X))$, $\mathscr{E}^{\mathbb{K}}(F,G) = 0$. □

Proof of Proposition 9. We first note that $(\mathscr{E}^{\mathbb{K}}, \mathscr{F}\mathscr{C}(\mathbb{K}(X)))$ is a pre-Dirichlet form form on $L^2(\mathbb{K}(X), \mu)$, i.e., if it is closable then its closure is a Dirichlet form. This assertion follows, by standard methods, directly from [13, Chap. I, Proposition 4.10] (see also [13, Chap. II, Exercise 2.7]).

For a fixed $\eta \in \mathbb{K}(X)$, the function $(s,x) \mapsto F(\eta+s\delta_x)$ is constant outside a compact set in $\widehat{X}$. Note also that, for each fixed $\eta \in \mathbb{K}(X)$, the function $x \mapsto \int_X \phi(x,x')\,d\eta(x')$ is differentiable on X and its gradient is equal to $\int_X \nabla_x \phi(x,x')\,d\eta(x')$. Hence carrying out integration by parts in formula (18), we get for any $F, G \in \mathscr{FC}(\mathbb{K}(X))$,

$$\begin{aligned}
\mathscr{E}^{\mathbb{K}}(F,G) &= \int_{\mathbb{K}(X)} d\mu(\eta) \int_{\widehat{X}} ds\,dx\, l(s,x)\, \exp\left[-s\int_X \phi(x,x')\,d\eta(x')\right] G(\eta+s\delta_x) \\
&\times \Bigg[-\frac{1}{s^2}\Delta_x F(\eta+s\delta_x) - \frac{1}{s^2}\langle \nabla_x \log l(s,x), \nabla_x F(\eta+s\delta_x)\rangle_X \\
&+ \frac{1}{s}\int_X d\eta(x')\,\big\langle \nabla_x\phi(x,x'), \nabla_x F(\eta+s\delta_x)\big\rangle_X - \Delta_s F(\eta+s\delta_x) \\
&- \big(\nabla_s \log l(s,x)\big)\big(\nabla_s F(\eta+s\delta_x)\big) + \left(\int_X \phi(x,x')\,d\eta(x')\right)\big(\nabla_s F(\eta+s\delta_x)\big)\Bigg].
\end{aligned}$$

Applying formula (14), we get (16), (17).

It easily follows from (16) that, for a fixed $F \in \mathscr{FC}(\mathbb{K}(X))$, there exist $\Lambda \in \mathscr{B}_0(X)$ and $C > 0$ such that

$$|L^{\mathbb{K}}F(\eta)| \le C(\eta(\Lambda) + \eta(\Lambda)^2), \quad \eta \in \mathbb{K}(X).$$

Hence, by (13), $L^{\mathbb{K}}F \in L^2(\mathbb{K}(X),\mu)$. Thus, the bilinear form $(\mathscr{E}^{\mathbb{K}}, \mathscr{FC}(\mathbb{K}(X)))$ has L^2-generator. Hence, it is closable and its closure is a Dirichlet form. The last statement of the proposition about Friedrichs' extension is a standard fact of functional analysis. □

4.2 Proof of Theorem 11

We will divide the proof into several steps.

Step 1. To prove the theorem, we will initially construct a diffusion process on a certain subset of the configuration space over $\widehat{X}$. So in this step, we will present the necessary definitions and constructions related to the configuration space.

We denote by $\ddot{\Gamma}(\widehat{X})$ the space of all $\mathbb{N}_0 \cup \{\infty\}$-valued Radon measures on $\widehat{X}$. Here $\mathbb{N}_0 := \{0,1,2,\dots\}$. The space $\ddot{\Gamma}(\widehat{X})$ is endowed with the vague topology and let $\mathscr{B}(\ddot{\Gamma}(\widehat{X}))$ denote the corresponding σ-algebra.

The configuration space over $\widehat{X}$, denoted by $\Gamma(\widehat{X})$, is defined as the collection of all locally finite subsets of $\widehat{X}$:

$$\Gamma(\widehat{X}) := \big\{\gamma \subset \widehat{X} \mid |\gamma \cap A| < \infty \text{ for each compact } A \subset \widehat{X}\,\big\}.$$

Here $|\gamma \cap A|$ denotes the cardinality of the set $\gamma \cap A$. One usually identifies a configuration $\gamma \in \Gamma(\widehat{X})$ with the Radon measure $\sum_{(s,x)\in\gamma} \delta_{(s,x)}$ on $\widehat{X}$. Thus, one gets the inclusion $\Gamma(\widehat{X}) \subset \ddot{\Gamma}(\widehat{X})$.

Let $\Gamma_{pf}(\widehat{X})$ denote the subset of $\Gamma(\widehat{X})$ which consists of all configurations γ which satisfy:

(i) if $(s_1,x_1),(s_2,x_2)\in\gamma$ and $(s_1,x_1)\neq(s_2,x_2)$, then $x_1\neq x_2$;

(ii) for each $\Lambda\in\mathscr{B}_0(X)$, $\displaystyle\sum_{(s,x)\in\gamma\cap(\mathbb{R}_+\times\Lambda)} s<\infty$.

