ON FRACTAL PROPERTIES OF NON-NORMAL NUMBERS WITH RESPECT TO RÉNYI f-EXPANSIONS GENERATED BY PIECEWISE LINEAR FUNCTIONS

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Abstract. The paper is devoted to the study of fractal properties of subsets of the set of non-normal numbers with respect to Rényi f-expansions generated by continuous increasing piecewise linear functions defined on $[0, +\infty)$. All such expansions are expansions for real numbers generated by infinite linear IFS $f = \{f_0, f_1, ..., f_n, ...\}$ with the following list of ratios $Q_\infty = (q_0, q_1, ..., q_n, ...)$.

We prove the superfractality of the set of Q_∞-essentially non-normal numbers, i.e. real numbers having no asymptotic frequencies of any digits from the alphabet $A = \{0, 1, ..., n, ...\}$, for any infinite stochastic vector Q_∞, independently of the finiteness resp. infiniteness of its entropy and independently of the faithfulness resp. non-faithfulness of the family of cylinders generated by these expansions.

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AMS Subject Classifications (2010): 11K55, 28A80, 60G30.

Key words: infinite IFS, Hausdorff dimension, Q_∞-expansion, fractals, singular probability measures, non-normal numbers.

1. Introduction

The notions of self-similar sets and self-similar measures are by now quite well known. They can be defined in a standard way by lists of similarity ratios $(q_0, q_1, ..., q_{n-1})$ and probabilities $(p_0, p_1, ..., p_{n-1})$ (see, e.g., [8, 9, 10]). Properties of such sets and measures are well investigated under the open set condition (see, e.g., [9]). On the other hand natural generalizations to the case of infinite (even linear) IFS (iterated function system) lead to a series of new phenomena. Let us mention only two interesting aspects related to the infinite IFS. The first one is naturally related to the possible divergence of the entropy of the stochastic vector of similarity ratios $Q_\infty = (q_0, q_1, ..., q_n, ...)$.

The second and rather unexpected aspect is related to the faithfulness (non-faithfulness) of the system of cylindrical sets from the coding space generated by infinite IFS ([3, 19]). To explain this phenomenon let us recall that a family Φ of coverings is said to be a fine covering system of the unit interval if for an arbitrary set $E \subset [0,1)$, and for each number $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of E ($E_j \in \Phi$, $|E_j| \le \varepsilon$). Let α be a positive number. The α-dimensional Hausdorff measure of a bounded subset E with respect to a given family of subsets Φ is defined by

$$H^\alpha(E,\Phi) = \lim_{\varepsilon\to 0}\left[\inf_{d(E_j)\le\varepsilon}\left\{\sum_j |(E_j)|^\alpha\right\}\right] = \lim_{\varepsilon\to 0} m^\alpha_\varepsilon(E,\Phi),$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of E, $E_j \in \Phi$. We remark that, generally speaking, $H^\alpha(E,\Phi)$ depends on the family Φ. The family of all bounded sets, the family of all open sets and the family of all closed sets all give rise to the same α-dimensional Hausdorff measure (see, e.g., [8]), which will be denoted by $H^\alpha(E)$.

Definition 1. The nonnegative number

$$\dim_H(E,\Phi) = \inf\{\alpha : \quad H^\alpha(E,\Phi) = 0\} \tag{1}$$

is called the Hausdorff dimension of the set E with respect to the family of subsets Φ.

Definition 2. A fine covering system Φ is said to be faithful if for any subset $E \subset [0,1]$ one has

$$\dim_H(E,\Phi) = \dim_H(E).$$

In [3] it has been shown that even for the simplest case of infinite IFS, leading to the classical Lüroth expansion, the family of corresponding cylinders is non-faithful for the determination of the Hausdorff dimension of subsets from the unit interval.

Let $Q_\infty = (q_0, q_1, ..., q_n, ...)$ be an infinite stochastic vector with strictly positive coordinates. Let us consider the infinite iterated functions system (IFS) generated by the following countable set of similitudes

$$F_0(x) = q_0 \cdot x, F_i(x) = q_i \cdot x + (q_0 + ... + q_{i-1}), \quad \forall i \in \mathbb{N}, \quad x \in [0,1),$$

(see, e.g., [8, 10] for details about IFS). It is clear that $[0,1)$ is invariant w.r.t. this IFS, and it generates a cylindrical expansion for real numbers from $[0,1)$. Indeed, for any real number $x \in [0,1)$ there exists a unique sequence $\omega = \omega(x) = (\omega_1, \omega_2, ..., \omega_n, ...) \in \{0,1,2,...\}^\infty$ such that

$$x = \bigcap_{n=1}^{\infty} F_{\omega_1} \circ F_{\omega_2} \circ ... \circ F_{\omega_n}([0,1)) =: \Delta^{Q_\infty}_{\omega_1\omega_2...\omega_n...}. \tag{2}$$

The expression $x = \Delta^{Q_\infty}_{\omega_1(x)\omega_2(x)...\omega_n(x)...}, \omega_k \in \mathbb{N}\bigcup\{0\}$ is said to be the Q_∞-expansion of $x \in [0,1)$. In the sequel, we will use the notation $\Delta_{\omega_1(x)...\omega_n(x)...}$ instead of $\Delta^{Q_\infty}_{\omega_1(x)...\omega_n(x)...}$ whenever no confusion can arise. Every point $x \in [0,1)$ has a unique Q_∞-expansion.

The above expansion is actually the f-expansion (see, e.g., [7, 22] for details), which is generated by the following strictly increasing continuous function f defined on $[0,+\infty)$ such that $f(0) = 0$ and f increases linearly on each interval $[n, n+1]$ with $f(n+1) - f(n) = q_n, \forall n \in \{0,1,2,...\}$.

This expansion can also be explained in the following geometric way (see [20]). Given a Q_∞-matrix we consecutively perform decompositions of the interval $[0,1)$.

Step 1. We decompose $[0,1)$ (from the left to the right) into the union of intervals Δ_{i_1}, $i_1 \in \{0,1,2,\ldots\}$ (without common points) of length $|\Delta_{i_1}| = q_{i_1}$,

$$[0,1) = \bigcup_{i_1=0}^{\infty} \Delta_{i_1}.$$

Each interval Δ_{i_1} is called a 1-rank interval.