We have $\Gamma_{pf}(\widehat{X})\in\mathscr{B}(\ddot\Gamma(\widehat{X}))$, and we denote by $\mathscr{B}(\Gamma_{pf}(\widehat{X}))$ the trace σ-algebra of $\mathscr{B}(\ddot\Gamma(\widehat{X}))$ on $\Gamma_{pf}(\widehat{X})$. Equivalently, $\mathscr{B}(\Gamma_{pf}(\widehat{X}))$ is the Borel σ-algebra on the space $\Gamma_{pf}(\widehat{X})$ equipped with the vague topology.

The following statement is proven in [6, Theorem 6.2].

Proposition 14 ([6]). *Consider a bijective mapping $\mathscr{R}:\Gamma_{pf}(\widehat{X})\to\mathbb{K}(X)$ defined by*

$$\Gamma_{pf}(\widehat{X})\ni\gamma=\{(s_i,x_i)\}\mapsto\mathscr{R}\gamma:=\sum_i s_i\delta_{x_i}\in\mathbb{K}(X). \tag{19}$$

Then the mapping $\mathscr{R}$ and its inverse $\mathscr{R}^{-1}:\mathbb{K}(X)\to\Gamma_{pf}(\widehat{X})$ are measurable.

Note that the pushforward of the completely random measure ν under $\mathscr{R}^{-1}$ is the Poisson measure on $\Gamma(\widehat{X})$ with intensity measure σ: if we denote this measure by π, the Fourier transform of π is given by

$$\int_{\Gamma_{pf}(\widehat{X})} e^{i\langle f,\gamma\rangle}\,d\pi(\gamma)=\exp\left[\int_{\widehat{X}}(e^{if(s,x)}-1)\,d\sigma(s,x)\right],\quad f\in C_0(\widehat{X}).$$

Here we denote $\langle f,\gamma\rangle:=\int_{\widehat{X}}f\,d\gamma=\sum_{(s,x)\in\gamma}f(s,x)$.

Let ρ denote the pushforward of the Gibbs measure μ under $\mathscr{R}^{-1}$. By Theorem 7 and (19), the measure ρ satisfies, for each measurable function $F:\widehat{X}\times\Gamma(\widehat{X})\to[0,\infty]$,

$$\int_{\Gamma_{pf}(\widehat{X})}\sum_{(s,x)\in\gamma}F(s,x,\gamma)\,d\rho(\gamma)$$
$$=\int_{\Gamma_{pf}(\widehat{X})}d\rho(\gamma)\int_{\widehat{X}}d\sigma(s,x)\,\exp\left[-\sum_{(s',x')\in\gamma}ss'\phi(x,x')\right]F(s,x,\gamma\cup\{(s,x)\}).$$

Let $\mathscr{FC}(\Gamma_{pf}(\widehat{X}))$ denote the set of functions on $\Gamma_{pf}(\widehat{X})$ which are of the form $F(\gamma)=G(\mathscr{R}\gamma)$ for some $G\in\mathscr{FC}(\mathbb{K}(X))$. Thus, $\mathscr{FC}(\Gamma_{pf}(\widehat{X}))$ consists of all functions F of the form

$$F(\gamma)=g\big(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle\big),\quad\gamma\in\Gamma_{pf}(\widehat{X}),$$

where the functions $g,\varphi_1,\dots,\varphi_N$ are as in (3). Thus, we may equivalently consider a bilinear form $(\mathscr{E}^\Gamma,\mathscr{FC}(\Gamma_{pf}(\widehat{X})))$ on $L^2(\Gamma_{pf}(\widehat{X}),\rho)$ which is defined by

$$\mathscr{E}^\Gamma(F,G):=\mathscr{E}^{\mathbb{K}}(F\circ\mathscr{R}^{-1},G\circ\mathscr{R}^{-1}),\quad F,G\in\mathscr{FC}(\Gamma_{pf}(\widehat{X})).$$

As easily seen, for any $F, G \in \mathscr{F}\mathscr{C}(\Gamma_{pf}(\widehat{X}))$, we have

$$\mathscr{E}^{\Gamma}(F,G) = \int_{\Gamma(\widehat{X})} \sum_{(s,x)\in\gamma} \left[\frac{1}{s}\langle \nabla_x F(\gamma), \nabla_x G(\gamma)\rangle_X + s\big(\nabla_s F(\gamma)\big(\nabla_s G(\gamma)\big)\right] d\rho(\gamma),$$

where $\nabla_x F(\gamma)$ and $\nabla_s G(\gamma)$ are defined analogously to formulas (4), (5). By Proposition 9, the bilinear form $(\mathscr{E}^{\Gamma}, \mathscr{F}\mathscr{C}(\Gamma_{pf}(\widehat{X})))$ is closable on $L^2(\Gamma_{pf}(\widehat{X}), \rho)$, and its closure, denoted by $(\mathscr{E}^{\Gamma}, D(\mathscr{E}^{\Gamma}))$, is a Dirichlet form.

Step 2. Our aim now is to construct a diffusion process on $\Gamma_{pf}(\widehat{X})$ which is properly associated with the Dirichlet form $(\mathscr{E}^{\Gamma}, D(\mathscr{E}^{\Gamma}))$. We will initially construct such a process on a bigger space $\ddot{\Gamma}_f(\widehat{X})$. In this step, we will define the set $\ddot{\Gamma}_f(\widehat{X})$ and construct a metric on it such that the set $\ddot{\Gamma}_f(\widehat{X})$ equipped with this metric is a Polish space.

For each $\Lambda \in \mathscr{B}_0(X)$, we define a local mass $\mathfrak{M}_\Lambda$ by

$$\mathfrak{M}_\Lambda(\gamma) := \int_{\widehat{X}} \chi_\Lambda(x) s\, d\gamma(s,x), \quad \gamma \in \ddot{\Gamma}(\widehat{X}).$$

We set

$$\ddot{\Gamma}_f(\widehat{X}) := \big\{\gamma \in \ddot{\Gamma}(\widehat{X}) \mid \mathfrak{M}_\Lambda(\gamma) < \infty \text{ for each } \Lambda \in \mathscr{B}_0(X)\big\}.$$

We have $\ddot{\Gamma}_f(\widehat{X}) \in \mathscr{B}(\ddot{\Gamma}(\widehat{X}))$, and let $\mathscr{B}(\ddot{\Gamma}_f(\widehat{X}))$ denote the Borel σ-algebra on the space $\ddot{\Gamma}_f(\widehat{X})$ equipped with the vague topology.

We will now construct a bounded metric on $\ddot{\Gamma}_f(\widehat{X})$ in which this space will be complete and separable. Let $d_V(\cdot,\cdot)$ denote the bounded metric on $\ddot{\Gamma}(\widehat{X})$ which was introduced in [14, Section 3]. Recall that this metric generates the vague topology on $\ddot{\Gamma}(\widehat{X})$, and $\ddot{\Gamma}(\widehat{X})$ is complete and separable in this metric.

For each $k \in \mathbb{N}$, we fix any function $\phi_k \in C_0^\infty(X)$ such that

$$\chi_{B(k)} \le \phi_k \le \chi_{B(k+1)}, \qquad \left|\frac{\partial}{\partial x_i}\phi_k(x)\right| \le 2\,\chi_{B(k+1)}(x),$$
$$i = 1, \dots, d,\ x = (x^1, \dots, x^d) \in X. \tag{20}$$

Here

$$B(k) := \big\{x = (x^1, \dots, x^d) \in X \mid \max_{i=1,\dots,d} |x_i| \le k\big\}.$$

Next, we fix any $q \in (0,1)$. We take any sequence $(\psi_n)_{n\in\mathbb{Z}}$ such that, for each $n \in \mathbb{Z}$, $\psi_n \in C_0^\infty(\mathbb{R})$ and

$$\chi_{[q^n,\, q^{n-1}]} \le \psi_n \le \chi_{[q^{n+1},\, q^{n-2}]}, \qquad |\psi_n'| \le \frac{2}{q^n - q^{n+1}}\,\chi_{[q^{n+1},\, q^n]\cup[q^{n-1},\, q^{n-2}]}. \tag{21}$$

For each $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, we define

$$\varkappa_{kn}(s,x) := \phi_k(x)\psi_n(s)s, \quad (s,x) \in \widehat{X}. \tag{22}$$

Note that $\varkappa_{kn} \in C_0^\infty(\widehat{X})$. For any $k \in \mathbb{N}$ and $\gamma, \gamma' \in \ddot{\Gamma}_f(\widehat{X})$, we define

$$d_k(\gamma, \gamma') := \sum_{n \in \mathbb{Z}} |\langle \varkappa_{kn}, \gamma - \gamma' \rangle|. \tag{23}$$

As follows from (20) and (21), for each $\gamma \in \ddot{\Gamma}_f(\widehat{X})$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \langle \varkappa_{kn}, \gamma\rangle &= \int_{\widehat{X}} d\gamma(s,x)\, \phi_k(x) \left(\sum_{n\in\mathbb{Z}} \psi_n(s)\right) s \\ &\le 4 \int_{\widehat{X}} d\gamma(s,x)\phi_k(x)s \le 4\,\mathfrak{M}_{B(k+1)}(\gamma) < \infty. \end{aligned} \tag{24}$$

Therefore, $d_k(\gamma, \gamma') < \infty$ for all $\gamma, \gamma' \in \ddot{\Gamma}_f(\widehat{X})$. Clearly, $d_k(\cdot, \cdot)$ satisfies the triangle inequality.