Step $k \geq 2$. We decompose (from the left to the right) each $(k-1)$-interval $\Delta_{i_1 i_2 \ldots i_{k-1}}$ into the union of k-rank intervals $\Delta_{i_1 i_2 \ldots i_k}$ (without common points)

$$\Delta_{i_1 i_2 \ldots i_{k-1}} = \bigcup_{i_k=0}^{\infty} \Delta_{i_1 i_2 \ldots i_k},$$

whose lengths

$$|\Delta_{i_1 i_2 \ldots i_k}| = q_{i_1} \cdot q_{i_2} \cdots q_{i_k} = \prod_{s=1}^{k} q_{i_s} \tag{3}$$

are related as follows

$$\left|\Delta_{i_1 i_2 \ldots i_{k-1} 0}\right| : \left|\Delta_{i_1 i_2 \ldots i_{k-1} 1}\right| : \cdots : \left|\Delta_{i_1 i_2 \ldots i_{k-1} i_k}\right| : \cdots = q_0 : q_1 : \cdots : q_{i_k} : \cdots .$$

For any sequence of indices $\{i_k\}$, $i_k \in \{0,1,2\ldots\}$, there corresponds the sequence of non-trivial embedded intervals

$$\Delta_{i_1} \supset \Delta_{i_1 i_2} \supset \cdots \supset \Delta_{i_1 i_2 \ldots i_k} \supset \cdots$$

such that $|\Delta_{i_1 \ldots i_k}| \to 0$, $k \to \infty$, due to (3). Therefore, there exists a unique point $x \in [0,1)$ belonging to all intervals Δ_{i_1}, $\Delta_{i_1 i_2}$, ..., $\Delta_{i_1 i_2 \ldots i_k}, \ldots$.

Conversely, for any point $x \in [0,1)$ there exists a unique sequence of embedded intervals $\Delta_{i_1} \supset \Delta_{i_1 i_2} \supset \ldots \supset \Delta_{i_1 i_2 \ldots i_k} \supset \ldots$ containing x, i.e.,

$$x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \ldots i_k} = \bigcap_{k=1}^{\infty} \Delta_{i_1(x) i_2(x) \ldots i_k(x)} = \Delta_{i_1(x) i_2(x) \ldots i_k(x) \ldots}. \tag{4}$$

In the sequel, $\Phi = \Phi(Q_\infty)$ will be the family of all possible cylinders of the Q_∞-partition of the interval $[0,1)$, i.e.,

$$\Phi = \{E : E = \Delta_{\alpha_1 \ldots \alpha_n}, \quad \alpha_i \in \mathbb{N}_0,\ i = 1,2,\ldots,n; \qquad n \in \mathbb{N}\}. \tag{5}$$

In [3] it has been shown that the fine covering system generated by the Q_∞-expansion is not necessarily faithful. In particular, if there exist a positive integer $m_0 > 1$ and real numbers A and B such that

$$\frac{A}{(i+1)^{m_0}} \leq q_i \leq \frac{B}{(i+1)^{m_0}}, \forall i \in \mathbb{N}, \tag{6}$$

then the fine covering system generated by the Q_∞-expansion is non-faithful.

In this paper we develop techniques and study fractal properties of subsets of non-normal numbers w.r.t. the Q_∞-expansions even for the case where the stochastic vector Q_∞ has infinite entropy and the corresponding family $\Phi(Q_\infty)$ is non-faithful.

2. Fractal properties of the set of Q_∞-quasinormal numbers

Let $N_i(x,n)$ be the number of digits «i» among the first n digits of the Q_∞-expansion of x.

Definition 3. If the limit $\lim\limits_{n\to\infty}\frac{N_i(x,n)}{n}$ exists, then its value $\nu_i^{Q_\infty}(x)$ is said to be *the asymptotic frequency of the digit «i» in the Q_∞-expansion of x.*

By the law of large numbers, for Lebesgue almost all real numbers from the unit interval the following equalities hold

$$\nu_i^{Q_\infty}(x) = q_i, \quad \forall i \in \mathbb{N}_0.$$

Definition 4. A real number x is said to be Q_∞-*normal,* if

$$\nu_i^{Q_\infty}(x) = q_i, \quad \forall i \in \mathbb{N}_0.$$

It is clear that the set $N(Q_\infty)$ of Q_∞-normal numbers is of full Lebesgue measure, and, therefore, it is a set of full Hausdorff dimension.

Definition 5. The set

$$W(Q_\infty) = \left\{x : \forall i \in \mathbb{N}_0, \lim_{n\to\infty}\frac{N_i(x,n)}{n} \text{ exists}\right\} \bigcap \overline{N(Q_\infty)}. \tag{7}$$

is said to be the set of Q_∞-quasinormal numbers.

Theorem 1. *The set of Q_∞-quasinormal numbers is of full Hausdorff dimension, i.e.,*

$$\dim_H(W(Q_\infty)) = 1.$$

Proof. Let $Q_\infty = (q_0\, q_1\, q_2\, \ldots\, q_j\, \ldots)$ be a given stochastic vector.

Let $M_k = \{x : \nu_i^{Q_\infty}(x) = \frac{q_i}{s_k}, \forall i \le k; \quad \nu_i^{Q_\infty}(x) = 0, \forall i > k\}$, where $s_k = \sum_{i=0}^{k} q_k$.

Thus $M_k \subset W(Q_\infty)$.