Let $(c_k)_{k=1}^\infty$ be a sequence of $c_k > 0$ such that $\sum_{k=1}^\infty c_k < \infty$. Below, in formula (35), we will make an explicit choice of the sequence $(c_k)_{k=1}^\infty$. We next define

$$d_f(\gamma, \gamma') := \sum_{k=1}^\infty c_k \frac{d_k(\gamma, \gamma')}{1 + d_k(\gamma, \gamma')}, \quad \gamma, \gamma' \in \ddot{\Gamma}_f(\widehat{X}).$$

Clearly, $d_f(\cdot, \cdot)$ also satisfies the triangle inequality. We finally define the metric

$$d(\gamma, \gamma') := d_V(\gamma, \gamma') + d_f(\gamma, \gamma'), \quad \gamma, \gamma' \in \ddot{\Gamma}_f(\widehat{X}).$$

Proposition 15. $(\ddot{\Gamma}_f(\widehat{X}), d(\cdot, \cdot))$ *is a complete, separable metric space.*

Proof. Let $\{\gamma_i\}_{i=1}^\infty$ be a Cauchy sequence in $(\ddot{\Gamma}_f(\widehat{X}), d(\cdot, \cdot))$. Then $\{\gamma_i\}_{i=1}^\infty$ is a Cauchy sequence in $(\ddot{\Gamma}(\widehat{X}), d_V(\cdot, \cdot))$. Since the latter space is complete, there exists $\gamma \in \ddot{\Gamma}(\widehat{X})$ such that $\gamma_i \to \gamma$ vaguely as $i \to \infty$. Denote

$$a_{kn}^{(i)} := \langle \varkappa_{kn}, \gamma_i \rangle, \quad a_{kn} := \langle \varkappa_{kn}, \gamma \rangle, \quad k \in \mathbb{N},\ n \in \mathbb{Z}.$$

As $\varkappa_{kn} \in C_0(\widehat{X})$, we therefore get:

$$\text{for each } k \in \mathbb{N} \text{ and } n \in \mathbb{Z} \quad a_{kn}^{(i)} \to a_{kn} \text{ as } i \to \infty. \tag{25}$$

Note that, for each $k \in \mathbb{N}$ and $i \in \mathbb{N}$, $a_{kn}^{(i)} \ge 0$ for all $n \in \mathbb{Z}$ and by (24)

$$\sum_{n \in \mathbb{N}} a_{kn}^{(i)} < \infty.$$

Hence, $(a_{kn}^{(i)})_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. As $\{\gamma_i\}_{i=1}^\infty$ is a Cauchy sequence in $(\ddot\Gamma_f(\widehat{X}), d(\cdot,\cdot))$,

$$\lim_{i,j\to\infty} \sum_{n\in\mathbb{Z}} |a_{kn}^{(i)} - a_{kn}^{(j)}| = \lim_{i,j\to\infty} d_k(\gamma_i, \gamma_j) = 0, \quad k \in \mathbb{N}.$$

Hence, $\{(a_{kn}^{(i)})_{n\in\mathbb{Z}}\}_{i=1}^\infty$ is a Cauchy sequence in $\ell^1(\mathbb{Z})$. Since the latter space is complete, the sequence $\{(a_{kn}^{(i)})_{n\in\mathbb{Z}}\}_{i=1}^\infty$ is convergent in $\ell^1(\mathbb{Z})$. In view of (25), we therefore conclude that the $\ell^1(\mathbb{Z})$-limit of this sequence is $(a_{kn})_{n\in\mathbb{Z}}$. This, in particular, implies that

$$\sum_{n\in\mathbb{Z}} a_{kn} = \sum_{n\in\mathbb{Z}} \langle \varkappa_{kn}, \gamma \rangle < \infty, \quad k \in \mathbb{N}. \tag{26}$$

By (21), $\sum_{n=1}^\infty \psi_n(s) \geq 1$ for all $s \in \mathbb{R}_+$. We therefore deduce from (26) that $\gamma \in \ddot\Gamma_f(\widehat{X})$. Furthermore,

$$d_k(\gamma_i, \gamma) = \sum_{n\in\mathbb{Z}} |a_{kn}^{(i)} - a_{kn}| \to 0 \quad \text{as } i \to \infty, \quad k \in \mathbb{N}.$$

Hence $d(\gamma_i, \gamma) \to 0$ as $i \to \infty$. Thus, $(\ddot\Gamma_f(\widehat{X}), d(\cdot,\cdot))$ is complete. The proof of the separability of this space is routine, so we skip it. □

Step 3. We will now consider $(\mathscr{E}^\Gamma, D(\mathscr{E}^\Gamma))$ as a Dirichlet form on $L^2(\ddot\Gamma_f(\widehat{X})), \rho)$ and prove that is is quasi-regular. For the definition of quasi-regularity of a Dirichlet form, see [13, Chap. IV, Def. 3.1] and [14, subsec. 4.1].

We consider the complete separable metric space $(\ddot\Gamma_f(\widehat{X}), d(\cdot,\cdot))$, and let $\mathscr{B}(\ddot\Gamma_f(\widehat{X}), d)$ denote the corresponding Borel σ-algebra on $\ddot\Gamma_f(\widehat{X})$.

Lemma 16. *We have* $\mathscr{B}(\ddot\Gamma_f(\widehat{X})) = \mathscr{B}(\ddot\Gamma_f(\widehat{X}), d)$.