Let us consider the random variable ξ which is defined by $\xi = \Delta_{\xi_1\xi_2\ldots\xi_j\ldots}$, where the random variables ξ_j are given by

ξ_j	0	1	2	...	k	$k+1$	$k+2$	$\ldots$
	$\frac{q_0}{s_k}$	$\frac{q_1}{s_k}$	$\frac{q_2}{s_k}$	...	$\frac{q_k}{s_k}$	0	0	$\ldots$

The above random variable ξ is known to be a random variable with independent identically distributed Q-digits ([24, P. 152]), where Q is given by

$$Q = (q_0, q_1, \ldots, q_k, 1 - \sum_{i=0}^{k} q_i)$$

and P is given by

$$P = \left(\frac{q_0}{s_k}, \frac{q_1}{s_k}, \ldots, \frac{q_k}{s_k}, 0\right).$$

So, we may apply Theorem 6 from [11], which states that the Hausdorff dimension of the distribution of any random variable ξ with independent identically distributed Q-digits can be calculated as follow:

$$\dim_H \mu_\xi = \frac{\sum\limits_{i=0}^{k} \frac{q_i}{s_k}\ln\frac{q_i}{s_k}}{\sum\limits_{i=0}^{k} \frac{q_i}{s_k}\ln q_i}.$$

The set M_k is the spectrum of the measure μ_ξ. Hence,

$$\dim_H(M_k) \geq \dim_H \mu_\xi = \frac{\left(\frac{q_0}{s_k}\ln q_0 + \ldots \frac{q_k}{s_k}\ln q_k\right) - \left(\frac{q_0}{s_k}\ln s_k + \ldots \frac{q_k}{s_k}\ln s_k\right)}{\frac{q_0}{s_k}\ln q_0 + \ldots + \frac{q_k}{s_k}} =$$

$$= 1 - \frac{\frac{\ln s_k}{s_k}(q_0 + \ldots + q_k)}{\frac{q_0}{s_k}\ln q_0 + \ldots + \frac{q_k}{s_k}} = 1 - \frac{s_k \ln s_k}{q_0 \ln q_0 + \ldots q_k \ln q_k}.$$

It is clear that $s_k \to 1$ and $s_k \ln s_k \to 0$ as $k \to \infty$. Therefore, $\dim_H(M_k)$ tends to 1 as $k \to \infty$. It easy to check that $\bigcup_{k=1}^{\infty} M_k \subset W(Q_\infty)$.

By properties of the Hausdorff dimension, we have

$$\dim_H\left(\bigcup_{k=1}^{\infty} M_k\right) = \sup_k \dim_H(M_k)$$

and

$$\dim_H(W(Q_\infty)) \geq \sup_k \dim_H(M_k) = 1,$$

which proves the theorem. □

Remark 1. The above proof is valid for the case where the stochastic vector Q_∞ has a finite entropy as well as for the case where $-\sum_{i=0}^{\infty} q_i \ln q_i = +\infty$, i.e. Q_∞ has infinite entropy.

3. Fractal properties of the set of Q_∞-partially non-normal numbers

Definition 6. The set

$$D(Q_\infty) = \left\{x : \lim_{n\to\infty} \frac{N_i(x,n)}{n} \text{ does not exist for at least one } i \in \mathbb{N}_0\right\} \tag{8}$$

is said to be the set of Q_∞-non-normal numbers.

It is clear that the set of Q_∞-non-normal numbers is of zero Lebesgue measure.

Definition 7. The set

$$P(Q_\infty) = \left\{x : \ \exists i \in \mathbb{N}\cup\{0\} : \ \nu_i^{Q_\infty}(x) \textit{ does not exist}, \ \exists j \in \mathbb{N}\cup\{0\} : \ \nu_j^{Q_\infty}(x) \textit{ exists}\right\}$$

is said to be the set of Q_∞-partially non-normal numbers.

Theorem 2. *The set $P(Q_\infty)$ of Q_∞-partially non-normal numbers is of full Hausdorff dimension.*

Proof. To prove that the set $P(Q_\infty)$ is of full Hausdorff dimension it is enough to show that for any $\varepsilon > 0$ there exists a subset $P(\varepsilon) \subset P(Q_\infty)$ such that $\dim_H P(\varepsilon) > 1 - \varepsilon$.

Let $Q_\infty = (q_0\, q_1\, q_2\, \ldots\, q_j\, \ldots)$ be a given stochastic vector. Let us consider the following set

$$\begin{aligned} P_{s,k} &= \{x: \quad x = \Delta_{\alpha_1^{(1)}(x)\alpha_2^{(1)}(x)\ldots\alpha_{2s}^{(1)}(x)01\alpha_1^{(2)}(x)\alpha_2^{(2)}(x)\ldots\alpha_{4s}^{(2)}(x)0011\ \ldots} \\ &\quad \alpha_1^{(m)}(x)\alpha_2^{(m)}(x)\ldots\alpha_{2^m s}^{(m)}(x)\underbrace{00\ldots0}_{2^{m-1}}\underbrace{11\ldots1}_{2^{m-1}}\ldots, \alpha_j^{(i)} \in \{0,1\ldots,k-1\}\} = \\ &= \{x: \quad x = \Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_j(x)\ldots}, \\ &\quad \alpha_j \in \{0,1,\ldots,k-1\}\,\text{if}\, j \in M_*, \alpha_j = 0\,\text{if}\, j \in M_0, \alpha_j = 1\,\text{if}\, j \in M_1\}, \end{aligned}$$

where $m \in \mathbb{N}$ and

$$M_* = \bigcup_m M_*^m,$$
$$M_*^m = \{r_m + 1, r_m + 2, \dots, r_m + 2^m s\}, r_m = (s+1)(2^m - 2),$$
$$M_0 = \bigcup_m M_0^m,$$
$$M_0^m = \{r_m + 2^m s + 1, r_m + 2^m s + 2, \dots, r_m + 2^m s + 2^{m-1}\},$$
$$M_1 = \bigcup_m M_1^m,$$
$$M_1^m = \{r_m + 2^m s + 2^{m-1} + 1, r_m + 2^m s + 2^{m-1} + 2, \dots, r_m + 2^m s + 2^m\}.$$

Actually, the set $P_{s,k}$ belongs to the family of Cantor-like sets $C[Q_\infty, \{V_j\}]$, where $V_j = \{0, 1, ..., k-1\}$ iff $j \in M_*$; $V_j = \{0\}$ iff $j \in M_0$; and $V_j = \{1\}$ iff $j \in M_1$. It is clear that for any $x \in P_{s,k}$ the limits $\lim\limits_{n\to\infty} \frac{\mathbb{N}_0(x,n)}{n}$, $\lim\limits_{n\to\infty} \frac{N_1(x,n)}{n}$ do not exist and $\nu_i(x) = 0$, $\forall i > k$. So,

$$P_{s,k} \subset P(Q_\infty), \quad \forall s, k \in \mathbb{N}. \tag{9}$$

Let $\gamma_k = \sum\limits_{i=0}^{k-1} q_i$, $h(k) = -\sum\limits_{i=0}^{k-1} \frac{q_i}{\gamma_k} \ln \frac{q_i}{\gamma_k}$, and let us prove that the Hausdorff dimension of the set $P_{s,k}$ is not less then

$$\dim_H(P_{s,k}) \geq \frac{2sh(k)}{2sh(k) - 2s \ln \gamma_k - \ln q_0 q_1}.$$

Let $\{\eta_i\}$ be a sequence of independent random variables with the following distributions:

if $i \in M_*$, then η_i takes values $0, 1, \dots, k-1$ with probabilities $q_0^*, q_1^*, \dots, q_{k-1}^*$, where $q_j^* = \frac{1}{\gamma_k} q_j$;

if $i \in M_0$, then η_i takes the value 0 with probability 1;

if $i \in M_1$, then η_i takes the value 1 with probability 1.

Let μ_ξ be the probability distribution of the corresponding random variable $\xi = \Delta_{\eta_1 \eta_2 \dots \eta_i \dots}$ with independent Q_∞-digits.

Let us determine the Hausdorff dimension of the measure μ_ξ. To this end we introduce an auxiliary random variable ψ with independent Q-digits (see, e.g., [5] for details) connected with the above Q_∞-expansion and defined as follows. If

$$Q_\infty = (q_0, q_1, ..., q_{k-1}, q_k, q_{k+1}, ...),$$

then

$$Q = (q_0, q_1, ..., q_{k-1}, \overline{q_k}), \quad \text{with } \overline{q_k} = \sum_{i=k}^{\infty} q_i,$$

i.e., this Q-expansion is generated by the following $k+1$ similitudes

$$F_0(x) = q_0 \cdot x, F_1(x) = q_1 \cdot x + q_0, ..., F_{k-1}(x) = q_{k-1} \cdot x + (q_0 + ... q_{k-2}),$$
$$F_k(x) = \overline{q_k} \cdot x + (q_0 + ... + q_{k-1}).$$

Then the random variable ψ is defined by

$$\psi = \Delta^Q_{\eta_1 \eta_2 \dots \eta_i \dots}, \tag{10}$$

where the random variables $\eta_1, \eta_2, ...$ are the same as above. From the construction of the random variables ξ and ψ it follows that they have the same probability distribution $\mu = \mu_\xi = \mu_\psi$, which is uniformly distributed on the set $A_{s,k}$.

Fine fractal properties of random variables with independent Q^*-digits were studied in [5]. In particular, an explicit formula for the determination of the Hausdorff dimension for probability measures with independent Q^*-digits was presented there.

Since any random variable with independent Q-digits belongs to the family of random variables with independent Q^*-digits (in such a case all columns of the matrix Q^* are the same), we may apply Theorem 1 from [5]. This theorem states that if $\inf\limits_{i,j} q_{ij} = q > 0$, then the Hausdorff dimension of the random variable with independent Q^*-digits is equal to

$$\varliminf_{n\to\infty} \frac{H_n}{B_n},$$

where

$$H_n = \sum_{j=1}^{n} h_j, \quad h_j = -\sum_{i=0}^{s-1} p_{ij} \ln p_{ij},$$

and

$$B_n = \sum_{j=1}^{n} b_j, \quad b_j = -\sum_{i=0}^{s-1} p_{ij}.$$

In our case $s = k+1$, $q_{ij} = q_i, i \in \{0, 1, ..., k-1\}$, and $q_{kj} = \overline{q_k}$, $\forall j \in \mathbb{N}$.

The probabilities p_{ij} give us the distributions of the above random variables η_j, i.e.,

if $j \in M_*$, then η_j takes the values $0, 1, 2, ..., k-1, k$ with probabilities $p_{0j} = q_0^*, p_{1j} = q_1^*, ..., p_{(k-1)j} = q_{k-1}^*, p_{kj} = 0$, respectively, where $q_j^* = \frac{1}{\gamma_k} q_j$;

if $j \in M_0$, then η_j takes the value 0 with probability $p_{0j} = 1$;

if $j \in M_1$, then η_j takes the value 1 with probability $p_{1j} = 1$.

So, if $j \in M_*$, then $h_j = h(k) = -\sum\limits_{i=0}^{k-1} \frac{q_i}{\gamma_k} \ln \frac{q_i}{\gamma_k}$, $\quad b_j = -\sum\limits_{i=0}^{k-1} \frac{q_i}{\gamma_k} \ln q_i$;

if $j \in M_0$, then $h_j = 0, b_j = -\ln q_0$;

if $j \in M_1$, then $h_j = 0, b_j = -\ln q_1$.

It is also clear that $b_j = h_j + \ln \frac{1}{\gamma_k}$ for $j \in M_*$.

Therefore,

$$\dim_H \mu = \varliminf_{n\to\infty} \frac{H_n}{B_n} = \frac{2sh(k)}{2sh(k) - 2s \ln \gamma_k - \ln q_0 q_1}.$$

Since the set $A_{s,k}$ is the support of the measure μ_ξ, we get

$$\dim_H P_{s,k} \geq \dim_H \mu = \frac{2sh(k)}{2sh(k) - 2s \ln \gamma_k - \ln q_0 q_1}.$$

For a given ε we can choose $k_0(\varepsilon) \in \mathbb{N}$ and $s_0(\varepsilon) \in \mathbb{N}$ such that $\forall k > k_0(\varepsilon)$, $\forall s > s_0(\varepsilon)$ the following inequality holds:

$$\frac{2sh(k)}{2sh(k) - 2s \ln \gamma_k - \ln q_0 q_1} = \frac{1}{1 - \frac{\ln \gamma_k}{h(k)} - \frac{\ln q_0 q_1}{2sh(k)}} > 1 - \varepsilon.$$

Set $k = k_0(\varepsilon) + 1$, $s = s_0(\varepsilon) + 1$. Then $P_{s,k} \subset P(Q_\infty)$ and $\dim_H(P_{s,k}) > 1 - \varepsilon$, which proves the theorem. $\square$

4. Fractal properties of the set of Q_∞-essentially non-normal numbers

Definition 8. The set

$$L(Q_\infty) = \left\{ x : \lim_{k\to\infty} \frac{N_i(x,k)}{k} \text{does not exist}, \forall i \in N_0 \right\} \tag{11}$$

is said to be the set of Q_∞-essentially non-normal numbers.