Proof. We have $d(\gamma, \gamma') \geq d_V(\gamma, \gamma')$ for all $\gamma, \gamma' \in \ddot\Gamma_f(\widehat{X})$. Therefore, $\mathscr{B}(\ddot\Gamma_f(\widehat{X})) \subset \mathscr{B}(\ddot\Gamma_f(\widehat{X}), d)$. On the other hand, it follows from the construction of the metric $d(\cdot,\cdot)$ that, for a fixed $\gamma' \in \ddot\Gamma_f(\widehat{X})$, the function

$$\ddot\Gamma_f(\widehat{X}) \ni \gamma \mapsto d(\gamma, \gamma') \in \mathbb{R}$$

is $\mathscr{B}(\ddot\Gamma_f(\widehat{X}))$-measurable. Hence, for any $\gamma' \in \ddot\Gamma_f(\widehat{X})$ and $r > 0$,

$$\{\gamma \in \ddot\Gamma_f(\widehat{X}) \mid d(\gamma, \gamma') < r\} \in \mathscr{B}(\ddot\Gamma_f(\widehat{X})). \tag{27}$$

But in a separable metric space, every open set can be represented as a countable union of open balls, see e.g. Theorem 2 and its proof in [12, p. 206]. Hence, (27) implies the inclusion $\mathscr{B}(\ddot\Gamma_f(\widehat{X}), d) \subset \mathscr{B}(\ddot\Gamma_f(\widehat{X}))$. □

We will now consider ρ as a probability measure on the measurable space $(\ddot{\Gamma}_f(\widehat{X}), \mathscr{B}(\ddot{\Gamma}_f(\widehat{X})))$, and $(\mathscr{E}^\Gamma, D(\mathscr{E}^\Gamma))$ as a Dirichlet form on the space $L^2(\ddot{\Gamma}_f(\widehat{X}), \rho)$.

On $D(\mathscr{E}^\Gamma)$ we consider the norm

$$\|F\|_{D(\mathscr{E}^\Gamma)} := \mathscr{E}^\Gamma(F,F)^{1/2} + \|F\|_{L^2(\ddot{\Gamma}_f(\widehat{X}),\,\rho)}.$$

We define a square field operator

$$S^\Gamma(F)(\gamma) := \sum_{(s,x)\in\gamma} \left[\frac{1}{s}\, \|\nabla_x F(\gamma)\|_X^2 + s\big|\nabla_s F(\gamma)\big|^2\right], \tag{28}$$

where $F \in \mathscr{F}\mathscr{C}(\Gamma_{pf}(\widehat{X}))$, $\gamma \in \Gamma_{pf}(\widehat{X})$, and $\|\cdot\|_X$ denotes the Euclidean norm in X. As easily seen, S^Γ extends by continuity in the norm $\|\cdot\|_{D(\mathscr{E}^\Gamma)}$ to a mapping $S^\Gamma : D(\mathscr{E}^\Gamma) \to L^1(\ddot{\Gamma}_f(\widehat{X}), \rho)$, and furthermore $\mathscr{E}^\Gamma(F,F) = \int_{\ddot{\Gamma}_f(\widehat{X})} S^\Gamma(F)\, d\rho$.

Lemma 17. *For each $\gamma \in \ddot{\Gamma}_f(\widehat{X})$, we have $d(\cdot,\gamma) \in D(\mathscr{E}^\Gamma)$. Furthermore, there exists $G \in L^1(\ddot{\Gamma}_f(\widehat{X}), \rho)$ (independent of γ) such that $S^\Gamma(d(\cdot,\gamma)) \le G$ ρ-a.e.*

Proof. Recall that $d(\cdot,\gamma) = d_V(\cdot,\gamma) + d_f(\cdot,\gamma)$. Using the methods of [14, Section 4] (see also [11, Section 6]), one can show that $d_V(\cdot,\gamma) \in D(\mathscr{E}^\Gamma)$ and there exists $G_1 \in L^1(\ddot{\Gamma}_f(\widehat{X}), \rho)$ (independent of γ) such that $S^\Gamma(d_V(\cdot,\gamma)) \le G_1$ ρ-a.e. Hence, we only need to prove that $d_f(\cdot,\gamma) \in D(\mathscr{E}^\Gamma)$ and there exists $G_2 \in L^1(\ddot{\Gamma}_f(\widehat{X}), \rho)$ (independent of γ) such that $S^\Gamma(d_f(\cdot,\gamma)) \le G_2$ ρ-a.e.

Analogously to the proof of [14, Lemma 4.7], we fix any sequence $(\zeta_n)_{n=1}^\infty$ such that $\zeta_n \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \zeta_n(t)\, dt = 1$, $\zeta_n(t) = \zeta_n(-t)$ for all $t \in \mathbb{R}$, $\operatorname{supp}(\zeta_n) \subset (-1/n, 1/n)$. We define

$$u_n(t) := \int_{\mathbb{R}} |t-t'|\zeta_n(t')\, dt' - \int_{\mathbb{R}} |t'|\zeta_n(t')\, dt', \quad t \in \mathbb{R}.$$

It is easy to check that, for each $n \in \mathbb{N}$, $u_n \in C^\infty(\mathbb{R})$, $|u_n(t)| \le |t|$, $u_n(t) \to |t|$ as $n \to \infty$ for each $t \in \mathbb{R}$, $u_n'(t) \to \operatorname{sign}(t)$ as $n \to \infty$ for each $t \in \mathbb{R}\setminus\{0\}$, and $|u_n'(t)| \le 2$ for all $t \in \mathbb{R}$.