Theorem 3. *Let Q_∞ be a stochastic vector such that*

$$\sum_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < +\infty. \tag{12}$$

Then the set $L(Q_\infty)$ of Q_∞-essentially non-normal numbers is of full Hausdorff dimension.

Proof. Let s and $l > 2$ be fixed positive integers. Let us consider the following set:

$$\begin{aligned} T_{s,\,l} \quad = \quad & \Big\{ x : \ x \in (0,1), \ x = \Delta_{\underbrace{0\alpha_{1,1}\alpha_{1,2}\ldots\alpha_{1,2s}}_{\text{first group}}\underbrace{001\alpha_{2,1}\alpha_{2,2}\ldots\alpha_{2,2^2s}}_{\text{second group}}} \\ & \underbrace{\overbrace{0\ldots0}^{2^{k-1}}\overbrace{1\ldots1}^{2^{k-2}}\ldots\overbrace{(k-2)(k-2)}^{2^1}\overbrace{(k-1)}^{2^0}\alpha_{k,1}\alpha_{k,2}\ldots\alpha_{k,2^ks}}_{k\text{-th group}}\ldots \, , \\ & \text{where } \alpha_{k,j} \in \{0,1,\ldots,l-1\}, \forall k \in \mathbb{N} \Big\}. \end{aligned}$$

Let us denote by $\mathrm{Fix}(j)$ the set of numbers of positions of the fixed digit «j» in the Q_∞-expansion of $x \in T_{s,\,l}$. Let us describe the set $\mathrm{Fix}(j)$ more precisely.

The fixed digit «j» firstly appears in the $(j+1)$-th group. The fixed digit «j» appears 2^{k-1} times in the $(j+k)$-th group.

The quantity of all positions before the $(j+k)$-th group is equal to

$$\begin{aligned} & (2^1 - 1 + 2^1 s) + (2^2 - 1 + 2^2 s) + \ldots + (2^{j+k-1} - 1 + 2^{j+k-1}s) = \\ & = (2^1 + 2^2 + \ldots + 2^{j+k-1}) + s(2^1 + 2^2 + \ldots + 2^{j+k-1}) - (j+k-1) = \\ & = (2^{j+k} - 2) + (2^{j+k} - 2)s - (j+k) + 1 = \\ & = (2^{j+k} - 2)(s+1) - (j+k) + 1. \end{aligned}$$

The $(j+k)$-th group starts with

$$\left.\begin{array}{rl} 2^{j+k-1} & \text{zeros} \\ 2^{j+k-2} & \text{digits «1»} \\ & \vdots \\ 2^k & \text{digits «}j-1\text{».} \end{array}\right\} \begin{array}{l} \text{in all: } \ 2^k(2^{j-1} + 2^{j-2} + \ldots + 2^1 + 1) = \\ \qquad = 2^k(2^j - 1) \text{ digits.} \end{array}$$

Therefore,

$$(2^{j+k} - 2)(s+1) - (j+k) + 1 + 2^k(2^j - 1) + i, \text{where } i \in \{1,2,3,\ldots,2^{k-1}\}$$

are numbers of the position of the fixed number «j» in the $(k+j)$-th group. Then for any $j \in \{0,1,2,\ldots\} = \mathbb{N}_0$ we get

$$\begin{aligned} \mathrm{Fix}(j) \quad = \quad & \{ n : \ n = (2^{j+k} - 2)(s+1) - (j+k) + 2^k(2^j - 1) + 1 + i, \\ & i \in \{1,2,\ldots,2^{k-1}\}, k \in \mathbb{N} \} \end{aligned}$$

Let

$$\text{Fix} = \bigcup_{j=0}^{\infty} \text{Fix}(j); \quad \text{Flex} = \mathbb{N} \setminus \text{Fix}.$$

Then the set $T_{s,\,l}$ can be defined by

$$\begin{aligned} T_{s,\,l} = \{x : \quad & x = \Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots}; \\ & \alpha_k(x) = j \text{ for all } k \in \text{Fix}(j),\ j \in \text{N}_0; \\ & \alpha_k(x) \in \{0, 1, \ldots, l-1\} \text{ for all } k \in \text{Flex}\}. \end{aligned}$$

The proof can be naturally splitted into a sequence of lemmas. Lemma 1 shows that $T_{s,\,l} \subset L(Q_\infty)$.

Lemma 1. *For any $s \in \mathbb{N}$ and for any $l \in \mathbb{N},\ l > 2$ the set $T_{s,\,l}$ consists of real numbers having no frequencies of any Q_∞-digit.*

Proof. Let us show that for any $x \in T_{s,\,l}$ the limit $\lim\limits_{k\to\infty} \frac{N_0(x,k)}{k}$ does not exist.

Let $m'_k(0) + 1$ be the number of the position at which the k-th group is started, i.e.,

$$\begin{aligned} m'_k(0) &= \\ &= (1 + 2s) + (2 + 1 + 4s) + \ldots + (2^{k-2} + \ldots + 2^1 + 1 + 2^{k-1}s) = \\ &= (2^1 - 1 + 2s) + (2^2 - 1 + 2^2 s) + \ldots + (2^{k-1} - 1 + 2^{k-1}s) = \\ &= (2^k - 2) - (k-1) + s(2^k - 2) = (2^k - 2)(s+1) - k + 1. \end{aligned}$$

Let $m''_k(0) = m'_k(0) + 2^{k-1}$.

$$N_0(x, m'_k(0)) = (1 + 2 + \ldots 2^{k-2}) + \tau_0(x, m'_k(0)) = 2^{k-1} - 1 + \tau_0(x, m'_k(0)),$$

where $\tau_0(x, m'_k(0))$ is the number of zeros among first $m'_k(0)$ digits.

$$N_0(x, m''_k(0)) = 2 \cdot 2^{k-1} - 1 + \tau_0(x, m'_k(0)).$$

$$\frac{N_0(x, m'_k(0))}{m'_k(0)} = \frac{2^{k-1} - 1 + \tau_0(m'_k(0))}{(2^k - 2)(s+1) - k + 1} = \frac{\frac{1}{2} - \frac{1}{2^k} + \frac{\tau_0(m'_k(0))}{2^k}}{(1 - \frac{2}{2^k})(s+1) - \frac{k-1}{2^k}};$$

$$\frac{N_0(x, m''_k(0))}{m''_k(0)} = \frac{2^k - 1 + \tau_0(m'_k(0))}{(2^k - 2)(s+1) + 2^{k-1} - k + 1} = \frac{1 - \frac{1}{2^k} + \frac{\tau_0(m'_k(0))}{2^k}}{(1 - \frac{2}{2^k})(s+1) + \frac{1}{2} - \frac{k-1}{2^k}}.$$

If $x \in T_{s,\,l}$ and the limit $\lim\limits_{k\to\infty} \frac{\tau_0(x,\, m'_k(0))}{2^k}$ does not exist, then the limit $\lim\limits_{k\to\infty} \frac{N_0(x,\, m'_k(0))}{m'_k(0)}$ also does not, and consequently the limit $\lim\limits_{k\to\infty} \frac{N_0(x,\, k)}{k}$ does not exist.

If the limit $\lim\limits_{k\to\infty} \frac{\tau_0(x,\, m'_k(0))}{2^k} = a(x)$ exists, then

$$\frac{N_0(x,\ m'_k(0))}{m'_k(0)} \longrightarrow \frac{\frac{1}{2} + a(x)}{s+1} = \frac{2a(x) + 1}{2s + 2},$$

but

$$\frac{N_0(x,\ m''_k(0))}{m''_k(0)} \longrightarrow \frac{1 + a(x)}{s + 1 + \frac{1}{2}} = \frac{2a(x) + 2}{2s + 3} > \frac{2a(x) + 1}{2s + 2}.$$

Therefore, for any $x \in T_{s,\,l}$ the digit «0» does not have a frequency in the Q_∞-expansion of x.

In the same manner we can see that for any $x \in T_{s,\,l}$ and for any digit «i» the limit $\lim\limits_{k\to\infty} \frac{N_i(x,\,k)}{k}$ does not exist. $\square$

Lemma 2.

$$\dim_H(T_{s,\,l}) = \dim_H(T_{s,\,l}, \Phi).$$

Proof. It is clear that $H^\alpha_\varepsilon(T_{s,\,l}) \leq H^\alpha_\varepsilon(T_{s,\,l}, \Phi)$.

Let us show that $H^\alpha_\varepsilon(T_{s,\,l}) \geq H^\alpha_\varepsilon(T_{s,\,l}, \Phi)$.

Let $E_j = [a_j, b_j]$ be an arbitrary closed interval from some covering of the set $T_{s,\,l}$.

Write $I_j = T_{s,\,l} \cap E_j$.

There exist cylinders containing the set I_j. Let $\Delta^j_{\alpha_1\ldots\alpha_k}$ denote the cylinder of the minimal length among all cylinders containing the set I_j.

Then $(k+1) \in$ Flex. Otherwise the cylinder $\Delta^j_{\alpha_1\ldots\alpha_k}$ should not be the smallest cylinder containing the set I_j. For example, $I_j \subset \Delta^j_{\alpha_1\ldots\alpha_k\alpha_{(k+1)}} \subset \Delta^j_{\alpha_1\ldots\alpha_k}$.

Let $c_j = \inf I_j,\ d_j = \sup I_j$.

Let

$$\Delta^j_{\alpha_1\ldots\alpha_k} := \bigcup_{i=0}^{\infty} \Delta^j_{\alpha_1\ldots\alpha_k i}.$$

Since c_j is the infimum of I_j, we have that $c_j \in \Delta^j_{\alpha_1\ldots\alpha_k 0}$.

Since $T_{s,\,l} \cap \Delta^j_{\alpha_1\ldots\alpha_k i} = \emptyset$, $\forall i \geq l$ and $T_{s,\,l} \cap \Delta^j_{\alpha_1\ldots\alpha_k(l-1)} \neq \emptyset$ and d_j is the supremum of I_j, it follows that $d_j \in \Delta^j_{\alpha_1\ldots\alpha_k(l-1)}$.

Hence $\Delta^j_{\alpha_1\ldots\alpha_k 1} \subset [c_j, d_j]$. Therefore

$$|(c_j, d_j)| > |\Delta^j_{\alpha_1\ldots\alpha_k 1}| = q_1 \cdot |\Delta^j_{\alpha_1\ldots\alpha_k}|.$$

So, $|\Delta^j_{\alpha_1\ldots\alpha_k}| < \frac{1}{q_1}(d_j - c_j) \leq \frac{1}{q_1}(b_j - a_j)$.

Thus for any closed interval E_j we can cover the set $T_{s,\,l} \cap E_j$ by one cylinder of length not larger then $\frac{1}{q_1}|E_j|$.

Therefore, for any $\varepsilon > 0$, any $\alpha \in (0,1]$ and any ε-covering of $T_{s,\,l}$ we get

$$H^\alpha_\varepsilon(T_{s,\,l}, \Phi) \leq \frac{1}{q_1^\alpha} \sum_j |E_j|^\alpha.$$

Hence,

$$H^\alpha_\varepsilon(T_{s,\,l}, \Phi) \leq \frac{1}{q_1^\alpha} H^\alpha_\varepsilon(T_{s,\,l})$$

for any $\varepsilon > 0$ and any $\alpha \in (0,1]$.

By letting ε approach 0 we can prove that

$$H^\alpha(T_{s,\,l}) \leq H^\alpha(T_{s,\,l}, \Phi) \leq \frac{1}{q_1^\alpha} H^\alpha(T_{s,\,l})$$

for any $\alpha \in (0,1]$.