Recall (22) and (23). For each $N \in \mathbb{N}$, we define

$$d_k^{(N)}(\gamma,\gamma') := \sum_{n\in\mathbb{Z}\cap[-N,N]} u_N(\langle \varkappa_{kn}, \gamma-\gamma'\rangle),$$

$$d_f^{(N)}(\gamma,\gamma') := \sum_{k=1}^N c_k\, \frac{d_k^{(N)}(\gamma,\gamma')}{1+d_k^{(N)}(\gamma,\gamma')}, \quad \gamma,\gamma' \in \ddot{\Gamma}_f(\widehat{X}). \tag{29}$$

Clearly, for a fixed $\gamma' \in \ddot{\Gamma}_f(\widehat{X})$, the restriction of $d_f^{(N)}(\cdot,\gamma')$ to $\Gamma_{pf}(\widehat{X})$ belongs to $\mathscr{F}\mathscr{C}(\Gamma_{pf}(\widehat{X}))$. Hence, $d_f^{(N)}(\cdot,\gamma') \in D(\mathscr{E}^\Gamma)$.

As easily seen, for each $\gamma \in \ddot{\Gamma}_f(\widehat{X})$, we have $d_f^{(N)}(\gamma, \gamma') \to d_f(\gamma, \gamma')$ as $N \to \infty$. Hence,

$$d_f^{(N)}(\cdot, \gamma') \to d_f(\cdot, \gamma') \quad \text{in } L^2(\ddot{\Gamma}_f(\widehat{X}), \rho) \text{ as } N \to \infty. \tag{30}$$

Note that, for $t \geq 0$, $\left(\frac{t}{1+t}\right)' = \frac{1}{(1+t)^2} \leq 1$. Hence, by (20)–(22), for each $\gamma \in \Gamma_{pf}(\widehat{X})$ and each $(s, x) \in \gamma$,

$$\begin{aligned}
\|\nabla_x\, d_f^{(N)}(\gamma, \gamma')\|_X &\leq \sum_{k=1}^{N} c_k\, \|\nabla_x\, d_k^{(N)}(\gamma, \gamma')\|_X \\
&\leq 2 \sum_{k=1}^{N} c_k \sum_{n \in \mathbb{Z} \cap [-N,N]} \|\nabla_x\, \varkappa_{kn}(x, s)\|_X \\
&= 2 \sum_{k=1}^{N} c_k\, \|\nabla\, \phi_k(x)\|_X \sum_{n \in \mathbb{Z} \cap [-N,N]} \psi_n(s) s \\
&\leq 4\sqrt{d} \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z} \cap [-N,N]} \psi_n(s) s \\
&\leq 16\sqrt{d} \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) s.
\end{aligned}$$

Hence, using the Cauchy inequality, we conclude that there exists a constant $C_1 > 0$ such that

$$\|\nabla_x\, d_f^{(N)}(\gamma, \gamma')\|_X^2 \leq C_1 s^2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x). \tag{31}$$

Analogously, using (20)–(22), we get

$$\begin{aligned}
&\left|\nabla_s d_f^{(N)}(\gamma, \gamma')\right| \leq \sum_{k=1}^{N} c_k \left|\nabla_s d_k^{(N)}(\gamma, \gamma')\right| \\
&\quad \leq 2 \sum_{k=1}^{N} c_k \sum_{n \in \mathbb{Z} \cap [-N,N]} \left| \frac{\partial}{\partial s}\, \varkappa_{kn}(x, s) \right| \\
&\quad = 2 \sum_{k=1}^{N} c_k \phi_k(x) \sum_{n \in \mathbb{Z} \cap [-N,N]} |\psi_n'(s) s + \psi_n(s)| \\
&\quad \leq 2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z}} \left(\frac{2}{q^n(1-q)} \chi_{[q^{n+1},\, q^n] \cup [q^{n-1},\, q^{n-2}]}(s) s + \chi_{[q^{n+1},\, q^{n-2}]}(s) \right) \\
&\quad \leq 2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z}} \left(\frac{2}{q^n(1-q)} \chi_{[q^{n+1},\, q^n] \cup [q^{n-1},\, q^{n-2}]}(s) q^{n-2} + \chi_{[q^{n+1},\, q^{n-2}]}(s) \right)
\end{aligned}$$

$$\leq 2\sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x)\left(\frac{8}{q^2(1-q)}+4\right).$$

Hence, there exists a constant $C_2>0$ such that

$$\left|\nabla_s F(\gamma)\right|^2 \leq C_2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x). \tag{32}$$

We define, for $\gamma \in \Gamma_{pf}(\widehat{X})$,

$$G_2(\gamma) := (C_1+C_2)\sum_{(s,x)\in\gamma} s\sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x). \tag{33}$$