So, $\dim_H(T_{s,\,l}, \Phi) = \dim_H(T_{s,\,l})$. $\square$

Lemma 3. *The Hausdorff dimension of the set $T_{s,\,l}$ is not less then*

$$\dim_H(T_{s,\,l}) \geq \frac{\sum\limits_{i=0}^{l-1} q_i \ln q_i - s_l \ln s_l}{\sum\limits_{i=0}^{l-1} q_i \ln q_i + \frac{s_l}{s} \lim\limits_{j\to\infty} \sum\limits_{k=0}^{j-1} \frac{(2^{j-k}-1)}{(2^j-2)} \ln q_k},$$

where $s_l = q_0 + q_1 + \ldots + q_{l-1}$.

Proof. Let us construct the singularly continuous probability measure such that the set $T_{s,\,l}$ will be the spectrum of the constructed measure.

Let $\xi(l)$ be a random variable with independent Q_∞-digits $\xi_k(l)$ defined by

$$\xi(l) = \Delta_{\xi_1(l)\xi_2(l)\ldots\xi_k(l)\ldots},$$

where $\xi_k(l)$ are defined by probability distributions:

If $k \in \mathrm{Fix}(j)$, then $\begin{array}{c|c} \xi_k(l) & j \\ \hline & p_{jk}=1 \end{array}$, $j \in \mathbb{N}_0$;

If $k \in \mathrm{Flex}$, then $\begin{array}{c|c|c|c|c} \xi_k(l) & 0 & 1 & \ldots & l-1 \\ \hline & p_{0k}=\frac{q_0}{s_l} & p_{1k}=\frac{q_1}{s_l} & \ldots & p_{(l-1)k}=\frac{q_{l-1}}{s_l} \end{array}$,

where $s_l = q_0 + q_1 + \ldots + q_{l-1}$.

Let $\mu_{\xi(l)}$ be the above defined probability distribution of the corresponding random variable $\xi(l)$ with independent Q_∞-digits.

The set $T_{s,\,l}$ is the spectrum of the measure $\mu_{\xi(l)}$.

So,

$$\dim_H(T_{s,\,l}) \geq \dim_H \mu_{\xi(l)}.$$

By Theorem 3.1 from [15], if $\sum\limits_{k=1}^{\infty} \frac{\sum\limits_{i=0}^{\infty} p_{ik} \ln^2 p_{ik}}{k^2} < \infty$ and $\sum\limits_{k=1}^{\infty} \frac{\sum\limits_{i=0}^{\infty} p_{ik} \ln^2 q_k}{k^2} < \infty$, then

$$\dim_H(\mu_{\xi(l)}, \Phi(Q_\infty)) = \varliminf_{n\to\infty} \frac{h_1 + h_2 + \ldots + h_n}{b_1 + b_2 + \ldots + b_n}.$$

In our case $\sum\limits_{k=1}^{\infty} \frac{\sum\limits_{i=0}^{\infty} p_{ik} \ln^2 p_{ik}}{k^2} < \infty$, because

$$\sum_{i=0}^{\infty} p_{ik} \ln^2 p_{ik} = \begin{cases} 0, & \text{if } k \in \mathrm{Fix}, \\ \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln^2 \frac{q_i}{s_l} = c_1(l), & \text{if } k \in \mathrm{Flex}. \end{cases}$$

Let us show that $\sum\limits_{k=1}^{\infty} \frac{\sum\limits_{i=0}^{\infty} p_{ik} \ln^2 q_k}{k^2} < \infty$.

Since

$$\sum_{i=0}^{\infty} p_{ik} \ln^2 q_i = \begin{cases} \ln^2 q_j, & \text{if } k \in \mathrm{Fix}(j), \\ \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln^2 q_i = c_2(l), & \text{if } k \in \mathrm{Flex}, \end{cases}$$

it is sufficient to prove the convergence of the series $\sum\limits_{j=0}^{\infty} \left(\sum\limits_{i\in \mathrm{Fix}(j)} \frac{\ln^2 q_j}{i^2} \right)$.

Let us denote by $m(j)$ the minimal element of the set $\mathrm{Fix}(j)$.

$$m(j) = (2^{j+1} - 2(s+1) - (j+1) + 2(2^j - 1) + 2 > s2^j.$$

Then

$$\sum_{j=0}^{\infty}\left(\sum_{i\in \mathrm{Fix}(j)} \frac{\ln^2 q_j}{i^2}\right) < \sum_{j=0}^{\infty}\left(\sum_{i=m(j)}^{\infty} \frac{\ln^2 q_j}{i^2}\right) \sim \sum_{j=0}^{\infty} \frac{\ln^2 q_j}{m(j)} \leq \sum_{j=0}^{\infty} \frac{\ln^2 q_j}{s2^j}.$$

By assumption, $\sum\limits_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < \infty$. Therefore, $\sum\limits_{j=0}^{\infty}\left(\sum\limits_{i\in \mathrm{Fix}(j)} \frac{\ln^2 q_j}{i^2}\right) < +\infty$.

Hence,

$$\begin{aligned}
\dim_H(\mu_{\xi(l)},\ \Phi)) &= \lim_{n\to\infty} \frac{h_1 + h_2 + \ldots + h_n}{b_1 + b_2 + \ldots + b_n} = \\
&= \lim_{j\to\infty} \frac{\sum\limits_{k=1}^{j-1} 2^k s \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln \frac{q_i}{s_l}}{\sum\limits_{k=1}^{j-1} 2^k s \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln q_i + \sum\limits_{k=0}^{j-1} (2^{j-k} - 1) \ln q_k} = \\
&= \lim_{j\to\infty} \frac{(2^j - 2)s \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln \frac{q_i}{s_l}}{(2^j - 2)s \sum\limits_{i=0}^{l-1} \frac{q_i}{s_l} \ln q_i + \sum\limits_{k=0}^{j-1} (2^{j-k} - 1) \ln q_k} = \\
&= \lim_{j\to\infty} \frac{(2^j - 2)\frac{s}{s_l}\left(\sum\limits_{i=0}^{l-1} q_i \ln q_i - \ln s_l \sum\limits_{i=0}^{l-1} q_i\right)}{(2^j - 2)\frac{s}{s_l} \sum\limits_{i=0}^{l-1} q_i \ln q_i + \sum\limits_{k=0}^{j-1} (2^{j-k} - 1) \ln q_k} = \\
&= \lim_{j\to\infty} \frac{\sum\limits_{i=0}^{l-1} q_i \ln q_i - s_l \ln s_l}{\sum\limits_{i=0}^{l-1} q_i \ln q_i + \frac{s_l}{s} \sum\limits_{k=0}^{j-1} \frac{(2^{j-k}-1)}{(2^j-2)} \ln q_k} = \\
&= \frac{\sum\limits_{i=0}^{l-1} q_i \ln q_i - s_l \ln s_l}{\sum\limits_{i=0}^{l-1} q_i \ln q_i + \frac{s_l}{s} \lim\limits_{j\to\infty} \sum\limits_{k=0}^{j-1} \frac{(2^{j-k}-1)}{(2^j-2)} \ln q_k}.
\end{aligned}$$