By the monotone convergence theorem,

$$\begin{aligned}\int_{\ddot{\Gamma}_f(\widehat{X})} G_2\, d\rho &= (C_1+C_2)\sum_{k=1}^{\infty} c_k \int_{\Gamma_{pf}(\widehat{X})}\sum_{(s,x)\in\gamma} s\chi_{B(k+1)}(x)\, d\rho(\gamma)\\ &= (C_1+C_2)\sum_{k=1}^{\infty} c_k \int_{\mathbb{K}(X)} \eta(B(k+1))\, d\mu(\eta).\end{aligned} \tag{34}$$

By (13), we have, for each $k\in\mathbb{N}$,

$$\int_{\mathbb{K}(X)} \eta(B(k+1))\, d\mu(\eta) < \infty.$$

So we may set

$$c_k := 2^{-k}\left(1+\int_{\mathbb{K}(X)} \eta(B(k+1))\, d\mu(\eta)\right)^{-1}, \quad k\in\mathbb{N}. \tag{35}$$

Then, by (34), we get $G_2 \in L^1(\ddot{\Gamma}_f(\widehat{X},\rho))$. Furthermore, by (28), (31)–(33), we get

$$S^\Gamma(d_f^{(N)}(\cdot,\gamma')) \leq G_2 \quad \text{point-wise on } \Gamma_{pf}(\widehat{X}). \tag{36}$$

Using (36) and the dominated convergence theorem, it is not hard to prove that

$$\mathscr{E}^\Gamma\big(d_f^{(N)}(\cdot,\gamma') - d_f^{(M)}(\cdot,\gamma')\big) \to 0 \quad \text{as } N, M\to\infty. \tag{37}$$

Hence, $\big(d_f^{(N)}(\cdot,\gamma')\big)_{N=1}^{\infty}$ is a Cauchy sequence in $(D(\mathscr{E}^\Gamma), \|\cdot\|_{D(\mathscr{E}^\Gamma)})$. Hence, by (30) and (37), $d_f(\cdot,\gamma')\in D(\mathscr{E}^\Gamma)$. Furthermore, since $d_f^{(N)}(\cdot,\gamma')\to d_f(\cdot,\gamma')$ in the $\|\cdot\|_{D(\mathscr{E}^\Gamma)}$ norm,

$$S^\Gamma(d_f^{(N)}(\cdot,\gamma')) \to S^\Gamma(d_f(\cdot,\gamma')) \quad \text{in } L^1(\ddot{\Gamma}_f(\widehat{X}),\rho) \text{ as } N\to\infty.$$

Hence, by (36), $S^\Gamma(d_f(\cdot,\gamma)) \leq G_2$ ρ-a.e. □

By [14, Proposition 4.1] (see also [17, Theorem 3.4]), Proposition 15 and Lemma 17 imply the following proposition.

Proposition 18. *The Dirichlet form* $(\mathscr{E}^\Gamma, D(\mathscr{E}^\Gamma))$ *on* $L^2(\ddot{\Gamma}_f(\widehat{X}), \rho)$ *is quasi-regular.*

Step 4. We will now construct a corresponding diffusion process on $\ddot{\Gamma}_f(\widehat{X})$.

Lemma 19. *The Dirichlet form* $(\mathscr{E}^\Gamma, D(\mathscr{E}^\Gamma))$ *has local property, i.e.,* $\mathscr{E}^\Gamma(F,G) = 0$ *provided* $F, G \in D(\mathscr{E}^\Gamma)$ *with* $\operatorname{supp}(|F|\rho) \cap \operatorname{supp}(|G|\rho) = \varnothing$.

Proof. Identical to the proof of [14, Proposition 4.12]. □

As a consequence of Proposition 18, Lemma 19, and [13, Chap. IV, Theorem 3.5, and Chap. V, Theorem 1.11], we obtain

Proposition 20. *There exists a conservative diffusion process on the metric space* $(\ddot{\Gamma}_f(\widehat{X}), d(\cdot,\cdot))$,

$$M^\Gamma = (\Omega^\Gamma, \mathscr{F}^\Gamma, (\mathscr{F}^\Gamma_t)_{t\ge 0}, (\Theta^\Gamma_t)_{t\ge 0}, (\mathfrak{X}^\Gamma(t))_{t\ge 0}, (\mathbb{P}^\Gamma_\gamma)_{\gamma \in \ddot{\Gamma}_f(\widehat{X})}),$$

which is properly associated with the Dirichlet form $(\mathscr{E}^\Gamma, D(\mathscr{E}^\Gamma))$. *Here* $\Omega^\Gamma = C([0,\infty) \to \ddot{\Gamma}_f(\widehat{X}))$, $\mathfrak{X}^\Gamma(t)(\omega) = \omega(t)$, $t \ge 0$, $\omega \in \Omega^\Gamma$, $(\mathscr{F}^\Gamma_t)_{t\ge 0}$ *together with* $\mathscr{F}^\Gamma$ *is the corresponding minimum completed admissible family, and* Θ^Γ_t, $t \ge 0$, *are the corresponding natural time shifts. This process is up to* ρ*-equivalence unique.*

Step 5. We will now show that the diffusion process from Proposition 20 lives, in fact, on the smaller space $\Gamma_{pf}(\widehat{X})$. This is where we use that the dimension d of the underlying space X is ≥ 2.

Proposition 21. *The set* $\ddot{\Gamma}_f(\widehat{X}) \setminus \Gamma_{pf}(\widehat{X})$ *is* $\mathscr{E}^\Gamma$*-exceptional. Thus, the statement of Proposition 20 remains true if we replace in it* $\ddot{\Gamma}_f(\widehat{X})$ *with* $\Gamma_{pf}(\widehat{X})$.

Proof. The proof of this statement is similar to the proof of [18, Proposition 1 and Corollary 1], see also the proof of [11, Theorem 6.3]. □

Step 6. We will now prove that the mapping $\mathscr{R}$ is continuous with respect to the $d(\cdot,\cdot)$ metric.

Proposition 22. *The mapping* $\mathscr{R}$ *acts continuously from the metric space* $(\Gamma_{pf}(\widehat{X}), d(\cdot,\cdot))$ *into the space* $\mathbb{K}(X)$ *endowed with the vague topology.*

Proof. Let $\{\gamma_i\}_{i=1}^\infty \subset \Gamma_{pf}(\widehat{X})$ and $\gamma \in \Gamma_{pf}(\widehat{X})$. Let $d(\gamma_i, \gamma) \to 0$ as $i \to \infty$. We have to prove that $\mathscr{R}\gamma_i \to \mathscr{R}\gamma$ vaguely as $i \to \infty$.

So fix any $f \in C_0(X)$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\operatorname{supp}(f) \subset B(k)$. Choose $N \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{Z},\, |n|\ge N} \langle \varkappa_{kn}, \gamma\rangle \le \varepsilon. \tag{38}$$

Since $d(\gamma_i,\gamma)\to 0$, we have $d_k(\gamma_i,\gamma)\to 0$. Hence, there exists $I\in\mathbb{N}$ such that

$$\sum_{n\in\mathbb{Z},\,|n|\ge N}\langle\gamma_i,\varkappa_{kn}\rangle\le 2\varepsilon,\quad i\ge I. \tag{39}$$

By (20)–(22), (38), and (39),

$$\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} s\,d\gamma(x,s)\le\varepsilon,$$
$$\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} s\,d\gamma_i(x,s)\le 2\varepsilon,\quad i\ge I.$$

Therefore,

$$\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} |f(x)|s\,d\gamma(x,s)\le\varepsilon\|f\|_\infty,$$
$$\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} |f(x)|s\,d\gamma_i(x,s)\le 2\varepsilon\|f\|_\infty,\quad i\ge I, \tag{40}$$

where $\|f\|_\infty$ is the supremum norm of the function f. Fix any $\xi\in C_0(\mathbb{R}_+)$ such that

$$\chi_{[q^N,\,q^{-N}]}\le\xi\le 1. \tag{41}$$

Since the function $f(x)\xi(s)s$ is from $C_0(\widehat{X})$, by the vague convergence

$$\int_{\widehat{X}} f(x)\xi(s)s\,d\gamma_i(x,s)\to\int_{\widehat{X}} f(x)\xi(s)s\,d\gamma(x,s)\quad\text{as } i\to\infty.$$

Hence, there exists $I_1\ge I$ such that

$$\left|\int_{\widehat{X}} f(x)\xi(s)s\,d(\gamma_i-\gamma)(x,s)\right|\le\varepsilon,\quad i\ge I_1. \tag{42}$$

By (40)–(42), for all $i\ge I_1$,

$$\begin{aligned}&\left|\int_{B(k)\times[q^N,q^{-N}]} f(x)s\,d(\gamma_i-\gamma)(x,s)\right|=\left|\int_{B(k)\times[q^N,q^{-N}]} f(x)\xi(s)s\,d(\gamma_i-\gamma)(x,s)\right|\\
&\quad\le\left|\int_{\widehat{X}} f(x)\xi(s)s\,d(\gamma_i-\gamma)(x,s)\right|\\
&\qquad+\left|\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} f(x)\xi(s)s\,d\gamma_i(x,s)\right|\\
&\qquad+\left|\int_{B(k)\times((0,\,q^N)\cup(q^{-N},\,\infty))} f(x)\xi(s)s\,d\gamma(x,s)\right|\end{aligned}$$

$$\leq \varepsilon(1 + 3\|f\|_\infty). \tag{43}$$

By (40) and (43), for all $i \geq I_1$,

$$\left| \int_X f(x)\, d(\mathscr{R}\gamma_i - \mathscr{R}\gamma)(x) \right| = \left| \int_{\widehat{X}} f(x) s\, d(\gamma_i - \gamma)(x, s) \right| \leq \varepsilon(1 + 6\|f\|_\infty).$$

Thus, the proposition is proven. □

Step 7. Finally, to construct the process $M^{\mathbb{K}}$ on $\mathbb{K}(X)$, we just map the process M^Γ from Proposition 20 onto $\mathbb{K}(X)$ by using the bijective mapping $\mathscr{R} : \Gamma_{pf}(\widehat{X}) \to \mathbb{K}(X)$. Proposition 22 ensures that the sample paths of the obtained Markov process are continuous in the vague topology on $\mathbb{K}(X)$.

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