So, $\lim\limits_{j\to\infty} \sum\limits_{k=0}^{j-1} \frac{(2^{j-k}-1)}{(2^j-2)} |\ln q_k| \leq \lim\limits_{j\to\infty} \sum\limits_{k=0}^{j-1} \frac{|\ln q_k|}{2^k} = \sum\limits_{k=0}^{\infty} \frac{|\ln q_k|}{2^k} < \infty$, because $\sum\limits_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < \infty$. □

Let $T_s = \bigcup\limits_{l=3}^{\infty} T_{s,\,l}$. taking into account lemma 2 and the definition of the Hausdorff dimension of a measure, we get

$$\dim_H(T_{s,\,l}) = \dim_H(T_{s,\,l}, \Phi) \geq \dim_H(\mu_{\xi(l)}, \Phi).$$

Then

$$\dim_H(T_s) = \sup_l \dim_H(T_{s,\,l}) \geq$$
$$\geq \sup_l \frac{\sum_{i=0}^{l-1} q_i \ln q_i - s_l \ln s_l}{\sum_{i=0}^{l-1} q_i \ln q_i + \frac{s_l}{s} \sum_{k=0}^{\infty} \frac{\ln q_k}{2^k}} = \frac{\sum_{i=0}^{\infty} q_i \ln q_i}{\sum_{i=0}^{\infty} q_i \ln q_i + \frac{1}{s} \sum_{k=0}^{\infty} \frac{\ln q_k}{2^k}}.$$

Finally,

$$\dim_H(L(Q_\infty)) \geq \sup_s \dim_H(T_s) = 1.$$

□

Remark 2. From the proof of the latter theorem it is clear that the assumption $\sum_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < \infty$ was necessary for using Theorem 3.1 from [15]. It is natural to ask whether this condition is just a technical one and whether it is possible (by using other techniques) to show that the Hausdorff dimension of the set $T_{s,l}$ is close to 1 for large enough s and l. Unfortunately this is not the case. Let us show that one can choose a stochastic vector Q_∞, such that $\sum_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} = \infty$, and for any fixed $s \in \mathbb{N}$ and any fixed $l \in \mathbb{N}$ the Hausdorff dimension of the set $T_{s,\,l}$ equals to zero.

Let for a fixed $s \in \mathbb{N}$ and a fixed $l \in \mathbb{N}$, the elements of the stochastic vector Q_∞ be defined by

$$q_i = \frac{A}{(l^{2^{i+1}s})^{i+1}},$$

where $\frac{1}{A} = \sum_{i=0}^{\infty} \frac{1}{(l^{2^{i+1}s})^{i+1}}$.

From the construction of the set $T_{s,\,l}$ it follows that this set can be covered by $l^{2s} \cdot l^{2^2 s} \cdot \ldots \cdot l^{2^{k-1}s}$ cylinders of rank m, where m is the order number of the position of the last fixed digit in the k-th group.

It is obvious, that $l^{2s} \cdot l^{2^2 s} \cdot \ldots \cdot l^{2^{k-1}s} = l^{(2^k-2)s} < l^{s \cdot 2^k}$.

The length of each cylinders of the covering is not larger than

$$q_0^{2^k-1} \cdot q_1^{2^{k-1}-1} \cdot \ldots \cdot q_{k-2}^2 \cdot q_{k-1} < q_{k-1} = \frac{A}{(l^{2^k s})^k}.$$

Therefore, for any positive α the α-volume of the latter covering of the set $T_{s,\,l}$ does not exceed the value

$$l^{s \cdot 2^k} \cdot \left(\frac{A}{(l^{2^k s})^k}\right)^\alpha = A^\alpha \cdot \frac{l^{s \cdot 2^k}}{(l^{s \cdot 2^k})^\alpha} = \frac{A^\alpha}{(l^{s \cdot 2^k})^{\alpha k - 1}},$$

which tends to zero as k tends to infinity.

So, $\dim_H(T_{s,\,l}, \Phi) = 0$.

For the completeness of the metric and topological classification of real numbers via the asymptotic behavior of their digits in $Q_\infty-$expansion, we mention the following result.

Theorem 4 ([3])**.** *The set $L(Q_\infty)$ of Q_∞-essentially non-normal numbers is of the second Baire category.*

Remark 3. The latter theorem shows that Q_∞-essentially non-normal numbers are generic from the topological point of view.

Summarizing, we have

	Lebesgue measure	Hausdorff dimension	Baire category
$N(Q_\infty)$	1	1	first
$W(Q_\infty)$	0	1	first
$P(Q_\infty)$	0	1	first
$L(Q_\infty)^*$	0	1	second

* — in the case where $\sum\limits_{j=0}^{\infty} \frac{\ln^2 q_j}{2^j} < +\infty$.

Some open problems.

1) We strongly believe that the set of Q_∞-essentially non-normal numbers is of full Hausdorff dimension for any choice of the stochastic matrix Q_∞, but up to now this conjecture is still open.

2) The superfractality of the set of essentially non-normal numbers has been proven for a series of different expansions for real numbers (see, e.g., [2, 1, 16, 17, 18, 21]). So, it is naturally to ask whether there exist expansions f such that:

2.1. the corresponding set $L(f)$ of f-essentially non-normal numbers is not of full Hausdorff dimension;

2.1. the whole set $D(f)$ of f-non-normal numbers is not of full Hausdorff dimension?

Acknowledgment

This work was partly supported by SFB-701, DFG KO 1989/6-1 project and by the Alexander von Humboldt Foundation.

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